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EDITOR'S FORWARD

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Special thanks must go from all of us who attended the conference to the organizers, and particularly to Vi Maeers who did so much before and during the meeting to make it such an enjoyable and profitable event.

I hope these proceedings will help generate continued discussion on the many major issues raised during the conference.

Martyn Quigley

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Lecture One

Understanding = Doing + Seeing?

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Analyzing Mathematical Understanding

1.1 The problem: what are the ingredients of mathematical understanding and how are they related to each other?

A few month ago I had the good fortune to meet one of the most accomplished of living mathematicians, a newcomer from the former Soviet Union to the United States. Well into his seventies, he immediately adopted a student-teacher mode of communication. This didn't surprise me, since this exceptional person is well known for his interest in education and for his many activities aimed at bringing mathematics to young minds as well as at bringing the young minds to the world of mathematics. He declared that before we start talking about teaching and learning he must put me to a mathematical test; and then he posed the question: *How many vertices and how many edges are there in a 4-dimensional cube?* "But don't you try to formulate the definition of n-dimensional cube," he warned. "Just think."

There was hardly anything else I could do. Had I remembered Davis and Hersh's (1981) remarks on this problem, I would immediately have struck upon the answer my interlocutor had in mind. Since, however, I had no recollection of the story, I resorted to the most obvious of methods: I tried induction. Starting with a straight-line segment, which may be regarded as a one dimensional cube, and proceeding through a square (2-dimensional cube) to the only "real" (3D) cube, I quickly found that the number of vertices is 2^N, where N is the dimension. The story of the edges proved a little bit more complicated. However, before I had time to look for a formula, the mathematician volunteered his method: "Start with 0-dimensional cube, with a point. Now, to get a one-dimensional cube from this one, all you have to do it to stretch the point to a line segment along, say, the *x*-axis." Immediately, I knew how to go on: "Good, so in order to get to the 2-dimensional cube we now move the segment along the y-axis, and then we make a transition to 3-dimensions by shifting the resulting square along the *z*-axis. One can now see clearly that the number of edges in (N+l)-dimensional cube is twice the number of edges in the N-dimensional cube plus the number of vertices in N-dimensional cube (you must add the number of vertices, because when you carry N-dimensional cube along a new dimension, each one of its moving vertices produces a new edge)."

"Good," said my instructor, "Now you have a good sense of what this 4-dimensional cube is all about; now you understand." But did I? I wasn't sure. True, I could do things that allowed me to get results and I could even do it very quickly. Even so, I didn't really feel I *understood* the concept of the 4-dimensional cube. Something was missing: I couldn't *see* this cube the way I see a segment or a square — not even with my mind's eye.

This brings me to the question of the meaning of the term 'mathematical understanding.' What does it mean to understand mathematical concepts? What are the ingredients of such understanding? How are these ingredients related to each other?

In mathematics, the notion of understanding seems to be one of the most stubborn unknowns of all. Numerous interpretations of the elusive term were offered in the past in relation to variety of contexts, mathematics being but one of many. The interesting thing is that most approaches are grounded in dichotomies, in the claim that, basically, there are two types of understanding. Or rather, that there is a full spectrum of "understandings," whereas the two extrema of such spectrum are distinct enough to be clearly contrasted with each other. It is only natural to look for two elementary ingredients which, when combined in different proportions, result in these different kinds of understanding. One glance on the approaches offered so far by psychologists of mathematical thinking suffices to notice a common thread going through most of the definitions: whether mathematical understanding is divided into *instrumental* and *relational* (Skemp, 1976), into *procedural* and *conceptual* (Hiebert and Lafevre, 1986), into *abstract* and *algorithmic* (Halmos, 1985), or into *operational* and *structural* (Sfard, 1991), there is the same kind of distinction underlying all these classifications: the distinction between *doing* and *seeing*. Or, to put it in different language, all the definitions imply that understanding mathematics is somehow related to two abilities: the ability *to perform* *mathematical processes* and the ability *to see the mathematical objects* to which the processes are applied. This is why the equation appearing in the title presents understanding as a combination of seeing and doing. This paper is devoted to a discussion of the role of both these ingredients. In the reminder of this section, I shall approach the subject theoretically, grounding my position in things said in recent past by philosophers, psychologists, and linguists. Then, in the second section, I shall take a closer look at the relative role of doing and seeing in understanding mathematics by three different populations: students, teachers and mathematicians. In the last part I shall close the discussion with some questions and suggestions regarding possible didactic implications of what has been said in the former two sections.

1.2 Is understanding anything more than an ability to do?

The problem of understanding, in general, and the question of its elementary ingredients in particular, are certainly not new. Since antiquity, philosophers have been devoting their writings to this exciting subject. That understanding entails ability to perform certain actions has always been considered obvious and indisputable. However, while many thinkers seemed to be in agreement that understanding means more than the ability to do, some others argued that the claim about anything as tangible and non-measurable as "seeing with one's mind's eye" must be dropped. Ludwig Wittgenstein, one of the most influential modern philosophers in recent history, belongs to the letter school of thought. In *Philosophical Investigations*, Wittgenstein (1953) begins his reflections on understanding with the story:

> Let us imagine the following example: *A* writes a series of numbers down; *B* watches him and tries to find a law for the sequence of numbers. If he succeeds he exclaims "Now I can go on!" — So this capacity, this understanding, is something that makes its appearance in a moment. So let us try and see what it is that makes its appearance here ... (151; p. 59).

Following a lengthy discussion in which he tries many different approaches to the issues of meaning, explanation, and understanding, Wittgenstein finally decides to stop looking for occurrences that take place inside our mind:

> Try not to think of understanding as a 'mental process' at all. — For that is the expression which confuses you. But ask yourself: in what sort of case, in what kind of circumstances, do we say, "Now I know how to go on," when, that is, a formula has occurred to me? *In the sense in which there are processes (including mental processes)* which are characteristic of understanding, understanding is not a mental process. (154, p. 62)

Instead of dealing with the elusive concept of mental state we should focus on externally observable factors, such as a person's ability to actually do things. According to Wittgenstein, grounding the concept of understanding in anything that is only accessible by introspection would be philosophically useless or even worse than that: it would be confusing. The refusal to think about understanding as a mental state was a natural thing to do for a philosopher who promoted a "third person view" of the world (as opposed to "first person view"), denied the possibility of "private language" and proposed to replace talks about the meaning of our verbal expressions with a discussion of the ways these expressions are used. According to Wittgenstein, the most useful thing we can do is to view understanding as something 'akin' (bearing 'family-resemblance') to the ability to do certain things. Thus, if Wittgenstein participated in our present discussion, he would probably object to talking about "seeing" as an important ingredient of understanding; after all, in the case of mathematics, "seeing" is a private experience which can only be approached through self-examination.

The very use of the term "seeing" in the equation appearing in title of this talk implies that I decided not to listen to Wittgenstein's exhortation to limit analysis of understanding to its more tangible and measurable manifestations. Before, however, I am accused of either ignorance or arrogance, let me explain my reasons. Wittgenstein's advice is methodological-pragmatic rather than anything else: we should not talk about the things we have no way to put our finger on. The question remains open, however, whether it would be useful to talk about anything more than the ability "to go on" for any practical reason. In this paper I will argue that from the point of view of teaching and learning it is important to realize that understanding does include more than an *ability to do*. *Ability to see* is the name given here to this other ingredient of understanding. It should be stressed that the "seeing" I am talking about should not be equated with the concepts of visualization, imagery or mental representations. Although my theme has a distinct bearing on these topics, I would not like to make unnecessary commitments by formulating exact descriptions. Clearly the term "seeing" is used here metaphorically, and this use is justified by the fact that we often substitute the expression "I understand" with the phrase "I see." The hope is that the word will become meaningful by contrast with *the ability to do*. To put it in a different language, *the term "ability to see" should be understood in the present context as referring to all these aspects of understanding which obviously are there, as I will try to show, but are not included in or explainable by the ability to do*. What the defining features of this "seeing" are will, hopefully, become gradually clearer as our discussion goes on.

That the phenomenon of understanding cannot be fully understood without considering more than behavioural aspects has already been signaled in my opening example. As I have pointed out, even though I could easily do whatever was necessary to solve the puzzle of the 4-dimensional cube, I clearly felt that my understanding of the central concept was far from satisfactory and that some crucial ingredients were missing. Paul Halmos (1985), a well known mathematician, provides us with an even more convincing illustration of the same phenomenon:

> ... I was a student, sometimes pretty good and sometimes less good. Symbols didn't bother me. I could juggle them quite well ... [but] I was stumped by the infinitesimal subtlety of epsilonic analysis. I could read analytic proofs, remember them if I made an effort, and reproduce them, sort of, but I didn't really know what was going on. (p. 47)

Thus, Halmos' ability to do things was clearly not enough. Even though he could "juggle symbols," read analytic proofs, remember these proofs and reproduce them, he still had a distinct feeling that he had not understood "epsilonic analysis" as well as he would wish to. And his longing for something more than the ability to do things was not just a fantasy. Some time later, things took a fortunate turn:

> ... one afternoon something happened. I remember standing at the blackboard in Room 213 of the mathematics building talking with Warren Ambrose and suddenly I understood epsilon. I understood what limits were, and all of that stuff that people were drilling in me became clear. I sat down that afternoon with the calculus textbook by Granville, Smith, and Longley. All of that stuff that previously had not made any sense became obvious ... (Albers & Alexanderson, 1985, p. 123)

Clearly, therefore, what Halmos calls the 'true' understanding or 'real knowing' involves something that goes beyond the operative ability to solve problems and to prove theorems. Even if difficult to describe, to analyze, and to measure, this additional constituent cannot be ignored by the students of teaching and learning. To defend this position even more strongly, let me summon the voices of contemporary thinkers, for whom education is not necessarily the main concern but who nevertheless make a similar claim. The thesis that the discussion on understanding should not be limited to behavioural aspects finds support in recent developments in cognitive science.

First, let me remark that by postulating the ability to see as one of the basic components of understanding I touched upon the perennial body-mind dilemma. This quandary has many disguises and it goes like a thread of scarlet through the history of human philosophical and scientific thought. Wittgenstein's logical behaviourism alone can be opposed to many past and recent schools of thought. Among others, it can be contrasted with the mentalist doctrine recently suggested by Chomsky and Fodor, and also with the intentionalism first formulated by Brentano and currently developed by Searle. The importance of deep understanding of the relationship between the mental and the physical in the context of human thinking and functioning was never of greater practical importance than it is today, in the view of its prospective impact on information processing technology. The historical disputes on the body-mind dilemma have been echoed in the ongoing discussion between proponents and opponents of "strong AI."

One of the most powerful and convincing attacks on the claim that it may be possible to program a computer in such a way that it will behave in a truly intelligent manner, as if it was endowed with human-like capacity for understanding, was launched by the American philosopher John Searle. Searle is fiercely opposed to the view that "the brain is just a digital computer and the mind is just a computer program" (Searle, 1984, p. 28). The reason for Searle's denial of this position is his deep conviction that, indeed, to understand means more than knowing how to behave and being able to apply an appropriate algorithm whenever necessary. To support this claim Searle concocted his famous Chinese Room allegory. This is a story of a person — in fact, Searle himself — sitting in a room full of Chinese written texts. Searle does not know Chinese, but being also endowed with English, purely syntactical, prescriptions for constructing Chinese responses ('answers') to various Chinese statements ('questions'), he is able to gradually develop an ability to behave as if he did understand Chinese. Indeed, his Chinese behaviour and his English behaviour are practically indistinguishable from the point of view of an external observer:

> Suppose that after a while I get so good at following the instructions for manipulating Chinese symbols ... [that] from the external point of view — from the point of view of somebody reading my "answers" — the answers to the Chinese questions and the English questions are equally good (after Hofstadter and Dennet, 1981, p. 33)

In spite of this striking similarity of behaviours, there is an important difference between the two situations. Searle stresses this difference time and again: "In the Chinese case, unlike in the English case, I produce the answers by manipulating uninterpreted formal symbols" (ibid). And he summarizes, leaving no doubt as to the necessity of considering more than behavioural aspects when talking about understanding:

> The whole point of the parable of the Chinese room is to remind us of a fact that we knew all along. Understanding a language, or indeed, having mental states at all, involves more than just having a bunch of formal symbols. It involves having an interpretation, or a meaning attached to these symbols. (Searle, 1984, p. 33)

The component of meaning that Searle stresses so forcefully here is an epistemological counterpart of the psychological phenomenon of "seeing" I am talking about in this paper. Once again, without making any explicit commitments as to the exact sense of the word, I include in the seeing anything that constitutes understanding but goes beyond the mere knowledge of rules or the ability to follow them.

1.3 Seeing and doing complement each other

In the light of the last section a crucial question arises: What does it mean "to see" in the context of mathematics, this most abstract of intellectual disciplines? Formalists would probably say that here, the only things to be observed with one's eyes — whether with those situated on both sides of one's nose, or with those placed in the mind — are mathematical symbols. For them, to understand a formula, $5x^2 + 3px$, for example, would just mean being able to manipulate it according to the rules of algebra. Although today many educators would instinctively shrug at this last sentence, there are psychologists who take the exhortation to restrict the discourse to the 'third person view' seriously, and there are mathematicians for whom manipulations on algebraic formulae are the only possible source of the meaning of formal symbols. The nineteenth century British mathematician D. F. Gregory (1840) became one of the leading representatives of this last group when he defined algebra as a discipline "which treats the combination of operations defined not by their nature, that is by what they are or what they do, but by the laws of combinations to which they are subject."

I already summoned Searle and his Chinese room to my support when I claimed that for the purpose of psychological analysis, as well as for many other reasons, it would be a methodological mistake, to say the least, to restrict the analysis of mathematical understanding to the scrutiny of people's ability to perform formal manipulations. Searle was talking about the necessity to consider "meaning attached to symbols." In the present section I will elaborate on the theme of the *meaning of mathematical symbols* and will try to explain why the verb "seeing" is pertinent to this context.

Since, however, this word in its original sense applies, first and foremost, to what belongs to the perceptibly accessible reality, let me begin with an analysis of the role of the sense of sight in our understanding of the physical world. In order to this, I shall use an example inspired by Searle's parable. Instead of the Chinese room, imagine entering an ordinary messy room, full of different pieces of furniture, with many odd objects scattered all over the place. Imagine also that your aim is to get from the door to the window. While doing so, you must be careful about your steps. Since, however, this messy room is a part of the world you have deeply engraved in your mind, your decisions are made in a most natural, non-reflective manner. Instinctively, you try to get to your target by following the shortest path. Sometimes, in order to keep the direction, you remove an object lying there on your route; in some other cases, when the object in your way seems too heavy to be easily moved, you just make a small detour. Whatever you do is not so much an effect of a reflection as a result of your deep understanding of the world in which you have to make your way. Your profound sense of the nature of the objects which furnish this world gives a clear direction to your actions and informs your operative decisions.

In order to be able to decompose understanding into its basic components I must now introduce a new element to the story: an external observer — another person, who actually does nothing except to watch you moving through the room and try to make sense of your actions. Let us concentrate on her understanding of the situation. Such understanding is not too difficult to attain. After all, the observer and you share the same world, the same experiences. It is easy to have a good grasp of your aims and the reasons for your actions just by looking at you moving through the room. We could test the observer's understanding of the situation by asking her to repeat your actions.

Now imagine, however, that rather than moving through a real messy room, you are engaged in playing a *virtual reality game*. Intricate computerized equipment — a helmet and a glove — allows you to feel *as if* you were, indeed, in a messy room. If the game is well-designed, you can see almost no difference between this situation and the former one, when you were in the real room. You see the same objects, you make the same decisions, you perform the same actions. How about the observer, however? For her, no doubt, the situation has changed dramatically. True, she can still see you doing things, but the moves of your empty hands and the sudden turns and bends of your body can no longer make much sense to her. In her eyes you just look funny, very funny. If asked to repeat your actions, she would probably feel utterly lost.

This time, one cannot say that the observer understood the situation. She did not understand it in spite of the fact that she was watching the same behaviour, the same actions as before. In the first case, her understanding could be confirmed by her description of the moves she would do herself in the same situation. This time, even though she has a full grasp of the rules of behaviour in a messy room, she would not be able to repeat your actions. What is missing, this time, is the element of seeing. The observer could not see the objects scattered on your way, the objects you moved or tried to circumvent, the objects which made you behave the way you did. Awareness of these objects, however, was indispensable to render your actions meaningful.

The example of the messy room made it possible to separate the component of seeing from the purely operative aspects of understanding. It has shown that being able to see — or, more generally, to grasp the existence of certain objects — is an important component in our ability to deal meaningfully with the physical world. Incidentally, by now it should be clear that the word "seeing" is used in this paper as a metonym for a perceptual or as-if perceptual grasp of a certain reality — either physical or virtual. In his insightful study of the experiential roots of understanding, Johnson (1987) states that

> ... *understanding is the way we 'have a world', the way we experience our world as comprehensible reality* ... our understanding *is* our mode of 'being in the world' ... Our more abstract reflective acts of understanding ... are simply an extension of our understanding in this more basic sense of 'having a world'. (p. 102)

Our example should clarify what 'having the world' is all about. In mathematics, where there are no palpable objects to guide a person through the maze of messy abstract spaces, one has to envision abstract objects in order to have a sense of direction. These abstract constructs, even though incommunicable to others, are often clearly seen by the mathematizing person with her or his mind's eye.

The importance of such virtual mathematical reality cannot be overemphasized. One can bring many examples showing why having the tangible world of even most sophisticated but empty formal symbols is definitely not enough to have a deep understanding of mathematics. Many such examples can be found in the paper dealing with learning and teaching of algebra by Liora Linchevsky and myself (see Sfard and Linchevski, 1994). In that paper we argue that being able to see different mathematical objects behind the same symbols and the ability to adjust one's perspective to the requirements of the problem at hand necessarily underlie every effective mathematical activity. Taking symbols at their face value would not be enough to tackle such an equation as $(p + 2q)x^2 + x = 5x^2 + (3p - q)x$. Here, one must see through symbols rather than just see symbols; before the problem-solver can undertake any concrete action, he or she has to decide whether the equation is numerical or functional — whether the objects hiding behind the two component formulae represent numbers or functions. And this is only one example out of many.

In fact, most mathematical activities require constant alternating between *seeing symbols* and *seeing through* symbols. One has to see abstract objects represented by the signs in order to make operative decisions, and one must forget all about these intangible entities if one is to capitalize on the possibility of fast mechanical symbol manipulations. Indeed, the semantic 'cargo' would put an unnecessary strain on the person's working memory and would thus make the whole algorithmic operation much less effective. The alternations between the different ways of looking at symbols, or rather between symbols' visibility and invisibility, is at the very heart of any effective mathematical activity. After Lave and Wenger (1991) I would say that "these two crucial characteristics are in a complex interplay, their relation being one of conflict and synergy." Lave and Wenger use a persuasive metaphor to describe the role of the visibility-invisibility alternations:

> It might be useful to give a sense of this interplay by analogy to a window. A window's invisibility is what makes it a window, that is, an object through which the world outside becomes visible. The very fact, however, that so many things can be seen through it makes a window itself highly visible, that is, very salient in a room, when compared to, say, a solid wall.

Even though Lave and Wenger refer in their writing mainly to technology and other physical tools, this metaphor seems adequate — and quite enlightening — when transferred into the context of mathematical symbolism.

Today, the majority of researchers seem to agree that in mathematics, the ability to do things must be complemented by some kind of "inner seeing." For example, among the eight modes of understanding postulated by Kieren and Pirie's "recursive" theory, there are "image making" and "image having." The researchers say that for the students "at the image having level … mathematics *is* the image they have and their working with that image." (Pirie and Kieren, 1994)

I wish to stress, once again, that the present discussion of the aspect of seeing in mathematical thinking goes well beyond the ability to create mental images. Imagining concrete pictures is certainly a part of the story, but it is not more than a part. Visualizing mathematical objects is often helpful, but it is not indispensable for understanding their nature and feeling their existence. On the contrary, even though mathematicians tend to agree that picturing mathematical structures is an important and quite common phenomenon, too concrete an image would hinder thinking rather than promote understanding. For instance, Hadamard (1949), who admits the importance of mental "drawings," stresses at the same time that the schemes he builds must always be "of vague character, as not to be deceptive." Similarly, Thurston (1994) warns that "words, logic, and detailed pictures rattling around can inhibit intuitions and associations" (p. 165).

In this paper, I include in seeing all the components of "virtual perception" — all the aspects of understanding which display characteristics similar to those of sensory perception of any kind: "to see" mathematical object means, among other things, to be able to think about many different components of the situation simultaneously and in a holistic manner and to feel as if there really was some permanent entity that exists independently of whether we think about it or not.

My recurring stress on the links and analogies between mathematical thinking and the perception of physical reality is not accidental. As I argued at length elsewhere (Sfard, 1991, 1994), I believe that sensory experience is what shapes our abstract reasoning and understanding. The intangible objects that populate the virtual realm of mathematics can be viewed as metaphorical reflections of our sensory experience. In their seminal work on the role of our perceptual capacities in shaping our vision of the world and of our imagination, Lakoff and Johnson (1980) explain why thinking in terms of abstract objects is a part and parcel of our understanding:

> Understanding our experiences in terms of objects and substances allows us to pick out parts of our experience and treat them as discrete entities or substances of uniform kind. Once we can identify our experiences as entities or substances, we can refer to them, categorize them, group them, and quantify them — and, by this means, reason about them. (p. 25)

I cannot conclude the theoretical reflections on the role of seeing and doing in understanding mathematics without stressing once again the complementary nature and the mutual dependence of these two abilities. Using the story of the messy room I have already shown that the ability to see underlies the ability to do. In mathematics, unlike in the physical world, the opposite is true as well: seeing mathematical objects would not be possible without intensive mathematical doing. According to the theory of process-object duality of mathematical concepts, mathematical objects are reified mathematical processes — they are entities that appear at those junctions in the development of mathematical concepts where certain well-known processes are to become "inputs" to some higher level processes; to fit this new role, the "input" processes have to be "squeezed" into object-like capsules. To put it differently, the things we see in the realm of mathematics with our minds' eyes are emergent phenomena brought into existence by the mathematical activity itself. Many kinds of evidence could be given to support this thesis. Since, however, I began this paper with a story of 4-dimensional hypercube, no example would be more pertinent than the one referring to the same concept. Let me remind: in the opening example I claimed that in spite of the ability to operate with the concept of 4-D cube, I did not feel that I fully grasped the idea — that I could see the hypercube with my mind's eye. Thanks to Davis and Hersh (1981) I can finish the story with a happy ending: one of the authors of *Mathematical Experience* was able to bring the hypercube into as-if tangible existence just by manipulating it — by turning around its different 3-dimensional projections presented to him by a computer:

> I tried turning the hypercube around, moving it away, bringing it up close, turning it around another way. Suddenly, I could *feel* it! THE HYPERCUBE HAD LEAPED INTO PALPABLE REALITY, AS I LEARNED HOW TO MANIPULATE IT, feeling in my fingertips the power to change what I saw and change it back again. The active control at the computer console created a union of kinaesthetic and visual thinking [of doing and seeing] which brought the hypercube up to the level of intuitive [structural] understanding.

This seems to me one of the most persuasive examples showing how mathematical objects can emerge out of a stubborn mathematical doing. There are many others.

Different Recipes for Understanding Mathematics

The focus of this section will be on personal attitudes of different people toward the issue of understanding. To put it more precisely, I will address the question of the requirements and expectations a person may have with respect to his or her own understanding of mathematics. It is important to stress that unlike in many traditional studies, the main interest will not be here in any externally measurable manifestations of understanding. Rather, it will be an attempt to bring 'first person's views': I will try 'to get into people's own heads' to learn as much as possible about those elusive mental states which can be described as states of understanding. In particular, I will be interested to find out the relative importance ascribed by different people to the ability to see abstract objects as opposed to the ability actually to do things. Just to remind ourselves, mental states were once banned from the philosophical discourse by Wittgenstein but have been recently brought back to grace by those interested in scientifically useful and technologically applicable conceptions of mind. In what follows I will turn to representatives of three different populations: research mathematicians presently active in the field, students for whom learning mathematics is not always a matter of their own decision, and teachers whose task it is to 'inculcate' mathematics into other people's head. As it will soon become clear, different people may have quite different conceptions as to what understanding mathematics is all about. Indeed, when talking to mathematicians, students, and teachers, I have found, basically, three 'prescriptions for understanding', each one of them offering a unique blend of seeing and doing.

2.1 The mathematician's formula: mostly seeing

My first object of interest is the mathematician — the person who may be expected to have more to say on mathematical understanding than anybody else. I interviewed quite a few representatives of this group, asking each one of them to describe as well as they possibly could the mental states that may be described as states of understanding (see more exhaustive account of these interviews in Sfard, 1994). In the absence of any other ways to get into mathematicians' heads, introspection was the only means of inquiry I could think about. Needless to say, self-examination is not the best "scientific" tool one can imagine. In his recent paper for *The Bulletin of AMS*, mathematician Thurstone (1994) warns that even for somebody as experienced as himself, talking about understanding mathematics may not be an easy matter at all. "Understanding is an individual and internal matter that is hard to be fully aware of, hard to understand, and hard to communicate." Nevertheless, Thurstone is able to make some helpful remarks about the components of mathematical understanding. He clearly emphasizes the abilities that are structural rather than operational or, to put it in our present language, that have to do with seeing rather than with doing. Indeed, he speaks about the importance of "vision, spatial sense, kinesthetic (motion) sense," all of which have their roots in the ways we interact with a physical reality.

The central role of the ability to "somehow grasp a structure" was a leading motif in all the mathematicians' testimonies I was able to collect. The ability to see seemed, therefore, the principal need of the mathematicians, the need much more obvious than their need to be able actually to do things. Nowhere else did this preference for seeing come through more forcefully than in this statement by one of my interviewees:

> There are mathematicians who try to see the general structure first of all. I sense that most mathematical activities don't require algorithmic thinking at all. The main thing is to see a structure, somehow. Only after you have this general picture you start thinking about details. I believe that the majority of good mathematicians proceed in this way. I am one of these people who do not think algorithmically. I don't use manipulations when I try to understand a new concept. Not even manipulations in the most abstract sense of the word. When I have a new concept, I need, first and foremost, a metaphor. A human metaphor — personification of the concept. Or a spatial metaphor. A new metaphor of a structure. Only when I have it can I answer questions, solve problems, perform manipulations. I can do all this only *after* I have the metaphor.

Although all the mathematicians I was able to talk to remarked that some of their colleagues may have different needs, none of them was able to produce an actual example of the "different type of mathematical mind." Neither did I find much evidence for the existence of this other type in the literature. I feel, therefore, that it would be justified to conclude that for the great majority of people who deal with mathematics on an everyday basis, having this as-if sensory perception of mathematical constructs is the most important, crucial component of understanding, whereas the ability to perform manipulations is somehow secondary, being but a natural outcome of the ability to see.

2.2 Student's formula: more often than not - mostly doing

The next group I shall turn to now is the largest: it is the group of those who learn mathematics as an obligatory part of their curricula. The mathematician I quoted in the last section was able to recall his experience as a student:

> Also as a child I couldn't manage symbolic manipulations and couldn't arrive from one thing to another without seeing the links between one entity to another. I knew what to do and how to do it only after I could see that important something that was hidden behind the symbols. It was because of this particular need of mine, this inability to manipulate empty symbols, that I didn't score too high in mathematics until I was 16 year old, until my last year in school. Only on the final examinations my grades jumped to the maximum.

My interviewee emphasized in the statement I quoted earlier that in his research work, he needs "to see" before he can do anything productive. Now it is clear that he displayed this preference already as a child. My observations and interviews with high school students show, however, that this attitude is rather rare among young people. Indeed, it may well be that it was precisely because of my interviewee's special ability (and need) to have as-if sensory perception of the virtual world of mathematical objects that he eventually became a mathematician. As I will try to show now, the cognitive preferences displayed by the mathematicians seem quite unique when compared with the formulae for understanding implicitly held by the majority of young learners.

My first example comes from the ongoing project on teaching and learning algebra carried out in Montréal by Carolyn Kieran and myself. In this project, twelve-year old children were introduced to algebra in such a way as to gain an ability "to see" certain abstract objects ("unspecified number," a function) well before any symbolism was offered. In this short talk it would be impossible to give the details of the teaching method or of the actual learning processes that took place during the two parallel thirty-hour long courses. The reader, therefore, has no choice but to rely on our preliminary observations and summaries, according to which the students had a pretty good grasp of numbers and functions as the referents of algebraic formulae well before they started to manipulate the formal expressions. Later, the basic rules of equivalence of linear formulae were discovered by the children themselves in the course of graphically supported activities with functions (adding functions, multiplying a function by a number, etc). As could thus be expected, when it came to manipulating algebraic expressions, the children had no difficulty with either doing what was necessary or with explaining their decisions in terms of functions and graphs. Needless to say, we were all quite pleased both with the children and with ourselves: at long, long last, here is a method that works, we thought. Judging from our own and other researchers' findings, never did young high-school students manifest so much understanding of algebra. Our joy, however, was short-lived. Final interviews, held only a few weeks later, brought a disappointment: the majority of children seemed to have lost their ability to see through symbols and the only thing they could do was to manipulate the symbols. There was not much difference anymore between our experimental groups and any standard algebra class. Here is a representative segment of an interview with Gillian, one of our subjects:

I: Could you write this [$3(x + 2)$] differently?
G: [writes: $3(x + 2) = 3x + 6$]
I: What makes these two things equivalent?
G: I don't know ...
I: But how could you explain this equality to somebody? Why didn't you write, say, $3(x + 2) = 3x + 2$, as some of you did sometimes?
G: Yeah ... cause they're just rules. They're there so you can follow them, so that everybody'll do the same thing ... Without [the rules] I'd do it one way, she'd do it another way.

Once again, arbitrary rules for doing things were what algebra seemed to be all about. There was no mention of functions behind the symbols, no hint of the world of abstract objects which was supposed to make the rules meaningful. The ability to see this abstract universe, which at a certain stage seemed to be here, disappeared after just a few weeks. When we complained about this regrettable change to one of our interviewees, he responded: *Once I know algebra, nothing comes to my mind. I just do it.*

Students' own expectations and attitudes toward learning mathematics may be the key to understanding the events which took place in our experimental classrooms. It seems that the children felt much less need to have "meaning attached to symbols" than we wanted them to have. Most of them quickly forgot the existence of "seeing-based" way of explaining equivalences; the best they could do was quoting the rules or repeating the process itself. We decided, therefore, that the majority of the children were *doers* rather than *interpreters*: they only expected to be able to do things, and had not much belief in (or ability to create, or awareness of) the realm of the abstract objects, in which the rules of algebra become "the laws of (virtual) nature."

That the degree of meaningfulness of mathematics is, to a great extent, the result of students' attitude toward understanding I was able to notice already in an earlier study on algebra carried out in Israel with Liora Linchevsky. A fuller account of the phenomena observed in this experiment are presented in Sfard and Linchevsky (1994). Here, let me only complete the picture by focusing on the findings relevant to our present topic. The study, in which six twelve-year old students had a series of individual one-hour long sessions devoted to the basic concepts of equations solving, provided us with an excellent opportunity to have a close look at children's behaviour and needs. We decided to focus on two pupils, Snir and Dana, who obviously differed in their aims and expectations. In our terms, Snir can be described as interpreter, while Dana was an example of a doer.

For Snir, there were always two dimensions in the problems he tackled: the dimension of means, namely of procedures to be performed, and the dimension of ends — of the purpose for which the procedure was employed. The purpose was something that had to be defined with no reference to the procedure. Thus, when asked to justify, say, the equivalence of $7x + 157 = 248$ and $7x = 91$, Snir would give a lengthy explanation:

> I do the same operation on both sides, and I substitute the same number on both sides ... if I substitute and compute the sides of the [new] equation, I get a different results [than in the old equation], but it is still the same structure ... the same ... how shall I put it? So the solution is the same solution ...

Snir obviously had some difficulty finding the right words, but what he seemed to be saying was that the permissible operations, although they altered the functions represented by the two sides of the equation, did not change the "input value" for which they both gave the same "outputs." In general, he clearly recognized the independent existence of a universe of mathematical entities, such as numbers and algebraic expressions, and whatever he did was a derivative of his knowledge of these entities and the relations between them. From the first moment he seemed to be in pursuit of links between the manipulations and the properties of the objects on which these manipulations have been performed.

For Dana, in contrast, the activity of equation solving had only one dimension: that of doing. In her eyes, the means were also the ends and the actions she performed were undertaken for their own sake. Dana seemed unable to understand the meaning of the request 'explain' the same way the teacher did. When asked to account for the equivalence of two equations, she would usually just repeat the operations that have been performed to transform one into the other: "$7x + 157 = 248$ and $7x = 91$ are equivalent because when we subtract 157 from both sides we get an equivalent equation." When prompted to say what it meant that the equations were equivalent, she would reply: "That they are equal." Dana's single-minded preoccupation with doing and manipulating persisted in spite of the instructor's efforts to entice her into a more reflective mode.

Interestingly, Snir's example has shown that a constant attempt to understand has its price: a struggle for meaning may mean postponement of an 'automatic mode' of dealing with algebra and a slow-down in the acquisition of skills. Indeed, after dealing for some time with the equations of the form $ax + b = c$ in a natural way of undoing, Snir was slower in adopting the algebraic method of solving this kind of problem. For a while, he refused to accept the idea of operation on both sides of an equation. It seemed to us that Snir's temporary resistance to the new technique stemmed from his inability to fully justify it. Such justification would require a fully-fledged functional view of algebra, which, as is well known from studies on the development of the concept of function, is not an easy thing to attain. As long as the undoing was more meaningful to him than the algebraic method, he would rather choose the former. By the same token, the relative easiness with which Dana 'switched allegiance' from the technique of undoing to the algebraic method stemmed from the urge to actually do something on the one hand, and from her relative indifference to the issue of "seeing through symbols" on the other hand.

The main moral from the stories I have just told is this: although some learners of mathematics are interpreters and include seeing in their implicit definition of mathematical understanding, the majority of students are doers, who will always rush to do things rather than reflect on them. For them, symbols are things in their own right and not peeping holes to a universe of intangible objects. You cannot entice a doer into the game of virtual mathematical reality.

2.3 Teachers' formula: undecided

Let me now turn to the teachers and ask the same question as before: what are the main ingredients of mathematical understanding, in their opinion? What kind of skills and abilities would they like to help their students to develop in order to make these students likely to come up with Wittgenstein's exclamation "Now I understand! Now I can go on!"

Data that can be extracted from the quickly growing bulk of research on teachers' attitudes and beliefs are ample and somehow confusing. The most important moral of these studies is that teachers' behaviour in the classroom may be quite different from the one he or she believes would be the most appropriate. Or, to put it in more formal language, teachers' espoused and enacted theories do not always agree with each other. Therefore, even if a teacher claims — as she often does — that her aim is to develop in her students more than a mere ability to perform mathematical procedures, her actual teaching may show little concern for anything but skills. In today's anti-behavioural atmosphere, when there seem to be a full consensus about the importance of 'conceptual understanding,' only a few have a courage to openly admit that they do not strive for anything more than their students' ability to do. For me, therefore, it was a special experience to meet a teacher who was prepared to voice his objections against the notion of "conceptual understanding" in public, in the presence of tens of other teachers and researchers. The event I am talking about took place during my recent meeting with a group of Israeli high-school inservice teachers. Following a discussion on what it means to understand algebra, one of the participants took the floor and declared that

> Teaching algebra should be like teaching the use of word processor. The students must know what to do and how to use it, and this is all they need to know. You don't have to 'see through symbols' to understand. Indeed, you wouldn't say that a person doesn't understand [a] word processor only because he never read the program that runs it or never saw the inside of a computer. (I reconstruct this statement from memory, helping myself with the notes I took immediately after the meeting).

Not every teacher would interpret the words "seeing through symbols" the way this teacher did, nor would everyone express his or her doubts about the role of 'seeing' in mathematical understanding in such an extreme manner. The teacher who participated as an observer in our Montréal study on algebra — let's call him John — had a more balanced view. He stressed many times how important it was that the students have 'meaning attached to algebraic symbols' and was pleased, therefore, with our intensive use of graphs which, so he believed, ran a good chance of bringing the missing element of seeing back into the play. On the other hand, John recognized the addictive power of successful manipulations. He was well aware of the fact that the aspects of doing would often dominate the vision and take over as the primary source of all mathematical wisdom. He could feel this himself and he acknowledged this attitude in his students. This can clearly be read in the answer he gave in our final interview to the question "How would you, as a teacher, describe, what it means that the students understand — or don't understand — an algebraic rule, like opening brackets, or like subtracting the same expression from both sides of an equation?" Indeed, John's response revealed his full awareness of the primacy of the aspect of doing:

> *[They understand if] they can do it*, would be my answer. [To those students who understand, the rules of algebra are] as clear ... as they are to me, without the words. Perhaps they won't be able to explain, they may not have the skills to explain, but it's just perfectly obvious to them. There is no point in asking them to justify why this is true because this is quite obvious ... so I don't feel the need with all my students to go around and say why $x + 3 = 7$ is equivalent to $x = 4$, you know ... For most of them it's as clear as that the sun rises in the East.

From this statement it is not yet clear whether John denied the very necessity of "seeing through symbols" for good understanding of algebra or was just saying that the seeing is not anything the students and the teachers should be expected to explicitly talk about. Knowing John and some other opinions expressed by him in the course of our work together, I would opt for the latter possibility. Even so, the conspicuous lack of any mention of the necessity to see beyond symbols indicates that John might have doubts as to his role as a teacher in helping students to develop this elusive ability.

John's doubts, if he indeed had them, would hardly be surprising. Whereas doing is public and thus relatively easy to describe, observe, regulate and measure, seeing is private, elusive, inaccessible to the inspection of another person, and not easy to detect even by indirect means. Therefore, even the teacher who recognizes the importance of the component of seeing in understanding mathematics may find it practically useless to talk about it or to try to do anything about it. What seems inaccessible to teachers' inspection may be instinctively put aside and never made an explicit objective of instructional effort.

Like in the former cases of mathematicians and students, it was not my intention to draw an exhaustive picture of teachers' attitudes toward the issue of understanding mathematics. In particular, I am not in a position to make any general statements on what mathematicians, students, and teachers usually think about mathematical understanding or what the latter are prepared to do about the learners' ability to see mathematical objects. Much research is still needed before one may start making this kind of claims. The little sketches I drew in this talk are but accidental snapshots of persons and events which caught my eye during my professional travels. I hope, nevertheless, that this impressionistic outline does point to important phenomena and conveys the essence of the problem. In the next section I will complete my sketchy descriptions by discussing the ways in which students could be helped in their efforts to turn the invisible mathematical objects into an integral part of their private mathematical universe.

Perennial Teachers Dilemma: What Comes First, Seeing or Doing?

When it comes to practical pedagogical implications of all that has been said in the former sections, one obvious question should be asked: how can we foster students' ability to see? Today, there is an immediate answer: this special ability may be massively supported with the help of computer graphing software. The graphics turn inner impalpable reality into external and tangible, and in the absence of the former they readily offer the latter, thus making up for what was missing. In our Montréal study, where learning algebra was accompanied by an intensive use of computer graphics, we found much evidence for the beneficial influence of this computer-generated counterparts of mental images. To give but one example, let me quote from the final interview with George, who was asked to solve the equation $7x + 4 = 5x + 8$.

G: Well, *you could see*, it would be like,... Start at 4 and 8, this one would go up 7, hold on, 8 and 7, hold on ... no, 4 and 7; 4 and 7 is 11 ... they will be equal at 2 or 3 or something like that.
I: How are you getting that 2 or 3?
G: *I am just graphing in my head.* (Emphases added)

The advantages of graphs as a means for making abstract into visible were explicitly stressed by some of the children in our final interviews. Here is a representative utterance: "It's good for kids who might have trouble just by numbers, because you can see something you could follow. *I mean, it's something to look at, not just abstract numbers.*" (Emphasis added)

It is hard to believe, however, that the perennial problem of mathematical understanding will be definitely solved by this new means of turning abstract into concrete. Firstly, not every abstraction is readily translatable into pictures. Secondly, the quandary of mathematical understanding requires yet deeper analysis before one starts to look for the ways to promote the seeing capacity of the learner. Let me explain.

Today, mathematics educators seem to be in an universal agreement that learning mathematics should be done with understanding, and that understanding means more that technical skills:

> Following Brownell (1935), mathematics educators oppose rote to meaningful learning ... Mathematics educators seem to be universally opposed to the drill and practice, skill-based conception that was given, perhaps, its firmest theoretical support by Thorndike (1922). In other words, there appears to be pretty good community agreement as to what "meaningful" is (or at least what it isn't). (Orton, 1994).

But is there, indeed, such universal consensus as to the nature of mathematical meaningfulness? It seems to me that for all the lengthy discussion that took place over the last decades (and, indeed, in this version or another — throughout history), and in spite of many insightful conceptual frameworks which deepened our comprehension of the subject, this question still remains wide open. As I tried to show in the preceding pages, even though there may indeed be a pretty general agreement as to the insufficiency of dealing exclusively with the operational aspects of mathematical behaviour, people seem still puzzled by the nature of this additional something that goes into the experience of understanding. In this talk I revisited this old issue and tried to scrutinize it in a "back-to-the-basics" manner. While dissecting the problem into its component elements I presented one possible theoretical route that can be taken in the search for a better understanding of the problem of mathematical understanding. But then, there is another question that waits to be answered — the question of ways in which this special non-operational component of understanding could be fostered in the student. This problem will be my theme in this closing section.

The recognition of the necessity to distinguish between conceptual and behavioural aspects of mathematical understanding leads inevitably to the old pedagogical dilemma regarding the relative importance of the different capabilities and the order in which they should be developed. Say Hiebert and Carpenter (1992):

> One of the longstanding debates in mathematics education concerns the relative importance of conceptual knowledge versus procedural knowledge or of understanding versus skill (Brownell, 1935; Bruner, 1960; Gagné, 1977; McLellan & Dewey, 1895; Thorndike, 1922). The debate was often carried out in the context of proposing instructional programs emphasizing one kind of knowledge over the other. The prevailing view has see-sawed back and forth, weighted by the persuasiveness of the spoke-person for each particular position. Although the arguments may have been convincing at times within the mathematics education community, we have not made great progress in our understanding of the issue. (p. 77)

The question of which comes first, seeing or doing, was apparently present also on the previous pages of this paper and was given confusingly diverse answers by different people. On the one hand, Davis and Hersh, in their hypercube example, were telling us that seeing comes with doing; on the other hand, the mathematician I quoted extensively made it a point that he, as well as many of his colleagues, must see before he is able to do. We seem to have a dilemma, which, as I will try to show in a moment, can only be solved by realizing that the problem "which comes first" is, in itself, ill-posed.

Those who ask the question of precedence present the distinction between meaning and skill (or between seeing and doing) as dichotomous in character, rather than dualistic. They conceive it as referring to two separate abilities which can be dealt with independently and opposed to each other, rather than viewing it as concerning two non-severable, complementary aspects of the same phenomenon. This, in my eyes, is a great theoretical and practical mistake. Moreover, the majority of those who propose the dichotomy between meaning and skills suggest, in fact, that mathematical manipulations have not much to do with meaning (and understanding), and that there is almost an opposition between the two. This seems to me an even graver mistake. As was emphasized time and again in this talk, whether we are engaged in a theoretical or pedagogical discussion, we must recognize that both seeing and doing are necessary components of understanding, and that being complementary they are inseparable from each other. Although in a slightly weaker sense, this last statement applies also to our dealing with the physical reality: I would say that the question whether we should foster seeing before or after we learn to perform actions on objects can be compared to the question whether we should take a look at the messy room before or after we actually make our way to the window (as opposed to looking at it as we are walking); or, paraphrasing William James we could say that making any definitive claims on the 'order of appearance' of seeing and doing in mathematics would be equally absurd as stating that, for some reason, our left leg should always be used exclusively before the right leg starts moving (compare Putnam, 1987, p. 77).

Thus, the mathematician I quoted earlier may have been deluding himself when claiming that seeing is, for sure, what usually comes first in his and his colleagues' mathematical learning and inventions. It is more plausible that the whole process consists in a subtle interplay between seeing and doing. The nature of this interplay is intricate and difficult to capture due to what was called elsewhere (Sfard, 1991, 1992) "the (vicious) circle of reification" — due to the fact that seeing mathematical objects seems a prerequisite to operating on these objects, while the objects themselves seem to be something that can only come into being out of these operations.

My conclusion from the preceding paragraphs is simple: since there is no understanding without doing, it would be a mistake to postpone any kind of mathematical activity (including the one aimed at the development of technical skills) until the students acquire the ability to see. I am not claiming that mechanical doing should precede an attempt to promote the seeing, either. My thesis is that these two intimately related abilities — the ability to see and the ability to do — should evolve together, the absence of one not being a good enough reason to stop fostering the other. After all, this is obviously the way mathematicians themselves arrived at understanding new mathematical concepts in the course of history. This is what may be learned, for example, from the rich biography of algebra. Says Kitcher (1988):

> Leibnitzians confidently set about using new algebraic techniques, vastly increased the set of problems in analysis, and postponed the task of attempting to provide a rigorous account of their concepts and reasoning. Their attitude is not only made explicit in Leibniz's exhortation to his followers *to extend the scope of his method without worrying too much about what the more mysterious maneuvers might mean*, but also in the acceptance of results about infinite series of sums that their successors would abandon as wrongheaded. Insofar as they were concerned to articulate the foundations of the new mathematics, the Leibnitzians seem to have thought that the proper way to clarify their concepts and reasonings would emerge from the vigorous pursuit of the new techniques. In retrospect, we can say that their confidence was justified. (p. 307, emphasis added)

Thus the solution that can be learned from history may be put into simple words: seeing may be expected to evolve from doing, and doing, in its turn, will become more and more effective thanks to this emergent seeing. How the development of the ability to do and the ability to see could be orchestrated in the most effective and secure way is quite a different question. The problem seems extremely difficult due to many factors. Let me name but three of them. Firstly, the ability to see, although indisputably necessary, is difficult to capture not only by an external observer, but by the cognizing person as well. It is certainly not easy to try to develop something which cannot be clearly delineated. Secondly, almost inevitable moments of imbalance between the two component abilities (as indicated, for example, in Ketcher's historical account) may have a lasting destructive effect on the learner's attitude toward mathematics. Thirdly, teachers and researchers are often bitterly disappointed and frustrated by their recurring failure to significantly improve students' understanding of mathematics. As I have argued earlier, however, while discussing the, case of Snir and Dana, the ability to see through symbols is, to great extent, a function of student's expectations and aims: the interpreters will struggle for meaning whether we help them or not, whereas the doers will rather rush to do things rather than to think about them. The problem with the doers stems not so much from the fact that they are not able to see the mathematical objects, as from their lack of urge to look for it. In a sense, they do not even bother about what it means to understand mathematics. The main moral of Snir and Dana's story seems to be this: our success in fostering student's ability to see depends, among others, on our ability to turn doers into interpreters. After we agree, in principle, that the ability to see and the ability to do should be promoted together, the above three difficulties should become a focus of intensive research.

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Lecture Two

A Collegiate Mathematical Experience for Non-Science Majors

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Introduction

At one level of reading, this article describes a collegiate-level course I developed at Colby College, in Maine, during the four-year period from 1989 to 1993, when I was professor of mathematics and chair of the Mathematics Department there. (The course continues to be offered at Colby, though perhaps the format has changed since I left. I intend to introduce the same course at Saint Mary's College starting in 1995.)

The course, which meets for three fifty-minute periods a week for a complete, thirteen-week semester, is not designed to develop any particular mathematical skills or knowledge; rather, the principal — and almost sole — goal is to develop in the student an awareness of the nature of mathematics as a part of human culture.

It is almost certainly not the only course of its kind, but discussions with mathematics educators from other institutions suggest that it may not be very common, and may have some unique features that could be transposed elsewhere. It is not clear whether a course that follows exactly the same procedure would be appropriate at other kinds of institutions; certainly, even moderately larger class sizes would entail changes to the format.

This report is not presented as a research paper within the discipline of mathematics education. I am trained as a research mathematician, and remain active in research. I have always taken teaching seriously, and have reflected on the activity over the entire 24-year span of my professional career, which has always involved teaching (usually two courses per semester).

My research over the past decade has been increasingly interdisciplinary, involving collaboration with linguists, cognitive scientists, philosophers, psychologists, sociologists, and computer scientists, and this has made me far more aware of issues directly relevant to education theorists than is the case for most mathematicians — issues of language acquisition and use, learning and concept formation, communication, and so forth. This has undoubtedly influenced the way I structure and present my courses, but not in any deliberate, overt manner.

Quite clearly, though I have strived to be objective, what you will find in this account is a fundamentally subjective view, by the individual who both developed and gave the course. Student reactions to the course were obtained from the anonymous assessment questionnaires I designed and handed out to the students during the semester, plus the feedback obtained from the anonymous all-college student evaluation forms at the end of the semester (which included a section for free-form comments). The procedures used for both kinds of evaluation ensured a near 100% completion rate.

In addition to being a description of a particular course, my article is also intended to make a case for a certain kind of mathematical experience to form part of the education of all non-science students. (Thus the format of the article is not at all that of a typical course report.) As such, it provides the view of at least one research mathematician on what I believe is an important aspect of collegiate mathematics education that is largely missing at most present-day institutions. (I could make a similar argument for an analogous course for science majors, but in this article I will restrict myself to the case for which I have concrete experience.)

Background

Colby College is a selective, private, four-year, baccalaureate institution. There are 170 faculty and 1700 students. Class sizes almost never exceed 30; mathematics classes never exceed that number. Colby students come predominantly from professional families with a fairly high disposable income. For the most part, their high school experience has been positive, either at a good public school or a private school. They are used to a high degree of personal attention in their schooling. Their most common career goals are business management, government, and medicine. Very few students go to Colby with the intention of pursuing a career in science, mathematics, or engineering.

Like many similar institutions, some years ago Colby instituted mathematics and science requirements for all students. In order to graduate from Colby with any major, a student has to successfully complete two science courses (at least one of them having a significant laboratory component) and one mathematical course (which does not have to be within the mathematics department, though most of the appropriate courses are). These courses have to be taken for full credit, not on a pass-fail basis.

The immediate result of the introduction of a mandatory mathematical course resulted in a large increase in the number of students enrolling for mathematics courses. For the most part, this new student population consisted of highly reluctant students, many of them with either math anxiety, hostility toward mathematics, or both. The new course I developed was aimed to provide these students with an educational experience that not only enabled them to fulfil their mathematics requirement, but had a genuinely positive effect on the way these students viewed mathematics.

That last remark is potentially controversial. Many would argue that the intent of the mandatory mathematics requirement was for the students to achieve a certain level of competence in "traditional" areas of mathematics, perhaps a mixture of algebra, trigonometry, and elementary calculus. As one of those directly involved in formulating Colby's mathematics requirement, I can accurately report that such was not the intent of all of us, but in any case, this issue is beside my current point. The new course I am reporting on was not designed as a "how to" course (though students inevitably picked up, or developed, some "how to" skills as the semester progressed). The significance of the mandatory mathematics requirement was that it provided the students in the class — students who, as remarked above, were often math anxious or math hostile. As the instructor, I knew *why* they were in the class; the question I addressed was what to do with them once they were there.

Aims of the course

Though the aims listed below really only make sense relative to an overall philosophy of mathematics and mathematics education, to be outlined in a later section, it seems only fair to the reader to present these aims now; likewise for the means adopted to achieve them, described in the following section.

- The principal aim of the course is to develop in the student an appreciation of mathematics as a part of human culture. (This aim notwithstanding, I should say at the outset that, as far as I can tell, the course is not a variant of the "math appreciation" or "math for poets" courses that can be found at many institutions. In fact, it was designed not so much as a "course" but to provide an "experience".)

- The course (or the "experience") is intended to bring out the creative, conceptual nature of mathematics, as opposed to the procedural, symbol-manipulative aspects that, for most of the students, constitute their entire view of the subject (at least when they come into the course).

- Emphasis is placed on mathematics as a "way of knowing", a "way of approaching", or a "way of understanding" the world, and a "means of communicating" that knowledge or understanding.

- Efforts are made to place mathematics within the whole framework of human culture and human understanding, to bring out both the similarities between mathematics and other branches of human learning and the differences.

- Efforts are made to demonstrate to the student, from the very outset, that mathematics has relevance to them. (This is not at all the same as saying that they will "need to know" some mathematics, either for their career or their future life in general.)

- The student should leave the course with far less math anxiety than at the beginning, and hopefully no math hostility. (My experience so far indicates that relatively few go so far as to say the course leaves them positively *liking* mathematics — though this does happen — but the more modest goals stated are achieved on a fairly regular basis.)

Before I proceed to describe the course, I'll let you know what the students are told in advance. (They tell me afterwards that what they *get* is not at all what they expected, but that reflects their initial expectation of what mathematics is, rather than a misleading course description.)

The Course Description

I have stated already that it would be more apt to use the phrase "experience" rather than "course".

The course description given to the students prior to their signing up for the course emphasizes that they will be required to do little "procedural mathematics", and that what work there is of this nature will count minimally (and in an essentially "just give it a good attempt" fashion) toward their final grade. (Incidentally, Colby students are strongly grade oriented. What pass-fail options are available to them are used ruthlessly to maximize their GPA. As remarked earlier, the mathematics requirement that brings them to this course cannot be taken on a pass-fail basis.)

The course description does, however, make it clear that the course is not at all an "easy option". The workload is high, and is not at all passive in nature. The students are required to carry out reading assignments, enter into class discussions, and complete two major written assignments, each one building on a series of preparatory written assignments. They are told that the written assignments will be graded on content, quality of exposition, and overall presentation, including the use of diagrams and pictures, where appropriate.

In addition, on the first day, the students are informed that the grade distribution for the course will almost certainly be much the same as for any other mathematics course, which means that there will be relatively few A's, an occasional F, with the bulk of the class earning a B or a C. I tell them that in order to obtain an A, they probably have to have an innate mathematical ability that, had they previously pursued a different educational career path, could have led to a mathematics major. I tell them that a serious effort on the part of anyone else will almost certainly result in a B. (For the vast majority of the class, the prospect of getting a B in a math class seems to exceed their wildest dreams! Many of them do just that.)

The course description does not include a syllabus. The reason for this is that there is no fixed syllabus. In addition to a description of the overall aims of the course, and its predominantly reading/discussion/writing format, there is a list of mathematical topics that *may* be covered (including properties of whole numbers, logic, infinity, geometry, dimension, statistics).

The course description does not indicate any required textbooks. I list a number of books that they may find useful. These are never textbooks; in particular, they are not "pedagogic" and they do not have exercises. Rather they are (low price) books that were written to be *read*, such as Ian Stewart's *The Problems of Mathematics,* my own *Mathematics: The New Golden Age*, or John Allen Paulos's *Beyond Numeracy.* Occasionally, I have required that they obtain a copy of a particular one of these. I generally require that they also obtain a copy of A. A. Abbott's *Flatland.*

The Start of the Course

One of the principal features of the course is that the students will have a significant role to play in the "design" of the curriculum. More precisely, they are told at the outset that the choice of topics will depend upon their interests (either their intended major or, perhaps, some extra-curricula pursuit). They are told that their final written assignment will be the production of a magazine that conveys to the general public that true nature of mathematics, and the ways it relates to everyday life. (I suggest 12-16 pages, but in practice many students exceed this guideline quite substantially.) They are told that many of the smaller assignments they carry out during the semester will build toward this final project.

I believe the initial two or three sessions are particularly important for the way the remainder of the course goes. Certainly, students who join the class after those first opening sessions usually have a tremendously difficult time adjusting to the course, and sometimes never succeed, despite the fact that almost no "mathematical content" is covered during this period. Accordingly, I shall describe those early sessions in some detail.

I regard the first day is purely an ice breaker. (Remember, there is a lot of math anxiety and math hostility about!) I start out by asking them all to write down a one sentence definition of mathematics. (I ask them to put their names on the sheet, as I collect them in a short while later. When I have looked through them all, I give them back and ask them to save them in a file in which they should keep all their written work.) Almost without exception, their response is some minor variant of the sentence "Mathematics is the study of numbers".

I also ask the class to write down, on the same sheet, five adjectives that they feel are appropriate to mathematics, and five adjectives they feel definitely do not apply to mathematics. Among the adjectives they produce as applicable to mathematics, the following are the most common: dull, boring, useless, tedious, frustrating, illogical. The ones most frequently chosen as being not representative of mathematics are: creative, interesting, stimulating, analytic, useful, and fun.

When I collect in their responses, I write up their chosen adjectives on the board. This inevitably breaks the silence, as they realize they are not at all alone in their views within the class.

I tell them that, although their two lists of adjectives are exactly the opposite way round from the ones I would choose, the result comes as no surprise to me. I am not surprised because I know that virtually none of the students has the slightest idea what mathematics is. How could they, I tell them, since it is almost never taught in our school system.

Of course, everyone sits through several years of "math classes", I continue, but they no more teach mathematics than a course in spelling will teach them what makes a great novel or a course on bricklaying will teach them anything of architecture.

I ask them (verbally now) why everyone studies mathematics at school. Their answer is invariably that mathematics is useful. I ask them in what way it is useful. They generally make vague comments about working out change when making a purchase, computing their taxes, and similar numerical tasks. I follow up by asking them to be definite: when was the last time they actually performed such a computation? When did they last observe one of their parents actually making such use of mathematics? Not surprisingly, almost no one can produce an actual instance when they or someone in their family actually performed a mathematical computation, other than making use of a calculator or a computer — for which uses they readily agree that the relevant mathematics was learned in kindergarten or soon afterwards.

Interestingly, so powerful is the belief that mathematics is useful in real life — that is to say, of real, direct use by ordinary citizens — that, even when faced with the evidence that practically no one in the class can supply a concrete instance, some of them invariably insist that it is useful in everyday life, that others need it, and that they themselves will one day need it.

I *up the ante* by asking them when was the last time they found themselves having to solve a quadratic equation; or how many times a year are they faced with having to calculate the time it will take a ball to land, when thrown from the top of a 40 foot building with an upward velocity of 55 feet per second. Do they remember their brother's age, say, by noting that 10 years ago he was half their age, but that fifteen years from now he will be 5/6 of their age?

They laugh, and start to wonder just what kind of a course this is going to be. I tell them that, to the best of my knowledge, almost nobody makes any direct use, in everyday life, of any of the procedures learned and practiced in the high school math class — ever.

The only people who do make use of some high school math are some trades and business people, who use some mathematics in their professional lives, and those who go on to careers in say science or engineering. This latter group, I point out, most definitely use mathematics, and lots of it, but for them, the stuff learnt at high school is at best a two-finger exercise for the symphony to be played later.

The point of this discussion is to provoke in them an expectation that, whatever the course has in store for them, it most certainly will not be numerical computation. It would be nice if they would also begin to reflect on just what was the purpose of large parts of their high school math class, but at this stage they rarely do. Later in the semester we sometimes revisit this issue in a discussion format, at which point some students generally produce a number of good observations.

I end the first session by general conversation with the students, finding out what their intended majors are, any mathematical background in their families, etc.

At the start of the second session, I hand back their initial one-sentence definitions, and write up on the board a crude statistical analysis of their responses. I then point out that the definition of mathematics as the study of numbers ceased to be accurate some two and a half thousand years ago. I make the following points.

In the days when a man's worth was measured in terms of the number of oxen he owned (and in those days ownership of property usually was a male perogative) or how many sacks of grain he had to sell or barter, numbers constituted an important intellectual tool. In those early days, competence in the language of numbers was as important as a facility with the language of words. Back then, mathematics was indeed the study of numbers.

By the dawn of Greek civilization, around 500B.C. the needs of architecture had led to the development of geometry and trigonometry, and from then on mathematics was not only the study of number but of shape as well.

In the mid-seventeenth century, mankind's desire to understand the universe in which we live led to the invention, by Isaac Newton and Gottfried Leibniz, of the calculus, one of the most significant intellectual creations of all time. With the dawn of the calculus, mathematics expanded yet again, this time to become the study of number, shape, motion and space.

Though the calculus was recognized to be a useful and powerful tool, mathematicians had difficulty understanding why the methods of Newton, Leibniz, and others worked. Using the calculus was in many ways a case of, "Do this, then this, and then, as if by magic, you get the answer you want." It took a further two hundred years of effort, and the development of an enormous amount of mathematics, to figure out what was going on. This period of development constituted a fourth era of mathematics.

At each stage in its development, mathematics concerned itself with the abstract structures that lay beneath the most pressing questions of the day.

And so it is today. In the present age of information and communication, mathematics has expanded — some might say exploded — to include the study of abstract structures involved in all walks of life: physics, chemistry, biology, geology, economics, communications, sociology, psychology, politics, music, computing, engineering, astronomy, linguistics, manufacturing, warfare, cryptography, sport, entertainment, the list goes on. If you ask a present-day mathematician for a definition of mathematics, the answer you obtain will almost certainly be some variant of "Mathematics is the science of patterns and abstract structures." Clearly, this definition requires considerable elaboration, and providing that elaboration will take the rest of the semester.

To accompany this brief account of the development of mathematics, I give the students a handout that gives the same overview in a little more detail. I do not want them trying to take notes. My aim is to give them a general sense of events.

I often ask the students to guess who, at the present time, is the largest single employer of PhD level mathematicians in the United States. Their first guess is the Internal Revenue Service — mathematics as facility with numbers is a difficult belief to crack. I tell then it is the National Security Agency, and that the single biggest area that these NSA mathematicians work in is cryptography. Other large employers of high-level mathematicians, I tell them, are the telecommunications companies such as AT&T, the large computer companies, the automobile industry, the aerospace industry, and the movie industry (in the computer graphics area).

I stress that mathematics, that is to say *real* mathematics, has never been something apart from the rest of life. It has always advanced hand in hand with society, as people seek ways to describe, understand, and deal with the abstract structures involved in the major issues of the day.

For Galileo, the major issue was the planetary universe. Galileo said "The great book of nature can be read only by those who know the language in which it was written. And this language is mathematics." Striking a similar note, in 1986 the Cambridge physicist John Polkinhorne wrote "Mathematics is the abstract key which turns the lock of the physical universe."

For these men, and many like them, the abstract structures of greatest interest were those of the physical universe. Today, there are different problems, giving rise to new kinds of mathematics. For instance, in our Information Age, there is a considerable interest in people, in the way we think, the way we learn, the way we communicate, and the way we live, work, and play together, and this interest has given rise to new kinds of mathematics, mathematics that is still not fully developed.

This is really the end of the introductory section(s) of the course.

Deciding the Syllabus

The next task is to determine the syllabus for the remainder of the course. To this end, we continue our earlier discussion of their intended majors or their interests. I try to indicate — very briefly at this stage — ways in which mathematics relates to their majors or interests, and what kind of mathematics it is.

For example, an intending student of law will likely be interested in ways that trial lawyers can — and do — use statistical techniques to choose juries likely to decide a case a certain way. This would lead in to an investigation of statistics later in the course.

A student of history or government would probably find it interesting to see the similarity — almost certainly not an accident — between the formulation of the United States Constitution and a set of axioms (and indeed, the culminating statement of a theorem based on those axioms). Or I could mention the way that historians often make use of statistical analyses to reconstruct past events, such as deciding the authorship of the *Federalist* papers.

Student interests in sports can give rise to a number of different mathematical themes: statistics, the analysis of baseballs in flight, the design of superfast running tracks, the development of high-performance equipment of various kinds, etc.

An intending airline pilot can lead to a discussion of what keeps an airplane in the sky (Bernoulli'sLaw), and how the design of a modern airplane followed the development of the appropriate mathematics. We might also talk about navigation.

Music can lead to a discussion of sound waves and Fourier analysis, and the twentieth-century electronic music-synthesizer technology that depends upon that nineteenth-century mathematics.

The idea is to discuss a number of such themes at a very superficial level, and then decide on on two or three to be examined in more detail. Though I believe it is important for the class to have the final choice (we search for a consensus, rather than moving to a vote), I try to steer them in such a direction that the first topic is "doable" to a suitable degree within a two week period.

My favourite initial topic of late has been mathematical linguistics, à la Chomsky. Since the class is invariably dominated by English majors, it is generally easy to "ensure" that this is chosen as the initial topic. It has a number of additional advantages . One is that it is an application of mathematics that comes as a complete surprise to the students, an advantage which I have found provokes a significant level of interest from the class. Another advantage is that it deals with a domain — the English language — with which everyone is familiar. It requires no deep mathematics. The importance of a symbolic notation is self-evident. It is easy to see both what the mathematics can tell you about language and what it cannot — I think it is important to convey an appreciation of the limitations of mathematics. Moreover, the level of abstraction that can be achieved by means of a mathematical analysis leads to significant insights, such as the deep connections between language and computation — see presently.

The following section outlines a treatment of mathematical linguistics that I have found to be of an appropriate length and depth.

Mathematics of Language

Take a look at A, B, and C below. In each case, without hesitation, tell me whether you think that what you see is a genuine sentence of English.

A. Biologists find the A-spinelli morphenium an interesting species to study.

B. Many mathematicians are fascinated by quadratic reciprocity.

C. Bananas pink because mathematics specify.

I am pretty certain that you decided, without having to give the matter any thought at all, that A and B are proper sentences but that C is not. And yet A involves some words that you have never ever seen before. How can I be so sure? Because I made up the two words "spinelli" and "morphenium". So in fact, in the case of example A, you happily classified as a sentence of English, a sequence of "words", some of which are not really words at all!

In the case of example B, all the words are indeed genuine English words, and the sentence is in fact true. But unless you are a professional mathematician, you are unlikely to have ever before come across the phrase 'quadratic reciprocity'. And yet again, you are quite happy to declare B to be a genuine sentence. On the other hand, I am sure you had no hesitation deciding that C is not a sentence, even though in this case you were familiar with all the words involved.

How did you perform this seemingly miraculous feat with so little effort? More precisely, just what is it that distinguishes examples A and B from example C ?

It obviously has nothing to do with whether the sentences are true or not, or even if you understand what they are saying. And it doesn't make any difference whether or not you know all the words in the sentence, or even if they are genuine words or not.

What counts is the overall *structure* of the sentence (or non-sentence, as the case may be). That is to say, the crucial feature is the way the words (or non-words, as the case may be) are put together.

This structure is, of course, a highly abstract thing; you can't point to the structure the way you can point to the individual words or to the sentence. The best you can do is observe that examples A and B *have* the appropriate structure but example C *does not.*

And that is where mathematics comes in. For mathematics is the science of abstract structure.

The abstract structure of the English language that we rely upon, subconsciously and effortlessly, in order to speak and write to each other and to understand each other, is determined by the *grammar* of English.

Since the pioneering work of the MIT linguist Noam Chomsky in the late 1950s, we have known that the most effective way to describe and study grammar is by means of mathematics. For example, the following are some of the rules of English grammar that enable us to tell sentences from non-sentences.

DNP VP ➔ S

V DNP ➔ VP

P DNP ➔ PP

DET NP ➔ DNP

DNP PP ➔ DNP

A NP ➔ NP

N ➔ NP

In words, the first of these says that a deterministic noun phrase (DNP) followed by a verb phrase (VP) gives you a sentence (S); the second says that a verb (V) followed by a DNP gives you a VP; the third that a preposition (P) followed by a DNP gives you a prepositional phrase (PP); the next that a determiner (DET), such as the word "the", followed by a noun phrase (NP) gives you a DNP. If I tell you that A stands for adjective and N stands for noun, you can figure out the meaning of the last three rules for yourself.

In order to use the grammar to generate (or analyze) sentences of English, all you need is a *lexicon*, a list of words, together with their linguistic categories. For example:

to ➔ P

runs ➔ V

big ➔ A

woman ➔ N

car ➔ N

the ➔ DET

Using this grammar, it is possible to analyze the structure of the English sentence

The woman runs to the big car.

Such an analysis is most commonly represented in the form of a *parse tree*, as shown in figure 1. At the "top" of the tree is the sentence. Then, each move you make from any point in the tree, down by one level, indicates the application of one rule of the grammar. For example, the very first step down, starting from the topmost point, represents an application of the grammar rule

DNP VP □ S

The parse tree represents the abstract structure of the sentence. Any competent English speaker is able to recognize (generally subconsciously) such a structure. You may replace each of the words in this tree with other words, or even nonwords, and, provided your substitutions "sound right" for each grammatical category, the resulting sequence of words will sound like an English sentence. By providing axioms that determine all such parse trees, the formal grammar thus captures some of the abstract structures of English sentences.

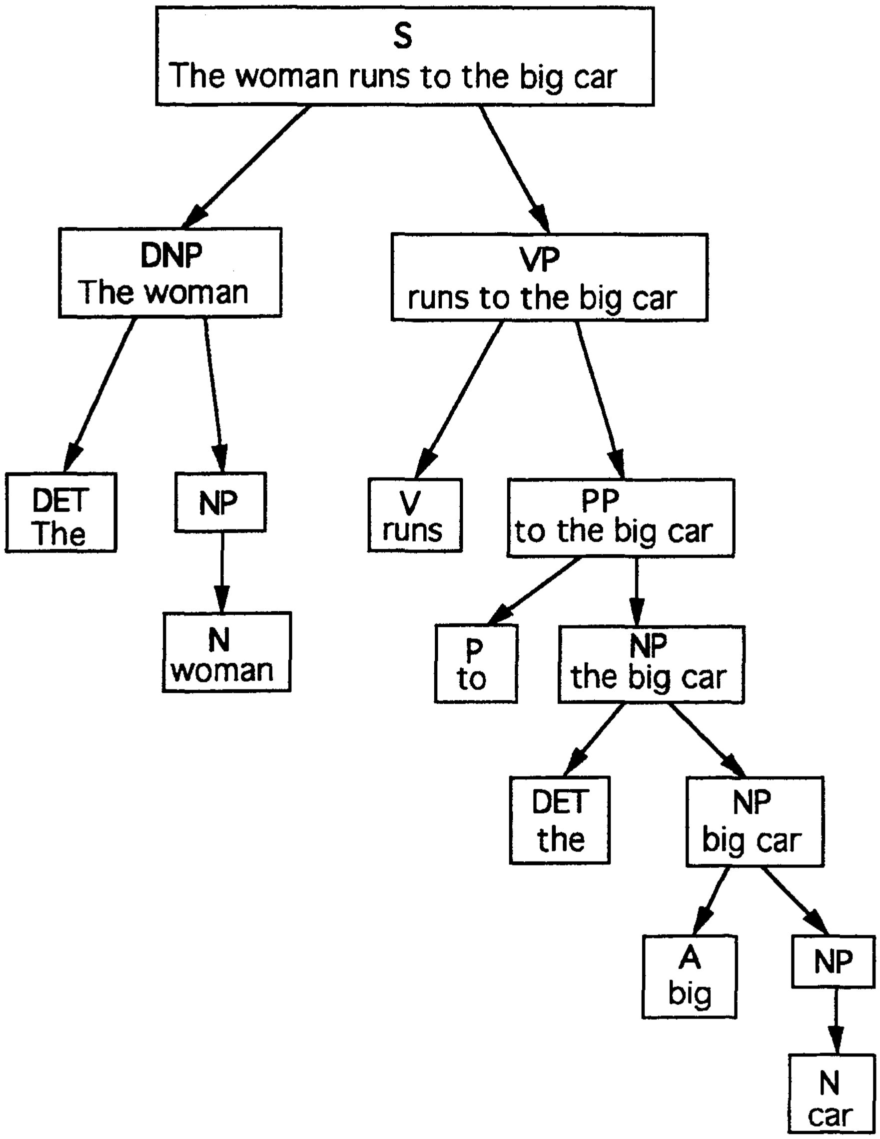

Figure 1

Of course, English is very complex, and these are just a few of the rules of English grammar. I just wrote enough of them to indicate the advantage of using a mathematical (algebraic) notation over a description expressed in ordinary English. For one thing, the symbolic notation is much briefer and easier to read (once you have remembered what the abbreviations mean). And, more significantly, since the whole point is to write down the rules of English grammar, it is surely better to avoid use of English sentences themselves at this stage.

In fact, neither of these is the real reason why a mathematical approach is so powerful in linguistics. The real reason is that English grammar determines a complex, abstract structure, and mathematics is simply the most precise intellectual tool we have to describe abstract structures.

Language is one huge, complex, abstract structure. When I write down, as I just did, some of the rules of grammar, what I am doing is representing, or describing, part of the abstract structure of the English language that lives in our minds.

What makes one particular sequence of sounds music to our ears and another a discordant noise is the abstract structure of that sequence of sounds. The notes of a tune may be written down on a sheet of paper, their order and position on the stave *representing* the structure of the tune, but the musical structure itself is an abstraction, something that exists in our minds when we hear the tune being played.

Of course, formal grammars of the kind just described can only capture part of the structure that constitutes an English sentence. What can be captured mathematically is captured in a very precise way — and indeed it is this mathematical precision that may be utilized in order to develop computer systems that can handle "natural language" — but mathematical formalisms do not capture all there is to know about sentence structures.

There is a great opportunity for discussion at this point. In principle, formal grammars could capture a considerable amount of sentence structure, but the resulting mathematical description would be so long and complicated as to be virtually useless. (One obvious question: how do you handle the relationship between active and passive sentences, such as *John greeted Sally* and *Sally was greeted by John*?) It is interesting to investigate whether there are features of language that absolutely could not be treated mathematically, and if there are such, what makes them resistant to mathematical analysis. Such an investigation can reveal much about both language and mathematics. What are the limitations inherent in applying mathematics to other domains besides language? What are the pluses and minuses of mathematical analyses in general?

The discussion of formal grammars can be taken a step further that leads to another illustration of the power of mathematical abstraction. Having found a way to use mathematical techniques in the study of language, Chomsky took his analysis a step further, examining the nature of formal grammars in general. By placing various restrictions on the kinds of rules that may appear in a grammar (i.e. the kinds of combination that can appear on the left or the right of the arrow), he introduced a whole hierarchy of grammars, known as the *Chomsky hierarchy*. The grammars in this hierarchy ranged from the simplest, called regular grammars, to the most complex, the phrase-structure grammars.

The *regular grammars* describe only very simple "languages". One example is the numerical language you use to release a combination lock. You probably have never thought of a combination lock as having anything to do with language, but it does. The "sentences" in the language are sequences of numbers. The "grammatical sentences" are the sequences that trigger the lock to open. For most combination locks, there is only one grammatical sentence.

At the other end of Chomsky's spectrum, the *phrase-structure grammars* are fairly complicated, and describe large parts of human-language sentence structure.

Having described, briefly, the various grammars in the Chomsky hierarchy, it is possible to demonstrate one of those remarkable moments in mathematics when sufficient abstraction leads to a significant new insight.

During the 1950s, when Chomsky was developing his theories at MIT, mathematicians at the same institute, and elsewhere, were developing a mathematical theory of computing. In particular, they proposed and studied a number of hypothetical models of computing devices.

Chomsky was able to prove a number of theorems that indicate fundamental connections between the grammars in his hierarchy and the various kinds of hypothetical computing device studied by the mathematicians, thereby demonstrating that, at a suitably abstract level, language and computation are two sides of the same coin.

The simplest of these hypothetical computers is the so-called *finite automaton*. Roughly speaking, this is a computer that can respond to input but has no memory. Chomsky showed that the languages that could be "understood" by such a device are precisely those whose grammar is a regular grammar. ("Understood" in this context means that the device will make an appropriate response to an input that is grammatical according to the particular grammar concerned. For example, in the case of a combination lock, the appropriate response is that the lock is released.)

Next in complexity are computers that have a fairly rudimentary memory (a single "stack", for those who know what this means). The languages these computers "understand" are precisely those determined by the *context-free grammars*, a class of grammars in Chomsky's hierarchy of particular interest to linguists.

Add still more memory to your computer and you find that the languages "understood" are exactly those determined by a *context-sensitive grammar*. The computers in this case are known as *linear-bounded automata*.

Add more memory still to your computing device, and you obtain what is known as a *Turing machine*. This hypothetical computer was introduced by the logician Alan Turing in 1935, in an attempt to capture the patterns of thought involved in human computation. The Turing machine is a hypothetical device having an unlimited, though rudimentary, memory. During the period from the 1930s to the 1950s, a large number of results demonstrated that, for all its simplicity and hypothetical nature, a Turing machine could, in principle, and given enough time, perform any computation that could ever be performed by any kind of computer, no matter how complex.

Chomsky tied language to computation in a very strong way when he proved that the languages that can be "understood" by a Turing machine are just those whose grammar is a so-called *phrase-structure grammar*, a particularly important class of grammars in linguistics.

Thus, an attempt to investigate the structure of English sentences (what it is that we recognize when we recognize a particular sequence of "words" as an English sentence) can lead, via an algebraic formalism, to the notion of a formal grammar, then on to an investigation of the nature of computation, and finally to some theorems that demonstrate a close connection between grammars for language and mathematical models of computers.

Curricula Fixed-Points

Though the initial topic covered can vary from semester to semester, there are a couple of regular themes that I always include — the only thing that varies is the manner in which the topics are introduced, since I always provide a link from one topic to the next.

Calculus is one such fixed-point. I do not think that such a major and far-reaching step in human intellectual advancement can be ignored. I do not attempt to train the students how to perform differentiation and integration, skills they will almost certainly never need. (The same can be said for about 90% of our science and engineering majors, but that is another argument.)

My emphasis is on the ideas behind the calculus and why it was important to develop a *calculus*. I motivate the former as a means of applying the essentially *static* objects of mathematics (points, lines, planes, etc.) to study *dynamic* issues of change and growth. The two main ingredients are (i) having a precise (static) description of the change (captured by means of a function), and (ii) capturing the pattern of a continuous changing quantity as a limit of a sequence of approximations by fixed, computable quantities. Finding the sum of a geometric series to resolve Zeno's Achilles and the Tortoise paradox is one way to introduce these ideas of working with the pattern or function and evaluating the limit of a sequence of approximations.

I usually ask the class to compute, from first principles, a couple of very easy derivatives, such as the derivative of x^3 (having dealt with x^2 myself in class).

Explaining why it was necessary to develop a calculus to compute analytic derivatives and integrals is easy: in the days when there were no electronic computers, there was no other option, and the rich calculus that was developed serves as a marvellous testament to the ingenuity of humankind in circumventing a major obstacle. Quite why so many of those techniques are still being taught today, when we do have enough computing power to evaluate the derivatives and integrals we need, escapes me, but once again that is another issue.

In addition to the coverage of the calculus, I also show an occasional video, to illustrate just how broad mathematics is. Among those I use frequently are twenty-minute sections of *A Mathematical Mystery Tour* (Time-Life Videos), which I made with the BBC some years ago, and *Scientific Visualization* (Films for the Humanities, Inc., of New Jersey).

A further theme that I always manage to weave in is to indicate how advances and trends in mathematics tend to be in keeping with developments elsewhere in human culture. Typical examples are the development of projective geometry in the Renaissance period, the trend toward great abstraction in art, music, and mathematics in the early part of this century, fascination with the notion of dimension in art, literature, and mathematics at about the same time, and the present-day use of graphical systems and (other) interactive media in mathematical research and education.

Finally, I want the students to come away from the course with an overall sense of how mathematics fits in to the broad spectrum of human knowledge and learning.

I try to indicate that mathematics and the various sciences are not fundamentally different from most other products of the human intellect. They are just ways of understanding ourselves and our environment, ways of describing, and ways of communicating. I generally make the following points.

- Humans are great toolmakers. We have hammers and nails to join pieces of wood, saws and knives to cut, drills to make holes.
- We have pens and pencils to make marks on paper.
- We have bicycles and automobiles to get from place to place.
- We draw a map to help people find their way around an unfamiliar town or region.
- We use blueprints to specify the way to assemble a machine.
- In addition to these physical tools, we develop and use conceptual tools to achieve various ends.
- If our aim is to capture and convey emotion and sensation, we can use poetry or music.
- If our aim is to record the main events of the day, we can use text, as in a newspaper.

If you want to try to change people's political views, or the way they run their lives, you would be far more likely to be successful if you use the rich conceptual framework of theatre or, the more modern variant, movies. (Oliver Stone is a successful master of this art, with his powerful anti-war movies, and his movie *JFK*. *In the Name of the Father* is another recent example that comes to mind.)

What medium would you use to try to understand animal life, and to communicate that understanding to others? Not movies. Not maps. Not blueprints. Not poetry or music. These are good for other things, not understanding life, just as saws are good for cutting wood but not for driving nails into wood. The appropriate medium to understand animal life is biology, or biological theory. Biology is the collection of conceptual tools we human beings have developed to help us understand animal life.

- Chemistry is the conceptual apparatus we have developed to help us understand matter.

- Physics is the conceptual apparatus we have developed to help us understand the forces that govern our environment.
- Psychology is the conceptual apparatus we have developed to help us understand people.
- Sociology is the conceptual apparatus we have developed to help us understand how societies function.
- Linguistics is the conceptual apparatus we have developed to help us understand how language functions as a medium of communication.
- Mathematics is the conceptual apparatus we have developed to help us understand various abstract patterns and structures that we perceive in the world.

And so on. The list can vary. I often ask the class for other suggestions, and allow a discussion to develop. I have also assigned homework on this theme, asking the class to suggest media particularly suited to conveying particular kinds of information or to achieving various ends, and to examine their relative advantages and disadvantages.

I also try to show the students that these various media are not separate and compartmentalized. Each has its own advantages and disadvantages, and we often have need to utilize several media at once.

Faced with a rose, we can approach it in various ways. We can see its beauty. We can feel its texture. We can smell its scent. We can study its biological properties. We can analyze its chemical composition. We can examine the physical forces that bind its molecules together. We can catalogue its mathematical symmetries. We can write a poem about it. We can paint a picture of it. Each one of these tells us something different about that rose. The more ways we can find to examine that rose and describe its properties, the deeper and richer will be our understanding of that rose.

The various sciences are just different ways to understand and describe aspects of our world. Since they are ways of understanding that have been developed by humans — since they are all products of the human mind — they are not fundamentally different from any of the so-called humanities. Both drama and biology tell us something about what it is to be human, to have the relationship to our environment that we do. Mathematics, the science of patterns, is a conceptual framework that humans have developed over three thousand years in order to describe, analyze, and understand various abstract features of ourselves and the world we live in.

Working Group A

Theories and Theorizing in Mathematics Education

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2. Introduction

1. What do you think of when you hear:
   "theories in mathematics education?"
   "theorizing in mathematics education?"

2. In what ways do you use theories in mathematics education?

3. What are the theories in mathematics education?

4. Into what categories can the theories of mathematics education be grouped?

5. Are there formal and informal theories? If so, how do they differ? What can each contribute to mathematics education?

6. What are the purposes of theories in mathematics education?
   In what sense are theories ""of" something?
   In what sense are theories "for" something?

7. What are some interesting theoretical constructs in mathematics education?
   Are such constructs useful independent of their theoretical framework?

8. In what ways are theories useful in mathematics education:
   to researcher?
   to students of mathematics education?
   for curriculum design?
   for material design?
   in classrooms?

9. In what ways are teachers' theorizing different from researchers' theorizing in mathematics education?

10. In what ways can theories and theorizing be a hindrance in mathematics education?

11. Are there alternative ways of knowing to theories/theorizing that are relevant to mathematics education?

12. Can we know anything about mathematics education without theorizing?

13. Are there theories not currently used in mathematics education that should be explored in terms of potential uses in mathematics education?

This is the list of questions that was prepared by the working group leaders to serve as the basis of discussion on theories and theorizing in mathematics education. However, the participants did not simply surrender themselves to this list but found ways of weaving in and out of it as they shared and resonated in each other's thinking and stories of experiences with respect to the nature and uses of theories in mathematics education. What resulted was a situation that seemed to be less of something to be recorded and more of something to be lived. However, for the purpose of this report, a collection of some of the ideas/thoughts that emerged in the group is being presented.

As seen below there was a variety of thoughts on the nature, uses and values of theories and theorizing in mathematics education. There is in these comments a sense that theorizing should have practical outcomes. Some put this in terms of the pragmatic; in those terms a theory is valued based on its projected effects on a desirable goal-for example, broader more integrated mathematical knowing by more of a broader cross-section of students. Cast in these terms of the pragmatic, a theory or theorizing need not be couched in directly practical terms, but its proposer must be able to defend it at least in terms of long term usefulness to the community.

One way of allowing for this practical/pragmatic outlook is to consider mathematics education theory or theorizing as a design theory. This in itself raises questions: if one sees students as constructing their own mathematics, then a design theory must take into account that teachers and students are fully implicated in curriculum and instructional design and not simply the recipients and users of the products of the design of others. The same kind of questions might be raised about the criterion of predictiveness. If one thinks that mathematics learning entails the building up of pre-determined "correct" images in one's mind and the capability of showing such images for others, then a predictive theory might be useful. But a student's mathematical actions are thought to be at once determined by that student's own structure and that a student is bringing forth a mathematical world with others which co-emerges with her or his space of actions, then "prediction is not a useful criterion. If mathematics education theories are "theories for" rather than "theories of", then they provide a platform for teacher or researcher observation of and action with respect to student mathematical behaviour without the necessity of being predictive of anything.

3. Thoughts on the Nature of Theories and/or Theorizing in Mathematics Education

** A theory in mathematics education is a design theory or an applied theory. It should have the possibility of generating some technology, some techniques or solve certain kind of practical problems. In some ways one doesn't know whether the theory is ever going to do that when one started working on the theory. But I would make it a requirement of a respectable, reasonable theory of mathematics education. If t doesn't do any of those things then we are wasting our time except for those who want to be philosophers. Mathematics education is about educating people, not about theorizing, philosophizing, tossing things around.

**Mathematics education has a very practical goal and we shouldn't lose sight of it. We have to have a pragmatic goal in mind and then speak of these theories in view of this goal.

** A theory must be based on a small, concise set of basic assumptions, from which all other statements of this theory could be derived. Any theory must be rich in possibilities, i.e., its basic set of assumptions has to be highly generative and give a possibility to drive conclusions covering broad areas and many phenomena. It must generate explanations that are global rather than local.

** A theory must explain how something is learned, how it is understood and how it could or should be taught. It should allow one to predict the outcome of one's actions.

** A theory should provide entry and exit into a situation, for example, when and how one should stop doing something.

** A theory should do something.

** A theory is a model for nurturing learning and/or teaching of mathematics.

**A theory is like a map or a guide book. It is designed for someone else as something one can use to travel a territory without getting to know it very well but get to where one wants to go, or like a guide book, for one to be able to explore the territory and get to know it oneself.

** A theory is a means of looking forward - let's try to get it right so that we know what to do.

** A theory should be considered more in terms of practice of mathematics education, the actual practice of helping people to learn mathematics through the action of teaching.

** A theory needs to explain something. If a theory wishes to explain something, it should go through the process of verification in some way. Otherwise it remains more of an opinion than a theory to someone.

**A theory is a conceptual framework of varying breadths that allow it to come up with a rationale for what to do and how to do it.

** A theory helps to make sense of experience, explaining things, explaining phenomena you have observed from teaching and so on. It should tell you what to do in the classroom. There are many theories that try to explain, but they don't tell you what to do. This creates a gap between theory and practice. But one can challenge the statement that the theory should tell you what to do in the classroom. The theory may be helpful to a teacher, not by telling the teacher what to do exactly, but helping the teacher to analyze what's going on - being more conscious and therefore taking action accordingly. This is different from, say, the theory that tells you how to do things.

** There are too many theories and they are too restricted. What we have so far are fragmentary (often, beautiful) conceptual frameworks good for working in restricted areas. What we don't have, as yet, is a theory which would provide us an overarching paradigm a framework that would be not so much be a guard of consensus (which is not a good thing at all) as a provider of a common language through which we could communicate (right now, we seem to talk past each other rather than communicate, only too often).

** Theories are not necessarily connected –an often fragmented series of formulations on a range of topics.

** Relationships among theories are not developed –no synergy.

** Is it possible to have a unified theory?

** A theory can be associated with a sort of distancing. Its more arrogant–if somehow we could get the theory right we would then know what to do.

** Theories can be used as weapons rather than intellectual tools.

** While theories could distance us from practice, they could in fact provide some different views of practice and change practice in the sense that one's common sense reaction to a situation may or may not be the most productive way to in fact practice it.

** Theories can make us know less.

** Theories are things you put in the way. They allow you to see things you can't always see. If you think of theory in terms of the sum of the parts, it can allow certain things to be seen that can't normally be seen because the overwhelming amount of stuff coming is too great.

** Theories allow one to make certain kinds of distinctions that one couldn't otherwise make, i.e., having made these distinctions what should one do now and since one doesn't have obvious follow-ups what can one do.

** A theory looks like something which is a product and a more public kind of phenomenon. It is something that would be known to some public.

** Small theories are theories with small themes.

** Theories are very large scale, complex, big things, not little things.

** Theory is jargon, specialized and often idiosyncratic use of terms, more of an interference than a source of insight.

** Pure unadulterated theories could not exist, because they could not be verified.

** Theorizing involves discussion and/or reflections that aim at developing theory or putting some flesh on a theory or buying some theory or even attempting to interpret a theory.

** Theorizing is post-facto. You look at what your experience is and what happened and what research is done and talk about that.

** Theorizing can be considered, at least, at two different levels. One is the context of, for example, the scientific community trying to build theory. In this case communication is important, i.e., the theory has to be communicable. The other level is, for example, when a teacher is using reflective analysis about what he or she is doing, reflecting upon phenomenon observed in the classroom, in a disciplined way, not just common sense, but in some disciplined way or with some tools. One could say that such a teacher is theorizing, trying to explain what is happening in the classroom. Maybe not with a big 'T' but at least a small "t" and that's a local theory which may be very helpful for the teacher. This can be called a personal theory. But very often big theories in the scientific community are built starting from personal theories. So there is a link between the two. But in the context of someone trying to explain what is happening in his/her particular situation, the person may say "I theorize for myself because I want to improve what I am doing, I want to be more aware" in which case communication is not essential.

** Theories are drawn from psychology, philosophy, sociology linguistic,.... We have few of our own in mathematics education. It seems that we have too many theories, but actually in mathematics education we suffer in that we don't have very many theories.

** A theory as multiple embodiments would be better than a singular embodiment.

**It is confusing as to what a theory is. Take, for example, this situation from my teaching of students teachers. Now is this just a principle (i.e., telling them "don't only use one material, you should use a lot of materials")? If it's only a principle then what theory was this embedded in?

** Consider Van Hiele's model, for example. Lots of people don't want to call this a theory. They feel it's a model. They say it gives the steps to go through. But there is something lacking in the whole thing - global explanation. It's the same thing with models of understanding. Many of them just say there are these and these and these kinds of understanding, but they never define what these understandings are. That's why I tend to not call them theories. Not yet.

** Principles would be much more specific than theories. But then there is the question about whether you could actually absolve the principles without dissolving the theory.

** Most things called theories are really paradigms and principles.

** Principles are not necessarily part of theory or are not big enough to be worthy of the name

** In classes lots of things that we use for theory, perhaps are more properly approached with paradigms or principles.

** There are things called principles that are not theories and are something separate.

** Principles based on theory are sort of part of the theory, not something separate, they are implications from the theory.

** If we think there are theories and principles, for me principles are embedded in theories or they speak about a thing or of teaching not learning.

** There don't seem to be a lot of theories to describe the role of the teacher.

4. Thoughts on the Uses of and/or Using Theories in Mathematics Education

** Theory can transform one's way of viewing something. It can provide one with different ways of looking at a situation one had accepted as being the only way it could be. One can use to stand back from what seems to be just obvious up front and see it in a different light.

** One can use theory to notice how one learns things then use this to help others help themselves.

** Theories can be used as a basis for explaining things in teaching; as object of study; for research.

** Theories promote one's own self awareness.

** Theories lead to personal growth.

** Theories allow one to see things one didn't see before.

** Theories help us appear to make sense; know what we are talking about.

** The only possible use of theory is to have something to talk about. It's a formulation. It gives a focus that can be discussed.

** Theories gives language to express a phenomenon.

** Theories are useful or maybe essential where common sense failed.

** Theories can be translated into practice in completely different ways.

** Theories are not prescriptions for teaching, they offer instruments to improve the teaching.

** There is no unique way of developing the technology or the teaching tools that fall from the theory, no matter how explicit the theory tries to be.

** Theories are used to inform thinking, but one personalizes them in the use of them. Is it still theory once personalized?

** Don't know if I use theories because what I use is part of me. I act out of strong beliefs. Therefore not a formal system.

** Theory seems too lofty/important for what I do, to talk about using theory.

** We don't notice where underlying ideas to theories come from, we just use them.

** In using a theory, we actually bring it into our embodiment, into our lives and want to make it part of our doing, It's hard to now call it theory. It wants to be called something else, a principle or a generalization. So in a way something happens in that this thing that was a theory just a minute ago now belongs to me because I have used it. It's part of who I am and it doesn't hold the label theory quite as well as it did just a minute ago.

** How much of a theory do you need to use to use it?

** I use theories in my teaching as a basis for explaining things. This is different from teaching the theories themselves as objects of study or using them as a basis for suggesting ways of teaching. I use them as a reference but I don't stop and explain the whole theory. My concern is as follows. If you take a look at, for example, Dienes' ideas about learning mathematics, he had a theory in which he claimed there were six stages to learning mathematical concepts, but there were also all the principles going with it. Now if you teach elementary teachers, at one moment you will refer to one of the principles, you won't refer to the whole theory. But it's not clear that if you refer to one principle that it mean's that you are buying or using the whole theory. What makes things even more difficult is that many theories incorporate ideas that come from others. Just following one element doesn't mean you are buying the whole theory, because the element can be common to other theories. So when we say using theory, does it mean using all of the theory or parts of the theory? It is a complicated matter but generally we don't use the whole theory, only the main features useful for one's purposes. Even in research very often we don't use the whole theory.

** Can we use ideas or concepts without buying into the theory? When we pick up a principle and actually use it, are we in some sense living out a kind of theory implicitly whether we know it or not?

5. Examples of Theories and Theorizers

** Skemp; Piaget; Vygotsky; Dienes; Van Hiele; Bruner; Ausubel; Gagné; Bloom and Carrol; Meyer; Wittgenstein; Perry; Gattegno; Minsky; Thorndike; Skinner.

** Constructivism; activity theory; critical pedagogy frames (e.g., feminist ways of knowing); concept image; Enactivism.

** Areas of influence: behavioural psychology; cognitive psychology; epistemological influence; mathematical influence; philosophy of mathematics; sociological influences; affective domain; gender; representations.

6. Thoughts on Categories of and/or Categorizing Theories

** What is the purpose of classifying theories? Our group could not find one.

** We would have to go back to the question of why we teach mathematics and answer that before we can start categorizing theory.

** One could categorize based on various areas borrowed from for example, psychology, philosophy, sociology, etc.

** One could categorize based on the underlying assumptions regarding mathematics, learning and teaching.

** One could categorize as useful and all the rest.

** One could categorize as older ones (learning theories) versus recent (shift in 70's from learning as primary focus to classroom) in research.

** One could categorize based on scope of theory, e.g., particular to mathematics education, more general, object of study.

** One could categorize as different theories of same thing versus different theories of different things.

** One could categorize based on pedagogical design.

** One could categorize based on the underlying assumption regarding mathematics, learning, epistemology, teaching.

** One could categorize by focusing on big theories, those that come from clearly outside mathematics education and those that are at least partly built by math educators –those partly developed within mathematics education for math educators.

** Can we come up with a unified theory from all of the pieces being listed? Is there some ultimate theory where it all comes together?

7. Questions Suggested to Add to List (at beginning of report)

** What areas are in need of theorizing?

** What areas are over theorized?

** What is the relationship between formulated theories and our unformulated actions; i.e., the way we behave and the way we theorize about our behaviours.

** We are borrowing theories from lots of places. Many people think we should have home grown theories in math education and of math education. So the question is what do we mean by home grown theories? Is it just a combination of ideas, or is it just an adaptation in some sense, adaptation to the specific case of mathematics? What is original in what you would call the home grown theory in the field of math education?

** What is important in order to call something a theory?

** Must or can all of mathematics education be explained by theory?

** Are all of our theories all at: the same level? Are they all theories?

** How does theory and theorizing enter into design of lesson sequences?

** How does theory and theorizing lead to technology?

** Can all practice be justified by theory?

** What would post-theoretical mathematics education look like?

** Can we have a more every day theory or theorizing about math education, down to level of classrooms?

** Take one or two theories and then critically analyze that particular theory to try to come to some agreement of the assumptions built into them. What questions would come out of the activity.

** How different would mathematics education be without any theories?

** What kinds of theories are possible in our domain and how can we overcome the problem of incompatible differences with science.

** What are instances of techniques that are derived from theories?

** What sort of techniques and things have come out that are now available for teachers, because of the theories that weren't there before the theory?

8. Folklore

The issue of folklore was raised by David Wheeler and resulted in a discussion from which the following excerpts are taken.

** In some ways, folklore very often fulfills the place of a theory. Folklore is what people have when they don't have theories.

** If you want to examine folklore and change it, I don't think you can do that empirically or just by changing beliefs or things like that. I think you need the apparatus of alternative theory in order to shift folklore and if you want, if you think folklore is mistaken in some respects, misguided, then it may become important to develop theories in order to have a groundwork on which to make the folklore change, force it to change. So again, is it quality of a good mathematics education theory that enables you to examine the folklore or change the folklore, or examine it or something?

** Common sense isn't what I'm calling folklore; there is quite the distinction between them. Nevertheless, common sense is related to folklore.

** Both common sense and folklore are difficult to define. But what are the characteristics of common sense? There is a sort of consensus level probably about folklore. An individual's belief would apply to folklore somehow. But of course every individual is embedded in some cultural situation. We do talk about the folklore of the classroom, don't we, or the folklore about teaching, about how this should be done.... Embodied in them, there seems to be a shared set of assumptions which has never really been put to critical evaluation and testing and it seems to be one of the purposes of science to examine folklore. Folklore may be right and/or elements of the folklore may in fact be correct, but still unexamined, still taken for granted and therefore we don't know whether or not they are really correct.

**Common sense may be my own experience, what you develop from your experience.

**Illustrations of folklore:

(1) Student teacher going into a class/school to perform practicum taken aside by the vice-principal or whoever looks after student teachers at the particular school and told, "Now forget all that nonsense they tell you in university, we want you to do a,b,c, and d." That's the induction to folklore.

(2) Limits are difficult.

(3) Multiplying by 7 is more difficult than multiplying by 5.

** Folklore is verified by centuries of experience, therefore making a big case of verifying them and making a big experiment doesn't make much sense.

** Folklore might have some kind of basis that's fact

** Is folklore knowledge isolated or tied into a lot of other things? Should someone challenge people's folklore?

** Folklore knowledge is often principled knowledge, things to do, ways to behave, and can help you get below the surface.

** Folklore is whatever comes from practice. It is known for a "fact".

** Some elements of folklore are in fact sound practical wisdom, but then there seems to be a lot of other stuff around.

** A piece of unsolved folklore: Girls cannot do mathematics. That should be challenged. Folklore should be challenged.

** Folklore obviously does have a variety of meanings, but one sense is clearly story telling. That's what lore is. It's the stories you tell about things and it tends to be shared stories that a group or culture has in common. Some of the stories may in fact serve very useful functions, others may in fact be obstacles to making certain things better.

** If you put folklore in the practical wisdom side, you might put opinion on the theory side and say there are opinions that pass for theories.

9. Concluding Remarks

Our group as a whole, as well as individual participants, worked hard to make the notion of theorizing in mathematics education problematic. Theorizing was challenged and questioned both in terms of its uses and its usefulness. It was studied to trace its sources and to consider its breadth and generality. It was challenged on the basis of its commonsense and "folkloric" alternatives. Perhaps the success of our working group is best measured by the wealth and breadth of thoughtful contributions which raised even more questions about theories and theorizing in mathematics education than were brought by its organizers.

Olive Chapman and Tom Kieren.

Working Group B

Popularizing Mathematics

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2. Introduction

In 1989, the International Commission on Mathematical Instruction (ICMI) organized at the University of Leeds (UK) a study seminar on the popularization of mathematics. This symposium was concerned with the general question of the nature and aims of the popularization of mathematics, as well as with the discussion of the problems faced in popularizing through particular media. The outcome of this seminar has been published in the ICMI Study Series (reference 1 provides a detailed description of the contents of these Proceedings). The purpose of this CMESG Working Group on "Popularizing Mathematics" was to identify and to examine some popularization activities that have been developed since the ICMI Study.

The Working Group started by attempting to reach a consensus on an informal definition and arrived at the following:

> *"Popularizing Mathematics" is any activity which makes mathematics more broadly accessible and acceptable to a wider lay audience, while also conveying a sense that mathematics itself is an area of human activity which extends far beyond school mathematics. The main objective in popularizing is not so much to teach content but rather is to change attitudes so that, as was expressed by one participant in the Group, the lamentation "I could never learn mathematics" would become as rarely heard as "I could never learn to read".*

Participants identified two different situations in which popularization activities can take place

- Popularization aimed at a CAPTIVE AUDIENCE (henceforth denoted by CA), that is individuals involved in a mathematical activity on a compulsory basis and over which the presenter has "full authority". Students in our classrooms (whatever the level of education) represent a typical instance of such a situation. Examples of such situations are presented in Appendices #2, and #4.
- Popularization aimed at WILLING PARTICIPANTS (henceforth denoted by WP), that is individuals involved in an activity on a voluntary basis. Typical of such situations are activities intended for the general public, such as Math in the Mall (see Appendix #1), Math Trails (see Appendices #5 and #1) or television programmes (see Appendix #6). In such a context there is no "audience" until the targeted individuals accept to participate.

The two situations above provided the general framework for the Working Group's discussions, which proceeded to explore them with respect to both the *aims* (Section 3 below) and the *process* (Section 4 below) of popularization. For the aims, the Group devoted its attention to the "WHY?" of popularization. For the process, the Group addressed the questions "HOW to do it?" and "WHO is involved?" When dealing with the latter the Group concentrated on the "FOR WHOM?" and did not have time to consider the "BY WHOM?"

3. Aims of Popularizing Mathematics

Even with considerable discussion the Group was not able to identify different aims for popularizing mathematics under the two CA and WP situations. Among the various aims identified by the participants, two were of a pragmatic and concrete nature. These are:

(1) from the point of view of the individual

Mathematics is required in a large segment of the job market. One of the central aims of popularizing mathematics should be to entice individuals to continue their mathematical education and thereby keep their career options open;

(2) from the point of view of society

The well-being of modern society relies heavily on the development and maintenance of technology, sciences both physical and social, etc., which in turn rely on individuals possessing a high level of mathematical understanding. Another central aim of popularizing mathematics should be to help meet this need of society by attracting a sufficient number of qualified individuals for these areas of study.

The other aims addressed the relationship of the individual to mathematics. These are:

(3) while most individuals might surround mathematics with an aura of respect, many have developed fear and even hostility for the subject (math phobia). Popularizing mathematics should aim to eradicate this fear;

(4) most individuals have experienced mathematics in a sterile and uninteresting environment. However, mathematics is much more of a cultural pursuit. Therefore another aim of popularization should be to engender appreciation of mathematics' long and rich history and cultural heritage, the role that it plays in today's life and mathematics as a human mental activity;

(5) mathematics, as a required component of school education, alienates a large portion of the population. These people do not feel part of "the club," they lack confidence in learning and using mathematics. Popularizing should aim to develop confidence in an individual's capacity for learning and using mathematics.

In conclusion the aim of popularizing mathematics is to stimulate people to engage in mathematical activity.

4. The Process of Popularizing Mathematics

Three process components of popularizing mathematics discussed by the participants. They were *FOR WHOM?*, *WHAT?* and *HOW?* In each case, both situations of CA and WP were discussed.

A — FOR WHOM?

i) CA situation In a certain sense all students must be taken into account when popularizing mathematics. However, because of their general negative attitudes towards mathematics and because of their future possible influence on others, students in three categories were identified as needing special attention of popularizing mathematics.

a) Elementary school teachers (both in pre- and in-service programmes). The majority of these teachers lack the appropriate mathematical knowledge and many have a genuine fear of the subject. Popularizing mathematics for these individuals should aim to help them develop a clear perception of the role and importance of mathematics in the development of humanity, and should underline the important role that they play in mathematics education at these introductory levels;

b) Pre-adolescent students. Many of the students in this age group drop out of mathematics, even though they continue to be physically present in mathematical classrooms they do not expand their limited mathematical horizons. The ultimate effect is to limit their career opportunities;

c) Students in university service courses. These students need to develop a mathematical understanding which empowers them to formulate the mathematics of the given situation. Courses for these students often tend to emphasize mathematical techniques, which are better performed by technology, and at the expense of mathematical concepts and understanding.

ii) WP situation While the whole population could eventually be seen as the target audience of popularization activities, the following categories were perceived by the Group participants as especially important.

a) Senior citizens. They have time, they are often very open-minded, and they can eventually exert some influence on their grandchildren;

b) Policy makers Many have opportunities to influence significantly the role that mathematics is to play in education;

c) Parents and children. The interaction between parent and child can provide positive reinforcement for the learning of mathematics;

d) Journalists. Because of their background and training journalists generally have little understanding of mathematics and yet they have the possibility of influencing many people;

e) Mathematicians. Few mathematicians have effectively popularized their activities or worked actively to explain their work and interest to others outside their profession. Mathematicians should be made aware of the importance of their role in this endeavour of popularizing mathematics.

B — WHAT?

The Group decided not to make any distinction between the CA and the WP situations in connection with this question. It recognized that no *a priori* restriction should be put on the mathematical topic used as a popularization activity. A variety of topics is important, since individuals may be "turned on" by different kinds of mathematical activities. Linking mathematics to social issues highlights its importance. Mathematical topics aimed at developing an individual's capacity for making decisions in non-deterministic situations such as the development of probabilistic thinking can be particularly useful. Finally, the topics chosen for popularization activities should convey the feeling that math is or can be fun. Popularizing mathematics ultimately aims at empowering citizens, at helping then to develop their critical judgment and to appreciate how mathematics can help in analyzing the world around them.

C — HOW?

Most of the following comments apply to the WP situation. Some also apply, with slight modifications, to the CA situation.

The medium used must reach the target audience wherever it is; in the video arcade, in the mall, at home in front of the television, in a bookstore, in a library, etc. To be appropriate, the material should have some of the following properties.

> The material should include some mathematical "big idea."
>
> The material should have a hook, a stimulus, to catch the attention of "spectators." This "attention getter" could be a particular device or instrument (the kaleidoscope, the Rubik's cube come to mind as examples), a pleasing pattern (visual, auditive, etc.), an attractive display, for example. Once the spectator is hooked, the activity should develop that attention.
>
> The material should convey an informal, pleasant and even fun atmosphere.
>
> The material should allow participants to access various levels of knowledge, allowing those who are more advanced to proceed deeper and deeper into the mathematics.

Some participants in the Group stressed the idea that a successful popularization module can be visualized as a multi-level procedure with the following steps:

1) capture the attention of the target audience;
2) convey some preliminary information;
3) get the people involved in some activities;
4) entice them to move on to another popularization sequence.

Individuals who have some experience of developing popularization materials believe that their success can be significantly enhanced when they are developed as a coordinated effort with policy makers, business people, media experts, etc.

5. Examples of Popularizing Mathematics

The Group included individuals with substantial experience in the development and implementation of popularizing activities. This is evident from the number of appendices attached to this report. The following examples provide reference to the appendices and other activities mentioned during the discussions.

ACTIVITIES FOR CAPTIVE AUDIENCES

1) **Backpack Math** Linking the Home and the School Through Math. Backpack math is a set of mathematical activities for children to take home and share with their parents. These activities are transported to and from school in backpacks, hence the name. This programme provides opportunities for parents to experience with their children the math that occurs in the classroom. The Family Math Program (developed at the Lawrence Hall of Science in Berkeley) is built on this idea of forming parent-child teams.

2) **Agriculture in the classroom** The use of food to develop math skills in geometry, measurement, estimation, graphing and problem solving. The Ontario Ministry of Agriculture and Foods has developed materials for such purposes.

3) **Mathematica's Mathshop** TV Ontario series of 10 fifteen-minute programmes for Primary Mathematics within the context of stories.

4) **Math Shop** The Open Learning Agency of British Columbia; see Appendix #6 by Tom O'Shea.

5) **Mathémathlon** See Appendix #3 by Bernard Hodgson.

6) **Mathematicians and their Society** See Appendix #6 by Tom O'Shea.

ACTIVITIES FOR WILLING PARTICIPANTS

1) **Math in the Mall** See Appendix #1 by Malgorzata Dubiel.

2) **Math Trails** See Appendices #5 by Eric Muller and #1 by Malgorzata Dubiel.

3) **Kaleidoscopes** See Appendix #3 by Bernard Hodgson.

4) **Newspaper articles** Keith Devlin described the work that he did for a column in England (see reference 4 below). He indicated that it was very difficult to get such a column, one has to be around just at the right time when an editor is looking for something specific.

5) **Popular mathematics books** for specific sectors of the general public. Keith Devlin compared the different approaches that were taken in the preparation of his two books *Mathematics: The New Golden Age* and *Mathematics: The Science of Patterns* (see references 3 and 5 below). After these experiences he is of the opinion that mathematicians writing a book for general audiences should involve someone else —a strong copy editor. This made the work far more demanding but it brought in a very different perspective.

6) **Television programmes** Keith Devlin spoke about his work on two different television projects developed in England. The first, *Mathematical Mystery Tour*, is aimed at sophisticated audiences and is now available as a Time Life Video. In this series mathematics is presented as human culture. The second, *The Johnny Ball Show* developed by the BBC, was aimed at children of 6–14 years. This was followed by another series in which Celia Hoyles was involved; called *Think of a Number*

it was not trying to impart mathematics but rather to bring awareness of mathematics in human culture (see the paper by C. Hoyles in reference 1 below). See also Appendix #6 by Tom O'Shea.

7) **Public lectures** Some departments of mathematics sometimes venture into organizing lectures intended for the general public. Two such talks were organized during 1993–4 by the Number Theory Group of Université Laval. The first was about Fermat's Theorem (a must in 1993!) while the other, given by Peter Hilton, concerned issues of code-breaking during the Second World War. The latter was publicized in particular among war veterans; they were really pleased both by the fact that organizers had thought of them and also by the lecture itself, which was given just at an appropriate mathematical level.

6. Towards a "didactical engineering" product

The group identified a number of characteristics which appeared to be linked to the success of a popularizing mathematics activity. The activities should be multi-level, an example of this concept being the *National Geographic* magazine — one can first be engaged by looking at the pictures; then if it is found to be interesting one can follow it up by reading the descriptions under each picture; and finally, one becomes completely engaged if one reads the articles.

In many cases the activities appear to be successful when the mathematician involves other specialists in such important areas as communication, child studies, writing, etc. Test trials with a segment of the target population also appeared to enhance the success of the activities.

References

The main report on the ICMI Study on Popularization of Mathematics is:

1. Howson A. G. and & Kahane J.-P. (1990) *The Popularization of Mathematics*, Cambridge University Press. (ICMI Study Series)

Its Table of Contents is as follows:

Sowing mathematical seeds in the local professional community (T. Shannon)
Mathematical news that's fit to print (L.A. Steen)
Christmas lectures and mathematics masterclasses (C. Zeeman)
Some aspects of the popularization of mathematics in China (D.Z. Zhang, H.K Liu and S. Yu)

Examples of books aimed at the popularization of mathematics are:

2. Davis, P. and Hersh R. (1980). *The Mathematical Experience*, Birkhäuser.
3. Devlin, K. (1988). *Mathematics: The New Golden Age*, Penguin.
4. Devlin, K. (1994). *All the Math That's Fit to Print*, Mathematical Association of America.
5. Devlin, K. (1994). *MathematicsThe Science of Patterns*. Freeman.
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12. Stewart, I. (1987). *The Problems of Mathematics*, Oxford University Press.

Appendix 1

A Pair of Examples of Popularizing Mathematics

Malgorzata Dubiel
Simon Fraser University

Loon Lake Trail

In August 1993 Science World BC organized a week long science camp for teachers at Loon Lake in the UBC research forest. Kathy Heinrich and I were invited to give a 3-hour workshop. We decided to organize our workshop as a mathematics trail (for more information on math trails, see the Appendix #5 by Eric Muller). The camp site is a peninsula which projects into Loon Lake. We went there earlier, to look at the terrain, find any interesting features, and make a map, as the organizers were not able to provide one.

In preparing the trail questions, we concentrated on three themes; measure, basic combinatorics, and patterns and shapes. Participants were strongly encouraged to add their own questions to the booklet. On the cover of the *Trail* booklet we put our map; the next page gave helpful hints and formulae. We decided not to give people a tape measure, just a piece of string, and asked them to devise their own way of measuring. One of the hints, illustrated by a famous sketch by Leonardo da Vinci, was that when you stretch your arms, the distance between the tips of your fingers is approximately equal to your height. Kathy, I, and two of our colleagues we took along, each took a small group on the trail. Each of us started at a different location — we designed the trail so that this was possible. My group found it easy to construct a measure. A woman said that she was exactly 150 cm high. By measuring the distance between the tips of her fingers, folding the corresponding piece of string into three and doubling the result, we had a length of string approximately 1 m long.

Combinatorics problems were mainly concerned with assigning people to their living quarters, and seating arrangements at the cafeteria tables.

The feedback from participants was very positive. They had fun with the trail—something most of them did not expect. They commented that the measuring exercises gave them a better insight into the concept of measure and the meanings of various units of measure. We wrote our trail using imperial measures. Teachers asked that the questions be changed to ones using metric units because the curriculum is based on the metric system. Many of them admitted to not feeling comfortable with it. After working through the questions and discussing them with us, they felt a sense of achievement.

The combinatorics/scheduling questions also elicited favourable comment. Teachers, especially those from the early grades, said they would try our ideas in scheduling their classroom activities.

Looking for patterns and shapes was the most novel activity we gave the participants. One of them said jokingly afterwards that we had spoiled the forest for her—now she keeps seeing patterns and shapes all around her.

This year the science camp will be repeated twice, and we have been invited back. I went to the orientation meeting for workshop leaders. The first thing I noticed was that our map was given to participants as part of the information package. The organisers cited our approach of connecting the workshop to the surroundings as one of those which had proved most successful in the past.

We are making some changes in the trail for this year. Metrization dictates the redesign of some of the problems. The trail will also be longer, to give teachers more ideas to take back to class. We will stress that they are not expected to finish answering the questions during the workshop.

The success of the *Loon Lake Trail* inspired us to introduce a mathematics trail exercise as a group assignment in the course *Mathematics for Elementary School Teachers* at Simon Fraser University. Students design a trail around the campus as the first part of this assignment. Each group is required to hand in two copies of their trail. The second part of the assignment consists of testing a trail designed by another group.

Math in the Mall

Kathy Heinrich and I discovered the need to popularize mathematics for general public while working with students taking Simon Fraser's *Mathematics for Elementary School Teachers* course (MATH 190). A very high percentage of students taking this course have an extremely poor attitude towards mathematics—they believe it is boring, difficult, and probably beyond their reach, not to mention that it is also completely removed from real life. We started to look for ways to help them to see a glimpse of what we see in mathematics—the beauty of it; the excitement and amazement upon seeing something unexpected; the mathematics around us; and the people of mathematics. We developed (or found in books) various projects to develop the students' interest in mathematics. Later, we asked them to prepare their own projects, as one of the course assignments.

The projects proved very succesful and we started to think about reaching younger children. Our first opportunity was the *Homecoming* event organized for the 25th anniversary of Simon Fraser University in September 1990. We mounted a display called "Is this Math?" We used some of our Math 190 projects as well as puzzles, games, books, geometrical models, and a videotape with a week's worth of "Square One TV" courtesy of PBS. The display was even more succesful than we expected. When colleagues from computing science complained that our display was diverting traffic from their exhibits, we knew we were on the right track. The president of SFU visited the display and was sufficiently impressed to promise funding for us to take the display to other sites.

While planning our exhibit for Simon Fraser's homecoming we were already discussing the possibility of taking our activities to a shopping mall. Malls are now centres of social life in most areas. You meet people there who would never go to a university event, people of all ages, and often whole families.

For our first foray outside the university we chose Lougheed Mall—a large mall in Burnaby, close to SFU and to several elementary schools. While preparing the event, we were contacted by organizers of the Science and Technology Week 91 and, at their request, agreed to repeat the display three weeks later in another Burnaby mall, Metrotown. Since then, our mall appearances have been restricted to Science and Technology Weeks and similar events, as this makes the organization much easier. We also take some of the activities to schools and occasionally have groups of children visit the Department.

From the outset our displays have been enormously popular with children. This has pulled in parents interested in activities for their children and teachers looking for ideas for classroom activities.

To attract attention to our mall display we have a large custom-made sign MATH & MAGIC (2.25 m high). Another large sign advertises Simon Fraser and the Canadian Mathematical Society. There are also colourful posters about the relation of mathematics to design, arts, crafts and other areas of human activity not usually associated with mathematics in people's minds. Other posters have a more humorous intent with lots of jokes about mathematics and mathematicians. We also have interesting models—a huge kaleidocycle, colourful models of platonic solids etc. The activities we have found most succesful include making and decorating kaleidocycles and hexaflexagons, Mobius bands, geometrical models from flexible drinking straws (we use up to 6000 straws a day!), and pentagonal stars made from strips of paper. These are especially popular with elementary school kids. The older children and adults enjoy puzzles—geometrical ones and these based on Gray codes are the best—and games. At some displays we also have a VCR and computers, but this is not really essential.

We recruit colleagues, graduate students, and undergraduates to assist with the display. At all times we have at least five people to talk to members of the public attracted to the display. Students who have taken, or are taking, MATH 190 like to help, partially because they have to demonstrate volunteer work for the admission to professional programmes in education. We never have difficulties in finding help when repeating our display. Those who help once usually come back; they have fun participating.

We have many wonderful memories from the displays, and some touching letters from children. It is difficult to assess whether we make any impact—we believe we do. Recently, for example, I met the father of a girl who came to our first display three years ago. He told me that MATH & MAGIC had sparked her wish to study mathematics and science when she eventually goes to University.

Appendix 2

Preparing Secondary School Teachers to aid in Popularizing Mathematics

Harvey Gerber
Simon Fraser University

The Working Group on Popularizing Mathematics at this CMESG conference felt that teachers should be educated about the broader aspects of mathematics. This paper looks at the work in this direction done at Simon Fraser University for secondary school teachers. In particular, it looks at one mathematics course in a master's programme.

Foundations of Mathematics is the first course in a series of six courses for the Master's Programme in Secondary School Mathematics Education at Simon Fraser University. The programme stresses the human aspects of mathematics. It emphasizes the role of mathematics in society and the natural development of mathematics as a growing, changing, entity. As the first course in this programme, *Foundations of Mathematics* sets the tone for the entire programme.

Foundations of Mathematics looks at various areas of the secondary school curriculum (including calculus) from a historical, and sometimes philosophical point of view. The emphasis is on the mathematical problems at certain moments of history, and how these problems were resolved. We are not only interested in the subject, but also in the way it evolved and the reasons for its evolution. Mathematics is shown as a subject created by people. The intent is to show mathematics in the making rather than as a finished product.

While the students entering the programme have a fairly strong mathematics background, their understanding comes from material they learned in isolated, seemingly unrelated, courses. One of the purposes of *Foundations of Mathematics* is to integrate the students' fragmented knowledge. We must make evident the connections between the supposedly separate topics.

Three things are taking place in the course. In the first place, the students are presented with the historical development of the real numbers (along with the mathematics required). We begin with the Pythagoreans and Eudoxus and continue to Cantor, and Dedekind's work. We close with a description of the nonstandard real numbers used in nonstandard analysis.

The second aspect of the course has the students read and present the material from *Journey through Genius* by William Dunham. This text presents certain great theorems in mathematics along with historical background to illuminate these theorems.

Finally the students have to write a major paper on the development of some topic from the secondary school curriculum (again including calculus). The papers are presented to the class. A brief sample of the papers presented include history of the influences leading to the development of analytic geometry, the background to Cantor's work on set theory, the development of trigonometry tables, and the development of algebra as a deductive science.

We feel that the courses and the programme have been very successful. The secondary school teachers walk away with the knowledge that mathematics has an influence on, and is influenced by, the culture of its time. Moreover, these teachers are actively engaged in learning and spreading the word. Witness the fact that two participants at this meeting have been students in that programme.

Appendix 3

Deux facettes de la vulgarisation des mathématiques
Bernard R. Hodgson
Université Laval

Comme il est clairement ressorti des discussions de ce Groupe de travail, la vulgarisation des mathématiques peut se réaliser dans des contextes et avec des moyens fort variés. Le but de ce texte est d'illustrer brièvement certains aspects de la vulgarisation à l'aide de deux activités, l'une conçue spécifiquement à l'intention d'élèves du primaire et l'autre pouvant être utilisée dans des situations diverses.

Le Mathémathlon

Les compétitions mathématiques sont souvent perçues comme des activités plutôt élitistes ne touchant qu'une portion infime de la population étudiante. De nombreux contre-exemples existent à ce propos, depuis les compétitions australiennes jusqu'au Kangourou mathématique français, en passant par le concours américain ASHME.

Au Québec, l'Association des promoteurs de l'avancement de la mathématique à l'élémentaire (APAME) a mis sur pied depuis une dizaine d'années une activité touchant de même un très grand nombre d'élèves du primaire. Connue sous le nom de Mathémathlon, cette "compétition" biennale s'inscrit tout à fait dans une démarche de vulgarisation mathématique, non pas seulement à cause de la quantité de jeunes impliqués, mais principalement en vertu de la philosophie qui la sous-tend. Il ne s'agit plus d'une compétition au sens classique, dans laquelle "le meilleur" (ou "la meilleure") gagne, mais d'une activité qui s'inscrit dans une vision pédagogique plus globale touchant même l'intervention de l'enseignante dans sa classe. En plus de fournir un cadre propice à une démarche en résolution de problèmes, on y cherche surtout à favoriser le développement d'habiletés et d'attitudes non traditionnelles dans la classe de mathématiques travail d'équipe, communication, analyse de stratégies possibles, identification d'objectifs spécifiques à une équipe, etc. L'enseignante qui inscrit son groupe au Mathémathlon se voit proposer une démarche qui l'accompagne pour une partie importante de l'année scolaire.

Qu'il suffise de mentionner que lors du dernier Mathémathlon (année 1993-94), 167 056 élèves du primaire (3e, 4e, 5e et 6e années) ont participé aux épreuves locales, dont 3 500 hors du Québec. (Ces épreuves étaient ensuite suivies d'épreuves régionales, avec des représentants de chaque classe, et enfin d'épreuves nationales.) Cela fait près de 7 500 enseignantes dont le travail quotidien a ainsi été influencé par la participation au Mathémathlon. Compte tenu du message que cherche à véhiculer le Mathémathlon quant aux mathématiques de l'école primaire, cela constitue certes une réalisation intéressante du point de vue de la vulgarisation mathématique.

Pour obtenir plus de renseignements sur le Mathémathlon, prière de s'adresser au Secrétariat de l'APAME à l'adresse suivanteAPAME, Case postale 300, Terrebonne, Québec J6W 3L5.

Le kaléidoscope

Très souvent, une activité de vulgarisation, et cela est certainement le cas avec un auditoire non captif, requiert une bonne amorce afin d'attirer l'attention du public visé et de susciter son intérêt. Cette amorce peut être de diverses natures situation choc, image fascinante, objet intriguant, etc.

Le kaléidoscope peut certainement être vu comme un objet à la source d'un cas assez unique de succès dans l'histoire de la vulgarisation scientifique. Dès son invention par sir David Brewster au début du XIXe siècle, il a suscité le plus vif intérêt et a même provoqué dans la population une excitation qui n'est pas sans rappeler, en des temps plus récents, la rage du cube de Rubik de la fin des années 70. Dans une lettre à sa femme, Brewster rapporte en effet ce qui suit "You can form no conception of the effect which the instrument excited in London; all that you have heard falls infinitely short of the reality. No book and no instrument in the memory of man ever produced such a singular effect." Rapidement, nous dit Brewster, des centaines de milliers de kaléidoscopes furent fabriqués, quoique bien peu respectaient, se plaint-il, les règles qu'il avait énoncées en vue de l'obtention d'un "bon" kaléidoscope (voir [B]).

La fascination exercée par le kaléidoscope a été décrite en termes on ne peut plus clairs par André Gide au tout début de son ouvrage *Si le grain ne meure* "Un autre jeu dont je raffolais, c'est cet instrument de merveilles qu'on appelle kaléidoscope. (...) Le changement d'aspects des rosaces me plongeait dans un ravissement indicible. (...) Bref, je passais des heures et des jours à ce jeu. (...) J'étais intrigué autant qu'ébloui, et bientôt voulus forcer l'appareil à me livrer son secret."

Même dans notre monde d'aujourd'hui, l'intérêt d'un instrument tel le kaléidoscope ne se dément pas (quoiqu'il ne suscite évidemment plus l'euphorie de l'époque de Brewster). La beauté de la rosace kaléidoscopique attire immanquablement l'attention, et l'instrument se prête très bien à une exploration visant à développer un modèle géométrique approprié. Brewster lui-même vantait d'ailleurs les mérites de tels objets qui, reposant sur des principes scientifiques, ne pouvaient, soutenait-il, qu'amener l'observateur vers une démarche et une connaissance systématiques.

Le kaléidoscope se prête évidemment bien à un activité libre du type "Les maths au centre commercial" (voir Appendice #1). J'ai moi-même eu à maintes reprises l'occasion de l'utiliser avec des auditoires captifs, dans le contexte de la formation des enseignants du primaire et du secondaire. Dans un cas comme dans l'autre, on y retrouve de nombreux ingrédients d'une vulgarisation fructueuse émotion soulevée par la beauté de l'image observée dans le kaléidoscope ; nombreuses possibilités d'expérimentation et de découvertes ; facilité de modification des données de base (par exemple, l'angle entre les miroirs, le motif à reproduire, le nombre de miroirs) de façon à mieux comprendre et contrôler la situation ; diversité des modèles, depuis des modèles très simples (miroirs réels, papier pointé) jusqu'à des versions plus sophistiquées (simulation sur ordinateur permettant à la fois la décomposition étape par étape du jeu d'interaction des miroirs et la mise au point de "kaléidoscopes fictifs" reposant sur des transformations autres que la réflexion axiale) ; liens avec la vie de tous les jours (miroirs parallèles ou à angle que l'on rencontre au salon de coiffure ou au magasin de vêtements, motifs géométriques parfois utilisés à la télé, etc.) On trouvera plus de commentaires sur certains de ces aspects dans les articles [H] et [GH].

Références

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Appendix 4

Popularizing mathematics in the classroom

Joanne McDonald
University of Regina

How can we popularize mathematics in the university classroom? Perhaps it can never be very popular with so many of our students taking the subject simply to satisfy a degree requirement. Many of them are afraid of mathematics, are convinced they can't do it, or believe it to be nothing more than number-crunching and manipulating formulas.

I have made several attempts to stimulate the interest of students who feel they are in a mathematics course through no fault of their own. Some of these attempts are described below. The success rate has not been uniformly high. While some students responded very well to the assignments, others sustained a high level of hostility in the face of all my efforts to improve their outlook.

Mathematical autobiographies

On the first day in each of my courses I assign a mathematical autobiography, usually to be submitted the next class day. The write-up is to be between a half-page and a page. Individual students may choose to write about their feelings toward mathematics, their perceived successes or failures, why they are taking the course and how they expect to use the knowledge they acquire, or whatever comes to mind when they hear the word 'mathematics'. Students are assured that I will not hold against them anything uncomplimentary they may say about mathematics.

I continue to assign autobiographies as I have found they are not only a good way for my charges to vent any hostility and clear the air for getting down to work on the course material, but they also give me a chance to get to know my students better. Another advantage for me is that there is no grading to be done; these assignments are read purely for my own enjoyment and information.

Term papers

Two years ago, in a class of 104 student teachers (elementary school), each of my students was required to submit a term paper worth 10% of the final grade. There was a great deal of 'math anxiety' in this class, as well as a conviction that mathematics never was and never would be something they could do. Many had the naive hope that since they were not interested in the subject they would never be called upon to teach it.

My reason for the assignment was to have these students feel more confident about mathematics. The class was informed on the first day of the course that a term paper was a required component. Within the first 3 weeks of the semester they were supplied with information on possible topics, suggested length of paper, the due date, and a list of reference sources available in the on-campus libraries. There was freedom to choose a topic other than a listed one, providing they checked with me to be sure the choice did actually have to do with mathematics. I did not require any in-depth understanding of mathematics, as these students often have very little background in the subject. Papers were due 2 weeks before the end of classes, that is, some 8 or 9 weeks after the detailed requirements were in the students' hands. Some of the suggested topics were:

- a short biography of a famous mathematician (several names were supplied, e.g. Gauss, Galois, Noether, Newton, etc.);
- a history of the development of the calendar or of numeration systems;
- a detailed explanation of why 'galley' multiplication works.

In this assignment, response exceeded any expectations. Of the 104 who were on my class list, 98 turned in papers and they were, for the most part, very well done. Some were of publication calibre. A prodigious amount of my time was required for grading, so the undertaking was overly ambitious on my part.

Should I teach this course again, the papers would be optional and they would make up a larger proportion of the final grade. On the trial run I had no idea the students would put in so much effort. They really deserved more than the agreed-on credit.

Micro-themes.

My first experience with assigning optional micro-themes was during a calculus class a year ago. There were many repeaters in the class. At least two students were taking the course for the *fourth* time, and for several it was the last chance to get the required math credit for a degree. The level of fear was extremely high.

Some of the pressure was relieved by allowing the students to choose any number from zero to all of the optional micro-themes, However, once an initial submission was made, there was no longer a choice about counting that micro-theme's result as part of the final grade. It would be counted.

There were 6 micro-themes in all. Each was a problem or procedure found in a typical first course in calculus. The student had to solve the problem without using derivatives, or explain the procedure used. The date for the initial submission for each project was chosen to correspond to material already covered in the course. Between the first and last submission dates for each project, the student could redo and resubmit it any number of times. To keep my marking time manageable, the assignment was given a mark of 0 (if anything was wrong) or 10, if done correctly. (I may indicate where an error occurs, but it is the student's responsibility to detect the error and correct it, before redoing the assignment for the next submission.) Each project was to include a brief but complete explanation, in good English, of what was done and why it was needed. A 2.5% block of the final grade was reserved for each micro-theme a student elected to do. Thus, if all six were completed correctly, the final exam was worth only 35 points instead of the usual 50% of the final grade.

One micro-theme was stated as follows *Guy wires are used to stabilize two poles that are of heights 35 feet and 15 feet respectively, and which are set 100 feet apart on level ground. One of the wires is attached to the top of each pole and to a peg at ground level between the two poles. Where should the peg be placed so the least amount of wire is used? Include a diagram and explain your reasoning carefully.* (Solve this problem *without* using derivatives.)

Another example was based on a problem solved in the text that found the 'most economical' proportions for a cylindrical can to have the diameter and the height equal. These questions were to be answered:

(a) What is the relation between the height of the most economical can and its diameter?
(b) Check some cans at home or on grocery shelves. What are some of the proportions? Do they match the answer in part (a)?
(c) If the results in (b) are different from those in (a), why are cans not made with these 'most economical' proportions? Provide a minimum of two distinct reasons. (Your reasons must be based on the idea of maximizing profits.)

(One student got so interested in this problem that he called the toll-free number for two food processors to ask the reason for their choice of can proportions. One answer was that "the size and shape of the can depends on the product inside it", the other was "I don't know, I never took calculus.")

None of these attempts was entirely successful, but student response was generally positive. The micro-theme projects were worthwhile for several reasons. The students had to really think about what they were doing. The written explanations required the use of correct sentence structure and clear, unambiguous statements. If submissions were carelessly done, it was soon obvious that it took precious time to redo the assignment correctly. It was clearly to the student's advantage to do a proper job the first time. I also hoped the message would get through that even if we were doing calculus, that did not mean calculus methods were always the best means for finding a solution.

Appendix 5

Mathematics Trails

Eric Muller
Brock University

What is a Mathematics Trail? A recent short publication [1] by the Association of Teachers of Mathematics in the UK says "Math trails are the mathematical version of nature trails". The aim of a nature trail is to observe and study nature. This suggests that a mathematics trail should point to applications of mathematics which are found in most situations and should also involve individuals in mathematical activities to stimulate appreciation of their surroundings. These possibilities inspired me to develop a set of mathematical activities that children, students, parents and teachers can experience in the environment in which they occur. School classes periodically take field trips as part of their history or geography courses. How often do students go on mathematics field trips? Rarely! We should not be surprised that most youngsters see mathematics as a human activity with little or no relevance to their daily life. If you are, like me, continually looking for activities to stimulate the interest of youngsters in mathematics then a Mathematics Trail may be what you are looking for.

This short article will take you through the various steps I used to develop two Mathematics trails. The first task in making a trail is to look for a location. This can be in the city, in the zoo, around the university (for freshmen orientation), in a provincial park or any location which attracts young people. Living in the Niagara Peninsula, I had two obvious sites—the Niagara Falls and the Welland Canal. These two areas have interesting natural history and are centres of human achievement. They attract people from all over the world including student groups from various parts of Canada and the United States. In both these cases I found the areas too vast to cover in a reasonable amount of time. I therefore concentrated the activities around major points of interest close to where the mathematics trails are distributed. Early in my planning I received support from the Niagara Parks Commission. This included the use of their materials and a commitment to have the *Niagara Falls Math Trail* booklets distributed from the terminal of their People Mover—a transportation system developed to ease car traffic around the Falls. For the *Welland Canal Math Trail* the City of St. Catharines offered to have the booklets distributed from their tourist facility at Lock 3.

A source of funding never hurts! I placed a high priority on funding those parts of the projects which would ensure the long term survival of the trails with little or no intervention on my part. I was fortunate to receive funding from a number of sources. To-date these include Science Culture Canada, SEED programme, Casio Canada Ltd., the Niagara Parks Commission and a number of Offices at Brock University. Further sponsorships are being sought to ensure that the trail booklets will be printed in large quantities.

There are a number of publications which suggest that youngsters appear to lose both their interest in and their confidence to do mathematics when they are between 10 and 14 years old. I therefore targeted the mathematical activities at this age group. A student was hired from Brock's concurrent BSc/BEd programme for mathematics teachers at the Junior and Intermediate grades in Ontario and together we worked on the activities for the booklets. What are appropriate Math Trail activities? To me, they should involve mathematics and should satisfy a majority of the following criteria: they should be fun, they should arise naturally from the situation, they should call attention to some of the remarkable natural phenomena or human achievements such as art, history, technology, etc. For the 10 to 14 year old age group we looked for applications of geometry, statistics and elementary algebra. By looking at existing Math trails one finds that their contents can vary in emphasis. Some, like the one developed by Blane and Clarke [2] in Melbourne and the one at ICME-7 in Quebec City by de Champlain, Gaudreault, d'Entremont [3] could be classified as mathematical classroom activities on a walk, while others, like the one developed by Fasanelli, Rickey and Thorington [4] for the Washington Mall, could be classified as observations of mathematics on a walk. The *Niagara Falls* and *Welland Canal Math Trails* are strongly influenced by both of these approaches. They are written in a dialogue between two youngsters expressing their knowledge of, and asking questions about their surroundings. They use mathematical ideas to explore possible solutions.

A preliminary draft of the Niagara Falls Math trail was tested with a young girls' soccer team and again with a grade 7/8 class. This experience was invaluable in completing the final version. Reactions expressing, dismay, enjoyment, bafflement etc., all contributed to a very different final trail booklet. The comments of the teacher and adults were very helpful. I strongly recommend this field testing. I was also happy to receive critical input from a writer in Brock's External Relations Office.

One of my aims was to develop a set of mathematical activities which would stand on its own, that is, I would not be involved each time a school group or family wished to do the trail. This has been partially achieved. It will become a reality when I have completed an introduction sheet for the teacher or group leader. I believe that a small manual of follow up activities with references to existing materials would also be helpful to the teacher. It is so much easier to generate interest in mathematics and its applications when they are based on experiences of everyone in the class.

This project has taken more time than I had originally estimated. However it is possible to develop a Math Trail on a smaller scale, for example, one of the university campus for visiting school groups. It does put a different perspective on one's surroundings! Good Walking.

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[2] Blane, D. and Clarke D, *A Mathematics Trail around the City of Melbourne.*

[3] de Champlain, D., Gaudreault, L.-P., et d'Entremont, M. M., *Mathematics Trail in Old Québec.*

[4] Fasanelli F., Rickey V.F., Thorington R., *Math on the Mall.*

This article first appeared in a slightly modified form in the Canadian Mathematical Society NOTES (1993)

Appendix 6

Two Examples of Popularizing Mathematics

Tom O'Shea
Simon Fraser University

From my point of view, two issues were of particular importance in this Working Group. The first concerned the problem of educating teachers about the nature of mathematics and the place of mathematics in our society. The second examined the broader issue of educating the general public about mathematics. For each issue, I have chosen one example of work that I have engaged in over the past several years.

Mathematicians and Their Society

Students in my mathematics methods course need to understand that mathematicians work within a social context; that mathematics does not develop in isolation—that it is a human enterprise. In my opinion, they cannot develop this understanding by reading a textbook on the history of mathematics nor a series of biographical sketches. They need to immerse themselves in the life and time of a mathematician.

Generally the mathematics methods class enrolls 30 to 40 teachers. For this assignment I chose the following mathematicians: René Descartes (1596–1650), Blaise Pascal (1623–1662), Isaac Newton (1642–1727), Évariste Galois (1811–1832), Ada Lovelace (1815–1852), Nikolai Lobachevsky (1792–1856), Sonya Kovalevsky (1850–1891), and Georg Cantor (1845–1918). This group spanned three hundred years of mathematical thought. I made sure to include female mathematicians to emphasize to the predominantly female group of student teachers that mathematics is becoming an equal opportunity employer.

I randomly assigned the students to mathematicians and presented them with a list of questions to answer for the first week. They were expected to work on their own and find their own resource materials. I gave them a second set of questions in the second week, and a third set in the third week. At the end of the sixth week, the students submitted written reports in response to the questions.

To complete the task, all students assigned to a particular mathematician formed a discussion group, and responded to a final set of questions. These discussions were tape-recorded and formed part of my evaluation of the students' work. They also provided feedback on how the assignment might be improved next time around. The assignments went as follows:

Let us designate the mathematician you will become intimate with by the letter M.

Week #1

In what year was M born? In what year did M die?
In what country was M born? In what country did M die?
Outline M's main contribution to mathematics in terms that you understand.
In what decade did M make this contribution? Upon whose work did M build?
Were M's mathematical ideas values at the time? If so, by whom? If not, when was recognition first given?
How would M exchange information with other mathematicians?
Who were M's acquaintances? Name some friends. Name some enemies. Why would they be enemies?
Did M marry? Have any children?
Was M happy? Why (not)?
Make up an epitaph for M's tombstone. What does it say?
Make up a picture to engrave on M's tombstone. Show it.

Week #2

The following questions are to be considered for the decade of M's most productive work.

In what country, call it C, did M reside?

Was this country the same as the one in which M was born? If different, why?

Name the countries or states having borders with C. Identify the most important of these.

Who was the head of state in these latter countries? How did M get along with his/her head of state?

Would M have been comfortable working in any of the neighbouring states? Why (not)?

What was M's professed religion? Did M practice that religion? How was that religion viewed at the time?

Did M's religious beliefs have any influence on M's mathematical thoughts or motivation?

After a hard year's work M decided to visit one of the recently discovered areas of the world. Where might M have gone? Why? If M had decided to go to "Canada" for a holiday, how would M have traveled? Sketch one of the vehicles used. How long would it take? Where might M have gone in "Canada"? Draw a sketch of the region which we might think of as "Canada" at that time.

Week #3

On Saturday night M went out for dinner. Would M have taken his or her spouse? Why (not)?

M met some friend at the restaurant and they spoke animatedly of the latest scientific developments. What topics might they have discussed? Would M have approved?

After dinner they all went out to the theatre to hear the latest musical entertainment. What did they go to? Did they like it? During the intermission they talked of art, and M expounded a length on his or her favourite artist (A) of the day. Who might that have been? Why would M have been attracted to A's work? Draw a sketch of a subject as A would have painted it.

Mathematicians Rap-up [for the group discussions]

1. Comment on the quality and quantity of information on M which you were able to find. Were there any surprises?
2. Did M contribute in areas other than mathematics to the development of human thought and understanding? If so, comment on the relative significance of those contributions.
3. What is your personal feeling about M? Affection? Admiration? Distrust? Contempt? Empathy?
4. Would you have liked to have been M? Given the benefit of hindsight, what would you do differently if you had been? Would you have liked to live at the time of M?
5. Would the world be different if M had been still-born?
6. Did this assignment contribute to your own understanding of mathematics? To your personal understanding of the world? Was it worth doing? How might if be reconstructed or improved for future use? Would an assignment of this type be useful in your own teaching? If so, how might it be modified for use with younger people?"

I intended in this assignment to move students week by week further into the details of M's life and times. Students could answer most of the first week's questions by consulting standard biographical sources. The second week's questions caught the students by surprise; they had not expected to deal with anything other than mathematics. Now they had to examine the political, religious, and geographical ideas of the time. By the third week, I think they were prepared to make subjective assessments of M's personal feelings with respect to matters of scientific, musical, and artistic importance of the day. The questions in the final discussion were designed to allow students to share their insights into the soul of the mathematician, to speculate on the significance of his or her contribution to mathematics, and to think about how they might modify the assignment for their own students.

Math Shop

The Open Learning Agency (OLA) of British Columbia provides opportunities for learning through the flexible delivery of courses and programmes using a variety of learning tools. It most commonly is thought of as a "distance education" institution. Means for learning include written course materials, telephone tutors, classroom-based lectures, audio tapes, videotapes, and television programming.

In 1993–4, I assisted OLA in developing a 16-part series of 30-minute television programmes to support its Adult Basic Education course Mathematics 11. The series was designed to illustrate mathematical concepts that can benefit from a visual treatment. The programme, *Math Shop*, uses a combination of real-life applications, computer graphics, and instructor demonstrations.

Because the series is televised on an open channel, anyone with access to basic cable service may view the programme. Thus the programme had to be designed with three audiences in mind students who were enrolled in OLA's ABE Mathematics 11 course, public school teachers who might want to tape the programmes off-air to use in their regular Mathematics 11 classes, and general viewers who watched as a matter of interest. It is the presence of the latter group that concerns us here, as the programme becomes a means to popularize mathematics and promote mathematical understanding in the general population.

The setting for the series is a "store-front" operation called the *Math Shop*. The two hosts are Christine (a technological expert) and Kanwal (a mathematics consultant). Euclid (Christine's super computer) assists in various novel ways. The series was designed to fit the Mathematics 11 curriculum, and the topics consist of the usual textbook-type descriptors: real numbers, equations and inequalities, data analysis, basic trigonometry, etc. Each episode begins with a mathematical problem, most commonly submitted by one of Christine's many cousins. Christine and Kanwal discuss the problem usually using graphics or video clips from the "field" camera. This leads into the main topic for that day. For example, in the programme on perimeter, area, and volume, Christine begins by deciding to landscape the grounds of Math Shop and discovers the importance of using the same units of measure when a truck delivers 3 cubic centimetres of soil instead of 3 cubic metres. Euclid provides computer graphics to help calculate the perimeter of the Math Shop yard, and the two hosts determine the relationship between the speedometer and odometer of a car and the circumference of its wheels. Euclid helps calculate the volume of soil needed to dress the lawn, and later in the programme Kanwal and Christine visit the beach to calculate the area of a triangle. Kanwal orders pizza to help calculate the area of a circle. The volume of two cans is calculated and Kanwal explains why that despite their different shapes the volumes are nearly equal. The programme ends with an environmental perspective by showing why cans of the same capacity may have different surface areas, causing one to use more metal than the other.

A Study Guide is being written for each episode, and it is expected that these will be completed during the summer of 1994. Viewers in British Columbia may tape the series off-air for later viewing or to use in instructional situations. It is expected that by Fall 1994 the video series and study guides will be available for sale singly or as a package. For up-to-date information on cost and availability, those interested may contact Craig Nichols, Marketing Department, Open Learning Agency, 4355 Mathissi Place, Burnaby BC, V5G 4S8.

Appendix 7

Hands-on activities through Dissection-Motion Operations (DMO). A physical justification for "The sum of the angles of any triangle is 180•"

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Lakehead University

It is known to all of us that in order to popularize mathematics, one should make mathematics less threatening and start right from the early levels of schooling. The construct of *mathematics anxiety* has received close attention in recent years, both among researchers, and mathematicians and mathematics educators (Chin & Henry, 1990). Richardson and Suinn (1972) defined mathematics anxiety as "feelings of tension and anxiety that interfere with the manipulation of numbers and the solving of mathematical problems in a wide variety of ordinary life and academic situations" (p 551). Fennema and Sherman (1976) defined it as "feelings of bodily symptoms related to doing mathematics" (p 4). Through my personal observations of the students' behaviours in my classes and my own experience as a mathematician and mathematics educator, I believe that more accessible mathematics through hands-on activities will reduce anxiety. It will make attitudes less hostile toward mathematics, mathematics practitioners, and mathematicians. The task of developing such accessibility is not easy, nor is it likely achievable by one individual. It needs a collaborative and continuous effort. Such a need should be more fully explored in the forthcoming conferences of the CMESG.

The following is an *example* of DMO activities (for further details, see Rahim, 1986, parts 1, 2, 3; Rahim & Sawada, 1986, 1989, & 1990) focusing on the idea of using hands-on manipulation and intuitive reasoning in justifying mathematical propositions. Through teacher-directed sessions, the aim here is to present a medium through which students will (i) be involved in a sequence of examining and exploring physical models, (ii) make appropriate dissections to the physical models, (iii) apply the transformation (motion) of translation, rotation, or reflection (or a combination of them) on the resulting pieces and (iv) make their own conclusions.

Activity

Main Objectives

Knowledge and Skills

- Understand and apply techniques in dissection theory on polygonal regions through paper folding.
- Apply transformations translation, rotation, and reflection.
- Experience working with simple spatial operations in geometry (DMO).
- Express in precise mathematical terms concepts or propositions which have been demonstrated concretely.

Attitudes

- Motivate students by providing a learning opportunity in a non-standard format hands-on activity.
- Have students experience a non-abstract application of mathematics.

Students appreciate that "learning" and "proof" may proceed through discovery, creativity and intuitive reasoning, as opposed to the usual deductive reasoning.

Purpose to justify the proposition that *the sum of the angles of a triangle is 180º*.
Materials 8.5 × 11 sheets of plain paper and scissors.
Procedure In a teacher-directed session, distribute an 8.5 × 11 sheet of paper to each student. Denote it as ABCD (see Figure 1).

Dissection Steps Instruct the students to:

1. Choose an arbitrary point on the top edge of the sheet of paper (not the vertices or the midpoint). Denote this point by P.

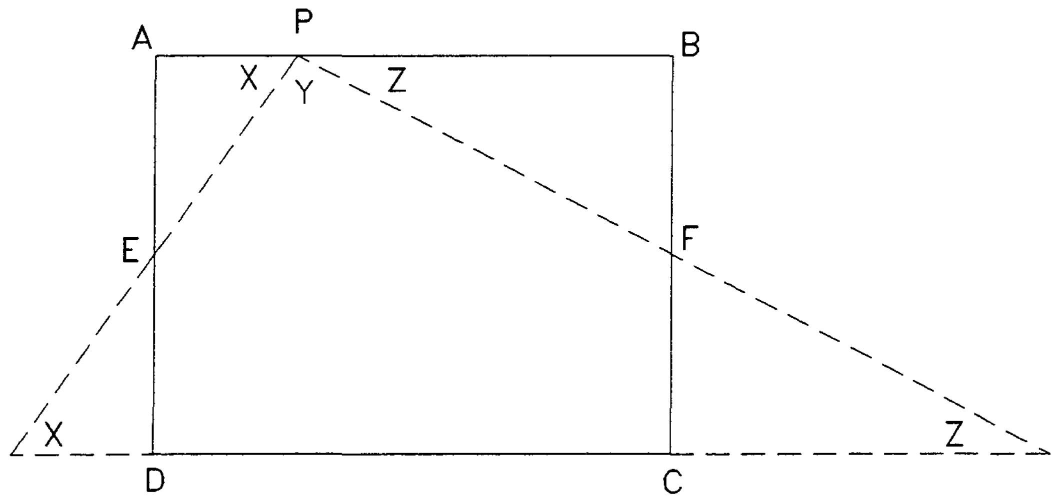

Figure 1

2. Fold corner A to corner D and make a crease. Mark the point where the crease intersects AD by E.
3. Fold corner B to corner C and make a crease. Mark the point where the crease intersects BC by F.
4. Using a ruler, starting from P, join P with E and F.
5. Label angles APE, EPF, and FPB by x, y and z respectively.
6. Find out the sum $x + y + z = \measuredangle$ APE + $\measuredangle$ EPF + $\measuredangle$ FPB = $\measuredangle$ APB = 180º.
7. Cut the rectangular paper ABCD along the line segments PE and PF.

Motion Steps Instruct the students to:

1. Use the following rules throughout (a) use all of the pieces; (b) no overlapping pieces; (c) do not lift a piece completely off the desk.
2. Find out if the rectangular paper ABCD can be transformed into a triangular region of equal area. [Answer: it can be transformed by a half turn to the left of triangle APE about E and a half turn to the right of triangle PBF about F.]
3. Identify that the angles of the resulted triangular region are x, y and z.

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Appendix 8

The Aim in Popularizing Mathematics

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One of the central concerns of our Working Group was to establish the aims of popularizing mathematics. There is a tendency to establish the aims of any project in an inclusive rather than exclusive manner. By doing so, all seemingly potential outcomes will be included. This tendency is inherently problematic as conflicting aims and outcomes may result within the list. Also, the aims as stated may claim to accomplish much more than was intended. We have included the following in the aims of popularization:

(a) increasing the knowledge of mathematics in the population so that more individuals will be prepared to pursue mathematics and science education and occupations;
(b) to increase the awareness and appreciation of mathematics as a human endeavour and as an integral part of our culture; and

(c) to improve self-confidence and reduce anxiety so that people will be more likely to engage in mathematical activity.

A conflict occurs between (a) and (c) when we consider the process involved in accomplishing these aims. For example, for individuals to pursue mathematics and science education and occupations they must have the knowledge and skills that are currently part of our mathematics programmes; however, the knowledge and skills currently emphasized are thought to be the primary cause of anxiety and frustration in mathematics. Until (or unless) there is a change to what is considered to be mathematical knowledge in our schools and universities, popularization activities may not improve the mathematical knowledge as it is currently defined.

Another conflict of aims occurs out of the premise that popularizing mathematics is based on the "freedom to choose". At the same time we realize that people have the freedom not to choose to engage in the activities that we lay in their path. This places (b) and (c) in conflict. In order for people to become aware of and appreciate mathematics as a human endeavour they must have the mathematical confidence to choose to become involved in the activities made available to them. Confidence and internal motivation then seem to be important prerequisites for engaging in popularized activities.

If we have been successful in tempting people to engage in a popularized activity, what can we expect to accomplish? For the vast majority of people it will be an enriching but isolated experience. Does this experience have the potential to overcome many years of boring and frustrating school math for a significant number of people? Or will those individuals who have had poor school experiences even be able to recognize popularized mathematical experiences as being related to their personal conception of mathematics? Isolated experiences in popularized activities overtly labelled as mathematics will conflict with many people's beliefs about the nature of mathematics. This conflict involving incompatible images of mathematics will drastically limit the number of people who are able to develop an appreciation and awareness of mathematics through popularized activities.

Stating broad general aims have initially allowed us to focus our attention on the importance of mathematics for individuals and for our society. The aims of popularization, although noble, have gone beyond what popularizing mathematics could hope to accomplish. In fact, these aims are very similar to the aims identified for all of mathematics education reform (see NCTM Standards, 1989). Our present list of aims represents an (over-simplified) philosophy of "understanding, awareness, and appreciation of mathematics for all." Perhaps it is time to move further and determine specific and attainable aims within the scope of popularizing mathematics.

One specific aim for popularizing mathematics was briefly mentioned, but deserves a more thorough discussion than was possible at the meeting. Keith Devlin, in his plenary address, referred to the aim of empowering students as one he had in mind when designing his course. By this he meant that he wanted his students to be in a position to sensibly critique the technology around them, rather than labouring under the false impression that technology, as a product of science and founded in mathematics, must be the way it is, on some logical or empirical grounds. Those of us in mathematics and the sciences are aware that these fields are products of human activity, and nothing has to be the way it is. Human beings are implicit in every aspect of science and mathematics. We stumbled across a good example of the human factors implicit in mathematics recently. In *The Mismeasure of Man*, Gould writes

> Readers who have done factor analysis for a course on statistics or methodology in the biological or social sciences ... will remember something about rotating axes to varimax positions. Like me, they were probably taught this procedure as if it were a mathematical

> deduction based on the inadequacy of principle components in finding clusters. In fact, it arose historically with reference to a definite theory of intelligence (Thurstone's belief in independent primary mental abilities) and in opposition to another (general intelligence and hierarchy of lesser factors) buttressed by principle components.
>
> (Gould, 1981, p 300, footnote)

Gould reports about a situation that uses the application of mathematics, in correct or misleading ways, to support claims about the nature of human intelligence. Empowering people, by fostering both an awareness of such misuses of mathematics, and the ability to understand and detect misuse, should be one of the aspects of popularizing mathematics. Davis & Hersh, in *Descartes' Dream*, suggest another important aspect of empowerment. They relate the growing mathematization of human thinking over the last four centuries and the damage that has been done as a result. This mathematization of thinking has facilitated the misuse of mathematics, such as Gould describes, but it is also a problem in itself. Mathematical thinking, by being positioned as the only correct mode of thinking, as supplanted by other modes, leave human beings less able to make sense of their world. (A more detailed description of one other mode of thinking, narrative thinking, can be found in Bruner, 1986.) Empowering people by making them more aware of this process of reducing thought to mathematical thinking is another aspect of popularizing mathematics.

The importance of being explicit about our aims in popularizing mathematics becomes clear when we consider the ways in which these aims might be addressed. Many of the suggestions made in the Working Group as to how we might popularize mathematics do not address the aim of empowerment, and some might actively interfere with this aim. In our classrooms we can popularize mathematics by being clear about the human origins of the mathematics we teach. We should touch especially on those areas we might wish to avoid; the origins of mathematics in war, and in social engineering. This may not be simple, as many of us were not taught this aspect of mathematics, and most historians of mathematics have chosen to downplay its darker side. In addition to referring to the human origins of mathematics, we can also make reference to the current applications of mathematics in circumstances which directly affect our students. The ranking of people on a linear scale according to mathematical principles (grading) is an important aspect of most students lives, and might be the easiest place to begin. Students are also keenly aware of the use of mathematics as a filter for determining career opportunities, and there are many interesting aspects to this issue. Larger social issues are also closely related to mathematics, and especially to the mathematization of thinking. As teachers we can make students aware of the ways that mathematics and mathematical thinking are used by government and business to the advantage of some and disadvantage of others. We can also make students aware of the ways they can use mathematics and mathematical thinking to reveal hidden features in social situations (as Frankenstein 1987, 1989, 1991 has done with differential mortgage rates in Boston).

Outside of the classroom, books such as *The Mismeasure of Man* and *Descartes' Dream* offer one model of popularizing mathematics to empower people. Similar projects could be undertaken through many media. Math trails and informative displays at historical sites can also serve to increase awareness of the human origins of mathematics, and the historical misuse of mathematics. As an example, major east coast ports might include displays on the relationship between the introduction of quantitative intelligence testing in the early twentieth century and the enacting of legislation to bar European immigrants from low scoring ethnicities entrance to the U.S. and Canada. Central European Jews were one group that suffered from this mathematically justified policy. Popularizing mathematics to empower may seem to be more likely to lead to mathematics becoming less popular, which is not the usual vision of popularization. An important part of popularizing mathematics to empower must be maintaining the distinction between mathematics itself and the use (and misuse) of mathematics. We should strive to make the misuse of mathematics as unpopular as we can, while preserving mathematics as a valuable part of human culture. Discouraging the misuse of mathematics in our society can only improve the cultural value of mathematics. Only when people begin to see mathematics as an inhuman force manipulating their lives, can they begin to see mathematics as a spectacular human invention of great beauty and complexity.

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[Note added by the Group leaders: More on the misuse of mathematics in societies can be found in the paper "Mathematics as propaganda" by N. Koblitz, in L. A. Steen, ed., *Mathematics Tomorrow*, Springer-Verlag, 1981, pp. 111-120.]

Appendix 9

Fractal Cards

Elaine Simmt
University of Alberta

Fractal cards are beautiful "pop-up" type cards that are reasonably easy to make and full of mathematics. Suitable activities which help popularize mathematics are important for all of us whether we work with young students, older students, or individuals who are long (and not long) out of school. Fractal cards are such an activity; they are appropriate for use both with mathematics students and the individuals in the general population (see the "Math in the Mall" activity described in Appendix #1 by Malgorzata Dubiel). Mathematics lessons that are easy enough for middle school students or challenging enough for secondary and post-secondary students can be developed around fractal cards. In this short paper I will explain how to make fractal cards and provide some suggestions for questions that point to the mathematics of the fractal cards.

Inherent in the construction of fractal cards are concepts such as self-similarity, recursion, scale, iteration, and infinity. When doing the activity "in the mall" it is possible to note some of the mathematical features of the process of generating the fractal card as well as some of the features of the fractal itself. If the fractal cards are used with mathematics students then an important part of the activity is the search for the mathematics of the fractal. Some of the obvious mathematics involve measure, number patterns, sequences, series, and limits. Questions and suggestions for mathematics students include:

- Describe the growth pattern.
- What happens to the number of boxes as the number of iterations goes to infinity?
- Write a sequence, then a series, to describe the growth pattern of the fractal.
- How many cuts are there at the tenth iteration?
- What is the surface area of the fractal generated?
- What is the volume contained by the fractal?
- How far away is the furthest box from the first box?

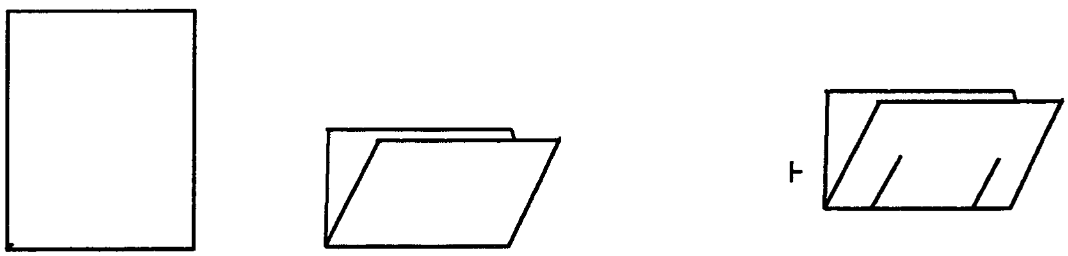
Create your own fractal card using fractal cuts and then find the mathematics in your fractal.

To make a card (see Figure 1) you define a simple rule and then repeat that rule until the physical properties of the paper prevent you from continuing at smaller scales. Once your "student" has created the card then the student can look for the mathematics in the fractal. This search can be informal (limited to observations) or it can be rigorous (involving mathematical notation, abstraction and formalization).

These fractal cards (Figure 2) provide an inexpensive, fun, and creative entry into mathematics.

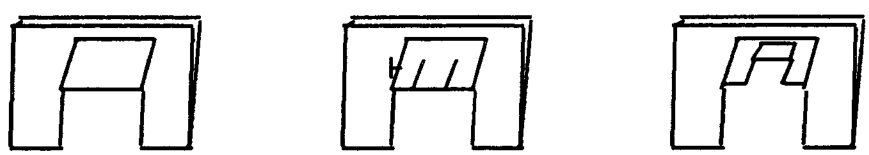
Directions for Making a Fractal Card

1. Take a sheet of paper and fold it in half.



2. Make two cuts, of f(1,2) the length of "a" and f(1,4) of the way in from each edge.

3. Fold along the line produced by the two cuts.



4. Repeat Steps 2 and 3 until it is too difficult to cut or fold the paper.

5. Once you stop cutting you must open the folds back out and push out the fractal.

Figure 1

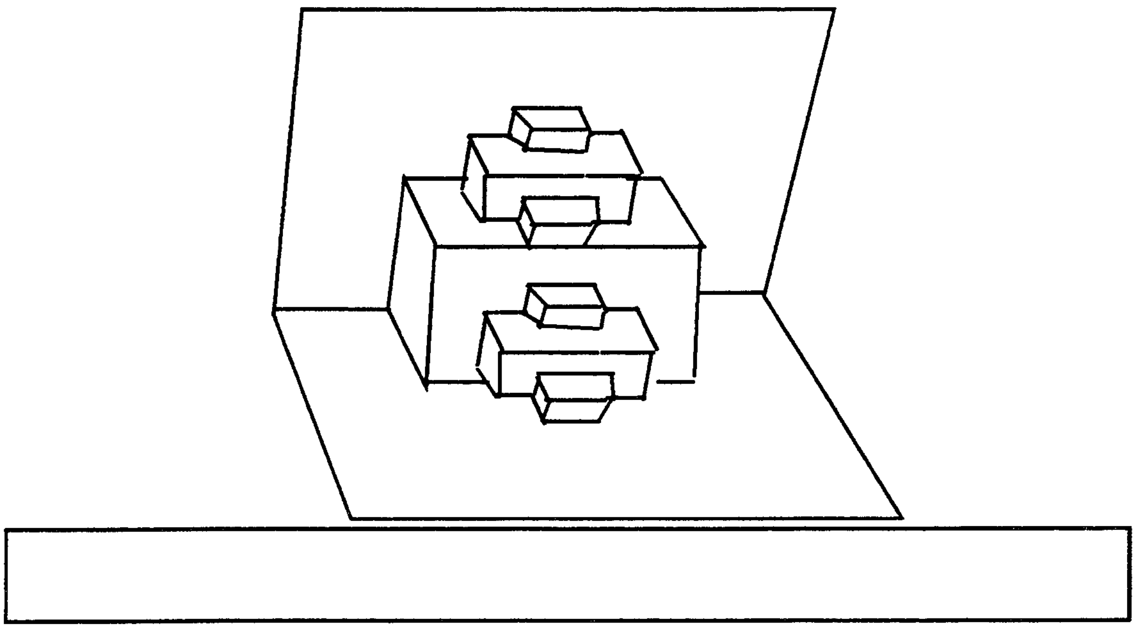


Figure 2 - Fractal Card

Reference

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Working Group C

Preservice Teachers as a Purposeful Learners: Issues of Enculturation

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Introduction

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Nel Noddings (1992, p 205) argues that "(p)rofessionals depend on the trust of their clients" and teachers "need opportunities to earn that trust, and such opportunities arise as teachers are entrusted by those now controlling their work." We need to trust teachers in the same way we trust other professionals, such as lawyers, doctors and architects (Fullan & Connelly, 1987, p 50). Without this trust, teachers are workers rather than professionals (McNeil, 1987, p 101). If teachers are to learn different ways of approaching their teaching, then their agency will play a crucial role (Connelly & Clandinin, 1988, pp 137-155). As Carl Rogers (1969) wrote, "the only learning which significantly changes behaviour is self-discovered, self-appropriated learning." It appears that without an active, interested role by teachers, change in education is likely to remain superficial.

Dewey (1938) wrote about personal agency in terms of democratic freedom and the power to "frame purposes and to execute or carry into effect purposes so framed." He saw freedom and power as a habit of mind based on scientific thinking (Dewey, 1965, p 67). This implies that thinking is problem and inquiry driven, requiring teachers "to reflect on the origins, purposes, and consequences of their actions" and "develop the pedagogical habits and skills necessary for self-directed growth" (Zeichner & Liston, 1987, p 31). Dewey (1965, p 171) saw these higher level thinking processes, these habits of mind which are an integral part of teacher agency, spilling over into classroom environments and changing perceptions of what constitutes education.

An opportune time for providing teachers with appropriate situations and support for exercising their agency is during the practicum component of their education as teachers. This could be a time when teachers develop the habits of mind associated with exercising their agency in growing as educators. However, in a review of such field experiences, Zeichner (1981, p 9) suggests that these experiences are presently fostering the development of utilitarian rather than inquiry-based perspectives as "(t)eachers for the most part do not seem to be especially reflective or analytic about their work."

The members of our Working Group shared an awareness of the need to improve the education of mathematics teachers. We joined the group with the hope of generating ideas for better teaching mathematics methods courses, for providing meaningful practicum experiences for new teachers, and for strengthening the link between course work and practicum experiences. We also made a conscious effort to understand mathematics teacher education in terms of the metaphor "teachers as purposeful learners."

Purposeful learning

In answering the question "What is purposeful learning?", in the context of mathematics teacher education, we identified a number of overlapping qualities:

- having a vision or goal of what is to be accomplished;
- visions are growth related and involve an improvement on the status quo;
- injecting one's self in the process and making it personally relevant;
- being committed to one's visions;
- being able to outline and carry out a plan of action to meet a goal;

- reflecting on the learning process and revising or adapting when necessary;
- making choices.

Our view of a purposeful learner/teacher is a person who:
(a) is committed to the improvement of the teaching of mathematics;
(b) creates visions for change based on personal experiences and knowledge;
(c) is able to work towards a goal by outlining and carrying out a plan of action; and
(d) reflects on actions and visions and adapts when necessary.

We considered the cases of Julie and Martin (Gadanidis, 1994), two preservice mathematics teachers whose thoughts and actions appear to exemplify purposeful learning. Julie and Martin were part of a teacher education program which provided them with the opportunity for developing personal visions of innovation in mathematics education and for experimenting with them in practice. That is, they were entrusted with the responsibility for improving on the status quo. Both were able to take advantage of the opportunity by drawing on their personal knowledge and goals to develop innovative visions and practice that reflected who they are as learners, teachers and people.

Julie learned best in informal, small group settings. A sense of belonging was important to her. She chose to attend a smaller university for that reason. Julie noticed a mutually beneficial relationship between tutor and tutee. Explaining concepts to others helped her understand the concepts better. This gave her an appreciation for the benefits of strong and weak students working together. Through tutoring, she also discovered that different students learn in different ways and that they need individual attention and help to overcome learning difficulties. This highlighted the importance of working on a one-to-one basis with students. These characteristics were part of who Julie was as a student and as a teacher.

Julie's practicum experiences with cooperative learning reinforced her personal beliefs about how students learn best.

> I think it's good for them, to be able to teach each other. ... I think you have to learn to let go of ... being the centre of attention ... that made it easier, seeing that they could actually accomplish it on their own ... instead of feeling that I have to do it for them ... So now I don't feel like I have to be up there guiding them all the time, they can be guiding themselves.

Martin learned best when he was active in the learning process. He found that material learned through lectures, such as concepts of Calculus, was soon forgotten. Martin had vivid memories of what he had learned while being involved in a two week school project that crossed curricular boundaries. He preferred one-to-one interactions as opposed to group interactions. He felt that he learned well when he had to interact with others, talking, explaining concepts in depth, as opposed to giving monosyllabic answers to predictable questions in teacher centred classroom formats. Martin felt that his schooling did not help him to develop personal interests and build on his strengths. He felt this was important. He wanted to do it for his own students.

Martin's practicum experiences reinforced his personal beliefs about student-centred learning.

> ... what really made it a success was the participation of the kids. You know, what really stands out in my mind ... was the kids' presentations, watching as they spoke for a mere 60 seconds or so and realizing that this was the first time I had heard them talk in complete sentences and express themselves in another way than responding in cryptic phrase to my questions. For the first time, I had a glimpse of the personalities of these kids, and their

> scientific 'sea-legs'. What a rush! ... Long live group work and individualized learning programs.

Balancing priorities

Our Working Group felt that making "purposeful learning" a goal for preservice mathematics teachers would mean a significant shift from current mathematics teacher education goals. It would imply focusing on changing teacher attitudes rather than providing them with teaching theories and utilitarian skills. It would mean allowing for much more choice and for constructive criticism of goals and processes. In our discussion we highlighted two opposing views on this matter.

On the one hand, we recognized that "survival" concerns of preservice teachers are strong. We wondered whether purposeful learning issues may be inappropriate concerns for them. There may be a danger in aiming too far down the road. Perhaps the best we can do for preservice teachers is to help them survive the practicum experience. We can do this by focusing more on commonly used techniques for teaching and less on ways of improving on the status quo. A successful practicum could give them the confidence to aim for higher goals in their future teaching.

On the other hand, we felt that focusing mostly on "survival" concerns may create strong pressures for perpetuating the status quo. The education styles that we model and preservice teachers rely on in their practicum help shape how they teach later in their careers. Some of our Working Group members reflected that when they were studying to be teachers they wanted more than just survival and its tactics. They had personal visions of how mathematics education could be improved. They wanted to inject creativity into their teaching and to bring their love of mathematics into their classrooms.

Fuller (1969) identified three stages of concern teachers go through as they learn to teach: survival concerns, teaching style concerns, and pupil concerns. Recognizing that, generally, certain concerns may tend to be more predominant in one stage than another is very helpful in supporting preservice teachers during their practicum experience.

However, stages of concern in teacher development are sometimes interpreted in terms of what preservice teachers can or cannot do at each of the stages identified. Such an interpretation questions the view that innovation, and an accompanying focus on teacher agency, is an appropriate concern for the majority of teachers in the preservice teaching "stage". Scardamalia and Bereiter (1989, p 43) concede that a case can be made for omitting such higher-level concerns "in preservice education and introducing them through inservice education, after teachers have passed through the 'survival' and 'mastery' stages and are ready to deal with impact." However, they also note that "it has been shown that once teachers are entrenched in problem-minimizing approaches it is very difficult to dislodge them."

There is a strong parallel between strictly linear interpretations of teacher development and traditional interpretations of student development. For example, it is usually assumed that students must first develop competency in individual facts and skills of mathematics, then move to understanding through meaningful application, and finally to problem solving. Unfortunately, given the myriad of facts and skills that are encompassed by the subject-matter of mathematics, students rarely get to experience the higher level thought processes involved in problem-based learning.

Through linear interpretations of human development, education becomes a process for preparing students for later, real-life stages or situations rather than a process which continuously involves them in real-life processes. However, as Dewey (1897, p 78) suggested, education is

> a process of living and not a preparation for future living. ... I believe that education which does not occur through forms of life, forms that are worth living for their sake, is always a poor substitute for the genuine reality, and tends to cramp and to deaden.

Perhaps by maintaining educational cultures where higher-level thinking is not supported or encouraged we are strengthening the forces which stunt student growth in this area, hence making the phenomenon of developmental stages culturally dependent. Papert (1980, p 20) suggests, in relation to the developmental stages attributed to Piaget, that such developmental differences may in fact be attributed to the poverty of the educational culture in which they are experienced. If we want students to learn to think at high cognitive levels when they do mathematics then we should, as Greeno (1988) suggests, be immersing them in classroom experiences where this is what they do.

An alternate view of teacher development is that the teacher concerns identified by Fuller define growth areas that are best dealt with in a parallel fashion rather than sequentially. This view is supported by the cases of Julie and Martin (discussed earlier), where the three types of teacher concerns identified by Fuller were intertwined. For example, Julie's search for a cooperative learning teaching style, based on her perception of the needs of students, led her to "survive" more effectively in the classroom. She not only noticed that students worked well together but also that it was easier on her as a teacher — she felt less tired after cooperative learning lessons. Also, Martin's use of CAL as a way of meeting individual needs led him to discover that CAL had a positive impact on classroom management. He noticed that not only were students able to work at their own pace and he was able to give them individual attention but also that they were on task throughout their work in the computer lab and often worked past the bell.

Redesigning mathematics teacher education

In our discussions of redesigning mathematics teacher education, our Working Group considered three issues: (a) working with associate teachers; (b) the curriculum of methods courses; and, (c) practicum structure.

Working with associate teachers

The current relationship between associate teachers and faculties of education is often one of the weakest links in teacher education programs. One exception to this appears to be in the Professional Development Schools organized in British Columbia. Both of the lower mainland universities, the University of British Columbia and Simon Fraser University, prefer working in this model if possible. A Professional Development School is a practicum environment with the following characteristics:

- the application to become a Professional Development School comes from the teaching staff, rather than the administration
  - this is a bottom-up approach
  - staff have ownership
  - it becomes a school project
- professional development goals are identified for both the associate teachers and the pre-service teachers
  - this is a win-win situation where both the participating school and the faculty of education use the practicum relationship as an opportunity to meet their goals

- the participating school and the faculty of education pool human and material resources to meet goals set
  - faculty of education staff may be used for teacher in-service
  - the university and the school district make professional development funds available
- preservice teachers are assigned to the school in groups
  - this allows a number of different associate teachers to cooperate in creating a learning environment for the preservice teachers
  - it also provides a peer support and reflection network for preservice teachers
  - clustered placements are cost-effective for faculties of education.

In our discussions of Professional Development Schools, we would also like to recommend the following:

- well-designed practicum handbooks for both associate teachers and preservice teachers
- a practicum newsletter
  - this could be edited and circulated by participating schools on a rotating basis
  - some of the items the newsletter could include are:
    - articles written by associates, preservice teachers or faculty of education staff
    - reports on exemplary programs
    - reflections of associates, preservice teachers, faculty of education advisors, administrators, students ...
    - pictures of classroom action
    - a bulletin board of upcoming events
    - ...

- better strategies for recruitment of associate teachers
  - consider what makes a good associate teacher
  - increase visibility of faculty advisors in schools

- greater involvement of teacher federations

- elevation of associate teacher status
  - a professionally oriented reward structure
  - increase awareness of the importance of the contribution made by associate teachers and the professional benefits of being an associate teacher.

Practicum structure

There is a wide variety of practicum structures across Canada. Some universities offer programmes which integrate practicum and course experiences. For example, concurrent programs offer practicum experiences throughout a preservice teacher's university career. Other universities offer less integrated programs, where practicum experiences alternate with faculty of education courses, but have no relationship with undergraduate courses. Some universities offer fairly fragmented experiences where practicum blocks are separate from faculty of education courses. A number of universities offer a variety of programs. Nonetheless, although practicum structure does have an effect on how well practicum and course experiences are integrated, the level of integration often depends on what professors do (course curriculum, assignments, etc.) within a particular structure. In our discussion of the practicum, we identified three aspects we value:

- the integration of faculty of education courses and practica

  - at present many preservice teachers fail to see a strong connection between what they learn in faculty of education courses and what they experience during the practicum
- extended practicum experiences
  - minimum 5-6 weeks at a time
- practicum supervision by methods course instructors
  - would give valuable feedback to instructors
  - it would help create a common language of experience.

The curriculum of methods courses

We felt that it is important to provide a more holistic experience in methods courses. To effect this purpose, we proposed a model for consideration. We suggested using a single theme or thread for the entire methods course. For example, the course could focus on the curriculum planning regularly done by teachers. The course goal could be to plan the curriculum for a particular unit (or units or course). This would be a communal effort. As the course progresses, the curriculum planning process can be refined and increased in sophistication by taking into account the following considerations:

- the rationale for the curriculum
- assessment and evaluation theory and techniques
- motivation and classroom management
- curriculum integration
- the use of problem-solving
- the use of computer technology
- the use of cooperative learning
- ...

In parallel to this class project of planning curriculum, small groups would also plan the curriculum for a different topic or unit of their choice for use in their practicum. This gives preservice teachers the opportunity to apply the curriculum planning skills in a new situation. It also gives a context for the "common language of experience" that may be used in supervision of the practicum.

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Topic Group A

Les didacticiens et les didacticiennes des mathématiques au Canada: un portrait de famille

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Introduction

Habituellement, les recherches en didactique portent sur les élèves ou sur les enseignants et les enseignantes. Pour une fois, il m'a semblé intéressant de faire une recherche sur le groupe auquel j'appartiens, soit celui des didacticiens et des didacticiennes des mathématiques. Ayant déjà mené une enquête sur les professeurs et les professeures de sciences mathématiques dans les universités canadiennes (Mura, 1990, 1991 et 1993), j'ai décidé d'effectuer une enquête semblable sur les personnes qui enseignent la didactique des mathématiques. Le but de l'enquête était de recueillir des données sur leur origine sociale, sur leur profil d'études et de carrière et sur leur vision des mathématiques et de la didactique des mathématiques afin de mieux connaître la communauté professionnelle qu'elles constituent.

Les résultats portant sur la vision des mathématiques et de la didactique des mathématiques sont analysés dans Mura (1994 et sous presse). Dans le présent article, après avoir décrit la méthode d'enquête, j'expliciterai d'abord les données d'ordre démographique, puis celles qui concernent les études (y compris l'origine de l'intérêt pour les mathématiques et pour la didactique des mathématiques) et la carrière. Tout au long de l'exposé, je signalerai au fur et à mesure les éventuelles différences, d'une part, entre les hommes et les femmes, et, d'autre part, entre les personnes travaillant en milieu anglophone et celles qui sont en milieu francophone. Enfin, avant de conclure, je comparerai les résultats de la présente enquête et ceux de l'enquête menée précédemment auprès des mathématiciens et des mathématiciennes.

1. La méthode d'enquête

Mon objectif était d'atteindre tous les professeurs et professeures de didactique des mathématiques dans les universités canadiennes. Dans ce but, j'ai envoyé un questionnaire aux personnes dont le nom apparaissait dans la liste d'envoi du Groupe canadien d'études en didactique des mathématiques ou dans le répertoire des recherches en cours produit par le même groupe (Kieran et Dawson, 1992). J'ai aussi demandé à chacune de ces personnes de nommer tous les didacticiens et les didacticiennes des mathématiques dans leur université et j'ai ensuite envoyé un questionnaire aux nouveaux individus ainsi repérés.

Aucune université canadienne n'ayant de département de didactique des mathématiques (la seule qui se rapproche de cette situation est l'Université de la Colombie-Britannique qui possède un département de didactique des mathématiques et des sciences), il revenait à chaque personne de se reconnaître ou non comme didacticien ou didacticienne des mathématiques. Compte tenu de cela, la page couverture du questionnaire contenait les deux questions suivantes: «Occupez-vous un poste permanent ou pouvant mener à la permanence dans une université canadienne ?» et «Est-ce que la didactique des mathématiques est votre domaine principal d'enseignement et de recherche ?» Ceux et celles qui répondaient «non» à l'une des deux questions étaient invités à retourner le questionnaire sans le remplir.

Quelques-unes des personnes qui ont reçu le questionnaire ont avoué avoir hésité avant de répondre à la deuxième question. L'une d'entre elles, membre d'un département de mathématiques, a écrit: «I almost returned your questionnaire with NO for mathematics education being my primary field, but I hesitated and now I am lost: I feel like I'm coming out of the closet! Confessing to mathematics education as my primary interest.»

En tout, 158 questionnaires ont été expédiés par courrier à la session d'hiver 1993. Après deux lettres de rappel, 106 questionnaires (67 %) étaient revenus. De ceux-ci, 63 ont été remplis par des personnes appartenant à la population cible et ont été retenus pour la présente étude. [1] Ces personnes sont au service de 28 universités différentes situées dans huit provinces (les deux provinces non représentées sont Terre-Neuve et l'Île-du-Prince-Édouard).

Deux des personnes qui n'ont pas retourné le questionnaire ont pris la peine d'écrire pour expliquer les raisons de leur réticence. La première se sentait mal à l'aise par rapport à certaines questions, surtout celles qui touchaient son CV, tandis que l'autre ne voyait pas l'intérêt de ce genre de recherche.

Le questionnaire comprend 54 questions, dont 7 sont «ouvertes». Plusieurs questions sont reprises du questionnaire utilisé pour l'enquête auprès des mathématiciens et des mathématiciennes afin de permettre une comparaison entre les deux groupes. Une version anglaise ou française du questionnaire a été utilisée selon la langue de travail du ou de la destinataire.

Contrairement à l'enquête précédente sur les mathématiciens et les mathématiciennes, la présente enquête n'a pas été planifiée dans le but principal de comparer les deux sexes. Ainsi, les hommes et les femmes n'ont pas été appariés, en aucune façon, et, comme on le verra, une plus grande proportion de femmes que d'hommes ont le rang d'adjointe, alors qu'une plus grande proportion d'hommes que de femmes ont le rang de titulaire. Lorsque des comparaisons entre les sexes seront établies, il faudra tenir compte de leur distribution différente dans les rangs universitaires.

Enfin, à noter que pour tous les tests statistiques, le seuil de signification est fixé à 0,05.

2. Les données démographiques

Sexe L'échantillon comprend 19 femmes (30%) et 44 hommes (70%). La proportion des femmes est égale à celle que rapporte Statistique Canada (1993a, p 12), pour 1990–1, pour le corps professoral du domaine de l'éducation, alors que le pourcentage correspondant est beaucoup plus faible du côté des mathématiques et sciences physiques (7%).

Langue de travail Le tableau 1 montre la distribution de l'échantillon selon le sexe et la langue de travail. Pour simplifier les analyses, deux personnes travaillant en milieu bilingue ont été classées ainsi: l'une dans le groupe anglophone et l'autre dans le groupe francophone selon leur langue maternelle.

Les personnes travaillant en milieu francophone constituent 35% de l'échantillon et sont ainsi légèrement surreprésentées par rapport aux francophones dans la population canadienne: en effet, d'après le dernier recensement, parmi les personnes qui parlent une seule langue à la maison, on trouve 69% d'anglophones, 23% de francophones et 8% d'allophones (Statistique Canada, 1993b, Tableau 1). Cette surreprésentation n'est pas due à une meilleure participation des francophones à l'enquête (sur 158 questionnaires distribués, 49 (31%) ont été envoyés à des francophones, et sur 106 questionnaires retournés, 31 (29%) ont été remplis par des francophones) et peut donc correspondre à une réelle plus grande importance accordée à la didactique des mathématiques dans les universités francophones.

[1] On peut donc estimer à une centaine de personnes la population des didacticiens et des didacticiennes des mathématiques dans les universités canadiennes.

Tableau 1
Distribution de l'échantillon selon le sexe et la langue de travail

Langue de travail	Femmes	Hommes	Total
Anglais	9	32	41
Français	10	12	22
Total	19	44	63

Note: χ^2 corrigé pour continuité = 2,7; non significatif.

Âge La moyenne d'âge pour l'ensemble de l'échantillon est de 50,5 ans (la médiane est de 50 ans, avec une étendue allant de 30 à 64 ans). La moyenne d'âge est de 46,7 ans pour les femmes (médiane = 47 ans) et de 52,1 ans pour les hommes (médiane = 54 ans); elle est de 51,1 ans pour les personnes qui travaillent en milieu anglophone et de 49,4 ans pour celles qui se trouvent en milieu francophone. La différence entre les sexes est significative ($t = -2{,}9$; $p < 0{,}01$; grandeur de l'effet $d = -0{,}8$), tandis que celle entre les deux groupes linguistiques ne l'est pas.

À nouveau, on constate que l'échantillon ressemble davantage au corps professoral des sciences de l'éducation qu'à celui des mathématiques et sciences physiques: en effet, Statistique Canada (1993a, p 43) rapporte, pour 1990–1, un âge médian de 46 ans pour les femmes en éducation et de 49 ans pour leurs collègues masculins, tandis qu'on observe 41 ans pour les femmes et 48 ans pour les hommes dans le domaine des mathématiques et sciences physiques.

Pays d'origine Les personnes qui ont participé à l'enquête viennent de 12 pays différents. La majorité cependant (N = 38, soit 60%) sont nées au Canada. Parmi les autres pays, le mieux représenté est les États-Unis (N = 11, soit 17%). La proportion de personnes nées au Canada est de 47% parmi les femmes et de 66% parmi les hommes; elle est de 51% parmi les personnes qui travaillent en milieu anglophone et de 77% parmi celles qui sont en milieu francophone. Les différences ne sont pas statistiquement significatives. À titre de comparaison, précisons que, en 1991, 83% de la population canadienne était née au pays (Statistique Canada, 1992a, Tableaux 1 et 2).

Citoyenneté Chez les personnes qui ont participé à l'enquête, 87% (N = 55) ont la citoyenneté canadienne (84% des femmes, 89% des hommes; 83% des personnes travaillant en milieu anglophone et 95% de celles qui se trouvent en milieu francophone). Cette proportion est égale à celle dans le corps professoral des sciences de l'éducation (87%), alors que, en ce qui concerne les mathématiques et sciences physiques, le pourcentage est un peu inférieur, soit 80% (Statistique Canada, 1993a, p 52). La même source rapporte des taux de 83% pour les femmes et de 81% pour les hommes, toutes disciplines confondues. Pour l'ensemble de la population canadienne, le taux est de 94% (Statistique Canada, 1992, Tableau 5).

Langue maternelle Pour ce qui est de la langue maternelle (la première langue apprise et encore comprise), les personnes interrogées ont nommé sept langues différentes; pour 39 d'entre elles (62%), il s'agit de l'anglais et pour 18 (29%), du français. Seulement 6 (10%) (trois femmes et trois hommes; quatre en milieu anglophone et deux en milieu francophone) ont donc mentionné une autre langue. Par ailleurs, deux personnes de langue maternelle anglaise travaillent en milieu francophone, alors qu'il n'y a aucun exemple de la situation inverse.

La proportion de l'échantillon qui est de langue maternelle anglaise ou française (90%) est supérieure à celle de l'ensemble de la population canadienne, soit 87% d'après le recensement de 1991 (Statistique Canada, 1992b, Tableau 1).

Scolarité et occupation des parents Le tableau 2 indique le plus haut niveau d'études atteint par la mère et par le père des personnes ayant participé à la présente étude.

Tableau 2
Scolarité des parents

Scolarité	Mère		Père	
	N	%	N	%
Primaire (terminé ou non)	22	35	21	34
Secondaires	25	40	27	44
Universitaire	15	24	14	23
Donnée manquante	1		1	
Total	63		63	

Le tableau 3 précise la catégorie socioprofessionnelle de l'occupation de la mère et du père des personnes ayant participé à la présente étude.

Tableau 3
Occupation des parents

Catégorie socioprofessionnelle*	Mère		Père	
	N	%	N	%
Strate supérieure	2	3	9	14
Strate intermédiaire	12	19	31	49
Strate inférieure	10	16	19	30
Hors travail ou autre	39	62	4	6
Total	63		63	

*Strate supérieure: haute administration publique et privée, grands propriétaires, membres des professions traditionnelles, universitaires;
Strate intermédiaire: cadres moyens, agriculteurs et agricultrices, semi-professionnelles et semi-professionnels, enseignants et enseignantes du primaire et du secondaire, artisans et artisanes;
Strate inférieure: contremaîtres, cols blancs, cols bleus.

Le niveau de scolarité et la catégorie socioprofessionnelle de l'occupation des parents sont semblables pour les femmes et pour les hommes ainsi que pour les personnes travaillant en milieu anglophone et francophone.

3. Les études

Diplômes universitaires

Cinquante-six personnes ayant participé à la présente enquête (89%) sont titulaires d'un doctorat. Les 7 autres ont une maîtrise. Parmi les 56 doctorats, 46 sont en sciences de l'éducation (y compris la didactique des mathématiques), 8 en sciences mathématiques et 2 en psychologie. Sans exception, ceux et celles (32) qui ont spécifié que leur doctorat était en didactique des mathématiques ont classifié leur diplôme comme un diplôme en sciences de l'éducation.

Parmi les sept personnes qui ne détiennent pas de doctorat, trois sont des femmes et quatre des hommes, deux travaillent en milieu anglophone et cinq en milieu francophone. Quatre de ces personnes ont une maîtrise en mathématiques, une a une maîtrise en éducation, une a deux maîtrises (en éducation et en mathématiques) et une dernière a une maîtrise en administration. Leur moyenne d'âge étant de 51,7 ans, elles ne constituent donc pas un groupe sensiblement plus vieux que celui des titulaires d'un doctorat (âge moyen = 50,4 ans), comme on aurait pu le supposer.

Ainsi, pour 75% des personnes interrogées, le diplôme le plus élevé est en sciences de l'éducation. Parmi ces personnes, 9 (14%) n'ont que des diplômes en sciences de l'éducation, 12 (19%) n'ont que des diplômes en sciences mathématiques, 35 (56%) ont des diplômes en mathématiques et en éducation, 16 (25%) n'ont aucun diplôme en éducation et 12 (19%) n'ont aucun diplôme en mathématiques. Quatorze personnes (22%), en plus d'un ou plusieurs diplômes en éducation ou en mathématiques, ou dans les deux à la fois, sont titulaires de diplômes dans d'autres disciplines (psychologie, sciences de l'administration, génie, physique, chimie, biochimie, biologie, histoire, philosophie, arts, littérature et latin).

Trente-six participants et participantes (57%) ont obtenu leur diplôme le plus élevé d'une université canadienne, 21 (33%) d'une université aux États-Unis, 2 du Royaume-Uni, 2 de Suisse, une de France et une d'Israël.

À titre de comparaison, en 1990-1991, dans l'ensemble du corps professoral, toutes disciplines confondues, la proportion de titulaires d'un doctorat était de 63% parmi les femmes et de 75% parmi les hommes (Statistique Canada, 1993a, p. 33).

Âge à l'obtention des diplômes universitaires

L'âge moyen auquel les participants et les participantes à l'enquête ont obtenu leur premier diplôme universitaire est de 22,7 ans (médiane = 22 ans, avec une étendue allant de 18 à 31 ans): il est de 22,8 ans pour les femmes, de 22,7 ans pour les hommes; et de 22,7 ans pour les personnes travaillant en milieu anglophone comme pour celles en milieu anglophone.

Quant à l'obtention du doctorat, l'âge moyen pour l'ensemble de l'échantillon est de 36 ans (médiane = 34,5 ans, avec une étendue allant de 25 à 60 ans): il est de 38 ans pour les femmes, de 35,2 ans pour les hommes; et de 34,9 ans pour les personnes travaillant en milieu anglophone et de 38,4 pour celles en milieu anglophone. Ces différences ne sont pas statistiquement significatives.

Origine de l'intérêt pour les mathématiques et pour la didactique des mathématiques

Le tableau 4 indique l'âge auquel les personnes qui ont répondu à l'enquête ont décidé de se spécialiser en mathématiques et en didactique des mathématiques. Comme on le voit, la majorité des personnes interrogées (67%) ont pris la décision de se spécialiser en didactique des mathématiques après l'âge de 21 ans, alors que 85% d'entre elles avaient déjà pris la décision de se spécialiser en mathématiques avant cet âge et que la moitié avaient même pris cette dernière décision avant l'âge de 17 ans. Les distributions sont semblables pour les deux sexes et pour les deux groupes linguistiques

Tableau 4
Moment de la décision de se spécialiser en mathématiques
et en didactique des mathématiques

Moment de la décision	Mathématiques		Didactique	
	N	%	N	%
Avant l'âge de 12 ans	4	6	3	5
Entre 12 et 17 ans	27	44	5	8
Entre 18 et 21 ans	22	35	13	21
Après l'âge de 21 ans	9	15	42	67
Donnée manquante	1			

Tableau 5
Raisons de l'attraction initiale pour les mathématiques

Raisons	N	%
Réussite	34	56
Défi, résolution de problèmes	27	44
Logique, rigueur, absence de débats*	8	13
Abstraction, imaginaire, fuite de la réalité**	7	11
Abstraction, imaginaire, fuite de la réalité**	14	23
Total***	90	

*Deux des personnes qui ont fait ce choix ont exclu «l'absence de débats d'opinions», en ne maintenant que «la logique et la rigueur».
**Une des personnes qui a fait ce choix a exclu «la possibilité de fuire la réalité», en ne maintenant que «la nature abstraite et imaginaire des mathématiques».
***Certaines personnes ont donné plus d'une réponse.

Raisons de l'attraction initiale pour les mathématiques
Le tableau 5 indique les raisons de l'attraction initiale pour les mathématiques. Les choix de réponses proposés étaient: 1) le fait de bien réussir; 2) le goût de relever des défis, de résoudre des problèmes; 3) la logique, la rigueur, l'absence de débats d'opinions; 4) la nature abstraite et imaginaire des mathématiques, la possibilité de fuir la réalité; et 5) autre (préciser). Il y a peu de différence entre les réponses données par les deux sexes et par les deux groupes linguistiques.

Une personne qui a opté pour les troisième et quatrième choix de réponses, en en exluant toutefois «l'absence de débats d'opinions» et «la possibilité de fuir la réalité», a ajouté l'explication suivante: «il me semble qu'on associe trop vite [ces aspects] à ce qui les précède». Une autre personne, par contre, a offert un commentaire qui me semble aller tout à fait dans le sens de la possibilité de fuir la réalité: «Math was a world I could escape to—a private, safe place my father could not steal from me.»

Les quatre thèmes suivants sont les seuls qui reviennent plus d'une fois parmi ceux qui ont été abordés dans les 14 réponses «autres».

1. L'impossibilité ou la difficulté de poursuivre d'autres études (N = 4).
Exemples

> «Le fait d'avoir été refusée en informatique.»
> «I was weak in literature and language arts.»
> «My first interest was Physical Education, but a serious knee injury prevented my studying Phys. Ed. at University. »

2. Un intérêt pour l'enseignement des mathématiques (N = 3).
Exemples:

> «The fact that mathematics was/is badly taught at the elementary level.»
> «I was able to help other students in doing math.»

3. La valorisation personnelle (N = 2).
Exemples:

> «Valorisation personnelle d'être une fille bonne en maths.»
> «The respect into which math was held by my pears.»

4. La possibilité de faire autre chose (N = 2).
Exemples:

> «Pour avoir du temps pour faire autre chose!»
> «Majoring in math at the university allowed me the greatest flexibility in opting into other electives. »

Raisons de l'attraction initiale pour la didactique des mathématiques
La question concernant les raisons de l'attraction initiale pour la didactique des mathématiques était une question ouverte formulée comme suit: «Qu'est-ce qui vous a attiré-e vers la didactique des mathématiques initialement?» Un espace de six lignes était disponible pour la réponse. Seulement 2 personnes sur 63 n'ont rien répondu.

Les raisons mentionnées le plus souvent peuvent se regrouper en dix thèmes. Le premier a trait aux mathématiques, sept autres portent sur l'enseignement et les deux derniers concernent des influences ou des contraintes externes.

1. L'intérêt ou l'amour à l'égard des mathématiques (N=12).
Exemples:

«J'ai beaucoup aimé «faire» les mathématiques comme étudiante au primaire et au secondaire.»
«I really enjoyed the subject.»

2. L'intérêt ou l'amour à l'égard de l'enseignement (des mathématiques) (N = 14).
Exemples:

«Le fait que j'aime enseigner.»
«Always (as far back as I remember) I've wanted to teach.»

3. Le désir de faire aimer, de faire comprendre les mathématiques. Le désir de les démythifier, de les humaniser, de les simplifier (N = 8).
Exemples:

«[...] vouloir que d'autres personnes comprennent les mathématiques.»
«Le besoin d'«humaniser» et de «démythifier» la mathématique.»
«As a student I always enjoyed helping my peers with mathematics and found satisfaction when I succeeded in letting them «see» some concepts that they could not «see» before.»

4. Le côté social de l'enseignement: le goût de travailler avec les gens ou avec les jeunes; le désir de faire quelque chose d'utile socialement, le souci de justice sociale (N = 8).
Exemples:

« I care about equity issues, fairness and humanizing the teaching process.»
«It seemed more socially useful than scientific mathematics.»
«[...] I like working with people.»
«Love of communication.»

5. L'expérience d'avoir aidé d'autres élèves lorsqu'on était soi-même aux études (N=4).
Exemples:

«Dès l'âge de 10 ans, j'aidais les plus faibles ...»
«I was good at mathematics and had success showing others how to do it. All through secondary school I tutored other students.»

6. La réussite dans l'enseignement (des mathématiques) (N = 7).
Exemples:

«Ma grande capacité et aptitude à pouvoir aider les autres à comprendre les mathématiques [...]»
«I was good at communicating abstract ideas [...]»
«[...] I was an excellent teacher of mathematics in my public school teaching experiences.»

7. Des questions ou des besoins suscités par la pratique de l'enseignement (des mathématiques): le désir de s'améliorer, le désir de comprendre les difficultés observées chez les élèves (N=11).
Exemples:

«L'enseignement que je réalisais. Le questionnement que cela suscitait en moi.»
«Interest in students' difficulties observed in my teaching.»

8. Le désir d'améliorer l'éducation en ce qui concerne les mathématiques, d'aider les enseignants et les enseignantes à mieux enseigner. L'intérêt pour la formation des maîtres (N = 5).

Exemples:

«[...] pour aider les enseignantes [*sic*] à enseigner une mathématique intelligente.»

«Challenge of trying to improve state of mathematics education.»

«Desire to train/educate good mathematics teachers.»

9. L'influence de certaines personnes (N = 7).

Exemples:

«Le contact avec d'autres personnes en didactique [...]»

«Des contacts avec des didacticiens de réputation internationale.»

«University people who worked in this area.»

10. Le hasard, des circonstances externes, la possibilité d'emploi (N = 9).

Exemples:

«L'offre d'un poste à une école secondaire.»

«Le hasard de l'emploi. »

«Free tuition.»

Plusieurs réponses, surtout parmi celles qui sont classées sous les cinq premiers thèmes, semblent expliquer l'attraction initiale pour l'*enseignement* des mathématiques autant que celle pour la *didactique* des mathématiques. Cette confusion se produit plus souvent, mais non exclusivement, en anglais, peut-être parce que l'expression «mathematics education» est moins précise que la tournure française «didactique des mathématiques» et peut désigner aussi l'enseignement (l'éducation).

4. La carrière

Emploi dans des établissements d'enseignement primaire, secondaire ou collégial

Cinquante-cinq participants et participantes à l'enquête (87%) ont déjà occupé un emploi dans une école ou dans un collège: 25 (40%) dans une école primaire, 45 (71%) dans une école secondaire et 23 (36%) dans un collège (plusieurs ont travaillé à plus d'un ordre d'enseignement). Dans la plupart des cas, il s'agit d'emplois comme enseignant ou enseignante, mais aussi comme conseiller ou conseillère (pédagogique ou d'orientation), comme orthopédagogue, comme personne-ressource ou dans des fonctions administratives. Proportionnellement plus d'hommes que de femmes ont travaillé au secondaire (80% c. 44%; χ^2 corrigé pour continuité = 6,1; $p < 0{,}02$) et plus de francophones que d'anglophones ont travaillé au collégial (55% c. 27%; χ^2 corrigé pour continuité = 3,6; $p < 0{,}06$). Cette dernière différence, marginalement significative, s'explique vraisemblablement par la structure particulière du système scolaire au Québec. Il n'y a pas d'autres différences statistiquement significatives.

Emploi dans des universités

Quarante-sept participants et participantes à l'enquête (75%) travaillent dans un département de sciences de l'éducation, 13 (21%), dans un département de sciences mathématiques, et 3 ont des postes combinés en sciences de l'éducation et sciences mathématiques.

Onze des treize personnes qui travaillent dans un département de sciences mathématiques sont au service de l'une ou l'autre des deux universités québécoises (l'une anglophone, Concordia, et l'autre francophone, l'Université du Québec à Montréal) qui confient à leur département de mathématiques les cours de didactique des mathématiques.

Rang universitaire

Sept des personnes ayant participé à l'enquête sont membres du réseau de l'Université du Québec et n'ont pas de rang universitaire. Le tableau 6 donne la distribution de l'échantillon selon le rang universitaire et le sexe, en excluant ces sept personnes. La distribution selon le rang universitaire est très semblable chez les deux groupes linguistiques.

Tableau 6

Rang universitaire selon le sexe

Rang	Femmes		Hommes		Total	
	N	%	N	%	N	%
Adjointe ou adjoint	7	47	8	20	15	27
Agrégée ou agrégé	4	27	11	27	15	27
Titulaire	4	27	22	54	26	46
Total	15		41		56	

Note: χ^2 = 4,7; non significatif.

À titre de comparaison, l'ensemble du corps professoral des sciences de l'éducation comprend 23% d'adjoints et d'adjointes, 44% d'agrégés et d'agrégées et 33% de titulaires, alors que les taux correspondants pour l'ensemble du corps professoral des mathématiques et sciences physiques sont 18%, 32% et 50% (Statistique Canada, 1993a, p. 32).

Cheminement dans la carrière universitaire

L'âge moyen au moment du premier engagement au rang d'adjoint ou d'adjointe est de 35 ans pour l'ensemble de l'échantillon: il est de 37,6 ans pour les femmes, de 33,8 ans pour les hommes; de 34,6 ans pour les personnes travaillant en milieu anglophone et de 35,9 ans pour celles en milieu francophone. Les différences ne sont pas statistiquement significatives.

L'âge moyen au moment du premier engagement au rang d'agrégé ou d'agrégée est de 37,7 ans pour l'ensemble de l'échantillon: il est de 41,5 ans pour les femmes, de 36,7 ans pour les hommes (t = 2,2; p < 0,04; grandeur de l'effet d = 0,8); de 37,1 ans pour les personnes travaillant en milieu anglophone et de 39,3 ans pour celles en milieu francophone. La dernière différence n'est pas statistiquement significative.

L'âge moyen au moment du premier engagement au rang de titulaire est de 42,9 ans pour l'ensemble de l'échantillon: il est de 42 ans pour les femmes, de 43,1 ans pour les hommes; de 41,9 ans pour les personnes travaillant en milieu anglophone et de 45 ans pour celles en milieu francophone. Les différences ne sont pas statistiquement significatives.

Obtention de la permanence

Quarante-sept participants et participantes à l'enquête (75%) ont obtenu la permanence dans leur emploi universitaire actuel. Les taux correspondants sont de 53% pour les femmes et de 84% pour les hommes (χ^2 corrigé pour continuité = 5,4; $p < 0,03$); cela s'élève à 73% pour les personnes travaillant en milieu anglophone et à 77% pour celles qui sont en milieu francophone.

L'âge moyen à l'obtention de la permanence est de 36,9 ans pour l'ensemble de l'échantillon: il est de 38,8 ans pour les femmes, de 36,4 ans pour les hommes; de 36 ans pour les personnes travaillant en milieu anglophone et de 38,5 ans pour celles en milieu francophone. Les différences ne sont pas statistiquement significatives.

Domaines de recherche autres que la didactique des mathématiques

Plusieurs des participants et des participantes ont fait (52%) ou font encore (22%) de la recherche dans un ou plusieurs domaines autres que la didactique des mathématiques. Les taux correspondants sont de 53% et 21% pour les femmes, et de 52% et 23% pour les hommes; de 46% et 17% pour les personnes travaillant en milieu anglophone et de 64% et 32% pour celles qui se trouvent en milieu francophone. Les différences ne sont pas statistiquement significatives.

Pour ce qui est du passé, les domaines mentionnés le plus fréquemment, par plus de deux personnes, sont les sciences mathématiques (mathématiques pures ou appliquées, statistique et probabilité, informatique) (19), les sciences de l'éducation (8) et la psychologie (6). Quant au présent, les domaines mentionnés le plus fréquemment sont les sciences de l'éducation (5), les sciences mathématiques (4) et la psychologie (3).

Donc, non seulement 81% des participantes et des participants à l'enquête ont-ils au moins un diplôme universitaire en sciences mathématiques, mais 30% affirment avoir fait de la recherche dans ce domaine dans le passé, voire en faire encore.

Réorientations

Cinquante-quatre participants et participantes à l'enquête (86%) affirment avoir connu des réorientations au cours de leurs études ou de leur carrière. Les pourcentages sont semblables pour les deux sexes et pour les deux groupes linguistiques. Dans le questionnaire, le sens du mot «réorientation» était précisé au moyen de deux exemples: le passage d'un emploi dans les écoles à un emploi universitaire ou des mathématiques à la didactique des mathématiques.

De plus, parmi les neuf personnes qui affirment n'avoir vécu aucune réorientation, six ont travaillé dans une école primaire ou secondaire ou dans un collège avant leur engagement dans une université et une autre ne détient que des diplômes en mathématiques, y compris un doctorat, et a fait des recherches en mathématiques pures. Si l'on considérait que ces sept personnes, contrairement à ce qu'elles affirment, ont en réalité vécu des réorientations, la proportion de l'échantillon ayant vécu des réorientations monterait à 97%.

La plupart des réorientations correspondent aux deux exemples donnés dans le questionnaire, c'est-à-dire le passage d'un emploi dans le système scolaire ou dans un collège à un emploi universitaire ou encore le passage des mathématiques à la didactique des mathématiques. Parfois ces deux types de réorientation se produisent en même temps ou l'un à la suite de l'autre. Un cas typique est celui de quelqu'un qui enseigne les mathématiques au secondaire ou au collégial, qui retourne aux études, obtient un doctorat en éducation et est engagé dans un poste universitaire en didactique des mathématiques.

Parmi les raisons qui ont poussé les participants et les participantes à l'enquête à laisser l'école ou le collège pour l'université, on trouve l'insatisfaction des conditions de travail en milieu scolaire, l'intérêt pour la recherche ou pour la formation des maîtres, le désir d'un poste plus prestigieux offrant plus de défis, de variété et de liberté ou encore l'espoir d'être en position de produire des changements dans le système éducatif. Quant aux réorientations des mathématiques à la didactique, elles sont dues surtout à l'intérêt personnel pour les questions touchant à l'enseignement, mais aussi à des circonstances externes (possibilité d'emploi) ou à une certaine insatisfaction à l'égard des mathématiques («Didn't enjoy the solitude of pure math»).

Ces réorientations ont généralement été vécues de façon heureuse. Ceux et celles qui ont quitté l'école ou le collège pour l'université s'en disent satisfaits, contents ou très contents, même si quelques personnes font allusion à la charge de travail plus exigeante ou à la nostalgie par rapport aux élèves. Une seule personne regrette d'avoir effectué ce type de changement dans sa carrière. Le passage des mathématiques à la didactique a été également, dans la plupart des cas, une expérience positive. Une petite minorité mentionne cependant une certaine amertume ou frustration (liées aux circonstances particulières de la réorientation ou aux réactions des collègues) et le regret de ne plus avoir la possibilité d'enseigner les mathématiques. Il est d'ailleurs ironique qu'une réorientation dictée par un intérêt pour l'enseignement des mathématiques ait parfois l'effet d'empêcher la pratique de l'enseignement de cette matière.

Satisfaction à l'égard du choix de carrière

Parmi les participantes et les participants à l'enquête, 86% ont affirmé que s'ils pouvaient recommencer leur carrière, ils choisiraient encore de devenir didacticien ou didacticienne des mathématiques. Les pourcentages sont très semblables pour les deux sexes et pour les deux groupes linguistiques.

Charge d'enseignement et de supervision de stages

À la session d'automne 1992, les participants et les participantes à l'enquête enseignaient en moyenne 5,5 heures par semaine (médiane = 5 heures, avec une étendue allant de 0 à 16) et accomplissaient une tâche de supervision de stages équivalente à 1,1 heure d'enseignement par semaine (médiane = 0, avec une étendue allant de 0 à 9 heures), pour un total de 6,6 heures par semaine. Si l'on ne tient pas compte de 17 personnes qui n'avaient pas de tâches de ce genre à la session d'automne 1992 (à cause d'un congé sabbatique, d'un congé de maladie, d'un dégagement pour accomplir des tâches administratives, etc.) la moyenne est de 7,6 heures d'enseignement et s'élève à 9,2 heures si on inclut la supervision des stages. Les statistiques sont semblables pour les deux sexes et pour les deux groupes linguistiques.

Direction de mémoires et de thèses

Le tableau 7 indique le nombre de mémoires de maîtrise et de thèses de doctorat terminés au cours des cinq dernières années sous la direction des didacticiens et des didacticiennes des mathématiques qui ont participé à l'enquête, selon leur sexe. Il y a peu de différence relativement à ces variables entre les deux groupes linguistiques.

Tableau 7
Direction de mémoires et thèses selon le sexe

Nombre de travaux dirigés	Mémoires de maîtrise			Thèses de doctorat		
	Femmes N	Hommes N	Total N	Femmes N	Hommes N	Total N
Aucun	10	10	20	15	29	44
Entre 1 et 5	7	26	33	4	16	17
Plus de 6	2	8	10	0	1	1
Donnée manquante	0	0	0	0	1	1
Total	19	44	63	19	44	63

Seulement 47% des femmes ont dirigé au moins un mémoire de maîtrise au cours des cinq dernières années, alors que 77% des hommes l'ont fait (χ^2 corrigé pour continuité = 4,2; $p < 0{,}05$). Toutefois, cette différence devient non significative si l'on contrôle le rang universitaire. [2] Quant aux thèses de doctorat, 21% des femmes en ont dirigé au moins une, comparativement à 33% des hommes; la différence n'est pas statistiquement significative.

Publications
Les questions posées aux professeurs et aux professeures de didactique des mathématiques au sujet de leurs publications visaient le nombres d'articles et d'ouvrages publiés ou acceptés pour publication au cours des cinq dernières années et le nombre de ceux qui avaient été écrits en collaboration.

En moyenne, le nombre d'articles publiés ou acceptés pour publication au cours des cinq dernières années est de 8,3 pour l'ensemble de l'échantillon: il est de 6,4 pour les femmes et de 9,2 pour les hommes ($t = -1{,}5$; non significatif); et de 8,4 pour les personnes travaillant en milieu anglophone et de 8,3 pour celles en milieu francophone. Au total, 44% de ces articles ont été écrits en collaboration. Les taux correspondants sont de 42% pour les femmes et de 44% pour les hommes; de 57% pour les personnes travaillant en milieu francophone et de 36% pour celles en milieu anglophone (χ^2 corrigé pour continuité = 19,6; $p < 0{,}001$).

En moyenne, le nombre d'ouvrages publiés ou acceptés pour publication dans les cinq dernières années est de 1,2 pour l'ensemble de l'échantillon: il est de 1,3 pour les femmes et de 1,2 pour les hommes; de 1,3 pour les personnes travaillant en milieu anglophone et de 1,1 pour celles en milieu francophone. Par ailleurs, 71% de ces ouvrages ont été écrits en collaboration. Les

[2] La correction pour la distribution différente des femmes et des hommes dans les rangs universitaires a été faite en calculant la moyenne pondérée des proportions d'hommes adjoints, agrégés et titulaires ayant collaboré à une recherche canadienne (respectivement 37,5%, 45,5% et 54,2%), en utilisant comme coefficients les pourcentages d'adjointes, d'agrégées et de titulaires parmi les femmes (46,7%, 26,7 % et 26,7%), ce qui donne 44%, un taux qui n'est pas significativement différent du taux observé chez les femmes (16%).

pourcentages correspondants sont à peu près les mêmes pour les deux sexes et exactement les mêmes pour les deux groupes linguistiques. On ne retrouve donc pas à propos des ouvrages la plus grande tendance à écrire en collaboration observée chez les francophones à propos des articles.

Travail pour des revues scientifiques ou des maisons d'édition
Le tableau 8 indique combien de fois, pendant les cinq dernières années, les personnes interrogées ont été appelées à évaluer des manuscrits pour une revue scientifique ou pour une maison d'édition. Les distributions sont semblables, aussi bien selon le sexe que selon la langue de travail.

Tableau 8
Évaluation de manuscrits

Fréquence	N	%
Jamais	7	11
Entre 1 et 5 fois	26	41
Entre 6 et 10 fois	14	22
Plus de 10 fois	16	25
Total	63	

De plus, 43% des participants et des participantes à l'enquête ont affirmé être ou avoir déjà été membre du comité de rédaction d'une ou de plusieurs revues scientifiques. Les pourcentages correspondants sont à peu près les mêmes pour les deux sexes. Ils sont de 34% parmi les personnes travaillant en milieu anglophone et de 60% parmi celles qui sont en milieu francophone; la différence n'est pas statistiquement significative.

Participation à des congrès
En moyenne, pendant les deux dernières années, les participants et les participantes à l'enquête ont assisté à 5,6 congrès, ont présenté une communication à 4,3 de ceux-ci et ont présidé une séance à 1,4. Les statistiques sont semblables pour les deux sexes et pour les deux groupes linguistiques.

Insertion dans les réseaux professionnels
Le tableau 9 donne la fréquence, au cours des deux dernières années, de diverses activités indicatives de l'insertion dans les réseaux professionnels nationaux et internationaux.

Il n'y a pas de différence significative entre les deux groupes linguistiques du point de vue de l'insertion dans les réseaux professionnels. Par ailleurs, la seule différence significative entre les sexes concerne la collaboration à des recherches avec des collègues du Canada. Au cours des deux dernières années, proportionnellement plus d'hommes que de femmes ont été engagés dans ce type d'activité (55% c. 16%; χ^2 corrigé pour continuité = 6,6; p = 0,01). Dans ce cas aussi, cependant, comme dans celui de la direction de mémoires de maîtrise, la différence devient non significative si l'on contrôle le rang universitaire. [3]

Tableau 9
Insertion dans les réseaux professionnels

Activité menée avec des collègues...	Collaboration à une recherche N	%	Rédaction d'articles N	%	Échange d'informations N	%
...du Canada	27	43	31	49	42	67
...des États-Unis	11	17	12	19	37	59
...d'Europe	11	17	9	14	33	52
...d'autres pays	11	17	10	16	29	46

5. Une comparaison avec les résultats de l'enquête auprès des mathématiciens et des mathématiciennes

Deux précautions ont été prises avant de comparer les résultats des deux enquêtes. Premièrement, puisque des didacticiens ou des didacticiennes des mathématiques peuvent travailler dans un département de mathématiques, comme on l'a vu plus haut, il a fallu vérifier si l'échantillon des mathématiciens et des mathématiciennes ne comprenait pas des didacticiens ou des didacticiennes. Effectivement, il s'en trouvait quatre. Les données qui les concernent ont été retirées de l'échantillon des mathématiciens et des mathématiciennes avant d'effectuer des comparaisons entre les deux groupes.

Deuxièmement, alors que la présente enquête s'adressait à l'entière population des didacticiens et des didacticiennes, la première enquête avait été menée auprès de toutes les mathématiciennes et d'un échantillon d'hommes jumelés à l'ensemble des femmes selon le rang universitaire, ce qui a entraîné, d'une part, une surreprésentation des femmes (50% au lieu de 6%) et, d'autre part, parmi les hommes, une surreprésentation des professeurs adjoints et une sous-représentation des professeurs titulaires. Afin de rendre comparables les données provenant des deux enquêtes, celles qui concernent les mathématiciens et les mathématiciennes ont été «ajustées», c'est-à-dire qu'elles ont été remplacées par des moyennes pondérées calculées en utilisant les pourcentages de femmes et d'hommes (6% et 94%) et les pourcentages d'hommes aux rangs d'adjoint, d'agrégé et de titulaire (16%, 39% et 45%) dans l'ensemble de la population. C'est pourquoi les données

[3] La correction pour la distribution différente des femmes et des hommes dans les rangs universitaires a été faite en calculant la moyenne pondérée des proportions d'hommes adjoints, agrégés et titulaires ayant collaboré à une recherche canadienne (respectivement 37,5%, 45,5% et 54,2%), en utilisant comme coefficients les pourcentages d'adjointes, d'agrégées et de titulaires parmi les femmes (46,7%, 26,7% et 26,7%), ce qui donne 44%, un taux qui n'est pas significativement différent du taux observé chez les femmes (16%).

rapportées ici pour les mathématiciens et les mathématiciennes diffèrent de celles qui sont publiées dans Mura (1990 et 1991).

Le groupe des didacticiens et des didacticiennes, comparé à celui des mathématiciens et des mathématiciennes, est environ quinze fois moins nombreux, il comprend une plus grande proportion de femmes (30% c. 6%) et une plus grande proportion de personnes travaillant dans un milieu francophone (35% c. 16%). Il s'agit aussi d'un groupe plus âgé (âge moyen de 50,5 ans c. 46 ans). Dans chacun des deux groupes, les femmes sont plus jeunes que les hommes (de 5,4 ans en didactique et de 4,3 ans en mathématiques). Parmi les didacticiens et les didacticiennes, on trouve également une plus grande proportion de personnes nées au Canada (60% c. 32%) et de personnes de langue maternelle anglaise ou française (90% c. 58%).

Parmi les mathématiciens et les mathématiciennes, les femmes viennent de familles plus scolarisées que les hommes, alors que cette différence ne se rencontre pas entre les didacticiens et les didacticiennes. Si l'on compare séparément, d'une part, les mathématiciennes avec les didacticiennes et, d'autre part, les mathématiciens avec les didacticiens, on observe que les premières viennent de familles plus scolarisées que les secondes, alors qu'il y a peu de différence entre les hommes des deux disciplines (tableaux 10 et 11).

Tableau 10
Scolarité des parents des femmes selon le domaine de travail

Scolarité	Mère Mathématiciennes %	 Didacticiennes %	Père Mathématiciennes %	 Didacticiennes %
Primaire	8	33	14	39
Secondaire	51	44	28	39
Universitaire	41	22	59	22

Tableau 11
Scolarité des parents des hommes selon le domaine de travail

Scolarité	Mère Mathématiciens %	 Didacticiens %	Père Mathématiciens %	 Didacticiens %
Primaire	34	36	19	32
Secondaire	56	39	52	45
Universitaire	10	25	29	23

En ce qui concerne les études, il y a peu de différence entre les deux groupes à propos du moment de la décision de se spécialiser en mathématiques et des raisons de l'attraction initiale pour cette discipline. Le pourcentage des personnes titulaires d'un doctorat est le même (89%) chez les deux populations.

Le tableau 12 résume la progression dans les études et dans la carrière universitaire des professeurs et des professeures de mathématiques et de didactique des mathématiques.

Tableau 12
Progression dans les études et dans la carrière selon le sexe et le domaine de travail

Âge au moment...	Didactique des mathématiques		Mathématiques	
	Femmes	Hommes	Femmes	Hommes
...du premier diplôme	22,8	22,7	22,0	22.6
...du doctorat	38,0	35,1	29,6	30,1
...de l'engagement comme adjoint ou adjointe	37,6	33,8	30,4	29,5
...de l'agrégation	41,5	36,7	34,6	33,6
...de la permanence	38,8	36,4	35,2	35,0
...de la titularisation	42,0	43,1	38,4	40,1
Années écoulées entre le premier diplôme et la titularisation	19,2	20,4	16,4	17,5

Même si l'âge d'obtention du premier diplôme est semblable, les didacticiennes et les didacticiens, comparés à leurs collègues de mathématiques, accumulent plusieurs années de retard (8,4 ans pour les femmes et 5,1 pour les hommes) avant d'obtenir le doctorat. Ce retard s'explique probablement en partie par le fait que 87% des didacticiennes et des didacticiens ont occupé un emploi dans une école ou un collège avant d'être engagés par une université. Peut-être aussi les études de doctorat sont-elles plus longues en éducation qu'en mathématiques, mais aucune donnée à cet égard n'a été recueillie dans la présente enquête. Le retard va en s'amoindrissant tout au long de la carrière universitaire: il n'est que de 3,6 ans pour les femmes et de 3 ans pour les hommes au moment de l'obtention de la titularisation.

Si l'on regarde les activités professionnelles, les professeures et les professeurs de didactique des mathématiques sont au moins aussi actifs que leurs collègues de mathématiques du point de vue des indicateurs suivants:

- les heures d'enseignement: 6,6 heures par semaine, y compris la supervision des stages, contre 6 pour les mathématiciens et les mathématiciennes;
- la direction de mémoires de maîtrise et de thèses de doctorat: dans les cinq dernières années, 68% ont dirigé au moins un mémoire de maîtrise et 29% ont dirigé au moins une thèse de doctorat, contre 40% et 29% pour les mathématiciens et les mathématiciennes;
- les publications: 8,3 articles et 1,2 ouvrages en moyenne au cours des cinq dernières années, contre 7,8 et 0,6 pour les mathématiciens et les mathématiciennes. Par ailleurs 44% des articles et 71% des ouvrages produits par les didacticiens et les didacticiennes ont été écrits en collaboration, contre 49% et 80% pour les mathématiciens et les mathématiciennes;
- le travail pour des revues scientifiques ou des maisons d'édition: 89% des didacticiennes et des didacticiens ont été appelés au moins une fois à évaluer un manuscrit pendant les cinq dernières années et 43% font ou ont fait partie d'un comité de rédaction, contre 82% et 13% des mathématiciens et des mathématiciennes;

- la participation à des congrès: en moyenne, au cours des deux dernières années, les didacticiens et les didacticiennes ont assisté à 5,6 congrès, ont présenté une communication à 4,3 de ceux-ci et ont présidé une séance à 1,4. Les données correspondantes pour les mathématiciens et les mathématiciennes sont de 3,1; 1,8 et 0,4.

Quant à l'insertion dans les résaux professionnels, comme le montre le tableau 13, les différences entre les deux groupes ne sont pas très grandes et ne vont pas toutes dans le même sens.

Tableau 13
Insertion dans les réseaux professionnels selon le domaine de travail

Activité menée avec des collègues	Collaboration à une recherche		Rédaction d'articles		Échange d'informations	
	Didact. %	Mathémat. %	Didact. %	Mathémat. %	Didact. %	Mathémat. %
du Canada	43	52	49	47	67	62
des États-Unis	17	30	19	28	59	71
d'Europe	17	35	14	16	52	46
d'autres pays	17	13	16	10	46	31

Enfin, 69% des mathématiciennes et des mathématiciens ont répondu inconditionnellement que s'ils pouvaient recommencer, ils referaient le même choix de carrière; les autres ont hésité ou répondu qu'ils feraient un choix différent. Parmi les didacticiennes et les didacticiens, 86% ont affirmé qu'ils referaient le même choix.

Pour ce qui est des différences entre les sexes concernant la carrière, une seule est commune aux deux groupes: les femmes sont moins souvent permanentes que les hommes. Des deux autres différences observées entre les didacticiennes et les didacticiens, l'une touche l'emploi dans une école secondaire et ne s'applique pas aux professeures et aux professeurs de mathématiques, tandis que l'autre (les femmes sont plus âgées que les hommes au moment de l'agrégation) apparaît entre les mathématiciennes et les mathématiciens comme une tendance mais non comme une différence significative. À l'inverse, quatre différences observées entre les mathématiciennes et les mathématiciens ne se retrouvent pas parmi leurs collègues de didactique, à savoir: les mathématiciennes sont plus souvent adjointes et moins souvent titulaires que leurs collègues masculins (c'est une tendance en didactique aussi, mais la différence n'est pas significative); au rang d'adjointes, les mathématiciennes publient moins d'articles que les hommes du même rang (c'est une tendance en didactique aussi, mais la différence n'est pas significative); leurs articles sont plus souvent écrits en collaboration et elles sont moins souvent appelées à évaluer des manuscrits pour des revues scientifiques ou des maisons d'édition (ce n'est pas le cas en didactique).

Conclusion

Au Canada, la didactique des mathématiques est plus souvent rattachée aux sciences de l'éducation qu'aux mathématiques: les trois quarts des professeurs et des professeures universitaires dans ce domaine ont obtenu leur diplôme le plus élevé en éducation et la même proportion travaille dans un département ou dans une faculté d'éducation. Il est donc normal de constater que, du point de vue de plusieurs variables, le groupe des didacticiens et des didacticiennes des mathématiques se rapproche davantage du corps professoral des sciences de l'éducation que de celui des mathématiques et sciences physiques. Ainsi, en faisant appel aux données fournies par Statistique Canada (1993a), on a observé que le présent échantillon, comme l'ensemble du corps professoral des sciences de l'éducation, comparé à celui des mathématiques et sciences physiques, constitue un groupe plus âgé et comprend une plus grande proportion de femmes ainsi que de personnes ayant la citoyenneté canadienne. Une comparaison avec les résultats d'une enquête effectuée précédemment auprès des mathématiciennes et des mathématiciens (Mura, 1990 et 1991) a mis également en évidence que, comparés à ces derniers, les didacticiennes et les didacticiens sont plus souvent nés au Canada et de langue maternelle anglaise ou française.

Peut-être la caractéristique la plus frappante du profil professionnel du groupe étudié est-elle que la grande majorité (87%) a vécu une carrière en deux phases: une première phase dans un établissement d'enseignement primaire, secondaire ou collégial et une seconde en milieu universitaire. La décision de se spécialiser en didactique survient tardivement par rapport à celle de se spécialiser en mathématiques — pour la majorité (67%), après l'âge de 21 ans. Cette situation entraîne un retard de quelques années dans l'obtention du doctorat et dans le début de la carrière universitaire comparativement à un groupe comme celui des mathématiciens et des mathématiciennes qui suit un cheminement professionnel plus linéaire. Malgré cela, la plupart des didacticiens et des didacticiennes ont vécu de façon heureuse la réorientation du milieu primaire, secondaire ou collégial au milieu universitaire et des mathématiques à la didactique: 86% des personnes interrogées affirment que, si elles pouvaient recommencer, elles referaient le même choix de carrière.

Mis à part le retard dans l'obtention du doctorat et dans le début de la carrière universitaire, les professeurs et les professeures de didactique se comparent favorablement à leurs collègues dans le domaine des mathématiques du point de vue de la formation et des activités professionnelles.

En ce qui concerne la comparaison faite, d'une part, entre les femmes et les hommes et, d'autre part, entre les personnes travaillant en milieu anglophone ou francophone, parmi les nombreuses variables considérées, si l'on contrôle le rang universitaire lorsqu'il est opportun de le faire, seulement quatre ont révélé une différence significative entre les sexes et deux entre les groupes linguistiques. Le peu de différences statistiquement significatives observées peut toutefois être dû, du moins en partie, à la petite taille de l'échantillon. Les quatre différences entre les sexes sont les suivantes:

- les femmes constituent un groupe plus jeune, en moyenne de 5,4 ans, que les hommes (grandeur de l'effet $d = -0,8$);
- proportionnellement moins de femmes que d'hommes ont travaillé dans une école secondaire;
- au moment de l'agrégation, les femmes sont plus vieilles, en moyenne de 4,8 ans, que les hommes (grandeur de l'effet $d = 0,8$). Cette différence est le résultat du fait qu'elles ont été engagées comme adjointes 3,8 ans plus tard que les hommes et du fait qu'elles demeurent au rang d'adjointe un an de plus que les hommes;
- proportionnellement moins de femmes que d'hommes ont obtenu la permanence.

Quant aux groupes linguistiques, les deux différences observées sont les suivantes:

- proportionnellement plus de personnes en milieu francophone ont travaillé dans un collège (ce qui s'explique vraisemblablement par la structure du système scolaire québécois);
- proportionnellement plus d'articles publiés par des personnes travaillant en milieu francophone ont été écrits en collaboration.

Si les sous-groupes étudiés ne se distinguent pas de façon marquée les uns des autres, le portrait de famille dessiné par les résultats de l'enquête est celui d'une communauté professionnelle diversifiée. Ainsi, chacun et chacune de nous pourra maintenant mieux se situer par rapport à ses collègues et savoir dans quelle mesure son expérience est singulière ou partagée par d'autres.

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Topic Group B1

A Look at Current Software Designed to Provide Different Representations of Functions

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Three software packages for the Macintosh which allow high school students to investigate the properties of functions were demonstrated. Following this the participants were able to carry out their own exploration of the software. The packages presented, CARAPACE, *Math Connections Algebra II* and *Function Probe* 2.0.3, all represent funtions using tables, graphs and expressions, but each does this in a different way. The criteria used for evaluation of these (and other) packages were based on needs emerging from a particular research project on introducing algebra to grade seven students.

Topic Group B2

Math in the Mall
The SFU Experience

Malgorzata Dubiel

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Introduction

Kathy Heinrich and I became interested in popularizing mathematics while teaching the course *Mathematics for Elementary School Teachers* (MATH 190). Our goal was—and is—to change the attitudes of student-teachers to mathematics. Our strategy was to prepare (and later let students prepare) various projects which would allow them to see a glimpse of what we see in mathematics; the beauty of it, the excitement and amazement upon seeing something unexpected, the mathematics around us, and the people of mathematics.

The projects proved very succesful and we started to think about reaching younger children. Our first opportunity was the Homecoming event organized for the 25th anniversary of Simon Fraser University in September 1990. We put up a display which we called "Is this Math?" using some of our MATH 190 projects; puzzles, games, geometrical models, videotape with a week's worth of *Square One TV* (with permission of PBS), and a display of our choice of books. The display was even more succesful than we expected.When colleagues from computing science complained that our display was diverting traffic from their exhibits, we knew we were on the right track. The president of SFU visited the display and was sufficiently impressed to promise funding for a shopping mall version.

For our first foray outside the university we chose Lougheed Mall—a large mall in Burnaby, close to SFU and to several elementary schools. While preparing the event we were contacted by organizers of the Science and Technology Week 91 and, at their request, we agreed to repeat the display three weeks later in another Burnaby Mall, Metrotown. Since then, our mall appearances have been restricted to Science and Technology Weeks and similar events, as this makes the organization much simpler. We also take some of the activities to schools or occasionallu have groups of children visit the department.

The most popular activities are:

1. Kaleidocycles and hexaflexagons We discovered them through Martin Gardner's books. We always have several models displayed, including one or two of "monster size" on the floor (each side the largest triangle you can make out of one piece of poster paper), and some with Escher designs on the faces. These are the "bait"—people start playing with them and want, and are encouraged, to make one themselves. We have designs ready for them to cut, glue and decorate. Decorating is an important part of the process as it allows the participants to discover symmetries of the object.

2. Platonic solids Forms to make them and lots of models.

3. Drinking straw models Flexible plastic drinking straws can be used to make geometrical models—the icosahedron being the most popular choice. You cut through the shorter end of a straw, squash it to make it narrower and insert it into the longer half of another. Polygons made in this way cnan be assembled using tape to make models of solids. This activity is very popular with small children, but everybody loves it, including our helpers, who always manage to build a strange object during slower moments.

4. Pentagonal stars A MATH 190 student from Singapore taught us how to make these. The basic idea is that if you tie a knot in a rectangular strip of paper and flatten it, you get a regular pentagon. When you fold the strip around itself several times and then push the edges in to pop it out you get a pentagonal star. We use gift wrapping paper cut into long strips of various widths.

5. Möbius band

6. Puzzles—especially ones based on Gray codes, and geometrical ones. They are especially good for participants twelve years and older—teenage boys often do not want to be seen "playing" with models, but will be tempted by puzzles. We are slowly building our own collection.

We also have lots of posters on geometry, Escher, mathematical careers, mathematical houmour; activities one does not usually associate with mathematics (for example, knitting, design, architecture etc.) Most of them we made ourselves, but there are many good posters available.

To help with the displays, we recruit colleagues and students, both graduate and undergraduate. Students from MATH 190 often like to help, partially because they have to demonstrate volunteer work for admission to the professional programmes in education. Many of those who help once come back—they have fun participating.

The list below includes both articles about mathematics displays and trails, and a selection of our favourite books.

Bolt, B. *Mathematical Cavalcade*, Cambridge University Press.
Bolt, B. *Mathematical Funfair,* Cambridge University Press.
Bolt, B. *More Mathematical Activities*, Cambridge University Press.
Bolt, B. *The Amazing Mathematics Amusement Arcade*, Cambridge University Press.
Gardner, M. *Mathematical Carnival,* MAA.
Gardner, M. *Mathematical Magic Show,* MAA.
Gardner, M. *Mathematical Circus*, MAA.
Gardner, M. *The Scientific American Book of Mathematical Puzzles and Diversions*, Simon and Schuster, 1959.
Gardner, M. *The Second Scientific American Book of Mathematical Puzzles and Diversions*, Simon and Schuster, 1961.
Heinrich, K. *Mathematics Education*, CMS Notes, 24(4), May-June 1992.
Muller, E. *Math Trails*, CMS Notes, 25(2), March 1993.
Schattschneider, D. and Walker, W. *M. C. Escher Kaleidocycles*, Pomegranate Artbooks, 1987. (A book with several very interesting ideas and wonderful models of kaleidocycles and solids to make, decorated with Escher pictures—definitely worth buying! We buy a new one every two years.)
Williams, E. R. *Math in the Mall*, CMS Notes, 25(1), January-February 1993.

Topic Group C1

"Booking", a Non-Traditional Approach to the Teaching of Mathematics in the Transition Years

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Booking, a Non-Traditional Approach to the Teaching of Mathematics in the Transition Years

Over the past five years a small group of mathematics educators has worked on producing a "different" series of middle school (Grade 7–9) student mathematics texts and associated teachers' resource books (Flewelling *et al.* 1991, 1993). This topic group focused on some of the decisions made by this writing team and some of the issues surrounding these decisions.

Author Beliefs

The authors brought the following beliefs to the writing task.

1. The learner constructs understanding based on his or her own experiences.

2. Traditional teaching methods provide the student with insufficient experiences upon which to create understanding.

3. Students in the middle years are "all over the map", with respect to experiences, skill levels, and so on.

4. Students need to take more control over their own learning.

It became the task of the team to create (rich) sources of experience, learning activities, for students.

Characteristics of Learning Activities

The authors tried to design the activities which

- placed stress on understanding
- aroused interest
- were accessible to all
- helped to broaden the student's view of what math is and what it means to do and to make mathematics
- built student confidence and willingness to take risks
- integrated problem-solving
- made the importance and relevance of mathematics self-evident
- fostered a liking for mathematics

- placed more emphasis on the student as teacher and evaluator
- placed a greater premium on the teacher as facilitator
- encouraged more student autonomy
- took greater advantage of technology
- encouraged an estimation mindset
- encouraged a spirit of enquiry
- focussed on different ways of doing something
- encouraged more reflecting on their experiences and thinking for themselves
- encouraged communication with more speaking, reading, writing, and reporting about ideas
- gave the students more opportunities to gather evidence upon which to make, verify, and revise generalizations

Texts and Resource Books

The student texts were designed to talk directly and solely to the student. The language used was informal and personal. Importance was placed on the student's ability to read. The aim was to get the student to accept ownership in their own learning and to think for themselves. The authors were able to share information, advice, and suggestions with the teacher through the teacher's resource book.

Activity Structure

An activity structure was decided upon because it was thought that such a format would increase student involvement in their own learning and provide the student with a richer source of experiences than they could get from a more traditional set of exercises.

An activity-based approach requires the teacher to use traditional teaching methods more for the setting up and summing up (or reflecting) phases of an activity. The teacher, during the activity, usually assumed the role of facilitator. Information, advice, and suggestions for the teacher in each of the three phases of an activity were placed in teacher resource books. Because of the open nature of many of the activities, suggestions concerning such things as time allotments and homework were often not included.

Many of the activities lent themselves to students working in small groups (for reasons listed above).

Many of the activities, by their nature, encouraged the teacher to use a broader range of assessment tools than is traditionally used.

Part of the problem of "booking" the above, that is, putting it into print in a student text was to decide on how much structure to wrap around any given activity, to decide on how much to say and direct and how much to leave unsaid and open. (The authors wanted students to exercise autonomy and learn to think for themselves.) The authors usually chose more structure than they cared for and gave suggestions to the teacher, in the teacher resource book, for modifying the structure or constraints. Tension and balance between autonomy and authority is an important issue.

Implementation

The non-traditional nature of the materials was not found to be a significant obstacle to effective implementation. Implementation problems, where they existed, were found to be more related to narrow views of mathematics and outmoded views of the roles of teachers and students.

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Topic Group C2

Student-Teachers' Conceptions of Mathematics: What They Are and How They Are Formed

R. Geoffrey Roulet

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In its simplest form, mathematics may be described as the study of numbers. Mathematics involves two components. The first component consists of numbers and the other component involves operations. These operations are then applied to one or more of these numbers resulting in another number. In a broader sense mathematics is also the study of lines, objects such as circles, squares and triangles. Mathematics involves the manipulation of equations and functions.

The most attractive attribute of mathematics that I enjoy is the fact that the answer is either right or wrong. There is no middle in mathematics. However, there is usually more than one way of doing a question or problem. I also enjoy mathematics because you can usually check your solution to your answer. Because mathematics is a logical process and has a step by step approach, it is easy to check where you went wrong. I also enjoy the manipulation of numbers and how mathematics can be useful in the real world; whether it be, being able to add up a bill or to check what the total cost of a product would be with the tax included before it is punched into a cash register. (Teacher Candidate, 1991)

The above was written by a secondary school mathematics teacher candidate in response to an initial course assignment asking students to describe mathematics and to identify those features of the subject that excited them. This image of mathematics, as a fixed body of indisputable facts and procedures, is not unique to this student but appears to be held by a majority of mathematics teacher candidates. It is also not the result of a weak academic background, for prior to beginning the one-year teacher preparation programme, this student had obtained an Honours BSc degree in Mathematics and Computer Science. Here he had completed the equivalent of nine full-year university mathematics courses, seven more than that required by the Ontario Ministry of Education and Training for certification as a teacher of mathematics.

Such a conception of mathematics may be contrasted with the philosophy underlying the curriculum guidelines describing courses that this candidate is preparing to teach and the view promoted by the leaders of the profession he will soon be joining. The introductory pages of the *Curriculum Guideline: Mathematics: Intermediate and Senior Divisions* (Ontario Ministry of Education, 1985) describe a programme that employs an experiential approach with concepts being developed out of applications. Mathematical modelling and problem solving, including the exploration of situations in which a strategy is not immediately evident, are identified as underlying themes. The recently released *Focus on Renewal of Mathematics Education* (Ontario Association for Mathematics Education/Ontario Mathematics Coordinators Association, 1993) sets out the principles of teaching and learning supported by the professional organizations for Ontario teachers of mathematics. Here is described a program in which pupils actively construct mathematical understandings through investigating, conjecturing, testing hypothesis, and the discussing and sharing of ideas.

Progress towards mathematics programmes which reflect the philosophies of these two Ontario curriculum documents has been slow. Provincial reviews examining the teaching and learning of mathematics at Grades 8, 10, and 12 summarize the data on instructional practice with, "The most commonly used methods for teaching ... mathematics are presentation of information to the class by chalkboard or overhead projector and assignment of individual work" (Ontario Ministry of Education, 1991a, 1991b, 1991c, p 1). Thompson (1992, August), while discussing this slow pace of educational change, suggests "the sharp contrast between these two images [of mathematics] may be the single greatest obstacle to achieving mathematics instruction as envisioned in many reform documents" (p 4). Recognition of the importance of subject images or beliefs has encouraged research exploring the conceptions of mathematics held by teachers and teacher candidates. This paper reports on two such studies conducted with student-teachers in their final year of preservice education.

Teachers' Conceptions of Mathematics

The research literature does not present a precise or consistent definition of belief and the boundary between belief and knowledge is not clear. However there is general consensus (Pajares, 1992) that beliefs impact on knowledge in that they filter new information and play a key role in interpretation. Both directly and through their influence on knowledge structures, beliefs strongly affect behaviour. Consistent with this view, surveys of the literature in mathematics education (Fennema, Carpenter & Lamon, 1991; Fennema & Franke, 1992) present models of teaching showing the impact of a teacher's beliefs on knowledge, decision making, and classroom practice. In particular, reflecting Thom's (1973) observation that "whether one wishes it or not, all mathematical pedagogy, even if scarcely coherent, rests on a philosophy of mathematics", Ernest (1989) and Higginson (1980) have developed models of mathematics education that give a central and critical role to teacher views of the nature of the discipline. Thompson (1992) uses the term *conception of mathematics* to collect together a teacher's beliefs, meanings, images, preferences and personal philosophy concerning the subject. This broader label will be employed in this paper.

Ernest (1989) identifies three possible conceptions of mathematics: instrumentalist, Platonist, and problem-solving. For the instrumentalist, mathematics is a collection of facts, rules and procedures that have their value in applications in other fields. Platonists view mathematics as a body of consistent absolute knowledge that exists independent of human thought while for those holding the problem-solving image the discipline is a cultural invention, created by human inquiry and thus continually growing and open to revision. In a more recent work Ernest (1991) collects the instrumentalist and Platonist conceptions under the label absolutist, indicating the view that mathematics is a collection of absolute truths established through logical deduction, while the problem-solving position is called fallibilist as it accepts the logical frailty of a system created by human thought.

Studies have shown a link between the pedagogical practices of high school and junior high school teachers of mathematics and their conceptions of the discipline. Thompson (1984) worked with three teachers of mathematics in an American junior high school. Although the author does not employ Ernest's categories, two of her participants possessed absolutist conceptions of mathematics taking Platonist and instrumentalist positions while the third teacher articulated beliefs that were closer to the fallibilist position and revealed a problem-solving orientation to her subject. Instructional methods reflected these conceptions of mathematics, with those holding the absolutist view adopting a transmission model of teaching. Students were expected to assimilate content and learn procedures from the teacher's examples, explanations and demonstrations. The only significant difference between these two teachers was that the Platonist stressed the reasons and logic underlying mathematical procedures and established links between concepts. Thompson's other participant demonstrated her problem-solving orientation by taking a more process approach to instruction. Students were encouraged to guess, make conjectures and produce their own problem solutions.

Through the use of a questionnaire, Lerman (1990) identified two pairs of mathematics teacher candidates that represented the extremes of the absolutist and fallibilist perspectives. After the presentation, via video recording, of a short mathematics lesson, interviews were conducted to elicit these four students' assessments of the teaching observed. The link between views of mathematics and orientations to teaching practice are drawn with the observations "that the two student teachers who were the most 'absolutist' felt that the teacher in the extract was not directing the students enough and was too open, whereas the most 'fallibilist' thought she was not open enough, and was too directed" (Lerman, 1990, p 59).

The four high school geometry teachers in McGalliard's (1983) study showed a dualistic (Copes, 1982; Perry, 1981) conception of geometrical knowledge. This view, that every question has a correct answer that can be defended by reference to authorities, was consistent with the observed instructional practice which focused on the preparation of pupils for the next mathematics course by careful compliance with the syllabus and the teaching of rules without explanations.

The Toolkit Conception of Mathematics

In September 1991 data was collected to help construct a picture of the conceptions of mathematics held by students beginning the one-year secondary school mathematics teacher preservice programme at Queen's University.

Method The 29 members of a mathematics curriculum and instruction class were asked after the initial course meeting to write position papers giving their images of Schwab's (1978) four "commonplaces of teaching"; the subject (mathematics), the student, the teacher, and society in general. In particular, students were asked to "[d]escribe the subject of mathematics to someone who has not studied the subject beyond the grade 8 level", and to relate what excited them about mathematics. Images of students and teachers were elicited by the questions: "How do students best learn mathematics?", "What makes a 'good' mathematics teacher?", and "What makes a 'good' mathematics lesson?" The questions concerning society at large asked class members to describe the benefits for students and the larger community that result from school mathematical studies. While the assignment was compulsory, to minimize anxiety and encourage students to report their own personal views, no assessment of the writing was made and no grades were assigned.

Analysis of the position papers involved three readings. An initial reading produced a general overview of student perspectives and a second identified common themes by noting frequently occurring words and ideas. During the third reading those themes addressed by each student and the particular positions taken were catalogued.

Results Two popular themes were identified: (a) the structure and logical completeness of mathematics, and (b) the utility or application of mathematics, with all but two students commenting on both issues. Students introduced these themes with their descriptions of the discipline and elaborated upon them in their discussions of teaching and learning and of the place of mathematics in the community at large. While the positions taken on the two themes were not unanimous, for both there was a strong majority view.

For two-thirds of the class "mathematics in its simplest form is only a set of rules", or "methodologies of how to work with numbers and symbols" which "demands from you patience and logical thinking to follow step by step procedures". These students find comfort in a system that "is rigorously precise" with "no grey areas or exceptions" and where one can "get the right answer just by following rules and methods."

This view of the structure and logical completeness of mathematics was continued in these students' descriptions of good teaching and of the best methods for learning the subject. Effective teaching involves "presenting the concepts in a clear and coherent manner without causing any confusion." The provision of "countless examples to explain and clarify concepts and methodologies" should be followed by student practice for "one of the best ways of learning mathematics is by repetition."

For these students, following mathematical procedures in a "step by step approach" is a "logical form of thinking" that may be applied in other situations. School graduates who have studied mathematics are "quick-thinking, reasonable individuals" capable of solving "many other problems through similar logical reasoning."

A smaller portion of the class (8 of 29) saw mathematics as a "language" for expressing "analytical thinking" and discovering "patterns and relationships." This image leads to a less absolutist conception of mathematics with some of these students noting that "two totally different solutions can end up at the correct answer" and in some cases one may "achieve multiple answers based on various assumptions." For these teacher candidates productive mathematics teaching and leaning involves more than the presentation of examples and the practice of routines. "It is important that the teacher invites [*sic*] the student to learn a more 'hands-on' and creative approach to the subject" which will "allow the students to 'own', through self-discovery and thorough knowledge, the concepts being taught." Through such a mathematics program "a student's imagination and creativity can greatly [*sic*] be enhanced" and one's "understanding of life" increased.

In a similar manner, when discussing the utility of mathematics the students generally adopted one of two positions. A majority of the class saw arithmetic as being central to mathematics and gave as a positive feature of the subject its usefulness in solving problems of "everyday life" such as calculating "how much of your entertainment money do you need to save to buy that new stereo for your car." A serious concern for these teachers-to-be is a lack of relevance of the secondary school mathematics curriculum that leads to pupils who "don't perceive the usefulness of a strong background in mathematics, and as such put forth little effort in trying to learn intangible concepts." Since "students best learn math by solving practical problems using concepts they have been taught," careful presentations of routines should be followed by applications.

The other one-third of the students also acknowledged the utility of mathematics but they saw applications beyond daily financial problems, for mathematics "allows us to explore the unseen and to experience the unexperienced." "In good measure, the value of math is more aesthetic than practical." Applications are an important component of successful lessons for this group also but the order is reversed with concepts introduced and developed "using key examples chosen from the real world."

In summary, instrumentalist and absolutist conceptions of mathematics were predominant in this class. More than half of these future mathematics teachers held a toolkit image where problems are generated in other disciplines, employment, or daily living and mathematics supplies fixed routines for calculating the answers. Beginning teachers who view their task as ensuring that school graduates possess full mathematical toolkits are not likely to contribute to the reform of mathematics education.

The Development of an Alternate Conception

While the toolkit image is the dominant conception of mathematics among student-teachers, not all candidates hold this view. Some arrive at their final preservice year possessing a more fallibilist and social contructivist (Ernest, 1992) philosophy of the discipline; one that views mathematics as a cultural product open to new interpretations, revisions and growth. Studies addressing Thompson's (1992) proposed research question, "How are teachers' beliefs formed?" might provide valuable insights when focused upon those who hold richer visions of mathematics. The second study to be reported here is an example of such a project.

Method In September 1992 Katerina's (a pseudonym) response to the initial course assignment revealed a conception of mathematics that stood in contrast to the toolkit view popular in her class. When a subsequent essay entitled "Why Teach Mathematics?" further developed this rich image she was recruited as a participant in an intensive study aimed at characterizing visions of mathematics, its teaching and learning, and identifying life experiences that might have contributed to these images.

Katerina's conceptions of mathematics and her images of how the subject may best be taught and learned were explored through unstructured interviews focusing on her writing, her teaching practice observed during a Grade 10 mathematics lesson, and repertory grids. Repertory grid technique (Beail, 1985) involves the identification of pairs of contrasting descriptors, the constructs, that are employed to form linear scales along which a set of elements are arranged. The sets of elements for the four grids built by Katerina involved: school subjects; teachers of mathematics, real and imagined; mathematics lessons, real and imagined; and pupils. Factor analysis, performed by the computer program, *Repgrid* (Shaw, 1990) gave measures of the clustering of elements and constructs and helped surface some of Katerina's more hidden images of mathematics and its teaching.

On the surface Katerina's academic career prior to the BEd year was not dissimilar from the norm: elementary and secondary education in Ontario public schools, four years of university leading to a BSc with concentrations in chemistry and mathematics, and one year of part-time teaching. Thus a search for experiences that might have contributed to the formation of her different conception of mathematics required a more detailed examination. In a series of free flowing interviews, Katerina provided narratives of her mathematical experiences from her early pre-school years to her present position, 24 years later, as a faculty of education student.

Katerina's writings, transcripts of all interviews, and the four repertory grids were analyzed to identify recurring themes. Repeated passes through the data were made to gather items under theme labels, to link features of Katerina's conception of mathematics to life experiences.

Results Katerina's mathematical life has not been a series of exclusively positive experiences but through reflection she has used the good and the bad to build her image of mathematics and its teaching. "I did well all the way through but I definitely, in terms of teachers, had ups and downs. It's just that having those bad experiences, almost drives me more. I have mental check lists that, try not to do this, try not to do that."

For Katerina mathematics is a language which "did not develop apart from other activities" but has been built up as a "needed tool for relating, communicating and expressing information." A consequence of this metaphor is that the meanings of mathematical terms must be personally constructed as in learning one's native tongue. "I think that a lot can come through without necessarily putting a definition up on the board. It helps to gauge how much the students already know by soliciting definitions from their words versus getting trapped in the terminology." During her practice teaching, Katerina put these principles into action when students were regularly called upon to flesh out their answers and fully explain their reasoning to the whole class, while their peers were expected to carefully analyze and comment upon the arguments.

This vision of vocabulary growth and Katerina's sensitivity to the exclusionary power of formal language have their roots outside of her mathematical experience. Being the child of recently arrived immigrants, Katerina's home language was Italian and she learned English through her introduction to Canadian schooling. In their interactions with the school system "my poor parents just felt totally inadequate. It is intimidating." Reinforcing these childhood recollections are Katerina's more recent experiences of similar emotions during an overly formal treatment of mathematics. "First year calculus, ... I just thought, oh my God, what's going on, it's a new language and I decided to go to chemistry."

The writing activities, repertory grids, and interviews make clear that for Katerina mathematics and problem solving are almost synonymous, but her definition of problem has evolved over the years. "I don't remember ever being excited about math until I hit Grade 7" when the teacher broke "the mould of the worksheets" and introduced problems; "mostly the word problems with money, you're going shopping, you've got so much money for candy, that kind of thing." This view was expanded by her experience in a Grade 11 enriched course with challenge problems, open-ended projects and no textbook.

While practical applications still hold interest for Katerina, her experience in an exciting university synthetic geometry course expanded this view to include problems within the discipline of mathematics itself. "He was trying to get us so we could see things we had just assumed were true in a different light often and trying to get us to question why we just accepted certain things." It taught me "don't assume that everything is as I'm told it is, because there is a lot more work to be done and a lot more work that can be done." Since "math is not just a cut and dried system of sign and procedure but an intellectual way of making sense of the world," it follows for Katerina that her students should be "mathematically seeking solutions, exploring patterns and formulating conjectures." However, Katerina realizes that there are rules to the game and she feels a tension between freedom and structure both in the subject and her planning of student activities. This view that there is an underlying logic that must be respected and her balanced position on student exploration appears to come from her Grade 11 teacher who, "had that binder, even though he seemed to be going everywhere." He "knew what he wanted to do with us but we always felt like we weren't structured. It was like he was discovering something new with us."

To teach in such a style requires enthusiasm and intellectual commitment and fortunately Katerina found such traits in her Grade 11 mathematics and Grade 13 calculus and algebra teachers, and her professor of synthetic geometry. Each made her feel that "he was almost learning it also and let's try to figure this out together." While specific teachers and their mathematics courses contributed to the development of Katerina's commitment to the discipline, the roots of this intellectual curiosity can be found in her supportive family. Her parents, while not well educated themselves, provided constant support and gave the message that the pursuit of knowledge not just schooling was valued. The immigrant experience of having to "look at a lot of things in different ways" set Katerina apart from her classmates in Grade 1 and now in her BEd year appears still to separate her intellectually from other teacher candidates.

Discussion

"The single most compelling issue in improving school mathematics is to change the epistemology of mathematics in schools, the sense on the part of teachers and students of what the mathematical enterprise is all about." (Romberg, 1992, p 433). Unfortunately it would appear that the proponents of mathematics education reform cannot count on the replacement of retiring senior staff by beginning teachers to change the epistemology of school mathematics. The toolkit view of the mathematical enterprise held by the majority of teacher candidates means the continuation of transmissive modes of instruction. Inservice work with these new teachers, once they have secured employment, is not likely to prove effective. Professional development activities are usually of short duration and reach only a limited portion of the teaching population. Underhill (1988) reports that there appears to be little change in teachers' conceptions with passing years in the classroom and research shows that even an extensive series of university level mathematics courses delivered to practising mathematics teachers resulted in no significant change in beliefs (Bush, Lamb & Alsina, 1990).

Teacher certification programmes must be altered to encourage graduates who begin their teaching careers with richer visions of mathematics. This task cannot be left to a single course in mathematics pedagogy in the final preservice year but must involve the mathematics studied prior to the BEd programme. Of the 29 students in the 1991 class studied, only three had majored in mathematics during their undergraduate studies. The rest of the class, mainly science, computer science and engineering graduates, had studied mathematics as support for their primary discipline. Only six students had experienced courses other than calculus, elementary differential equations, introductory statistics, linear algebra, and numerical methods. Service courses addressing such topics are likely to promote a toolkit image of mathematics. Katerina's university experience with calculus, statistics and linear algebra were counter-productive to the generation of excitement about mathematics as a creative field. Only the synthetic geometry course had a positive impact.

Increasing the Ontario mathematics teacher certification requirements from the present two full credits and further specifying a range of topics would broaden student-teachers' experiences and could hopefully generate new images of the discipline. To ensure some familiarity with the syntax and substantive structures of mathematics (Schwab, 1964), the processes of enquiry and knowledge production in the discipline, regulations might require prospective mathematics teachers to study a course addressing historical and foundational issues of mathematics (Shenitzer, 1987). However change will require more than just an increase in university mathematics credit requirements.

Mura (1993) has found that university faculty generally hold formalist views of mathematics that encourage presentations of the subject emphasizing abstraction, logic, rigour and symbolism. Such approaches are likely to encourage absolutist conceptions of mathematics among the student population, including those who will eventually be teaching at the secondary school level. University mathematics instruction must change to reflect a social constructivist philosophy before such views can be expected on the part of teacher candidates.

Katerina's story and the work of Bush et al (1990) suggest that student-teachers' views of mathematics are based more on pre-university experiences than on their university courses. Positive influences such as Katerina's supportive family and her problem solving focused Grade 11 course are not the results of personal choice and are not available to all those planning careers as secondary school mathematics teachers. Present elementary and secondary school programmes mean that potential mathematics teachers will continue to arrive at first year university with absolutist views of mathematics. It will require active promotion of alternate conceptions of the discipline by both those who teach undergraduate mathematics and those who teach mathematics education to graduate beginning teachers for whom the present reform proposals are a natural way to proceed.

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Ad Hoc Group 1

"Can we follow what we preach?" Teaching According to Constructivist Principles

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Introduction

There are many books and articles on constructivism, but little on how this theory may be transposed into actual classroom practice. In this discussion group, some characteristics of a teaching/learning method which is consistent with constructivist principles was discussed. We shall call such a method AC (for Applied Constructivism). Of course there may be many such methods, so let AC stand for what is common to all of them.

The following are some principles which seem to me to be an essential part of AC. To emphasize the applied aspect of this report, I have chosen to address it to an imaginary teacher.

1. People learn best by *doing* and then *reflecting* on what they did. (Papert, Piaget, Dewey, Montessori).

Verbal explanations can be effective mainly when they organize and summarize knowledge that students have already intuitively gleaned through previous or ongoing experiences. In general, listening to a lecture is an ineffective way of learning new stuff.

2. A rich source of activities can be found in appropriate *computer environments*.

For example, turtle geometry, function machine, Cabri or Geometric Supposer, Logo or INSETL programming activities.

3. Good activities are engaging, open-ended and allow for many different solutions and answers.

Such activities encourage creativity, exploration, problem posing and problem solving on many levels and in many directions. Poor activities are narrowly prescriptive, have only one route to the solutions and have only correct or incorrect answers. Contrary to the dictum found in many books on didactics, good activities often defy a simple clear-cut answer to the question "What is the goal of this activity?"

4. Learners express their creativity and individuality through their work on the activities.

It is important that we should educate ourselves to cherish this variety and self-expression. What we view as "the best" solution may often just happen to be the one that agrees best with *our* way of thinking.

5. Cooperative learning is an excellent way to stimulate reflection on the activities.

This includes small-group work, peer help and classroom discussions. In most cases, learning through activities is more engaging and more effective when done in a group rather than individually. It is very important to create a supportive climate in the classroom, in which students feel safe to explore, make guesses and learn from their errors.

6. **Activities over explanations:** When preparing to teach some non-trivial piece of mathematics, it is more important for mathematics teachers to think up some good activities rather than just good explanations.

The activities are meant to build up an experiential base for the students. When the explanations come after the activities, and address the experiential base explicitly, there is a better chance that students will be able to make sense of them. There seems to be a strong temptation in all of us to explain too much too early. We should strive to resist this temptation.

7. **The principle of effective minimal help**: The more independent the students become in finding errors in their own work and in correcting them, the more they will benefit from coping with these errors.

This principle is especially powerful when applied to computer activities, where feedback from the computer often renders the teacher's intervention unnecessary. When you observe an error in your students' work, resist the temptation to point out the error prematurely and how to fix it; in most cases the students will come to a solution on their own a little later.

Even when students explicitly ask for help during computer activities, try not to solve their problems for them, Instead, help them gently and sensitively to solve the problem on their own, by helping them clarify their thinking or by giving appropriate heuristic advice. A good rule of thumb is that in such a dialogue most of the talking is to be done by the students, not the teacher. Another rule of thumb is that asking questions is preferable to making statements. When appropriately practiced, this principle enhances students' learning as well as their independence and self-confidence.

(On the other hand, not all errors are worth fussing about. In some cases, when a particular error seems to distract students unduly from their main task, you may decide to help them more actively to get rid of the distraction.)

8. **The principle of successive refinements**: Complex concepts are not learned in one shot but through a sequence of successive refinements.

Finding errors and fixing them is where most of the learning occurs. We should educate ourselves and our students to believe that errors and their analysis are the stepping stones to the next stage in the learning process.

More generally, how should we approach the teaching (or learning) of a complex piece of mathematics, such as a concept or a proof? The standard model of dealing with complexity advocates decomposing a topic into a linear sequence of tiny "atoms," then proceeding along the sequence, mastering the atoms one at a time. The problem with this approach is that students in most cases will not see the forest for the trees: it is difficult for them to know where they are going and why they are learning this particular atomic piece at this particular time; and when they finish, they seldom have an integrated view of the complex topic that originated the whole sequence. Clearly this way of dealing with complexity is not conducive to humane and insightful learning.

The principle of successive refinements offers a viable alternative for dealing with complexity. One starts with a simplified version of the phenomenon under study, and refines it successively to include more and more details, subtleties and precision. Through the entire process, the student constantly deals with the whole picture, though it may be vague or imprecise in the intermediate stages.

This principle has its roots in the work of Piaget and Papert, and in computer science. Piaget has contributed the notion of mental schemes and their successive updating by assimilation and accommodation; Papert has demonstrated the great power of computer programming as a medium for learning by successive refinement; and computer scientists have explicitly talked about program development by successive refinement, though with a much more restricted meaning than the one used here.

9. **Constructing versus discovering**: Learning by construction is not the same as learning by discovery. The activities are meant for the student to establish an experiential base in some topic, not to discover the correct answers.

Through the activities, students become familiar with the complexities and intricacies of a problem or a concept, even if in a vague and incomplete way. When the topic is finally discussed in the class, it will appear to the students as a summary of a familiar situation, rather than as a totally unknown and alien topic.

In this approach, it is not expected that all students will always discover the "correct" answers. However, it should be acknowledged and explicitly discussed with the students, that this state of affairs may cause frustration; indeed that a certain amount of "constructive frustration" is inherent in any serious learning effort. Experience shows that when frustration is acknowledged and legitimised-and held with reasonable bounds-then students are willing to struggle.

10. **The recursive call**: *Learning to teach with the constructivist approach-constructively*: Educating yourself to use the constructivist approach in your learning and teaching is a non-trivial and lengthy learning experience which can only happen through a sequence of successive refinements.

When things do not quite work the way you expected, remember that errors are just the stepping stones to the next refinement. Likewise, having someone read this document is not a good way to introduce them into AC; it would be better to go through some learning experience conducted according to the principles of AC, and have these principles emerge by reflection on the learning experience. This document is therefore more appropriate as a summary than an introduction.

Let us end with a meta-principle: Use the principles in this document (including this one) only as heuristic advice on the way to finding your own individual expressions as a teacher. Beyond everything else, your educational philosophy is a state of mind, not a way of behaving. Avoid paying behavioural lip-service to this or any other set of rules.

Ad Hoc Group 2

Enacting a Chaos Theory Curriculum

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Introduction

Goldenberg (1989) describes a view of the mathematics educational experience that encourages knowledge generation by Teamers. From this view mathematics

> *can* be the most freeing of subjects, a game in which the player is free to invent any set of playing pieces, rules, and constraints, and then reason out or observe the consequences of these choices. It is a game whose players frequently use the words elegant and beauty, and whose beauties are both visual and intellectual. Yet we show little or none of this to our students. (p 170–1)

Underlying this approach to mathematics education is a different view of cognition than that which is generally associated with mathematics education, that is, the representationist stance. From the representationist stance "The world is pregiven... its features can be specified prior to any cognitive activity. Then to explain the relations between this cognitive activity and a pregiven world, we hypothesize the existence of mental representations inside the cognitive system" (Varela, Thompson and Rosch, 1991, p 135). Action in the world is based upon these cognitive representations. The test of these cognitive representations is correspondence to the appropriate features of the world.

In discussing the representationist point of view Varela, Thompson and Rosch (1991) state that "the realist naturally thinks that there is a distinction between our ideas or concepts and that which they represent, namely, the world" (p 136) while "the idealist, on the other hand, quickly points out that we have no access to such an independent world except through our representations" (p 137). Both stances require a search for a ground. For the realist the ground is external while the idealist searches for an internal ground. The idealist, by rejecting the idea of an independent world, is left without a sense of an outer ground.

Varela, Thompson and Rosch (1991, p 140) describe an alternative approach to this searching for a ground internally or externally. From this perspective, called "enaction", the world is enacted "as a domain of distinctions that is inseparable from the structure embodied by the cognitive system". The world is not completely independent nor is it dependent. It is not purely the construct of our own thoughts or perception but is enacted by our structure. Thus it is not an independent entity 'out there' nor is it the internal construction of individuals. It is the interplay between the internal constructions and the structure.

This stance challenges the Cartesian dualism separating mind and body, student and task which is a very representationist one relying on the notion of an external and an internal. This idea is at the core of numerous research programmes in regard to mathematics and computers in education. But what if we were to see education differently? What if we were to envision some way without this separation?

Varela, Thompson, and Rosch (1991, p 149) propose embodied action or enaction in which "knowledge depends on being in a world that is inseparable from our bodies, our language, and our social history—in short, from our *embodiment*... We must see the organism and environment as bound together in reciprocal specification and selection" (p 174). The organism and world mutually specify each other, each enacting the other. They are said to codetermine each other. The system's (organism's) behaviour is determined by its structure and history of interaction in the environment. A path of action is laid down by the mutual specification of an organism acting on and in the world.

Implications for Education and Research

The meaning of instruction is codetermined by students in context. The structure and meaning of the 'information' is codetermined by students in the setting. Students structure their activities according to their perceptions of the medium and based upon their experiences and personal history.

> There is no "transmittal information" in communication. Communication takes place each time there is behavioral coordination in a realm of structural coupling... Each person says what he says or hears what he hears according to his own structural determination... From the perspective of an observer, there is always ambiguity in a communicative interaction. The phenomenon of communication depends on not what is transmitted, but on what happens to the person who receives it. (Maturana & Varela, 1987, p 196)

The focus of educational encounters then must be the actor in the educational context. The student who takes action brings forth a world. The meaning of the actions is codetermined by the students' structures and history in the medium. The medium codetermines with the student a sphere of behavioural possibilities in which the student may take action (or may not, dependent upon the student). Together, through educational contexts and codetermined actions, students and teachers can bring forth worlds. These worlds might seem exploratory, expressive, manipulative or mathematical to an observer. But it is the actors in the media who codetermine their true meanings.

In the chaos theory study students saw themselves in the process of generating mathematical ideas. The nature of these students' activities was codetermined by students' perceptions of themselves in the roles they created for themselves, through their perceptions of the activities, and through students' perceptions of the environment through which they generated their experiences. The students brought forth these meanings through their activities and interactions within the computer-enhanced medium. Ihde (1979, 1993) described technical experiences of technology as 1) relating through a machine to the world, and 2) relating to a machine as something in the world. Students in the chaos theory environment related to their mathematical explorations through the computer and related to the computer interface as the focus of their attention watching the cobwebs generate and iterate gaphs being drawn, in short they had technical experiences of the computer. Ihde also describes nontechnical experience of technology in which the computer is active as backgound. However, later in the activities students primarily used the computer as an environment in which to create mathematical ideas (in generating a chart of values which led them to generate bifurcation diagrams), and thus this nontechnical experience of the computer is evident.

Ihde (1979, p 56) discussed the "amplification-reduction-transformation" of technology through which "technologies bring with it a simultaneous amplification of certain possibilities of experience while at the same time reducing others". The technology, in this case the computer, selects from the human sphere of behavioural possibilities and amplifies some activities while reducing others. For the students in this study the experience of generation of mathematical ideas was amplified through their experiences with the cobwebber and spreadsheet. Educators exploring computer environments with their students need to examine what types of activities are amplified by interaction with the environment and what types of activities are reduced. This determination can only be done by interacting with students as they explore the medium. This will allow educators to balance the activities, hopefully providing a more inclusive view of computing and broader applications and usage of skills and concepts than traditional narrow definitions of computer applications.

The chaos theory study traced students' interactions with the cobwebber and spreadsheet progams. The way students interacted in this enviroment were through/with the computer and with/

in the mathematical context. Students enacted the computer environment in this context as an exploration environment through which they could develop mathematical ideas. The computer environment, including hardware and software, facilitated particular types of explorations and activities by its structure, like the generation of a series of cobwebs from which generalized statements about the action of a cobweb could be made. It enframes students' activities to a particular range of activities delimiting the types of input students could enter. However within that range the computer environment freed students to try many values and to draw repeated cobwebs they may not have tried if the computer environment were not available. Thus the computer environment both empowers and enframes students' activities.

The chaos theory study provided an overview of several computer programs for the exploration of mathematical concepts in the field of non-linear dynamics and chaos theory. The environment provided by these software packages was brought forth by Barnes (1994) through interactions with it. Each package provided particular features which made aspects of the mathematics salient. The computer programs, taken together, generated a space for exploring and generating mathematical concepts consistent with the field of non-linear dynamics. Students demonstrated taking action within computer environments building mathematical meanings. By interacting with these students, by watching what they did, how they explored, and by having them record their activities, findings and perceptions, it was possible to get some sense of their enacted meanings in the context. Their writings revealed an experience of this computer medium through mathematical actions. The medium seemed to be of mathematical significance to them. They noticed patterns and the lack of patterns, developed understandings of the context, and enacted this environment through explorations and interactions. They noticed the approach of a fixed point, cycles of activity when iterating, and chaos. However these students wondered what happened when $\mu > 4$. Was another type of behaviour other than chaos evident? They also wondered about values of $\mu < 0$. What effect would this have? They wondered what effect input values of x, other than $0 < x < 1$, might have. In particular they wondered about negative values. Students were introduced to fractal geometry in the study. They were interested in the connection of their non-linear dynamics mathematical explorations to the study of fractal geometry. Many fractal packages exist which use iterative functions to generate fractals. These could also be explored by students to extend their mathematical activities.

If the patterns revealed through each of the studies characterizes human-computer interaction in this context, then the role of the teacher, the student, the environment, the curriculum and the computer changes. The effect of each is codetermined by the context in which activity takes place. The teacher's role changes from provider of information to facilitator of exploration, becoming a learner and explorer with the students. The students' activities take on new importance as these are the ways that students experience and generate new meanings. The environment mediates the activities of students so it must be considered in educational decision making and educational research. The curriculum becomes that which is enacted by students immersed in a context. New pedagogic concerns will undoubtedly emerge from student and teacher involvement in computer-enhanced contexts. These become places for further research.

Conclusion

An enactivist stance is an alternative to a representationist stance. "The autonomy of the living being [must] be given its full place" (Maturana and Varela, 1987, p 253). This stance is a view of knowledge which is quite different from that usually found in the domain of cognitive science. This stance to cognition has broad implications. If we cast out our representationist views then what is research? What is instruction? What is educational experience? If this non-representationist view takes the sense-making capacity of autonomous living systems as its focus then research agendas and educational endeavours must take new forms. This stance implies the need to look deeply at our own practices and change them to encompass this broader view. This focus toward change will engage us in action, bringing forth a world of research and education from an enactive stance.

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Ad Hoc Group 3

Mathematical Modeling

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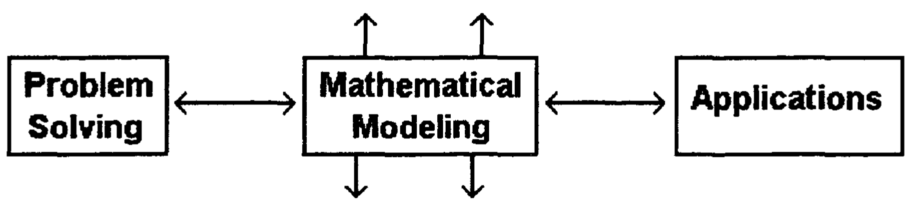
Mathematical Modelling

The principle thrust of this paper is that mathematical modeling is one of the main links between problem solving and applications of mathematics. Problem solving is like an empty vessel and people fill it with anything they want. It really means different things to different people. However, the process of analysis and synthesis is the central notion in the strategy of problem solving.

Curriculum and Evaluation Standards for School Mathematics (NCTM, March 1989) states problem solving is one of the primary goals of teaching mathematics at all grade levels. Standards also states that curriculum should include broad applications of mathematics to the real world. How can these two objectives be achieved? In my view further refinement and extension of problem solving should include the Process of Mathematical Modeling as shown below.



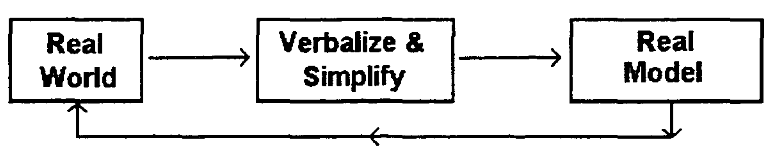
This will make applications of mathematics real and reachable in the context of school curriculum

What is Mathematical Modeling?

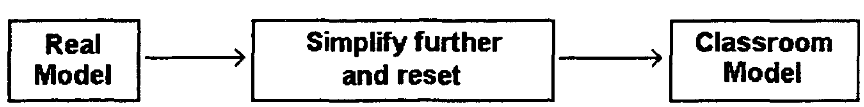
The process of forming and using Mathematical models is an evolutionary process that takes place in steps. The suggested steps are:

1.Identification of a real-world problem.

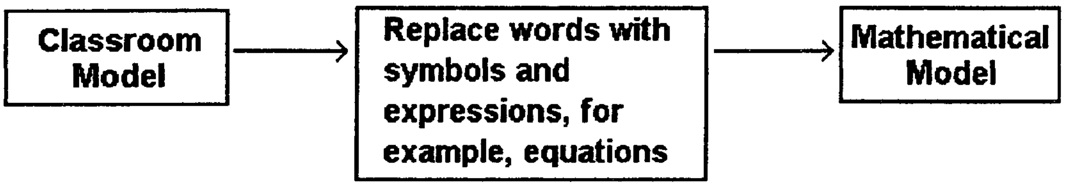
2. From the real world to the real model—some real world realizations of the problem.



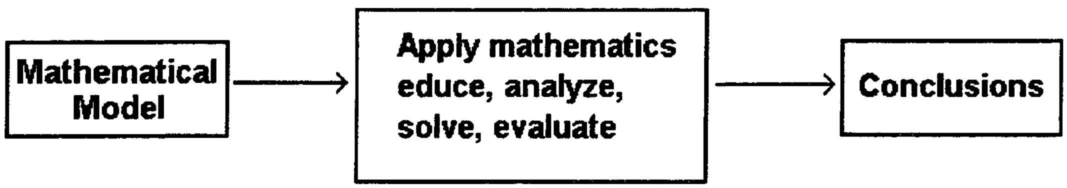
3. From the real world to the classroom model.



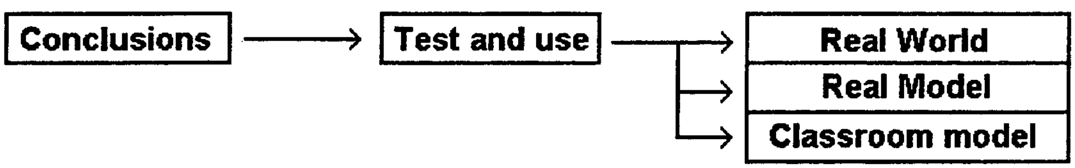
4. From the classroom model to the Mathematical model



5. From the mathematical model to the conclusions.

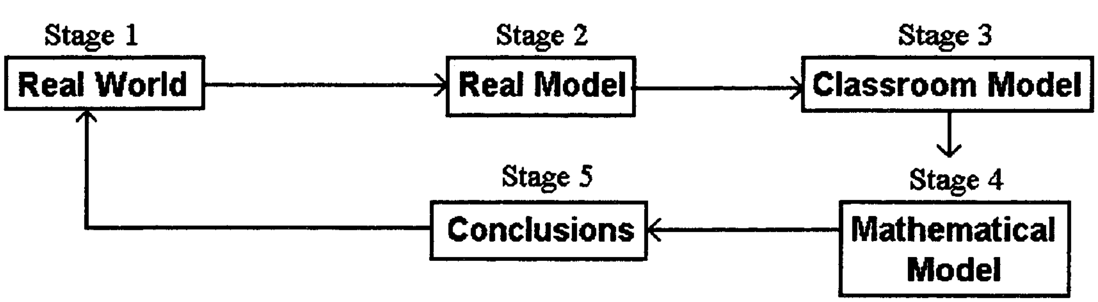


6. From conclusions to wherever you started.



The usefulness of a model depends on the appropriateness of the simplifications and on the accuracy of the mathematical applications. Sometimes models lead to false conclusions based on faulty data or observations. Some models, though not perfect, prove to be useful. Most people use a very crude and inadequate model for predicting the weather. They look up at the sky and make a judgement, for example, about whether or not to take an umbrella to work. Meteorologists collect data on present and past weather conditions to make predictions which can prove fallible. The real message is to *refine and improve models* frequently.

The six stages of mathematical modeling are illustrated below.



Problems in the real world are often too complicated to deal with mathematically. A very important step in the model-making process is to decide which aspects of the problem can be ignored in order to make the problem simpler. Of course, there is always the danger of ignoring something important, and then the resulting model may not be useful.

What is the difference between problem solving and mathematical modeling?

Mathematical modeling is a type of problem solving. Although mathematical modeling shares characteristics with problem solving, it is distinctly different. Frequently, in a mathematical modeling situation, a phenomenon that is seemingly non-mathematical in context must be modelled. This may be an event in the realm of politics, such as predicting election results; or even of ecology, such as future growth patterns of a forest. These events should be interpreted as problems in the context of mathematics.

Thus, mathematical modeling is a systematic process that draws on many skills and employs the higher cognitive activities of interpretation, analysis, and synthesis.

The modeling process is composed of four main stages:

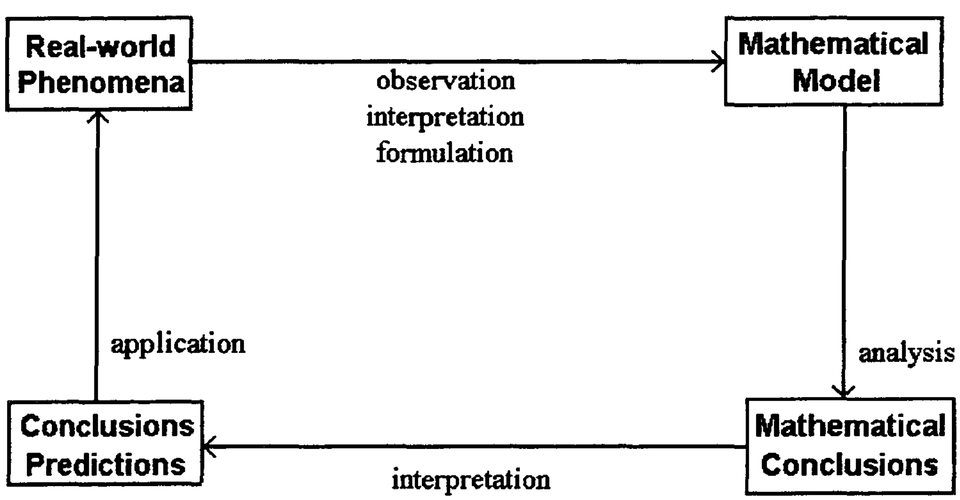
1. Observing a phenomenon, delineating the problem situation inherent in the phenomenon, and discerning the important factors (variables/ parameters) that affect the problem.

2. Conjecturing the relationships among factors and interpreting them mathematically to obtain a model for the phenomenon.

3. Applying appropriate mathematical analysis to the model

4. Obtaining results and re-interpreting them in the context of the phenomenon under study and drawing conclusions.

These stages can be schematically represented in the form of a closed cycle:



A fifth stage could also be added to this process, the testing and refinement of the model. Are the conclusions usable? Do they make sense? If not, a re-examination of the model's factors and structure is called for and a possible reformulation of the model may be necessary.

Broad Levels of Applications

Pamela Ames of the University of Chicago Laboratory Schools has categorized mathematical applications into four broad levels as follows:

Level Zero These are a large collection (mostly mental) of very short statements consisting mostly of references or allusions to uses of mathematics. They are used in class when we are dealing with a particular idea. Just make plain remarks to use even if applications are not actually being considered or planned for a lesson. Examples:

(i) Lesson on vector sum
Remark: Airplane pilots use vector sums for everyday trips.

(ii) Lesson on congruence
Remark: congruence is the basis of all mass production.

Most of the level zero statements are complete in themselves; once you have mentioned them there is essentially nothing more to be said or done. The purpose is to keep the real world in contact with the classroom. However, these short statements do not just arise automatically; you have to constantly look for them and plan their appearance during instruction. Try this resolution: one remark per unit or lesson. With effort and time, one can develop a large enough bank of these statements to make at least one such remark a day.

Level One These are the so-called story or word problems. They are short, self-contained, artificial, real-world [*sic*] situations that usually pose a question that has a single solution or an easily obtainable definite number of solutions. Normally the "search model" is a linear or a quadratic equation which is readily available. Textbooks and teachers do a fairly adequate job in this area though most of the problems posed are rather artificial and remote.

Level Two These are entire lessons built around a single real world situation and may take from one to five class periods to complete. A single situation is investigated in depth using many different mathematical techniques. It is this level of application which is most important and the most neglected. In this application stage, the plan is to work long enough within one situation to see mathematics as a resource to build understanding of the situation rather than mathematics as a tool to carve out answers to specific questions. An example of a Level Two application may be:
(i) building a garage or swimming pool
(ii) cost estirmtes, drawing floor plans, etc.

Level Three These are open-eneded investigations. They are simply fertile ideas that I have not really dealt with yet. Most of these I have never used with a class as I rarely take a whole class on an entirely open-ended investigation. I rather save them for individuals who are ready to strike out alone or in small groups.

As a teacher I place these situations that I would like to do something with. But when these ideas do get investigated, I will get some Level Zero, One, and Two materials from the results [*sic*].

The purpose of this paper is to focus on *Level Two and/or Three Applications* through the process of mathematical modeling. This goal can be accomplished if we take a problem solving strategy which utilizes a wide variety of applications and uses of mathematics. You have to constantly look for ideas. The problems given below are chosen to show how mathematics can be applied, problem solving skills developed, and incidental learning in another area incorporated.

An Illustration of Mathematical Modeling

PROBLEM #1 SIZE OF WILDLIFE POPULATIONS

Stage 1 - Identification of a Real World Problem
How to Estimate the Size of Wildlife Populations

Stage 2 - From the Real World to the Real Model
Some realisation of this Problem

1 How many fish in a pond?
2 How many trees in a given forest area?
3 How many chips in an envelope?

Stage 3 - From the Real Model to the Classroom Model
Some Ways to Solve the Problem

1 By actual counting. But this method is tedious, time consuming and impractical. Example: catch all the Eve fish in a pond and count them. One way is to drain the water and count the fish. However, the end result is dead fish. Ends would not justify the means.
2 Reducing the problem to a situation which is solvable (that is, situation closer to reality).

Stage 4 - From the Classroom Model to the Mathematical Model
Defining Parameters of the Problem
How many chips in this envelope?
What is n? (n = number of chips)
Do the following tasks:

1 Select 10 chips, mark them and return them to the envelope.
2 Shake the envelope, select any 15 chips; record the number of marked ones; and return all of them to the envelope.
3 Shake the envelope and repeat (a total of 10 times).
4 Record your observations in the form of a table.

Stage 5 - From the Mathematical Model to the Conclusions
A Suggested Solution:

Defined Variable:
n : total number of chips in an envelope
y : number of chips marked

q : number of chips taken out each time
$\bar{y}$: mean of the marked chips for x trials

Given y, q and $\bar{y}$, we can calculate n using:

$$\frac{y}{n} = \frac{\bar{y}}{q}$$

The table gives the values for y, q and $\bar{y}$ for ten trials.

Trial	Number Marked : $y = 10$	Number Selected : $q = 15$
1	3	15
2	2	15
3	2	15
4	2	15
5	1	15
6	1	15
7	1	15
8	1	15
9	1	15
10	2	15
Total 16		
$\bar{y} = 1.6$		

Using $\frac{y}{n} = \frac{\bar{y}}{q}$, we have

$$\frac{10}{n} = \frac{1.6}{15}$$

$\Rightarrow n = 93.75 = 94$

It is interesting to note that the exact number of chips in the envelope was 100. The answer of 94 is a reasonable estimate to the real situation. However, if the experiment is repeated a number of times, the answer will be closer to 100.

Stage 6 - From Conclusions to Wherever You Started
Can This Problem Help Us to Solve the Original Problem?
Surely, applying the same strategy on a pond problem or taking samples of the various regions of a forest.

Some Questions for Class Discussion:

1. What concepts do children learn in solving the problem?
2. Can this strategy be applied to other problems? Brainstorm
3. Is this strategy too specific or too general? Some variations or applications of this strategy?

Exercises for Discussion

1. Problem: Yellow Traffic Lights

What is the duration of the yellow light at a traffic intersection?

How is the duration of the yellow light detemined?

Math concepts required. velocity, acceleration

Specific Problem

Find a minimum safe duration for a yellow light in an 80-foot intersection. The intersection lies in a 45 kph speed zone.

Model

Two equations from physics will be needed to solve this problem:

Equation #1: $$x = vt + \frac{1}{2}at^2$$

Equation #2: $$v^2 = v_0^2 + 2ax$$

x = *distance*
a = *acceleration*
v = *velocity*
t = *time*
v_0 = *initial velocity*

2. Problem: Facility Location

Description of the problem

A big controversy erupted in Carlisle, Pa., because the ABF Trucking Company wanted to build a new terminal that was to be the biggest in the nation. The residents of the area were fighting the new terminal because they felt that it would create too much air and noise pollution. This is an example of one problem that many businesses face when establishing a new location.

Many factors must be considered when a company opens a new store or plant. One of the most important considerations is where to locate the facility so that the distances travelled by suppliers and customers, or the distances its product must be shipped, are kept to a minimum. A company can save thousands of dollars every year by properly locating its facility, since the cost of shipping products today is so high.

Mathematics concepts required: Simple geometry and graphing

Specific problem

This type of problem requires geometry in its solution. To get an idea of the mathematics involved, we will consider a simpler type of problem in which we decide where to locate a school-bus shelter for a group of seven students living along a road. Suppose one student lives in each house. The distances between their houses are given below. A school-bus-stop shelter is to be erected for the students to share while they wait for the bus. Determine where the shelter should be located so that the total distance the seven students have to walk is the minimum amount.

A B C D E F G

500' 100' 200' 150' 300' 50'

Questions for Discussion

1. Does mathematical modeling mostly involve discrete mathermtics?

2. Why incorporate mathematical modeling into the secondary school curriculum?

3. How can mathematical modeling be incorporated into secondary school mathematics teaching?

4. How can a teacher prepare to undertake modeling exercises with students?

Ad Hoc Group 4

Asian, American, and Albertan High School Mathematics Comparisons

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University of Alberta

Introduction

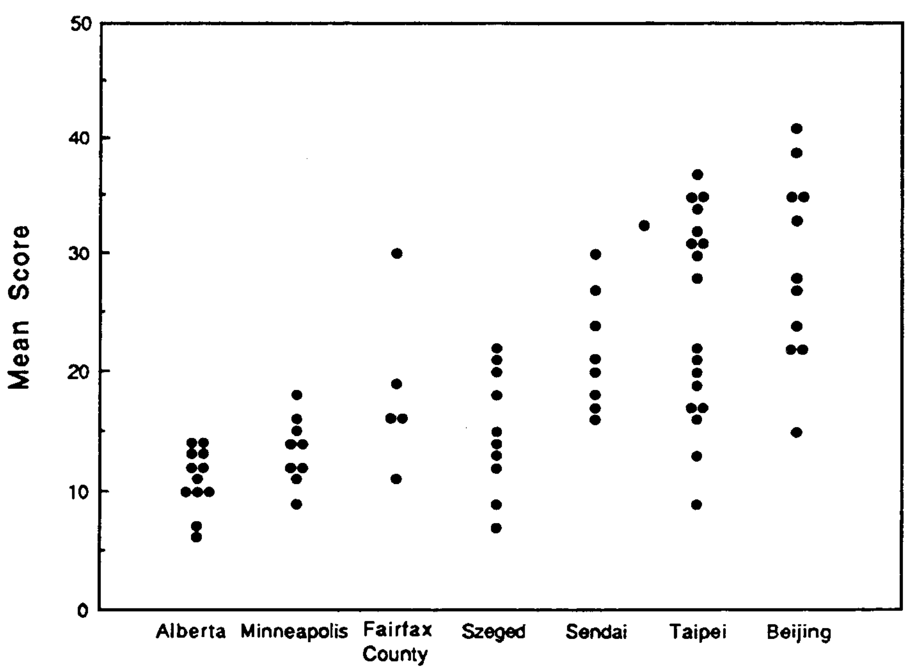
Recently, Alberta Education invited Harold W. Stevenson to administer his Asian-American high school mathematics tests to a sample of Alberta schools. In retrospect, the sampling reflected practical necessities and it could be questioned. However, the Alberta sample approximated the numbers of students in our three different levels of high school mathematics. It also represented an appropriate urban-rural mix. Teachers in the sample schools claimed, one year later, that the students seemed to take the test seriously and they had no reservations about the test accurately representing the students' best efforts.

An important concern about the test itself is that Alberta students might have given up in the face of so many questions that they could not answer. However, as we shall see, the first ten questions of the test were mainly arithmetic and would have allowed most students to get a good start. Even on questions that students did incorrectly, they may well have thought that they had done them correctly. These were questions that could easily have been misinterpreted.

I make these initial statements because the results on the tests do not show Alberta mathematics education in a very good light. Although the testing and sampling situations, as in many international studies, are not ideal, they are reasonable. We are not alone but part of a North American phenomenon, as seen by the distinctly parallel nature of the Alberta and American data (see Graph 3). This parallelism also gives confidence that the North American mathematics education scene was fairly represented.

An Interpretation

On seeing the achievement data, the reader is immediately drawn to asking why there are such differences. There is no simple message. Stevenson (1992), of course, realizes this as much as anybody. To this end, he included in the testing student self-report data on the students' family background, study habits, out-of-school activities, views on schools and teaching, and general demographic information. The present paper presents some of the data that is already fairly well known in the literature but more importantly the intention is to interpret what this data means to teachers, schools, educational ministries and our society.



Graph 1: Mathematics Achievement (Grade 11)

Mathematics Achievement

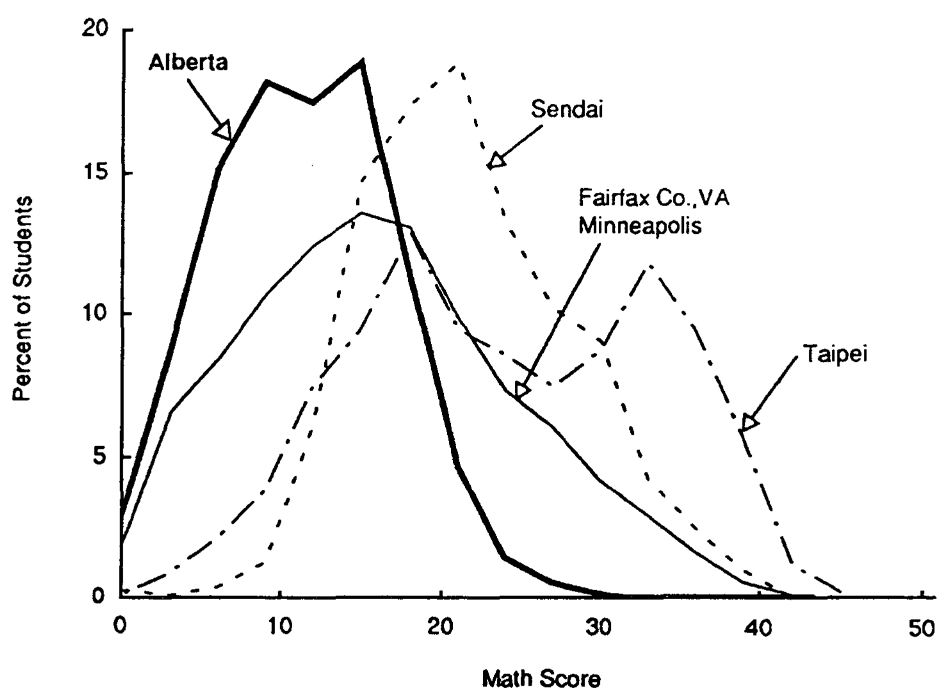
All testing was done at the end of the grade eleven year. The questions asked represented a negotiated selection of questions that were relevant to the North American and Asian scene. For example, few questions on calculus or three-dimensional geometry were asked, although these are a significant part of the Asian curriculum. Although the graph in Figure 1 includes school averages from six different sites, our observations are limited to Alberta, Sendai, Taipei and the two American sites. In this graph, the distribution of school averages, namely the Alberta cluster versus the wide range of Sendai schools, results from every Alberta school having a full range of students while Sendai schools are differentiated academically. The more important pattern in the graph is the average scores for each site. Table 1 shows the scores out of a possible forty-seven questions.

Site	Alberta	Minneapolis	Sendai	Taipei
School Average	11	14	22	23

Table 1: Average number of mathematics questions correct out of forty-seven

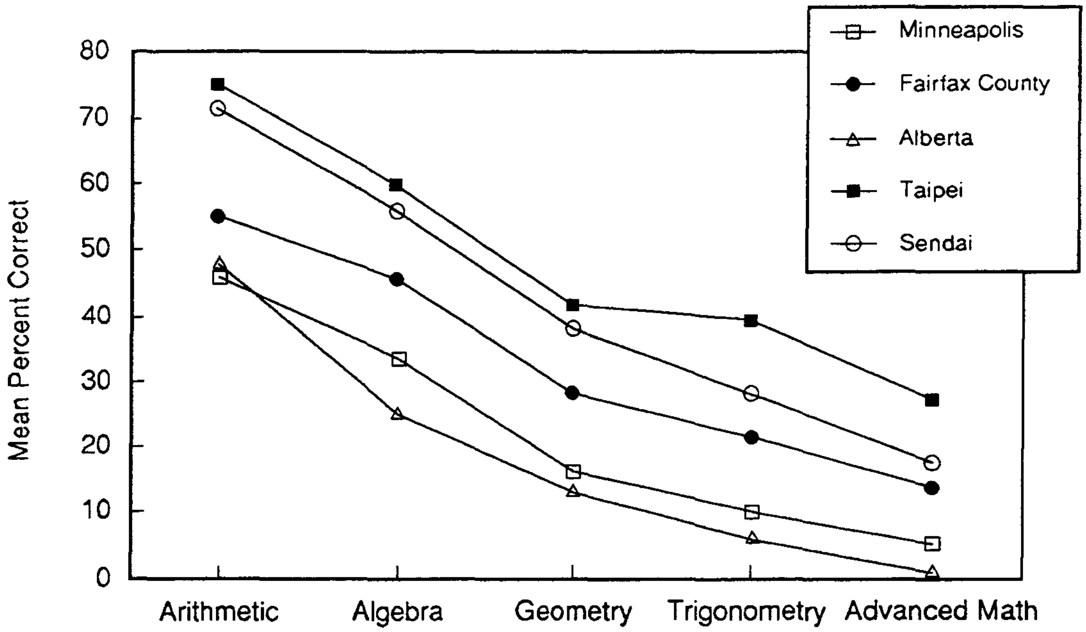
Alberta scored around eleven, Minneapolis around fourteen while Sendai and Taipei were approximately twenty-two and twenty-three respectively. If, as in traditional scoring of items, the first ten questions (easiest) were weighted one, the next ten, two, the third ten, three, and the final seventeen questions, four, then the Sendai and Taipei scores would have been around thirty-three. In this case, the achievement differences would have looked even greater.

In Graph 2, two observations are noteworthy. The steepness of the right-hand side of the Alberta curve means there were few high-achieving (even for Alberta) students. Although the Taipei curve is bimodal (representing students in non-academic and academic schools) the non-academic group in Taipei has a mean higher than both Alberta and the American sites.



Graph 2: Distribution of Students

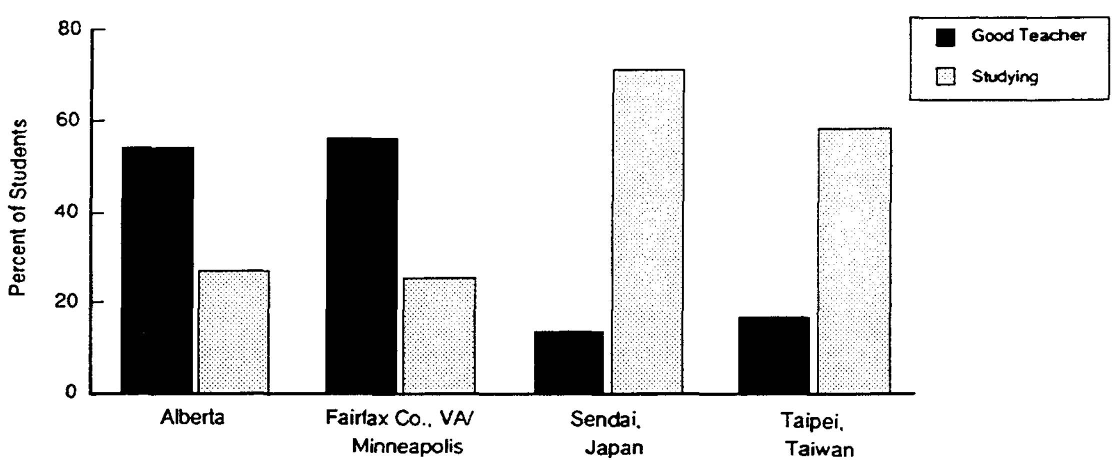
Graph 3 shows the consistency not only across five mathematical topics but also between the North American sites and the Asian sites. (Fairfax County is not typically North American but it also fits the pattern.) The conclusion from the three views of the data presented above is that Asian students score consistently and significantly higher on these mathematics tests. We now look at what these differences mean.



Graph 3: Achievement in five domains

Social, cultural and schooling comparisons

Many social and cultural factors represent differences between countries that are independent of schooling. However, many differences between Asian and North American students are in areas in which the school might be able to exercise some control.

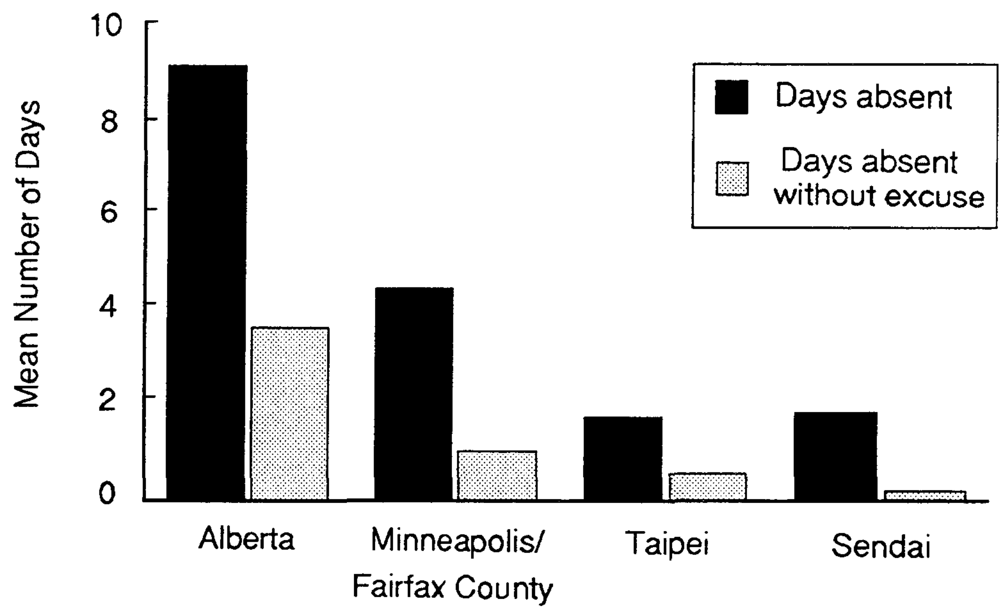


Graph 4: Most important factor influencing achievement

Student responsibility for learning Graph 4 shows that North American students believe that teaching rather than students' individual effort contributes more to student achievement. Where do they get this idea from? Similarly the number of hours students spend on studying suggests that Asian students in general take this aspect of learning more seriously (Table 2). Besides this, Asian students also spend more time in organized tutorials outside class time. Graph 5 shows that North American students (especially Alberta students) are much more likely to miss school.

Site	Alberta	Minneapolis	Sendai	Taipei
Hours studying outside school	9.3 hours	12.4	11.4	16.6

Table 2. Mean hours of study outside school per week

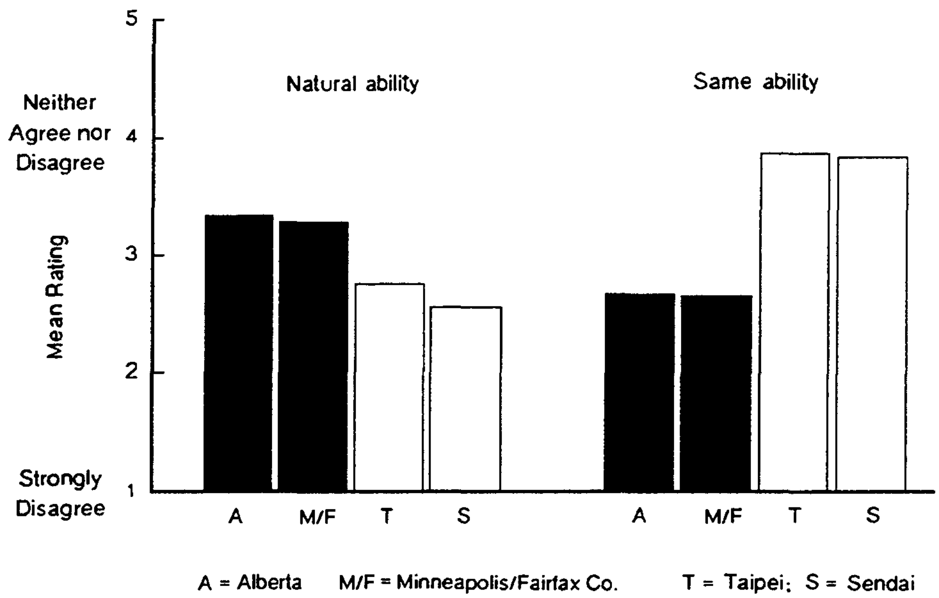


Graph 5: Number of days absent in the semester

A final item in this category refers to ability. Two questions were asked:

Natural ability: The tests I take in school can show how much or how little natural ability I have?
Same ability: Everyone in my math class has about the same natural ability in math?

Graph 6 shows that North American students, more so than Asian students, think both mathematics test scores reflect the *natural ability* of the student and that not everyone in their class has the same mathematical ability. While these are not absolute measures, both beliefs show North American students think "natural ability" plays a more significant role in mathematics achievement than other factors.



Graph 6: The perceived importance of ability

These four sets of data suggest that North American students do not take their own responsibility for learning very seriously. They are more likely to attribute success to good teaching and ability rather than to the individual's time and effort in study. While some of this lack of assuming responsibility is due to societal factors such as students having to work, some part of it can be attributed to the student's view of learning, a view that is, perhaps, reinforced by their teachers, namely that individual effort is not a key factor in learning. This whole area is one in which classroom teachers might influence students.

Learning Expectations The Stevenson data in the area of student attitude in interesting. Surprisingly, in view of the achievement scores, Stevenson finds that only 30 percent of North American students are dissatisfied with their academic performance while approximately 80 percent of Asian students are dissatisfied. Also North American students are more likely to think that the school and the teacher are "doing a good job." When asked how they are "doing in mathematics," North American students respond very positively compared to Asians. As well, they rate their mathematics teacher much more highly that do Asians.

The student attitude data suggests that North American students are living in an unrealistic world of feeling they know a lot of mathematics and that things are going along well in their mathematics education. In comparison to the Asians, North Americans have set low expectations for their students. In fact, one general conclusion from this and other Asian comparisons is that our North American students could be learning much more mathematics than they currently are. This begs the question of whether our teachers are indeed as satisfied as are our students with the mathematics students learn in our high schools.

Other factors Although we have singled out student responsibility and learning expectations factors from the Stevenson data, we will mention other data from this and other studies which are not as directly related to the classroom. The list comes from Lynn (1989):

1. In Asian cultures, families provide a "nuturant and protected atmosphere" for learning. Elsewhere, Stevenson (1992, p 11) has said that Asians believe that the "avenue to success, as it has for hundreds of years in Asian cultures, lies in becoming a learned person."
2. External incentives. Entrance to the best high schools and universities are based on achievement.
3. Curriculum is centrally organized in a country like Japan. In this way, expectations for students are higher.
4. Greater time is spent in schools, in hours per day and in days per year, in learning mathematics and much more out-of-school tutoring occurs for students at all levels of schooling.
5. Teaching methods. While teaching methods were not part of the present study, Stevenson's video "Polished Stones" reports that teachers spend much time preparing lessons and teach in whole group sessions to larger classes than is typical in North America (Stigler and Stevenson, 1991).

The first four "other" factors suggest that learning in general and mathematics learning in particular are highly valued in Asian society. The teaching-methods factor may simply be a consequence of Asian students coming to school prepared to learn and to take some responsibility for that learning.

Mathematics Learning

In presenting the above data, the parallels between the Alberta and American students have served to reinforce the consistencies of the comparison. I would now like to look at only the Alberta results on the mathematics achievement test in an effort to understand qualitative differences between the mathematics learning of Alberta and Asian students. It isn't simply that Asians did, at least, twice as many questions correctly but rather that there are qualitative differences in the questions on which Alberta students did poorly. Our hypothesis is that many questions required a certain "mathematical maturity" which was beyond Alberta students. The 42 of the 47 items, that we have chosen to comment on from the achievement test, fall into four categories:

A. Questions judged not to be in the Alberta curriculum — 14 questions.

B. Questions which 60% or more of Alberta students did correctly — 10 questions.

C. Questions which 40% to 60% of Alberta students did correctly — 8 questions.

D. Questions which fewer than 10% of Alberta students did correctly — 10 questions.

The remaining 5 questions (for which achievement levels are between 10% and 40%) will not be commented upon here.

A. Non-curricular questions These questions underscore the reality that some of the poor achievement of Alberta students is due to neither the students nor the teachers but rather to the curriculum expectations. An examination of the 14 questions in this category indicates that Asian students in the eleventh grade study calculus, three-dimensional geometry, plane geometry of the circle, slope of a quadratic curve, the sum of a geometric series, some aspects of the trigonometry of the right angle triangle, the intersection of two parabolas, vectors, complex numbers, logarithms, and the trigonometric functions.

Even this partial list is a very impressive segment of mathematics to which our grade eleven students are not being exposed. The low scores on the achievement tests which we have just reported, generally, indicate that the quality of mathematics learned is low. This list of topics indicates that the quantity of mathematics learned is also low. To some extent this situation is due to less time being spend on mathematics in Alberta schools than in Asian schools. It does serve to reinforce the point that our expectations, that is our curriculum standards, for our grade eleven students are low compared to other countries. Our provincial department of education, responsible for setting these standards, must assume responsibility for this state of affairs.

B. Questions with 60 % or better achievement For the most part, these questions are arithmetical in nature. Typical examples are

6. Cloth is sold by the square metre. If 6 square metres of cloth cost $4.80, how much does 17 square metres of cloth cost?

9. Write 3.225×10^5 in expanded form

The only algebra question in this category was to evaluate a quadratic at $x = 2$. For the most part all of these questions could be done by average grade nine students.

C. Questions with between 40% and 60% achievement Questions in this category were of four types.

1. More difficult arithmetic questions such as

13. The price of a product was $100. The price was first raised by 10% and then lowered by 10% of the new price. What is the price of the product now?

This is a simple percent question requiring a two step procedure. At least one grade eleven student, interviewed separately, was able to multiply 100 by 0.1 in his head. Many cannot.

2. Simple algebra questions given in a verbal statement

12. On a number line two points A and B are given. The coordinate of A is -3 and the coordinate of B is 7. What is the coordinate of point C if B is the midpoint of line segment AC?

Mathematically, this is a very simple question requiring knowledge of only the numberline and midpoint of a line segment. This question provides some problem in reading. First of all, finding the midpoint between two points is a standard type of exercise in beginning coordinate geometry. This interpretation would lead students astray. A second difficulty would be in incorrectly assuming the unknown point, C, was to be the midpoint rather than the end point.

3. An algebra question given in an unusual format:

19. If $\frac{a^x}{a^3} - 1 = 0$, then $x = ?$

A more straight forward question involving the same mathematics such as,

19. If $\frac{a^x}{a^3} = 1$, then x =?

would have been much easier. Seeing the unusual expression, a ratio of exponents, in an unusual circumstance made the question more difficult to answer.

4. A fourth type of questions which was answered by only half of the students was a sketch of a basic quadratic equation.

29. Sketch the graph of the equation $y = x^2 + 1$

The low score of this question suggests that grade eleven students cannot sketch simple graphs. Perhaps if they had been asked to *plot* the graph, they may have done better. Informal methods such as sketching are difficult for students who are used to performing only mathematical procedures.

In this category, the mathematics (algebra) was not the difficulty. The multi-step nature of the questions, somewhat lengthy verbal statements of algebraic relationships, atypical contexts for the algebra questions and informal methods contributed to half of the students incorrectly answering the questions. Although sketching a graph is not given high priority as a mathematical outcome in grade eleven, it is normally discussed as a method for understanding quadratic equations. If students have exposure to this method, the sketch question should not have been a difficult question.

D. Curriculum questions answered correctly by fewer than 10% of students These questions are very difficult for these grade eleven students. Although one or two of the questions in this category could be identified as difficult mathematics. Most of these question require some interpretation and transfer of knowledge to situations that were non-routine. We have categorized these questions into three types.

1. Simple questions that were mis-read. The most interesting of these is

5. If it takes 12 minutes to saw a piece of copper pipe into 3 pieces, how long would it take to saw it into 4 pieces?

An unthinking response is to generate ratios based on the numbers in the question. A possible explanation for the low number of correct responses to this question is that Alberta students do not expect to have to interpret questions on mathematics tests. If they had stopped to draw the picture many more would have done the problem correctly. Another of these is

25. Find all the roots of $x^3 + 2x^2 + x = 0$

The two factoring procedures in this question are dealt with in the introductory stages of factoring. We surmise that the cubic term signified a situation for which students could not identify a procedure. Again this suggests that students do not expect to have to interpret questions on mathematics exams. (One observer has suggested that Alberta students are unfamiliar with the expression "find all the roots." However a popular textbook uses the expression "find the roots" throughout the quadratic equation section (Kelly et al, 121).)

2. Questions that require non-procedural responses

I will give three questions of this type because they are important in that they represent an element of mathematical maturity that Alberta students seem not to possess. The first,

> 27. For the two points A = (1, 2) and B = (1, -2), find the equation of the line that is perpendicular to segment AB and bisects it.

can be easily solved visually by plotting on the coordinate plane. The answer is simply the x-axis. Not visualizing the problem and trying to solve it by a mathematical procedure is much more difficult. The second question is, again, best understood visually:

> 30. What is the shortest distance from the point (3,4) to the circle $x^2 + y^2 = 1$?

Once it is recognized that the circle is the unit circle about the origin, the student must recognize that the shortest distance is along the line from the point (3, 4) to the origin. Finally the student must recognize that the point (3, 4) indicates a 3–4–5 Pythagorean relationship. Once visualized, the problem is simple subtraction.

The final question of this type is less a matter of visualizing than of using knowledge in new ways. This question involves problem solving.

> 35. The graph of $y = ax^2 + bx + c$ (*a, b, c* are constants) is given below. What are the signs of *a*, *b*, and *c*?

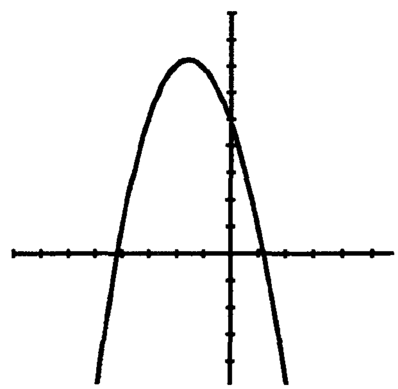


Figure 1

If the graph opens down then a is negative. The constants b and c can be determined by noting that c must be positive since y is positive when x is zero and by making use of the formula: the vertex of a quadratic is F$(-b, a)$. The question requires rather unusual uses of this particular knowledge. Certainly most students know these facts and formulas but none were unable to use them in this non-routine manner.

3. Complex symbolic manipulation

The third type of question with which students had extreme difficulty is illustrated by

> 34. Solve the equation: $3x = \dfrac{1}{1 + \dfrac{1}{x}}$

In isolating x, there are several opportunity to make mistakes. The student needs to be careful.

Summary of mathematical learning First we must not overlook the major differences in curricular expectations between Asian countries and Alberta. The larger Asian curriculum requires students continually to use the more elementary mathematics, which is, then, continually reinforced and becomes more meaningful through usage. Why is mathematical learning given such low priority in Alberta and North American Schools?

Secondly, Alberta students did reasonably well on what is essentially ninth grade arithmetic and algebra. They did less well on multi-step arithmetic problems, simple algebraic relationships given in verbal form, symbolic algebra in an unusual format, and simple informal methods of algebra. These suggest a lack of exposure to a variety of verbal, symbolic, and informal uses of algebra. Better than average students can work in these ways but others cannot.

Thirdly, Alberta students have extreme difficulty with any situation that could be misinterpreted. They are unwilling, unable, and are not predisposed to read a question with the intention of understanding it. They are used to responding quickly and procedurally to all questions. Geometric visualizing also gives great difficulty. While this relates to failing to understand the question, students seem incapable of visually seeing questions even when that visualization is specifically indicated. A third area of extreme difficulty is using well-known knowledge in non-routine ways. Again this relates to the students' reluctance to try to understand the problem but also identifies an inability to use procedural and propositional knowledge in non-routine ways. This means that they are unable to engage in problem solving, even in this simplest of ways. The final difficulty concerns complex manipulations which probably results from lack of practice in symbolic manipulation. Of all the weaknesses mentioned above, this latter would be the simplest to remediate.

What do we in Alberta learn from the Asian comparisons?

There is no reason to doubt the general findings of this study. At the least, we have no reason to be complacent about the mathematics learning in our high schools. The comparison has major implications for our expectations of students not only in high schools but also at earlier levels. Provincial curriculum standards from kindergarten to grade 12 must be called to our attention. In Alberta, especially, teachers have a large input into the development of provincial mathematics programs. Are teachers, like their students, satisfied with student performances? Do our 65% averages on Provincial diploma exams have any validity? Do these and other tests, predispose our students to respond procedurally to test items? The adequacy of our examinations is especially important in Alberta because of the renewed emphasis in this area in the mathematics program.

Implications from this study for teaching practices are less obvious. Are teachers satisfied with their students' learning? Are they and their students assuming responsibility for student learning? How, indeed, are teacher expectations maintained? Do our teachers have an adequate vision of what it means to know mathematics or are they solely dependent on curriculum guides for this expectation? The current study does not directly comment on classroom practice. However, the indicated lack of mathematical maturity of Alberta students should give every teacher cause to reflect on his or her teaching practices.

More than anything, this recent comparison of Alberta and Asian mathematics should provide an impetus for those of us interested in mathematics education. Our 17-year olds are learning much less mathematics than is possible. Is mathematics a foundational study for many careers in our post-industrial society? If our time with mathematics teaching is to be limited, as it currently is, we must be selective. Is the mathematics that we are selecting most appropriate and is our limited time most wisely used? A senior high school mathematics teacher recently commented that the extreme emphasis on data management in Alberta most rightfully belongs in the area of social studies. The Asian comparisons are certainly a wake-up call. Out students have a right to the best possible mathematics education that we can provide.

The interpretation offered here calls for us to go beyond comparisons. Whether our students are better than Sendai or St. Petersburg, they should be able—through their own efforts and expectations, through a curriculum which places different emphasis, and through teaching which provides occasions for appropriate student activity—to work in ways that allow them to overcome the weaknesses identified here.

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ICMI Reports: 1

Conference on Gender and Mathematics Education

Höör, Sweden
October 7-12, 1993

Gila Hanna

Ontario Institute for Studies in Education

Introduction to ICMI Studies

This report describes the seventh in the series of ICMI studies, each dedicated to a specific theme. The series began in 1986 with a study devoted to the question of the influence of computers and informatics on mathematics education.

The work of an ICMI study is organised in a special way. The executive committee of ICMI appoints an international programme committee whose first task is to write a discussion document. This document outlines the themes, aims and scope of the study and is published in several international journals and newsletters. Mathematics educators are invited to comment on the document and to apply for participation in the study conference.

An ICMI study conference is held with a limited number of participants (50-100). It constitutes a working forum of both experts and novices who meet to exchange ideas on the theme of the study. The conference consists not only of presentations, but also of group discussions of significant issues identified in the discussion document.

The overall aim of a study is to provide to the participants and to the interested community at large a picture of the state of the art in the topic under study. Accordingly many of the group discussions are summarized and published, together with a selection from the papers presented, in a book which appears under the general editorship of the President and the Secretary of ICMI.

To date ICMI has sponsored eight studies, on the following topics: computers and informatics, school mathematics in the 1990's, mathematics and its teaching, the popularization of mathematics, mathematics and cognition, assessment in mathematics education, gender and mathematics education, and research in mathematics education. Seven books have already been published, based on the first six studies. Books based on the last two conferences will appear in 1995.

The Seventh ICMI Study: Gender and Mathematics Education

This study was originally planned for 1990 or 1991. An initial background document written by Gila Hanna and Gilah Leder was prepared in January 1989, and disseminated to various mathematics education groups during 1990 and 1991. It was published in 1991 in both the ICMI Bulletin and in the IOWME newsletter. This was followed by the official discussion document titled "Gender and Mathematics Education: Key issues and questions," which was published in 1992 in a number of international mathematics education journals, in the ICMI Bulletin, and in several newsletters of mathematics and mathematics education groups.

The conference was held in Höör, Sweden, from October 7 to October 12, 1993. It was attended by eighty participants (sixty-eight women and twelve men) from some twenty-three countries. Most of the participants were scholars actively involved in one or more of the following areas of action and investigation: mathematics, mathematics education in general and gender issues in mathematics education in particular, the psychology of gender and learning, feminist issues, educational policy, and the active promotion of women's participation in mathematics.

It was generally acknowledged at the conference that the issues of gender and mathematics education are open to multiple interpretations and subject to examination from a number of different perspectives. Indeed, the International Program Committee conceived the conference as a forum in which the participants would not necessarily reach a consensus but would make some progress toward a better understanding of the field and of the often radically disparate existing positions.

The overall task of the opening panel was to set the stage of the study conference. The following extracts from the panel statements will give an idea of the direction of the conference.

Gila Hanna: Opening remarks

"First, let me say that it is not the aim of the present conference to race towards universal solutions to the problems of gender and mathematics education. Instead, I see this conference as an opportunity to explore differing and often opposing views, and in the end to deepen our understanding of the important issues through this exchange of ideas. I presume we hold many different views of mathematics education in general. As far as I know, we also have different ideas on gender and mathematics education in particular: on how to look at the issues, on which issues are more important, and on an agenda for action. During this conference we should be prepared to argue for the positions we hold. However, we should also be prepared to listen to others, and to abandon some of our present beliefs if during the discussions we find them to have been unfounded. Since I see no virtue in premature theoretical closure, I look forward to a few days of fruitful exchanges and lively discussions."

Gilah Leder: Future directions

"There is much that can be learnt from previous research. To maximize the chances of doing so we must engage in constructive debate, particularly with those in other countries and with researchers working in paradigms and disciplines different from our own. It is essential that as part of this debate due recognition be given to the context in which data were gathered and findings were obtained. We should tackle the difficult task of exploring how research questions might sensibly be refined further so that the many subtle, indirect, and less readily quantifiable factors are not ignored. We should welcome and foster diversity in the formulation and exploration of relevant issues. We also need to consider seriously the challenge of identifying and discussing gender differences in mathematics learning without at the same time further perpetuating these differences."

Structure of the Conference

The scientific program of the conference consisted of plenary sessions, panel discussions, two streams of working groups, and paper presentations, and is summarized below.

Plenary Sessions:

Elizabeth Fennema: Scholarship in gender and mathematics education: past and future

Karin Beyer: A gender perspective on mathematics and physics education: similarities and differences

Mary Gray: Recruiting and retaining students in mathematics.

There was also an evening plenary with Bent Christiansen speaking on aspects of mathematics teaching which promote gender imbalance.

Panels:

Panel 1: Gender and mathematics education (Moderator: Mogens Niss) Gila Hanna, Carl Jacobson, Christine Keitel, Anna Kristjansdottir, Gilah Leder

Panel 2: Feminist perspectives (Moderator: Elizabeth Fennema) Leone Burton, Suzanne Damarin, Ann Koblitz, Beth Ruskai, and summary by Jeremy Kilpatrick

Panel 3: Role of organisations (Moderator: Christine Keitel) Josette Adda, Gerd Brandell, Kari Hag, Cora Sadosky

Panel 4: International perspectives (Moderator: Jean-Pierre Kahane) Elfrida Ralha, Hanako Senuma, Teresa Smart, Maria Trigueros

Panel 5: Research perspectives (Moderator: Maria Trigueros) Karin Beyer, Helga Jungrith, Meredith Kimball, Else-Marie Staberg

Panel 6: ICMI and equity in mathematics education (Moderator: Gila Hanna) Jean-Pierre Kahane, Miguel de Guzman, Jack van Lint

Working Groups (Streams A and B)

A1: Students: Personal and psychological factors (Gilah Leder)
A2: Mathematics as a discipline (Gila Hanna)
A3: Social, economic and technical developments (Lesley Jones)

B1: Assessment and curriculum (Mogens Niss)
B2: Teachers: Personal and psychological factors (Christine Keitel)
B3: Sociological and cultural factors (Maria Trigueros)

Paper Presentations

Thirty papers were presented (three at a time, in parallel sessions). The presenters discussed research on a number of topics: the role of attitudes, ethnicity and gender differences, applications of selected feminist theories to classroom organisation, policy and equity, gender inclusive teaching, the role of methods of assessment in gender imbalance, and the place of values in teaching and learning. In addition, several presenters reported on the situation in their own country as to the differential participation of boys and girls in mathematics.

Despite the diverse and often conflicting perspectives on gender and mathematics education expressed at the conference, there were several important points of similarity. All participants attached great importance to the continuation of research in gender and mathematics education, and all seemed to agree that both an awareness and an understanding of positions different from their own were an important outcome of this ICMI study.

Appendix A

This is what Barbro Grevholm, chair of the Local Organising Committee and member of the International Program Committee wrote about the site of the conference:

> "Welcome to Åkersberg
>
> From all over the world we are gathered in Höör. We do hope that you will find Åkersberg and its surroundings comfortable and inspiring. Åkersberg was officially opened by the Bishop of Lund on September 19, 1993. So we are among the first guests after the opening and we are the first international conference to be here.
>
> We wish that our work during the conference will be successful and influence the situation in mathematics education in many countries and for many years to come. ... If you need some physical exercise to balance the intellectual activities in the conference you will find a beautiful park for walking or, if you are sporty, you will find jogging paths a bit further away. ...If you are looking for a peaceful, quiet place where you can hear your own thoughts you will find a chapel at Åkersberg. It is always open and you can test your eyes on some beautiful Swedish design, glasswork, woodwork, pottery and architecture. There will be short services every day. Anyone is welcome to join. If you would like to have a service in English, talk to the priest."

Appendix B: International Program Committee

Gila Hanna (Chair), Canada
Carlos Bosch Giral, Mexico
Barbro Grevholm (Chair of the local organising committee), Sweden
Geoffrey Howson, UK
Christine Keitel-Kreidt, Germany
Gilah Leder, Australia
Mogens Niss (member ex-officio), Denmark

Appendix C

We were fortunate to have the financial support of the following Swedish institutions:

The National Agency of Education,
Department of Education,
Swedish Council for Planning and Coordination of Research,
Swedish Mathematical Society,
The Crafoord Foundation, Lund
University, Malmö School of Education,
Sydkraft AB,
Astra-Draco AB.

ICMI Reports: 2

What is research in Mathematics Education and what are its results?

Washington, USA
May 1994

Anna Sierpinska

Concordia University

Some logistics

Between January and May 1993 a discussion document for the study was published in various journals: *L'Enseignement Mathématique*, *Bullettin of ICMI*, *Educational Studies in Mathematics*, *Recherches en Didactique des Mathématiques*, *Zentralblatt für Didaktik des Mathematik.* Moreover, the *Journal for Research in Mathematics Education* published a revised version of the document as an article (Sierpinska, 1994). The discussion document called for papers with a deadline of September 1, 1993.

In May 1994, a conference was held in Washington, grouping some 80 people from around the world. Three kinds of activities were held at the conference: (I) Plenaries addressing general questions; (II) Working groups addressing five more specific questions from the discussion document, and (III) "Paper sessions", in which specific examples of research were discussed in view of the general theme of the study.

There were five plenary sessions devoted to the following themes:

1. General questions: What is mathematics education as a field of research?
   The speakers addressing this topic were: G. Brousseau, J. Mason, F. Lester, E. Wittmann.
2. Balancing theory and practice in research.
   Speakers: C. Margolinas, B. d'Ambrosio, E. Müller, G. Vergnaud, J. Sowder, C. Shiu and G. Hatch.
3. Training researchers in mathematics education.
   Speakers: C. Laborde, G. Gjone, C. Gaulin, D. Johnson.
4. Perspectives on mathematics education research:
   (a) Mathematics education through the eyes of mathematicians
   Speakers: L. Blum, R. Brown, M. Artigue, W. Dörfler.
   (b) Mathematics education research through the eyes of researchers
   Speakers: N. Presmeg, E. Pehkonen, P. Boero.

The working groups were supposed to discuss the following questions:

Group 1. What is the specific object of research in mathematics education?
Leader: J. Confrey. Main speakers: A. Sfard, C. Keitel. Reporters: A. Kristjansdöttir, D. Blane

Group 2. What are the aims of research in mathematics education?
Leader: Ole Bjorkqvist. Main speakers: G. Leder, J. Szendrei. Reporters: P. Gomez, T. Romberg.

Group 3. What are the specific questions / themes / problématiques of mathematics education?
Leader: M. G. Bartolini-Bussi. Main speakers: N. Balacheff, E. Silver. Reporters: B. Hodgson, I. Osta.

Group 4. What are the results of research in mathematics education?
Leader: S. E. J. Pirie. Main speakers: C. Kieran, F. Arzarello. Reporters: T. Dreyfus, J. Becker.

Group 5. What criteria should be used to evaluate research in mathematics education?
Leader: B. Johansson. Main speakers: G. Hanna, H.-G. Steiner. Reporters: M. Brown, M. Blomhøj.

There will be no proceedings of the conference; instead, a book will be published which will add to the series of ICMI Study Publications. The book will contain contributions by people who decide to contribute to the Study, not just from those who were at the conference. There will be a reviewing process to ensure a high level of quality.

In the sequel, I shall give some information concerning mainly the first two "plenary" themes as they were discussed at the Washington conference.

Some themes and questions discussed at the Washington conference.

One (unexpectedly) much debated question was "What is mathematics?" A serious point of disagreement was the relationship between research and the activity of teaching. Two things on which everybody seemed to agree were: (1) it is through confrontation with a different research paradigm that we come to a better awareness of our assumptions and standpoints; (2) we need some sort of "forum" for discussion of the issues related to a definition of the domain of research in mathematics education.

Theme 1: What is mathematics education as a field of research?
Guy Brousseau gave the following definition, in French and English:

> La didactique des mathématiques est la science des conditions spécifiques de la difusion imposée des savoirs mathématiques aux gens et leurs institutions.
>
> Mathematics education is the science of the specific conditions of teaching mathematics in educational institutions.

The main message of Guy Brousseau's address was that mathematics education as a field of research is part of mathematics. His argument went, approximately, along these lines (I hope I am not distorting Guy's ideas). Problems related to mathematics teaching contain an irreducibly mathematical part; choice of problems, organisation of mathematical contents for different didactic purposes, structuring of the mathematical discourse, analysis of mathematical understanding, identification of specifically mathematical activities and "their modeling in order to cause them to happen in students"... All these problems are related to communication of mathematics. And communication of mathematics is part of mathematics.[1] Calling upon the authority of W. Thurston, Guy Brousseau said:

> The activity of mathematicians is not restricted to the production of definitions, conjectures, theorems and proofs. It includes also the communication of results, the reorganisation of theories and knowledge, the formulation of questions and problems and all that "enables people to understand mathematics" .[2]

One of Brousseau's points was that mathematicians do not usually recognize activities such as communication of results, organisation of knowledge, etc. as mathematical activities. One is not a mathematician unless one produces original mathematics, i.e. new (and significant, I would add)

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[1] Although a safer conclusion would have been that communication of mathematics and mathematics have a non-empty intersection.

[2] "*We are not trying to meet some abstract production quota of definitions, theorems, and proofs. The measure of our success is whether what we do enables people to understand and think more clearly and effectively about mathematics*" (W.P. Thurston).

theorems. The challenge of mathematics education as a field of research, he said, is to find scientific means of legitimizing these activities; but, he added, "cela demande une sérieuse reconsidération par les mathématiciens et les autres de ce que sont les mathématiques" ("this requires a serious reconsideration by mathematicians and others of what is mathematics").

Thus, we are down to this fundamental question: What is mathematics? This is a serious question which is implicated in more practical issues such as: What should count as mathematical knowledge for curriculum designers or educational policy makers? Is "communication of mathematics" a mathematical activity that should be explicitly taught (and assessed)? More generally, what is a "mathematical activity"? Many people agree that the core of research in mathematics education should be related to the identification of "mathematical activities", their study and the design of ways in which students can be motivated to engage in such activities. Views start to diverge at the question of what counts as a mathematical activity.

In mathematics education we often broaden the meaning of "mathematics". Brousseau wanted to include communication of results and organisation of mathematical knowledge. Erich Wittmann included into mathematics all the societal uses and modes of expression that are mathematical in nature but are not studied at the universities.

> Work in the core [of mathematics education] must start from mathematical activity as an original and natural element of human cognition and must conceive of 'mathematics' as a broad societal phenomenon whose diversity of uses and modes of expression is only in part reflected by departments of mathematics at the universities. As a consequence, mathematics educators need a lively relationshhip with mathematics and with applications of mathematics and they must devote an essential part of their professional lives to stimulating, observing and analyzing genuine mathematical activities of children, students and student teachers.

For Wittmann, the core of research in mathematics education should consist of, among others:

- development of local theories (for example, of mathematizing, problem solving, proof),
- mathematical analysis of content and the identification of possible contents of mathematics teaching focussed at making them accessible to learners,
- development and evaluation of "substantial teaching units..."

In the discussion following Erich Wittmann's presentation this notion of "teaching unit" was debated for a while as a term which was not very clear.

Both Brousseau and Wittmann mentioned that approaches and methods as well as problems of neighbouring disciplines (such as psychology or sociology) cannot be directly applied to the core problems of mathematics education. A specific theorizing effort is necessary, and borrowing from the neighbouring disciplines can even harm the development of didactics of mathematics as a scientific discipline. Wittmann wrote in his paper:

> ... approaches, methods and standards taken over from related disciplines are more easily applied to problems in the neighbourhood of these disciplines than to problems in the core. As a result, a great deal of didactic research sticks to mathematics, psychology, pedagogy, sociology, history of mathematics, etc. Thus the holistic origin of didactic thinking, namely mathematical activity in social contexts, is dissolved in single strands, and the specific tasks of the core are neglected. In my view, this is a dilemma that presently inhibits major progress in mathematics education.

Wittmann's main thesis was that mathematics education should be regarded as a design science.

As a side comment, let me mention that this postulate is strangely remindful of the words that can be read in the 1980 NCTM publication "Research in Mathematics Education", written by T.J. Hummel:

> As educators, we are primarily involved with design science. A design science deals with the principle of constructing human artifacts, entities that do not occur naturally in the environment... Educators use information from state sciences to understand better the potential of students and the boundary conditions within which they must work. Although educators must sometimes carry out state science research when certain data are not available, they are primarily concerned with the design function, without which curriculum could change only through an unplanned evolutionary process. (Hummel, 1980, p 66).

Things looked so much simpler back in 1980! We have become, since then, more sceptical about research being able to provide us with "missing data", and more aware that we have to find original methods and theories and not just "use information from state sciences" to understand how people learn mathematics or teach mathematics. We became so deeply involved in these activities that we almost forgot about the design function of research in mathematics education, and we have to remind ourselves of it (at least some of us, not to offend didactic engineers and educational technologists of the world!).

John Mason's view on research in mathematics education had an "introspective" flavor: a significant result in mathematics education, according to Mason, is one that brings about the transformation of the being of the researcher, an awareness which enables others to alter their practice. He proposed "researching from the inside", reflecting on, for example, what it is like to be stuck, to be turned on or off from mathematics. Such research, he assured the audience, can also be systematic and epistemologically sound. For Mason, the aim of the research in mathematics education is to enable us to be mathematical with and in front of the students, to make us more sensitive to the experience of others, to provide us with a broader range of choices in the moments of decision. These moments of choice, which are the teaching acts, are the main objects of study for research in mathematics education. Mason proposed a few criteria of validity of research: research is valid if (a) it explains, organizes the past; (b) fits with present experience; (c) informs future practice.

Frank Lester spoke about criteria for evaluation of research reports in mathematics education. He noticed changes in the field by looking at the *Journal for Research in Mathematics Education*. In 1973, the average length of a paper was a bit more than nine pages with statistical analysis as the predominant method, in 1983 — about eleven pages with one-third of analyses non-statistic in nature. In 1993 the average length became twenty pages with three-eighths of statistical techniques, one-half non-statistical techniques and one-eighth some combination of techniques. We have now papers with rich story telling and more narrative in the stories. After the enumeration of criteria currently in use by the *Journal*, and some discussion of them, Lester concluded that it may not be useful to have one set of criteria, but it is necessary for the community of mathematics education researchers to discuss the criteria for the evaluation of research reports. Indeed, the question of criteria of evaluation raised quite a discussion in the audience. Participants were not so much proposing new sets of criteria, as they were stating the fact of the extreme difficulty in mathematics education of coming up with a coherent and universal set of criteria. It is not rare, in reviewing processes, to obtain contradictory opinions, one rejecting a paper, the other praising it.

Theme II: Relations between theory and practice

There were two distinct positions with respect to this theme:

Position I: *There is a sharp dichotomy between theory and practice.*

Guy Brousseau said "The mixing up or confusion between the research on teaching and the activities of teaching can be a subject of criticism."

Claire Margolinas proposed the view that the line which most clearly separates theory from practice is the line between *facts* and *phenomena*. A fact is an isolated statement that can be verified. For example, "pupils have difficulties with the concept of limit" is a fact. It can be verified by statistical methods. However, to become a phenomenon, this statement must be embedded into a theory that will explain it. She suggested that "research must take fact into account and practice must take phenomenon into account".

In a discussion, Nicolas Balacheff very strongly criticized the use of such "hybrids" as "teacher-researcher" or "action-research". An interesting analogy he used was the following: you cannot be a teacher-researcher just as you cannot be your own psychoanalyst.

Position II: There is a *continuum* of practical and intellectual activities between the questions *What is the case?* (theory) and *What is to be done?* (practice) (cf. Begle, 1980).

Christine Shiu and Gill Hatch showed an example of how a teacher is likely to reflect upon a piece of lesson transcript, how a researcher would reflect upon it, and how, indeed, one kind of reflection can feed the other.

Beatriz d'Ambrosio showed how a reflection of a teacher on her own practices, supported by readings and discussion, improved her understanding of the phenomenon of teaching.

The issue of relations between research and practice is important in mathematics education at two levels: (a) the level of research and (b) the level of curriculum development. At the level (b) it leads to questions like "How is mathematical thinking related to practical thinking?" or "Can mathematical thinking be developed through 'learning mathematics in contexts', or 'anchored instruction', or 'realistic mathematics'?" Is "realistic mathematics" still mathematics? (cf. Sierpinska, 1995, to appear).

This way, again, we are asking ourselves "what is mathematics"? This question had been left for us as homework. As it came out very clearly in Anna Sfard's presentation in Group I, it is to no avail that we shall be looking up to philosophers to solve this problem for us.

References

Begle, E. G. (1980). Why do research? in R.J. Shumway (ed), *Research in mathematics education*, Reston, Virginia: National Council of Teachers if Mathematics, pp 3–19

Hummel, T. J. (1980). The randomized comparative experiment and its relationship to other research procedures. In R. J. Shumway (ed), *Research in Mathematics Education*, Reston, Virginia: National Council of Teachers if Mathematics, pp 66–95.

Sierpinska, A. (1995). Mathematics: "in contexts", "pure", or "with applications"? A contribution to the question of transfer in the learning of mathematics, *For the Learning of Mathematics* (to appear).

Appendix B

Previous Proceedings

The following is the list of previous proceedings available through ERIC.

Proceedings of the 1980 Annual Meeting	ED 204120
Proceedings of the 1981 Annual Meeting	ED 234988
Proceedings of the 1982 Annual Meeting	ED 234989
Proceedings of the 1983 Annual Meeting	ED 243653
Proceedings of the 1984 Annual Meeting	ED 257640
Proceedings of the 1985 Annual Meeting	ED 277573
Proceedings of the 1986 Annual Meeting	ED 297966
Proceedings of the 1987 Annual Meeting	ED 295842
Proceedings of the 1988 Annual Meeting	ED 306259
Proceedings of the 1989 Annual Meeting	ED 319606
Proceedings of the 1990 Annual Meeting	ED 344746
Proceedings of the 1991 Annual Meeting	ED 350161
Proceedings of the 1993 Annual Meeting	Not yet assigned

There was no Annual Meeting in 1992 because Canada hosted the Seventh International Conference on Mathematical Education that year.