CANADIAN MATHEMATICS EDUCATION
STUDY GROUP

GROUPE CANADIEN D'ÉTUDE EN DIDACTIQUE
DES MATHÉMATIQUES

PROCEEDINGS
2001 ANNUAL MEETING

University of Alberta
May 25 – 29, 2001

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25th Annual Meeting
Canadian Mathematics Education Study Group /
Groupe Canadien d'Étude en Didactique des Mathématiques

University of Alberta, 2001

PROCEEDINGS

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Acknowledgements

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The executive would like to thank the Department of Secondary Education, University of Alberta for hosting the meeting and for providing excellent facilities. Special thanks to our hosts in Edmonton for making the local arrangements: Diane Boyko, Lynn Gordon Calvert, Renee Jackson, Rick Johnson, Mary Lee Judah, Peter McCarthy, Joyce Mgombelo, Immaculate Namukasa, Coreen Pangle, David Pimm, Sue Ratti, Elaine Simmt, Dave Wagner, and Brenda Wolodko.

Finally, on behalf of our membership, we would like to thank the guest speakers, working group leaders, topic group and ad hoc presenters, and all of the participants. You are the ones who made this meeting an intellectually stimulating and worthwhile experience.

Supplementary materials to some of the contributions in these *Proceedings* are posted on the CMESG/GCEDM website (http://www.cmesg.math.ca), maintained by David Reid.

Schedule

	Friday May 25	Saturday May 26	Sunday May 27	Monday May 28	Tuesday May 29
AM		**09h00 - 12h15** Working Groups	**09h00 - 12h15** Working Groups	**09h00 - 12h15** Working Groups	**9h00 - 11h00** In memory of David Wheeler **11h30 - 12h30** Closing Session
Lunch		**12h15 - 13h30**	**12h15 - 13h30**	**12h15 - 13h30**	
PM	**13h30 - 15h00** Math Fair **15h30 - 16h15** Ted Lewis & Andy Liu **16h15 - 17h30** Registration / Friends of *FLM*	**13h45 - 14h10** Small group discussion of Plenary I **14h15 - 15h15** Questions for Skovsmose **15h45 - 16h45** Topic Groups	**13h45 - 14h45** Plenary II: Rousseau **15h00 - 16h00** Topic Groups **16h30 - 17h30** New PhDs	**13h45 - 14h10** Small group discussion of Plenary II **14h15 - 15h15** Questions for Rousseau **15h45 - 16h45** Ad hoc sessions **17h00 - 18h00** Annual General Meeting	
Supper	**17h30** BBQ	**17h15** Depart for downtown, supper and theatre / concert	**18h15** Muttart Conservatory	**18h15** Lister Hall	
Evening	**20h00 - 21h00** Plenary I: Skovsmose **21h00** Social			Social	

Introduction

Malgorzata Dubiel - President, CMESG/GCEDM
Simon Fraser University

It is a great pleasure to write an introduction to the *CMESG/GCEDM Proceedings* from the 2001 meeting held at the University of Alberta in Edmonton.

A necessary part of the introduction to the *CMESG/GCEDM Proceedings* is an attempt to explain to readers—some of whom may be newcomers to our organization—that the volume in their hands cannot possibly convey the spirit of the meeting it reports on. It can merely describe the content of activities without giving much of the flavour of the process.

To understand this, one needs to understand the uniqueness of both our organization and our annual meetings.

CMESG is an organization unlike other professional organizations. One belongs to it not because of who one is professionally, but because of one's interests. And that is why our members are members of mathematics and education departments at Canadian and other universities and colleges, and school teachers, united by their interest in mathematics and how it is taught at every level, by the desire to make teaching more exciting, more relevant, more meaningful. Our meetings are unique, too. One does not simply attend a CMESG/GCEDM meeting the way one attends other professional meetings, by coming to listen to a few chosen talks. You are immediately part of it; you live and breathe it.

Working Groups form the core of each CMESG/GCEDM meeting. Participants choose one of several possible topics and, for three days, become members of a community that meets three hours a day to exchange ideas and knowledge. Through discussions that often continue beyond the allotted time, they create fresh knowledge and insights. Throughout the three days, the group becomes much more than a sum of its parts—often in ways totally unexpected to its leaders. The leaders, after working for months prior to the meeting, may see their carefully prepared plan ignored or put aside by the group, and a completely new picture emerge in its stead.

Two plenary talks are traditionally part of the conference, at least one of which is given by a speaker invited from outside Canada, who brings a non-Canadian perspective. These speakers participate in the whole meeting; some of them afterwards become part of the Group. And, in the spirit of CMESG/GCEDM meetings, a plenary talk is not just a talk, but a mere beginning: it is followed by discussions in small groups, which prepare questions for the speaker. After the small group discussions, in a renewed plenary session, the speaker fields the questions generated by the groups.

Topic Groups and Ad hoc presentations provide more possibilities for exchange of ideas and reflections. Shorter in duration than the Working Groups, Topic Groups are sessions where individual members present work in progress and often find inspiration and new insight from their colleagues' comments.

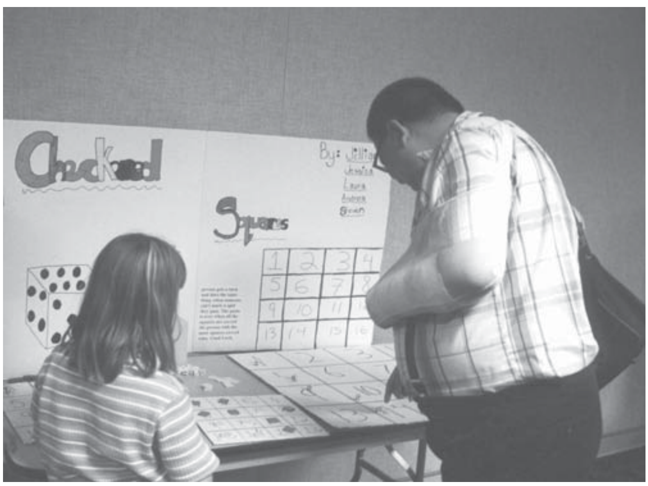
Ad hoc sessions are opportunities to share ideas, which are often not even "half-baked"—sometimes born during the very meeting at which they are presented. A traditional part of each meeting is the recognition of new PhDs. Those who completed their dissertations in the last year are invited to speak on their work. This gives the group a wonderful opportunity to see the future of mathematics education in Canada.

But the Edmonton meeting was also a look at our past. This was the first meeting since the death of David Wheeler, one of the founders of our organization, and the creator of our journal, *For the Learning of Mathematics*. A special session in memory of David was held Tuesday morning, with David's colleagues and students sharing their stories and remembrances.

We are grateful to Ted Lewis and Andy Liu (pictured below), who demonstrated "live" a math fair. Andy and Ted have been organizing math fairs at local elementary schools for the past several years. It was a great experience to be part of one and see the excitement and the pride of the students demonstrating their projects.

Late night pizza runs have been a tradition of our meetings. After trying pizza in Edmonton, at least one reason for this became clear: Pizza is really good here. And, since so many of our members have been students at the University of Alberta, no wonder they learned the habit here.

The 2001 was a memorable meeting, in large part thanks to the local organizers: Elaine Simmt and her team. In addition to the great program, the participants had the opportunity to play a mean game of volleyball, attend a concert and hear our colleague David Pimm sing in the choir, visit the Muttart Conservatory and have a dinner there, at which math puzzles were competing with food for our attention. Thanks, Elaine, for the great job!



Andy Liu, Ted Lewis, and a few of the Math Fair participants

Plenary Lectures

Mathematics in Action: A Challenge for Social Theorising

Ole Skovsmose
Aalborg University, Denmark

Mathematics: Insignificant or Crucial?

Is it true that mathematics—and we talk about 'real' mathematics and not about, say, school mathematics—has no social significance? Or does also 'real' mathematics provide a crucial resource for social change?

In *A Mathematician's Apology*, G.H. Hardy discusses the usefulness of mathematics, and his general conclusion is: "If useful knowledge is […] knowledge which is likely, now or in the comparatively near future, to contribute to the material comfort of mankind, so that mere intellectual satisfaction is irrelevant, then the great bulk of higher mathematics is useless" (Hardy, 1967, p. 135). Could mathematics, nevertheless, do any harm? Hardy concludes: "[…] a real mathematician has his conscience clear; there is nothing to be set against any value his work may have; mathematics is […] a 'harmless and innocent' occupation" (Hardy, 1967, pp. 140–141). In the final pages of his *Apology*, Hardy draws conclusions about his own work in mathematics: "I have never done anything 'useful'. No discovery of mine has made, or is likely to make, directly or indirectly, for good or for bad, the least difference to the amenity of the world" (Hardy, 1967, p. 150). Hardy provides a picture of 'real' mathematics as an intellectual enterprise that cannot be judged by its effects on society, for the simple reason that there are no such effects.[1] Mathematics is *insignificant* in the sense that mathematics does not have any structuring impact on social development. Therefore, social theory is well justified in ignoring the possible social functions of mathematics.[2]

The philosophy of mathematics has been occupied by investigating the foundations of mathematics. What are the sources of this knowledge? What is the nature of mathematical objects and of mathematical truths? This preoccupation easily leads to the claim that an adequate understanding of mathematics can be obtained by studying the logical architecture of mathematics from 'within' the edifice of mathematics. The classical position in the philosophy of mathematics, thus, seems to align nicely with the assumption of insignificance: mathematics has no influence on social affairs and, therefore, it can be adequately interpreted in terms of its internal and logical structures alone.[3]

Let me contrast this perspective on mathematics with the following claim made by Ubiratan D'Ambrosio in 'Cultural Framing of Mathematics Teaching and Learning': "In the last 100 years, we have seen enormous advances in our knowledge of nature and in the development of new technologies. […] And yet, this same century has shown us a despicable human behaviour. Unprecedented means of mass destruction, of insecurity, new terrible diseases, unjustified famine, drug abuse, and moral decay are matched only by an irreversible destruction of the environment. Much of this paradox has to do with the absence of reflections and considerations of values in academics, particularly in the scientific disciplines, both in research and in education. Most of the means to achieve these wonders and also these horrors of science and technology have to do with advances in mathematics" (D'Ambrosio, 1994, p. 443). D'Ambrosio strongly indicates that mathematics is positioned in the nucleus of social development. The role of mathematics is *crucial* and must be considered in the investigation of a wide range of social phenomena.

However, what is the response in the most overall social theories to the question of whether mathematics is indeed insignificant or crucial for social development? Naturally no simple answer is found, but if we study works such as *The Constitution of Society* and *Social Theory and Modern Sociology* by Anthony Giddens, and *The Theory of Communicative Action* by Jürgen Habermas, we do not find any reference to mathematics.[4] We do, of course, find suggestions for basic categories to interpret social development. So, judged by the silence about mathematics, the conception in much social theorising appears to be effectively that of Hardy's: The social impact of this science is insignificant. There is no reason to consider mathematics in particular in order to interpret social affairs.

In what follows, I shall discuss how mathematics can be interpreted as an integrated part of technological planing and decision making, and how mathematics therefore operates as an integrated part of technology. Therefore, I find that an *understanding of mathematics in action is crucial for interpreting basic aspects of social development*. This idea has recently been discussed with particular reference to critical mathematics education, but although it concerns social theorising, it has not got solid ground in sociology.[5]

Reflexivity

In *Reflexive Modernization*, Ulrich Beck, Anthony Giddens and Scott Lash present (in individual written chapters) a discussion of modernisation. According to Beck, we now face "the possibility of creative (self-)destruction for an entire epoch: that of industrial society. The acting 'subject' of this creative destruction is not the revolution, not the crisis, but the victory of Western modernization" (Beck et al., 1994, p. 2). In fact, it does not seem possible to identify more specifically any acting subject for this creativity. And Beck continues: "This new stage, in which progress can turn into self-destruction, in which one kind of modernization undercuts and changes another, is what I call the stage of reflexive modernization" (Beck et al., 1994, p. 2). So, *reflexive modernisation* is not about radical changes taking place as a result of certain critical dysfunction of modernity. Beck does not follow a variant of Karl Marx's analysis, that "capitalism is its own gravedigger"; instead he finds that it is "the victories of capitalism which produce a new social form" (see Beck et al., 1994, p. 2ff.). So this new social form is born within the existing social structures. Reflexive modernisation includes an unplanned change of industrial society which harmonises with existing political and economic orders. Nevertheless, reflexive modernisation breaks up the contours of industrial society and opens 'paths to another modernity'. Although there will be no revolution, there will be a new society.[6]

If we want to understand the dynamics of social development, then we should not seek for that understanding from within the institutions which represent this development. The mechanisms of reflexivity bypass the democratic institutions and operate as part of the social subconsciousness. This problem is significant to sociology: "The idea that the transition from one social epoch to another could take place unintended and unpolitical, bypassing all the forums for political decisions, the lines of conflict and the partisan controversies, contradicts the democratic self-understanding of this society just as much as it does the fundamental convictions of its sociology" (Beck et al., 1994, p. 3). Beck indicates that sociology has not been able to grasp the basic principles of reflexivity. In what follows, I shall try to explain in what sense I agree with this. However, before we embark on this analysis we need to follow Beck in one more step.

Beck introduces the notion of *risk society* which "designates a developmental phase of modern society in which the social, political, economic and individual risks increasingly tend to escape the institutions for monitoring and protection in industrial society" (Beck et al., 1994, p. 5).[7] Risk society is symbolised by many events such as the Chernobyl disaster, financial crises, pollution of food, etc. According to Beck: "Society has become a laboratory where there is absolutely nobody in charge" (Beck, 1998, p. 9). In this return of uncertainty a new frame of social life is established. Risk society is however formed by basic elements of industrialised society: "One can virtually say that the constellations of risk society are produced because the certitudes of industrial society […] dominate the thought and action of people and institutions in industrial society. Risk society is not an option that one can choose or reject in the course of political disputes. It arises in the continuity of autonomized modernization processes which are blind and deaf to their own effects and threats" (Beck et al., 1994, pp. 5–6).[8] Industrial society accumulates its own products, including their effects and side-effects, and eventually this turns society into a new form. In particular, due to 'certitude', industrial society produces risks, which transform the industrial society into a risk society. But how might the nature and the process leading to the emergence of new risk structures be understood?

Mathematics! Let us take a look at the index of *Reflexive Modernization*: No reference to mathematics. However, we find the following sentence in Beck's chapter: "Risks flaunt and boast with mathematics" (Beck et al., 1994, p. 9).[9] In *Reflexive Modernization* this sentence is left as a passing remark. If reflexive modernisation can be discussed and analysed in depth, without any reference to mathematics, then the thesis of insignificance appears justified. But I want to illustrate that this is not the case. The recent development of the industrialised society—establishing a reflexive modernisation, a risk society, or maybe a network society—is linked to a mathematical resourced development. Mathematics makes part of that 'certitude', which transforms industrial society into a risk society.

Mathematics in Action

By means of a couple of examples, I hope to illustrate the importance of considering how mathematics may be operating as part of a technological planning and decisions processes, and how mathematics becomes part of technology itself.[10]

My first example of *mathematics in action*[11] refers to a model presented by Dick Clements in 'Why Airlines Sometimes Overbook Flights'.[12] Airlines deliberately overbook?! Why? Naturally, in order to maximise profit or, to put it more gently, to make sure that the prices of tickets are kept to a minimum. It is essential to try to prevent flying with empty seats. The costs associated with flying a full airplane or one with empty seats are approximately the same: "The airline must pay its pilots, navigators, engineers and cabin staff regardless of whether the airplane is full or not. The extra fuel consumed by a full airplane compared to that consumed by a half empty one is very little as a percentage of the gross fuel load […] The take-off, landing and handling fees charged by airports are independent of the number of passengers carried by an aircraft" (Clements, 1990, p. 325). For every departure, it is most likely that some of the passengers who have already booked will fail to turn up ('no shows'): "The standard conditions of carriage for airline passengers allow full fare passengers to do this without penalty. They can turn up at the airport later and their tickets will be valid for another flight" (Clements, 1990, p. 326). As a consequence, it appears possible to overbook flights. Certainly, there must be an upper limit to this, as the company is going to compensate those passengers who might be refused, or 'bumped', if more than the expected number of passengers turn up. Furthermore, it must be considered that the probability of a passenger being a 'no show' depends on, for instance, the destination, the time of the day, the day of the week, and, as we shall see return to later, the type of his or her ticket.

All this can be incorporated into a mathematical model containing parameters such as the cost of providing a flight, the fare paid by each passenger, the capacity of the airline, the number of passengers booked on a flight, the costs of refusing a passenger who has booked, the probability of a booked passenger arriving being a 'no show', the surplus generated by a flight, etc.[13] With reference to the model, it becomes possible to plan the overbooking in such a way that revenue is maximised. Essential information, of course, is the probability, p, that a booked passenger will in fact be a 'no show'. If this probability is equal to 0, then there is no point in overbooking, but if p is greater than 0, then we can devise an overbooking strategy. The actual value of p for particular departures can be estimated by means of statistical records concerning previous departures, and in this way the degree of overbooking can be graduated according to a set of relevant parameters. For instance, the degree of overbooking the last evening flight from Copenhagen to London should be kept lower than that of an afternoon flight, as the compensation for bumping a passenger in the first case would include hotel costs.

This example illustrates that mathematics may serve as a basis for planning and decision-making. The traditional principle: 'Do not sell any more tickets than there are seats' becomes substituted with the much more complex principle: 'Overbook, but do it in such a way that revenue is maximised, considering the amount of money to be paid as compensation, the destination, the time of departure, the day of the week, as well as the long term effects of having sometimes to bump passengers who in fact have made valid bookings.' This new principle cannot be created or come to operate without mathematics. Its complexity presupposes that applications of mathematical techniques are 'condensed' into a booking programme. The principle illustrates what, in general, can be called *mathematics-based action design*.

A mathematical booking-model does not only *describe* a certain situation, in this case, patterns of reservation, cancellations and 'no shows'. Mathematics does not only provide a 'picture' of reality, as suggested in several philosophies of mathematical modelling. In fact, many descriptions of mathematics as language assume a picture-like theory of what mathematics does. In this way the descriptions embark on the metaphysics from Ludwig Wittgenstein's *Tractatus Logico-Philosophicus*. However, should mathematics be compared with language, then the speech act theory, as suggested by John L. Austin and John R. Searle, invites the following question: What is in fact *done* by means of mathematics? This question introduces also the idea of linguistic relativism as presented by Edward Sapir and Benjamin Lee Whorf: What world view is provided by a specific language? Applied to the language of mathematics, the question becomes: What world views are made available by means of mathematics? Or: How is the world constructed, according to mathematics?[14]

A booking model does not just describe some principles of queuing. It actually establishes new types of queues. And it might create a situation in which some people suddenly have to make new travel plans. Mathematics becomes part of a technique, here represented by the management of booking of flights. But this is just a particular example illustrating the fact that mathematics of all possible kinds and complexities operates in a wide range of modern management systems. Mathematics becomes part of reality, as mathematics-based design is put in operation.

An adequate understanding of the actions carried out in the process of selling tickets is not possible unless we pay attention to the existence of the booking-model. What interpretation to make of the airline assistant's exclamation: "Oh, I'm so sorry, but unfortunately we have some problems with the computer system.…" How would a sociological interpretation of this particular situation look like? Without awareness of the existence of a booking model, the assistant's explanation may appear plausible. But this explanation does not capture the fundamental rationality of the situation. In many cases, 'bumping' a passenger is not a computer mistake. Instead it is a well-calculated consequence, occurring when the passenger in question comes to represent a statistical 'deviation' from the expected norm. If we want to interpret the episode, we need to understand how mathematics operates behind the desk. This is the case as well with many other situations where mathematical models provides rationales (or pseudo-rationales) for decision making. The example of overbooking is not a unique example of mathematics-based action design. Instead it can be seen as paradigmatic of any (complex) business management. Without being aware of mathematics being in place, sociological explanations of such enterprises will become superfluous, if not misleading. To me sociology must be aware of mathematics-based action design in order to interpret a wide range of social phenomena.

Mathematics is certainly involved in grand scale economic management. This can be illustrated by the Danish macro-economic model ADAM (Annual Danish Aggregated Model), which is used by the Danish Government as well as by other institutions (private as well as public).[15] One of the principal aim of ADAM is to promote 'experimental reasoning' in political economy. In this way, ADAM provides a basis for political decision-making. One way of doing so is to provide economic prognoses. Another, maybe even more important application of the model, is to provide different scenarios. Experimental reasoning tries to address the question: If a certain set of decisions is made and the economic circumstances develop in a particular way, what would be the consequence? Implications of a scenario can be investigated by a comparison between applications of the model to different sets of values of the parameters in question. In this way it becomes possible to observe the implications of a political action without having first to carry out the action. Naturally, such reasoning is basic in politics. However, by relying on the model, the political discourse changes because the experimental reasoning which refers to the model acquires a new authority. Experimental reasoning can help to discover which economic initiatives are 'necessary' in order to achieve some economic aims, say, within a definite time limit. (Certainly, 'necessary' has to be put in inverted commas, as necessity refers to the space of possibilities produced by the model.)

As emphasised by the builders of the model, the quality of the scenarios provided by the model depends on the accuracy of the estimations of the variables providing the foundation for the calculations. It naturally has to be added that the quality of the presented scenarios also depends on the quality of the model itself. What, then, does a model like ADAM consist of? An awful lot of equations! These equations can be summarised in different ways, one possibility is to group them into seven clusters having to do with commodity demands, commodity supply, labour market, prices, transfers and taxes, balance of payments, and income. In fact, ADAM can be considered as a set of sub-models addressing certain aspects of the Danish economy. The system of equations in ADAM is constructed around different types of variables, exogenous and endogenous. The value of an exogenous variable is determined from outside the model; the population of Denmark is as an example of such a variable. To estimate the employment-unemployment ratio, this number is essential. Endogenous variables are those which are determined by the model itself, and many variables, which appear exogenous in some part of the ADAM-complex, are determined by other parts of the model, so when ADAM is considered in its totality, they become endogenous.

When such a system of equations is constructed and accepted, experimental political reasoning can be carried out. The problem is, of course, how to present such reasoning. Obviously, the detailed structure of the model cannot be presented, nor grasped, in actual political discussions. A possibility is to let experimental reasoning take the particular form of a multiplier analysis. Let us assume that the equation $y = f(x_1,\dots, x_n)$ belongs to the model. If the variable x_1 is multiplied by a certain factor c, the result would be $y_c = f(cx_1,\dots, x_n)$. By calculating $d = y_c/y$, we it can be claimed that when the input x_1 is multiplied by c, the output y will be multiplied by the factor d. Questions inviting multiplier analysis are raised everywhere in political discussions. For instance, if the government tries to carry out an expansive finance policy, and expand public demand, what effect would such a policy have on the degree of unemployment? In particular, if the government increases its public demand by 5%, how much would the unemployment then decrease? A multiplier analysis would provide an estimation.

ADAM is certainly not merely providing a description of some part of socio-economic reality. It also imposes certain theoretical assumptions about this reality. Taken together, ADAM "displays features which are characteristically Keynesian" (Dam, 1986, p. 31). Thus, the choice of the basic equations, which supply the model with a 'soul', does not simply reflect certain economic reality; it also prescribes a particular perception of economic affairs. Also in this case, the phenomena of linguistic relativism must be kept in mind. ADAM provides a new example of mathematics-based action design. By being a resource for actions, the model becomes part of economic reality. It even comes to dominate this reality, to the extent that its assumed economic linkages establish real linkages. ADAM was created by mathematics, but ADAM got life. And, as we all know, ADAM did not stay alone.[16]

Since 1981, ADAM has been connected to the international LINK project, through which a huge number of national macro-economic models are structured into a world model. In 1995, 79 nations and regions participated in the LINK Project, organised by the United Nations. The connection of different models makes it possible to estimate many of the exogenous variables of particular national, macro-economic models. With reference to a 'connecting structure of models' such exogenous variables can be regarded as endogenous variables. In this way, our world gets enveloped in calculations.

Human beings become part of a reality structured by economic principles formulated in mathematical terms. We observe the same phenomenon associated with the booking-model: the mathematical model becomes part of a social reality. Therefore, we must again raise the question: How is a sociological interpretation of economic decision-making possible without an understanding of the nature of the economic world as represented (and, therefore, reworked and constructed) by an ADAM or other macro-economic models? In *Social Theory and Modern Sociology*, Giddens discusses the problems of macro-economics in relation to social theorising. One of the issues he raises is that such models include assumptions, for instance in terms of a 'rational expectation theory', which may compromise the descriptive value of such models. I am sure Giddens is right: macro-economic models cannot be justified by their descriptive relevance for sociology. But this is not the point. Whatever the macro-economic models might do or not do, they are in fact *used*, and *this use* is of critical importance for social theorising, as understanding this example of mathematics in action is one of the conditions for understanding political and economic decision making.

Mathematics does not only influence the economic part of our reality. In 1995, the Danish Council of Technology (Teknologirådet) published the report, *Magt og Modeller* (*Power and Models*), discussing the increasing use of computer-based models in political decision making. The report refers to 60 models, which cover the following areas: economics, environment, traffic, fishing, defence, population. The models are developed and used by public as well as private institutions in Denmark.[17]

The authors of the report *Magt og Modeller* emphasise that political decision-making concerning a wide range of social affairs is closely linked to applications of such models. They also emphasise that this development may erode conditions for democratic life: Who construct the models? What aspects of reality are included in the models? Who have access to the models? Are the models 'reliable'? Who is able to control the models? In what sense is it possible to falsify a model? If such questions are not clarified in an adequate way, traditional democratic values may be hampered. As an illustration of this problem, I shall summarise the comments of the report related to traffic and environmental issues. In this case models are often used in support of decisions which cannot be changed, like the construction of a bridge between two major Danish islands. Decisions concerning traffic are almost exclusively based on models developed in private companies. It is not usual to develop more than one model to illuminate a certain issue. Finally, it happens that models are used in order to legitimate *de facto* decisions, in the sense that a model-construction provides numbers and figures which justify a decision already made.

Beck claims that the process of reflexivity, which leads to a risk society, occurs outside of democratic control, and that it eludes contemporary sociology. The extensive use of mathematical modelling, as discussed in *Magt og Modeller*, exemplify this claim. How to obtain a democratic access to decision-making, which refers to mathematical modelling processes? The conditions for democratic life may be eroded by the spread of mathematical based action design.[18] Thus, it becomes difficult to ignore the role of mathematics, if we want to establish a sociological discussion of conditions for democracy regarding the nature of technological development.

Three Aspects of Mathematics in Action

The philosophy of mathematics has interpreted mathematics as abstract and has tried to study sources for abstraction. By talking about mathematics in action, I concentrate on the inverse process: seeing how mathematical abstractions are projected into reality. When we use mathematics as a basis of technological design, we bring into reality a technological device that has been conceptualised by means of mathematics. First, it exists in the world of mathematics, later it is brought into reality by an actual construction. A mathematical 'speech act' has been carried out.

In order to specify aspects of this particular act, let us consider the notion of *sociological imagination*, which expresses a capacity to separate what is necessary from what is contingent and, therefore, possible to change. A fact is not only a fact but also a (social) necessity, when it is impossible to imagine that the fact is not the case. If we consider a particular culture where a certain work process is carried out in a particular way (maybe obeying some ceremonial traditions), and no alternative to that approach is identified, then this process would appear to be a (social) necessity.[19] The existence of an imagination that describes alternatives to an actual situation makes a difference. In this case, the fact is 'reduced to' a contingent fact. The experienced necessity is revealed as an illusion when an alternative is conceived. This is the power of sociological imagination: A social given has been identified as available to change.[20]

A process of design includes the identification and the analysis of hypothetical situations, and mathematics helps by providing material for constructing such situations. By means of mathematics, we can represent something not yet realised and therefore identify technological alternatives to a given situation. Mathematics provides a form of technological freedom by opening a space of hypothetical situations. In this sense, mathematics becomes a resources for *technological imagination* and, therefore, for technological planning processes including mathematical based action-design. However, as we shall come to see, all the attractive qualities associated to sociological imagination are not simply transposed to technological imagination. This is important to keep in mind.

The space opened by a technological imagination might very well contain hypothetical situations which are not accessible via common sense. A mathematical framework provides us with new alternatives. For instance, when a booking model is established, it becomes possible to specify 'special fare schemes' like the APEX.[21] Thus, the model makes clear the relevance of creating certain groups of passengers, where it becomes easy to predict the probabilities of 'no show'. In order to do a more detailed planning (How many APEX are going to be offered? By how much should the price be reduced?), it becomes essential to have a booking model available. The set of equations in ADAM also constitutes hypothetical situations. The ADAM makes it possible to establish political thought experiments; this means conceptualising details of situation, which is not possible to identify by common sense. In other respects, the space of hypothetical situations might be very limited. Certainly, ADAM does not support political thought experiments which contradict the political priorities, installed in ADAM in terms of its basic equations. When a technological imagination relays on mathematics, it may provide a very particular space of hypothetical situations.

Political and economic interests can express themselves in the set of technological alternatives that are established as mathematically well-defined. Therefore, mathematics as part of a technological imagination can interact with other power structures. As mentioned previously with reference to models for traffic planning, the set of alternatives established by mathematics can be so limited that the modelling in fact serves as a legitimisation of a *de facto* decision. By providing one and only one alternative, this alternative appears to be a necessity within the space of hypothetical situations provided by the model. This situation helps to establish credibility in the political claim that a certain political decision is a 'necessary' decision.

Thus, the first aspect of mathematics in action concerns technological imagination: *By means of mathematics, it is possible to establish a space of hypothetical situations in the form of (technological) alternatives to a present situation. However, this space may contain serious limitations.*

Mathematics provides the possibility for *hypothetical reasoning*, by which I refer to analysing the consequences of an imaginary scenario. By means of mathematics we seem to be able to investigate particular details of a not-yet-realised design. Thus, mathematics constitutes an important instrument for carrying out detailed thought experiments. Because of ADAM, it is possible to carry out hypothetical reasoning related to economic policy. This reasoning is counterfactual, as it address implications of the form: 'p implies q, although p is not the case'. A representation of p is provided by ADAM in terms of equations including the values of the relevant parameters. The hypothetical reasoning can then address a particular situation 'realised' by ADAM. Some conclusions of the hypothetical reasoning can then be simplified and expressed in multipliers that are easily included in the common political discussion. Without mathematically based hypothetical reasoning, the political discussion would take a completely different form. It would lose a great deal of so-called 'precision'. Hypothetical reasoning represents an essential element in the mathematics-bases analysis of *particular* implications of *particular* actions.

The strength of the hypothetical reasoning is demonstrated by the level of details to which the hypothetical situation is specified. However, hypothetical reasoning, supported by mathematics, also lays a trap, because we are investigating details represented only within a specific mathematical construction of a given alternative. Furthermore, the actual hypothetical reasoning is limited by the fact that the reasoning itself is supported by mathematics. As clearly illustrated by ADAM, the weakness of the hypothetical reasoning is that the decisions made on the basis of hypothetical reasoning will operate in a real life situation, not grasped by ADAM. So when q is found attractive, and p is realised, we will see that the ADAM-supported hypothetical reasoning, does not operate straightforward in a real life context. The hypothetical situation, p, is an imaginary situation created only by the model, and it need not have much in common with any actual situation. The problem of hypothetical reasoning is caused by the 'gap' between the model-constructed virtual reality and the 'complexity of life'.

The second specification of mathematics in action concerns hypothetical reasoning: *By means of mathematics, it is possible to investigate particular details of a hypothetical situation, but mathematics also cause a severe limitation of the hypothetical reasoning*. This means that the quality of mathematically-based thought experiments might be highly problematic. Here we touch upon an aspect that can help to explain the emergence of risks.[22]

A particular aspect of carrying out investigations of hypothetical situation concerns the choices between alternatives. One option is to let 'formal' reasoning do the job. This is based on the assumption that, in some way, we can measure 'pain' and 'pleasure' ('cost' and 'benefit') related to the realisation of each of the alternatives in question. This utilitarian assumption makes it possible to transform a political discussion into a management discourse. This transformation can be illustrated by the use of ADAM to provide justification for political actions. If, say, it is a political aim within five years to decrease unemployment by a certain percentage, then a multiplier analysis could indicate necessary political actions. The 'necessity' of such actions of course refers to the model, but when this model-reference is forgotten, and this reference seems immediately forgotten when policy is discussed in public, then the political actions can be referred to as merely 'necessary'. And then there is only a small step to be taken in order to introduce a 'technological necessity' in politics: We have to do so and so, because this is the only possibility feasible! When 'technological necessity' is acted out this way, reality becomes structured in accordance with the perspective of ADAM. The gap between model and reality tends to diminish. The distinction between 'reality' and the 'virtual reality' of the model becomes blurred.

When an alternative is chosen and *realised*, our environment changes. What is the nature of this new situation? The point here can be illustrated by the ADAM and also by many micro-economic models. As already emphasised, the model that structures airline bookings is certainly not simply a description of what takes place when tickets are booked and sold. When introduced, the model becomes part of the passengers' reality. And this story can be continued: Insurance companies also offer insurance for APEX tickets. They, therefore, need a model telling about the likelihood that a 'sure passenger' will in fact become a 'no show'. In this sense, models create models, and one layer after another of mathematics sinks into our social reality.[23]

Thomas Tymoczko has summarised this point in the following point:

> Business does not just apply various already existing mathematical theories to facilitate an activity that is, in principle, independent from such mathematical application (although it

> can do that). Business could not exist in anything like its historical form without some mathematics. Certainly we cannot imagine a modern economy struggling along without mathematics then suddenly becoming more efficient because of the introduction of mathematics! (Tymoczko, 1994, p. 330)[24]

That mathematics becomes part of reality is a general phenomenon. At his lecture at the 7th International Congress on Mathematical Education in Québec, Tymoczko mentioned the relationship between mathematics and war. His point was that war and mathematics are interrelated in an intimate way. We may talk about modern warfare as constituted by mathematics. Not in the sense that mathematics is the cause of war; but we cannot imagine a modern warfare to take place without mathematics as an integrated part. The same statement can be made if we, instead of 'war' or 'business', talk about 'travel', 'management', 'communication', 'architecture', 'insurance', 'marketing', etc. In their present form such types of social phenomena are modulated if not constituted by mathematics.[25]

Whenever we talk about mathematics-based design, we have to remember that the realised situation need not have much in common with the hypothetical situation presented and investigated in mathematical terms. Any technological design has implications not identifies by the hypothetical reasoning. This is a basic problem related to any kind of mathematical based investigation of counterfactuals. When p is represented by a mathematical based vision, and the implications of p is identified by a hypothetical reasoning as q, and found attractive, then the realisation of p may nevertheless contain heavy surprises. Risks emerge in the gap between the mathematical based reasoning related to the hypothetical situations and the really functions of the contextualised realisation. Certitude turns into risk.

Still, the realisation maintains mathematics as an operating element. In this sense we come to live in a environment, produced by integrating a model-supported virtual reality with an already constructed reality.[26] For instance, much information technology materialise in 'packages'. Such packages can be installed and come to operate together with other packages, and they contain mathematics as a defining ingredient. In particular, Hardy's research has made a significant contribution to the area of cryptography, which addresses the question of 'trust' and security of electronic communication. Knowledge about the distribution of prime numbers and about the efficiency of mathematical algorithms, is essential for estimating the likelihood of maintaining privacy. Also in this case mathematics has become inseparable from other aspects of society.[27]

This bring us to the third aspect of mathematics in action which concerns realisation: *Mathematics modulates and constitutes a wide range of social phenomena, and in this it becomes part of reality*.

Put together, the three aspects of mathematics in action send the following message: By means of mathematics it is possible to establish a space of hypothetical situations in the form of possible (technological) alternatives to a present situation. However, this space may have serious limitations. By means of mathematics, in the form of hypothetical reasoning, it is possible to investigate particular details of a hypothetical situation, but this reasoning may also include limitations, and therefore also uncertainties for justifying technological choices. As part of the realisation of technologies, mathematics itself becomes part of reality and inseparable from other aspects of society. Being part of this process, mathematics is positioned in the centre of social development, in the production of wonders as well as of horrors.

Social Theorising

In his study 'The Information Society', Daniel Bell emphasises that "information and theoretical knowledge are the strategic resources of the postindustrial society, just as the combination of energy, resources and machine technology were the transforming agencies of industrial society" (Bell, 1980, p. 545). In his impressive work, *The Information Age: Economy, Society and Culture I-II-III*, Manuel Castells both develops and modifies this idea. He describes knowledge and information as "critical elements in all modes of development, since the process of production is always based on some level of knowledge and in the processing of information" (Castells, 1996, p. 17). Such statements are certainly crucial to understanding the information age. However, the significance of these statements rests upon an specification of what can be understood as information and as knowledge. Castells adds a footnote to this part of his text: "For the sake of clarity of this book, I find it necessary to provide a definition of knowledge and information, even if such an intellectual satisfying gesture introduces a dose of the arbitrary in the discourse, as social scientists who have struggled with the issue know well." Following these preliminaries he characterises knowledge as set of organised statements, which includes some kind of justification, and which is transmitted to others. 'Information' he described as a concept even broader than knowledge. It is clear that Castells does not take this intellectual gesture seriously, and he does not apply this definition in any profound way later in his work. Instead he lets 'knowledge' and 'information' stay as cloudy concepts throughout his whole study of the information age. (I am sure that Castells has realised this.) But I find that it is essential to make a much stronger specification of the notion of knowledge in order to get a deeper understanding of some of the basic social process of the information age (and I am afraid that Castells has not realised this).

By being kept on a general level, the discussion of knowledge and information makes it difficult to raise questions about the particular roles different types of knowledge and information might play in the construction of new technologies. In this way, the thesis of mathematics being insignificant regarding social affairs becomes incorporated in the sociological discussion of the information age. However, I simply do not think that any kind of knowledge and information operate as 'strategic resources'. Quit contrary, I find that particular types of knowledge operate in particular ways as resources for developing and realising technologies. Thus, the use of 'knowledge' and 'information' as dummies obstruct the possibility of an interpretation of social development. Beck did emphasise that the risk society is produced because the certitudes of industrial society dominates thought and action. As I have tried to argue, this phenomenon is related to mathematics-based action design and, in particular, to the application of mathematics in investigating counterfactuals. To me, *a basic challenge to social theorising* is to grasp the nature and scope of mathematics in action. I conceive this as a condition for any adequate interpretation of the basic processes which brings about reflexive modernisation, and for interpreting how 'certainty' turns into free growing risk structures, which are going to accompany us into the future.

One more aspect of the challenge to social theorising has to be mentioned. This also concerns the philosophy of mathematics. Implicitly, in our discussion of mathematics in action and of the apparatus of *reason*, we have been dealing with reason. Following the 'modern condition' and the spirit of the Enlightenment, reason can be interpreted as a powerful resource for progress. Reason, in the shape of science and of mathematics, represents an 'ultimate good'. Following logical positivism the trust in rationality evolves into a trust in scientific methodology. However, critical voices have indicated that reason, in the shape of instrumental reason, reveals its problematic nature. In *One-Dimensional Man*, Marcuse tried to show how instrumental reason, associated with logical positivism and instrumentalism and specified by a scientific methodology, could increase in scale and manufacture social development in a particular form. Operating outside its proper domain, the natural sciences, instrumental reason becomes problematic. It comes to exercise an illegitimate power. It facilitates suppression and social manipulation. However, instead of concentrating on instrumental reason as basis for an interpretation of how science becomes involved in social affairs, I find it necessary to broaden the scope of investigation considerable. We have to study the role of reason, in particular as manifested by mathematics.

Do we like mathematics-based action design? For instance, do we like the booking-model? If we think of the situation as a passenger who has just been bumped, then we will surely have a negative impression. The principle of not selling anything more than you have seems to represent 'honest business'. But it is also possible to see the model in a different light. It ensures that the total number of flights are kept to a minimum, ensuring that, as far as possible, airplanes do not travel with empty seats. By a slight reformulation of

'Kranzberg's First Law', my claim is: *What mathematics is doing is neither good nor bad, nor is it neutral.*[28]

According to classic philosophy of mathematics, mathematical thinking was a model for human thought. However, this glorification of the queen of science is no longer the object of all philosophies of mathematics.[29] In particular, *aporism*, as a philosophy of mathematics, acknowledges that 'pure reason', in terms of mathematics, can turn into 'disastrous reason'.[30] Aporism sees mathematics as an essential element in social and technological development; at the same time aporism realises that the presence of mathematics does not provide any guarantee for the 'quality' of this apparatus. Therefore, the certainty of mathematics can transform into uncertainty regarding the construction of our future. Wonders mix with horrors.

Previously it might have been appropriate for sociology to ignore the social role of mathematics. Mathematics might *recently* have disarmed social theorising from grasping the basic processes of reflexivity. The theoretical task now is to provide a framework for grasping mathematics in action, in particular to identify how mathematics supports a technological imagination (which might be problematic and narrow), how it establishes possibilities to investigate particular aspects of possible technological constructions (and ignores other aspects), and how mathematics becomes installed in society and starts operating as part of technological devices. The functioning of mathematics cannot be ignored by social theorising. In order to cope with this, sociology may get inspiration from recent studies of mathematics and of mathematics education, which have tried to reconsider mathematics in action.

Notes

1. Hardy makes a distinction between 'real' mathematics and practical applied—or trivial—mathematics which may have such effects. I do not make a sharp distinction between pure and applied mathematics, or between real and trivial mathematics. All areas contribute to the mix, which I call mathematics, and in the rest of this paper, I will simply talk about mathematics.
2. Many studies have revealed that a social structuring of mathematics takes place. See, for instance, Wilder (1981). However, this issue is not going to be discussed in the following.
3. As an illustration of classical concerns in the philosophy of mathematics, see, for instance, Benacerraf and Putnam (eds.) (1986).
4. See also, for instance, Giddens (1990, 1998) and Habermas (1987). Surprisingly, mathematics is not referred to in Castells (1996, 1997, 1998). However, Lyotard (1984) includes mathematics in his discussion of the post-modern condition.
5. For a discussion of critical mathematics education and related ideas see Borba and Skovsmose (1997); Keitel et al. (1989); Niss (1994); Skovsmose (1994); and Skovsmose and Nielsen (1996).
6. The notion of reflexive modernisation has come to play a crucial role in recent sociology. By this concept, Giddens emphasises that the consequences and the implications of any action become part of the process of acting itself. Giddens seems to rephrase reflexivity as part of the 'conscious' level of social dynamics, while Beck relegates reflexivity to a deeper level of social processes.
7. See also Beck (1992, 1995a, 1995b); Franklyn (1998); and Hiskes (1998).
8. Richard P. Hiskes expresses this as follows: "Risk is the product of our lives together, and to fully understand risk's emergent character is to realize that most of the efforts to either explain risk or to cope with it within an individualistic political framework are doomed to failure because they do not acknowledge the 'togetherness' of our risky present" (Hiskes, 1998, p. 13).
9. In other parts of his work, Beck refers to mathematics. See, for instance, Beck (1995b, 20–22) where he talks about the calculus of risks. See also the discussion of 'hazards' in Beck (1995a, 73–110).
10. I include a variety of aspects within the notion of technology: the artefacts of technology (be it a car, a computer or any other device) as well as strategies for action (a plan of production or any other product of 'systems development'). Tailorising is one classic example, and computer-based systems development has produced all kinds of examples.

11. The expression 'mathematics in action' is inspired by the title of Latour's book, *Science in Action*. However, while Latour follows scientists and engineers through society, I try to follow mathematics into society. In other contexts I have developed this idea in terms of the *formatting power of mathematics*. See, for instance, Skovsmose (1994).
12. Clements (1990) does not claim that his model is identical to any actually used model (such models are 'commercial in confidence'), but certainly it is similar to such models: "The purpose […] is to develop a model of the decisions facing an airline and, from this, to acquire an understanding of why it may indeed be beneficial to an airline to book more passengers onto a particular flight than the capacity of the flight that is to make the flight" (Clements, 1990, p. 324). Booking strategies may have developed considerably since Clements constructed his model; nevertheless this model illustrates several basic aspects of mathematics in action. Clements's model has been further discussed by Hansen, Iversen and Troels-Smith (1996).
13. For more details, see Clements (1990, p. 325).
14. See Austin (1962, 1979); Sapir (1929); Searle (1969); and Whorf (1956).
15. ADAM is presented in Dam (1986) and Dam (ed.) (1995). For a critical examination of ADAM, see Dræby, Hansen and Jensen (1995).
16. The Institute for Learning and Research Technology, Bristol University has provided a Virtual Economy, which is an on-line model of economy based on the Treasure's model: "Users can try out policies [...] The program provides extensive feedback on how the economy would perform over the next ten years if those policies were actually implemented. Users can also see the impact of their policies on a range of sample families" (*Newsletter*, University of Bristol, 22 April 1999). The Virtual Economy can be found at: http://www.bized.ac.uk/virtual/economy.
17. The authors are Per Kongshøj Madsen, Bent Andersen, Jørgen Søndergaard, Ruth Emerek, Hans Frost, Poul Lübcke, Kim Viborg Andersen and Rolf Ask Clausen. Besides ADAM, the economic models referred to in *Magt og Modeller* include: the SMEC (Simulation Model of the Economic Council), which operates in a similar way to ADAM but is used first of all by the Economic Council; GEMIAE (General Equilibrium Model of the Institute of Agricultural Economics), which emphasises economic aspects related to agriculture; GESMEC (General Equilibrium Model of the Economic Council); HEIMDAL (Historically Estimated International Model of the Danish Labour Movement), which emphasises Nordic relationships; MONA (Model Nationalbank), which is used by the Danmarks Nationalbank as a tool of forecasting and analysis making; and MULTIMOD (Multi-region Econometric Model). The environmental models referred to in *Magt og Modeller* include: ARMOS (Areal Multiphase Organic Simulator For Free Phase Hydrocarbon Migration and Recovery); HST3D, which provides simulation of heat and solute transport in three-dimensional groundwater flow system. Among the models related to defense is SUBSIM (Small Unit Battle Simulation Model).
18. For a discussion of how mathematics may influence different spheres of practice, see, for instance, Appelbaum (1995); Dorling and Simpson (1999), and Porter (1995).
19. It is, naturally, possible to specify further the notion of necessity by distinguishing between 'logical necessity', 'physical necessity', 'social necessity', etc., depending on the possibilities of conceptualising alternatives. Thus, a fact constitutes a physical necessity, if it is impossible to imagine it to be different without also imagining some physical laws to be different. Similarly, a fact constitutes a social (or cultural) necessity if it is impossible to imagine it to be different without also imagining some (deeply rooted) cultural traditions and social norms to be different.
20. The importance of sociological imagination to sociology has been emphasised by Wright Mills (1959) and repeated by Giddens (1986).
21. With the APEX "... the passenger is offered tickets valid only for a specified flight but at a reduced fare. If the passengers fail to arrive for that flight the ticket is void and the passengers lose their money. Obviously some passengers (chiefly business travellers requiring some flexibility in their planning [and not paying for the tickets themselves]) will still be prepared to pay full fare to retain that flexibility, whilst others (chiefly holiday makers) will accept the restriction in return for the reduced fare. The second category of passengers will not miss their flight lightly so we can assume that their 'no show' probability is virtually zero. These passengers then form a solid base of passengers who can be relied on to turn up for the flight" (Clements, 1990, pp. 335-336).
22. For an indication of how risks can be related to mathematical formalisation, see Booss-Bavnbek (1991).

23. In a similar way many other services, public and private, are based on models linking to models. For instance, the many new forms of services and special offers provided by telecompanies cannot be established without careful mathematically based planning.
24. A historical study of how mathematics constitutes and modulates economic affairs is discussed in Swetz (1987).
25. See Tymoczko (1994) and Højrup and Booss-Bavnbek (1994).
26. The notion of 'frozen mathematics', which refers to mathematics as part of social and cultural life, has been discussed in, for instance, Keitel (1989, 1993). The prescriptive use of mathematics, also illustrating mathematics in action, is discussed in Davis and Hersh (1988).
27. For a discussion of mathematical foundation for 'trust' and security in the electronic transmission of information, see Skovsmose and Yasukawa (2000).
28. See Kranzberg (1997).
29. See, for instance, Bloor (1976); Ernest (1998); Hersh (1998); and Kitcher (1984). The social role of mathematics in technology has been discussed by many authors, for instance, Booss-Bavnbek (1995); Højrup and Booss-Bavnbek (1994); Keitel (1989, 1993); Keitel, Kotzmann and Skovsmose (1993); and Restivo et al. (1993).
30. The Greek word *aporia* refer to 'being without direction' or 'being lost'. In the present *aporia* refers to the basic uncertainty in identifying the role of rationality, as exercised by mathematics in action. Aporism has been presented in Skovsmose (1998, 2000). It can serve as a working philosophy of critical mathematics education.

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Mathematics, A Living Discipline within Science and Technology

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The purpose of this paper is to present a new course « Mathematics and technology » which was created at the Université de Montréal and has been taught once during the winter term of 2001. The students in the course were for the most part future high school teachers. A few students in applied mathematics also attended the course.

The objective of the course is to introduce to several applications of mathematics in technology. The applications chosen are:

- very modern for the most part;
- using relatively elementary mathematics;
- but some sophistication is needed to get extra power.

In the course the students have to do:

- mathematical modeling;
- problem solving;
- use of computers (but not for all applications);
- a project (similar to the projects done in science fairs).

Around 12 applications are studied, for usually five hours each:

- Elementary theory (one hour);
- Exercises (2 hours);
- Advanced theory (2 hours).

For the evaluation the students have to do:

- 3 half-exams half take-home (mostly on the elementary parts);
- a project (report of 25 pages) plus an oral presentation (30 minutes). The students work by teams of 2.

Throughout the course the students find the following messages:

- mathematics are everywhere present in new technologies;
- mathematics are alive and new developments occur all the time;
- with mathematical tools and problem solving skills anyone can contribute to technology, BUT programming is also an essential tool.

A guided tour of some applications (not all elementary)

The purpose of the guided tour is to show how numerous are the applications of mathematics.

Applications in health

- Cardiac arrythmias and chaotic dynamics: Mathematicians and cardiologists work together to better understand the mechanisms of the heart and the onset of chaos. The hope is to be able to control arrythmias with pacemakers;
- Pharmacy: how to better control the diffusion of drugs so as to be able to give smaller quantities and minimize side effects;

- Medical imaging: wavelets allow to "clean" an image to get a better diagnosis;
- Medical imaging: reconstruction of 3D-images from 2D-images.

Applications in molecular biology

- Knot theory is used to explain the action of enzymes on DNA. [R]

Shape optimization

- Shape of a plane wing (aeronautics);
- Shape of a boat shell;
- Shape of a column. Let us recall the old problem posed by Lagrange: "find the shape of the stronger revolution column with fixed height and volume, under pressure from above". Lagrange "proved" that the strongest column is the cylinder. However Lagrange made a mistake and the strongest column was finally found by Cox and Overton in 1992. [C] If one reads Lagrange's work one cannot find the error as all his mathematical deductions are OK. The error lies in the fact that Lagrange erroneously supposed that the profile of the column was given by a differentiable function.

FIGURE 1: The optimal solution of Cox and Overton

Operational research

- Optimization in transport networks;
- Optimization in the distribution of cellular phone frequencies.

Shape recognition

- Reading of postal codes;
- Reading the amount of a check in an automatic teller;
- Recognition of voice;
- Recognition of finger prints;
- Vision of computers.

Financial mathematics

- Conception of derivatives.

Image compression

- Use of fractals.

Structural rigidity in architecture

Mathematics and music

- Clean a sound (for instance an old record);
- Compose new sounds on a synthetizer.

Cryptography

- Public key cryptography (RSA code for bank cards, internet);
- Quantic cryptography;
- The use of Penrose tilings for cryptography.

Engineering

- The movements of a robot.

Error correcting codes

DNA computers

Etc. …

We will discuss in more details the following subjects:

- GPS;
- Public key cryptography;
- Error correcting codes;
- Image compression;
- Vision of computers;
- Movements of a robot.

There exist many more.

We start with a flash-science.

A remarkable property of the parabola

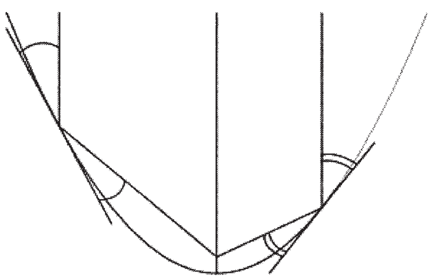


FIGURE 2: A remarkable property of the parabola

- When vertical rays are reflected they all meet in the same point (elementary).
- The parabola is the only curve with that property (more advanced: differential equation).
- Applications:

  Parabolic antenna;
  The mirror of a telescope;
  The shape of a radar;
  The shape of head-lights.

The GPS (Global positioning system)

The system was completely developed only in 1995 by the Ministry of Defense in United States, which allows the public to use it. 24 satellites move on orbits around the world, so that anyone on earth can catch the signals of at least 4 satellites.

The GPS gives one's position on earth. The principle is that the small receiver in one's hand measures the time necessary for a signal emitted by the satellite to travel from the satellite to the receiver. Given that the signal travels at the speed of light this allows us to measure the distance from the satellite to the receiver: from this we know that the receiver is located on a sphere centered at the satellite. As three spheres intersect in 2 points the knowledge of the distance from the receiver to 3 satellites yields the position of the receiver since one intersection point is unrealistic.

This is theory. In practice the satellites have expensive atomic clocks which are perfectly synchronized while the receiver has a cheap clock. Then there is a fourth unknown: the clock offset, additional to the three unknowns for the position. Then the receiver needs a fourth measurement of the travel time from a fourth satellite to the receiver (the clock offset is the same for the 4 satellites). Here again we have a system of 4 equations with 4 unknowns which has 2 solutions, one of which is unrealistic.

This is the elementary theory. More advanced topics can be studied inside a project.

Examples:

- The use of differential GPS to get more precision (we compare with the travel time of the signal from the satellite to a second GPS located not too far and whose position is known to calculate exactly the speed of the signal since it does not travel in vacuum);
- The type of signal generated by the satellite: they are generated by shift registers using finite fields and have the property that they are very badly correlated to any other signal or to the signal translated;
- For other topics and details, see [GPS].

Applications:

- Finding one's way in wilderness;
- Drawing a map;
- Driving a plane in clouds and fog, etc.

Public key cryptography (RSA code 1978)

The basic ingredient is number theory, more precisely arithmetic (+,.) modulo n. We use the small Fermat theorem generalized by Euler.

The method works because theory and practice in number theory are very different:

- It is difficult (for a computer) to factor a large number;
- It is easy to create large prime numbers;
- It is easy to decide if a large number is prime.

Advantages of a public key system:

- There is no danger that the code becomes known! Hence it is the only possible code with millions of users.
- It is possible to "sign" a message in order to be sure it has been sent by the person who pretends having sending it.

The principle [RSA]:

- We choose p and q large prime numbers (more than 100 digits).
- We calculate $n = pq$. The number n, the "key", is public while p and q are kept secret.
- We calculate $\varphi(n)$, where φ is the Euler function defined as follows: $\varphi(n)$ is the number of integers in $\{1,2,...,n\}$ which are relatively prime with n. Then $\varphi(n) = (p-1)(q-1)$.
- Computing $\varphi(n)$ without knowing p and q is as hard as factoring n.
- We choose $e \in \{1,...,n\}$ relatively prime with n. e is the *encryption* key. It is public and allows the sender to encode the message.
- There exists $d \in \{1,...,n\}$ such that $ed \equiv 1 \pmod{\varphi(n)}$ (i.e., the rest of the division of ed by $\varphi(n)$ is 1. The existence of d follows from Euclid's algorithm to find the GCD of e and $\varphi(n)$. d is the *decryption* key. It is secret and allows the recipient to decode the message.
- The sender wants to send a message m which is a number in $\{1,2,...,n\}$, relatively prime

with n.

- He codes $m^e \equiv a \pmod{n}$, i.e., $a \in \{1,...,n\}$. He sends a.
- The recipient decodes. He calculates $a^d \pmod{n}$. The small theorem of Fermat, generalized by Euler, ensures that $a^d \equiv m \pmod{n}$.

Theorem of Euler:

If m is prime with n, then $m^{\varphi(n)} \pmod{n}$.
(Fermat had proved the theorem when n is prime.)

Consequence:

$$m^e \equiv a \pmod{n},$$
$$a^d \equiv (m^e)^d = m^{ed} = m^{b\varphi(n)+1} = m^{b\varphi(n)}.m = (m^{\varphi(n)})^b.m \equiv 1.m = m \pmod{n}.$$

Signature of a message: 2 public keys are necessary.

- Sender: n_A, d_A, public, e_A secret.
- Recipient: n_B, e_B, public, d_B secret.

To send a message m relatively prime with n_A and n_B:

$$m \mapsto m^{e_A} \equiv m_1 \pmod{n_A} \mapsto m_1^{e_B} \equiv m_2 \pmod{n_B}.$$

Then m_2 is sent.

To decode the message

$$m_2 \mapsto m_2^{d_B} \equiv m_1 \pmod{n_B} \mapsto m_1^{d_A} \equiv m \pmod{n_A}.$$

We have claimed that it is easy to construct large prime numbers. This follows from the prime number theorem which gives the asymptotic distribution of primes. To construct a prime number of 100 digits we generate random natural numbers with 100 digits and we test if they are prime. The prime number theorem ensures that after a mean of 125 trials we should get a prime (if we generate only odd numbers).

This means that there is a test for primality of a natural number n which is easier than to factor n. The test is technical and will not be discussed here. The underlying principle is that n leaves its "finger prints" everywhere so that if n is not prime then at least half the numbers in $\{1,...,n\}$ "know" that n is not prime. The test uses the Jacobi symbol. If k numbers $m_1,..., m_k \in \{1,...,n\}$ fail the test then n has a high probability of being prime (this is an exercise with Bayes formula). The number k need not be very high to yield a very large probability that n is prime (details in [RSA]).

Error correcting codes

Principle: We lengthen a message so that the information is contained in several places.

Example: We repeat each bit 3 times. If the 3 bits received are different we correct using the law of the majority, i.e., as if only one error has occurred. Then we recover the message if zero or one error has occurred. We say that the code corrects one error.

If we want to send a word of 8 bits we send 24 bits. As two different words have at least three different bits we get the right word if one error or less occurred.

We can do much better!

Hamming code:

We want to send a word of 4 bits: $u_1u_2u_3u_4$. We send a word of seven bits. We add

$$u_5 = u_1 + u_2 + u_3$$
$$u_6 = u_2 + u_3 + u_4$$
$$u_7 = u_1 + u_2 + u_4$$

This codes corrects one error. Indeed

No error	u_5, u_6, u_7 compatible
1 error in u_1	u_5, u_7 incompatible
1 error in u_2	u_5, u_6, u_7 incompatible
1 error in u_3	u_5, u_6 incompatible
1 error in u_4	u_6, u_7 incompatible
1 error in u_5	u_5 incompatible
1 error in u_6	u_6 incompatible
1 error in u_7	u_7 incompatible

We can do much better but with more sophisticated tools!

Reed-Solomon codes [RS]:

They use finite fields. The elements are words of n bits with an addition and a multiplication.

Example: The field K with 8 elements

The 8 elements can be identified with the 3-tuples whose entries are 0 and 1.

The addition of two 3-tuples is the 3-tuple whose entries are given by the addition of the respective entries modulo 2, i.e.,

$$(a_1, a_2, a_3) + (b_1, b_2, b_3) = (a_1 + b_1 \bmod 2, a_2 + b_2 \bmod 2, a_3 + b_3 \bmod 2,)$$

For multiplication we identify a 3 tuple (a_1,a_2,a_3) with the polynomial $a_1 + a_2x + a_3x^2$.
To reduce the product of two polynomials of degree ≤ 2 which is a polynomial of degree ≤ 4 we use the rule $x^3 = x + 1$. We deduce:

$$x^4 = x(x + 1) = x^2 + x$$
$$x^5 = x(x^2 + x) = x^3 + x^2 = (x + 1) + x^2 = x^2 + x + 1$$
$$x^6 = x(x^2 + x + 1) = ... = x^2 + 1$$
$$x^7 = x(x^2 + 1) = x^3 + x = 1$$

With this rule it is clear that any nonzero element of the field can be identified to one of the x^i with $i \in \{1,...,7\}$. (Note that $x^3 + x + 1$ is an irreducible polynomial over Z_2: this is the essential ingredient to get a field.)

Principle of the coding with a field K having elements:

We code words of m letters, the letters being elements $k_1,...,k_m$ of K by transforming them in words of 2^n elements. As before the non zero elements of K can be written in the form $\{x,x^2,...,x^{2^n-1}\}$. The first letter is k_1, while the $2^n - 1$ remaining letters are given by

$$k_1 + k_2x^i + ... + k_mx^{i(m-1)}\ ,\ i = 1,...,2^n - 1\ .$$

This codes corrects:

$$\frac{2^n - m}{2} \text{ errors if } m \text{ even}$$

$$\frac{2^n - m - 1}{2} \text{ errors if } m \text{ odd}$$

In particular if $n = 3$ (K has 8 elements) and $m = 4$, a word of 4 letters is encoded in a word of 8 letters and the code corrects 2 errors.

Applications:

This code is usually applied with a field of 256 elements (polynomials are multiplied modulo an irreducible polynomial of degree 8 ever z_2). Important applications are, for instance, the communication with satellites. Also Reed-Solomon codes are used when recording music on compact disks.

Image compression

The simplest way to keep an image in memory is to give the color of each pixel. *An enormous memory is needed as soon as we deal with many images!*

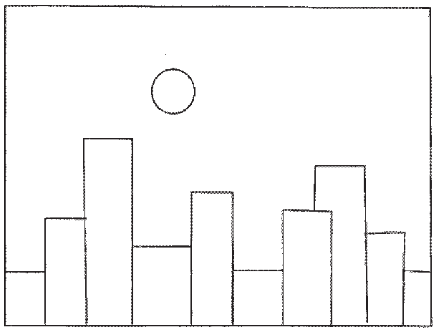
How to do better?

Suppose we have drawn a city. We keep in memory ...

Line segments;
Circles arcs;
etc.

... which approximate our image.

FIGURE 3:
A city



We have approximated our image with known geometric objects.

To keep a line segment in memory it is more economical to keep in memory

- the two ends of the segments;
- a program which tells the computer how to draw the line segment joining two points.

The geometric objects are our *alphabet*.

How can we keep in memory a complex landscape?

We use the same principle with a larger alphabet, i.e.,

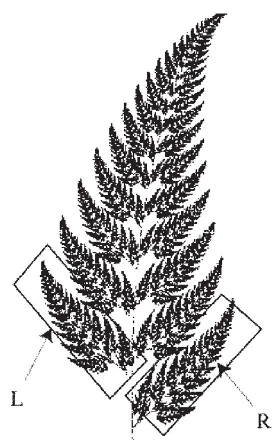
- we approximate our landscape with fractals, for instance the fern;
- we keep in memory the program for drawing the fractals, for instance the fern. Because the fern is auto similar the program is less than 15 lines long.

Principle to draw the fern:

The fern is a union:

- of a tail,
- of 3 smaller ferns.

FIGURE 4:
The fern



We can reconstruct the fern from 4 affine transformations:

- the transformation T_1 which sends the large fern to the fern without the two smaller branches;
- the transformation T_2 which sends the large fern to the small left fern;
- the transformation T_3 which sends the large fern to the small right fern;
- the transformation T_4 which sends the large fern to the tail.

It suffices to keep this information in memory to reconstruct the fern. The method is called "Iterated functions systems" [B].

Algorithm:

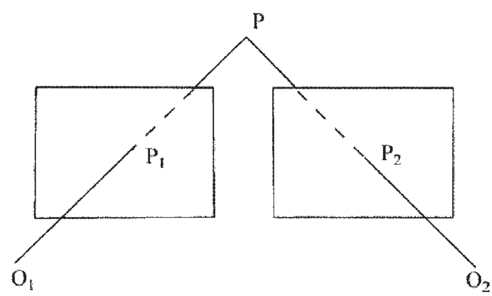
- we start with P on the fern;
- we choose at random $i_1 \in \{1,2,3,4\}$; we draw $P_1 = T_{i_1}(P)$;
- we choose at random $i_2 \in \{1,2,3,4\}$; we draw $P_2 = T_{i_3}(P_1)$;
- etc. …

Vision of computers

We treat here just one small aspect which consists in understanding 3D-space from 2D-images.

We have two pictures taken by two different observers located at O_1 and O_2. In our model the images of P are respectively P_1 and P_2. These points are located at the intersection of the lines D_1 and D_2 joining respectively P to O_1 and O_2 with the projection planes (in our figure we took the same projection plane for the two pictures).

FIGURE 5:
The two pictures



- From the knowledge of P_1 we know that the observed point is on D_1.
- From the knowledge of P_2 we know that the observed point is on D_2.
- The lines D_1 and D_2 have only one intersection point. Hence we know the position of P.

This is what we do all the time: we need two eyes to evaluate deepness: our brain makes the calculation from two images. We need to understand the mechanism to teach computers to do the same.

Exercises:

The exercises done in the course had to do with the images of straight lines and circles in the picture and with perspective.

The movements of a robot

A 3-dimensional robot: *six degrees of freedom are necessary to bring the grip to its position*.

Reflection on the number of degrees of freedom:

- movements 1, 2, and 3 bring P to its position;
- movements 4 and 5 bring the axis of the grip to its position;
- movement 6 brings the grip to its final position by a rotation around its axis.

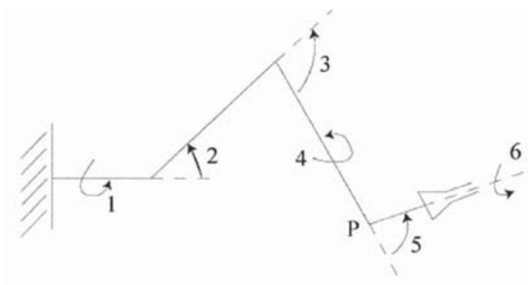


FIGURE 6: Example of a robot with 6 degrees of freedom

Exercises: reflection!

- The construction of the robot is not unique but 6 degrees of freedom (so at least 6 independent movements) are necessary to reach any point in a given region of the space with the grip properly oriented. So 6 degrees of freedom are also necessary for the handles with which one controls the robot.
- Try to imagine other models of robots with 6 degrees of freedom.
- How many degrees of freedom are necessary for a robot moving only in the plane (the answer depends on the problem, namely the different positions of the grip which are necessary to achieve the job)?
- Depending on the length of the different parts of the robot, what points of the plane (space) can be reached by the grip?

The underlying mathematics:

- Each movement is a rotation $R_i(\theta_i)$ in coordinates (x_i,y_i,z_i) centered in P_i.
- It is represented by a matrix $M_i(\theta_i)$.
- We change from one coordinate system to another by a translation followed by a rotation.
- This allows us to know the coordinates of a given point Q in each coordinate system.
- In particular we can calculate the position of Q in the original system after rotations $R_i(\theta_i)$, $i \in \{1,2,3,4,5,6\}$. This involves matrix multiplications.
- Hence we know the effect of a composition of movements on any point. All operations can be inverted.

Exercises:

Imagine problems for an engineer. For instance:

- There exists several sequences of movements bringing the robot to the same final position. Which is best? Some "small" movements lead to "large" displacements of the grip, while some "large" movements lead to "small" displacements of the grip. The latter are better when doing precision work.
- We may add extra pieces and movements in order to allow the robot to go around obstacles. What is the effect of adding pieces and increasing the number of possible movements?
- What is the effect of changing the length of some of the pieces?
- Inverse problem (difficult!): Given a final position of the grip give a sequence of movements to bring the grip to this position. This yields to solving a system of nonlinear equations.

Application:

The Canadian arm for the international spacel station.

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Working Group Reports

Report of Working Group A

Considering How Linear Algebra is Taught and Learned

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Participants

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Introduction

The working group leaders came to the working group with the assumption that teaching introductory linear algebra at university is a particularly vexing experience. In their abstract they wrote:

> Introductory Linear Algebra courses seem to have all the right ingredients—linearity is a basic and simple concept; the underlying theory is built from just very few notions (linear combinations, linear (in)dependence); there are plenty of interesting applications; and many of the concepts can be nicely illustrated geometrically. Yet, experience shows that students have a much more difficult time with linear algebra than with the calculus. In the Working Group, we will examine why this is the case, both from teaching and learning perspectives, and discuss some of the ways in which the current situation can be improved.

Although the difficulties in teaching linear algebra provided much of the impetus for the discussion in our group, it became clear that there was not universal agreement that the subject was especially hard to teach. Moreover, the interests of some of the participants ranged outside the context of teaching linear algebra to first or second year university students. About half the participants were faculty and graduate students who teach linear algebra at university. The others included people who did not currently teach the subject but were interested in issues of teaching and learning that they hoped the working group would consider: how to relate mathematics to the reality of high school students' lives; the connection between what is taught in high school and what will be demanded in university; how to bring personal relevance and connection to mathematical interactions and how to create a sense of community.

Although our opening discussion revealed differing perspectives and interests, it also elicited a consensus that teaching (and studying) introductory linear algebra evokes special tensions that do not arise in other introductory courses. These tensions, and participants' reports and reflections on strategies for resolving them, provide a useful framework for reporting on the activities of our working group. The working group discourse in our three sessions revisited issues identified during the first day, including questions of interest to participants for whom linear algebra was a vehicle for probing wider issues about teaching and learning.

Tensions for teachers and students in introductory linear algebra

Selecting material. Linear algebra is a rich subject for a university teacher because it embodies many elements of mathematical thought, and the teaching and learning problems associated with these mathematical ingredients. It is also a large subject, and confronts the teacher with questions such as: "From all this, what do I have to teach? What is the real basis for the applications I want to or need to investigate?"

Competing "structural conceptions" of the subject. Most introductions to linear algebra lie between the poles of a matrix algebra course and an exposition of vector spaces with everything else of secondary importance. Hence there are computational and structural threads vying for the students' attention. Moreover, geometry can also be used to illustrate, motivate or explain the theory, and this introduces an element that instead of being an aid to students, often becomes something else they have to learn and integrate into the computational and structural ingredients.

Conflicting expectations by teacher and students. One of the surprises that first year university students encounter in algebra is that explaining things is an integral part of the course. They want to be able to produce the answer to a question via a computation and be done with it. Is the high school preparation a problem, in that the need to teach algebraic skills leaves no time to convey to students that there is more to algebra than computation? Is the migration of linear algebra to first year of university a complementary part of this phenomenon?

Preparing students for later vs. dealing with the "here and now." Linear algebra is often seen as a microcosm of a substantial part of current mathematical thought and practice (see Appendix A, particularly the attributions to Cowen, Messer and Tucker). This can translate into a desire to use it as a foundation for later mathematics study by students, and into a keen desire to "cover the prerequisites" for that study. Students can be overwhelmed. The issue we are signalling is distinct from that of trying to use the same course as a service course and as preparation for a math programme. Even when a linear algebra course has a well-defined audience or purpose, the tension we have in mind here is likely to arise. We will return to this in our later discussion.

Is time and effort spent on technology rewarded by better learning? The issues here are similar to those in calculus, with notable differences. Graphical computer representations in calculus are of the objects being studied. In linear algebra the software available to aid geometrical insight is limited to two-dimensional instances of much more general phenomena being studied. And judging by experiences recounted in the working group, the software does not always guide the student to the intended conceptualization.

Coping with the tensions

Selecting material. Among the working group participants, introducing applications seemed to be an expected aspect of teaching an introductory linear algebra course. This is not surprising: the theoretical and practical utility of the subject is responsible for linear algebra having migrated to first and second year university mathematics offerings (Cowen 1998; Tucker 1993). There seemed to also be an implicit consensus that focusing on a few applications was sufficient, and even necessary, and an explicit and unchallenged sense that it was important to choose the applications to be coherent with the course "narrative."

When it came to considering the importance of the applications to the course as a whole, there was considerable difference in the approaches followed by working group members. In the courses that people described the role of applications ranged from being sidelights to the theory, to being equal partners with the theory, to being the vehicles through which students are led to the theory through a process of "inevitable discovery" on the way to solving a problem. (See Peter Taylor's contribution, Appendix C.)

The phrases "on the way to solving a problem" and "inevitable discovery" in connection with applications merit amplification. Several of the people teaching linear algebra include in their course a significant component of having students solve substantial problems, even if the theory is developed before-hand, rather than en route to a solution. Scepticism surfaced about the project of unveiling concepts and theory by making them seem "natural" ways of organizing information or phenomena that accompany a problem. Problems can often be solved in several ways, using different theoretical foundations. Nevertheless, the approach does give a better basis for building a problem-based course, and helps to answer to question of what topics and concepts are actually needed to use linear algebra to analyse and interesting and substantial problem.

Competing "structural conceptions" of the subject. We will set aside for another section the competing conceptual outlooks that teachers and students bring to the subject. Our interest here is on the phenomenon that linear algebra supports radically different "expository layouts."

"Matrix analysis vs. vectors spaces" is a familiar demarcation between different approaches to linear algebra. Both approaches are well represented in main-line textbooks. The discussion in the working group was interesting because it brought out that conventional associations or assumptions that people sometimes make about these approaches were not necessarily represented in the courses that the working group members knew about in their own institutions. One of these assumptions is that the matrix-based approach is appropriate for a service course not apt to be needed as a building block for further mathematics, or as part of an engineering program. Another is that a matrix-focused course is a more computational one, whereas vectors spaces suggests more emphasis on conceptual development.

The situation at the University of Regina provides an interesting case study that points a flaw in these assumptions. The first linear algebra course there is a matrix analysis course. This provides a vehicle for starting with very concrete computational material. But computations are only a starting point for a continuing and probing conceptual development of matrices. Geometry is not played up. Instead the focus on matrices is maintained and their properties become the hook for drawing students in. Quite early in the course students make contact with research questions that instructors are engaged with, for example, questions related to properties of signed matrices. This course is quite successful, and instructors at the University of Regina do not find it a particularly difficult course to teach.

One of our working group participants (Shannon) who is not well versed with linear algebra conducted an "interview" with the instructor (Judi) who teaches at the University of Regina, asking her to "micro-teach" an introduction to the course. What Shannon noticed is the effective way in which she felt invited to uncover the interesting properties of matrices, occasionally having to suspend questions which could not be answered at that point, but which would be treated later (see Appendix B).

We conclude this section by mentioning treatments of linear algebra that depart from conventional ones in challenging ways. One is Axler's (1997) treatment without determinants. Edward's (1995) approach leads to a development that involves concepts that are quite different from those in any standard treatment. Tucker's book gives an treatment based on a few somewhat unusual applications pursued in depth.

Conflicting expectations by teacher and students. There are many reasons why a mathematicians might appreciate linear algebra, and most mathematicians subscribe to more than one of them. Just the mention of "linear algebra" is capable of evoking for mathematicians its manifold connections to other mathematics, its essential representation of the power of the axiomatic approach, its balanced stance between powerful computational and theoretical knowledge bases. Students come to an introductory linear algebra course with none of these sensibilities, nor even with the sense that they exist.

Other issues

Language. Reliance on precise definitions is an important characteristic of mathematical practice, at least in writing. This alone creates a problem in teaching most mathematical subjects to novice students where they are expected to explain things. Linear algebra presents additional problems. There is a lot of terminology and notation to keep track ofæ students have trouble not only with its quantity, but with using it in a semantically sensible way. Before providing illustrations supporting these assertions we hasten to add that we do not have research-based evidence that language is linked to special problems in learning linear algebra; and even if it is, we do not know whether alternative approaches are available that preserve the subject in a way acceptable to the community.

We take up first the proliferation of terminology. A significant aspect of learning linear algebra involves learning to recognise the same concept in different guises. Hence, alternative characterisations become as important as definitions in working problems and in explaining things. Transitions that the teacher makes without a second thought, such as between row-rank and column-rank (each of which can be characterised as a dimension, or as a count of linearly independent vectors) can be a stumbling block for students. Another example is *linear independence* of a set of vectors, which might equally well arise as the zero vector being only the trivial linear combination of the vectors in the set, or as the set being a basis for the subspace that it spans. *Invertibility* of a square matrix A is worse: it is equivalent to A having a two-sided (or just a left inverse, or just a right) inverse; or to det(A) being nonzero; or to every linear system $AX = C$ having a solution (or a unique solution). To complicate matters further, *invertibility* is sometimes referred to as *nonsingularity*, for reasons generally not explained to the student (since there is little to be gained by such an explanation in an introductory course). That students seldom ask why this additional nomenclature is introduced indicates that rhyme or reason for mathematical terminology is something they see as beyond their ability to fathom. Then there are minor variations in words and phrases: *row-echelon form* and *reduced row-echelon form*; *row-reduction* and *elimination* and *Gaussian elimination* and *Gauss-Jordan elimination*; *matrix of a linear system* and *augmented matrix of a linear system*. These arise at the beginning of a typical course, and interfere with students focusing on the conceptual issues.

Something that adds to the language problem, and that leads to student problems in using terminology in a semantically sensible way, is the layer of set terminology and notation that intrudes in the discussion of vector spaces. Set terminology is a foundation for much mathematical discourse, but it is questionable whether it helps students learn linear algebra. This question is explicitly raised by Edwards (see Appendix A).

In fact, when mathematicians talk to each other informally about linear algebra they often ignore niceties of set terminology: they are quite apt to refer to the linear independence of "the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$" rather than of "the set of vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$." It is difficult to argue with Edwards' sense that not much is gained by insisting on the formalities of set terminology before students understand the concepts that we wish to organise into sets.

In some situations insistence on correct language can work against a process of mathematical construction that students typically demonstrate. It is not uncommon for students to start talking about "an $|^2$" to refer to a plane in some higher dimensional space. What seems to be arising is a nascent and somewhat crude notion of isomorphism. It is a pity that conventional presentations of linear algebra demand that we correct students, seemingly working against what appears to be a natural development of a sophisticated mathematical idea.

Can research help in the linear algebra classroom? Suggestions for what and how to teach introductory linear algebra courses are more plentiful than accounts (even anecdotal) of how particular topics and approaches have successfully overcome barriers to student understanding. The Working Group did not attempt to systematically identify research issues about the learning of linear algebra that would likely impact on how the subject is taught. However, the tenor of some questions asked in the WG suggested that participants would be receptive to implementing changes on the basis of evidence of benefit. Although some of these questions have already been mentioned in this report, it seems worthwhile to repeat them here with more pointed attention on the possibility of resolving them.

- How crucial is it to teach introductory linear algebra as a separate subject? Does moving it to later in student's mathematical development make the subject easier to teach and learn? An investigation into how technical knowledge and viewpoints from linear algebra are used in other courses might reveal that the shift of the subject as a whole to earlier years is not as necessary as current practice suggests. Discussion in the WG provided anecdotal support for the notion that teaching linear algebra after a year of calculus circumvents some problems arising from mathematical immaturity.

- Can students be led to develop some important notions (such as eigenvector or isomorphism) themselves, say through well-chosen problems, rather than by being presented with formal definitions?
- What kind of visualisation tools do we need; how do we use them effectively; what pitfalls must we learn to avoid?
- Does reducing the quantity of terminology affect how students develop conceptual understanding?

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Appendix A – Comments about teaching linear algebra

"Students who have learned how to learn linear algebra have learned how to learn mathematics!" — Carl Cowen (1998)

Linear Algebra. Must the Fog Always Roll In? — David Carlson (1993)

"Accordingly the emphasis throughout the book is on algorithms. A by-product of this emphasis is the complete disappearance of set theory, a disappearance that will greatly disturb teachers accustomed to the standard linear algebra course but will, and should not, disturb students in the least. The material in the book will be helpful, in addition to its other uses, in learning the language and the peculiar habits of mind that are set theory.... The standard linear algebra course attempts to reverse the order and to use set theory to teach linear algebra—an approach that is as silly as it is unsuccessful." — Harold Edwards (1995)

"What sort of concept images do students build in their linear algebra courses? Tables 1 and 2 present slightly edited results of a survey of 25 students.... These results suggest that students do not build effective concept images.... They manage to remember concept definitions until the final exam is over but are unable to retain them for an extended period of time." — Guershon Harel (1998)

"The teaching of linear algebra at university level is almost universally regarded as a frustrating experience and many among those who teach this course have resigned themselves to the fact that it is simply 'the nature of the beast' and not much can be done to change things." — Joel Hillel (2000)

[It is interesting to contrast this comment with the following one.]

"... the material is presented with an eye toward making it easy to remember, not just for the next hour test, but for a lifetime of diverse uses." — Tucker (1988)

Linear Algebra – Gateway to Mathematics — Robert Messer (1993)

"Linear algebra takes students' background in Euclidean space and formalizes with vector space theory that builds on algebra and the geometric intuition developed in high school. Then this comfortable setting is shown to apply with unimagined generality, producing vector spaces of functions and more.... A further pedagogical strength of linear algebra is that it joins together methods and insights of geometry, algebra, and analysis .... Linear algebra really is a model for everything a mathematical theory should be!" — Alan Tucker (1993)

"Linear algebra as rendered in many textbooks is a reflection of mathematical aesthetics and romanticism, a subject constructed to be appreciated and to be taught—but not to be learned." — Morris Orzech

Appendix B – Math immersion: A narrative look at incomplete mathematical understanding and its place it the classroom (Contributed by Shannon Sookochoff)

A long time ago

It was 1980. One hundred high school math students had gathered at the University of Saskatchewan for a conference aimed at enriching their understandings of mathematics. I was in grade 10. I was one of those students. And until then I had never been in a math classroom and not understood *everything*. For many, this might seem the ideal.

In one session of the conference, I remember sitting in a lecture theatre. (I had never sat in a lecture theatre.) I remember a professor with a turban. (I had never seen, much less been taught by, a man in turban.) And I remember variables approaching infinity. This was my introduction to limits. I recall *g(x)* and *h(x)* when until that point I had seen only *f(x)*. Although the session planted seeds which would assist my calculus education a few years later, very little stuck with me. There is one thing, however, which I remember vividly: the feeling of sitting absolutely upright, of focusing all of my powers on the board and the man at the front of the room. I recall breathing more quickly, my tummy unsettled, as ideas rushed toward me, just skimming the top of my Olivia Newton John hair. I nodded slowly. And realized, with the kind of sobriety I reserved only for funerals, "Math is vast." *And this was not for a moment unappealing*.

A more recent vastness

Now flash to the Canadian Mathematics Education Study Group, 2001. It was May and I was just winding up a year of teaching Grade 9 and 10 mathematics when I found myself in the working group focused on teaching and learning introductory linear algebra. I was the only one who had not even studied linear algebra. Before joining the group, I must confess to a completely inappropriate amount of humility: I thought, "Well, I know algebra ... and I know what "linear" means.... What's all the fuss about?" Indeed. It turned out, as the working group progressed, that linear algebra was at the very least ... kinda complicated. Over the course of four days of intense discussion, I was privy to serious questions about how to talk about a linear transformation in terms that a student can access. I heard that "we have a 'line', a 'plane', and a ... a ... a ... *what*?!" Hmm. I heard about students who talk about "a 2-space in 3-space" and that Morris at Queen's credits them for formulating a complicated mathematical idea called "isomorphism." I saw an example of a problem using affine transformations. I heard about vector space. I learned that the Fibonacci sequence is linear. (And this I learned just two days after formulating a definition for linear functions with my Grade 10s that included the words "straight" and "line" and "graph". I'm still quite puzzled.)

From Judith McDonald, I heard about the matrix as a beautiful creature, its reduced echelon form, that "the leading ones tell all", and that "Eigenvalues gush forth like children from the womb." Judith took a half hour during a break to offer me an introduction to her approach to linear algebra at the University of Regina. She began with giving me what I would call an object—the matrix. It had features I could cope with: it was a coded form of algebra; it had knowable rules; it had usefulness. Implicitly, she asked me to suspend my own objects and grounding metaphors: algebraic expressions as geometric space, equations as graphs, and vectors as indicators of force and direction. Now, I ordinarily resist giving up "my" knowing. So it might be useful to share with you why I think Judith was able to coax me into this more vulnerable state.

☒ Judith gave me the matrix to hold onto. The *matrix offered me a micro-world*, a cyber space in which to play. There were *explicit rules*.

☒ Judith presented the material in an organized and open way. She was clear about sharing the features of the matrix and about what I could expect during our "class". This *openness and sequencing honoured my structure as a mathematics learner*.

☒ Judith *made links to what I knew* about manipulating two variables in two equations. She gave me a context in which to situate these new ideas: $x + 2y = 3$ and $3x + (-y) = 5$.

☒ Then Judith coded the equations into a matrix, *temporarily suspending my context*.

☒ She manipulated the codes with the rules she had stated upfront.

☒ Then she *compared the results* from the matrix with that which we would have reached had we approached the equations by isolating a variable, substituting in, and solving. As a result, I *gained some trust in the matrix and its rules*. It became a tool for me.

☒ And like all the teachers of linear algebra in our working group, Judith paid *close attention to language*, understanding how powerful words are in constructing mathematical understanding. To define linear independence, Judith tells her 18 year old students, "you are *independent* when you write home for money and you get nothing back—you get back the *zero solution*."

☒ As well, Judith demonstrated *careful listening*, responding to my questions like they energized her teaching. My questions provoked moments of pause, prompting me to feel the privilege that it was to converse with a practiced mind, prompting me to feel proud that I could formulate questions with answers I was not ready to hear.

On one level, the linear algebra working group talked about good teaching; all of the things I noticed about Judith's lesson, could be noticed about any good teacher. On another level, it was about linear algebra: What are the special challenges of the linear algebra classroom? On yet another level, the working group was about learning. How does the learner construct mathematical knowing? And isn't this the way: we cannot talk about mathematics education without talking about the tangled mess that is teaching, mathematics, and learning.

Incomplete learning

By the end of CMESG 2001, my understanding of linear algebra was certainly incomplete. I took notes for the group and I heard words that I had no idea how to spell (like affine and Eigen). I listened for the structure of what was said because very few of the specifics were mine to access. I heard the pedagogical insights. I heard the curricular reflections. I saw parallels to the curricular revision going on in all levels of school mathematics.

I'm tempted to apologize at this point, saying that my experience of the linear algebra working group was quite different from the other participants. I could clarify, too, that I focused on the nature of mathematics learning, the nature of the questions and concerns of this group of dedicated post secondary educators, and the interconnectedness of the arbitrary and the necessary (Hewitt, 1999) in mathematics curriculum. I could clarify that my grasp of linear algebra is less than weak, that at best I have some linear algebra seeds planted inside me.

But, no.

Instead, I will say that indeed my experience of the linear algebra working group was quite different from the other participants, but that this is true of all the participants, that it is impossible to ensure a specific or complete understanding for all, further, for any. My grasp of linear algebra is NOT less than weak. My grasp of linear algebra is neither strong nor feeble. My grasp of linear algebra just is. *And this is not for a moment unappealing.*

Features of successful incomplete learning and mathematics immersion

So why, in this world where many students leave math class demoralized and alienated from mathematics, was I able to feel stimulated and satisfied by discussion so far from my own mathematics? Perhaps the following chart will help.

Features of CMESG Linear Algebra Working Group	*Ways to generate successful incomplete learning and mathematics immersion in a traditional classroom setting*
☒ The "learning outcomes" were not a prescribed set. I was not accountable to a pre-written test.	☒ We could allow the test to co-emerge with the established curriculum, the classroom exploration, and the students' insight. ☒ The test might be written and taken in community. The test might include opportunities to ask questions. ☒ The test might be a one-on-one discussion. ☒ The test might be different for different students.
☒ My questions mattered. They came out of the group discussion and they contributed to it.	☒ All questions could matter. For example, fundamental questions prompt a deeper understanding, even for the mathematically fluent.
☒ Mathematics prompts were selected with a sense of the variety of people in the room. The problems were offered not to prompt "answers" but instead to generate discussion and exploration.	☒ Variable entry prompts (Simmt, 2000) could be selected to honour the variety of people in the room. ☒ Prompts needn't be offered to elicit answers but instead could generate discussion and exploration.
☒ Private tutorials were available. All the teachers in the room were willing to talk in more detail about any part of our co-emerging curriculum.	☒ Private tutorials are already available in the traditional mathematics classroom. This is nothing new. However, if the tutorials live in a space where questions generate deeper understanding for all, then the investment in the discussion could be more equal, more satisfying, and more motivating.
☒ I was included, as a listener and a speaker. I was included with no caveats. No one whispered to me that I might be better off in another group. I was given membership simply because I wanted it.	☒ What if students were not streamed? ☒ What if the mathematics classroom were a community to which students would choose to belong? ☒ How could we assess without shaming? ☒ What if we focused on what is each student's understanding rather than what *should be*? What if we offered links to what *could be*?

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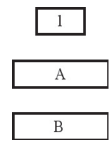
Appendix C – Trains (Contributed by Peter Taylor)

Example 1. I want to construct a train of total length n using cars which are either of length 1 or of length 2, except there are two kinds of cars of length 2, type A and type B. How many different trains can I build?



Solution. Let t_n denote the number of trains of length n. We can work out the first few terms of the t_n sequence simply by listing the possibilities. There is clearly only 1 train of length 1, and 3 trains of length 2, etc. There are 11 trains of length 4 and these are displayed at the right. Note carefully that a train is an *ordered* list of cars, that is, there is a front and a back. Thus the train 1-1-B is different from the train B-1-1.

The 11 trains of length 4
1-1-1-1
1-1-A
1-1-B
1-A-1
1-B-1
A-1-1
B-1-1
A-A
A-B
B-A
B-B

The t_n sequence is tabulated below. Can we find an arithmetic "rule" that would allow us to continue the sequence easily?

Length of train n	*Number of trains* t_n
1	1
2	3
3	5
4	11
5	21
6	43

This is actually a nice problem for high school students and there are a number of nice patterns they will find in the data and there are elementary ways of analyzing them. But our objective here is to uncover an important and powerful approach to the analysis of such sequences.

We begin by looking for a "recursive" formula for each term in terms of the previous term (or terms). For example let's try to count the 6-trains in terms of the numbers of shorter trains. Well there are three kinds of 6-trains:

those trains that begin with a 1-car;
those trains that begin with an A-car;
those trains that begin with a B-car.

It's clear that every 6-train is in one and only one of these sets and that means that the number of 6-trains is the sum of the numbers in each of those sets. But it's clear that the first set has t_5 trains (the different trains that can be added onto the one car), and it's equally clear that the last two sets have t_4 trains. We have shown that:

$$t_6 = t_5 + 2t_4 .$$

The same argument works in general:

$$t_n = t_{n-1} + 2t_{n-2} .$$

Check that the above sequence conforms to this rule. We can now employ it again and again to extend the table as far as we like.

Our objective here is to use this recursion to find a general formula for t_n. Mathematically, what we are doing is "solving" the recursion. I start by restating the problem.

Solve the recursion

$$t_n = t_{n-1} + 2t_{n-2}$$

with the initial conditions, $t_1 = 1$ and $t_2 = 3$.

The crucial idea is that we ignore the initial conditions and seek solutions of the equation on its own. Now that means there will be lots and lots of solutions and the idea is that maybe some of them will be really simple—simple enough that they are easy to describe and find general formulas for.

This may sound like a strange thing to do—rather like looking under the street lamp for the earring you lost in the dark alleyway—but it will turn out that these "simple" solutions hold the key to finding all the solutions of the recursion, and in particular, the one we wanted in the first place. What we are going to do in fact is "build" the train sequence we want using simple sequences as building blocks.

Some solutions to the recursion:

1, 1, 3, 5, 11, 21, ...
2, 4, 8, 16, 32, 64, ...
1, 3, 5, 11, 21, 43, …
4, 6, 14, 26, 54, …
1, 6, 8, 20, 36, ...
2, 0, 4, 4, 12, 20, ...
2, 1, 5, 7, 17, 31, …

But what good will that do us?—no matter how simple the solution is, if it doesn't satisfy the initial conditions, it won't be a solution to the trains problem. Ah, but maybe it will. Read on.

One might at this point simply list a few solutions and see what we get. They are easy enough to construct: start with any choice of t_1 and and t_2 then the rest are determined.

Well once t_1 and and t_2 are chosen, the rest of the sequence is specified. So all we have to do is to take different possibilities these first two terms. A number of these are listed at the right.

Now if you wanted to choose a solution which was easy to describe, what would it be? Well the second one is simply the powers of 2:

2, 4, 8, 16, 32, ...

This is the sequence $t_n = 2^n$. We have a formula for the general term.

This sequence is called *geometric* because each term is obtained from the previous term by multiplying by a fixed "ratio" r which in this case is 2. All such sequences are easy to describe—they are just powers of r. So we wonder whether there are any other sequences of this kind which satisfy the recursion.

Okay. Any geometric sequence will have the form $t_n = r^n$. If we plug this into the recursion:

$$t_n = t_{n-1} + 2t_{n-2}$$

we get:

$$r^n = r^{n-1} + 2r^{n-2}$$

and this has to hold for all $n \geq 3$. That is, we want:

$$r^3 = r^2 + 2r$$
$$r^4 = r^3 + 2r^2$$
$$r^5 = r^4 + 2r^3$$
etc.

It seems as we have an infinite number of equations which have to hold, but in fact if r satisfies the first equation, it will satisfy all of them (multiply repeatedly by r) so we only need the first to hold, and in fact we can even divide that by r (since we are not interested in the solution with $r = 0$) to get:

$$r^2 = r + 2 .$$

This is a quadratic equation which factors:

$$r^2 - r - 2 = (r - 2)(r + 1)$$

and the roots are $r = 2$ and $r = -1$. This tells us that there are exactly *two* geometric sequences which satisfy the recursion, one for $r = 2$ (and that's the one we found already) and the other for $r = -1$. That gives us the sequence:

–1, 1, –1, 1, –1, 1, –1,...

which has the nice formula: $t_n = (-1)^n$.

From a mathematical point of view, there's really no reason not to allow negative numbers (or even fractions). We are after all interested in finding nice solutions of the recursion and if we have to expand our palette of numbers to do that, so be it.

Now here's the point. Neither of these simple solutions is the train sequence, but it's possible to get the train sequence by using these as building blocks. The reason is that sums of solutions are solutions and scalar multiples of solutions are solutions. In general:

Linear combinations of solutions are solutions.

And the reason for this is that the recursion is linear.

Okay. Can we find a linear combination of the two special solutions which gives us the train sequence? That is, can we find a and b so that

a times the first sequence plus b times the second sequence gives us the train sequence?

$$\begin{bmatrix} 1 \\ 3 \\ 5 \\ 11 \\ 21 \\ \vdots \end{bmatrix} = a\begin{bmatrix} 2 \\ 4 \\ 8 \\ 16 \\ 32 \\ \vdots \end{bmatrix} + b\begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \\ -1 \\ \vdots \end{bmatrix}$$

Here we write the sequences vertically using the more usual vector notation. This makes it easier to see that what we have is a system of infinitely many linear equations in the unknowns a and b

Now this gives us an infinite number of equations in a and b which have to hold, but here's a key observation—since all of these sequences (solutions of the recursions) are determined by their first two terms, if we get the first two equations to hold, they will all hold. Just to emphasize that, let's write the first three equations:

$$1 = 2a - b$$
$$3 = 4a + b$$
$$5 = 8a - b$$

Observe that the third equation is obtained as the sum of the second plus twice the first. Similarly, the fourth equation is the sum of the third plus twice the second. Etc. So we only need:

$$1 = 2a - b$$
$$3 = 4a + b$$

and these solve to give $a = 2/3$ and $b = 1/3$. We get our train-sequence formula:

$$\boxed{t_n = \frac{2}{3}2^n + \frac{2}{3}(-1)^n}$$

What this means is that the vector space of solutions of the recursion has dimension 2. Essentially what we are doing here is using the special sequences as a basis for this vector space and a and b are the coefficients we need to write the train sequence in terms of this basis.

This is an instance of one of the fundamental "methods" of linear algebra: find a special basis for the vector space we are working with, and seek to write all vectors in terms of that basis. For example, this is what Fourier analysis is all about—expressing the sound of a violin in terms of pure sine waves.

Well? Do you see the shape of things to come? We have a strategy for counting trains (and maybe other things as well!):

1) Find the recursion for t_n .
2) Ignore the initial conditions and find lots of solutions.
3) Look for special (geometric) solutions that are easy to describe.
4) Write t_n as a linear combination of these special solutions.

For extensions of this approach to Binet's Formula, and problem sets, etc., contact peter taylor at taylorp@post.queensu.ca.

Report of Working Group B

Children's Proving

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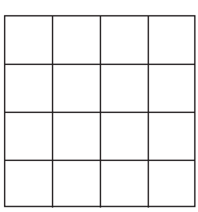
Introduction

Proofs and proving are usually addressed at the high school level, but what role should proving play in the elementary or middle years classroom? The intention of this working group was to begin to understand the role that proving might have in the classrooms of children and to discuss how teachers might recognize and promote proving as a form of reasoning and discourse. As a way into the topic, we engaged in the chessboard problem (below) and focused our discussions on our own solutions and on the solutions and explanations provided by fifth grade students as viewed on videotape. These interactions and observations all served as possible objects, ideas, acts, symbols, and words to point to as we struggled to identify, describe and clarify the nature of proving at the elementary school level.

Chessboard Problem



Count the squares:

What if … this were a 5 by 5 square?
How many squares would you have?

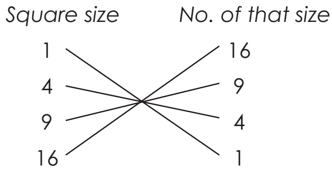
Extensions:

What if this were a 10 by 10 square? How many squares would there be?
What if this were a 60 by 60 square? How many squares would there be?

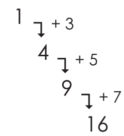
Observing Interactions

Vicki Zack shared two segments of videotape from her fifth grade class working on the chessboard problem. As was the usual practice in class, the children first worked individually on the problem in their Math Logs and then moved into groups of two or three to share their ideas and expand on them further.

As Will worked individually on the 4 by 4 chessboard problem, he wrote the following chart in his Math Log:



Criss-Cross Pattern



Pattern of Differences

In the first segment of videotape, we viewed Will, Lew and Ross coming together to share their solutions. Here, Will had an opportunity to share the criss-cross pattern with his two partners. In his explanation he said, "I was pretty sure there would be a pattern so I was keeping my eyes open and I found one." Although he had not yet tested other sized squares he was already assuming that the pattern of number of squares could be generalized to chessboards of various sizes: "The chances are if it works for those it works for others." Lew was very impressed: "That's clever. That's very clever." While working in the group of three, Will applied his pattern to a 5 by 5 chessboard and found that his conjecture held true: "So, it's basically the same. I never realized [the pattern] would be so helpful."

Will's pattern was used by the group of three as the basis for further exploration into a 60 by 60 chessboard. It was Gord who recognized it as a pattern of square numbers and together they used this information to determine a solution method. They claimed that an answer could be found by adding 60 x 60 + 59 x 59 + 58 x 58 ... and so on. They were thrilled with their creation and exclaimed, "We're a genius!"

In the second segment of videotape, which occurred two days later, Will, Lew and Gord joined Ross and Ted to further expand on their understanding of the problem. Although the two groups used different methods, the five children together confirmed several solutions: The number of squares in a 4 by 4 was 30, 5 by 5 was 55, and 10 by 10 was 385. Although Will, Lew and Gord had a method for determining a solution for the 60 by 60 they had not yet computed the answer. Ross and Ted stated that the answer was 2310 squares. They explained that they solved it by taking the answer for the 10 by 10 and multiplying it by 6; therefore, a 60 by 60 was 385 x 6 = 2310. Will and Lew emphatically disagreed with that solution method. Lew said, "That doesn't work … I'll make you a bet … I'll bet you anything in the world." Ross replied, "I'm not betting. You have to prove us wrong." The group of three then set out to prove that the answer of 2310 was incorrect through a series of counterarguments. The first argument was that since an 8 by 8 was not double the number of squares in a 4 by 4 chessboard their solution method was inappropriate; second, Will pointed to the pattern identified earlier which, he said, must continue to grow in the same way; and the third argument was that since a 60 by 60 had 3600 little squares it had to be bigger than 2310. While Ross was still hesitant, Ted finally agreed, "Yah, right. That's true." (See Zack, 1997 and Zack, 1999 for a fuller description of the activities and the transcripts.)

What are we looking at? What are we looking for?

Vicki introduced the videotape to our working group with the statement that the tasks were not assigned with the intention or expectation of invoking or provoking acts of proving; however, the climate for proving—the expectations for mathematics and language—appeared to support and perhaps encourage that form of reasoning. While we had many questions as to how these expectations were put in place, there was acknowledgement throughout our discussion that a culture of proving or, at least, of mathematical reasoning needed to be in place to support the thinking and discourse required for proving. The children's interaction as viewed on the videotape raised many questions and became the basis for our discussion into the nature of children's proving: Are the children just playing with numbers or are they actually proving? If so, in what ways? What were the elements of proving that occurred? What forms of discourse supported the acts of proving? Are the children equating proof with finding a pattern? If so, what should a teacher do? Did the children understand the need to prove that the pattern continues? Is observing the pattern enough in elementary school? Children are easily convinced by empirical evidence and this will get worse with the available technology. Again, what is a teacher to do about it?

It is important to note that the diverse membership in our working group represented a range of experiences and beliefs about the nature of mathematics and of mathematics teaching and learning. We were classroom teachers, consultants, mathematicians, teacher educators and researchers. Each of us also brought different lenses through which to view the activities. Members raised questions and contributed to the discussion from mathematical, pedagogical, psychological, philosophical and linguistic frameworks. The questions and topics raised and the underlying tensions in our discussions were inherently based on the multiple perspectives brought forth.

What is proof/proving? What are the elements of proof/proving?

"Do we have enough to trust?" "Is it compelling?"
"Starting from an assumption and moving beyond it."
"Coherent articulation."
"A coherent string of reasons."
"What is needed: ideas of necessity, efficiency, sufficiency"
"A move from patterns to something more formal or more abstract."
"What is true?"
"True for all."
"A type of understanding."
"Deductive and inductive reasoning."
"Reputation—Whose ideas get taken seriously?"
"Community standards."
"Why … that … how …." "Prove why … prove that … prove how."

We spent a lengthy but necessary amount of time struggling with our own understanding of proof and proving, and attempting to develop a working definition or description of the elements that may be involved. However, we continued to have difficulty distinguishing between proving and other forms of mathematical reasoning.

"Are these not simply elements of understanding?"
"What is the difference between understanding and proving?"
"Does proof not represent a way of thinking rather than a physical thing?"

Developing such a definition or framework appeared necessary to some members as it served as a means to identify the assumptions under which we worked. It also seemed to be an important prerequisite for determining how or whether proving had a role to play in the elementary classroom. Even further, "[w]ithout a working definition, how can we begin to provide meaningful learning activities for students?" On the other hand, many people resisted attempts to define the nature of proof and proving as such a definition appeared to result in an oversimplification of a multifaceted process; there was also a fear that due to the inherent complexity of the nature of proving, a working definition would very likely be too abstract and divisive to be of much use … particularly to classroom teachers. Regardless of whether individuals were searching for or resisting a definition, a dominant topic of discussion across the three days was listing the potential elements of proving that were or were not displayed by the children in the videotape.

As we observed the children engaged in mathematical activity we pointed to particular features of their reasoning and understanding as potential elements of proving. As markers of this form of reasoning we looked for and recognized acts of *conjecturing* and *verifying* and we saw *pattern noticing* and *property noticing* (Pirie & Kieren, 1994); for example, we heard Will's declaration, "I was pretty sure there would be a pattern so I was keeping my eyes open and I found one." His criss-cross pattern was extended and clarified further to a pattern of square numbers and a pattern of differences.

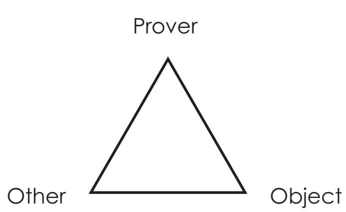
There was agreement that a significant component of proving was the ability to *generalize*. (However, the point was made that while all proofs are generalizations, not all generalizations are proofs.) How capable are children of making generalizations? What are the markers of such acts? In part, generalizing carries with it a sense of "for all"; one example of this was when Will conjectured that his pattern could be generalized to all chessboards when he stated, "[t]he chances are if it works for those it works for others." He verified that his pattern held true for a 5 by 5 chessboard and said, "So it's basically the same." In acts of generalizing there is a belief that the conjecture is true for all numbers, true for all time or true under particular assumptions or logical and mathematical circumstances. Once this belief or generalization is formed, it goes into the "truth box" (Kieren) for future reference.

We know that children can recognize patterns, make conjectures, verify predictions, generalize within and across contexts, but can they make such understanding explicit to themselves and others? If "proof is just a convincing argument" (Hersh 1993, p. 389) in which its primary function is that of explanation (Hanna, 1995), then the markers of proving as observed between individuals sharing their explanations and the listeners attempting to understand or be convinced, needed to be addressed. It was clear to us as observers that the children in the videotape had many previous experiences describing patterns, presenting explanations, justifying their thinking, debating solutions and providing counterarguments. "The giving of reasons" appeared to be an expectation of their interactions and an important form of discourse supporting acts of proving.

Through our discussion we explored the nature of the discourse within the community and attempted to identify linguistic markers of proving; one of which was the sequence: "If …., then …, because …". Within this sequence an individual potentially articulates assumptions or constraints, a solution, and a justification of why that solution is true or correct. Other markers such as "must be …" or "have to be …" in response to the question "Why?" are further indicators of efforts to prove.

The listener is also implicated in acts of proving. As observers we can listen to and for questions, clarification, challenges, refuting statements, even counterexamples. Both the explanation and the artifact produced as a re-presentation of that explanation (e.g., through physical objects, diagrams or symbols) require interpretation by another member of the community. The communication between the prover and the audience and between the object of proof and the audience requires a specific set of language tools to express mathematical thoughts. Providing opportunities for children to become aware of and to develop mathematical language tools also appear necessary to support acts of proving in elementary classrooms.

Throughout our working group discussions, we raised a number of important elements of proving addressing reasoning, language tools, linguistic structures and community interaction. However, each time we turned to highlight potential features of proving, there was unease as we simultaneously neglected other essential elements. On the third day it became clear that each element needed to be viewed in relationship to the whole: as a dynamic between the individual(s) as the prover(s), the other(s) as the audience or community, and the object of proof as a representation of reasoning.



As we viewed the children's interactions, we noted that aspects of the triad were simultaneously implicated in the acts of proving: in the person(s) mathematical reasoning and explanation; in the ways in which the listener took up or critiqued the re-presentation; and in the actual form or structure of the re-presentation. Even with a broader understanding of the dynamic there was full awareness that our understanding and description was not sufficient for or restricted to the act of proving.

What of proving in elementary schools?

There was a general consensus that children should be asked "why?" with respect to mathematical activities. The concern, however, was that proof may be different, perhaps a "lesser" form or as a "prooflet" (Watson) at the elementary school. However, given our tentative framework for viewing acts of proving we witnessed and generally agreed that children can produce arguments that have many of the elements of proving that we described. Children have the capacity to prove, if not the verbal structure/language/communication tools to present the argument concisely.

Even if children are capable, why should they engage in activities that promote proving as a form of reasoning? "What goes into the math tool box from this kind of activity?" "What mathematics do students learn?" "What is the mathematics?" Or, as our colleagues were exploring down the hall, "Where is the mathematics?" Through our discussion we addressed the goals of proving as a means to provide opportunities for children to internalize "theorems of/for thinking" (Pimm) and encounter "mathematical structures" (Watson) (e.g., square numbers, sequences, shape-dissection, algebraic ways of looking at shape, geometric ways of looking at number). Through repeated experiences children develop the mental power of mathematical thinking (which may or may not be similar to thinking and discussion elsewhere in their lives). Mathematics provides us with a "way of seeing" and a means for representing; "it is personally empowering to know that mathematics is not accidental."

What is the role of the teacher?

> "What might a teacher look for in the process? What might a teacher look for in the thing? How are these different?"
>
> "What theorems of/for thinking might we look at/for? How are these useful in analyzing discourse among students?"
>
> "How does already knowing the proof, affect our ability to help children also become knowers of the proof?"
>
> "How do elementary teachers prepare to take on this role?"
>
> "What tools do they have to evaluate whether the proofs produced by children are valid?"
>
> "What might/ought the teacher do to promote proving? How might she intervene?"
>
> "What are the implications of a particular intervention for a student who sees things differently or doesn't see at all?"

We had many questions, but little opportunity to explore specifically what the teacher's role might be in recognizing and promoting proving as a form of reasoning and discourse in the classroom. It was, however, clear that we felt it was unnecessary and possibly detrimental to teach the rules of logics or the technical language of proof as a prerequisite to engaging in acts of proving. What is needed is further exploration of classroom expectations which encourage and support elements of reasoning and discourse including conjectures, justifications, explanations and arguments. We need to continue to explicate markers for acts of proving including linguistic markers such as "if-then-because", "therefore," "works for all," and so on. Finally, we need to find ways to make explicit the meta-cognitive components of student thinking including the types of reasons and arguments (logical, social) used in proving (Pallascio).

Many More Questions

There were many more questions raised that we could not even begin to address in our working group or in this report. However, given the lack of literature in the area of children's proving, our working group made a significant beginning by generated many fruitful starting points for further research.

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Inservice Mathematics Teacher Education

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Participants

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Our working group displayed one feature that tends to be common to CMESG working groups, i.e., going with the flow and not limiting ourselves to the script of the goals and activities described in the abstract that preceded it. Our abstract described the working group as follows:

> *This working group will focus on ways to facilitate mathematics teacher professional development. The underlying theme will be teacher growth and change. Thus the group discussion will include teacher change, the change process, measuring change, and facilitating change. However, the group will focus on activities on which inservice teacher education can be based, emphasising a humanistic perspective in which the personal is valued. These activities will include narrative inquiry, investigations, concept maps, case inquiry, action research, self-reflection, reform-oriented mathematics resources/programs. These activities are not necessarily mutually exclusive, but for the purpose of investigating their nature, they will be looked at individually and discussed in terms of their interconnections, similarities, differences, advantages, limitations, potential to offer opportunities to examine knowing in teaching and the possibility for supporting the construction of new knowledge for teaching and opportunities to examine and discuss familiar mathematics represented in unfamiliar ways, … . The activities will provide a basis to draw conclusions on viable models of inservice mathematics teacher education.*
>
> *An experiential approach will be adopted in the working group to allow participants to engage in some of the activities in a "hands-on" way. The activities will be spread over the three days. Experience with the activities will form the basis of reflection and discussion of their nature and potential for facilitating meaningful growth and change.*

While we did not follow this description in a literal way, we were able to deal with many aspects of it directly or indirectly and on occasions branched off into other related domains pertaining to teaching and learning mathematics. This made the working group sessions over the scheduled three days an experience in which participants' voices were an important component to what unfolded. However, while it is not possible to fully capture the warm interactions and sharing of personal excitement and visions, we attempt to report on the spirit of our discussions in the following overview of the activities of the working group.

Day 1: Drawing the Big Picture

The investigation into preservice mathematics teacher education has permeated working groups at CMESG over the past 25 years. Recently, Stuart and Higginson (1996) focused on preservice and inservice teacher programs. Their working group participants first described the characteristics of a 'developed' mathematics teacher, then directed their attention to strategies that could help teachers develop into the image that they had described. These strategies supported the group's belief that reflective teaching is important in professional development. The group also agreed that learning to reflect on action and on learning was important (Stuart & Higginson, 1996, p. 53).

The themes of professional development for preservice mathematics teachers (Bednarz & Gattuso, 1998) and teaching practices and teacher education (Gattuso, Evans, & O'Shea, 1999) were recent contributions to mathematics teacher education discussion at CMESG. In the working group of 1999, three models of teacher education were described in great detail. This group ended their discussion by highlighting three different teacher education programs. However, they did not have the time to critique the assumptions of each program.

It is with this historical perspective that the 2001 Working Group began their discussion of inservice teacher education. After a brief introduction, the participants were asked to work in small groups to create a concept map of all of the elements they considered important in conceptualizing teacher education. The groups presented their concept maps, drawn on large flip-chart pages, for feedback and discussion. While each map was unique, there were many similarities conceptually in terms of focusing on the major components of a teacher education program, concepts/content to be taught, specific contextual issues, and the interconnections between program elements. The large group discussion also dealt with the similarities and differences between the various components. It was evident that many issues exist for mathematics teacher education. However, teacher knowledge about content and instructional strategies was the central issue that emerged from the concept maps.

The participants next worked on a mathematical activity called Tally Ho. In this activity, the participant is to take the digits one through nine and place them on a grid with six mathematical operations. The goal is to determine the largest possible "score". The format of the page is set and contains addition, subtraction, multiplication and division. The activity is used to explore mathematical operations, with a focus on multiple answers and the teacher defines the "best" answer. The students can also define the "best" answer. In the working group, we used this activity to identify features of mathematics that involve a dynamic approach and to discuss the role of such activities in teacher education. We also focused on the types of questions that might be used to encourage teachers to think about their own understanding of mathematics. For example, does multiplication make the result larger? If not, when does multiplication result in smaller numbers? The same number? Through discussion of such questions, the teachers could reflect on their concept of number operations with different number systems. Finally, we talked about surface mathematics and the personal feeling we have about learning mathematics. We described activities that are created for the teacher to use in the classroom that might lead to mathematics understanding and motivation.

The Tally Ho activity was followed by a lengthy discussion on the role of reflection in mathematics teaching and teacher education We considered reflection on object, action and thought. Some participants focused on the importance of the relationship between the person and the object. They felt that reflection should be experience-related and it is determined by one's relationship with the object. We also discussed the value of self-reflection. As Sandy Dawson explained, the value of self-reflection depends mostly on how deeply one is able to go in the reflection. The goal is to try to push teachers' reflection to another level, e.g., beyond the taken-for-granted level. Sandy Dawson contributed another interesting component to our discussions in terms of cultural differences in the interpretation of reflection and publicly sharing personal experiences. His experience in the South Pacific was the basis for discussing these differences. This allowed us to explore teacher culture, South Pacific culture and cultural perspectives that can influence reflection in different/unique ways. For example, in some cultures, men must speak first and then women can speak, i.e., a hierarchical structure of talk/conversation that could restrict the type of activities used to facilitate reflection. We also considered mono-logical and dialogical societies and the relationship between public reflection and assessment. We concluded that the building of mathematics communities was dependent in some ways to the social attributes of a culture.

To end the day, we worked in small groups to identify and reflect on the goals or factors in which to frame meaningful inservice mathematics teacher education. Some key goals/factors identified were positive experience, mathematical understanding through exploration of mathematical concepts, reflection on mathematics content and pedagogy, and reflection on self. Our sharing of these factors led us to further discussion of reflection. Our collective view was that reflection was key to teaching (e.g., in the context of being a reflective practitioner) and learning (e.g., as a way of making sense of a situation/experience).

Day 2: A Case for Teacher Education

We began day 2 with a 3-2-1 activity. Participants were asked to list three ideas about reflection, two questions about inservice teacher inquiry, and one challenge for inservice mathematics teacher education. The list was used to review the working group discussion from the first day and to provide continuity for the discussion on the second day. After sharing and discussing our lists, we shifted to a different medium and instructional strategy for exploring mathematics teaching and learning—the case study. Doug provided a 20-minutes video titled "Good morning, Miss Toliver" which served as a case to illustrate a perspective of mathematics teaching in which elementary students were actively engaged throughout the lesson. Before viewing the video, we formed 4 small-groups and each group was assigned a different task that dealt with focusing their observation on a particular component of the teaching portrayed in the video. These components were—classroom management, assessment practices, types of mathematical activities used in the classroom, and student communication. Each group discussed their findings then reported back to the large group, providing a synopsis of their discussion. The ensuing conversation was rich and embedded in context. We felt that the use of video case studies could be beneficial for teacher education programs.

Throughout the discussion on case study, we agreed that exemplary activities would need to be added to the curriculum to enrich teacher's mathematics exploration and reflection. To illustrate this point, participants were asked to represent 1 mathematically in as many ways as possible conceptually. We formed three small groups to work on this task, then each group shared and discussed their findings. This question facilitated discussion on three dimensions: looking at a curriculum area, thinking of connections across mathematics and multiple representations. Through this type of discussions, we can get teachers to reflect on their own conceptions of mathematics.

Another way in which we explored reflection was through the use of narratives or stories. Teachers' stories provide a basis to help teachers to explore their own thinking of mathematics teaching and learning and their behaviours in the classroom. Olive Chapman led a brief demonstration on one way to conceptualize narrative inquiry in this context of self-reflection using a story written by a classroom teacher that was supplied by João Ponte. João read the story titled "the glory of knowing how to use a calculator". Participants were invited to react to the story in whatever way that was meaningful to them. This was followed by a discussion of the nature of the participant's response, i.e., a focus on the teacher in the story. Joao re-read the story as participants now listened to it focusing on themselves and their own story that was stimulated by the story that was being read. Participants were invited to share their stories. This was followed by a discussion of the shift in tone of the sharing and the focus on resonating in the stories of others. We also discussed the characteristics of these stories to allow them to facilitate deep as opposed to surface reflection. We raised concerns of feeling vulnerable during such story-sharing sessions and how we could facilitate them to make them positive experiences. Good facilitating includes working with a group of participants of the narrative inquiry to establish a non-judgmental context to share and think about the stories. The facilitator can also help the participants to synthesize particular themes emerging from the stories for further consideration in a different way, e.g., through paradigmatic analysis.

To end the day, Olive provided each participant in the working group with a collection of abstract/excerpts from selected articles and a bibliography on inservice mathematics teacher education. The bibliography also included selected articles/books on teacher development in general. This 39-page document was prepared specifically for the working group. A sample of the bibliography pertaining to mathematics teachers follows:

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Day 3: Building Models for Teacher Education

Doug McDougall made a presentation on research on teacher change. Doug also suggested that research shows that teachers' beliefs about mathematics conflict with reform conceptions of mathematics as a fluid, dynamic set of conceptual tools. As noted in other working group meetings, some teachers lack the mathematical knowledge required to make full use of rich problems. An added challenge is that reform ideals are not consistent with policy statements and curriculum in many parts of Canada. Doug also stated that teachers need to reflect on their teaching to make the required adjustments to their teaching style and subject content. Finally, the role of the teacher is paramount to the success of implementation of mathematics curriculum.

An activity from the Ontario Mathematics Impact Math project was used to investigate exploratory problems. The participants worked in groups to identify the key elements of the activity. The discussion focused on the role of the teacher instructor in working with preservice and inservice teachers. How much support should the instructor give? How do adult learning theories assist the instructor to provide support for teacher candidates? We moved from these discussions to the concept maps created on the first day to make more connections between the instructors and the students.

Tom O'Shea described a large-scale inservice teacher program in BC he is involved in creating through Simon Fraser University. The program consists of a two-year model that connects the school district with the university in designing a basis of professional development that matches the goals of the district. We engaged Tom in a lively discussion of the program by posing a number of interesting questions to him. For example: What do you expect to happen in classrooms after the inservice course? Suppose the goals of the school district do not match the goals of the university for the course – how will you handle the situation? How might these professional development programs evolve over time? What motivates people who are in the top salary grid to attend such programs? While we posed many questions to Tom to facilitate discussion of his project, we felt that university-school district partnership is a potentially powerful way of enhancing mathematics teacher education programs.

In the final hour of this the last day of the working-group sessions, models of inservice mathematics teacher education were explored. We also worked in small groups to suggest a model of preservice mathematics teaching. While past working groups have explored this topic, the members of our working group felt that we had made some progress on the topic. The various models were sent back with the proposers to identify what we might do next. We agreed that we would have an informal discussion at the 2002 CMESG conference.

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Where is the mathematics?

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Some Initial Intentions

As teachers, we give students tasks of various kinds. As educators, we expect novice teachers to develop skills in using and presenting tasks to students. If we construct tasks through didactic engineering starting from a knowledge of the situation, with emphasis on the 'mathematics' then there is an issue in acquainting others about the aims, intentions, and means so that potentials are actualized. As Dick Tahta observed (1981), in addition to an outer task (to perform what you are asked to do) there is an inner task (to make contact with mathematical ideas, to experience mathematical themes, to employ mathematical heuristics and powers, etc.). The aim of the Working Group was to approach a number of questions including: How does mathematics emerge from playing games, from using apparatus and from mathematical instruments? Is it possible to identify the properties of such instruments which motivate and facilitate the student's transition from the outer task to the inner task? How can this be planned for, enhanced, and exploited? What is the role of such instruments in the teaching and learning of mathematics?

How the Time Was Spent

A summary of how the Group spent its time may provide some additional insight into this report. As the number of participants was fairly large, it took some time for everyone to introduce themselves, their interests and their experiences in the Working Group topic. As an ice-breaker, the game of SKUNK (see Appendix A) was played. In the first two sessions participants also worked with apparatus, which included the MIRA, hinged mirrors, leapfrog, Chinese jigsaw, and other games besides SKUNK that included Brock Bugs and Four-Bidden. The Group worked both in plenary, in small discussion groups and in pairs. The latter allowed for the sharing of reflections, team playing and the development of game strategies.

Reflecting on Experiences

The first task proved very popular with most participants, to the extent that some wanted to carry on playing. An element of competitiveness entered, as well as interest in how an extreme strategy seemed to work rather better than a more considered one. Certainly it got the group off to an active and energetic start. The initial energy release may have made it difficult for more cerebral tasks later.

A large group of participants increases the possibility of divergent views, of experiencing shared situations in different ways and of not responding and reacting in unison. This was certainly the case for this group and tensions were generated that were not resolved. Within the context of the given situation individuals and groups reflected on the mathematics they used or developed, the links that facilitated their connection to the mathematics, the barriers that they perceived in accessing the mathematics, the role, timing of, and necessity of interventions, in order to move to the mathematics, and many other related points that arose during the activities. With manipulative-apparatus, including the MIRA, hinged mirrors, leapfrog and the Chinese jigsaw, points recorded were categorized as follows:

Goals

In the case of Leap Frogs and Chinese Jigsaw, goals were suggested at the beginning. For MIRA and hinged mirrors participants were free to set their own goals. Some teams felt that the goals got in the way and distracted from the process of doing mathematics. In the Chinese Jigsaw some pairs indicated that they would have benefited more if the goals had been set in broader terms. For example if it had been explicitly stated that all patterns and their relationships should be explored. Some groups expressed boredom with MIRA because no goals or mathematical problems had been set. Some teams changed the goals or set up new goals once they had reached the preset goals. Others were satisfied in reaching the goals and moved their attentions to other matters. Some discussions ensued about possible conflicts between the goals of the students in the activity and the goals of the teacher. How explicit should the teacher express his/her goals to the student? How and when should a teacher intervene when she/he realizes that the student is moving away from the intended mathematical goals?

Awareness

Participants, as individuals or as teams, were asked to note whenever they became aware of positive or negative reactions or experiences, of changes in strategies or approach, and of mathematization. Participants recorded negative reactions to situations where they had used the manipulative-apparatus previously, and therefore had already gone through some of the mathematization. There was very little enthusiasm shown about extending previous experiences, preferring to engage in another activity. They noted positive reactions when they found the activity particularly interesting or when the mathematical modeling development was not obvious. Some of the groups did not even complete the activity but went straight to the mathematization, satisfied that they had a mental image of the activity. In the mathematization process participants used words such as abstraction, replacing the manipulative-apparatus by symbols, paper and pencil activities, looking for order, searching for patterns and trying to generalize. Others reflected that they had reached the goal with very little awareness of any mathematics.

The reactions reflect those of a group that brought together a very substantial set of experiences and mathematical power, a group that was disposed to approach the tasks, and individuals that were looking for the mathematical potential within those tasks. One can expect similar responses for activities that students enjoy but depending on the age and mathematical ability of students one can predict a need to motivate the mathematization of the situation.

The Group spent some time isolating properties of 'good' games and apparatus, in terms of providing a rich environment in which to mathematize. A good game or apparatus prompts reflection, is amenable to different approaches, motivates conjectures, is aesthetic and sensorial, is attention grabbing (it runs when you chase it), has the potential for generalization.

Is there a comparable list for students who bring a lower level of maturity, experience and mathematics background? We distinguish between the 'task' as introduced by the teacher (which may or may not be what they envisaged when planning nor what the author or other source had in mind), the 'task' as constructed by the participant, and the activity engaged in by the participant. The Group suggested that a good game for the classroom would require the following:

- the task as constructed by students needs a hook to engage them, and that the hook should be the mathematics itself and not the game;
- the task needs to provide enough impetus that students can break through the imposed structure and get hooked on the mathematics;
- the mathematics needs to be appropriate for that age group and not too deeply imbedded in the activity (i.e., it needs pointers to the fact that mathematics may be helpful), it needs some element of surprise in the mathematics and some unexpected results that challenge intuition (accessible but bothersome);
- the activity needs to provide various levels of mathematization allowing the better students to progress beyond the average.

Perhaps it is useful to extend the notion of 'the mathematics' to include the pleasure that comes from using one's mathematical powers. Thus, a hook can be that I find myself enjoying being called upon to imagine something, to express what I see or think in some way (through gesture and movement, through drawing and displaying, through words and symbols), to make conjectures but then find I want to modify them, to try to convince others to see what I see, to get a sense of what might be going on through trying some examples and then seeing through the particulars to a generality. A task which affords access to becoming more aware of such powers, and developing and honing them, contributes to mathematical development just as well as a task which prompts the student in rehearsing and refining a mathematical technique, or appreciation of a mathematical idea or topic.

A distinction was made between 'jeopardy' type games which provide an environment for practice—the game is more the motivator, and other games where the mathematics itself motivates the students to continue to examine it. Participants felt that the 'distance' between the game and the mathematics is shorter for Brock Bugs than it is for SKUNK. There is a notion that one can at the design/deployment stage modify this 'distance'. For example in Leap Frogs, adding a quest for minimality to the stated rules decrease the 'distance' (making game-play more directly engaged in problem solving); yet suppressing such a rule keeps the structure more flexible and open. In SKUNK the 'distance' was found to be huge, which does not diminish the value of the game but seriously changes the intervention model: where low-distance games like Brock Bugs and Leap Frogs are amenable to punctuated intervention. This may be due to the sophistication of the mathematical concepts required for analyzing SKUNK.

Appendix A

Skunk

A description of this game can be found in pages 28 to 32, of the April 1994 issue of the journal "Mathematics Teaching in the Middle School". A short description follows. On the blackboard, the teacher draws five columns each headed with one of the letters of the word SKUNK, where each letter represents a different round of the game. Students play in teams of two or three and their aim is to accumulate as many points as possible over the five rounds of the game. For a team to acquire the points resulting from the sum of the values on the roll of two dice, all team members must be standing. Before each roll a team must decide

whether to stand or to sit as a group. If at any stage a team decides to sit it has to remain seated until the end of that round. When standing a team gets the total of the dice unless a one comes up, then the play is over for that round and the standing teams loose the points accumulated for that round. If a double one comes up at any time all standing teams loose the points accumulated for all the rounds up to and including the present round.
The game provides a blend of experiences in probability and decision making.

MIRA

The MIRA is a plastic two-way mirror made for classroom use. It is available commercially together with various print materials developed for teacher and student use. Geometric properties of planar objects and constructions that involve translations, rotations, angle and line bisectors, etc. can be motivated by using this apparatus/manipulative.

Hinged Mirrors

The name of this apparatus describes it fully. Eric Muller brought number of pairs of mirrors hinged at one edge. No print or other materials were supplied to the Group.

Brock Bugs

The game of Brock Bugs was developed by Eric Muller for use in the teaching and learning of probability. The game aims to provide experiences in three specific concepts, namely relative frequency and probability distributions, expectations, and the use of the binomial probability distribution. Each of these concepts are explored in one of three levels of the game which allow students to progress through the game as they develop their understanding. Separate instructions are provided for the teacher and for students. The teacher's notes include suggestions about classroom management, interaction with group of students, intervention to motivate understanding, etc. There is a game board with spots marked 1 to 14 and two teams are issued with different colour chips. To start the game the two teams take turn to place one of their chips on an unoccupied position of their choice. The positions of the chips stay fixed for 25 rolls of a pair of dice. For each roll the team whose chip is on the position corresponding to the sum on the faces of the two dice gets one point. The team with the most points after 25 rolls wins the game. Chips are now removed from the board and the game can be repeated. The objective is for students to first realise that strategies are in the appropriate placing of their chips and then to develop their optimum strategy to win the game. Level 2 is played on the same board but positions on the board carry different number of points. In other words a 7 may be worth two points while an 11 may be worth seven points. These points are shown in a square below each position. Level 3 asks students to explore reasons why the number of repetitions of the level 1 game were set at 25.

Leapfrogs

This is a classic Lucas problem that became the name of a group of mathematics teachers in the late 60s and early 70s. They met annually to work on mathematics and to design resources which required little or no introduction (more like stimuli perhaps, or phenomenon to attract attention and mathematization).

You are given a number of green frogs, and yellow frogs, lined up with greens together and the yellows together and a single space in between. Frogs can jump over another frog into a (the) vacant space, or slide into an (the) adjacent vacant space. The challenge is to interchange the green and yellow frogs and top predict the (minimum) number of moves required. Dudeney (I believe it was) extended this into two dimensions: you have two squares of frogs that overlap in just one square. This square is vacant. Again the challenge is to interchange the frogs and to predict the minimum number of moves.

Chinese Jigsaw

Nine coins are placed in a 3 by 3 array, with all but the centre coin showing a head. You are permitted to flip all the coins in a row, or all the coins in a column. The aim is to get all the coins facing the same way.

Eventually participants begin to conjecture that there are some difficulties. Perhaps if you are also permitted to flip all the coins in one or even both main diagonals? What if the two-state coins are replaced by s-state objects (so after s-flips they are back to their starting state)?

The name comes from a harder three-dimensional version, found in Chinese toy-stores. Nine cubes have been laid out in a 3 by 3 array and a picture has been pasted on the full set, then slit along the edges of the cubes to give one-ninth of the picture on one face of each cube. The cubes are rolled to display another face each and again a picture is pasted. When six pictures are pasted, so that each cube has one-ninth of each of six pictures, you have a six-fold jigsaw puzzle. Fortunately, the picture gluing is achieved by rolling all the cubes in each row in the same way (that permits four pictures) then in each column (for the remaining two pictures). In other words, once you have solved one picture, you can achieve the others by rolling all the rows, or all the columns, the same way and the same amount.

Now, suppose you displace just the centre cube by showing a different face. Can you roll the rows and or the columns and restore the picture? The non-commutativity makes analysis rather more difficult!

(*Comment*: The coin version of this task generated considerable interest. It seemed to be at an appropriate level mathematically, and various ways of thinking afford access (groups, linear algebra, combinatorics), yet no really deep theorems or techniques are required. Mainly you have to reach the conjecture that it is not possible, and then to justify why.)

Four-Bidden

Packs of cards (produced by ATM in UK) were offered. Each card has a technical term from secondary school, and four other terms which are 'forbidden'. Participants draw a card and give the others clues as to the technical term, without ever using the four forbidden terms.

Many different variants are possible such as only using the 'forbidden' terms, trying to work out the forbidden terms given the main term, using only diagrams or drawings to give clues, acting clues out in silence, team collaborations etc.

(*Comments*: This game generated considerable reaction. Some felt that students should not be restricted in how they try to express themselves, especially when the essence of the term is best expressed using some of the forbidden terms. In some cases it was not clear why certain terms were forbidden and not others.)

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Topic Sessions

Professional Narratives in Mathematics Teacher Education

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In recent years, narratives gained prominence in many educational fields, especially in teacher education and in research on teaching. In this paper I discuss why narratives may be of interest both for mathematics teachers' professional development and for researchers that investigate teachers' professional knowledge.

The narrative representation of experience

A narrative or story is a way of representing experience for oneself or for others. A narrative involves three basic elements: (i) a situation involving some conflict or difficulty, (ii) one or more agents that act on that situation with their own intentions, and (iii) a plot, that is a temporal sequence of related events in which the conflict is resolved in a certain way. It involves people, settings, and events that take place in a given time frame.

According to Bruner (1991), we organize our experience and our memory of past events in the form of stories. In this sense, we live through stories, that is, we think, perceive, imagine, and make moral choices using narrative structures. Every human being is a storyteller, perceiving reality in a narrative way. Telling stories enables us to establish order and coherence in our experience and to make sense of the events in the world around us (Carter, 1993).

Stories are imbedded in the culture. A story is a conventional way of thinking, culturally transmitted, and is as much constrained by the social and institutional context as by the capacities of each person (Bruner, 1991). Also, the culture speaks through stories, stories that are constructed around themes that yield the projection of human values (Carter, 1993; Riessman, 1993). To ask whether a story is true or false is to ask the wrong question. The issue is what the teller is trying to say, the interpretation that is being offered, the interpretation that the reader may draw of that story, and the understandings that it may lead to. The acceptance of a story is ruled by convention and by "narrative necessity" (Bruner, 1991).

Teachers' stories

Stories are one powerful way that humans use to make sense of their experiences. This includes the experiences that teachers have in their professional activity. Indeed, teachers tell each other about classroom events, about the professional meetings they attend, about their ongoing projects. They share their experiences with children, with parents, administrators, and colleagues. Professional stories, besides being a natural way of registering teachers' experience, may serve formative goals, notably for those who tell them, for other practicing teachers, and for prospective teachers.

Stories constitute a way of knowing closely related to action. According to Carter (1993), stories are ways of knowing emerging from action that attempt to explain human intentions in the context of action. Allowing for a multiplicity of meanings, they are a suitable way of expressing knowledge related to the complexity of action. As teaching is an intentional activity in a given situation, teachers' professional knowledge is inextricably related to teachers' practice. Therefore, to understand teachers' knowledge we may begin by scrutinising the stories that provide structure to their thinking about classroom events—their practical theories. However, one must do that in the understanding, that in their stories, teachers do not just recall and report their experiences. They recreate their own stories, reconstructing meanings, redefining their personal and professional self (Cortazzi, 1993).

The thinking, the perceptions and the experiences of teachers are integral elements of their culture and, therefore, the mark of the cultural contexts is present in their thinking: "what teachers tell us about their practices is, most fundamentally, a reflection of their culture, and cannot be properly understood without reference to that culture which is interpersonal" (Olson, 1988, p. 169). Stories select in a special way the richness, the *nuances* of meaning, the ambiguities and the contradictions in human affairs, contrarily to paradigmatic or scientific thinking, that requires consistency and absence of contradiction (Bruner, 1991). They have a strong ability to represent life and promote linkages between it and the educational experiences. Stories are ways of capturing the complexity, specificity, and internal and external connections of phenomena, overcoming the limitations of atomist and positivist approaches. They are, therefore, a way of knowing and thinking particularly apt to deal with the issues that we face in educational research (Carter, 1993).

Stories may fulfill many roles in teacher education and professional development. Of course, teachers' stories are a fundamental part of the profession's heritage and constitute rich resources for teacher education activities. But the most important role of stories in professional development involves teachers constructing their own narratives about past experiences. This autobiographical work may focus on single events or classroom episodes or reflect the complex movements of an extended career. An autobiography is not just a description of a professional career—it is a means by which teachers may become aware of key issues that may lead to substantial changes in their professional activity.

Although much less than in other educational fields, narratives are present in some mathematics education work. For example, in a theoretical paper, Burton (1997), discusses how to combine an emphasis of individual agency of constructivism with the prevalence of external authorship in sociocultural theories, proposing to regard mathematics learning as a narrative process in which learners have agentic control in authoring. Mostly concerned with methodological issues, Love (1994) noted that teachers' accounts about their practices, in interview settings, should be regarded as narratives and, therefore, the analysis of such accounts should take in consideration the specific features of narratives. Chapman (1999) uses the process of storying and restorying to help preservice mathematics teachers to reflect on their thinking and actions, in relation to mathematics and mathematics teaching and learning, aiming to broaden their understanding of new curriculum orientations. And Shifter and Simon (1996), used teachers' narratives as starting points to explore issues related to the teaching of particular topics, to enacting new teaching approaches, and to the challenges posed by curriculum reforms to teachers' professional identities.

Constructing narratives

In education, stories can take myriads of forms. For example, teachers' anecdotes are simple episodes teachers tell each other about classroom events or other events such as their own history as learners. A personal history is an extensive account of first-hand experiences of learning and of being in a school. An autobiography is a reconstruction that involves a conscious and reflexive elaboration of much of the author's life, including personal and professional experiences. A collaborative biography is the joint description and interpretation of a teacher's life experience carried out by the teacher working together with a researcher (Cortazzi, 1993). Narrative inquiry, as carried out by Connelly and Clandinin (1990), is the process of making meaning of personal experience through collaborative storytelling.

A narrative always involves a narrator that produces a text, in oral or written form. A teacher may produce it on his or her initiative (such in autobiography), but more often it is a joint production of a teacher and a researcher. Such production may come about in a rather standard interview setting, through open-ended questioning, in a more informal reflection on past experiences, or as a deliberate construction in sessions devoted to "narrative inquiry". In a joint production, there is a range of possible roles of both participants—narratives may be just produced orally by a teacher and them passed into writing by a researcher or may evolve in oral and written steps as a collaborative activity.

The sources of narratives may be unstructured interviews, journal entries, field notes of shared experience, etc. Many stories are first expressed orally and then in a written way. The construction of a narrative in educational research involves several steps. Riessman (1993) describes them in the following way: (i) attending, that is, living the experience; (ii) telling the experience; (iii) transcribing the experience, (iv) analysing the experience, which implies the elaboration of a new text (usually written); and (v) reading, involving a new recounting. Steps (i) and (ii) must be undertaken by the person that lived the experience; steps (iii) and (iv) may be carried out just by the researcher or by the teacher and the researcher, as a joint production; and step (v) involves all possible audiences of the narrative.

For Riessman (1993), these steps are different levels of representation of an experience. First, one must note that there is an inescapable gap between the lived experience and the telling and writing that is done about it. Telling an experience also implies the creation of an identity—a way how one wants to be known by the others—as every narrative is inescapably an auto-representation. Transcribing is (as the other levels of representation) necessarily incomplete, partial and selective—it is an interpretative action as much as it is photographing reality. Decisions about transcribing, as well as about speaking and listening are guided by theory and rhetoric rules. Analysing implies to select, emphasize, relate and compare. As in any research activity this is a most critical step in the creative activity of research. Such analysis should not pervert the voice and the meaning of professional practices, but enrich and clarify them as it draws on further experiences and perspectives. And, in its final form, the narrative is still open to different readings and interpretations as the meaning of a text is always a meaning for someone. A story, once told (orally or in written form), does not belong anymore to the narrator. It gets an existence independent of his or her will, intentions or interpretations and becomes the property of all of the educational community (Clandinin and Connelly, 1991).

Narratives carry a strong cultural and historical load. The "truths" that we construct are meaningful for specific interpretative communities in well-defined historical circumstances. As Riessman (1993) underlines, each level of her model involves a reduction, but also an expansion—each teller selects to narrate the aspects of his or her experience and adds other interpretative elements.

There are several ways of analysing narratives. One of the most used models in education was designed by Labov (see Cortazzi, 1993; Riessman, 1993), who proposes that a narrative is made up of six fundamental parts: (i) abstract, with the summary of the substance of the story; (ii) orientation, providing information about place, time, context, participants; (iii) complicating action, that is, the sequence of events; (iv) evaluation, indicating the meaning of the action for the narrator; (v) resolution, stating what finally happened; and (vi) *coda*, through which one returns to the perspective of the present. Next, I use this model to analyse a teacher' story.

A professional story

The following story—entitled *The glory of knowing how to use a calculator*—was written by Maria João Simões, a teacher is a secondary school[1] in Lisbon, Portugal, as part of an activity in an in-service teacher course held in 1996. It was included in a book containing several stories constructed in the same way, published by the Association of Teachers of Mathematics (Ponte, Costa, Lopes, Moreirinha, and Salvado, 1997).

> This year I have two tenth-grade classes and they're very different from each other. In class B almost half the pupils have already bought a graphic calculator, while in class C there isn't a single one—and this has nothing to do with economical differences…
>
> I've had a graphic calculator for almost two years, but I must admit that I hardly know

how to do anything on it because I have dedicated very little time to become acquainted with it. In one of the first classes on quadratic functions, I decided to take my calculator (the school doesn't have one!) to show the pupils in class C what happens when we changed the coefficients. But I don't know what I did, you couldn't see anything! This was right at the end of the class and next I was going to have a class with the ninth grade. But I didn't even leave the classroom because a load of tenth-grade B pupils came in. Among them was André, of whom I shall talk in particular.

André has a hearing impairment, as recognized by the Decree 319/91. He is a weak pupil who had a 10 in the first period[2], not because that was "his" grade, but because he is "different", so he has a right to a different assessment. He is devoted to the graphic calculator, which he took just one day to decipher. As he puts it: "Teacher, pressing these buttons is how you learn!"

So this group, with André at the lead, came barging in, filled with self-confidence: "Hey teacher, what's the problem? Can't you handle this?" And they picked up the calculator. Meanwhile I'd already understood my mistake—the values of the axes' scale weren't very good. But I resisted and didn't steal their glory. They did everything, got the graphics "working" and said goodbye: "If you need us we're here!"…

André is sitting just in front of me (naturally) and he has a rather condescending attitude (which really amuses me) whenever I need to use the calculator. He gives me advice discreetly or else when he constructs something he thinks is interesting, he calls me and explains what he did in detail. This happens often for sometimes I have to take his calculator away so that he pays attention to the class.

What happened in the last test, which was without calculators, got me thinking. André didn't pass but everything concerning function graphics he got right.

This episode shows that we do learn from pupils—and not just indirectly, as we usually think. But do we make this clear enough for them? André's pride when he teaches me something shows how important this is to him; probably it's just as important to the others… And they don't respect me less for it!

Drawing on Labov's model of analysis, we can see that this short teacher story is filled with different complications. At a first sight it is just a story about a teacher who was not quite well prepared to use the calculator in the classroom. Dealing with something unexpected, arising from a new instructional material or from any other source, is a common situation that teachers face in the classroom. But, on a second level, this appears to be much more a story about a teacher with a special pupil who has an officially recognized handicap and is weak in mathematics, but likes to show off in front of his colleagues. On a third level, one sees the complication regarding the questions that are puzzling this teacher: why a pupil who is able to explore so well with the calculator and make good use of it keeps getting poor marks in a test?

In this story, not all three complications get a resolution. This teacher could easily fix the calculator problem and did not have trouble in finding a strategy to deal with André. Much more difficult—in fact unsolved—was finding a way to make this pupil have success in mathematics. It is quite apparent that the teacher makes a positive evaluation regarding the way she relates with André, but she is much more ambivalent regarding wider issues in her practice.

There are several issues in this story. One concerns the relationship of the teacher with the calculator, at a quite basic level of operating with it. Another is the relationship of the teacher with a pupil that is behaving just borderline regarding what may be the teacher's tolerance for outspokenness. Comments such as those André is making may be acceptable for some teachers but not for others, and they may be acceptable once in a while but not constantly. A third issue concerns what is wrong with mathematics teaching and assessment that leads this pupil to fail when he shows interest and ability for mathematics. Still a fourth issue concerns the graphic calculator. If it must be used as a tool for experimentation and exploration, why does it not lead this pupil to a better achievement?

In pre-service and in-service teacher education stories such as this provide good starting points to discuss issues faced by a teacher in making curriculum decisions and conducting classroom instruction. Also, they may be the starting point for participants to tell and/or write their own stories dealing with related issues. In the work carried out in our research group, stories have also been produced to study teacher's knowledge in innovative teaching practices, such as dealing with mathematical investigations in the classroom (Ponte, Oliveira, Cunha, and Segurado, 1998).

Conclusion

This paper argued that narratives are a powerful tool for professional development and a useful research methodology for those interested in the study of teachers and teaching, including the teachers themselves. The story of Maria João was made in an inservice course—providing this teacher an opportunity to reflect on several issues concerning her practice—and has been used as a basis of discussion with preservice teachers. Stories such as this illustrate several aspects of mathematics teachers' professional knowledge and may be used in research. However, the use of stories in teacher education and research raises a number of issues. Questions regarding the quality, the value, and the ethics of work in narratives have to be addressed. We need to pay attention to the desirable and undesirable features of professional narratives and adequate and non-adequate ways of constructing and reporting them. We also need to find strategies to encourage teachers and mathematics educators alike to write professional narratives and to share them within their professional communities.

Notes

1. Escola Secundária do Lumiar Nº 1.
2. 10, in the grading scale 0-20, in use in Portugal at this school level, is a "just pass" mark.

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From Kindermath to Preservice Education: Some Connections with Content and Student Responses

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The Kindermath program was designed in response to requests of parents who wanted some enrichment opportunities for their elementary children. Its creation pre-dated the new Ontario elementary curriculum, so initially it seemed to be quite unrelated to the curriculum children were experiencing in school. However, with the publication of the new curriculum in 1997 Kindermath became quite relevant to the new curriculum especially in the areas of patterning, geometry and spatial reasoning.

The format for Kindermath is small group after-school or evening sessions, for children aged 6 to 10. Children work for 45 minutes together or individually, and then are joined by a parent for a 15-minute "sharing time" at the end. During sharing time children either explain their work from that day, or work on an activity with the parent. The take home project is also explained at this time. The idea of sharing time is to connect the Kindermath class work with things parents can do at home to extend the ideas.

One thing that is different about Kindermath from school is that it is a voluntary program that parents are paying for, and as such competes with a huge myriad of other after school activities. Thus it must be fun and interesting.

While the idea of Kindermath is not based on any existing programs, the inspiration of the activities themselves reads like a "who's who" of CMESG. For example, Kindermath activities include a version of Ralph Mason's cube figures, Elaine Simmt's fractal cards, Brent Davis's wool and cardboard factor circle, and Bernard Hodgson's kaleidoscope work. For further details of the activities, see Kajander (forthcoming).

Activities were chosen for Kindermath based on the interest level they were deemed capable of generating, as well as their ability to convey something of the excitement and aesthetic quality of mathematics often felt by mathematicians, and often missed in school. Elements of discovery and creativity are also included wherever possible. For example, children cut out paper pieces to verify a visual version of the Pythagorean theorem, and then cut out cubic boxes from cardboard and fill them with cereal to decide if (visually) the contents of the two boxes on the legs equal the contents of the cube on the hypotenuse. This leads to an informal discussion of proof, in the form of "how can we know if it will always (not always) be true". At each of the three levels of Kindermath, activities are provided to include the following list of criteria:

- math can be fun and engaging;
- we can 'play' with math;
- mathematical patterns can be beautiful;
- there is awe, mystery in math;
- child's own creativity is encouraged;
- mathematical communication;
- mathematical collaboration;
- mathematical reasoning;
- infinity, limits;
- 'big picture' mathematical ideas;

- how can we know something is true.

As there is no formal evaluation in Kindermath, children are freer to take risks than in traditional school. This also makes the program more difficult to evaluate in a formal way.

Plenty of anecdotal evidence exists however to support the benefits of the program. When we ask the question "Does it 'work'?" of Kindermath, we have to ask "work to do what?" Here are some possible criteria to consider:

- Do kids like it?
- Does it appear to effect attitudes longer term?
- Does it help kids to see the 'big picture'?
- Does it promote the learning of technical skills?

It seems to me kids like it a lot. No child has ever left Kindermath due to lack of interest (one left due to baby sitting difficulties, and one due to a divorce/custody situation). As one eight year old put it somewhat to my dismay, "it's a lot more fun than real [he meant school] math".

As to the second question, long term data is not really available, but I do regularly run into a high school girl at mathematics contest sessions who tells me she is going to be a mathematician because of Kindermath.

While Kindermath has never attempted to teach technical skills, there have been students whose attitudes to mathematics were changed so noticeably that their grades in school went up dramatically. For example, a student who joined us in grade 5 was, according to her teacher, "failing" school math. After some very successful experiences in Kindermath with difficult problems such as the 'handshakes' problem, which she subsequently presented to her teacher at school, her self esteem and interest levels rose remarkably, according to her mother. She received A's in grade 8 and had a motivation level that increased a great deal after her Kindermath experience.

Other parents have reported that they refer back to Kindermath activities, such as creating and using a balance, when learning more sophisticated ideas such as solving equations (as was suggested to them at the time).

If Kindermath is successful with children, a natural extension is to ask what is its role in teacher education? How do preservice teachers respond to learning math in a class that makes strong use of similar activities?

Preliminary data seems positive. Out of 114 students starting a mathematics course for elementary teachers in 1999–2000 at Lakehead University,

- 24% said they had a positive attitude to math;
- 20% said they found math difficult but were willing to work at it;
- 46% said they were very nervous or unhappy to be taking a math course and used words like "terrified" or "dread";
- 10% did not respond or answered with something unrelated (i.e., 66% had some form of negative attitude about mathematics initially and nearly half a very strong negative attitude).

At the end of the course, 89% of the students reported a noticeable change in attitude for the better. Many made comments like "it feels like the first time I've taken math".

For example, a student who professed at the beginning of the year to have "barely passed grade 10 general math" came up with a general solution to the sum of n integers problem (the 'staircase problem') with tiles, which did not require two different solutions for odd and even numbers of stairs (as most student solutions do).

Another student, a native woman who could not even work with fractions at the beginning of the course and was terrified to the point of tears, was working with complicated fractal patterns on the computer (among other things) by the end. She felt the visual hands-on approach was particularly effective with the Native children she worked with on the reserves.

The relationship between directed 'play' as in the Kindermath experience and 'inquiry' as in the sense of the new secondary curriculum in many provinces is still uncertain.

Can enough mathematical 'play' early on create a mindset in which 'inquiry' is more easily attainable? Other questions also remain unanswered. What about unmotivated learners? What is 'different' about Kindermath that could possibly be continued at higher levels?

Some characteristics are that Kindermath is fun, students work in one small group with one teacher, an emphasis is placed on personal creativity and students' own ideas with less emphasis on 'answers' and none on grades. Clearly, some of these criteria are not possible in school, but perhaps some are.

I would like to conclude with some questions and observations related to applying these ideas to school mathematics classrooms, both elementary and secondary.

'Play' in the sense of Kindermath activities may be restricted or unavailable the majority of the time in school, but I find that even secondary kids still love it. Does a background in 'play' make 'inquiry' more comfortable? What exactly is inquiry anyway? I find that 'conjecturing' is not an easy concept for either elementary or secondary students – would more of a background in exploration help with this? How much support/scaffolding is needed/ideal?

What might help the inquiry process at the secondary level? Here is a list of suggestions:

- a background in 'play', giving students a positive attitude and a willingness (if not the skills) to 'mess around';
- a reduced list of skills to be learned;
- (fewer) small groups;
- more curricular freedom to get 'sidetracked';
- environments with a little more fun, imagination, visualization, and creativity … and magic.

I look forward to future sessions of CMESG to help to provide more discussion on these important issues!

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Mathematics and Literature I: Cross Fertilization

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When one thinks of Mathematics and Literature one often evokes the image of *Alice in Wonderland* and the logic games and paradoxes that permeate Alice's voyage. It is indisputable that Lewis Carroll's use of mathematics in his novels was profoundly original and groundbreaking. However mathematics did not play a determining role in the plot or creation of the work, nor can it be said that *Alice in Wonderland*, or most other novels containing mathematics, have in any way contributed to mathematics research. Indeed, the relationship between mathematics and literature is most often seen as a one way relationship, literature using mathematical ideas a symbolism, and not a profoundly deep one at that. In this article I will tell the story of deep cross-fertilization between literature and mathematics, both directions: research level mathematics guiding novel creation at the profoundest levels and literature spurring active and important research.

1. Leonhard Euler, Oulipo and Georges Perec

The story begins with a game, very like what might be found in *Alice in Wonderland*. Leonhard Euler asked the following question in 1782:

> *Thirty-six officers are given, belonging to six regiments and holding six ranks (so that each combination of rank and regiment corresponds to just one officer). Can the officers be paraded in a 6 × 6 array so that, in any line (row or column) of the array, each regiment and each rank occurs precisely once?* [4]

Smaller versions of this problem are fun to try with a deck of cards. From the deck of card remove the 1,2 and 3 of three suits and try to arrange them in a 3 × 3 square such that every row and column has each suit and each number just once (also try with the 1,2,3 and 4 of all suits and if you have a deck of cards for playing the French card game *Tarot* then you can also try with the 1, 2, 3, 4 and 5 of the five suits). Since he represented the regiments with Greek letters and the ranks with Latin letters these squares became known as *Græco-Latin bi-squares*.

Euler believed that the answer to his question was "no", and that no $n \times n$ Græco-Latin bi-square existed when $n \equiv 24$. There is a simple construction for a $q \times q$ Græ co-Latin bi-square where q is a prime power: Index the rows and columns by the elements of the Field of order q and the "officer" in cell (i,j) is in "regiment" $I + j$ and has "rank" $I + 2j$. MacNeish proved that if a solution exists for an $n \times n$ square and an $m \times m$ square that a solution can be constructed for a $nm \times nm$ square [3]. The prime power field construction and the 4 × 4 solution that you found with your deck of cards now confirm that a solution exists for all $n \times n$ Græco-Latin bi-squares when $n \equiv 24$. In addition to this, in 1900, G. Tarry, a math teacher in a girl's schools was able to prove in 1900 that in fact Euler had been correct for the 6_6 square; it is not possible. He did this by dividing the problem into many cases and then given the checking of each case to his students as homework. This is possibly the first instance of distributed computing to solve a computationally hard problem, predating the Internet by 90 years!

For the first half of the century it looked as if all the evidence was supporting Euler's conjecture until 1960 when Bose, Shrikhande and Parker used Wilson's construction to prove that Græco-Latin bi-squares exist for all n except $n = 2$ and 6. In particular they exhibited a 10 × 10 Græco-Latin bi-square [5].

00	47	18	76	29	93	85	34	62	51
86	11	57	28	70	39	94	45	02	63
95	80	22	67	38	71	49	56	13	04
59	96	81	33	07	48	72	60	24	15
73	69	90	82	44	17	58	01	35	26
68	74	09	91	83	55	27	12	46	30
37	08	75	19	92	84	66	23	50	41
14	25	36	40	51	62	03	77	88	99
21	32	43	54	65	06	10	89	97	78
42	53	64	05	16	20	31	98	79	87

FIGURE 1: A 10 × 10 Græco-Latin bi-square.

This was such an unexpected and exciting result that a color format of the solution was on the cover of *Scientific American*.

At this same time, in France, the mathematician François Le Lionnais and the writer Raymond Queneau were founding the *Ouvroir de Littérature Potentielle* (Oulipo), or Workshop for potential Literature which focused on the the question "what are the possibilities of incorporating mathematical structures in literary works? " and eventually included in it's scope all writing that was "subjected to severely restrictive methods" [8]. Several years after the publication of the 10 × 10 square, three members, George Perec, Claude Berge and Jaques Roubaud, devised a method of applying Græco-Latin squares to literature. George Perec describes the method in simple terms:

> *Imagine a story 3 chapters long involving 3 characters named Jones, Smith, and Wolkowski. Supply the 3 individuals with 2 sets of attributes: first headgear – a cap (C), a bowler hat (H), and a beret (B); second, something handheld – a dog (D), a suitcase (S), and a bouquet of roses (R). Assume the problem to be that of telling a story in which these 6 items will be ascribed to the 3 characters in turn without their ever having the same 2. The following formula:*

	Jones	Smith	Wolkowski
chapter 1	CS	BR	HD
chapter 2	BD	HS	CR
chapter 3	HR	CD	BS

> *– which is nothing more than a very simple 3 × 3 Græco-Latin bi-square – provides the solution. [8]*

The plot of the story can simply be read off the table: In chapter one Jones is wearing a cap and holding a suitcase, and so on. In *Life, a User's Manual* [9], Perec did the same thing using 22 10 × 10 Græco-Latin bi-squares corresponding to such things as who to plagiarize in the chapter, fabric types, furniture, shapes, and even instructions to remove some of the plot details generated by some of the other bi-squares! Additionally the chapters are in the order of a re-entrant chess knights path on top of the square. He permuted the rows and columns of the 10 × 10 Græco-Latin bi-squares in a way which he did not reveal and so we are still in the dark about all the sets of attributes that he did use. It would be a wonderful graduate project in either mathematics or literature to do the detective work and reverse-engineering required to deduce the squares and attributes that he used! Perec is also well known for the longest Palindrome ever written, 5000 letters and for having written a novel entirely without the letter **e**.

2. Dante's Purgatorio, Beckett and Gray-Codes.

In the previous section, I gave one direction of influence: from math to literature by telling the story of Euler's 36 Officers problem, the history of its development and its impact on the creation of Georges Perec's Novel *Life, a User's Manual*. This gives a potent example of math can be applied to literary creation at a deep and structural level. In this section I want to provide you with an example of the influence proceeding in the other direction: Literature motivating important and difficult mathematical questions.

One of the most influential religious works of all time is Dante's *Divine Trilogy* documenting Virgil and Dante's travels through Hell, Purgatory and Paradise. This work has influenced many modern writers, most notable T.S. Elliot, Ezra Pound, James Joyce and Samuel Beckett. One aspect of the religious symbolism in the *Divine Trilogy* is the meaning of movement. Individuals move in very different ways depending on whether they are damned or saved. As John Freccero discusses,

> the directions given by Dante on the purely literal level are entirely consistent and that in imitating the further traditional pattern of "descent" before "ascent" the pilgrim's leftward journey spiraling clockwise down through Hell is, with respect to "absolute up" (the Southern Hemisphere), actually a movement upward and to the right and continues after the turn-around (conversion) at the earth's center, in the same absolute spiral direction to the right up the Mount of Purgatory. [7]

The connection between Dante's symbolism of movement and Samuel Beckett has been made in regards to his late play *Quad* [2]. In *Quad* there are four actors, 1, 2, 3, and 4. They traverse a square and follow the following paths:

actor 1	AC	CB	BA	AD	DB	BC	CD	DA
actor 2	BA	AD	DB	BC	CD	DA	AC	CB
actor 3	CD	DA	AC	CB	BA	AD	DB	BC
actor 4	DB	BC	CD	DA	AC	CB	BA	AD

The figures then follow the following schedule of who is on stage ...

1	13	134	1342	342	42
2	21	214	2143	143	43
3	32	321	3214	214	14
4	43	432	4321	321	21

... where the play then begins to repeat. Each actor traverses the square once in his prescribed pattern of sides and note that the actors leave stage in the same order that they arrived and thus it is always the actor who has been on the longest who leaves, although any actor my enter.

The figures are always turning left, moving in an anti-clockwise direction. But they arrive at the sides in a clockwise order. Thus they can be seen to be moving both clockwise and counter-clockwise at the same time. This movement in both directions simultaneously was first pointed out a possible connection to Dante by Antoni Libera. "Anti-clockwise and clockwise are the directions moved by the inhabitants of the inferno and purgatory respectively in Dante's Divine Comedy, to signify movement away from and towards God, or, to put in another way, away from or towards freedom"[6] Thus the characters in this play at once move towards and away from freedom.

This can be viewed in a more combinatorial way. We equate freedom with achieving all the combinations of a finite system, as Beckett does in the following humorous passages from his novel *Murphy*:

> He took the biscuits carefully out of the packet and laid them face upward on the grass, in order as he felt of edibility. They were the same as always, a Ginger, an Osborne, a Digestive, a Petit Beurre and one anonymous. He always ate the first-named last, because he liked it the best and the anonymous first, because he thought it very likely the least palatable. The order in which he ate the remaining three was indifferent to him and varied

> irregularly from day to day. On his knees now before the five it struck him for the first time that these prepossessions reduced to a paltry six the number of ways in which he could make this meal. But this was to violate the very essence of assortment, this was red permanganate on the Rima of variety. Even if he conquered his prejudice against the anonymous, still there would only be twenty-four ways in which the biscuits could be eaten. But were he to take the final step and overcome his infatuation with the ginger, then the assortment would spring to life before him, dancing the radiant measure of its total permutability, edible in a hundred and twenty ways! Overcome by these perspectives Murphy fell forward on his face in the grass, beside those biscuits of which it could be said as truly as of the stars, that one differed from another, but of which he could not partake in their fullness until he had learnt not to prefer any one to any other. [1]

We similarly equate the restrictiveness of the "first-in first-out" nature of the actors' schedule as the opposite of freedom then the characters of *Quad* move "away from and towards freedom" both combinatorially and geometrically.

This raises and interesting mathematical question: When can all subsets of an *n*-set be arranged in a circular list such that each one appears just once, two adjacent subsets differ by the inclusion or removal of just one element and the only element that may be removed is the one which has been in the most previous consecutive subsets in the list. Such an object is a type of Gray-Code, a powerful mathematical representation of objects in a circular list such that consecutive members differ in only small ways [4]. Gary-Codes are extremely useful for Digital to Analogue conversion as well as efficient storage and generation of lists on computer. Usually they do not require the "first-in first-out" restriction, but such a restriction always implies that there is a two times cost saving for storing and generating the lists! We call such lists *Beckett-Gray codes* and their power for efficient storage makes them a potent new area of research.

Unfortunately it seems very difficult to find Beckett-Gray Codes. I have been able to prove that for sets of 3 and 4 actors it is impossible to have each combination appear exactly once (notice that Beckett repeats some combinations) but for sets of 5 and 6 actors such a schedule is possible.

∅	0	0 1	1	1 2	1 2 0	2 0	2 0 3
0 3	3	3 1	3 1 0	3 1 0 4	1 0 4	1 0 4 2	0 4 2
4 2	2	2 3	2 3 4	3 4	3 4 1	4 1	4 1 2
4 1 2 3	1 2 3	1 2 3 0	1 2 3 0 4	2 3 0 4	3 0 4	0 4	4

FIGURE 2: A Beckett-Gray Code for a set of 5 actors.

The existence is unknown for all larger sets!

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Elementary Number Theory: (Some Issues in) Research And Pedagogy

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Prelude

> *Mrs. Times went grocery shopping (in a tax-included outlet). She bought 6 bottles of coke for $1.75 each, 5 kg of apples for $2.40/kg and some lemons for $0.60 each. The cashier counted the lemons and said, "$27.50 please."*
>
> *"There must be a mistake," replied Mrs. Times.*
>
> *How did she know?*

The majority of people attempting to solve this problem calculate the totals for coke and apples (10.50 and 12 respectively), subtract these amounts from 27.50, to get the amount to be paid for lemons (5 dollars), divide this by the price of each lemon, (after my assurance that there are no "special deals", like buy 5 get the sixth one free), that is, divide 5 by 0.6 to get the number of lemons, get 8.333333333... on the calculator display and agree with Mrs. Times, understanding the convention that lemons are sold in wholes.

But Mrs. Times didn't have a calculator and, even if the above calculation were to be performed mentally, she had no time to engage in it. How could she know *immediately* that the price was wrong? A possible approach, that provokes immediate response, is to attend to divisibility of numbers. Considering the numbers representing the price paid in cents (in order to avoid decimal points), the price of a coke is divisible by 3, as there are 6 bottles, the price of apples is divisible by 3, as 240 is, and the price of the lemons is also divisible by 3 , as 60 is. In this case, the total, calculated as the sum of 3 numbers divisible by 3, must be divisible by 3. But 2750 is not divisible by 3, which makes Mr. Times (and us) believe that there must be a mistake. Similar argument can be made with respect to divisibility by 6.

The purpose of this problem is not to convince students that number theory can help in grocery shopping, but to increase appreciation of the beauty of an argument derived by considering some properties of numbers.

Introduction

"Mathematics is the Queen of Sciences and the Theory of Numbers is the Queen of Mathematics"—suggested Carl Friedrick Gauss (1777–1855). However, he continued, "The enchanting beauties of this sublime study are revealed in their full charm only to those who have the courage to pursue it."

Thinking of a queen, provokes several associations. She could be beautiful, elegant and feminine. She could be strong, powerful and dominant. She could also be perceived as unnecessary remains from the past, having no useful role.

Number Theory is not getting attention in school curriculum that a queen deserves. Students spend weeks on "factoring", but many of them do not understand the meaning of the concept of a factor. They spend weeks dividing polynomials, but many of them do not understand the idea of divisibility.

When I started to teach preservice elementary school teachers, I was surprised with their level of understanding of concepts related to introductory Number Theory. I read that their mathematical background wasn't strong. I read about their specific struggles with some specific concepts, but not about concepts related to Number Theory. Today, 8 year after I started the research, I believe I have some results and some insights. My main result is that we still know very little. Different theoretical perspective may phrase this in different ways, but the core doesn't change: Students' understanding of / learning of / conceptual schemas (field) of / constructing objects of / web of knowledge in / ... elementary number theory is a fascinating area of research.

Participants in my research are preservice elementary school teachers. The context is the course "Principles of Mathematics for Teachers", which is a core course for teaching certification at the elementary level. I have explored students understanding of and connections among specific concepts and topics, such as factors, divisors and multiples, greatest common divisor and least common multiple, prime decomposition and the Fundamental Theorem of Arithmetic, divisibility and divisibility "rules". In what follows are examples of questions used in my research to investigate students' understanding of specific concepts and relations. Furthermore there are examples of tasks used in my classrooms to help students strengthen their understanding of concepts and relations underlying elementary Number Theory.

What do we "see" in different representations of numbers?

Consider for example the following list of numbers:

$$a = 216^2$$
$$b = 36^3$$
$$c = 3 \times 15552$$
$$d = 5 \times 7 \times 31 \times 43 + 1,$$
$$e = 12 \times 3000 + 12 \times 888$$

What do we know about each one of the numbers from its representation?

Without performing any calculations, we "see" that a is a square number, b is a cube, c is divisible by 3 as well as by 15552, d leaves a remainder of 1 in division by 5, 7, 31 and 43. We "see" that e is divisible by 12, and if we look harder, we notice it is divisible also by 3888.

What we may not "see" is that $a = b = c = d = e = 46{,}656$. Following Mason (1998), we say that each representation shifts our attention to different properties of the number. For example, the property of divisibility by 3 is transparent in c, is derivable from a, b and e, but cannot be determined from d without further calculations; the property of being a cube number is explicit in b, but not in other representations of the number.

Research shows that students do not attend to different representations, they have a strong tendency to calculate the number in order to make any claims about its features and do not take advantage of what is offered by different representations. Helping students "see" what is there to be seen is one of my instructional goals.

Divisibility and Prime Decomposition

> Consider the number M, $M = 3^3 \times 5^2 \times 7$
> Is M divisible by 7?
> Is M divisible by 5? by 2? by 11? by 15? by 81? by 14?

This task proved rich in investigating the relationship between divisibility, division and prime decomposition. The question was chosen to assure success, that is, correct answer, for all of the participants. However, the chosen approach rather that the correct answer illuminated student's thinking about divisibility. The prime decomposition of M clearly indicates the divisibility of M by some primes, where the divisibility (or lack of it) by other prime and composite numbers can be derived by considering the numbers in the prime decomposition.

However, the research showed that significant number of participants do not relate to $3^3 \times 5^2 \times 7$ as a number. They need to calculate the value of M in order to discuss its properties, only after finding out that $M = 4725$ and dividing it by 7, they can conclude that M is divisible by 7. Even students who recognize that divisibility of M by 7 is assured by the prime decomposition, often do not take advantage of the prime decomposition when considering divisibility by other numbers. This is exemplified in the following excerpt:

Interviewer: *Is* M *divisible by 7?*
Bob: *Yes, it is.*
Int.: *And would you explain why?*
Bob: *Well if 7 (pause), let's see (laugh),* M *is, or let's see, so 7 is a factor of* M*, therefore, it's divisible by* M*, pardon me, by 7.*
Int.: *And how about 5?*
Bob: *5 is also a factor of* M.
Int.: *Okay, and would* M *be divisible by 2?*
Bob: *No, it would not, since 2 is um, (pause) since 2 is not seen here, it's not a factor of* M.
Int.: *Hmm, okay, and why do you feel that that's the case?*
Bob: *Um, explain this clearly (pause), since 2 is not one of the numbers that's being multiplied, the product therefore, can't be divided by 2.*
[...]
Int.: *Okay. How about 15? Is* M *divisible by 15?*
Bob: [pause] *Um, well since there's 5, 5^2 in this problem, we know that the, that the units digit will be 5, now 15 obviously has a 5 in it as well, therefore quite possibly 15 will go into* M*, and once again I'd have to solve for that.*
[...]
Int.: *Would you think that* M *is divisible by 81?*
Bob: *I'd want to find out what* M *would be, I guess that's the, the best thing, that's what I'd prefer.*

This excerpt shows that Bob's strategy changes when moving consideration from prime to composite factors of M. For prime factors Bob justifies his conclusion of divisibility of M by 5 and indivisibility of M by 7 by considering whether or not those numbers are "seen" in M's prime decomposition. However, when the question is posed about composite numbers, Bob prefers to "find out what M would be." Bob clearly "sees" several factors, but he didn't take further advantage to manipulate what is "seen" in order to derive further conclusions.

Interviewer: *OK. And will it (*M*) be divisible by 2?*
Pat: *I would multiply each one and find out what the total number is. So 3* x *3 is 9* x *3 is 27, and this 25 is* x *7, (pause) it's not, 2 doesn't go into it evenly.*
Int.: *So you computed the number and you got 4,725, and now you are sure that it is not divisible by 2.*
Pat: *Right.*
Int.: *But you were able to conclude about divisibility by 7 before you knew what was the number …*
Pat: *Um hmm.*
Int.: *So how is it?*
Pat: *Because 7 is a factor of it, so it's, what is it, the commutative law or associative law—7 is a factor of it …*
Int.: *And what about divisibility of* M *by 11?*
Pat: *I would divide 4725 by 11 to find out.*

In this except Pat chooses different approaches to draw conclusions about factors and non-factors of M. She was able to conclude that M was divisible by 7 "just by looking" at the number's prime decomposition. However, "looking" didn't help Pat to derive indivisibility of M by 2 or by 11, she had to perform a calculation to make up her mind.

Why, one may ask, are divisibility and indivisibility treated differently? The following claims are similar in the structure of the argument:

(a) 7 is a prime factor of M, therefore M is divisible by 7;
(b) 11 is not a prime factor of M, therefore M is not divisible by 11.

Why, if so, claim (a) is easy for students and claim (b) is difficult? The answer is, that (b) takes into account the Fundamental Theorem of Arithmetic, this issue is addressed in the next section.

Uniqueness of Prime Decomposition

The Fundamental Theorem of Arithmetic claims that decomposition of a composite number into its prime factors *exists* and is *unique*, except for the order in which the prime factors appear in the product. While existence of such decomposition is intuitively taken for granted, its uniqueness presents a challenge. This uniqueness was not recognized by Pat (see excerpts in the previous section). She clearly understood that *M* was divisible by all the primes listed in its prime decomposition. However, she implicitly believed that other primes could be listed in some other way of representing *M* as a product of primes.

In order to investigate further understanding of uniqueness, the following question has been posed:

> K = 16199 = 97 x 167 , where 97, 167 are both primes.
> Is K divisible by 13?

The numbers were carefully chosen such that occasional consideration of last digits did not provide a hint. Majority of participants preferred to answer this question using a calculator. When asked whether it was possible to draw a conclusion without performing division, with or without calculator, a common claim was that it would be "quite possible" for *K* to be divisible by 13, since *K* ended with 9. Some students claimed that 13 was not a factor of 97 or 197 because those were prime. No one referred to prime decomposition and its uniqueness, despite the fact that all the participants could exemplify the uniqueness of prime decomposition by building different factor trees of the same "large" composite number, and ending up with the same list of primes.

Divisibility and Parity

Is 391 divisible by 46?

Acknowledging that 391 = 23 x 17, uniqueness of prime decomposition could be an argument in this question as well. However, a parity argument—an odd number cannot be divisible by an even number—is a simpler one in this case. A variety of other arguments were preferred by participants. In what follows some of these arguments are presented in order of increasing sophistication.

Interviewer: *391, is it divisible by 46?*
Armin: [pause] *Um, I guess I'd just have to guess out of the blue, I would say, no, but, I mean, I would never trust my own opinion, I always have to work it out just to see.* [laugh]

Int.: *Would 391 be divisible by 46?*
Anita: *Yes.*
Int.: *And why so?*
Anita: *Oh, maybe not.*
Int.: I*'m, I'm interested in both of those things that just happened to you. I'm interested in the 'yes' and the 'maybe not'.*
Anita: *Well, first I said yes because I thought 46 is, well 23 is a factor, is a factor of 46, it's 23* x *2, um, but then again, I thought the 5 is a factor, like, for example, 5 is a factor of 25 but 10 isn't, and so just because it's doubled doesn't mean it's a factor of, so I'm not too sure. I think I'd have to say no.*

Int.: *Okay. How about 46, would 391 be divisible by 46?*
Bob: [pause] *No, it wouldn't because uh in 46 the unit digit is 6, and the units digit of 391 is 1, and 6, knowing the multiples of 6, I know that there will not be a units digit of 1 after being multiplied by 6. For example, 6* x *6 is 36, units digit and that is obviously 6.*

Pam: *Because 46 ends with an even number and 391 is an odd number.*
Int.: *Um hmm.*
Pam: *And 6 is even, it won't fit into an odd number.*

Int.: *Would 391 be divisible by 46?*
Anabelle: *No, because it* [46] *was an even number, and this one* [391] *is an odd number.*

Int.: *Would you say 391 is divisible by 46?*
Tanya: *I guess gives me the idea that those parts are themselves going to be small.*

Excerpt with Tanya suggests that the intuitive belief in "small primes", that is, the belief that every composite number has a small prime factor, is grounded in experience of factoring numbers with small prime factors. It seems that exposure to a variety of "large" primes ("large" here means grater than 20), made painless with the availability of a calculator, could help in adjusting this intuitive belief.

Questions

- What instructional activities can help students in constructing meaning of concepts related to elementary number theory?
- What instructional activities can help students in making connections among these concepts?
- What are the effects of students' prior knowledge on their learning?
- What are the effects of students' "intuitive beliefs" on their learning? What instructional activities can help resolve the conflict between knowledge and intuition?
- Is there a link between the language students use and their understanding of the concepts involved?

These questions have driven my research. There are still more questions than answers. The only thing that I'm convinced of, more than ever, is that students' learning of elementary Number Theory is a fascinating area of research. Little work has been done in this area. However, some seeds have been planted. I hope that the forthcoming monograph (Campbell & Zazkis, in press) will generate further inquiry into students' learning of concepts underlying Number Theory and new ideas for classroom implementation.

Partial Answer: "Look!"

Can we help students to see what is there to be seen? Sometimes just saying "look!" could be sufficient. Many of preservice teachers believe that there should be a "procedure" or a "formula" to address every mathematical problem, or at least every mathematical problem posed to them. In a way, invitation to "look!" is a distraction from a search for a standard procedure. It is an invitation to apply prior knowledge and reasoning, rather than a learned algorithm. Further, in order for students to attend to what can be deducted from different number representations, I suggest to focus on questions that are easily answered if one attends to the structure, and require lots of "dirty" calculations if one does not. Several such questions are presented above.

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New PhD Report

Difficultés liées aux premiers apprentissages en théorie des groupes

Caroline Lajoie
Université du Québec à Montréal

Ma recherche doctorale visait à décrire et interpréter les difficultés rencontrées par des étudiants et étudiantes universitaires alors qu'ils étudient les notions de groupe, isomorphisme de groupes, sous-groupe et groupe cyclique. Dix étudiants et étudiantes ayant terminé la deuxième année d'un baccalauréat de trois ans en mathématiques ont été interviewés individuellement. Toutes ces personnes avaient suivi un cours obligatoire d'introduction aux structures algébriques, et la moitié d'entre elles avaient suivi en plus un cours optionnel de théorie des groupes. Les entrevues ont été analysées suivant une démarche qui s'apparente à la méthode d'analyse par théorisation ancrée.

Préalablement aux entrevues, une analyse historique et une analyse conceptuelle ont été réalisées en vue d'anticiper un certain nombre de difficultés. Les difficultés qui ont ainsi pu être appréhendées étaient liées pour la plupart au formalisme inhérent à l'algèbre abstraite.

L'analyse des entrevues a permis quant à elle de mettre en lumière quatorze difficultés, parmi lesquelles se trouvent les suivantes : difficulté à discerner les propriétés essentielles des groupes ; à considérer des transformations géométriques comme des éléments d'un groupe ; à donner à l'idée que des groupes isomorphes sont des groupes semblables l'interprétation que lui donnent les experts ; à voir l'isomorphisme comme une relation d'équivalence ; à établir une relation entre l'ordre d'un groupe et l'ordre de ses éléments ; à accepter que certains groupes peuvent être isomorphes à certains de leurs sous-groupes ; à accepter que des groupes infinis peuvent être cycliques.

Cette même analyse empirique a permis d'identifier quelques facteurs qui exercent une influence importante sur la manière qu'ont les étudiants et étudiantes universitaires d'aborder la structure de groupe, et qui permettent par le fait même de mieux comprendre les difficultés que ces personnes rencontrent dans la manipulation des premières notions de la théorie élémentaire des groupes. Parmi ces facteurs se trouvent le formalisme inhérent à l'algèbre abstraite, certaines divergences entre le langage mathématique et le langage naturel, la tendance à assimiler les nouveaux concepts à des images et exemples familiers, de même que le dédoublement d'objets mathématiques.

Le lecteur ou la lectrice intéressé(e) par ma recherche doctorale pourra consulter ma thèse de doctorat (Lajoie, 2000) ou encore des publications portant sur certaines des difficultés examinées dans le cadre de cette recherche (Lajoie, 2001; Lajoie et Mura, 2000; Lajoie et Mura, 1996a; Lajoie et Mura, 1996b).

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Ad hoc Sessions

The 'AHA Moment'

Peter Liljedahl
Simon Fraser University

> *In that moment when the connection is made, in that synaptic spasm of completion when the thought drives through the red fuse, is our keenest pleasure.*
>
> – Thomas Harris (2000, p. 132)

How is mathematics made? What are the necessary conditions and the mental processes that allow mathematicians to solve new problems? As mathematicians, we know that, given the right circumstances, the solution will come to us; there will be an illumination. The 'AHA moment' is both an accepted and expected part of the problem solving process. We go through private rituals to bring it on, and then we wait patiently for it to come. Yet, we know very little about this phenomenon.

The sharp moment of illumination is full of mysticism and wonder. Within a field where concise definitions of terms like *learning* and *understanding* are non-existent there is still a certainty that during the split second that it takes for the illuminated thought to come to completion *learning* has taken place and *understanding* has been gained. To obtain insight into the mental processes that facilitate this wondrous transformation and organization of ideas would give us greater understanding of what it means to learn and to understand.

Motivated by a lecture given by Henri Poincaré in the early part of the 20th century, Jacques Hadamard, a prominent French mathematician, searched for insight into the 'AHA' phenomenon. The results, published as *The Psychology of Invention in the Mathematical Field* (1945), is an eclectic account of the various theories espoused by mathematicians in Hadamard's time as to what causes the 'AHA moment'. Everything from the shape of a person's skull to the effect of taking two baths in a row to the more serious interpretations of Henri Poincaré were examined. Although there exists no contemporary equivalent of Hadamard's work such a survey would prove to be invaluable in the search to understand the moment of learning.

Session participants offered their own views as to the conditions necessary to invite and, perhaps, induce an 'AHA moment' as well as their knowledge of existing literature on the phenomenon.

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Multicultural Math on the Menu

Irene Percival
Simon Fraser University

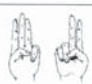
The NCTM call for "numerous and varied experiences related to the cultural, historical, and scientific evolution of mathematics" (NCTM, 1989) encouraged the publication of many elementary level historical and multicultural math texts (e.g. Irons and Burnett, 1995; Reimer and Reimer, 1992, 1993, 1995; Zaslavsky, 1994). As an advocate of the inclusion of such material into elementary school classes, I presented a short play, "Multicultural Math on the Menu", which I had written to introduce the 'pros and cons' of multicultural mathematics to elementary school teachers. I played the part of an enthusiastic teacher who spends her lunch break proclaiming the advantages of culturally based material to her rather sceptical colleagues, and answering their attacks upon this approach.

The issues raised in the play are those I have encountered during my research into the use of historical and multicultural mathematics. These include such advantages as the human interest angle, the increased motivation of doing 'different' mathematics and alternative methods of reinforcing basic skills. A knowledge of the historical development of such problem areas as negative numbers can also raise teachers' awareness of difficulties that their students may experience. My protagonists in the play point out problems of teacher preparation, such as lack of time and resources: in my evangelical role in the play, as in real life, I attempt to provide solutions to these issues.

After the play, an audience member commented on her positive experience of letting a Chinese student teach his classmates to write the number ideograms of his native land. I mentioned my own experiences in teaching Hindu mathematics to classes including students of Indian background. As this topic was new to most of the audience, I gave a brief explanation of the 'deficit' method of multiplication (Joseph, 1991), one of many techniques found in ancient Hindu texts, and suggested a modern adaptation which allows students to find products from 6 x 6 to 9 x 9 just by looking at their fingers.

number	deficit
7 *(given)*	3 *(10-7)*
8 *(given)*	2 *(10-8)*
5 *(7-2 or 8-3)*	6 *(3x2)*
So 8x7 = 56	

Left hand (7) shows deficit of 3
Right hand (8) shows deficit of 2



Product of raised fingers gives unit digit of 7x8 (6)
Sum of lowered fingers gives tens digit of 7x8 (5)

The Egyptian method of multiplication was also discussed, and a comparison made to modern methods, showing how the distributive property of real numbers is a common feature of these seemingly different approaches.

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Special Session

David Wheeler Memorial Session

Malgorzata Dubiel, *Simon Fraser University*
David Pimm, *University of Alberta*

More than fifty conference members assembled at 9 a.m. on Tuesday morning in the atrium lounge of Education North for a session of remembrance for David Wheeler.

Dave Lidstone started the session with a projected image of an Inukshuk.[1]

Dave then offered the following words:

> It was a tradition of the Inuit in our North to construct human-like statues called Inukshuk. These served to offer direction and safety for travellers on a tree-less tundra that otherwise offered no markers. More recently, the roads and hiking trails of our country have been populated with Inukshuk, erected by travellers as records of their journeys. The Inukshuk pictured above is located on the shore of English Bay in Vancouver. It was at this site that, in early November of 2000, a small group of friends of David Wheeler scattered his ashes into the Pacific waters.
>
> Our choice of this site was rather spontaneous and dictated by a desire for some marker by which we could launch David's final material journey. However, I now find considerable

> comfort in the images Inukshuk offer as a testament to his life. His legacy includes many Inukshuk, among them *For the Learning of Mathematics* and CMESG/GCEDM, sites that offer direction for us on our professional journeys. The English Bay Inukshuk serves to mark his journey among us and the proliferation of Inukshuk elsewhere serve as reminder that his influence continues to be very widespread.

A number of different people present then offered words to us all, while others contributed their attention and care. Some words were more prepared, others seemed more spontaneous; some were more formal, while others were more personal. But it was always talk of memories and influences: reminiscences, tales and observations about the man and his work, his passionate involvement with music, a far-ranging correspondence, an intense engagement with food and people, always people.

From these words and from many others spoken in smaller gatherings or privately recalled, a mere glimpse of the life of David emerged. Unsurprisingly, perhaps, opinions did not always completely mesh, even about whether David himself would have approved of today's session.

An emailed message from Dick Tahta in England (who was unable to attend) was read out near the end.

> I have been enjoying the thought of what David would have written to me about the things people may be saying about him at this CMESG meeting.
>
> He would have squirmed. And be secretly pleased.
>
> I am reminded of a time when he wrote that he preferred my having called him admirable to an occasion when I had said that he was lovable. But I think he was also secretly pleased to know that people found him lovable. As he was.
>
> One Christmas, he sent me a poem about opening doors. I replied with one about crossing bridges—written by a sixteenth-century Armenian troubadour, Nahabed Koutchag. I would like to have this read for him today.
>
> Even though you take the mountain roads
> someday you'll fall into my palm.
> And when you fall into my hands I'll make you confess
> how you go over bridges without paying tolls.
> You'll show me the bridges you have already crossed.
> You'll take me on the one you still have to pass.

The session ended with an extract (the first *Kyrie*) from a recording made in 1961 of the choruses from Bach's *B-minor Mass* by the Philharmonia Chorus and Orchestra conducted by Otto Klemperer.[2] David Wheeler had been delighted to discover that this recording had finally been issued on CD in 1999, as he had been a member of the Philharmonia Chorus singing on these occasions. For us listening during this session, there was added poignancy in knowing that, in some important sense, David was singing his own *Requiem*, despite him having now crossed that final bridge.

Notes

1. An image of this Inukshuk can also be seen on the cover of *For the Learning of Mathematics*, volume 20, number 3, which announced to readers David's recent death.
2. As the CD notes in detail, this project was never completed. This particular CD appears in the TESTAMENT series (SBT1138). Further information is available at: http://www.testament.co.uk.

Appendices

APPENDIX A

Working Groups at Each Annual Meeting

1977 *Queen's University, Kingston, Ontario*
- Teacher education programmes
- Undergraduate mathematics programmes and prospective teachers
- Research and mathematics education
- Learning and teaching mathematics

1978 *Queen's University, Kingston, Ontario*
- Mathematics courses for prospective elementary teachers
- Mathematization
- Research in mathematics education

1979 *Queen's University, Kingston, Ontario*
- Ratio and proportion: a study of a mathematical concept
- Minicalculators in the mathematics classroom
- Is there a mathematical method?
- Topics suitable for mathematics courses for elementary teachers

1980 *Université Laval, Québec, Québec*
- The teaching of calculus and analysis
- Applications of mathematics for high school students
- Geometry in the elementary and junior high school curriculum
- The diagnosis and remediation of common mathematical errors

1981 *University of Alberta, Edmonton, Alberta*
- Research and the classroom
- Computer education for teachers
- Issues in the teaching of calculus
- Revitalising mathematics in teacher education courses

1982 *Queen's University, Kingston, Ontario*
- The influence of computer science on undergraduate mathematics education
- Applications of research in mathematics education to teacher training programmes
- Problem solving in the curriculum

1983 *University of British Columbia, Vancouver, British Columbia*
- Developing statistical thinking
- Training in diagnosis and remediation of teachers
- Mathematics and language
- The influence of computer science on the mathematics curriculum

1984 *University of Waterloo, Waterloo, Ontario*
- Logo and the mathematics curriculum
- The impact of research and technology on school algebra
- Epistemology and mathematics
- Visual thinking in mathematics

1985 *Université Laval, Québec, Québec*
- Lessons from research about students' errors
- Logo activities for the high school
- Impact of symbolic manipulation software on the teaching of calculus

1986 *Memorial University of Newfoundland, St. John's, Newfoundland*
- The role of feelings in mathematics
- The problem of rigour in mathematics teaching
- Microcomputers in teacher education
- The role of microcomputers in developing statistical thinking

1987 *Queen's University, Kingston, Ontario*
- Methods courses for secondary teacher education
- The problem of formal reasoning in undergraduate programmes
- Small group work in the mathematics classroom

1988 *University of Manitoba, Winnipeg, Manitoba*
- Teacher education: what could it be?
- Natural learning and mathematics
- Using software for geometrical investigations
- A study of the remedial teaching of mathematics

1989 *Brock University, St. Catharines, Ontario*
- Using computers to investigate work with teachers
- Computers in the undergraduate mathematics curriculum
- Natural language and mathematical language
- Research strategies for pupils' conceptions in mathematics

1990 *Simon Fraser University, Vancouver, British Columbia*
- Reading and writing in the mathematics classroom
- The NCTM "Standards" and Canadian reality
- Explanatory models of children's mathematics
- Chaos and fractal geometry for high school students

1991 *University of New Brunswick, Fredericton, New Brunswick*
- Fractal geometry in the curriculum
- Socio-cultural aspects of mathematics
- Technology and understanding mathematics
- Constructivism: implications for teacher education in mathematics

1992 *ICME–7, Université Laval, Québec, Québec*

1993 *York University, Toronto, Ontario*
- Research in undergraduate teaching and learning of mathematics
- New ideas in assessment
- Computers in the classroom: mathematical and social implications
- Gender and mathematics
- Training pre-service teachers for creating mathematical communities in the classroom

1994 *University of Regina, Regina, Saskatchewan*
- Theories of mathematics education
- Pre-service mathematics teachers as purposeful learners: issues of enculturation
- Popularizing mathematics

1995 *University of Western Ontario, London, Ontario*
- Anatomy and authority in the design and conduct of learning activity
- Expanding the conversation: trying to talk about what our theories don't talk about
- Factors affecting the transition from high school to university mathematics
- Geometric proofs and knowledge without axioms

1996 *Mount Saint Vincent University, Halifax, Nova Scotia*
- Teacher education: challenges, opportunities and innovations
- Formation à l'enseignement des mathématiques au secondaire: nouvelles perspectives et défis
- What is dynamic algebra?
- The role of proof in post-secondary education

1997 *Lakehead University, Thunder Bay, Ontario*
- Awareness and expression of generality in teaching mathematics
- Communicating mathematics
- The crisis in school mathematics content

1998 *University of British Columbia, Vancouver, British Columbia*
- Assessing mathematical thinking
- From theory to observational data (and back again)
- Bringing Ethnomathematics into the classroom in a meaningful way
- Mathematical software for the undergraduate curriculum

1999 *Brock University, St. Catharines, Ontario*
- Information technology and mathematics education: What's out there and how can we use it?
- Applied mathematics in the secondary school curriculum
- Elementary mathematics
- Teaching practices and teacher education

2000 *University du Québec à Montréal, Montréal, Québec*
- Des cours de mathématiques pour les futurs enseignants et enseignantes du primaire/Mathematics courses for prospective elementary teachers
- Crafting an algebraic mind: Intersections from history and the contemporary mathematics classroom
- Mathematics education et didactique des mathématiques : y a-t-il une raison pour vivre des vies séparées?/Mathematics education et didactique des mathématiques: Is there a reason for living separate lives?
- Teachers, technologies, and productive pedagogy

APPENDIX B

Plenary Lectures at Each Annual Meeting

1977	A.J. COLEMAN	The objectives of mathematics education
	C. GAULIN	Innovations in teacher education programmes
	T.E. KIEREN	The state of research in mathematics education
1978	G.R. RISING	The mathematician's contribution to curriculum development
	A.I. WEINZWEIG	The mathematician's contribution to pedagogy
1979	J. AGASSI	The Lakatosian revolution*
	J.A. EASLEY	Formal and informal research methods and the cultural status of school mathematics*
1980	C. GATTEGNO	Reflections on forty years of thinking about the teaching of mathematics
	D. HAWKINS	Understanding understanding mathematics
1981	K. IVERSON	Mathematics and computers
	J. KILPATRICK	The reasonable effectiveness of research in mathematics education*
1982	P.J. DAVIS	Towards a philosophy of computation*
	G. VERGNAUD	Cognitive and developmental psychology and research in mathematics education*
1983	S.I. BROWN	The nature of problem generation and the mathematics curriculum
	P.J. HILTON	The nature of mathematics today and implications for mathematics teaching*
1984	A.J. BISHOP	The social construction of meaning: A significant development for mathematics education?*
	L. HENKIN	Linguistic aspects of mathematics and mathematics instruction
1985	H. BAUERSFELD	Contributions to a fundamental theory of mathematics learning and teaching
	H.O. POLLAK	On the relation between the applications of mathematics and the teaching of mathematics
1986	R. FINNEY	Professional applications of undergraduate mathematics
	A.H. SCHOENFELD	Confessions of an accidental theorist*
1987	P. NESHER	Formulating instructional theory: the role of students' misconceptions*
	H.S. WILF	The calculator with a college education

1988	C. KEITEL	Mathematics education and technology*
	L.A. STEEN	All one system
1989	N. BALACHEFF	Teaching mathematical proof: The relevance and complexity of a social approach
	D. SCHATTSNEIDER	Geometry is alive and well
1990	U. D'AMBROSIO	Values in mathematics education*
	A. SIERPINSKA	On understanding mathematics
1991	J .J. KAPUT	Mathematics and technology: Multiple visions of multiple futures
	C. LABORDE	Approches théoriques et méthodologiques des recherches Françaises en didactique des mathématiques
1992	ICME-7	
1993	G.G. JOSEPH	What is a square root? A study of geometrical representation in different mathematical traditions
	J CONFREY	Forging a revised theory of intellectual development: Piaget, Vygotsky and beyond*
1994	A. SFARD	Understanding = Doing + Seeing ?
	K. DEVLIN	Mathematics for the twenty-first century
1995	M. ARTIGUE	The role of epistemological analysis in a didactic approach to the phenomenon of mathematics learning and teaching
	K. MILLETT	Teaching and making certain it counts
1996	C. HOYLES	Beyond the classroom: The curriculum as a key factor in students' approaches to proof*
	D. HENDERSON	Alive mathematical reasoning
1997	R. BORASSI	What does it really mean to teach mathematics through inquiry?
	P. TAYLOR	The high school math curriculum
	T. KIEREN	Triple embodiment: Studies of mathematical understanding-in-inter-action in my work and in the work of CMESG/GCEDM
1998	J. MASON	Structure of attention in teaching mathematics
	K. HEINRICH	Communicating mathematics or mathematics storytelling
1999	J. BORWEIN	The impact of technology on the doing of mathematics
	W. WHITELEY	The decline and rise of geometry in 20th century North America
	W. LANGFORD	Industrial mathematics for the 21st century
	J. ADLER	Learning to understand mathematics teacher development and change: Researching resource availability and use in the context of formalised INSET in South Africa
	B. BARTON	An archaeology of mathematical concepts: Sifting languages for mathematical meanings
2000	G. LABELLE	Manipulating combinatorial structures
	M. BARTOLINI BUSSI	The theoretical dimension of mathematics: A challenge for didacticians

NOTE

*These lectures, some in a revised form, were subsequently published in the journal *For the Learning of Mathematics.*

APPENDIX C

Proceedings of Annual Meetings

Past proceedings of CMESG/GCEDM annual meetings have been deposited in the ERIC documentation system with call numbers as follows:

Proceedings of the 1980 Annual Meeting .... ED 204120

Proceedings of the 1981 Annual Meeting .... ED 234988

Proceedings of the 1982 Annual Meeting .... ED 234989

Proceedings of the 1983 Annual Meeting .... ED 243653

Proceedings of the 1984 Annual Meeting .... ED 257640

Proceedings of the 1985 Annual Meeting .... ED 277573

Proceedings of the 1986 Annual Meeting .... ED 297966

Proceedings of the 1987 Annual Meeting .... ED 295842

Proceedings of the 1988 Annual Meeting .... ED 306259

Proceedings of the 1989 Annual Meeting .... ED 319606

Proceedings of the 1990 Annual Meeting .... ED 344746

Proceedings of the 1991 Annual Meeting .... ED 350161

Proceedings of the 1993 Annual Meeting .... ED 407243

Proceedings of the 1994 Annual Meeting .... ED 407242

Proceedings of the 1995 Annual Meeting .... ED 407241

Proceedings of the 1996 Annual Meeting .... ED 425054

Proceedings of the 1997 Annual Meeting .... ED 423116

Proceedings of the 1998 Annual Meeting .... ED 431624

Proceedings of the 1999 Annual Meeting .... not available

Proceedings of the 2000 Annual Meeting .... not available

NOTES

1. There was no Annual Meeting in 1992 because Canada hosted the Seventh International Conference on Mathematical Education that year.
2. *Proceedings of the 2000 Annual Meeting* and *Proceedings of the 2001 Annual Meeting* have been submitted to ERIC.