CANADIAN MATHEMATICS EDUCATION STUDY GROUP

# 40 YEARS OF CMESG

CMESG

GCEDM

# LES 40 ANS DU GCEDM

GROUPE CANADIEN D'ÉTUDE EN DIDACTIQUE DES MATHÉMATIQUES

EDITED BY: Peter Liljedahl • Simon Fraser University | Darien Allan • Collingwood School | Olive Chapman • University of Calgary | Frédéric Gourdeau • Université Laval | Caroline Lajoie • Université du Québec à Montréal | Susan Oesterle • Douglas College | Elaine Simmt • University of Alberta | Peter Taylor • Queen's University

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*Canadian Mathematics Education Study Group /*
*Groupe Canadien d'Étude en Didactique des Mathématiques*

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*In memory of the colleagues who are no longer with us.*

*En mémoire des collègues qui ne sont plus parmi nous.*

# ABOUT THIS SPECIAL ISSUE

Peter Liljedahl
*Simon Fraser University*

The 40th anniversary of an organization is of great significance. It is the year that, for the most part, the people who started the organization are, unfortunately, no longer part of the organization. With one exception, this is true of CMESG. As such, it is an important time to look back at our collective history, to reflect on where he have been, and who we have become as an organization.

This special issue of the CMESG/GCEDM Proceedings is such a reflection. Through a selection of excerpts from past proceedings we have stitched together a partial history of our time as an organization. This serves not only as a summary of our collective history, but also serves as an introduction to the activities of CMESG/GCEDM through our first 40 years.

To be clear, the selections included in this special issue are not to be seen as the *best of* CMESG/GCEDM. The *best of* CMESG/GCEDM already exists in the comprehensive archives of all CMESG proceedings located on our website. The pieces selected for inclusion in this special issue stand, instead, as signposts along our history – as exemplifications of what CMESG/GCEDM is as an organization as well as the activities of CMESG/GCEDM at our annual meetings.

The charge of selecting what was to be included in this special issue was given to a representative team CMESG/GCEDM members.

Peter Liljedahl • *Simon Fraser University*
Darien Allan • *Collingwood School*
Olive Chapman • *University of Calgary*
Frédéric Gourdeau • *Université Laval*
Caroline Lajoie • *Université du Québec à Montréal*
Susan Oesterle • *Douglas College*
Elaine Simmt • *University of Alberta*
Peter Taylor • *Queen's University*

Among this group we have CMESG/GCEDM presidents, proceedings editors, mathematicians and mathematics educators, and academics and practitioners. The list also includes members who joined CMESG/GCEDM in every decade of our first 40 years, including Peter Taylor who was present at the first meeting and every meeting since.

Together, we chose the selections to be included in the special issue as well as crafted introductions to each piece articulating the ways in which that piece exemplifies the valued aspects of CMESG/GCEDM.

We hope you find enjoyment in reading again, or for the first time, the selections contained herein.

# À PROPOS DE CE NUMÉRO SPÉCIAL

Peter Liljedahl
*Simon Fraser University*

Le 40e anniversaire d'une organisation revêt toujours une grande importance. Généralement, plusieurs des personnes fondatrices ne font alors malheureusement plus partie de l'organisation. À une exception près, les membres fondateurs du GCEDM sont toujours parmi nous. En ce sens, le moment nous semble bien choisi pour examiner en rétrospective notre histoire collective, et pour réfléchir à ce que nous avons accompli et à ce que nous sommes devenus en tant qu'organisation.

Ce numéro spécial des actes du CMESG/GCEDM est une telle réflexion. Il s'agit en fait d'une histoire partielle de notre organisation, construite à partir d'une sélection d'extraits d'actes de rencontres passées. Ce numéro spécial peut aussi être vu comme une introduction aux activités du CMESG/GCEDM réalisées au cours de nos 40 premières années d'histoire.

Il est important de comprendre que les extraits qui ont été retenus pour ce numéro spécial ne doivent pas être vus comme étant le *meilleur du* CMESG/GCEDM. Le *meilleur du* CMESG/GCEDM existe déjà dans les archives du groupe, soit dans les actes de nos rencontres, tous disponibles sur notre site web.

Les textes retenus pour ce numéro spécial doivent plutôt être considérés comme des points repères de notre histoire, des exemples à la fois de ce que le CMESG/GCEDM est comme groupe et des activités qui caractérisent nos rencontres annuelles.

La responsabilité d'élaborer ce numéro spécial a été donnée à une équipe représentative du CMESG/GCEDM.

Peter Liljedahl • *Simon Fraser University*
Darien Allan • *Collingwood School*
Olive Chapman • *University of Calgary*
Frédéric Gourdeau • *Université Laval*
Caroline Lajoie • *Université du Québec à Montréal*
Susan Oesterle • *Douglas College*
Elaine Simmt • *University of Alberta*
Peter Taylor • *Queen's University*

Cette équipe regroupe des président(e)s du CMESG/GCEDM, des membres ayant édité les actes, des mathématicien(ne)s et didacticien(ne)s, des académiques et des praticien(ne)s. La liste inclut aussi des membres ayant joint les rangs du CMESG/GCEDM à chacune des quatre décennies de son histoire. En particulier, on y trouve Peter Taylor, qui était présent à la première rencontre ainsi qu'à toutes celles qui l'ont suivie.

Nous avons choisi ensemble les extraits à inclure dans le numéro spécial et nous avons rédigé collectivement des introductions à ces extraits de manière à mettre en évidence pour chacun les particularités du CMESG/GCEDM qu'il exemplifie.

Nous espérons vivement que vous aurez du plaisir à relire, ou à lire pour une première fois, nos sélections.

SPECIAL 40TH ANNIVERSARY ISSUE OF THE

# CANADIAN MATHEMATICS EDUCATION STUDY GROUP / GROUPE CANADIEN D'ÉTUDE EN DIDACTIQUE DES MATHÉMATIQUES

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# FOREWORD

Peter Taylor
*Queen's University*

Fall 1977 – that was the beginning. We didn't have an organization or a name and we weren't quite sure what our objective was, other than the fact that John Colman's report, commissioned in the early 70's by the Science Council of Canada, made it clear that important work needed to be done to bring mathematics – real mathematics, indeed the *joy* of mathematics – into the lives of our children and our citizens. Coleman and David Wheeler, who deserve to be called the prime movers of CMESG, decided to convene a national meeting of mathematicians and math educators to come together and talk about next moves, and they recruited the young Bill Higginson to help them organized this event in the Fall of 1977. By the following year, 1978, the christening of CMESG/GCEDM had occurred and a few years later, the locus of the annual meeting of this national body left Kingston and began to move around Canada.

From that point on, the rest of those 40 years went by in a whistle. CMESG grew in size and flowered in a way that made it stand out from most of the other fine organizations and meetings all over the world that devote all or some of their energies to mathematics education. I base this assertion on the comments of the many members and visitors from other places who are most often impressed and delighted by what they find in their few days with us. I feel that there are a number of factors behind this.

The first is our size. We are much bigger than we were at the beginning but not nearly as big as almost all the other national and international math education conferences. Thus we have that rare Goldilocks size. Secondly I point to the flowering, and there are two aspects, what we do and who does it. For the "what," our working groups, our *ad hocs*, our new PhD talks all have the sense of new beginnings. And for the "who," the most striking feature for me is the growth of our graduate student numbers; at our annual meeting they radiate optimism and awe - awe that they have found a space where such important issues are talked about in such a caring way. And before you know it they are playing prominent roles in shaping the organization.

Finally, everything we do, we do together, and it is our size and composition which allows this to work so well. We eat together, in a cafeteria or a pub or late at night in a pizza joint, we go on trips together and we never know who will be sitting beside or in front of us. We have two languages and we work hard to make them both work. And we have two disciplines, mathematics and mathematics education, and we work hard to learn about them from one another and to extend their dimensions.

On this 40th anniversary, as we come back to Queen's, it is miraculous to me to see what we have become. Coleman and Wheeler would be as proud of us as we are of them.

# PRÉFACE

Peter Taylor
*Queen's University*

Automne 1977 : c'est alors que tout a commencé. Nous n'avions ni organisation, ni nom, et nous n'étions pas certains de nos objectifs, si ce n'est que le rapport de John Coleman, réalisé à la demande du Conseil des Sciences du Canada au début des années 70, nous indiquait clairement qu'il fallait amener les mathématiques – de vraies mathématiques, et même le plaisir d'en faire – dans la vie de nos concitoyens, de nos enfants. Coleman et David Wheeler, les véritables initiateurs du GCEDM, avaient décidé de tenir une rencontre nationale de mathématiciens et d'éducateurs mathématiques afin de discuter des suites à donner au rapport, et ils avaient recruté le jeune Bill Higginson afin de les aider à organiser cette rencontre à l'automne 1977. L'année suivante, 1978, le groupe s'était donné un nom et, quelques années plus tard, la rencontre annuelle quittait Kingston pour se tenir en différents endroits au Canada.

Les quarante années suivantes ont filé à vive allure. Le GCEDM a grandi tout en s'épanouissant, et se distingue sur la scène internationale parmi les excellentes organisations se préoccupant d'éducation mathématique. Je fais cette affirmation en me basant sur les commentaires de nombreux membres et de visiteurs d'outre frontières qui sont souvent impressionnés et charmés par ce qu'ils découvrent en quelques jours avec nous. Je pense que plusieurs facteurs expliquent ce succès.

En premier lieu, la taille du groupe. Nous avons grandi mais nous demeurons plus petits que la plupart des autres conférences nationales et internationales en éducation mathématique. En ce sens, comme Boucle d'Or, nous avons la bonne taille, ce qui est rare. Puis, en revenant à notre épanouissement, il y a deux aspects importants: ce que nous faisons, et ceux qui le font. Qu'il s'agisse de nos groupes de travail, de nos séances ad hoc ou des présentations des nouveaux doctorants, ce que nous faisons a toujours un aspect de nouveauté, l'élan d'un nouveau départ. Quant à ceux qui le font, je suis frappé par l'augmentation du nombre d'étudiants des 2$^{e}$ et 3$^{e}$ cycles à nos rencontres; à nos rencontres annuelles, ils apportent optimisme et émerveillement – un émerveillement d'avoir trouvé un lieu d'échange dans lequel des sujets si importants sont abordés avec autant de respect. Et rapidement, ils jouent à leur tour un rôle important dans l'évolution de l'organisation.

Finalement, tout ce qu'on fait, on le fait ensemble, et c'est notre taille et la composition de notre groupe qui permet que cela fonctionne si bien. Nous mangeons ensemble, dans une cafétéria, un pub, ou tard le soir devant une pizza, ne sachant jamais à côté ou en face de qui on sera. On a deux langues, et on travaille fort pour que ça fonctionne dans les deux langues. On a deux disciplines académiques, les mathématiques et la didactique des mathématiques, et on travaille fort pour en apprendre sur l'une et sur l'autre, en enrichissant l'une et l'autre.

En ce 40$^{e}$ anniversaire, alors que nous sommes de retour à Queen's, je trouve miraculeux de voir ce que nous sommes devenus. Coleman et Wheeler seraient aussi fiers de nous que nous le sommes d'eux.

# PREFACE TO THE 1977 PROCEEDINGS

# PRÉFACE AUX ACTES DE 1977

The decision to organize a conference to discuss the universities' responsibilities in the preparation of mathematics teachers sprang from two related desires. One was to achieve some discussion of the issues concerning mathematics education raised in *Mathematical Sciences in Canada*[1]; the other was to bring together a group of mathematicians and mathematics educators across Canada to explore the possibility of improving inter-provincial contact and communication.

Although many of the people consulted in the preparation of the Back Ground Study had a great deal to say about mathematics education in Canada, and particularly about its shortcomings, this aspect of the report itself has received very little public discussion. One of the contributory reasons may be the lack of a national organization with any responsibility to consider and speak about mathematics education in Canada. Although there are a number of provincial associations of mathematics teachers, the only professional organization with a national membership is the National Council of Teachers of Mathematics, and this is understandably more concerned to speak for mathematics education in the United States where the bulk of its membership resides.

A small conference seemed more likely to achieve the initial contact and communication that we wanted, so we decided to restrict the conference membership to university mathematicians and mathematics educators. The subject of teacher preparation immediately suggested itself as the appropriate part of mathematics education to focus on. We drew up a programme and an invitation list for a meeting at Queen's University, Kingston, from August 31st to September 3rd, 1977. The Science Council of Canada generously agreed to sponsor the conference and meet the expense.

The report that follows covers most of what can be reported of the conference proceedings, and it is published as a contribution to the national discussion of mathematics education in Canada. The conference was short, the participants had to get to know each other, and many of the discussions that took place did not lend themselves to being written-up in detail, so the final report should be seen as an indication of the issues that were discussed, not a definitive statement on them.

We were cheered beforehand by the ready acceptance by most of the people who were invited, and by their assurances afterwards that the conference had been worthwhile. Seen as a first step in the direction of more professional contact and more public discussion, we think the Kingston conference has a future.

We are indebted to the Science Council of Canada for financial assistance, to conference participants for their enthusiastic response, particularly to Speakers and Working Group Chairmen and reporters, and to Noreen Mills, Torn Racey, Patricia Whitaker and Eileen Wight for their unstinting, high quality technical support.

A. J. Coleman
W. C. Higginson
D. H. Wheeler

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[1] K.P. Beltzner, A. J. Coleman, and G. D. Edwards, *Mathematical Sciences in Canada, Background Study No. 37*. Ottawa, Ontario: Science Council of Canada, July, 1976.

# THE ORIGINS AND ACTIVITIES OF CMESG

David Wheeler
*Concordia University*

This history was original written for the monograph Current Research on the Teaching and Learning of Mathematics in Canada — Les Recherches en Cours sur l'Apprentissage et l'Enseignement des Mathématiques au Canada edited by Carolyn Kieran and A. J. (Sandy) Dawson and published for ICME-7 in 1992.

À l'origine, ce texte a été écrit en guise d'introduction à la monographie Current Research on the Teaching and Learning of Mathematics in Canada — Les recherches en cours sur l'apprentissage et l'enseignement des mathématiques au Canada, éditée par Carolyn Kieran et A. J. (Sandy) Dawson et publiée pour ICME-7 (Université Laval) en 1992.

## INTRODUCTION

The Science Council of Canada sponsored a mathematics education conference at Queen's University, Kingston, Ontario, in September 1977. Thirty mathematicians and mathematics educators from across Canada accepted an invitation to join the three organisers of the conference (Professors A. J. Coleman and W. C. Higginson of Queen's University, and D. H. Wheeler of Concordia University, Montréal) in discussing the general theme: "Educating teachers of mathematics: the universities' responsibility." The encounter generated a demand from many of the participants for further opportunities to meet and talk. The Science Council supported a second invitational meeting in June 1978 at which the decision was taken to establish a continuing group, to be called the CANADIAN MATHEMATICS EDUCATION STUDY GROUP / GROUPE CANADIEN D'ÉTUDE EN DIDACTIQUE DES MATHÉMATIQUES (CMESG/GCEDM – sometimes referred to as the Study Group). The fifteenth annual meeting of CMESG/GCEDM was held at the University of New Brunswick in Fredericton in May 1991.

The history of this professional group is very short but it seems worth presenting here, partly to give some context to the accounts of research that follow, but also because the special character of CMESG/GCEDM may be found to have some instructive features.

## BEGINNINGS

The introduction to the programme for the 1977 meeting reads:

> *The Conference has been convened as part of the follow-up to the Council's Background Study No. 37 (Mathematical Sciences in Canada) [1] to consider the place and responsibility of Canadian universities in the education of teachers of mathematics. The participants are university mathematics educators and mathematicians, but the organisers do not intend to imply that only universities are or should be concerned in the education of teachers. Universities have traditionally played a principal role, however, and will certainly continue to be involved in teacher education for the foreseeable future even though the forms of their involvement may change. The Conference is an opportunity to make a contribution, related to one particular aspect and from one particular point of view, to the public discussion of mathematics education in Canada. The Conference has no official status and is in no*

> *sense a policy-forming or advisory body. It is not the intention of the Conference to seek consensus or to make recommendations to anyone.*
>
> *One purpose of the Conference is served by the mere fact of bringing participants together and the consequent pooling of ideas and information by those who have overlapping interests but seldom meet. It is meant to have other, tougher, purposes too. At a level above that of information-sharing there are questions to be formulated, problems to be isolated and tendencies identified, maybe even achievements to be acknowledged; in other words, an attempt to get a grasp on the present situation and an orientation on the future. At a still higher level belongs the task of studying together how the questions may be answered and the problems resolved. Independent of this hierarchy is the job of communicating something of value to other professionals and to the public. How much of this can be achieved in such a short time remains to be seen. At least a start can be made.*

The faintly apologetic tone of all this is characteristically Canadian, but the sense it conveys that the organisers were stepping warily is quite genuine. One good reason was that the Background Study referred to had been badly received by the mathematical community, at least as represented by the Canadian Mathematical Congress (later to rename itself the Canadian Mathematical Society), which did not enjoy the many explicit and implicit criticisms made by the writers of the Study. A reviewer of Mathematical Sciences in Canada summarised its general argument in the following terms:

> *Mathematics plays a commanding role in modern technological societies, yet many professional mathematicians have little interest in its applications, and government and business are often unsure how best to use the mathematicians they employ. Mathematics is taught to Canadians in one of the most generous and accessible educational systems in the world; yet only a minority of students gain much competence in it, and only a minority of those more than a routine grasp. Mathematical research is published in daunting quantities; yet most papers do no more than dot i's and cross t's well inside the frontiers. The output of Canadian PhD's in mathematics has increased tenfold in the last fifteen years; yet a large majority of them still expect to remain in academia and do little but produce more of their kind. Mathematical Sciences in Canada elaborates on a situation that might once have been described as productive redundancy, but which in these less easy-going times seems more like capricious and conspicuous waste. [2]*

Another reason for the organisers' caution can be found in the statutory division of responsibilities for education in Canada between the federal and provincial authorities. The provinces have total authority for the organisation and governance of primary and secondary education. To obtain federal support for the 1977 conference, which was necessary if participants were to be drawn from all parts of Canada, the organisers had to make sure that the objectives did not infringe on the application of provincial powers. Direct examination of the school curriculum, for example, had to be carefully avoided, and the conference had to refrain from making recommendations that might appear as an attempt to interfere with provincial rights.

The programme of the 1977 conference included three keynote lectures:

- The state of research in mathematics education (T. E. Kieren)
- Innovations in teacher education programmes (C. Gaulin)
- The objectives of mathematics education (A. J. Coleman)
- and four working groups:
- Teacher education programmes
- Undergraduate mathematics programmes and prospective teachers
- Research in mathematics education
- Learning and teaching mathematics.

The organisers felt that it was important for the meeting to give a substantial amount of attention to mathematics education research. Without this component it would be only too easy for the discussions in the meeting and the conclusions that might emerge to do no more than recycle familiar folklore about the shortcomings of mathematics teaching in Canada.

Conference proceedings were published by the Science Council [3]. One of the organisers, penning some "Reflections after the Conference," which are included in the Proceedings, began by quoting from the Background Study:

> *It no longer seems possible for any component of the mathematical ecosystem to function effectively in isolation. Awareness and communication seem to be the key issues. [1, p. 86]*

and continued:

> *They were the underlying themes of the Conference too. Bringing university mathematicians and mathematics educators together involved an interaction between two groups which tend to be somewhat suspicious of each other. The assumption by the universities of the responsibility for training teachers has not led, in general, to greater mutual understanding or cooperation by those who teach university mathematics and those who teach would-be teachers of mathematics. Both groups have other interests and responsibilities and it may be that the lack of common ground in these other areas contributes to the suspicion. But it also extends into that part of their work where they might be expected to find a shared cause — the preparation of specialist mathematics teachers. University mathematicians look at education courses and see an apparent lack of structure and rigour together with a plenitude of non-refutable theories; university mathematics educators look at the students emerging from undergraduate mathematics programmes and see the apparently deadening effects of a training dominated by structure and rigour. Both sides, when apart, tend to stereotype each other. [3, p. 56]*

The generally favourable response to the 1977 meeting led Coleman, Higginson, and Wheeler to propose a continuation. Their first plan was to work toward meetings in 1978 and 1979 which would culminate in the production of documents; these might form the basis for a Canadian contribution to the Fourth International Congress on Mathematical Education (ICME-4) to be held in Berkeley USA in August 1980. This focus on the production of documents led them to suggest meetings covering five working days, which would allow for some writing to take place during the meetings. But the overwhelming response was a rejection of five days as impossibly long and, in the event, the 1978 meeting set a pattern which has become the standard for all subsequent meetings: three full working days sandwiched between arrival and departure half-days.

The programme for the 1978 meeting included two lectures:

- The mathematician's contribution to curriculum development (G. R. Rising)
- The mathematician's contribution to pedagogy (A. I. Weinzweig)

and three working groups:

- Mathematics courses for prospective elementary school teachers
- Mathematization
- Research in mathematics education.

The working groups were scheduled simultaneously for a total of 18 hours. Although this proved to be too much time — it took so large a chunk of the time available that it squeezed out other activities, such as up-dating the work done at the previous meeting — it symbolised the considerable significance that the organisers gave to this activity: the working groups were always intended to be the core activity of the meetings. From the 1979 meeting onward, working groups have met for nine hours, but they have retained their centrality, in many ways setting the tone of the meetings and distinguishing them from most other scholarly conferences in Canada. (A list of the working groups for the first fifteen meetings is given in Appendix 1.) Less distinctive, perhaps, has been the effect of putting the keynote lectures in the hands of "guest" speakers, usually non-Canadians. The intention here was to enrich the input to the meetings by inviting speakers who would bring fresh perspectives to the discussion of mathematics education. The guest speakers over the years make a diverse and distinguished bunch, as the list in Appendix 2 shows.

Unfortunately, the ambition to produce significant discussion documents for ICME-4 was not realised. The published evidence of the Study Group's activities is largely confined to the proceedings of its annual get-togethers, and even these do not always manage to convey a good idea of the real transactions of the meetings. (Appendix 3 lists the ERIC numbers of available CMESG/GCEDM proceedings.)

At the close of the 1978 meeting the participants voted to give CMESG/GCEDM a continuing existence and an acting executive committee. A formal constitution was approved at the 1979 meeting and the first elections under the terms of the constitution took place in 1980. Although a few changes in the organisational structure have occurred, and although the annual programmes have evolved to some extent, the main characteristics of the Study Group were settled in the first few years.

## CHARACTERISTICS

Canada's size, location, and federal structure pose special problems for any organisation aiming at nationwide status. Travel distances and costs for regular face-to-face meetings are simply enormous. Whatever purpose a Canadian group might espouse, there is almost certainly a group in the USA with a similar purpose whose meetings are as easy (or difficult) to get to. The province-based organisation of primary and secondary education tends to lock up some of the money that would otherwise be available to support attendance at meetings. Given a different context, the original animators would have tried to establish a national group open to educators in all parts of the system: elementary school teachers, school administrators, university professors of mathematics, and so on. But it never seemed realistic in the Canadian setting to attempt to cast such a wide net.

Furthermore, the first meeting of what was to become CMESG/GCEDM chiefly involved university mathematicians and university mathematics educators. These populations seemed the most appropriate to target for a number of reasons. The meetings could then be kept small enough to facilitate the kind of personal interactions the organisers wanted to promote; they could focus on some of the scholarly questions in the field; and they could help to bridge the professional and ideological gaps between mathematicians and teacher educators and researchers. So with some regret the decision was made to develop a programme to attract university teachers in departments and faculties of education and in departments of mathematics. The trade-off under this restriction would be, it was hoped, a greater involvement of university professors of mathematics. CMESG/GCEDM can report some success in attracting to its ranks a number of Canadian mathematics professors (to the extent of approximately a third of the active membership). A higher rate of participation, even if desirable, is not likely given the fact that a serious involvement in education is, for university mathematicians, an additional demand on their time and energy, a commitment rarely recognised or rewarded by their departmental colleagues. In any case, the regular interaction and cooperation of professors from education and mathematics departments within the Study Group remain a significant and treasurable feature.

From the beginning, as can be seen from the lists of working groups and lectures in Appendices 1 and 2, the two main interests of CMESG/GCEDM have been teacher education and mathematics education research, with subsidiary interests in the teaching of mathematics at the undergraduate level and in what might be called the psycho-philosophical facets of mathematics education (mathematization, imagery, the connection between mathematics and language, for instance). There are obvious overlaps with the interests of other Canadian groups. An early decision was made to resist integration with the Education Committee of the Canadian Mathematical Congress (later "Society") even though a group bringing together university mathematicians and mathematics educators might seem to have fitted well there. The original animators felt it was important for CMESG/GCEDM to establish an identity and a professional credibility before getting too closely involved with CMC (CMS), whose Executive Committee, in the 1970s at least, was not noticeably interested in, informed about, or sympathetic to, mathematics education. Subsequently CMESG/GCEDM developed good relations with a revitalised CMS Education Committee and in 1985, 86, and 87, the Study Group met in the same locations as the CMS so that a few of its sessions could be co-sponsored by the two organisations. In 1990 CMESG/GCEDM co-sponsored a day's activities with the Canadian Society for the History and Philosophy of Mathematics (CSHPM).

Many scholarly and academic associations in Canada hold their annual meetings on the same site during the same period, at an event called the Learned Societies Conference. Some of the people who would have liked to be involved in CMESG/GCEDM were accustomed to attend meetings of the Canadian Society for the Study of Education (CSSE), which always participated in the "Learneds," and it was natural for them to suggest that CMESG/GCEDM should hold its meetings there too. Again the initial organising group resisted a move toward immediate integration, though for a different reason. It seemed to them that if CMESG/GCEDM was to develop a distinctive character, and particularly if it was to develop a genuine working atmosphere, it needed to be able to persuade people to commit themselves entirely to the Study Group for the whole of a meeting. Setting the meeting in a situation where n fascinating lectures were always on offer in adjacent buildings would make that dedication difficult if not impossible to realise. So, to the annoyance of a few, CMESG/GCEDM did not join the collection of societies in the "Learneds." (It must be noted here, with considerable gratitude, that the Social Sciences and Humanities Research Council of Canada, which gives a block grant to the "Learneds," has never used its muscle to insist that CMESG/GCEDM belong in order to qualify for financial help.)

Attendance at CMESG/GCEDM meetings has varied between 30 and 70, with most in the 50-60 range. This is a good size for the kind of meetings the Group organises: small enough to give a feeling of community while large enough to ensure a mix of interest and experience. Two-thirds of this number are usually regulars who attend most of the meetings. Membership is predominantly but not exclusively Canadian. The Group benefits a lot from the presence of a few non-Canadians, though it is watchful that the proportion does not grow too large.

## INNOVATIONS

The emphasis on spending a substantial amount of time at meetings in working groups has already been mentioned. The "philosophy" behind this is more than an acknowledgment that "two heads are better than one," or that multiple perspectives on important issues are potentially more illuminating than unitary ones. It goes further and says that it is possible for people to work collaboratively at a conference on a common theme and generate something fresh out of the knowledge and experience that each participant brings to it. This is not easy to achieve, it must be said, perhaps because people are not used to working this way and have not yet learned the techniques. Newcomers sometimes feel that the first 3-hour session allotted to a working group is "wasted" because the group has come together without a common view on the topic, so everyone has to sit through the expression of a lot of different opinions before the group can actually "start." Ways have been proposed to overcome this problem: giving each member of the group papers to read before the meeting, making a clear presentation of the group's programme before members have chosen which group to attend, and so on. But of course the ideal picture of a working group, in which everyone wants to work in exactly the same way on exactly the same questions, is a fiction. The task of the group leaders (there are usually two) is to capitalise on the diversity of expectation and experience within the group while fostering the pursuit of those emergent sub-themes which appear to be going somewhere. It is not surprising that this activity does not always lead to the sort of outcomes that can be immediately written down and polished into a conventional scholarly paper. A well-run working group handles complexity very effectively, but effective ways of recording its achievements are difficult to develop.

The emphasis on working groups influences other aspects of the CMESG/GCEDM meetings. People are not divided disjointly into a set of those who present and a set of those who sit and listen. There are presentations of a quite conventional kind, but in the context of the meeting they also become subjects for discussion. An innovation which symbolises this is the "discussion hour" scheduled on the day following a plenary address at which the members discuss the talk with the speaker.

CMESG/GCEDM programmes always have at least one slot in the timetable for "ad hoc groups." Any person may volunteer to make a presentation or lead a discussion, and these items are added to the programme (subject to the availability of time and facilities).

The intention of these various opportunities is to encourage members to take an active part in the meetings. The policy would be ineffective if it did not deliver, and if it were not situated in a relatively relaxed and accepting atmosphere. As in school, people would soon stop making contributions if these kept getting shot down in flames. A CMESG/GCEDM meeting is free of the point-making and competitiveness that are features of many academic gatherings. People listen to other people, with respect if not necessarily agreement.

## IMPACTS

With fifteen annual meetings to its credit, and a core of active members, CMESG/GCEDM now certainly exists. Although the first group of enthusiasts in 1978 may have hoped for more, they probably expected less: the Canadian environment for innovation is notoriously harsh. The Study Group hasn't produced the discussion documents, or made the public and political pronouncements, or developed the regional and local networks, or achieved any of the partial agendas that people have at times proposed for it. But it exists. And it holds annual meetings. And these are, to judge from the comments of regulars and of newcomers, appreciably different from, and more involving than, other meetings that the same people go to. In an important sense CMESG/GCEDM is its annual meetings since these are where what is characteristic of CMESG/GCEDM actually takes place — its study-in-cooperative-action. (For Plato, philosophy was to be found at its best in the serious talk of friends rather than in the story of it that someone might write up afterwards.)

CMESG/GCEDM now exists in Canada alongside the CMS Education Committee, whose natural interest inclines more to the teaching of mathematics at the tertiary level. Both are small, national groups catering mainly to university teachers. Each province in Canada has its own separate association of teachers of mathematics (Québec has four: three francophone, one Anglophone). Two provinces, Ontario and Québec, have associations of mathematics advisers (alternatively called "coordinators" or "consultants"). Many high school teachers and advisers belong to the National Council of Teachers of Mathematics (NCTM) and attend its annual meetings. The NCTM claims coverage of Canada, indeed, and always has a Canadian on its Board of Governors, but rarely interests itself in particularly Canadian issues. Many Canadian mathematics educators belong to the American Educational Research Association (AERA) or its subgroup SIG/RME (Special Interest Group for Research in Mathematics Education), just as many university and college professors of mathematics belong to the American Mathematical Society or the Mathematical Association of America. (And it is likely that a majority of school, college, and university teachers of mathematics are not active in any of the above.) This is a uniquely fragmented situation. There is no body in Canada able to deal with the whole of mathematics education at all levels, no national voice speaking about mathematics education to governments and the public — though perhaps this matters little in a country which has no national educational policy.

When it comes to impact and influence, though, who can be sure what Canadians lose by not having a powerful voice speaking on behalf of mathematics education? The USA and France, for example, both have very powerful professional organisations able to talk to governments, but it is by no means certain that their influence is always good, judged from the viewpoint of the "consumers" of mathematics education in the schools. (National medical associations, to consider a possible parallel, do not always seem to be arguing or advancing the cause of the sick.) CMESG/GCEDM lacks a powerful voice, but it has influenced, perhaps changed, a number of individuals.

The Study Group takes as its essential position that the teaching of mathematics and all the human activities that are connected to it can, and should, be studied, whether the study has the form of an individual's reflections, the reasoned argument of professional colleagues, or the more formal questioning of empirical or scholarly research. By putting this emphasis CMESG/GCEDM has signalled to Canadian mathematics educators the importance of scholarship and research in a field that often seems dominated by folklore. The Study Group has provided a forum where research plans can be discussed and an encouraging atmosphere where novice researchers can find out how to begin. Mathematics teaching may go back to "the dawn of history," as the journalist might say, but mathematics education as a field of study is only a few decades old. It has no traditions of research and scholarship: these are only now being developed.

In brief, through its activities CMESG/GCEDM has given some mathematics educators a taste for research and shown them how to get started. It has shown them that their puzzlement about some aspects of mathematics is shared by many mathematicians. It has shown some mathematicians that learning can be studied and that teaching might be made into something more than flying by the seat of the pants. A sufficient number of such small victories could launch a revolution.

## JOBS TO DO

As suggested above, CMESG/GCEDM has already played a strengthening and encouraging role in the Canadian effort on research in mathematics education and it seems clear that it should continue and perhaps extend its work in this direction. There is a long way to go, as is generally acknowledged, before mathematics education research becomes a resource that everyone in the teaching business will be glad to be able to draw on, but nothing less should be demanded of it. Understandably most teachers find that most research to date is immature: it fails to convince because it cannot yet match the complexity of a good teacher's professional insights. Nevertheless it is extremely important that the work go on. Research takes a significant stand against an extraordinarily widespread but destructive belief: that the teaching of mathematics is essentially unproblematic. That this might not be so was perhaps first signalled by Poincaré when he asked why some people never manage to acquire mathematical knowledge. Research, of course, in common with other theoretical positions, can as readily be used to "explain away" as to explain; nevertheless its insistence on inspection, reflection, and trial, is an important corrective to the naive view that teaching and learning are transparent activities.

On a less broad front, CMESG/GCEDM still needs to work on improving the amount and quality of the interaction between mathematicians and mathematics educators. There is a job to be done while there are still mathematics educators involved in teacher education and research who have only a tenuous acquaintance with genuine mathematical activity, and while there are still mathematicians who think that all questions belonging to the field of mathematics education are intrinsically trivial. University mathematicians as a class are not noticeably modest. It is probably not too much of a caricature to say that in general they seem happy to admit–grace à Descartes–the god-like character of their main activity. They are not in general reluctant to take advantage of the universities' traditional favouring of academic over professional knowledge. Moreover, mathematicians have been deemed successful in what is recognised by everybody as a difficult intellectual discipline. Given all these advantages, they sometimes fail to recognise that the skills and sensitivities that have served them well in working on mathematics are not necessarily the ones that can meet the challenges presented by mathematics education.

There is a need in Canada to make public a more accurate picture of mathematics education, one which admits that its development has only just started, but which also shows that its heuristic is effective and its arguments capable of being made, within reason, rigorous and disciplined. If some real substance can be put into such an account, a greater respect for mathematics education must follow. CMESG/GCEDM is in a good position to work with mathematicians on improving the image of mathematics education as a field of study.

These are long-term goals — ideals, perhaps — which could point CMESG/GCEDM in a certain direction but do not spell out in detail how it might reach them. Probably the future of CMESG/GCEDM, in any case, will be shaped by a combination of internal and external forces most of which cannot now be predicted.

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# Plenary Lectures

# Conférences plénières

# MATHEMATICS EDUCATION RESEARCH IN CANADA: A PROSPECTIVE VIEW

Tom Kieren
*University of Alberta*

Quite apart from the fact that these three talks gave us a sense of why we started, what we felt were the issues of the time, and what we hoped for, they remain, for the reader of today, absorbing and timely. All three speakers were leaders in their areas at that time and continued as key participants in CMESG activities for decades afterwards. Together they examined the big three aspects of our work, mathematics education research (Kieren), teacher education (Gaulin) and mathematics education (Coleman), and over the following 40 years, these three areas danced forward and intertwined along many rich axes.

Non seulement ces trois conférences nous avaient permis de donner un sens à notre action initiale et de décrire ce que nous percevions comme les principaux défis à cette époque ainsi que ce que nous espérions pour l'avenir, elles demeurent, encore aujourd'hui, captivantes et pertinentes. Les trois conférenciers étaient des experts de leurs domaines, et ils ont continué à jouer un rôle clé au GCEDM pendant des décennies. Ensemble, ils ont exploré les trois pôles de notre travail, soit la recherche en didactique des mathématiques (Kieren), la formation des enseignants (Gaulin) et l'éducation mathématique (Coleman), et, depuis 40 ans, ces domaines ont pu poursuivre leur développement, se liant de multiples manières dans des interactions riches.

## INTRODUCTION

### WHAT IS OUR VENUE?

Mathematics education research, like much of educational research, has not been given an entirely charitable construction in the past. Its value has been questioned, and even when it contained solid advice regarding theory as well as practice, this advice was ignored in favour of the fad of the moment or the comfort of old ways in the face of the problems of the day.

Still, the document *Mathematical Sciences in Canada* cites a general dissatisfaction with mathematics programmes and instruction in the schools and universities and other tertiary institutions as well. There is a strong call for improvement of programmes and practices. What might be the bases of this improvement? At least some of the input for these bases should come from sound educational research.

Mathematics education research makes use of mathematical ideas, but certainly differs from research in pure mathematics both in method and content. The issues of concern for a mathematics educator - for example, "How does a learner build up the idea of function?" - may be of little interest to the mathematics research community (although it could be argued that real insight into mathematics *per se* comes from studying its learning). Similarly, psychological researchers, although sometimes using mathematical settings, are not generally interested in the mathematical development of an individual or the psychological aspects of mathematics acquisition or use. Thus, the researcher in mathematics education has a unique sphere of interest: the development of mathematical constructs in persons, the mechanisms used in this development, and the conditions necessary for this development.

### NECESSITY FOR CONNECTEDNESS

The unique sphere described above is not one with closed or smooth boundaries. Because the problems of studying mathematical constructs and their growth and development is complex, this research must be internally and externally connected. These external connections might be with mathematical or psychological research. But it is as likely that they will be with a broader spectrum involving other areas of endeavour, such as research on learning of science or the development of higher level constructs, or general research on teaching. The complexity of problems facing the mathematics education researcher suggests that single isolated studies will yield very limited results, hence internal connectedness and cooperative efforts are needed. Perhaps the critical comments referred to above stem from the lack of such connectedness in much previous mathematics education research.

### OVERVIEW

It is the purpose of this paper to develop a picture of the potential for mathematics education research in Canada. Although the next section of the paper attempts to give snapshots of past and current mathematics education research, the thrust of the paper will be prospective and not retrospective. To give a framework to a general research scheme, Section 3 will deal in some detail with the notion of a "construct" and the ways in which constructs grow and are developed by human learners. Suggestions for major types of research efforts as well as suggestions for mechanisms for fostering such research in a Canadian context are found in Section 4.

## THE STATUS OF MATHEMATICS EDUCATION RESEARCH

### COGNITIVIST VS. BEHAVIOURIST

Over the past fifteen to twenty years, research in mathematics education has been influenced by one of the two sides in a more general conflict in educational thinking. One side, the behaviourists, have sought immutable cause-and-effect laws relating the sequencing of instructional stimuli and predictable student responses. This camp has sought to develop instructional sequences individualized on the basis of the learner's current learning history and has made use of hierarchies of behaviourally-stated objectives. The cognitivist camp has sought to discover the schema which individuals have and use in dealing with their environment. They are interested in an individual's development over time and the tailoring of instructional settings congruent with the learner's stage of development and mental structures.

These educational positions (greatly oversimplified) are but a recent manifestation of an age-old philosophical controversy. This controversy revolves around the question, "To what extent is a human a being who simply responds to the environment for his own or the general good?" This question has been central in the fields of ethics, religion, and science, as well as in education. While it is doubtful that this question will ever be resolved, in the sense of a consensus position, it is almost certain to continue to influence the search for knowledge about human endeavours. It is certainly true that mathematics education research has moved beyond the behaviourist-cognitivist dichotomy suggested above. Nonetheless, the question of the nature of human behaviour continues to influence research and does influence the suggested course of this paper.

### TRENDS IN RESEARCH

In what way does mathematics education research fit within or go beyond the dichotomy discussed above? Bauersfeld (1976) suggests a number of types of research which have recently been done, some of which transcend the specific behaviourist-cognitivist conflict and some of which represent a departure from traditional experimental methodology.

There is still an immense number of studies done using the experimental paradigm of comparing the effects of two (or more) treatments or states on mathematical achievement or affective variables. Some of these have taken into account interaction effects which can give hints for matching treatments and groups of students (Bauersfeld, p. 5), but even these have very limited contributions to make to knowledge. This is due to the complexity of the teaching-learning environment, which can easily conceal or distort experimental effects.

A second style of research is typified by the Soviet practice of "teaching experiments". Here mathematics learning is studied in a group or class over an extended period of time through variation of conditions of instruction. There is less emphasis on psychometric measures, and outcomes are reported more in terms of dynamic process descriptions (Bauersfeld, p. 6).

A third trend is seen in research which deliberately involves teachers as co-investigators. Such research studies the decision-making efforts of teachers and effects on the teaching-learning environment. Much of such research is very informal and introspective, but some has involved sophisticated study of teacher-student interaction, though there has been very little on student-student interaction.

A fourth trend, which represents a clear transcendence of the cognitivist- behaviourist polarization, is the increasing number of studies involving an information-processing approach. Here there is an attempt to describe internal mental functioning and yet to give a time sequence of actions to describe processes.

A fifth trend (and one of which this conference, and particularly this paper, is both a symptom and a part) is a search for frames of reference for knowledge about mathematics learning and development. This involves a search for statements about the nature of the mathematical sciences, about models for teaching and learning, about the nature of mathematical abilities and the interaction of these with learning environments. These more philosophical studies have been followed, particularly in the latter case, by numerous attempts to identify and trace the abilities of students across time and situations.

## A NOTE ON COMPLEXITY

One of the conclusions drawn from a consideration of Bauersfeld's (1976) trends, is that mathematics learning is being viewed as a more complex phenomenon and there is a movement away from research questions, paradigms, and methodologies which ignore, mask, or try to oversimplify the situation. Indeed two hypothesized theorems pertinent to this conference might be:

<u>Complexity Theorem (C.T.):</u>

$$\text{C(learning)} \rightarrow \text{C(instruction)}$$

<u>Teacher Education Corollary</u>

$$\text{C.T.} \rightarrow \text{C(teacher education)}$$

The complexity of the task and its attendant richness are heightened as one moves from a narrow frame of reference for mathematics to a broader construct of mathematical science and its position and interconnections.

## CANADIAN CONCERNS

A central concern of Beltzner *et al.* (1976) with respect to mathematics education in Canada is growth. This is seen in its personal sense in the call for an education in and the opportunity to practise "mathematization". In a collective sense this growth emphasis appears in a desire for a more extensive view of mathematics, and particularly in a renewed emphasis on applications of mathematics.

This new emphasis on growth is, perhaps, a call for the renewing of and broadening of contact between the mathematics and mathematics education communities and the societal and personal dimensions of the broader Canadian community. From an "employment" point of view there is a simultaneous need for persons skilled in technology and for persons able to fill diverse service positions. Because of unique Canadian problems in communications, transportation, and resource management, Canadian solutions to these problems may be prototypic for general human problems in these areas. Because these demands are non-trivial there should be a sense of mission in the mathematics education community. Because of the technico-mathematical components of society, the goals of personal growth in mathematics should enhance the acuity with which a person can view the contemporary Canadian scene.

These growth goals call for changes in the mathematics curriculum at all levels. These changes cannot easily be incorporated within the framework of a textbook and have broad implications for teacher education as well.

Of course the above changes suggest many changes in mathematics education research. Among these is the need for a deeper and broader understanding of mathematical notions in persons of all age levels, and the patterns of growth of such notions. Beyond this broad research need, one specific area of study is the impact of computational technology on both the curriculum and learning in the mathematical sciences.

Evaluation is currently being carried out on the effects of current practice on mathematics achievement. This work, of large scale, and ongoing in many provinces, seeks to answer diverse questions. It can, should, and does, serve as a stimulus to mathematics education research.

The mathematics education research community in Canada does not have a long history or tradition. However, the recent work of this community has relevance for the concerns expressed in *Mathematical Sciences in Canada* as well as forming a basis for the work still to be done that is described in the remainder of this paper. A portion of this work falls directly in the category of variable relationships noted above, the particular merits and relevance of which must be judged in each individual case.

There has been substantial research work and writing in Canada on cycles within mathematics learning. These have focused particularly on the variety of personal activity and related curriculum experiences involved in what Beltzner *et al.* (1976) would call "mathematizing".

There has also been considerable recent research on cognitive development as it affects and effects mathematical development. This work has, in part, derived from the work of Piaget in content and in method. It has also been concerned with differences in structural learning across various ages.

A fourth category of Canadian mathematics education research has concerned itself with the structure, style, and manner of mathematical knowing. Some of this work has been philosophical in nature and sought either to describe aspects of personal mathematical knowing or the curriculum antecedents generative of such a process. Other work has entailed the detailed observation of persons, particularly young children, as they worked within situations with mathematical content. This work has sought to define the character of mathematical knowing as seen in the patterns of behaviour of children.

As suggested above, much of this work is closely related to the concern for personal growth. This research has gone on in a milieu of a great deal of curricular experimentation, some of which, at least, has been creative and carefully studied. This aspect of Canadian mathematics education research, informal though it may be, cannot be ignored and indeed needs strengthening.

## ON CONSTRUCTS

Important questions raised by Beltzner *et al.* (1976) are:

- What is the contemporary view of the mathematical sciences?
- What is a personal view of mathematics in general, and of one's own mathematics?
- How does one build up mathematical notions?
- How does one use mathematical notions? How does this use affect society?

Before proceeding to discuss possible research directions, this section of the paper gives a general characterization of mathematical knowing and the ways in which this is developed.

### MARGENAU'S IDEA OF CONSTRUCT

In trying to characterize scientific epistemology, the philosopher Margenau (1961) divides phenomena into two categories. The first of these comprises the elements of physical reality, facts, or, as he chooses to term them, "protocols". These are seen as phenomena which are not dependent upon human construction. The second category contains "constructs", the deliberate ideas which a person builds up about phenomena and which he or she can ultimately test against other constructs or the plane of protocols. It should be noted from Figure 1 that some constructs are in close proximity to the P-plane and offer limited explanatory power and control. Other constructs are more "abstract" - that is, further from the P-plane; these can be more powerful and give the person broader control.

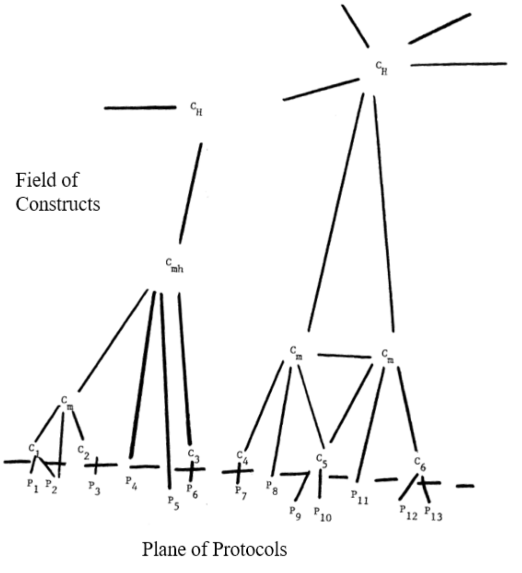


Figure 1

A person's total mathematical construct consists of his or her network of sub-constructs, some very narrow, others much broader in their perspective. While it is difficult from this point of view to speak of the construct of mathematics or, even more difficult, the mathematical sciences, these notions result in part from societal consensus but more from the product of a combination of tests and mathematical argument.

## IMPORTANT CHARACTERISTICS OF CONSTRUCTS

Margenau (1961) describes a number of important characteristics of constructs, two of which are especially useful for the purposes of this paper and for mathematics education in general. The first of these has been alluded to above and is termed the *extensibility* of the construct. This refers to the breadth or variety of phenomena to which the construct addresses itself. It has been suggested, for example, that the rote learning of computation leads to constructs which have very little power or breadth of applicability. One might say that a goal of the modern mathematics movement has been to broaden a person's constructs through the understanding of mathematical structure.

Particular mathematical constructs do not and should not stand in isolation from one another. Further, they should not stand in isolation from a person's broad range of constructs of reality. Thus constructs which are *connected* are of particular value. This connection may be internal or external. For example, the sub-construct of additive inverse is internally connected to the other notions about the domain of integers in a variety of ways. It is externally connected to the construct of inverse transformation in the geometric sense, and to a broader and more extensive notion of inverse in general.

Thus it can be conceived that mathematics education has a professional responsibility to provide experiences which are generative of extensive and connected mathematical constructs in our clients, our students.

## ON CONSTRUCT FORMATION AND BUILDING

### On cycles

As suggested, a prominent theme of Canadian mathematics educators has been the description and study of cycles. Dienes (1961), for example, uses cycles of "play" to describe the building up of mathematical ideas - a movement from object or element play to symbolic play and hence to applicational or extensive play which may, in turn, be a foundation to a new cycle. Dawson (1971) uses the epistemology of Popper and Lakatos for a base and defines viable cycles of observation, testing, and proving (e.g., O T P, P O T) in the development of mathematical ideas. Sigurdson (1976) sees six phases in problem solving (or construct development). These are: the perception of the mathematical content of a situation; the posing of an answerable mathematical question; the making of a model or theorem to help answer the question; the validating of the theorem; the generalizing of the theorem; and, finally, perceiving and/or developing the axiomatic supports for the model or theorem. While probably not unique to mathematical construct development, all three accounts describe formalizing and generalizing processes which are part of the mathematical milieu.

The cycles described above might be termed micro-cycles in that they pertain to the development of a single subconstruct or the solution to a single problem. However, they are suggestive of a cycle of macroscopic construct development which may be pertinent to larger mathematical constructs. As seen in Figure 2, this cycle has three general stages. In the first, the person encounters a construct in a variety of representations and particularly explores the elements of its mathematical variates. While representation theorems in mathematics are designed to produce logical economy through isomorphisms, it may not make constructive or peda-logical sense to subsume construct development under a single variate.

The second stage of the cycle involves formal development. This involves the ability to work with the construct quickly and easily using standard forms, notations, etc.

The third level entails advanced exploration using the construct as a basis or tool. This may involve more advanced mathematics (e.g. rational numbers → rational expressions) or may involve some special technical application (rational approximation to the internal circumference of a metal tube).

This development cycle can be applied to a variety of mathematical constructs. In school mathematics, particularly at the upper elementary and secondary school levels, we have concentrated on the second level to the detriment of the complete cycle of construct development and the consequent broadening of the scope of an individual's view of mathematics.

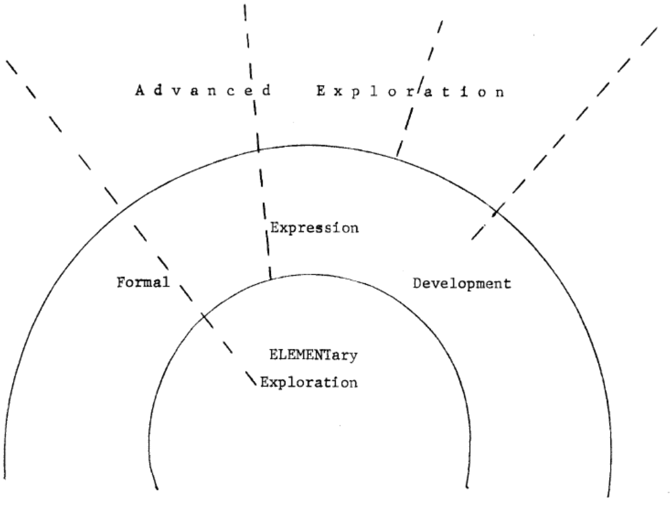


Figure 2

<u>On a "types" problem</u>

One of the effects of this almost exclusive concentration on the formal development level of construct formation is the accompanying view of a construct entirely as a behavioural surface of formal manipulation, frequently computation. While not denying the importance of such manipulation, we notice that this has led to the formation of empty or sterile constructs. Margenau (1961) saw a similar problem in a science which emphasized experimentalism without supporting theory, the results of which were shallow and subject to collapse. Similarly, empty mathematical constructs collapse, as seen in poor personal performance in later mathematics or in its application.

In a way this collapse suggests an analogy in the instruction-learning field to the classic Russellian theory of types. In that theory confusion of types led to paradoxes. In construct development, and curricular experiences designed to that end, the confusion of a formal surface with a complete construct leads to meaninglessness for the learner (Olson, 1977).

<u>On mechanisms</u>

How are mathematical constructs built up by children and adults? This still remains a puzzling question which should be a focus for research. It is apparent that a person, consciously or unconsciously, uses a variety of mechanisms and schemes in exploring, developing, and using mathematical constructs. One category of such mechanisms, *developmental mechanisms*, although partly the product of experience are not the product of any formal learning experience and are not dependent upon such experience. Examples of such are conservation of various sorts, class inclusion, and proportionality. The second category of mechanisms, *constructive mechanisms*, although general and in a sense "natural", are likely to be the product of some type of instruction. Examples are counting, partitioning, and algorithmic thinking. Such mechanisms deserve much more detailed study and their potency needs to be recognized in our curriculum-making efforts at all levels.

Building Mechanisms

| Developmental | Constructive |
|---|---|
| Conservation | Counting |
| Simultaneous Comparison | Partitioning |
| Reversibility | Applying Structure |
| INRC group | Using Inequalities |
| | Transformations |

Figure 3

### ON "WHAT IS MEANT" AND "WHAT IS LEARNT"

Bauersfeld (1976) claims that there are important distinctions to be made between what is meant, taught, and learnt. It has been the purpose of this rather extended section to characterize in general "what is meant" in mathematics, using Margenau's notion of construct. This notion entails something built up by individuals in their own minds. Thus there can be at least a rough parallel between what is meant and learnt provided there is *an appropriate form of mathematical analysis*, that *the individual's construct is not a behavioural surface* without support, and that *curriculum and instruction are based on broad mathematical constructs*.

## IMPLICATIONS AND DIRECTIONS FOR RESEARCH

The discussion in the previous section contains a wide variety of researchable hypotheses. Bauersfeld's (1976) excellent analysis suggests a spectrum of potential for mathematics education research. The suggestions given in this section are by no means the "whole cloth" of research. Yet they represent a rather broad but hopefully cohesive direction and dimension of research. Further, this research has obvious links to much successful personal work already ongoing in Canada. Further, it can have some direct if not immediate (and maybe this possibility is underestimated) results for mathematics learners at all levels.

### BASIC CONSTRUCTS

Some of the needed curriculum research is analytic and philosophical in nature. Given today's world, what are the basic constructs to be included in a mathematics curriculum? This question was asked in a very limited way by the reform movement of the last 25 years, but the answers seemed to dwell more on the depth of the constructs than on the kinds of construct. The question was also answered in a speculative way for a limited range of students in the Cambridge report (1963), and the work at CEMREL which has followed from it. There have also been curricula (e.g. Papy (1970)), or parts of curricula (e.g., geometry in Ontario) which reflect certain definite answers to this kind of question.

At the early elementary school level and perhaps in university honours curricula there is less need for this kind of study. However, in upper elementary school, secondary school, many university programmes (e.g., teacher education), and in other tertiary education programmes, answers to such basic construct questions are overdue.

In answering such questions, the nature of society will have to be considered. For example, in what ways does the availability of computing devices enter into deliberations on basic constructs? Similarly the nature and content of mathematics and the basic knowledge about human development *vis-à-vis* mathematics will also be bases for answering such construct questions.

BASIC MECHANISMS

A question similar to the above can be asked about mechanisms. Within our selected constructs what are the mechanisms useful for their development? Are there mechanisms which have a broad range of functioning (e.g., counting) and deserve a central curriculum role of their own?

While such a question calls for philosophical and psychological analysis, it also calls for active research with persons at various age levels. This research will entail the observations of persons in situations designed to "trigger" the particular mechanism and would attempt to ascertain how the mechanism functioned and developed.

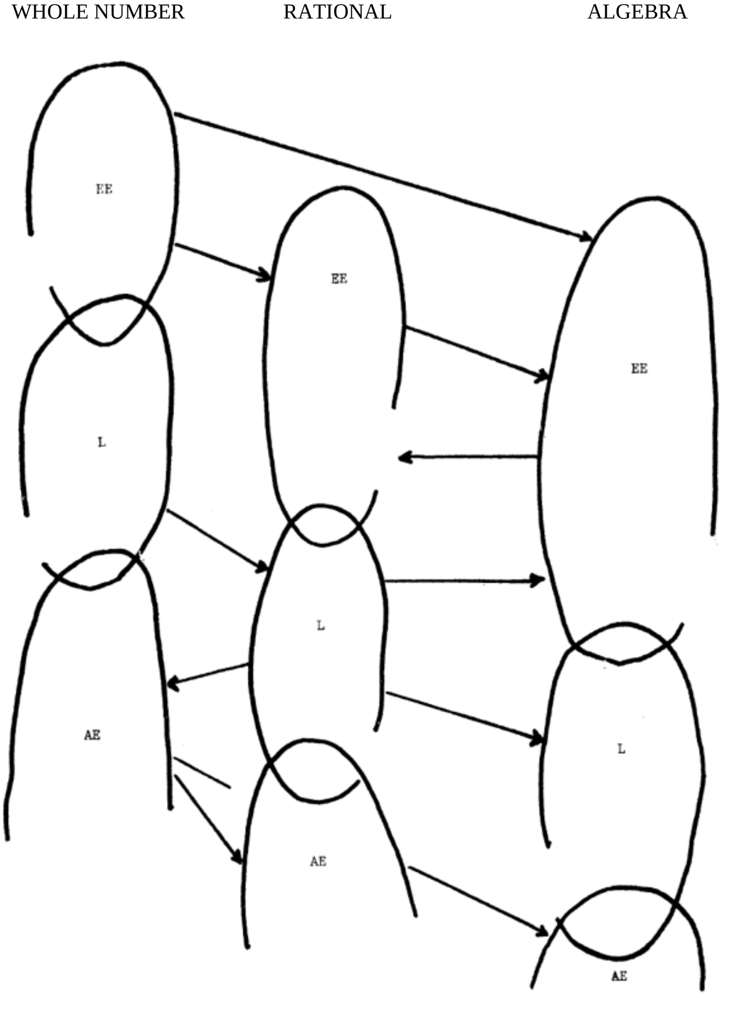


Figure 4

Construct validation

There are many ways in which a construct can be tested. One way is to lay its sub-constructs against the qualities of a maturely functioning person within the domain of consideration and see analytically (and empirically if this is desired) if the construct—based curriculum meets functional needs.

A second validation is to test whether the developed construct is generative of learning activities appropriate to the group of intended learners. Further, do such activities also induce the development of constructs in a vast majority of the intended learners? (This has been a serious "textbook" problem in the past.)

On extensibility and connectedness

One important test of a developed curriculum is an assessment of its constructs and mechanisms. To what extent does the curriculum highlight powerful constructs and mechanisms? (This has been partly done in past searches for "unifying" mathematical concepts.) If a curriculum is to be useful today, it must be based on constructs of broad importance which enlarge the scope of the learner's exploratory and explanatory power.

CYCLE RESEARCH

The notion of "cycle" has been important in this paper and in recent Canadian mathemat1es education research. There are a variety of researchable questions which fall in this category.

For discussion purposes, this paper has posed a three-level construct development cycle. Given a particular basic construct, what are the characteristics of each of the three levels or stages? In some senses this is a very "nitty-gritty" question. Yet it is central to the development of learning experiences. If there is no answer to such a question, the foregoing philosophical discussion remains only that. Although answering this question has an "armchair" component, it should also have a large component of work with appropriate learners in particular experiences.

A related question pertains to mechanisms. Which mechanisms contribute to development at which stages? Answering this question allows a different way of studying the validity and particularly the extensibility of particular mechanisms.

A third category of "cycle" questions concerns micro-cycles. How are the cycles suggested by Dawson (1971), Dienes (1961), or Sigurdson (1976) pertinent to construct and/or mechanism development at particular macro-cycle stages? For example, are they more pertinent at the two exploratory levels than at the formal development level? Are they (the micro-cycles) different in character at various macro-cycle levels? Are they developable and are they unique to each construct or mechanism? These questions present a rich field for study both at an experience development and experience testing level. They, with other questions in this section, allow researchers, and indeed force researchers, to be precise about their intents, transactions, and outcomes - to use Stake's (1971) terms.

On technology

Technology, particularly computing technology, will have a profound effect on mathematics learning and instruction as it affects and becomes part of the basic constructs. The kinds of activities which relate to the use of such technology will also have an impact on instruction. Four such activities are:

- algorithm design
- coding

- machine application
- data organization and study

One might consider the first of these to be representative of a profound mechanism. Engel (1976) suggests that the mathematics curriculum centre on this basic mechanism.

A less controversial matter is suggested by the figure below.

| | E.E. | F.D. | A.E. |
|---|---|---|---|
| Algorithm Design | | | |
| Coding | | | |
| Machine Application | | | |
| Data Study | | | |

Figure 5

How do these informatic activities contribute to levels of construct development? Because of the computational power provided by a machine, some areas of advanced exploration of mathematical constructs become feasible and convenient. It may be that algorithm design and coding are key personal activities in the formal development aspect of construct building. Of course, these statements are but two of many testable hypotheses.

<u>On language cycles</u>

A major concern in mathematics learning is the use of language and the formality of this language. A question with respect to construct development is whether there is a language-use cycle which parallels the development cycle. One hypothesized cycle is given below.

Informal codes

Formal language

Technical language

Figure 6

The first level is suggestive of learner-developed expressions about the mathematical phenomena being explored. There may be different codes pertaining to different variates of a construct, for example.

The second level relates to the standard language used with a construct. Learning such language may well present a connotation problem with a single standard code now applying to a wide variety of construct variates (Hillel, 1976).

The third level pertains to certain "standardized" uses of language which are peculiar to an application of a construct (e.g., rational numbers applied to measuring devices in a millwright's trade). Here the user must relate this language to both the standard language and his or her construct. This proposed cycle and its relationship to construct development contains numerous testable hypotheses for researchers and developers.

## MECHANISM RESEARCH

There has been considerable research to date in the area of developmental mechanisms. There is considerable Canadian research on the growth of such mechanisms with respect to mathematics (Harrison, 1976; Drost, 1977). There has been some (Bourgeois, 1976), but much less, attention paid to the growth of constructive mechanisms (counting, partitioning, algorithm design). This area of study needs considerable research with attention paid to the *choice of mechanisms* and to the development of useful measuring devices and techniques.

A second question is the relationship between mechanism growth and construct development. This has been studied on a limited basis - for example counting and whole numbers (Steffe, 1976), and measurement and fractional numbers (Owens, 1977; Babcock, 1977). There are many important questions yet to be asked in this area.

## BRAIN PHYSIOLOGY

Research on brain-functioning is just reaching a stage where it can have impact on mathematics education research. How is brain-functioning a basis for construct and mechanism growth and use? Questions of this nature will likely prove an interesting field of basic research in the near future.

## A NOTE ON TEACHER EDUCATION

Research activity such as that suggested above has implications for teacher education. Some of these are direct in the sense that they concern the necessary mathematical constructs and mechanisms for teachers. Perhaps more important is to think about mathematics learning in terms of the learner's constructs and mechanisms. For the teacher of younger children, this likely means a more intensive mathematical education than is currently acquired in Canadian teacher education programs. For other teachers this likely means a broadening of their education in significant ways, both in terms of applications to science, commerce, social science, etc., and of extensible constructs and mechanisms.

## SUMMARY

The research problems suggested above are far from being clean and simple. They represent a recognition of the complexity of mathematics and its learning. In general, solutions to these problems will give explanatory assistance to those dealing with mathematics learning in the raw, the teachers, but certainly do not offer a panacea for currently perceived ills in our field.

By design and by necessity the research problems suggested above are interconnected. It is only by a network of research that the complex problems posed can be studied effectively.

Beyond the research already suggested, and included in it, is a need for studying learning relationships in mathematics. What is the nature and impact of teacher-student and student-student interaction with respect to the learning cycles, and to construct growth and use by individuals?

There is a need for *much more* interrelated mathematics education research to tackle these problems. Perhaps our small numbers in Canada and our personal interrelationships will allow us to engage in such interrelated research.

## RECOMMENDATIONS

What can be done to effect the cooperation needed in Canadian mathematics education research? In the short run two things suggest themselves. Since we need better information as a base, it would be useful to have a bibliography, briefly annotated, of work done in the last five years. This should include university and school-sponsored research and should include various graduate-level theses, as well as research done by professionals in the field. Such a bibliography would outline our current strengths, weaknesses, and personal resources, *vis-à-vis* the task suggested above. It would also give some indication of potentials for cooperative effort.

An active newsletter describing current work and supporting interpersonal research communication is a second short-term need. This would be a specialized informal document and should complement the more formal organs already available.

In the longer run there is a great need for cooperative research. Because the problems are complex, several persons are needed to investigate parts of these problems in a pre-planned way using a language which is understandable to all working in an area. At a first level such cooperation needs to occur among professional mathematics education researchers. But because the problems have many facets and levels, this cooperation needs to include the broader academic community, including linguistics experts and philosophers, for example, as well as mathematicians, computer scientists, and psychologists.

Howson (1976) states that an increasing number of teachers are active in curriculum development on a worldwide basis. There is need for cooperation among researchers and teachers (who could be the same persons). The former can give the latter advice about the framework and parameters of the curriculum. The teachers can provide dynamic feedback about various situations to the researcher.

Finally, there is a need for groups of researchers and teachers to meet regularly on problems in mathematics education in Canada. Because of our geography, it may be well to look at the French IREM as a model of regional groups and centres. It would be hoped that such centres would provide the support and life necessary to tackle the problems outlined above in a substantial and ultimately practical way.

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# INNOVATIONS IN TEACHER EDUCATION PROGRAMMES

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Quite apart from the fact that these three talks gave us a sense of why we started, what we felt were the issues of the time, and what we hoped for, they remain, for the reader of today, absorbing and timely. All three speakers were leaders in their areas at that time and continued as key participants in CMESG activities for decades afterwards. Together they examined the big three aspects of our work, mathematics education research (Kieren), teacher education (Gaulin) and mathematics education (Coleman), and over the following 40 years, these three areas danced forward and intertwined along many rich axes.

Non seulement ces trois conférences nous avaient permis de donner un sens à notre action initiale et de décrire ce que nous percevions comme les principaux défis à cette époque ainsi que ce que nous espérions pour l'avenir, elles demeurent, encore aujourd'hui, captivantes et pertinentes. Les trois conférenciers étaient des experts de leurs domaines, et ils ont continué à jouer un rôle clé au GCEDM pendant des décennies. Ensemble, ils ont exploré les trois pôles de notre travail, soit la recherche en didactique des mathématiques (Kieren), la formation des enseignants (Gaulin) et l'éducation mathématique (Coleman), et, depuis 40 ans, ces domaines ont pu poursuivre leur développement, se liant de multiples manières dans des interactions riches.

First, I wish to thank the organizers of this conference for inviting me to give this lecture. I also wish to congratulate them for having succeeded in organizing such a meeting of mathematics educators and mathematicians from all over Canada and for having done it so well. Such an opportunity to gather, to share information, to discuss current problems and issues in mathematics education, and to plan concerted activities for the future has long been badly needed in Canada and I sincerely hope follow-up activities will be organized on a more permanent basis.

Like Tom Kieren, I shall attempt to focus on *general trends and issues* in the field of teacher education, more particularly on those which might be relevant in Canada in the near future. I must confess that my knowledge of the present state of teacher education in many Canadian provinces is deficient, and I apologize in advance for possibly omitting to mention important realizations, concerns, or trends in some parts of the country.

My presentation will follow the following lines:

1. The traditional organization of teacher training
    - a. Preservice teacher training (PRESET)
    - b. Inservice teacher training (INSET)
2. Some innovations and new directions developing in teacher training
    - a. A study by Coutts and Clarke on the future of teacher training in Canada
3. One current programme showing several innovations in inservice teacher training: the PERMAMA programme
4. Same thoughts about the role and responsibility of universities in teacher education in the future.

Although many remarks will relate to teacher education in general, it remains understood throughout this presentation that the innovations and trends reported here specifically concern the training of teachers of mathematics.

## THE TRADITIONAL ORGANIZATION OF TEACHER TRAINING

In order to better appreciate some of the new directions developing in teacher training which will be mentioned in Part 2, I shall first briefly point out some features of the traditional organization of teacher training which is still quite common today.

### PRESERVICE TEACHER TRAINING (PRESET)

Traditionally teacher education in universities has been chiefly conceived and organized in terms of *preservice* teacher training. In most places, inservice teacher training has been subordinated to PRESET or treated as a second-order priority.

Generally PRESET in Canada is done *in the universities*. Prospective *elementary* teachers are trained as generalists within a B.A./B.Ed. programme, with *little mathematics* and *some mathematics methodology*. On the other hand, prospective *secondary* teachers are trained as specialists within a B.Sc./B.Ed. programme, with *a great deal of mathematics* and *some mathematics methodology*.

In most universities, programmes of preservice teacher education have an *extensive common core* of *compulsory* courses or activities and do not allow much opportunity for flexible individually-tailored programmes for students. The philosophy underlying this is essentially that PRESET should prepare every teacher *for his whole career* and accordingly should include many courses and activities considered to be fundamental and essential.

To a large extent, decisions about the structure of PRESET programmes and about the objectives of the courses they include are made *a priori* by government and university people, with *little participation* by the students concerned and/or by people actually teaching in schools. Moreover little continuous evaluation of such programmes is usually done.

Traditionally, except for the practice teaching period and a few workshop-style activities, a majority of the courses are of the *lecture type*, followed by exercises, assignments, etc. This seems true for both mathematics courses and foundations courses in education.

There are at least three classical problems in preservice teacher education which still persist and deserve special mention:

1. Lack of integration of the various components of PRESET programmes
   Many PRESET programmes, whether aiming at preparing elementary school generalists or secondary school specialists, look like *mere juxtapositions of many components* for which a *heterogenous group of people* is responsible. Any kind of genuine integration seems to be missing, even between the components in education or between many of the courses offered in mathematics. No wonder, then that so many criticisms are heard about the way teachers are trained, since "the learner himself is expected to integrate in his learning all the knowledge his teachers were not able to integrate in their teaching: a high expectation, a vain expectation," as Hans Freudenthal so properly pointed out during the Pécs conference last month.
2. Lack of balance and the gap between theory and practice in PRESET
   Many criticisms are still heard about many PRESET programmes being too theoretical. Specific reference is often made, for example, to the inadequacy of the practice teaching component, and to some courses (in education, in mathematics, or in

mathematics education) whose objectives and methodology are too remote from the needs and concerns of a school mathematics teacher. Bridging the theory-practice gap remains very difficult because of well-rooted attitudes among university people, such as the "*first* you learn it, *then* you apply it" axiom, or the belief expressed by Boileau's classical, "*Ce que l'on conçoit bien s'énonce clairement et les mots pour le dire arrivent aisément*". Another difficulty arises from the fact that many teacher trainers, including mathematics educators or mathematicians, have themselves little knowledge of what is actually going on in the schools. (In the United States, competency-based teacher education is gaining popularity, but I am afraid that, considering the way it is implemented in some colleges, it is likely to go too far the other way and put too much emphasis on practice.)

3. Lack of coordination between the "mathematics" and "mathematics education" components of PRESET programmes

   In many PRESET programmes, more particularly in those for secondary school mathematics teachers, a big gap still exists between courses in mathematics and courses in mathematics education (often called "methods courses" although they usually cover much more than teaching methods). There are several reasons for this—for example both types of courses are not often given concurrently. But the main reason seems to be a lack of communication and cooperation between mathematicians and mathematics educators, representing two groups of professional people with different specializations, basic concerns, and types of activities, and which in addition are often located in different places. To improve the situation, some mathematics departments offer a few mathematics courses especially devised for prospective teachers. However, many recommendations made by mathematicians about the mathematical training of a would-be teacher overemphasize content and disregard any related didactical problems, under the implicit assumption that "*first* you learn mathematics, *then* you (eventually) learn to teach it!" Of course, such an attitude does not help to bridge the existing gap.

In the mathematics courses which are part of PRESET programmes, most of the teaching continues to be *product*-oriented, with little explicit emphasis on *processes* characteristic of mathematical activity (e.g., mathematization, heuristics, etc.).

## INSERVICE TEACHER TRAINING (INSET)

During the past fifteen years, a great number and diversity of INSET courses and activities have been organized, partly by *universities* and partly by *other organizations*: school boards, teacher associations, ministries of education, and even private organizations. Some have been *credit-bound*, while many others have not. Some have been *university-based*, while many others have rather been *school-based* (e.g., off-campus university courses or professional development days). Such INSET activities often give participating teachers the opportunity to eventually obtain an increase in salary.

During the last fifteen years, the majority of inservice teacher training courses and activities in mathematics have been of the updating type, and to some extent of the *RE-training* type. During the "new math revolution" of the sixties, for example, most practising teachers had to be literally re-trained in terms of the content and the methodology which were characteristic of the "new mathematics" curricula and textbooks. More recently, many INSET courses and workshops have had to be organized to prepare inservice teachers for the implementation of the Syst't3me International (SI) in the schools.

In recent years, there has been a growing awareness among university people that inservice teachers constitute a clientele with *specific* needs and expectations, background and experience, attitudes etc. Accordingly, it is now more widely accepted that in many respects INSET should be conceived and organized differently from PRESET, with much more flexible entrance requirements (e.g. analogous to those of the Open University in England). Many university professors, however, are still reluctant to accede to this and fear that the consequence might well be an unacceptable lowering of academic standards (in the traditional sense).

The majority of INSET courses and activities, particularly those offered or sponsored by universities, are *not part of the professional task* of the teachers concerned.

In the mathematical component of INSET programmes, most of the teaching continues to be *product*-oriented, with little explicit emphasis on *processes* characteristic of mathematical activity (e.g. mathematization, heuristics, etc.).

## SOME INNOVATIONS AND NEW DIRECTIONS DEVELOPING IN TEACHER TRAINING

During recent years, interesting innovations have been tried in teacher training, more particularly in North America, in England, and in the Scandinavian countries. Such innovations reflect some *general medium term or long-term directions* at present developing in PRESET and INSET in many countries. I shall now attempt to sketch such current trends.

### "CONTINUING EDUCATION" AS A CONCEPTUAL FRAMEWORK FOR PRESET AND INSET

The general concept of "continuing education" ("education permanente") as applicable to the education of every individual throughout his or her life, is quite fascinating, but it still needs clarification and more agreement about its meaning and ways to make it operational enough. There are strong indications, however, that this concept can advantageously serve as a conceptual framework for PRESET and INSET.

In broad terms, in the case of any individual, education may be viewed as a life-long process of which the development includes the following phases:

(a) basic general education, acquired in school (compulsory schooling period) as well as outside school;
(b) education in preparation for a career, which may be acquired in various ways and places (this includes further general education as well as professional preparation and apprenticeship);
(c) further education during a career, which may also be acquired in various ways and places (this includes general as well as professional further education or training).

Of course this should be refined with "loops" (to account for changes in career orientation), and with a provision for skipping phase (b) in some cases, but I shall stick to the above rough model for the purpose of the discussion here.

In the particular case of a school teacher, PRESET in universities is clearly *part of* phase (b), while INSET is *part of* phase (c). It must be clearly kept in mind, however, that phases (b) and (c) include not only activities *related to the professional task of a teacher*, but also other kinds of activities which may be *educational in a general sense* and contribute to the personal development of an individual (e.g. getting information or experience in other subjects or occupations through personal study or involvement).

In my opinion, one of the features of a genuine concept of continuing education is that the above phases (a), (b), and (c) are *not merely juxtaposed*, but are *conceptually and practically interlocked through deliberate planning and action*.

Using continuing education as a conceptual framework for PRESET and INSET therefore implies in particular that:

(i) PRESET and INSET are *conceptually inseparable*, complementary parts of a continuous process, *with many interdependent components*.

An immediate consequence of this is that PRESET should no longer be thought of nor organized as if it were aimed at preparing a teacher for a whole career. If properly organized, INSET should allow any inservice teacher to eventually take any course he may have "missed" and which was optional in PRESET. This in turn implies that in preservice teacher training the *common core of compulsory courses* and activities for all students should be *reduced to a minimum*, in order to allow more opportunities for flexible individual programmes (see above). Considerable efforts should then be made to integrate at least the components of the compulsory common core in PRESET through concerted work of *teams* of university specialists (see above)

Another consequence is that INSET should be viewed *primarily as a continuation of PRESET throughout the career of a teacher*, allowing him or her to discuss problems actually met in the classroom or in the school, to learn more about fundamental relevant subjects, to share experiences or initiate concerted action with other teachers, to get up-to-date information about current trends, new teaching methods and media, etc. From this point of view, INSET should *include, but not be focused chiefly on RE-training* activities that may be made necessary by sudden, often carelessly planned changes in curricula or textbooks or teaching methods, or by a significant change in career orientation (see above). Moreover INSET should include a much *greater range of activities* which are relevant and 'worthwhile for teachers: courses, workshops, discussion periods, projects, participation in professional conferences, participation in a research, etc. (Of course this is not easy in a university credit-bound INSET programme. However, the more INSET becomes part of the professional task of the teacher in the future, the more such variety may be possible.)

(ii) Nevertheless, preservice teachers and inservice teachers constitute *different clienteles*, each with its *specific* needs and expectations, background and experience, attitudes, etc. Accordingly the objectives and methodology of many INSET activities and the overall structure of INSET programmes offered in universities are likely to be very different from those in PRESET (see above).

In my opinion, there is a *long-term trend* slowly emerging in the direction I have just sketched, particularly where training teachers of mathematics is concerned. I feel, however, that there is still a long way to go before a genuine concept of "continuing education" becomes clear and operational enough in our universities (even if many already offer so called "continuing education courses"!).

### MORE DELIBERATE COOPERATION IN PRESET AND INSET BETWEEN UNIVERSITIES, SCHOOLS, AND PEOPLE TAKING PART IN TEACHER TRAINING ACTIVITIES

A medium-term trend which is increasingly noticeable is towards a more deliberate cooperation in PRESET and INSET between three groups: universities; school representatives; people taking part in preservice and inservice teacher training activities.

On the one hand, this means that student-teachers in PRESET and inservice teachers in INSET are likely to play an every-increasing role in decision-making concerning teacher training: for example, as members of programme committees or as active participants in surveys conducted about their needs, expectations, and evaluations of current teacher training activities (see above).

On the other hand, there are already strong indications that a much closer cooperation will be established in the near future between universities and school representatives in the organization of teacher training, particularly along the following lines:

(i) In PRESET, *more efficient practice teaching or internship schemes* allowing more deliberate interaction between certain theoretical courses and classroom experiences, and accordingly narrowing the classical gap between theory and practice (see above).

(ii) *School-focused INSET* with increased responsibility for school boards and schools and with new roles played by universities. (An international "Conference on Strategies for School-Focused Support Structures for Teachers in Change and Innovation" was held in Stockholm in October 1976, sponsored by O.E.C.D., and follow-up international conferences have already been planned for November 1977 and 1978.)

(iii) Creation of many *local "professional centres" or "teachers' centres"* (school- or school board-based preferably) serving many purposes, but considered chiefly as *privileged places for a variety of INSET activities*. (In the U.S.A. there already exist many such centres of different types, and in England the "James Report", published in 1972, recommended the creation of a country-wide network of "professional centres", although many such centres have existed for years.) Universities might contribute in many ways to the realization of INSET activities in such locally-run "teachers' centres" and in particular play non-traditional roles in the organization of INSET activities which may not be credit-bound but which will be part of the professional task of the teachers concerned. Moreover, it might well be a worthwhile idea for universities to plan and realize a few PRESET activities in close cooperation with "professional centres" in their area.

The above medium-term directions are particularly noticeable as far as training teachers in mathematics is concerned.

### OTHER POSSIBLE DIRECTIONS IN PRESET AND INSET IN MATHEMATICS

I would like to point out two new directions which might develop in teacher training. They however remain more problematical than the preceding trend because they presuppose significant changes in deeply-rooted traditions and attitudes among university people—in particular among mathematicians and mathematics educators. These two new possible directions apply to both PRESET and INSET.

The first one is a *greater emphasis on processes* characteristic of mathematical activity (e.g. mathematization, heuristics, etc.) both in mathematics courses and in mathematics education courses, which are still much too product-oriented. This is certainly highly desirable in both preservice and inservice courses, particularly in today's climate where mathematics curricula tend to be biased by an excessive emphasis on specific behavioral *content* objectives (see above).

The second one is a *significant change in the way research in mathematics education is viewed, planned, and conducted*. In my opinion, much more research and development in this field should be planned and conducted in cooperation with practising teachers *as part of INSET activities*. Schemes for training preservice teachers might also allow the involvement of more (undergraduate) prospective teachers in some research projects. I feel there should be a strong interdependence between (1) the evolution of mathematics curricula and teaching methods and media in schools; (2) inservice training activities in mathematics; (3) much of the research and development done in mathematics education which is not of the fundamental type.

A study by Coutts and Clarke on the future of teacher training in Canada

Let us pause to look at some of the conclusions of a study conducted by H.T. Coutts and S.C.T. Clarke on "The future of teacher education" and presented at the American Educational Research Association Convention in New York City, February 1971.

According to this study, a sample of chief administrators of the English speaking teacher education institutions in Canada estimated that teacher education in the foreseeable future (1975-1980) would move in the following directions:

1. Teacher education would be centred around an extended intern, ship.
2. Teacher education would continue throughout the teacher's career, with frequent use being made of sabbatical leave for one or two semesters to be spent at a university.
3. Candidates for teacher education, both for admission to preparatory programmes and for first certification, would be required to exhibit a satisfactory standard of excellence in: speech, English usage, mental health, and human relations.
4. Teachers would be prepared more intensively as subject specialists.
5. Although there would be a common core of learning for all, each candidate's programme would be individually tailored.
6. The common core learning required by all teachers would include:
    a. preparation in working as a member and as a leader of a group or team which might be a mixture of superordinates and subordinates, or persons all at one professional level;
    b. a great deal of attention to ethics, morals, attitude development, and character formation;
    c. preparation in the use of the latest education technology and media.
7. Teacher education would be about half "common core" for all candidates and about half specific to specialization in terms of: function, level, and staff discrimination.
8. Teacher education would emphasize the *process* of learning (observing, clarifying, inferring, inquiring, reasoning, remembering) as contrasted with the *product* (information, knowledge, concepts, generalizations).

## ONE CURRENT PROGRAMME SHOWING SEVERAL INNOVATIONS IN INSERVICE TEACHER TRAINING: THE PERMAMA PROGRAMME

In order to illustrate some innovations which are taking place in INSET at present, I would like to sketch one particular INSET programme which I have been associated with and which is increasingly popular in Québec. It is called the "PERMAMA programme", where PERMAMA stands for "*PERfectionnement des MAîtres en MAthématique*".

I shall give here only a very general description of the programme and anyone interested in more specific information may consult a few papers on the subject or contact the Director of PERMAMA.

Preliminary remarks:

(a) PERMAMA is an *inservice* teacher training programme run by *Télé-université du Québec* (a branch of the Université du Québec). Most collaborators work in Montréal.
(b) PERMAMA is a programme primarily intended for *high school mathematics teachers from all over the Québec territory*.
(c) At present PERMAMA is a *credit-bound* programme leading to a "certificate" and eventually to a bachelor's degree. To be admitted to the programme, one must have taught in schools for at least three years.
(d) About 1300 teachers are currently registered in the programme.

(e) PERMAMA started in 1972. It has been built upon the experience and the understructure which have grown out of a previous Government- run inservice training programme for high school mathematics teachers (1966-1971), called "C.R.P.M." (*Cours de Recyclage et de Perfectionnement en Mathématique*). As a matter of fact, since 1966, the philosophy and the type of organization of inservice training of secondary mathematics teachers have gone through three distinct phases in Québec, giving rise to three types of INSET programmes: (1) C.R.P.M. from 1966 to 1971; (2) PERMAMA 1st generation, from 1972 to 1975; PERMAMA 2nd generation, since 1975.

(f) Teachers registered in the PERMAMA programme participate in courses and activities in their leisure time. This type of inservice teacher training is currently not part of the teacher's professional task.

## A NETWORK OF TEACHERS' CENTRES FOR INSET ACTIVITIES IN MATHEMATICS

Teachers registered in the PERMAMA programme participate in courses and activities in *local or regional teachers' centres*. There are, at this time, 97 such PERMAMA centres spread over the Québec territory. *In each* centre there is one so-called "*moniteur-animateur*", generally a mathematics teacher himself; his job consists mainly in doing some organization and in serving as an "animator" during the working and discussion periods of "permamists" (N.B. his role is not to teach!). Periodically all moniteurs-animateurs meet in order to share their experiences, to prepare for new PERMAMA courses and coming activities, to give feedback about recent courses and activities offered, and to participate in decision-making about the continuation of the programme.

Remark: *moniteurs-animateurs* are paid a salary for their work.

## A BANK OF "MODULES" OF VARIOUS TYPES ALLOWING PERSONALIZED INSET PROGRAMMES

Up to now a bank of about 60 "*modules*" has been established. Each module is a learning unit, generally using various media, provided with a guide for the *moniteur-animateur*. Supposing a group of teachers using a module meets once a week on the average, and that each member does required work at home or in schools every week, then finishing the module may require at least four to eight weeks. Modularization of previous "courses" (offered in the PERMAMA 1st generation programme) has been very successful and has given much more flexibility to PERMAMA.

At the present time five types of modules may be found in the bank:

(a) *modules focused on mathematical content* (elementary algebra, geometry, algebraic structures, statistics, vectors, Boolean algebras graph theory, derivative, programming, integral, number systems, etc.)

(b) *modules focused on mathematical activity* (problem solving, mathematization, etc.)

(c) *modules focused on didactical problems* (concept learning in mathematics, student-teacher relations, learning through problem solving, teaching geometry, laboratory activities in mathematics, using worksheets for teaching mathematics, etc.)

(d) *modules focused on the realization of "projects" in schools*: after identifying a problem or a need in mathematics teaching in their schools, a group of teachers think of a relevant "project" to realize, plan it carefully, realize it, and evaluate the results (N.B. supervision is provided by the PERMAMA "equipe-pedagogique" for such projects)

(e) *modules permitting teachers to plan personalized sequences of PERMAMA modules with appropriate information and cooperation* ("management modules").

To some extent teachers registering in the PERMAMA programme have the opportunity of tailoring *personalized* sequences of modules. Only three modules are compulsory. There are, however, a few constraints which may limit such an individualization of INSET programmes. For example, some modules are offered only if a minimum number of teachers want to use it simultaneously (because group work and discussion are considered essential to make such modules profitable) and accordingly negotiation and cooperation between teachers in the same PERMAMA centre may be necessary to find an optimal compromise.

During 1977, 400 teachers who were registered in the PERMAMA programme initiated 141 "projects" between January and August. Such projects may play a tremendously dynamic role in promoting better teaching of mathematics in schools and in making more interdependent: (1) research and development in mathematics education; (2) inservice teacher training; (3) the evolution and the improvement of mathematics teaching in schools.

PERMAMA full-time staff includes the *équipe pédagogique*, consisting of a few mathematicians and mathematics educators and many experienced high school mathematics teachers. This group mostly works on the preparation and testing of modules, keeping close contact with schools and with the *moniteurs-animateurs* of the 97 PERMAMA centres spread over Québec. A few other full-time people are more specifically concerned with management so that the network of PERMAMA local centres functions properly and that information about PERMAMA activities is disseminated appropriately.

### PARTICIPATION OF TEACHERS IN DECISION-MAKING AND IN THE ORGANIZATION

Participation of the teachers concerned in the PERMAMA programme is insured in various ways. For example, many experienced high school mathematics teachers are part of the *équipe pédagogique*. On the other hand, periodic surveys are made in order to determine the degree of satisfaction of the "permamists" in regard to existing modules, their suggestions for improvement, and their desires concerning the production of new modules. Many teachers also cooperate in pre-experiments with modules in preparation, or make a more systematic evaluation of existing modules. Finally, the group of all "permamists" has a representative in the *Comité directeur* of the programme.

During 1976-1977, registrations in the PERMAMA programme have increased by 80%, showing the degree of satisfaction of inservice teachers with respect to this second generation programme. It is obviously much more flexible, relevant, and stimulating than the previous (first generation) PERMAMA programme which was much more uniform and too exclusively content-oriented.

## SOME THOUGHTS ABOUT THE ROLE AND RESPONSIBILITY OF UNIVERSITIES IN TEACHER EDUCATION IN THE FUTURE

I have already sketched a few possible medium-term and long-term directions developing in teacher education, and made allusions to changes they might imply as far as the role and responsibility of universities in PRESET and INSET is concerned. I do not wish to add much more to that. Let me therefore finish with four short remarks:

In PRESET, it is quite certain that universities will keep the largest responsibility. In INSET, however, they are likely to lose a significant part of the responsibility they have traditionally had; this may be taken over by regional or local school communities.

In order to keep their leadership in PRESET and to improve the quality of preservice teacher education, the universities must first of all continue to "put their own house in order". It is clear, for example, that to make improvements in PRESET programmes, with respect to part 1 of this presentation, is primarily *our* job as university people.

It is urgent that universities establish closer permanent connections and share some responsibilities with schools, both in order to improve the training of prospective teachers and to offer more relevant INSET courses and activities.

The really big challenge for universities in the future may well be to show enough imagination and initiative in offering *new types of services and contributions to INSET* (going far beyond offering creditbound courses!), particularly if trends continue towards the establishment of "teachers' centres" and towards a greater integration of INSET activities with the professional task of the teachers. If they can achieve that successfully, I am convinced that universities will continue to play a key role in INSET although their responsibility will inevitably be somewhat diminished in this area.

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# THE OBJECTIVES OF MATHEMATICS EDUCATION

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Quite apart from the fact that these three talks gave us a sense of why we started, what we felt were the issues of the time, and what we hoped for, they remain, for the reader of today, absorbing and timely. All three speakers were leaders in their areas at that time and continued as key participants in CMESG activities for decades afterwards. Together they examined the big three aspects of our work, mathematics education research (Kieren), teacher education (Gaulin) and mathematics education (Coleman), and over the following 40 years, these three areas danced forward and intertwined along many rich axes.

Non seulement ces trois conférences nous avaient permis de donner un sens à notre action initiale et de décrire ce que nous percevions comme les principaux défis à cette époque ainsi que ce que nous espérions pour l'avenir, elles demeurent, encore aujourd'hui, captivantes et pertinentes. Les trois conférenciers étaient des experts de leurs domaines, et ils ont continué à jouer un rôle clé au GCEDM pendant des décennies. Ensemble, ils ont exploré les trois pôles de notre travail, soit la recherche en didactique des mathématiques (Kieren), la formation des enseignants (Gaulin) et l'éducation mathématique (Coleman), et, depuis 40 ans, ces domaines ont pu poursuivre leur développement, se liant de multiples manières dans des interactions riches.

It would appear rather obvious that there could be only one objective of mathematics education. Clearly, it is to teach mathematics!

Agreed! But this having been said, many questions press upon one. What kind of mathematics? How much? To whom? Why teach any to anyone? Should the programme be the same for all pupils? If not, according to what criteria can pupils be distinguished? After all, in a democracy everyone is equal, so should we not aim to achieve a common universal mediocrity? Is there any percentage in considering objectives in a vacuum? Must we not link with such consideration an evaluation of the practical possibility of achieving them, given the mathematical competency (or incompetency) of the corps of teachers and the strong societal pressures which denigrate intellectual excellence or mental effort? In other words, is there any use discussing "objectives" without considering whether they can be achieved?

As with any issue of basic existential import, once you start thinking about the aims of mathematics education, questions, flood into your mind. Clearly, I cannot in a brief article deal with them all or, indeed, adequately with even one of these many issues.

You will recall that fairly recently an OECD study of education in Canada expressed amazement at the extraordinarily high proportion of the GNP which goes into education in a society which seems to have no explicit statement of what it expects the school system to accomplish. Evidently most other OECD countries have clearer aims for education than has Canada. So perhaps my topic is timely.

## THREE OBJECTIVES

I have read discussions in which as many as twelve aims of mathematics education were distinguished - or, at least, the author thought he could distinguish this number, though my mind was not sufficiently acute to grasp the subtleties of his thought. In any case, I consider such long lists pedantic nonsense and psychologically ineffective. It is best to concentrate on essentials. I fix on three aims which I regard as crucial.

1. The average citizen should be enabled to master the minimum of mathematics needed for ordinary commercial transactions and for understanding the functioning of our society.
2. All people should be given the possibility of enriching themselves culturally and intellectually by extending their basic human capacity for abstract thought.
3. A supply of thoroughly trained mathematical "practitioners" - engineers, economists, research mathematicians - sufficient for the needs of society should be assured.

Let us look at these objectives in order.

Of course, there cannot be a satisfactory dogmatic definition of the precise set of mathematical facts which every Canadian should know. A minimum would surely comprise arithmetic, including percentage; mensuration, including change of units; and simple geometry. However, anyone who has noted the extraordinary rate at which mathematics has increasingly been applied in our society in recent decades would argue that much more than this is needed. The proper functioning of democracy requires a universal ability to interpret, and not be misled by, a wide variety of statistics which various interest groups thrust upon us. Presumably the hand-held computer will be universally available for the generation of students which is now in the schools. They will need to learn how to use it effectively.

No matter how we define the contents of the curriculum required to accomplish the first objective, the material can properly be called *basic*. However, I am not prepared to campaign under the slogan "Back to Basics". We have never done an adequate job in inculcating basic mathematics in the average Canadian, and what might have been sufficient twenty years ago is not enough now and will be abysmally inadequate in another ten years. So I lift high my banner which reads: "Forward to the New Basics!"

In addition to enabling people to function as citizens or as economic agents in society, mathematics education should contribute to personal and cultural enrichment. David Wheeler has argued correctly that anyone who can use language effectively thereby demonstrates an ability to apprehend structure. But mathematics is the study of abstract structure, so one can argue that to mathematize is to be truly human. If, with Aristotle and St. Thomas Aquinas, we characterize man as the "rational animal", we must recognize that as we increase our ability to reason we enhance our humanness. Mathematical puzzles can be enjoyable and relaxing. But the more mathematics one understands and can use, the easier it is to understand and control the technological environment in which all of us are now immersed. Without such understanding the feeling that one's life is dominated by mysterious unknown forces must be overwhelming. So, to feel at ease with mathematics can enhance one's sense of freedom, as well as opening up the limitless and fascinating literature of mathematics and its applications.

All citizens should know enough mathematics to be able to manoeuvre in our society. All citizens should be given the opportunity to experience the joy of developing their innate capacity to mathematize and to exult in this power. But at a more mundane and practical level, Canadian society needs a supply of competent mathematical practitioners. Engineers, physicists, economists, biologists, and social scientists are increasingly making use of more and more sophisticated types of mathematics. They need to be properly trained. Many Canadians are proud that Bell Northern Research Company has a high reputation internationally as one of the few research-oriented companies in Canada which competes effectively for international markets. The Science Council of Canada Background Study, *Mathematical Sciences in Canada*, reports that of 126 professionals in BNR, two had doctorates, 26 had master's and 46 bachelor's degrees in mathematics. It is not a coincidence that the Canadian company with the highest research effectiveness is the one with the highest concentration of mathematically trained personnel.

If Canada wishes to remain in the forefront of the technological age, as it has in the past, it is desirable, indeed essential, that our programmes of mathematics education encourage gifted students to push their mathematical training forward as fast and as soundly as possible.

## ARE THE OBJECTIVES ACHIEVABLE?

I expect that most of my readers would agree that the three objectives sound fine. However, doubts might be raised as to whether they are realizable, and questions will be asked about how they are to be achieved.

It is my belief that, at least in Western countries, our past methods of mathematics education have been abysmal. We have barely scratched the surface when it comes to developing and unleashing the power of human beings to mathematize. This is chiefly because for centuries the study of mathematics has been overshadowed by a powerful inhibiting factor which reveals itself in the widespread fear or awe of mathematics. Perhaps the root source of this fear is the manner in which the rote learning of Euclid was drilled into many successive generations of European, American, and Canadian children. Or perhaps its source is the stance of superiority which the mathematically-gifted, particularly university professors, have often assumed. Whatever the cause, I believe that a fear of mathematics and a feeling that "higher mathematics" (Le. anything beyond 2 + 2 = 41) is accessible only to a gifted elite have been the chief factors in preventing the majority of Canadians from entering with joy and satisfaction into the pleasant fields of mathesis.

This conviction was reinforced by my visit to the Soviet Union in April and May of 1977, when I had the opportunity to learn something about the "Kolmogorov reform" and to observe mathematics lessons in two schools at grade 3, 9, and 10 levels. Already in grade 3 (but not in 1 and 2), mathematics is taught by a specialist mathematics teacher. The system is not divided into elementary, junior, and senior schools. The child enters at the age of seven and continues in the same school for 10 years. All pupils study mathematics every year; six periods per week in grades 3 through 8, and five in grades 9 and 10. The textbooks and, in principle, the programme are uniform throughout the USSR. Apparently the reformed curriculum which has been gradually introduced during the past twelve years is fully implemented only in the cities. In School No. 169 in Leningrad, which specializes in English and not in mathematics, *every* pupil covers in *ten* years a mathematics syllabus which goes well beyond the total mathematics syllabus which is offered in *thirteen* years in Ontario to less than one-third of our children. I was particularly struck by the assigned homework on inequalities which was more difficult than we would dare set for a first year student at Queen's!

Are all children in Leningrad more intelligent than the top thirty percent of Ontario youth? I think not. In my view, the difference is that in Leningrad young people are better motivated and work harder. There is a basic confident feeling that everyone can and will learn mathematics. To be able to do mathematics is a natural human capacity which can be developed if one merely tries! The children are greatly helped by support and pressure from parents, reflecting the insistence of Lenin and his successors on the vital importance of the study of science in general, and of mathematics in particular, for the attainment of the social goals of the Communist Party. There has been a great debate in North America about the so-called "New Math". In my view much of this has been ill-informed and misdirected. There is some evidence - but little which is statistically convincing - that students coming into the universities in recent years do not have as confident a control of the manipulative aspects of arithmetic and algebra as they did twenty years ago, This, it is claimed, proves that the New Math is a total disaster!

I readily admit that some publishers who rushed on to the market with poorly written textbooks did a considerable disservice to mathematics education. Further, some teachers were ill-prepared to adopt the new approach to the teaching of mathematics which was introduced universally in the Soviet Union, Europe, and North America in the late 1950's. Some served up for their pupils a confusing mish-mash of poorly digested jargon.

However, many of the highly emotional critics of the "New Math" and the proponents of "Back to Basics" have merely added to the confusion without making any constructive contribution to the solution of the many important tasks of mathematics education in Canada.

They overlook key elements of our situation. The percentage of the school-leaving age group seeking entrance to university has increased markedly, so we are comparing the performance of the current freshmen with a smaller and more selected group. It has been estimated that nowadays the average child spends 15,000 hours watching TV. No one claims that this will improve mathematical competence! In Ontario, the abandonment of Grade XIII Examinations has had a greater influence - for good or ill - on the preparation of university freshmen than any other single factor with which I am familiar. I suspect that some of the older critics of the New Math tend to recall their youth through rose-coloured spectacles. From 1953 to 1960 I taught Freshman Calculus to the students of the Honour Course in Mathematics and Physics at the University of Toronto. They were the intellectual cream of Ontario - hardly excelled by a similar group anywhere in North America. In recent years I have taught analogous courses at Queen's. I cannot honestly say that during my 25 years experience I have observed any significant difference in the types of difficulties which students have had in understanding mathematics. I do recall a freshman at Toronto, in about 1955, who had come from the University of Toronto School with an average of 92 percent on the Grade XIII examinations, who thought that $(a+b)^{-1} = a^{-1} + b^{-1}$.

In Ontario, and I believe elsewhere in Western nations, there have been no essential changes in the mathematics curriculum in school between 1910 and 1960. However, in that period there was a total revolution in the role of mathematics in society. The aim of the old mathematics education was to inculcate the rote understanding of certain manipulative skills. In the 1950's we began to realize that this was not enough. In addition to basic manipulative skills the average citizen now needs to have some grasp, however dim, of what mathematics is, what you can expect of it and - equally important - what you *cannot* expect of it. Thus the aim of the new programmes is to convey an *understanding* of some mathematical ideas. Of course, this is much more difficult. The transition involves pain. It is far from complete. This is the contemporary challenge.

## WHAT CAN WE DO?

If our three objectives for mathematics education are accepted as necessary and desirable, then our first task is to make sure they are understood and accepted by the Ministers of Education, teachers, parents, and students. Only then can we hope to mobilize the forces needed to realize them effectively in Canada.

We must seek to dissipate the anxiety feelings towards mathematics, especially among elementary school teachers. This might be done by extensive in-service training programs and by improved pre-service courses. In order to ensure the latter, it is necessary for many university professors of mathematics to change their attitudes and redirect some of their energies. Possibly, TV can be used to good effect - as has been done by PERMAMA in Québec and by Professor Z. Semadeni in Poland.

The crucial factor is that teachers should be competent and know and feel that they are competent. Then they will be psychologically free and able to open to their students the experience that to mathematize is to joy.

# CONFESSIONS OF AN ACCIDENTAL THEORIST

Alan Schoenfeld
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Alan Schoenfeld is known for many things in mathematics education – two of which are that he is a mathematician working in mathematics education and that he is one of the pioneers of research on mathematical problem solving. And this plenary lays forth the story of how both of these came to be. In considering which plenaries to include in this volume his came to the forefront both because of the pioneering spirit of the work and the relevance of the research even 30 years later.

Alan Schoenfeld est bien connu en didactique des mathématiques, et ce pour plusieurs raisons – parmi celles-ci, le fait qu'il soit un mathématicien travaillant en éducation mathématique, et qu'il soit l'un des pionniers de la recherche sur la résolution de problèmes mathématiques. Cette plénière nous explique l'avènement de ces deux aspects de son travail. Sa conférence plénière se distingue tant par l'esprit novateur du travail accompli que par la pertinence actuelle des recherches, trente ans plus tard.

David Wheeler had both theoretical and pragmatic reasons for inviting me to write this article. On the theoretical side, he noted that my ideas on "understanding and teaching the nature of mathematical thinking" have taken some curious twists and turns over the past decade. Originally inspired by P6lya's ideas and intrigued by the potential for implementing them using the tools of artificial intelligence and information-processing psychology, I set out to develop prescriptive models of heuristic problem solving – models that included descriptions of how, and when, to use P6lya's strategies. (In moments of verbal excess I was heard to say that my research plan was to "understand how competent problem solvers solve problems, and then find a way to cram that knowledge down students' throats.") Catch me talking today, and you'll hear me throwing about terms like metacognition, belief systems, and "culture as the growth medium for cognition;" there's little or no mention of prescriptive models. What happened in between? How were various ideas conceived, developed, modified, adapted, abandoned, and sometimes reborn? It might be of interest, suggested David, to see where the ideas came from. With regard to pragmatic issues, David was blunt. Over the past decade I've said a lot of stupid things. To help keep others from re-inventing square theoretical or pedagogical wheels, or to keep people from trying to ride some of the square wheels I've developed, he suggested, it might help if I recanted in public. So here goes ...

The story begins in 1974, when I tripped over Pólya's marvelous little volume *How to Solve It.* The book was a *tour de force*, a charming exposition of the problem solving introspections of one of the century's foremost mathematicians. (If you don't own a copy, you should.) In the spirit of Descartes, who had, three hundred years earlier, attempted a similar feat in the *Rules for the Direction of the Mind,* Pólya examined his own thoughts to find useful patterns of problem solving behavior. The result was a general description of problem solving processes: a four-phase model of problem solving (understanding the problem, devising a plan, carrying out the plan, looking back), the details of which included a range of problem solving *heuristics*, or rules of thumb for making progress on difficult problems. The book and Pólya's subsequent elaborations of the heuristic theme (in *Mathematics and Plausible Reasoning,* and *Mathematical Discovery)* are brilliant pieces of insight and mathematical exposition.

A young mathematician only a few years out of graduate school, I was completely bowled over by the book. Page after page, Pólya described the problem solving techniques that he used.

Though I hadn't been taught them, I too used those techniques; I'd picked them up then pretty much by accident, by virtue of having solved thousands of problems during my mathematical career (That is, I'd been "trained" by the discipline, picking up bits and pieces of mathematical thinking as I developed). My experience was hardly unique, of course. In my excitement I joined thousands of mathematicians who, in reading Pólya's works, had the same thrill of recognition. In spirit I enlisted in the army of teachers who, inspired by Pólya's vision, decided to focus on teaching their students to think mathematically instead of focusing merely on the mastery of mathematical subject matter.

To be more accurate, I thought about enlisting in that army. Excited by my readings, I sought out some problem-solving experts, mathematics faculty who coached students for the Putnam exam or for various Olympiads. Their verdict was unanimous and unequivocal: Pólya was of no use for budding young problem-solvers. Students don't learn to solve problems by reading Pólya's books, they said. In their experience, students learned to solve problems by (starting with raw talent and) solving lots of problems. This was troubling, so I looked elsewhere for (either positive or negative) evidence. As noted above, I was hardly the first Pólya enthusiast: By the time I read *How to Solve It* the math ed literature was chock full of studies designed to teach problem-solving *via* heuristics. Unfortunately, the results – whether in first grade, algebra, calculus, or number theory, to name a few – were all depressingly the same, and confirmed the statements of the Putnam and Olympiad trainers. Study after study produced "promising" results, where teacher and students alike were happy with the instruction (a typical phenomenon when teachers have a vested interest in a new program) but where there was at best marginal evidence (if any!) of improved problem solving performance. Despite all the enthusiasm for the approach, there was no clear evidence that the students had actually learned more as a result of their heuristic instruction, or that they had learned any general problem solving skills that transferred to novel situations.

Intrigued by the contradiction – my gut reaction was still that Pólya was on to something significant - I decided to trade in my mathematician's cap for a mathematics educator's and explore the issue. Well, not exactly a straight mathematics educator's; as I said above, math ed had not produced much that was encouraging on the problem solving front. I turned to a different field, in the hope of blending its insights with Pólya's and those of mathematics educators.

The first task I faced was to figure out why Pólya's strategies didn't work. If I succeeded in that, the next task was to make them work – to characterize the strategies so that students could learn to use them. The approach I took was inspired by classic problem solving work in cognitive science and artificial intelligence, typified by Newell and Simon's (1972) *Human Problem Solving.* In the book Newell and Simon describe the genesis of a computer program called General Problem Solver (GPS), which was developed to solve problems in symbolic logic, chess, and "cryptarithmetic" (a puzzle domain similar to cryptograms, but with letters standing for numbers instead of letters). GPS played a decent game of chess, solved cryptarithmetic problems fairly well, and managed to prove almost all of the first 50 theorems in Russell and Whitehead's *Principia Mathematica* – all in all, rather convincing evidence that its problem solving strategies were pretty solid.

Where did those strategies come from? In short, they came from detailed observations of people solving problems. Newell, Simon, and colleagues recorded many people's attempts to solve problems in chess, cryptarithmetic, and symbolic logic. They then explored those attempts in detail, looking for uniformities in the problem solvers' behavior. If they could find those regularities in people's behavior, describe those regularities precisely (i.e. as computer programs), and get the programs to work (i.e. to solve problems) then they had pretty good evidence that the strategies they had characterized were useful. As noted above, they succeeded. Similar techniques had been used in other areas: for example, a rather simple program called

SAINT (for Symbolic Automatic INTegrator) solved indefinite integrals with better facility than most M.I.T. freshmen. In all such cases, AI produced a set of *prescriptive* procedures – problem solving methods described in such detail that a machine, following their instructions, could obtain pretty spectacular results.

It is ironic that no one had thought to do something similar for human problem solving. The point is that one could turn the man-machine metaphor back on itself. Why not make detailed observations of expert human problem solvers, with an eye towards abstracting regularities in their behavior – regularities that could be codified as prescriptive guides to human problem solving? No slight to students was intended by this approach, nor was there any thought of students as problem solving machines. Rather, the idea was to pose the problem from a cognitive science perspective: "What level of detail is needed so that students can actually use the strategies one believes to be useful?" Methodologies for dealing with this question were suggested by the methodologies used in artificial intelligence. One could make detailed observations of individuals solving problems, seek regularities in their problem solving behavior, and try to characterize those regularities with enough precision, and in enough detail, so that students could take those characterizations as guidelines for problem solving. That's what I set out to do.

The detailed studies of problem solving behavior turned up some results pretty fast. In particular, they quickly revealed one reason that attempts to teach problem solving *via* heuristics had failed. The reason is that Pólya's heuristic strategies weren't really coherent strategies at all. Pólya's characterizations were broad and descriptive, rather than prescriptive. Professional mathematicians could indeed recognize them (because they knew them, albeit implicitly), but novice problem solvers could hardly use them as guides to productive problem solving behavior. In short, Pólya's characterizations were *labels* under which families of related strategies were subsumed. There isn't much room for exposition here, but one example will give the flavor of the analysis. The basic idea is that when you look closely at any single heuristic "strategy," it explodes into a dozen or more similar, but fundamentally different, problem-solving techniques. Consider a typical strategy, "examining special cases:"

> *To better understand an unfamiliar problem, you may wish to exemplify the problem by considering various special cases. This may suggest the direction of, of perhaps the plausibility of, a solution.*

Now consider the solutions to the following three problems.

**Problem 1**. Determine a formula in closed form for the series

$$\sum_{i=1}^{n} k/(k+1)!$$

**Problem 2.** Let P(x) and Q(x) be polynomials whose coefficients are the same but in "backwards order:"

$P(x) = a_0 + a_1 x + a_2 x^2 + .,. a_n x^n$ , and

$Q(x) = a_n + a_{n-1} x + a_{n-2} x^2 + ... a_0 x^n$.

What is the relationship between the roots of P(x) and Q(x)? Prove your answer.

**Problem 3.** Let the real numbers $a_0$ and $a_1$ be given. Define the sequence $\{a_n\}$ by

$a_n = ½ (a_{n-2} + a_{n-1})$ for each $n \geq 2$.

Does the sequence {an} converge? If so, to what value?

I'll leave the details of the solutions to you. However, the following observations are important. For problem 1, the special cases that help are examining what happens when where the integer parameter n takes on the values 1, 2, 3, ... in sequence; this suggests a general pattern that can be confirmed by induction. Yet if you try to use special cases in the same way on the second problem, you may get into trouble: Looking at values n = 1, 2, 3, ... can lead to a wild goose chase. It turns out that the right special cases of P(x) and Q(x) you to look at for problem 2 are easily factorable polynomials. If, for example, you consider

$$P(x) = (2x + 1)\,(x + 4)\,(3x - 2),$$

you will discover that its "reverse," Q, is easily factorable. The roots of the P and Q are easy to compare, and the result (which is best proved another way) is obvious. And again, the special cases that simplify the third problem are different in nature. If you choose the values $a_0=0$ and $a_1=1$, you can see what happens for that particular sequence. The pattern in that case suggests what happens in general, and (especially if you draw the right picture!) leads to a solution of the original problem.

Each of these problems typifies a large class of problems, and exemplifies a different special cases strategy. We have:

> **Strategy 1:** When dealing with problems in which an integer parameter n plays a prominent role, it may be of use to examine values of n = 1, 2,3, .. , in sequence, in search of a pattern.
>
> **Strategy 2:** When dealing with problems that concern the roots of polynomials, it may be of use to look at easily factorable polynomials.
>
> **Strategy 3:** When dealing with problems that concern sequences or series that are constructed recursively, it may be of use to try initial values of 0 and 1 – if such choices don't destroy the generality of the processes under investigation.

Needless to say, these three strategies hardly exhaust "special cases." At this level of analysis – the level of analysis necessary for implementing the strategies – one could find a dozen more. This is the case for almost all of Pólya's strategies. In consequence the two dozen or so "powerful strategies" in *How to Solve It* are, in actuality, a collection of two or three hundred less "powerful," but actually usable strategies, The task of teaching problem solving *via* heuristics – my original goal – thus expanded to (1) explicitly identifying the most frequently used techniques from this long list, (2) characterizing them in sufficient detail so that students could use them, and (3) providing the appropriate amount and degree of training.

> **Warning:** It is easy to underestimate both the amount of detail and training that are necessary. For example, to execute a moderately complex "strategy" like "exploit an easier related problem" with success, you have to (a) think to use the strategy (non-trivial!); (b) know which version of the strategy to use; (c) generate *appropriate* and *potentially useful* easier related problems; (d) make the right choice of related problem; (e) solve the problem; and (f) find a way to exploit its solution to help solve the original problem. Students need instruction in all of these.

Well, this approach made progress, but it wasn't good enough. Fleshing out Pólya's strategies did make them implementable, but it revealed a new problem. An arsenal of a dozen or so powerful techniques may be manageable in problem solving. But with all the new detail, our arsenal comprised a couple of hundred problem solving techniques. This caused a new problem, which I'll introduce with an analogy.

A number of years ago, I deliberately put the problem

$$\int \frac{x}{x^2 - 9}\, dx$$

as the first problem on a test, to give my students a boost as they began the exam. After all, a quick look at the fraction suggests the substitution u= $x^2$ - 9, and this substitution knocks the problem off in just a few seconds. 178 students took the exam. About half used the right substitution and got off to a good start, as I intended. However, 44 of the students, noting the factorable denominator in the integrand, used partial fractions to express $x/x^2$-9 in the form [A/$x$-3 + B/$x$+3] – correct but quite time-consuming. They didn't do too well on the exam. And 17 students, noting the ($u^2$ - $a^2$) form of the denominator, worked the problem using the substitution $x = 3\sin\theta$. *This* too yields the right answer – but *it* was even more time-consuming, and the students wound up so far behind that they bombed the exam.

Doing well, then, is based on more than "knowing the subject matter;" it's based on knowing which techniques to use and when. If your strategy choice isn't good, you're in trouble. That's the case in techniques of integration, when there are only a dozen techniques and they're all algorithmic. As we've seen, heuristic techniques are anything but algorithmic, and they're much harder to master. In addition, there are hundreds of them – so strategy selection becomes even more important a factor in success. My point was this. Knowing the strategies isn't enough. You've got to know when to use which strategies.

As you might expect by now, the AI metaphor provided the basic approach. I observed good problem solvers with an eye towards replicating their heuristic strategy selection. Generalizing what they did, I came up with a prescriptive scheme for picking heuristics, called a "managerial strategy." It told the student which strategies to use, and when (unless the student was sure he had a better idea). Now again, this approach is not quite as silly as it sounds. Indeed, the seeds of it are in Pó1ya ("First. You have to understand the problem."). The students weren't forced to follow the managerial strategy like little automata. But the strategy suggested that heuristic techniques for understanding the problem should be used first, planning heuristics next, exploration heuristics in a particular order (the metric was that the further the exploration took you from the original problem, the later you should consider it), and so on. In class we talked about which heuristic technique we might use at any time, and why. Was the approach reductive? Maybe so. But the bottom line is that this combination of making the heuristics explicit, and providing a managerial strategy for students, was gloriously successful.

The final examinations for my problem solving courses had three parts. Part 1 had problems similar to problems we had worked in the course. Part 2 had problems that could be solved by the methods we had studied, but the problems did not resemble ones we had worked. Part 3 consisted of problems that had stumped me. I had looked through contest problem books, and as soon as I found a problem that baffled me, I put it on the exam! The students did quite well even on part 3; some solved problems on which I had not made progress, in the same amount of time.

Thus ended Phase I of my work. At that point – the late 1970's to 1980 – I was pretty happy with the instruction, and was getting pretty good results. Then something happened that shook me up quite a bit. Thanks to a National Science Foundation grant I got a videotape machine, and actually looked at students' problem solving behavior. What I saw was frightening.

Even discounting possible hyperbole in the last sentence, one statement in the previous paragraph sounds pretty strange. I'd been teaching for more than a decade and doing research on problem solving for about half that time. How can I suggest that, with all of that experience, I had never really looked at students' problem solving behavior? With the videotape equipment, I brought students into my office, gave them problems (before, after, and completely independently of my problem solving courses), and had them work on the problems at length.

Then, at leisure, I looked at the videotapes and examined, in detail, what the students actually did while they worked on the problems. What I saw was nothing like what I expected, and nothing like what I saw as a teacher. That's because as teachers (and often as researchers) we look at a very narrow spectrum of student behavior. Generally speaking, we only see what students produce on tests; that's the product, but focusing on the product leaves the process by which it evolved largely invisible. (There's a substantial difference between watching a 20-minute videotape of a student working a problem and reading the page or two of "solution" that student produced in those 20 minutes. The difference can be mind boggling.) In class, or in office hours, we have conversations with the students, but the conversations are directed toward a goal – explaining something the student comes prepared to understand, and knows is coming. The student is primed for what we have to say. And that's the point. When we give students a calculus test and there's a max-min problem in it, students know it's a max-min problem. They've just finished a unit on max-min problems, and they expect to see a max-min problem on the exam. In other words, the context tells the students what mathematics to use. We get to see them at their very best, because (a) they're prepared, and (b) the general context puts them in the right ballpark and tells them what procedures to use. By way of analogy, you don't discover whether kids can speak grammatically (or think on their feet) when you give them a spelling test, after they've been given the list of words they'll be tested on. (Even when I taught the problem solving class, I was showing students techniques that they knew were to be used in the context of the problem solving class. Hence they came to my final prepared to use those techniques.)

In my office, problems come out of the blue and the context doesn't tell students what methods are appropriate. The result is that I get to see a very different kind of behavior. One problem used in my research, for example, is the following:

> **Problem 4:** Three points are chosen on the circumference of a circle of radius R, and the triangle with those points as vertices is drawn. What choice of points results in the triangle with largest possible area? Justify your answer as well as you can.

Though there are clever solutions to this problem (see below), the fact is that you can approach it as a standard multivariate max-min problem. Virtually none of my students (who had finished 3rd semester calculus, and who knew more than enough mathematics to knock the problem off) approached it that way. One particular pair of students had just gotten A's in their 3rd semester calculus class, and each had gotten full credit on a comparably difficult problem on their exam. Yet when they worked on this problem they jumped into another (and to me, clearly irrelevant) approach altogether, and persisted at it for the full amount of allotted time. When they ran out of time, I asked them where they were going with that approach and how it might help them. They couldn't tell me. That solution attempt is best described as a twenty-minute wild goose chase.

Most of my videotapes showed students working on problems that they "knew" enough mathematics to solve. Yet time and time again, students never got to use their knowledge. They read the problem, picked a direction (often in just a second or two), and persevered in that direction no matter what. Almost sixty percent of my tapes are of that nature. But perhaps the most embarrassing of the tapes is one in which I recorded a student who had taken my problem solving course the year before.

There is an elegant solution to Problem 4, which goes as follows. Suppose the three vertices are A, B, and C. Hold A and B fixed, and ask what choice of C gives the largest area. It's clearly when the height of the triangle is maximized – when the triangle is isosceles. So the largest triangle must be isosceles. Now you can either maximize isosceles triangles (a one-variable calculus problem), or finish the argument by contradiction. Suppose the largest triangle, ABC, isn't equilateral. Then two sides are unequal; say AC $\neq$ BC. If that's the case, however, the isosceles triangle with base AB is larger than ABC – a contradiction. So ABC must be equilateral.

The student sat down to work the problem. He remembered that we'd worked it in class the previous year, and that there was an elegant solution. As a result, he approached the problem by trying to do something clever. In an attempt to exploit symmetry he changed the problem he was working on (without acknowledging that this might have serious consequences). Then, pursuing the goal of a slick solution he missed leads that clearly pointed to a straightforward solution. He also gave up potentially fruitful approaches that were cumbersome because "there must be an easier way." In short, a cynic would argue that he was worse off after my course than before. (That's how I felt that afternoon.)

In any case, I drew two morals from this kind of experience. The first is that my course, broad as it was, suffered from the kind of insularity I discussed above. Despite the fact that I was teaching "general problem solving strategies," I was getting good results partly because I had narrowed the context: students knew they were supposed to be using the strategies in class, and on my tests. If I wanted to affect the students' behavior in a lasting way, outside of my classroom, I would have to do something different. [Note: I had plenty of testimonials from students that my course had "made me a much better problem solver," "helped me do much better in all of my other courses," and "changed my life." I'm not really sanguine about any of that.] Second and more important, I realized that there was a fundamental mistake in the approach I had taken to teaching problem solving – the idea that I could, as I put it so indelicately in the first paragraph of this paper, cram problem solving knowledge down my students' throats.

That kind of approach makes a naïve and very dangerous assumption about students and learning. It assumes, in essence, that each student comes to you as a *tabula rasa,* that you can write you problem solving "message" upon that blank slate, and that the message will "take." And it just ain't so. The students in my problem solving classes were the successes of our system. They were at Hamilton College, at Rochester, or at Berkeley because they were good students; they were in a problem solving class (which was known as a killer) because they liked mathematics and did pretty well at it. They come to the class with well engrained habits – the very habits that have gotten them to the class in the first place, and accounted for their success. I ignore all of that (well, not really; but a brief caricature is all I've got room for) and show them "how to do it right." And when they leave the classroom and are on their own ... well, let's be realistic. How could a semester's worth of training stack up against an academic lifetime's worth of experience, especially if the course ignores that experience? (Think of what it takes to retrain a self-taught musician or tennis player, rather than teach one from scratch. Old habits die very very hard, if they die at all.)

Well, the point is clear. If you're going to try to affect students' mathematical problem solving behavior, you'd better understand that behavior. That effort was the main thrust of what (linear type that I am) I'll call phase 2. Instead of trying to do things to (and with) students, the idea was to understand what went on in their heads when they tried to do mathematics. Roughly speaking, the idea was this. Suppose I ask someone to solve some mathematics problems for me. For the sake of a permanent record, I videotape the problem solving session (and the person talks out loud as he or she works, giving me a verbal "trace" as well.). My goal is to understand what the person did, why he or she did it, and how those actions contributed to his or her success or failure at solving the problem. Along the way I'm at liberty to ask any questions I want, give any tests that seem relevant, and perform any (reasonable) experiments. What do I have to look at, to be reasonably confident that I've focused on the main determinant of behavior, and on what caused success or failure?

The details of my answer are xvi+409 pages long. The masochistic reader may find them, as well as the details of the brief anecdotes sketched above, in my (1985) *Mathematical Problem Solving.* In brief, the book suggested that if you're going to try to make sense of what people do when they do mathematics, you'd better look at:

A. "Cognitive resources," one's basic knowledge of mathematical facts and procedures stored in LTM (long term memory.) Most of modern psychology, which studies what's in a person's head and how that knowledge is accessed, is relevant here.
B. Problem solving strategies or heuristics. I've said enough about these.
C. Executive or "Control" behavior. [For the record, this behavior is often referred to as "metacognition."] I discussed this above as well. It's not just what you know (A+B above), it's how you use it. The issue in the book was how to make sense of such things. It's tricky, for the most important thing in a problem solving session may be something that doesn't take place – asking yourself if it's really reasonable to do something, and thereby forestalling a wild goose chase.
D. Belief systems. I haven't mentioned these yet, but I will now.

Beliefs have to do with your mathematical *weltanschauung,* or world view. The idea is that your sense of what mathematics is all about will determine how you approach mathematical problems. At the joint CMS/CMESG meetings in June 1986, Ed Williams told me a story that illustrates this category. Williams was one of the organizers of a problem solving contest which contained the following problem:

*"Which fits better, a square peg in a round hole or a round peg in a square hole?"*

Since the peg-to-hole ratio is $2/\pi$ (about .64) in the former case and $\pi/4$ (about .79) in the latter, the answer is "the round peg." (Since the tangents line up in that case and not in the other, there's double reason to choose that answer.) It seems obvious that you have to answer the question by invoking a computation. How else, except with analytic support, can you defend your claim?

It may be obvious to us that an analytic answer is called for, but it's not at all obvious to students. More than 300 students – the cream of the crop – worked the problem. Most got the right answer, justifying it on the basis of a rough sketch. *Only four students* out of more than 300 justified their answer by comparing areas. (I can imagine a student saying "you just said to say which fit better. You didn't say to prove it.") Why? I'm sure the students could have done the calculations. They didn't think to, because they didn't realize that justifying one's answer is a necessary part of doing mathematics (from the mathematician's point of view).

For the sake of argument, I'm going to state the students' point of view (as described in the previous paragraph) in more provocative form, as a belief:

> **Belief 1:** If you're asked your opinion about a mathematical question, it suffices to give your opinion, although you might back it up with evidence if that evidence is readily available. Formal proofs or justifications aren't necessary, unless you're specifically asked for them – and that's only because you have to play by the rules of the game.

We've seen the behavioral corollary of this belief, as Williams described it. Unfortunately, this belief has lots of company. Here are two of its friends, and their behavioral corollaries.

> **Belief 2:** All mathematics problems can be solved in ten minutes or less, if you understand the material. **Corollary:** Students give up after ten minutes of work on a "problem.")

**Belief 3:** Only geniuses are capable of discovering, creating, and understanding mathematics. **Corollary:** Students expect to take their mathematics passively, memorizing without hope or expectation of understanding.

An anecdote introduces one last belief. A while ago I gave a talk describing my research on problem solving to a group of very talented undergraduate science majors at Rochester. I asked the students to solve Problem 5, given in Fig. 1. The students, working as a group, generated a correct proof. I wrote the proof (Fig. 2) on the board. A few minutes later I gave the students Problem 6, given in Fig. 3.

In the figure below, the circle with center C is tangent to the top and bottom lines at the points P and 0 respectively.

a. Prove that PV = QV. Confessions.
b. Prove that the line segment CV bisects angle PVQ

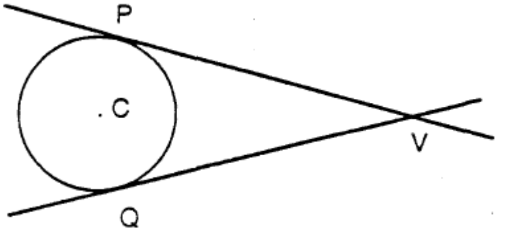
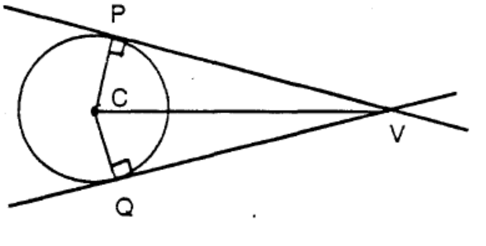



Figure 1




**Proof:** Draw in the line segments CP, CQ, and CV. Since CP and CQ are radii of circle C, they are equal; since P and Q are points of tangency, angles CPV and CQV are right angles. Finally since CV=CV, triangles CPV and CQV are congruent.

a. Corresponding parts of congruent triangles are congruent, so PV = QV.
b. Corresponding parts of congruent triangles are congruent, so angle PVC = angle QVC.

Figure 2

You are given two intersecting straight lines and a point P marked on one of them, as in the figure below. Show how to construct, using straightedge and compass, a circle that is tangent to both lines and that has the point P as its point of tangency to the top line.

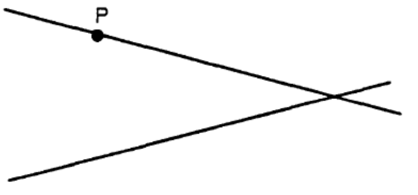



Figure 3

Students came to be bored and made the following conjectures, in order:

a. Let Q be the paint on the bottom line such that QV = PV. The center of the desired circle is the midpoint of line segment PQ. (Fig. 4a).
b. Let A be the segment of the arc with vertex V, passing through P, and bounded by the two lines. The center of the desired circle is the midpoint of the arc A. (Fig. 4b).
c. Let R be the point on the bottom line that intersects the line segment perpendicular to the top line at P. The center of the desired circle is the midpoint of line segment PR. (Fig. 4c).
d. Let $L_1$ be the line segment perpendicular to the top line at P, and $L_2$ the bisector of the angle at V. The center of the desired circle is the point of intersection of $L_1$ and $L_2$. (Fig. 4d).

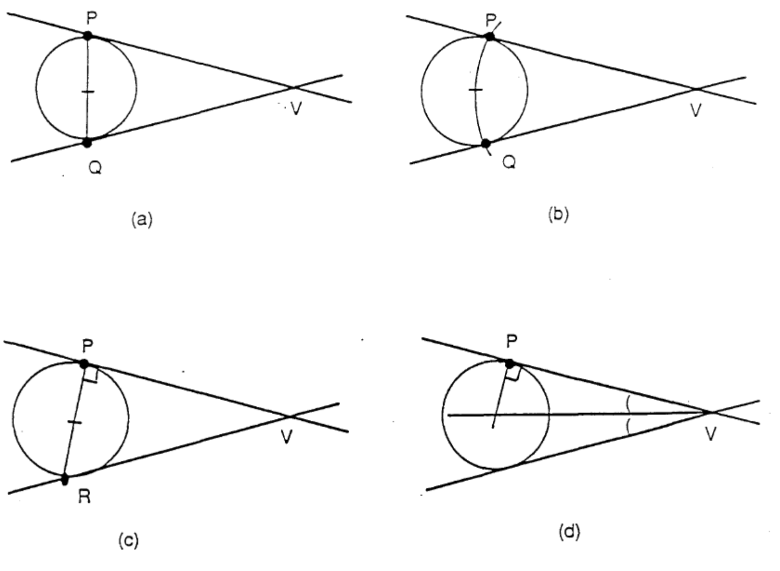



Figure 4: Students' conjectured solutions
(Short horizontal lines denote midpoints.)

The proof that the students had generated – which both provides the answer and rules out conjectures a, b, and c – was still on the board. Despite this, they argued for more than ten minutes about which construction was right. The argument was on purely empirical grounds (that is, on the grounds of which construction looked right), and it was not resolved. How could they have this argument, with the proof still on the board? I believe that this scene could only take place if the students simply didn't see the proof problem as being relevant to the construction problem. Or again in provocative form.

> **Belief 4:** Formal mathematics, and proof, have nothing to do with discovery or invention. **Corollary:** The results of formal mathematics are ignored when students work discovery problems.

Since we're in "brief survey mode," I don't want to spend too much time on beliefs *per se*. I think the point is clear. If you want to understand students' mathematical behavior, you have to know more than what they "know." These students "knew" plane geometry, and how to write proofs; yet they ignored that knowledge when working construction problems. Understanding what went on in their heads was (and is) tricky business. As I said, that was the main thrust of phase 2.

But enough of that; we're confronted with a real dilemma. The behavior I just described turns out to be almost universal. Undergraduates at Hamilton College, Rochester, and Berkeley all have much the same mathematical world view, and the (U.S.) National Assessments of Educational Progress indicate that the same holds for high school students around the country. How in the world did those students develop their bizarre sense of what mathematics is all about?

The answer, of course, lies in the students' histories. Beliefs about mathematics, like beliefs about anything else – race, sex, and politics, to name a few – are shaped by one's environment. You develop your sense of what something is all about (be that something mathematics, race, sex, or politics) by virtue of your experiences with it, within the context of your social environment. You may pick up your culture's values, or rebel against them – but you're shaped by them just the same.

Mathematics is a formal discipline, to which you're exposed mostly in schools. So if you want to see where kids' views about mathematics are shaped, the first place to go is into mathematics classrooms. I packed up my videotape equipment, and off I went. Some of the details of what I saw, and how I interpreted it, are given in the in-press articles cited in the references. A thumbnail sketch of some of the ideas follows.

Borrowing a term from anthropologists, what I observed in mathematics classes was the *practice* of schooling – the day-to-day rituals and interactions that take place in mathematics classes, and *(de facto)* define what it is to do mathematics. One set of practices has to do with homework and testing. The name of the game in school mathematics is "mastery:" Students are supposed to get their facts and procedures down cold. That means that most homework problems are trivial variants of things the students have already learned. For example, one "required" construction in plane geometry (which students memorize) it to construct a line through a given point, parallel to a given line. A homework assignment given a few days later contained the following problem: Given a point on a side of a triangle, construct a line through that point parallel to the base of the triangle. This isn't a problem; it's an exercise. It was one of 27 "problems" given that night; the three previous assignments had contained 28, 45, and 18 problems respectively. The test on locus and constructions contained 25 problems, and the students were expected to finish (and check!) the test in 54 minutes – an average of two minutes and ten seconds per problem. Is it any wonder that students come to believe that any problem can be solved in ten minutes or less?

I also note that the teacher was quite explicit about how the students should prepare for the test. His advice – well intentioned – to the students when they asked about the exam was as follows: "You'll have to know a" your constructions cold so that you don't spend a lot of time thinking about them." In fact, he's right. Certain skills should be automatic, and you shouldn't have to think about them. But when this is the primary if not the only message that students get, they abstract it as a belief: mathematics is mostly, if not all, memorizing.

Other aspects of what I'll call the "culture of schooling" shape students' view of what mathematics is a" about. Though there is now a small movement toward group problem solving in the schools, mathematics for the most part is a solitary endeavor, with individual students working alone at their desks. The message they get is that mathematics is a solitary activity.

They also get a variety of messages about the nature of the mathematics itself. Many word problems in school tell a story that requires a straightforward calculation (for example, "John had twenty-eight candy bars in seven boxes. If each box contained the same number of candy bars, how many candy bars are there in each box?"). The students learn to read the story, figure out which calculation is appropriate, do the calculation, and write the answer. An oft quoted problem on the third National Assessment of Educational Progress (secondary school mathematics) points to the dangers of this approach. It asked how many buses were needed to carry 1128 soldiers to their training site, if each bus holds 36 soldiers. The most frequent response was "31 remainder 12" – an answer that you get if you follow the *practice* for word problems just described, and ignore the fact that the story (ostensibly) refers to a "real world" situation.

Even when students deal with "applied" problems, the mathematics that they learn is generally clean, stripped of the complexities of the real world. Such problems are usually cleaned up in advance – simplified and presented in such a way that the techniques the students have just studied in class will provide a "solution." The result is that the students don't learn the delicate art of mathematizing – of taking complex situations, figuring out how to simplify them, and choosing the relevant mathematics to do the task. Is it any surprise that students aren't good at this, and that they don't "think mathematically" in. real world situations for which mathematics would be useful?

I'm proposing here that thorny issues like the "transfer problem" (why students don't transfer skills they've learned in one context and use them in other, apparently related ones) and the failure of a whole slew of curriculum reform movements (e.g. the "applications" movement a few years back) have, at least in part, cultural explanations. Suppose we accept that there is such a thing as school culture, and it operates in ways like those described above. Curricular reform, then, means taking new curricula (or new ideas, or ...) and shaping them so that they fit into the school culture. In the case of "applications," it means cleaning problems up so that they're trivial little exercises – and when you do that, you lose both the power, and the potential transfer, of the applications. In that sense, the culture of schooling stands as an obstacle to school reform. Real curricular reform, must in part involve a reform of school culture. Otherwise it doesn't stand a chance.

Well, here I am arguing away in the midst of – as though you haven't guessed – phase 3. There are two main differences from phase 2. The first is that I've moved from taking snapshot views of students (characterizing what's in a student's head when the student sits down to work some problems) to taking a motion picture. The question I'm exploring now is: how did what's in the student's head evolve the way it did? The second is that the explanatory framework has grown larger. Though I still worry about "what's going on in the kid's head," I look outside for some explanations – in particular, for cultural ones.

And yet *plus* ça *change, plus* ça *reste le même.* I got into this business because, in Halmos' phrase, I thought of problem solving as "the heart of mathematics" – and I wanted students to have access to it. As often happens, I discovered that things were far more complex than I imagined. At the microlevel, explorations of students' thought processes have turned out to be much more detailed (and interesting!) than I might have expected. I expect to spend a substantial part of the next few years looking at videotapes of students learning about the properties of graphs. Just how do they make sense of mathematical ideas? Bits and pieces of "the fine structure of cognition" will help me to understand students' mathematical understandings. At the macro-level, I'm now much more aware of knowledge acquisition as a function of cultural context. That means that I get to play the role of amateur anthropologist – and that in addition to collaborating with mathematicians, mathematics educators, AI researchers, and cognitive scientists, I now get to collaborate with anthropologists and social theorists. That's part of the fun, of course. And that's only phase 3. I can't tell you what phase 4 will be like, but there's a good chance there will be one. Like the ones that preceded it, it will be based in the wish to understand and teach mathematical thinking. It will involve learning new things, and new colleagues from other disciplines. And it's almost certain to be stimulated by my discovery that there's something not right about the way I've been looking at things.

Are there any morals to this story – besides the obvious one, that I've been wrong so often in that past that you should be very skeptical about what I'm writing now? I think there's one. My work has taken some curious twists and turns, but there has been a strong thread of continuity in its development; in many ways, each (so-called) phase enveloped the previous ones. What caused the transitions? Luck, in part. I saw new things, and pursued them. But I saw them because they were there to be seen. Human problem solving behavior is extraordinarily rich, complex, and fascinating – and we only understand very little of it. It's a vast territory waiting to be explored, and we've only explored the tiniest part of that territory. Each of my "phase shifts" was precipitated by observations of students (and, at times, their teachers) in the process of grappling with mathematics. I assume that's how phase 4 will come about, for I'm convinced that – putting theories and methodologies, and tests, and just about everything else aside – if you just keep your eyes open and take a close look at what people do when they try to solve problems, you're almost guaranteed to see something of interest.

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# THE ROLE OF EPISTEMOLOGY IN THE ANALYSIS OF TEACHING/LEARNING RELATIONSHIPS IN MATHEMATICS EDUCATION

Michèle Artigue
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In 1995 we had the honour of hosting Professor Michèle Artigue, French mathematician and mathematics educational researcher, former president of ICMI and Felix Kline medalist for life-time achievements in mathematics education. Artigue's plenary was chosen for inclusion in this volume because it represents the world class scholarship that Canadian mathematics educators and researchers have had access to through our plenary lectures. Her lecture, "The role of epistemology in the analysis of teaching/learning relationships in mathematics education" introduced those of us who had little exposure to French didatiques, including notions of epistemological obstacles, tool – object dialetic and alternate contexts for mathematical objects. Not only did she illustrate their theoretical value but she offered a critique on the limitations of these concepts. With concrete examples from empirical work she illustrated these deep theoretical notions.

En 1995, nous avons eu l'honneur de recevoir la professeure Michèle Artigue, mathématicienne et chercheure française en didactique des mathématiques, ancienne présidente de l'ICMI, récipiendaire de la médaille Klein de l'ICMI pour sa carrière exceptionnelle dans le champ de la recherche en didactique des mathématiques. La conférence plénière de Michèle Artigue a été retenue pour ce volume parce qu'elle nous rappelle qu'au fil du temps, les rencontres annuelles du GCEDM nous ont donné accès, par le biais de leurs conférences plénières, à de nombreux chercheurs de renommée internationale. Sa présentation, "Le rôle de l'épistémologie dans l'analyse des relations enseignement/apprentissage en didactique des mathématiques", a initié les membres du groupe peu exposés à la didactique française à des concepts clés comme l'obstacle épistémologique, la dialectique outil-objet et les jeux de cadres. Non seulement a-t-elle insisté sur leur valeur théorique mais, adoptant un regard critique, elle a mis en évidence quelques limites de ces concepts. Aussi, puisant à des recherches empiriques, elle a illustré ces notions théoriques à l'aide d'exemples.

## INTRODUCTION

More and more, mathematics education is considered as an autonomous scientific field, capable of defining its own problems and methodologies. Nonetheless the question of developing or maintaining relationships with related scientific fields remains an essential one. In this text, we focus on the relationships with epistemology which, in my opinion, are fundamental.

First we clarify our use of the word "epistemology" by underlining what we feel is of crucial interest in epistemological work from a didactical point of view, namely, reflection on the nature of mathematical concepts, on the processes and conditions for their development, on the characteristics of present as well as past mathematical activity, on what constitutes the specific nature of one mathematical domain or another.

Such a reflection is necessary in mathematics education, for several reasons, the following being of particular importance:

- our work in mathematics education is governed, implicitly if not explicitly, by our epistemological representations and we have to be as clear as possible about them,

- a strong and privileged contact with mathematics via the educational world tends to distort epistemological representations, to shape them in order to make them compatible with the way mathematics is living in this educational world, and to reduce our mathematics to the taught mathematics.

So epistemological work is necessary to be able to look at this educational world from the outside, to make its epistemological choices apparent and questionable.

By making its epistemological choices apparent and open to discussion, epistemological work helps us to gain an extrinsic view on education. This epistemological work has taken different forms in the didactics of mathematics. In the first part of this text, without pretending to be exhaustive, we will give some idea of its variety and richness. In the second part, we will evoke its limits.

## THE ROLE OF THE THEORY OF EPISTEMOLOGICAL OBSTACLES IN THE DIDACTICS OF MATHEMATICS

For a French didactician, the word "epistemology" immediately evokes the theory of "epistemological obstacles" initially developed by the philosopher G. Bachelard (Bachelard, 1938) and transported into the didactics of mathematics by G. Brousseau, 20 years ago (lecture given at the CIEAEM Conference, in Louvain la Neuve, in 1976). As stressed by A. Sierpinska in her book "Understanding in mathematics", the idea of "discontinuity" inherent to this theory can be found in many philosophers before and after Bachelard from Bacon and Husserl to Lakatos and Kuhn, not to mention others. Perhaps more than others, this theory dramatically put to the fore discontinuity, by considering that new knowledge is always founded, in some part, on a rejection process.

Bachelard (1938) wrote in "La formation de l'esprit scientifique."

> *Reflecting on a past of errors, the truth is found in a real intellectual repentance. In fact, one knows always against some previous knowledge, by destroying ill built knowledge, by overcoming that which in the mind itself is an obstacle to spiritualization (p.13).*

This dramatic position is specially questioning for mathematics educators as it radically disqualifies the still dominant illusion that mathematics learning can be organised along a smooth path, where knowledge increases gently step by step, with some necessary reorganisations, of course—everyone has heard about Piaget's theories of assimilation and accommodation-but fundamentally in a process capable of avoiding major ruptures and the disturbing paradoxes of the didactic contract they induce (Brousseau, 1986).

At the opposite end, the theory of epistemological obstacles is based on the fact that ruptures are the normality, that we cannot directly learn definitive forms of knowledge, that progress necessarily requires some kind of rejection of what has been for a time, often a long time, a motor of progress.

Initially, Brousseau exploited this notion to analyze the persistent errors of pupils in the extension of numbers from whole numbers to rationals and decimals and to question the dominant status of errors in the educational world (Brousseau, 1983). Later, the field of mathematical analysis and especially the notion of limit became a field of interest for the development of this theory within the didactics of mathematics, through the works of B. Cornu first (Cornu, 1983) and then A. Sierpinska (Sierpinska, 1985). After an in-depth historical study, Sierpinska produced a structured list of obstacles[2] which marked out the historical evolution of the concept of limit and she proved that such a list could be used in order to interpret persistent difficulties encountered by present students. Beyond that, Cornu's and Sierpinska's research clearly showed the existence of different kinds of epistemological obstacles:

- some could be linked to common and social knowledge about limits,
- some could be traced to the under-development of crucial notions such as that of function,
- some were linked to the over-generalization of properties of familiar finite processes to this infinite process according, for instance, to the "continuity principle" stated by Leibnitz,
- some, not of minor importance, could be linked to more philosophical principles and beliefs about the nature of mathematical objects and mathematical activity, for instance about the status of infinity.

At a theoretical level Sierpinska, with reference to Wilder and Hall, (Sierpinska, 1988 and 1994) integrated this diversity by considering "mathematics as a developing system of culture and a sub-culture of the overall culture in which it develops." This cultural conception, of mathematics leads to the identification of three levels in mathematical culture, in on-going interaction:

- the formal level, that of unquestioned principles and beliefs,
- the informal level which is the level of "tacit knowledge, of unspoken ways of approaching and solving problems," "of canons of rig our and implicit conventions,"
- the technical level which is the domain of rationally justified, explicit knowledge.

According to Sierpinska, epistemological obstacles are situated at the first two levels and this location has some important consequences for the strategies we have to develop in order to overcome them.

This approach in terms of epistemological obstacles is often associated with a search in the history of mathematics, for significant and fundamental problems which permit an organization of the teaching process that would be epistemologically more adequate than the usual ones. Research developed in Louvain la Neuve under the direction of N. Rouche and, especially M. Schneider's thesis entitled "Des objets mentaux aires et volumes au calcul des primitives" is typical of that direction (Schneider, 1989). It is based on a close analysis of students' behaviour when faced with a field of problems, mainly adapted from historical ones. This analysis tends to prove that the perception of surfaces (respectively volumes) as the piling up of segments (respectively surfaces), similar to that developed by Cavalieri and others in the seventeenth century, although not explicitly taught, *is* present in the mental representations and informal mathematical culture of today's students. Schneider has shown that this fact can explain some frequent and persistent errors in the calculation of areas and volumes, as well as some difficulties in understanding the modem process of integration.

From there, a teaching strategy is designed where problems are first chosen to highlight the productive character of this perception of geometrical objects and make explicit corresponding informal reasonings. For instance, as shown in Figure 1, students have to explain why the area

[2] The list was structured around four categories: "horror infiniti" grouping both obstacles linked to the rejection of the status of mathematical operation for the limit process and obstacles linked to the automatic transfer of methods and results of finite processes to infinite ones; obstacles linked to the concept of function; obstacles linked to an over-exclusively geometrical conception of limit; obstacles of a logic nature.

oft be parallelogram is the same as the area of the rectangle, or why the volume of the solid delimited by the cylinder and the half-sphere is the same as that of the cone.

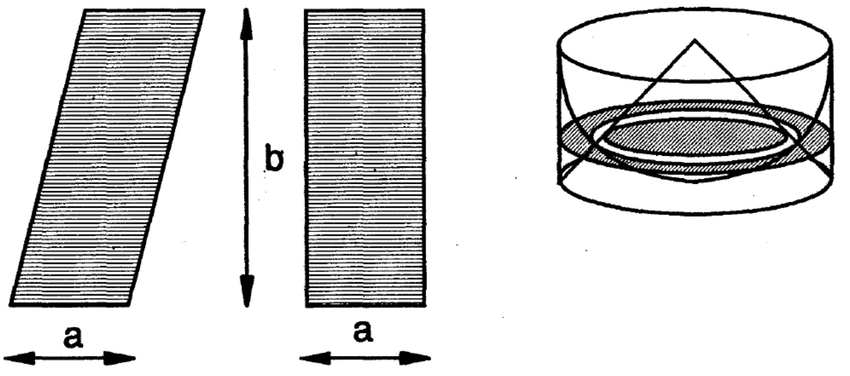
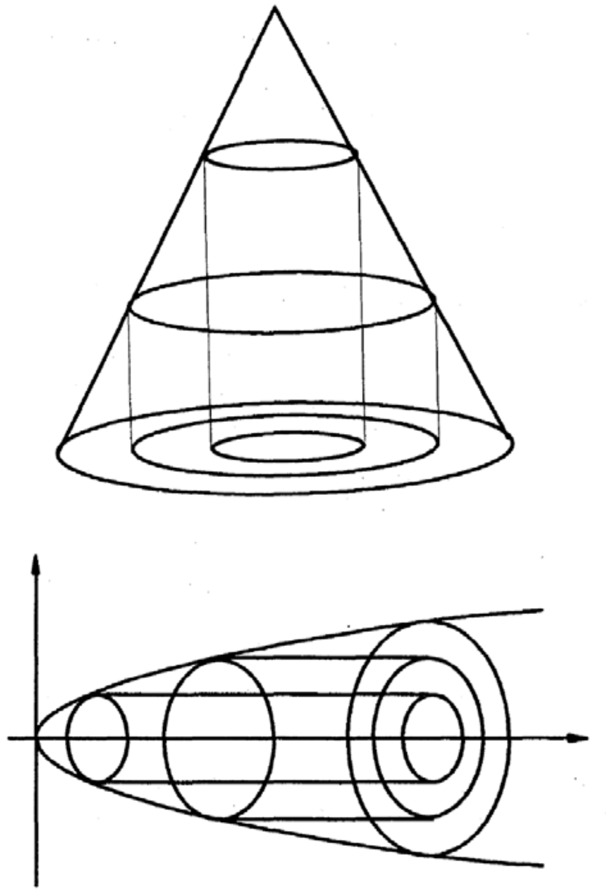



Figure 1: The productive character of an intuition in terms of indivisible.

In the next step, problems which fired controversies in the seventeenth century were used to attack the epistemological obstacle (called by the author "the obstacle of heterogeneity of dimensions") derived from this productive perception: It results from the simultaneous, uncontrolled and often unconscious use of geometrical objects of different dimensions in the calculation of areas and volumes.

For instance, students have to explain why the lateral area of the cone is different from its base area even though there is a one-to-one correspondence between the circles which compose each of them. Or they have to explain why the formula obtained by summing up the lateral areas of the cylinders which compose the solid of revolution, in Figure 2, does not give a correct value for the volume of this solid.

Before leaving this point, let us stress that such a focus on strong and necessary ruptures in the growth of knowledge, is not necessarily subordinated to historical analysis in didactical research. For instance, M. Legrand (1993) recently developed a strong epistemological analysis in order to understand the relationships between algebra and analysis, referring only to the actual prevalent vision of the field of analysis, as the field of "approximation, majoration, minoration", as expressed by Dieudonne. He showed that students must construct their knowledge in mathematical analysis both on and against their previous algebraic knowledge. For instance, in the same way these students have to radically modify their relationship to equality when passing from numerical thinking developed in our elementary schools to the algebraic thinking developed in junior high schools, they have once more to radically modify this relationship when passing from algebraic thinking to analytic thinking. For example, in order to prove that two objects *a* and *b* are equal, most often in analysis, we do not use equivalence processes directly as in algebra-such a strategy is often out of reach and, even if it can be used, generally it is not the most effective. Instead, we try to prove that for an adequate norm or distance, the norm of the difference between *a* and *b* or the distance between *a* and *b* is less than any positive real number.



$$\int_0^4 2\pi\sqrt{x\,(4-x)}\,dx$$

Figure 2: The epistemological obstacle of heterogeneity of dimensions.

Entering analytic thinking means understanding that this kind of detour will generally be worthwhile. At a more advanced level, students will have to understand that, in analysis, in order to prove that a family FI of objects possess a given property, a similar detour will often be used: prove the property for a family of simpler objects F2, then prove that each element of FI can be considered as the limit for an adequate topology of elements of F2 and that the property at stake is conserved through the limit process. In the same way, students have to change their habits with the treatment of inequalities and many other objects, familiar but in another world-the algebraic world.

Legrand points out how traditional teaching, shaped by the illusion of continuity, is insensitive to these problems and tends to leave it up to the students to deal with these crucial ruptures and reconstructions.

## EPISTEMOLOGICAL ROOTS OF GLOBAL THEORIES IN DIDACTICS

Beyond attracting attention to ruptures and discontinuities in knowledge, epistemology also plays an essential role in the more global theoretical frameworks we develop. Many examples can illustrate this point. I would like to deal with this aspect through one example: the framework in terms of "tool-object dialectic and setting games" developed by R. Douady in the last decade (Douady, 1984) and widely used in French didactics.

### THE TOOL OBJECT DIALECTIC

The "tool-object" dialectic is based on the following epistemological distinction: a mathematical concept can be attributed two different status:

- a "tool" status when it is thought of and used as a tool in order to solve specific problems,
- an "object" status when it is considered in its cultural dimension, as a piece of socially recognised scientific knowledge, and studied for its own sake.

The study of the history of mathematical concepts suggests that most of the time (but not always) mathematical concepts come into being first as tools and the tool is the basis for the construction of the object.

This distinction appears as a strong characteristic of mathematical concepts and mathematical activity. One could also evoke the distinction between the structural and operational dimensions of mathematical concepts introduced by A. Sfard (Sfard, 1991). Here, the term "structural" refers to a treatment of mathematical concepts "as if they referred to some abstract objects" and the term "operational" to a description in terms of "processes, algorithms and actions." The distinctions introduced by Sfard and Douady do not match exactly. In other words, one cannot equate the "tool" and "operational" dimensions nor equate the "object" and "structural" dimension.

Sfard's analysis is essentially set up around the cognitive processes linked to the transition between processes and objects. Douady's analysis refers more to an analysis in terms of mathematical problems. Thus, when speaking of the tool dimension of the concept of function for instance, one refers to the use of this concept in order to solve problems internal or external to mathematics; when speaking of the process or operational dimension of this concept, one refers to a procedural view of the concept in terms of input-output system as opposed to the static and structural vision developed within set theory. Nevertheless both approaches stress the complementarity and duality of the dimensions they identify in mathematical activity; both point out the historical anteriority of one of these dimensions (the tool dimension for Douady, the operational dimension for Sfard) and the necessity to pay some attention to these characteristics in didactic transposition processes.

This is again a challenging distinction for mathematics educators as the traditional teaching tends to reverse this natural order and introduce objects which only later have to be used as tools in different contexts.

In Douady's research, the tool-object distinction does not appear as a consequence of an historical epistemological study. It is essentially induced by her analysis of contemporary mathematical activity. Nevertheless, history of mathematics shows the relevance of this distinction. Let us take the example of the history of complex numbers. Indeed, what appeared on the mathematical scene during the sixteenth century, in the work of Italian algebraists such as Ferrari, Cardan, and Bombelli was first a tool and even an implicit tool, via the audacious extension of a technique for the solving of equations of the third degree, known as Cardano's technique[3]. This technique was extended to an a priori illegal case (negative square), which led to the introduction of new operational signs such as the "piu di meno" and "meno di meno" notations introduced by Bombelli in his "Algebra" (1572) to deal with the addition and substraction of square roots of negative quantities.

These new expressions, of course, did not immediately take on the status of number. They remained for a long time, without any meaning, pure syntactical objects whose manipulation was governed by the principle of permanence stated later by Leibnitz. Thus, during the seventeenth century, they took on the status of convenient intermediaries in calculations, but only calculations which started with ordinary numbers and led to ordinary numbers.

Slowly, however, this status evolved. Tools, explicitly identified, and named (a specific article is devoted to them in the Encyclopedia by d'Alembert), the imaginary quantities went beyond the single context of equations to enter other domains such as trigonometry with Moivre (1722). They also began to be treated as autonomous variables in functional expressions, especially developments in series, which were so important at the time. Moreover, imaginary quantities became necessary instruments in the formulation of such important results of the eighteenth century such as:

- the fundamental theorem of algebra,
- the unification of spherical and hyperbolical trigonometries, via Euler's formulae.

The tool thus started to present undeniable characteristics of a mathematical object, but it remained a purely symbolic one, defined but not constructed, not susceptible to real interpretation. It was only during the nineteenth century that imaginary quantities acquired their present status of fully legitimate objects. This was achieved in two distinct stages: first via the geometrical interpretation proposed independently by Wessel, Argand, Gauss and others, then via Cauchy's and Hamilton's constructions which finally founded them algebraically.

The kind of analysis this brief presentation attempts to summarize has, in our opinion, an important role to play. It helps the mathematics educator to become more aware of the denaturations didactic theories often suffer when, from a tool for understanding the functioning of teaching/learning relationships, they become tools for acting on educational systems and are more or less consciously shaped in order to become compatible with them. For instance, the above historical analysis reminds us that the relationships between the tool and object dimensions of mathematical concepts are complex and dialectic, much more complex and dialectic than they ordinarily appear in the very simplified versions of the tool-object dialectic proposed in most educational papers. The complex number object is not of the tool-object dialectic proposed in most educational papers. The complex number object is not created suddenly by some miraculous institutionalisation process on the basis of activities of problem solving in which it only had a tool status. Before being fully legitimised, it already produces generality and is engaged in more complex processes.

The dialectic between the tool and some pre-object is established very early and plays an essential role in the evolution of the two dimensions. A restrictive interpretation of the theory such as: first comes the tool, then comes the object and they both develop dialectically, appears rather inadequate. Moreover, the proof of effectiveness in problem solving is not enough to

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[3] In order to solve the equation $x^3 = a + bx$ by searching

$$x = \sqrt[3]{u} + \sqrt[3]{v}$$

as Italian algebraists did at the time, one is led to find $u$ such that the square of $u$-$a$/2 is equal to $(a/2)^2$-$(b/3)^3$. When the initial equation has three real roots, this quantity is negative.

guarantee the status of object: the mechanisms of acquiring an institutional legitimacy are much more subtle.

Beyond this function of epistemological vigilance, such an historical work help us to question our theoretical categories necessarily built in a limited context. For example, Douady formulates the hypothesis that most concepts obey the tool-object dialectic process. What does "most" means exactly? Is it necessary to introduce other categories and what could be the consequences of other distinctions on our didactics theories?

The research we carried out on quaternions (Artigue and Deledicq, 1992), clearly shows that these new numbers entered the mathematical scene in the middle of the nineteenth century, directly as objects. This was achieved in a generalisation process whose aim was to extend to geometrical three dimensional spaces the possibility of an algebraic calculus opened up by the use of complex numbers in plane geometry.

Also the recent work on the history of linear algebra (Dorier, 1990) shows that fundamental concepts of modem linear algebra such as that of vector space in axiomatic form, dimension and rank result more from an unifying process than from a tool-object process. This unifying process was aimed at connecting different problems already solved by mathematicians and was to give the means for elaborating this connection formally. This unifying character is not without influence on their development. Dorier points out for instance that it took a very long time for mathematicians, working in this area and used to solving problems with non-intrinsic methods even in infinite dimension, to really appropriate these concepts and understand their importance. He also stresses that the development of functional analysis and theory of Banach spaces played an essential role.

No doubt other epistemological distinctions could be made about the status of concepts that we teach, beyond the tool-object distinction. For example the notion of "proof generated concept" introduced by Lakatos (1976), could be of some inspiration to analyze the problem mentioned above and reflect on adequate ways for introducing concepts such as the fundamental concepts of linear algebra. For instance, Robert and Robinet (1993) formulate the hypothesis that teaching processes for what they call "generalizing and unifying concepts," have to obey specific strategies and that metamathematical dimension can play an important role.

## SETTING GAMES

Another facet of Douady's thesis is the notion of "setting games." Beyond the tool-object dialectic, she identifies another characteristic of mathematical activity which seems to play an essential role in the growth of knowledge: the fact that mathematical concepts function in different settings (complex numbers, for example, are algebraic objects which can function both in algebraic and geometrical settings). What clearly appears when we observe the research work of mathematicians is that they often play with these different settings, in order to progress in the problems they have to solve (cf. Douady and Douady, 1994, for such an analysis). Roughly speaking, we can schematize the process as follows:

> The initial problem stems in one setting (say setting A) and the work inside this setting allows to attain some state, say state 1A where it seems to be stopped. The translation in another setting: say setting B (necessarily imperfect) allows to transform the problem or some sub-problem considered at state 1A into a new one and thus pass from state 1A to state 1B. Then the work in setting B allows to progress until state 2B and a translation back to setting A allows to get a state 2A which one could not get directly.

This analysis leads Douady to theoretically organize the didactic transposition of the tool-object dialectic around situations which can be worked in several settings and to consider the changes in settings, carefully managed by the teacher, as an essential means for solving the eternal problem of the filiation between old and new knowledge. Her thesis illustrates the potentials of this theory in a long term engineering work at elementary school.

Before leaving this point, let us stress that the description given above of "setting-games" is a very schematic description. Most often, mathematicians do not switch from one setting to another and the interplay between settings is better described in terms of changes in dominant roles: one setting is devoted to technical and formal mathematical work; others are used at a more heuristic level in order to plan, help some choices, control mathematical activity... This dialectic interplay looks evident for instance if we come back to the construction of quaternions evoked above and analyze the long research process-it took 13 years – described by Hamilton in the preface of his book "Lectures on quaternions" (1853).

Even if Hamilton's problematics had evident geometrical roots at the beginning of the process, the dominant setting was an algebraic one. Hamilton wanted to generalize to triplets the construction he had previously made with couples of reals in order to algebraically define complex numbers. During this phase, the geometrical setting seems to appear punctually, with a heuristic role:

- it guides the choice of undetermined products for the six different unities at play as Hamilton simultaneously introduces two products; an internal and an external one:

> *There still remained five arbitrary coefficients [ ... ] which it seemed to be permitted to choose at pleasure: but the decomposition of a certain cubic function combined with geometrical considerations, led me, for the sake of securing the reality and rectangularity of a certain system of lines and planes, to assume the three following relations between those coefficients (Preface, p.25).*

- it helps to understand why the different trials made always give a product which is not regular:

> *The foregoing reasonings respecting triplets systems were quite independent of any sort of geometrical interpretation. Yet it was natural to interpret the results and I did so, by conceiving the three sets of coefficients [ ... ] which belonged to the three triplets in the multiplication, to be coordinate projections, on three rectangular axes, of three right lines drawn from a common origin (preface, p.22).*

After transposing the algebraic product in terms of line products, Hamilton was able to interpret nul products in terms of orthogonality of systems of planes and lines. Not finding the way of solving this problem of non-regularity, Hamilton changed his mind and gave the dominant role to the geometrical setting, trying to generalize to three dimensional space the geometrical interpretation of complex number product: angles, rotations, and lengths were then the main tools. But once more, algebraic setting remained present with a controlling role. Several different generalizations were for instance rejected as they did not obey the distributive principle of algebraic product. And, when, once more blocked, Hamilton came back to algebra, the geometrical setting remained present in order to guide the definition of the product of units and then to find a geometrical global interpretation of the product, first defined algebraically.

Obviously, when we decide to capture such essential features of mathematical activity within our theoretical frameworks, we have to be epistemologically vigilant to the risk we encounter of loosing their essence by over-simplifying them for our educational purposes.

## LIMITS OF EPISTEMOLOGICAL WORK FROM A DIDACTIC POINT OF VIEW

Up to this point, we have stressed the importance of epistemological work. This epistemological work has evident limits.

The educational genesis of concepts cannot match their historical genesis. Obviously, the cognitive functioning of present students can hardly be identified to the cognitive functioning of present or past mathematicians. Historical problems which led to the construction of one or other mathematical concept most often cannot easily be transposed to current teaching. In order not to distort epistemological values and adapt to our present students, a difficult transposition work turns out to be necessary.

Changes in mathematical culture are also evident. Thus, epistemological obstacles identified in history are only candidates for obstacles in the present day learning processes and, conversely, non-historical formal and informal forms of knowledge can act as obstacles for our students.

As far as complex numbers are concerned, for instance, the situation of today's students for whom complex numbers are directly introduced as legitimate objects which are endowed from the start with punctual and vectorial geometrical representations, cannot be compared with the situation of Italian algebraists of the sixteenth century and even with that of their successors. What obstacles are resistant to these differences? In fact, what we can immediately transpose from the theory of epistemological obstacles is the fundamental question: why and how do our students have to change their conception of numbers, their numerical and algebraic informal habits in order to cope efficiently with complex numbers?

The answer to such questions cannot avoid taking into account, beyond the cognitive and epistemological dimension, the social and cultural aspects of present mathematical education. We will illustrate this point with two examples.

### TEACHING DIFFERENTIAL EQUATIONS AT UNIVERSITY LEVEL

The first example refers to personal research on differential equations. In 1986, I had been involved for several years in mathematical research in differential equations and I was struck by the epistemological inadequacy of teaching in this area for students in their first two years at university. Teaching focused solely on the methods of algebraic resolution typical of the functioning of the field in the eighteenth century. It appeared impervious to the epistemological evolution of the field towards geometrical and numerical approaches. What constraints could explain such an obsolete stability of teaching? Was it possible to find another equilibrium which would be epistemologically more adequate? Was this possible at a price acceptable by the didactic system, and, if so, how?

Research began by an epistemological analysis. It showed that, historically, the field of differential equations had developed in at least three settings:

- the "algebraic" setting where the fundamental problem is mainly that of finding exact solutions (in finite or infinite terms) or discussing the possibility or such solutions,
- the "numerical" setting where the fundamental problem is to find approximate solutions and control these approximations,
- the "geometrical" setting where the fundamental problem is the geometrical and topological study of flows associated with equations or families of equations.

Teaching for French beginners was focused on the first setting and gave students an erroneous image of the field. They were convinced that every equation could be exactly solved and that researchers in this area were only looking for the missing recipes.

Epistemological analysis clearly showed that epistemological constraints contributed to explain the characteristics of present teaching, mainly:

- the long domination of the algebraic setting in the historical development of the field,
- the more recent appearance at the end of the nineteenth century in Poincare's work of the geometrical approach,
- the difficulty of problems associated with the geometrical approach such as structural stability problems as well as of problems of exact integrability which marked the development of the algebraic setting past the eighteenth century, and, last but not least,
- a development of the three settings without strong interactions which contributed to their relative isolation.

But soon it appeared that these epistemological constraints were reinforced by cognitive constraints (Artigue, 1992b), as, for instance:

- the flexibility between the graphical register of representation with drawings of flows and associated curves such as isoclines, and the algebraic and symbolic register of representation with the differential equation, associated equations, inequations and functions required by the qualitative resolution of differential equations; indeed qualitative resolution requires a permanent interplay between these two registers and obliges students to coordinate various levels of flexibility, by taking into account not only functions but also their first or second order derivatives,
- the delicate use of elementary tools of analysis required by qualitative proofs when they were presented in their academic form.

What appeared also, was the fact that both types of constraints were strengthened by didactic constraints among them:

- the ordinary tendency of didactic systems to avoid cognitive difficulties in mathematical analysis by favouring algebraic and algorithmic processes; usual algebraic resolution of differential equations, which is essentially algorithmic at this level of teaching, is in accordance with this tendency; on the contrary, qualitative resolution cannot be reduced to algorithmic processes,
- the devalued status of the graphic register of representation in the French educational system which created a didactic obstacle to the necessary acceptance of graphical reasonings, at this level of teaching.

In order to answer the questions at stake, it was necessary to identify and understand the real strength of all these constraints as well as their interrelations. We tried to do this and then exploited the analysis in order to build a realistic teaching strategy which better respected the current field's epistemology. The engineering product was then experimented and progressively adapted with undeniable success (Artigue, 1993).

Undoubtedly, epistemological reflection was an essential part of this engineering work, in some way its starting point, but we had to go far beyond it in order to reflect on the possible ways for acting effectively on the current educational system and succeed.

### TANGENT CONCEPTIONS AND THEIR EVOLUTION AT SECONDARY LEVEL

The second example we will use is a research on the notion of tangent by C. Castela (Castela, 1995). Its aim was to clarify the development of secondary school pupils' conceptions of tangent and to explore the effects following this development in teaching.

At the beginning of secondary school, pupils first encounter the tangent to a circle. This object is a geometrical object endowed with specific properties:

- it does not cut through the circle,
- it touches it at only one point,
- it is perpendicular to the radius at the contact point.

All these properties are global and do not bring into play the idea of common direction. Moreover, in order to help the pupils to become aware of the abstract status of the figure, teachers often insist on the fact that although to the eye, circle and tangent seem to merge locally, they have exactly one common point. In. the same way, this tangent is linked to secants but secants of a given direction which, when moved, help to visualize the change in the number of intersecting points.

In high schools, the teaching of analysis introduces another point of view on the tangent:

- it is a local object with which the curve tends to merge locally,
- it is also the line whose slope is given by the derivative.

Obviously there is no direct relationship between these two objects and it is legitimate to wonder how students manage the transition, if it does work, from "circle" conceptions to "analytic" conceptions. We can wonder also whether, in this transition process, the tangent to the circle is itself a posteriori reconstructed, by integrating the characteristics of more general tangents and becoming the prototype of the tangent to close convex curves, or if it remains s it was before, isolated from the analytic tangent.

Research was carried out through the analysis of text-books, students' and teachers' questionnaires, the students" questionnaire. We will focus here on the students' questionnaire in which various curves and lines were proposed to the students and they were asked to judge for each of them whether: "the line is tangent to the curve at point A" and then to justify their answers.

About 400 students completed the questionnaire. They were at different levels of schooling and from more or less scientific orientations. The answers obtained highlight local adaptation processes which are set up in the long term. Indeed, as could be expected, in high school, before learning analysis, the large majority of students demonstrated coherent conceptions linked to what we have called above the "circle" conception. Differences occurred as some of them blended all the properties of the circle tangent while others seemed to focus on one of them. After the teaching of derivatives, landscape became more chaotic, even though all but one of the items (a curve locally merged with its tangent) obtained high rates of success. This exception scored only a 50% success in "terminale", the final year of secondary school, even though the derivative had been introduced from the notion of linear approximation.

It appears as though, progressively, while remaining an anchor point, the circle conception gradually gave way through various processes:

- by admitting prototypical exceptions such as inflexion points,
- by rejecting prototypical cases such as angular points,
- by integrating in a more or less coherent way some elements from the analytic conception.

Justifications such as the following, for instance, attest it:

> *There is only one point of intersection and it is a maximum.*
>
> *There is one common point and the curve approaches the line tangentially.*

All these processes continued until a swing towards an analytic conception of the tangent could occur. This analytic tangent in turn became the dominant object in relation to which the cognitive network was reorganised. Note that, however, even in scientific "terminales", only 25% of the students taking the questionnaire presented reasonings sufficiently homogeneous to allow one to suppose that such a swing had occurred. Perfectly correct answers were still accompanied by justifications globally incoherent, from one item to another.

The description given above is expressed in cognitive terms. Historical identified conceptions can be used in order to interpret, at least partially, the answers obtained and it was effectively done.

I am not convinced that this is the more pertinent approach. Both epistemological and cognitive approaches encourage us to set the behaviour of our students in a cognitive rationality. Thus they tend to underline the fact that adaptation processes at play in schools are as much adaptation processes to the educational institution as cognitive mathematical adaptation processes.

What conditions the observed adaptations and their limits? The cognitive characteristics of students? The kind of situations they have encountered about tangents? The way they perceive the demands of their teacher? The status of the notion of tangent in current secondary education?

Data collected provide us with elements for answering these questions. On one hand, analysis of textbooks shows that the question of the relationship between circle conceptions and analytic ones is completely absent from teaching: either the tangent to the circle is not mentioned when introducing derivatives or it is considered as a transparent example. Answers to teachers" questionnaire show that they are no more sensitive to this problem. On the other hand, one of the "terminale" classes, a non-scientific one, appeared to be non-typical. It was clearly better than all the other classes in its results: this was a class in which the cognitive reorganisation was not left entirely up to the individual student.

Finally, what this research mainly shows is not the way in which students might construct the concept of tangent and the conceptual difficulties linked to this learning process. More essentially it shows the game the students play with this notion in school and the way they optimize this game.

The adaptations carried out, although chaotic and globally incoherent, are quite adequate to allow these students to play their role of students correctly. Indeed, for the marginal object the tangent is in current secondary teaching, this role simply consists in:

- knowing how to recognize simple cases of non-derivability: vertical tangents, angular points,
- knowing how to determine the equation of a tangent, eventually with given constraints,
- knowing how to draw particular tangents on a graphical representation of function, especially tangents corresponding to extreme and inflexion points, vertical tangents.

It is clear that epistemological and cognitive approaches have to be articulated with approaches which include much more detailed analysis of the didactic situations proposed to students, of the way mathematical adaptations combine in these situations with more contractual and institutional adaptations, and of their possible effects on the learning process.

We also need more macroscopic approaches allowing analysis at an institutional level, to understand how the didactic transposition processes we observe function and how one can effectively act on them, to understand how transition processes are managed by institutions and why, to understand how the personal relationships to knowledge our students articulate with the institutional relationships which define the norm.

In French didactics, the theory of didactic situations, first developed by G. Brousseau (Brousseau, 1986) and the anthropological approach developed more recently by Y. Chevallard (Chevallard, 1992) have these aims.

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# ALIVE MATHEMATICAL REASONING

David Henderson
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Cette conférence plénière, qui place l'humain au centre des mathématiques, est étroitement liée au groupe de travail sur le rôle de la preuve. David Henderson y traite du formalisme et de l'impact très négatif que peut amener une formalisation omniprésente. Se basant sur son expérience personnelle et sur celle de mathématiciens de sa génération, il plaide pour le rôle de l'intuition, et pour une réappropriation des mathématiques par ceux qui l'apprennent. On note aussi l'ajout d'une section de questions et réponses. Celles-ci sont issues de la séance de questions qui suit la plénière, le lendemain de celle-ci, un autre aspect important de nos rencontres.

This plenary, which places human beings at the center of mathematics, is closely linked to the working group on proofs. David Henderson looks at formalism and at the negative impact that an excess of formalism can have. Based on his personal experience and on that of other mathematicians from his generation, he argues for the role of intuition and for an ownership of mathematics by its learners. We can also note a section with questions and answers, at the end of the paper. These came from the questions and answers session, which is always held the day after the plenary, another distinctive feature of our meetings.

## INTRODUCTION

Cecilia Hoyles in first plenary told us that whenever we examine someone's conceptions of proof we should learn something about their background-what have they been taught about mathematics. So, I thought it was appropriate that I start by telling you something of my background, since I am going to talk with you for the next hour about my views of proof as gleaned from my experiences as student, teacher, mathematician and general experiencer of the world.

I have always loved geometry and was thinking about geometric kinds of things since I was very young as evidenced by drawings that I made when I was six which my mother saved. But I did not realize that the geometry, which I loved, was mathematics. I was not calling it geometry-I was calling it drawing or design or not calling it anything and just doing it. I did not like mathematics in school because it seemed very dead to me-just memorizing techniques for computing things and I was not very good at memorizing. I especially did not like my high school geometry course with its formal two-column proofs. But I kept doing geometry in various forms in art classes, out exploring nature, or by becoming involved in photography. This continued on into the university where I was a joint physics and philosophy major and took only those mathematics courses which were required for physics majors. I became absorbed in geometry-based aspects of physics: mechanics, optics, electricity and magnetism, and relativity. On the other hand, my first mathematics research paper (on the geometry of Venn diagrams for more than 4 classes) evolved from a course on the philosophy of logic. There were no geometry courses except for analytic geometry and linear algebra, which only lightly touched on anything geometric. So, it was not until my fourth and last year at the university that I switched into mathematics because I was finally convinced that the geometry that I loved really was a part of mathematics. This is not an uncommon story among research geometers. Since high school, I have never taken a course in geometry because there were no geometry courses offered at the two universities which I attended. So, in some ways I may have had the advantage of not having taken a geometry course!

But I was educated in a very formal tradition-in fact, mathematics was, I think, the most formal that it has been about the time that I was studying at the university in the late 1950's and early 1960's. One of the evidences for this was the number of geometry courses offered at colleges and universities- there were practically none anywhere at that time except for a few geometry courses for prospective school teachers and still at many institutions such courses are the only geometry courses offered.

I am the same generation as most of the faculty now in mathematics departments in North America (most of us are 50-65 years old) because we were hired to teach the baby boomers in the 1960's. So now my generation is clogging up most of the tenure faculty positions all over North America, and in the USA we will not be required to retire because the Supreme Court has ruled recently that it is unconstitutional to have mandatory retirement ages. Almost all of the mathematicians in my generation had a very formal training in mathematics. This has affected us and affected mathematics and will continue to affect mathematics because my generation now has the positions of authority in mathematics.

I want to mention specifically one mathematician in my generation and that is Ted Koscynski, the suspected Unabomber. He had very much the same kind of university mathematics education that I had. Both of us at the time were socially inept and it was difficult for us to get to know people. In fact, in some ways, this was encouraged in our training all the way through graduate school and certainly was not a hindrance in any way. My thesis advisor talked to me and a few of the other men graduate students and said to us that it would be very important for us to find wives who would take care of all of our social responsibilities, so that we would not have to deal with social things and could put all of our energy into mathematics. Fortunately, one of the things that helped save me was that I did not take his advice-I got married but not to such a wife. The suspected Unabomber talks about similar things which happened to him.

Both Ted Koscynski and I accepted tenure track professorial positions at major research mathematics departments (Berkeley and Cornell). And we were initially both successful with professional mathematics. Then, in the early 1970's, both Ted Koscynski and I quit mathematics. I got angry with mathematics – I got very furious about what mathematics had done to me. It is too complicated to go into all my feelings then (even if! could retrace them accurately) – but if someone came up to me at that time and called me a mathematician I felt strongly like punching them in the face! The evidence indicates that Ted Koscynski had a different more violent reaction but his writings express feelings very similar to the ones I had at that time. He and I both went into the forest and built a cabin and lived there alone and we isolated ourselves. But, there was a huge difference – I made a constructive positive breakthrough and Ted Koscynski didn't.

It was geometry, the many friends I made, and my family that brought about this breakthrough and in many ways saved my life. I got back into geometry. Before this, I had not been teaching geometry-I had been teaching geometric topology and such courses but all of my teaching up to then had been very formal. There was one geometry course at Cornell at the time-the one for prospective secondary school mathematics teachers. It was not considered to be a *real* mathematics course and I considered myself to be a *real* mathematician so I did not have any interest in teaching it. But at that time, when I thought I was quitting mathematics, I needed to teach a little in order to have enough money to survive in my cabin, so I took a leave without pay and then occasionally came back and taught for some money. (Fortunately I did not bum any bridges.) So I started teaching this geometry course for prospective teachers. In the first three years that I taught the course, while living in the forest, three mathematics educators familiar to most of you were in the class, David Pimm (Open University), Jere Confrey (Cornell), and Fran Rosamond (National University, San Diego). This geometry course was essentially all that I was teaching for a few years. A lot of what I am going to talk about are my experiences with that course and what happened since then. This course (and my new friends) pulled me out of the fire that Ted Koscynski never got out of.

## FORMAL DEDUCTIVE SYSTEMS

Global formal deductive systems can be very powerful and are important in certain areas (for example in the study of computer algorithms and in the study of questions in the foundations of mathematics). Local formal deductive systems can be important and powerful in many areas of mathematics (for example group theory.) But many people hold the belief that mathematics is only the study of formal systems. These beliefs are wide-spread especially, I find, among people who are not mathematicians or teachers of mathematics. Let me give some descriptions of formal mathematics. For example, in *FOCUS: The Newsletter of the Mathematical Association of America* a professor of computer science wrote:

> *... one of the most remarkable gifts human civilization has inherited from ancient Greece in the notion of mathematical proof [and] the basic scheme of Euclid's Elements ... This scheme was formalized around the turn of the century and, ever since ... mathematicians have rested assured that all their ingenious proofs could, in principle, be transformed into a dull string of symbols which could then be verified mechanically. One of the basic features of this paradigm is that proofs are fragile: a single, minute mistake (e.g., an incorrectly copied sign) invalidates the entire proof. (Babai 1992)*

This is the kind of view of mathematics that I learned when I was in school and the university.

Here's a more recent description that just appeared in the past year in the *American Mathematical Monthly* in an article (by another professor of computer science and member of my mathematical generation) about a new reform teaching technique and text for discrete mathematics which is based on a "computational" formal approach which uses uninterpreted formal manipulations which have been stripped of meaning:

> *... most students are troubled by the prospect of uninterpreted manipulation. They want to think about the meanings of mathematical statements. Having meanings for objects is a "safety net", which students feel, prevents them from performing nonsensical manipulations. Unfortunately, the use of the "meaning" safety net does not scale well to complicated problems. Skill in performing uninterpreted syntactic manipulation does. (Gries 1995)*

He literally means to get rid of the meaning. He takes literally the formalist view of mathematics that the meaning is not important. He goes further to say that the meaning actually gets in the way. I was at one of his talks when he was explaining his new teaching method and he gave a proof of some result in discrete mathematics and I tried to follow the meaning through from the hypothesis to the conclusion, because the hypothesis and conclusion did have meaning. I tried to follow that meaning through the proof in order to see the connections, but I failed to do so. I brought this up at the end and he said something close to: *"Yes!* That's precisely the idea! We have managed to get the meaning out of the way so that it doesn't confuse the students so they are now better able to do mathematics."

Now let me give another description of mathematics. This was written by Jean Dieudonné in an article which was written in response to an article by Rene Thom in which Thom was talking about intuition and how it was important to bring in and foster intuition in the schools.

> *I am convinced that, since 1700, 90 per cent of the new methods and concepts introduced in mathematics were imagined by four or five men in the eighteenth century, about thirty in the nineteenth, and certainly not more than a hundred since the beginning of our century. These creative scientists are distinguished by a vivid*

> *imagination coupled with a deep understanding of the material they study. This combination deserves to be called "intuition." ...*
>
> *In most cases (the transmission of knowledge) will be entrusted to professors who are adequately educated and prepared to understand the proofs. As most of them will not be gifted with the exceptional "intuition" of the creators, the only way they can arrive at a reasonably good understanding of mathematics and pass it on to their students will be through a careful presentation of their material, in which definitions, hypotheses, and arguments are precise enough to avoid any misunderstanding, and possible fallacies and pitfalls are pointed out whenever the need arises. (Dieudonné, 1973)*

Both of the first two quotes were from computer scientists who down-play the role of meaning and intuition in mathematics. Now, Dieudonné who certainly *is* a mathematician and a very good one, pointed out that intuition and imagination are very important but that there are only a few people (apparently, men) who have that intuition and that for the rest of us it is necessary for mathematics to be put down in a very precise formal way. Dieudonné has two claims to fame that are connected to this. One is he was the founder of the Bourbaki movement which was an attempt (which was never finished) to formalize all of mathematics. The other which is more significant for this gathering is that about the time of this article he was chair of the ICMI (International Commission on Mathematics Instruction) and chair of it at the time that the "New Math" was being spread around the world. He has always been involved in education.

Another example of descriptions of mathematics: *Mathematica*©, the computer program, when it frrst came out was advertised as a program that can do all of mathematics – remember those early ads? If you believe a strictly formal view of mathematics then that claim was believable and many people did believe it.

## CONFINING MATHEMATICS WITHIN FORMAL DEDUCTIVE SYSTEMS IS HARMFUL

Now I want to talk about how I see the view (which I take as starting roughly a hundred years ago) that mathematics is just formal systems has been harmful. I see this view as harmful because:

- **It encourages what I think are incorrect beliefs**. For example, those beliefs mentioned above that mathematics is only the study of formal systems. Of course, people can have disagreements as to what mathematics is, but I think that most of the people in this room do not believe that mathematics is just formal systems. And let me make it clear that I believe that formal systems do have a place in mathematics and that they are very useful and very powerful in many ways. Formal systems are very important in computer science because that is what a computer does – deal with formal systems. So it is not surprising that it was professors of computer science who made the statements that I have put here. Formal systems have certainly been very important in various parts of algebra and analysis and topology (which was my area of research) which flourished in this century. But geometry virtually disappeared as evidenced by the fact that there were almost no undergraduate geometry courses in 1970. That trend has now reversed. For example, now at Cornell there are eight undergraduate geometry courses and of those eight there is only one-half of one of them that deals with axiomatic systems. So things are changing.
- **Much interesting and useful geometry is either not taught at all or is presented in a way that is inaccessible to most students.** For example, spherical geometry was in the high school and university curriculum (or, at least in the textbooks) of 100 years ago. Of course, high schools in those days were more elite institutions than they are today, but spherical geometry is almost entirely absent from our courses and textbooks

now. Why is it that it disappeared? It is not because it is not useful: Spherical geometry is very applicable-navigation on the surface of the earth, the geometry of visual perception, the geometry of astronomical observations, surveying on a scale of several kilometers. Spherical geometry is a very useful geometry, but we do not teach it anymore-why? I think the reason is that spherical geometry is very difficult to formalize-there is no convenient axiom system for spherical geometry. There is an axiom system for spherical geometry (Borzuk did it just before the Second World War)-it is in a book that is in many mathematics libraries but it rarely has been used because it is not a useful axiom system. "Non-Euclidean" geometry has been often taught in under-graduate geometry courses, but it has always been "the" non-Euclidean geometry, hyperbolic geometry, which has a relatively simple axiom system and which has only been around for about 160 years. Spherical geometry which is very old (the Babylonians and Greeks studied it) is rarely taught. I cannot think of any reasonable explanation for why spherical geometry disappeared except that it does not fit into formalism. This is one of the reasons that I have it in my geometry course. When freed from the confines of formal systems it is possible to present spherical geometry in ways that are based on geometric experiences and intuitions. (See Henderson, 1996a)

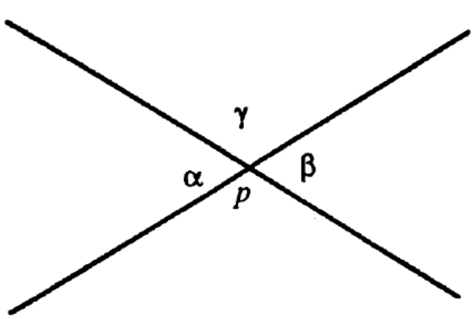
- **Important notions in mathematics are formally defined in ways that separate them from the students' experiences**. For example, the new Chicago Mathematics Curriculum for American secondary schools (which has many good things in it and is now the fastest growing curriculum in the USA) defines a rotation as the product of two reflections. Now that is an interesting fact (theorem) about rotations. But what does a student think when he or she comes to that as the definition of what a rotation is? It is very difficult to relate the product of two reflections with our experiences of rotations such as opening a door or riding a merry-go-round. One of the problems is that our intuition of rotations is a dynamic thing-it is actually a motion. Whereas to think of rotation as the product of two reflections is a static thing-it is the result of the rotation motion that is equal to the product of two reflections. If I were a student and saw this definition in the textbook I would say that this geometry is not relating to what I know geometry is and I would feel that the text is telling me that my experiences and intuitions are not important. It appears that the main reason for using this definition is that it is convenient formally in the deductive system in which the geometry in the text is confined. Also, differential geometry (the geometry of curves and surfaces, the geometry of the configuration spaces of mechanical systems, the geometry of our physical space/time) has extremely difficult formalisms which make it inaccessible to most students and even, I suspect, most mathematicians are uncomfortable with the formalisms of differential geometry. Some people have called it the most complicated formalism in all of mathematics. But, yet, differential geometry is basically about straight lines and parallelism-very intuitive notions. When we insist on formalizing differential geometry then it becomes inaccessible – even more so because there is no agreed upon formalism. My second book, (Henderson, 1996b), is an attempt make differential geometry accessible by basing it on geometric experiences and intuitions, as opposed to basing it on standard algebraic and analytic formalisms.
- **Many important and useful questions are not asked.** This was something that really surprised me when I started teaching this geometry and started listening to the prospective teachers who were taking the course. There are a lot of questions that students have that we never ask in mathematics classes. For instance, the reliance on a formal Euclidean deductive system rarely allows for questions such as "What do we mean when we say that something is straight?" We normally don't ask that in any classes, even though we talk about straight lines all the time. We just write down some axiom or we just say "everyone knows what 'straight' is." In differential geometry the

formalism has attempted to get at what the meaning of straight is, but in a way that is not accessible. But one can ask the question about what it means to be straight; you can ask that of students. I've done it with first graders-they can come up with good discussions. One of the results of this is that when spherical geometry or other geometries are talked about, usually they are just presented with some statement like: "We will define the straight lines to be the great circles on the sphere." But that is ridiculous, the great circles are the straight lines on the sphere, you do not have to define them. If you have a notion of what straightness is, then you can imagine a bug crawling around on the sphere and ask how would the bug go if the bug wanted to go straight. You can convince yourself that it is the great circles. But we cannot even ask those questions in a formal context. Also, the connections between linear algebra, geometric transformations, symmetries, and Euclidean geometry are very difficult to talk about in a formal system (in fact I don't know if I want to say impossible or not), but it is not conveniently done and often not done at all in a formal system. Remember the example above of defining a rotation to be the product of two reflections. (Other questions which we ignore include: "Why is Side-Angle-Side true on the plane (but not on the sphere)?", "What is the geometric meaning of tangency?", and "How do we experience the connections between?")

- **Mathematicians are being harmed.** I have already talked about how mathematicians are being harmed-I was harmed by the over emphasis on formalism – so was Ted Koscynski. And I'm sure that you know of examples (at your own university or around your own university) of mathematicians, roughly my generation, who have more or less dropped out of society. There are a lot of them around-people who have been good mathematicians, who had been successful in the system back in 1950's and 1960's. So it has been harmful to mathematicians.
- Students are being harmed. When a student's experiences lead her/him to understand a piece of mathematics in a way that is not contained in the formal system, then the student is likely to lose confidence in her/his own thinking and understanding even when it is backed up by what I will call alive geometric reasoning. Deductive systems do not encourage alive mathematical reasoning (which in my experiences with students and teachers is a natural human process) and thus they serve to deaden human beings whose thinking and understandings are forced to reside in these systems. We now have machines that can do the computations and formal manipulations of deductive systems: we need more alive human reasoning.

Here are some examples:

One of the things that I clearly remember from the beginning of my teaching of the geometry course is the following: I was teaching the Vertical Angle Theorem and its standard proof:




$\gamma + \alpha = 180$ degrees
$\gamma + \beta = 180$ degrees

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I can still remember one of the students who was very shy and wouldn't speak up in class, but I was having the students do writing. She wrote on her paper something like the following:

> *All you have to do is do a half-turn. Take this point here (P) and rotate everything about this point half of a full revolution. We have already discussed that straight lines have half-turn symmetry and so each line goes onto itself and α goes onto β.*

I don't know what your reaction is now but my reaction then was "That's not a proof" and I told her so. Fortunately, though she was shy, she was persistent and stubborn and she kept coming back and insisting that that was a proof. She worked on me for about two weeks and I kept listening to her and struggling with the question, Is that a proof?, because it did not seem like proofs that I had been accustomed to and that I would accept. Finally, she convinced me and now I think it is a *great* proof and much better than the standard proof which is in most of the textbooks. The standard proof has a lot of underlying assumptions that need to be cleared out and many formal treatments do that – they put in the "Protractor Postulates" which state the appropriate connections between angles and numbers and then you can do the standard proof. But the proof with the half-turn is just connected to a certain symmetry of straight lines. You can use other symmetries of straight lines to prove this result also, but this proof is the cleanest, the simplest. And this proof is not possible in a formal system and it is particularly not possible in a formal system if (because you want to insist on putting everything in a formal system) you define a rotation as the product of two reflections. That particularly won't work here because, if you take one of the lines and reflect through the line and then reflect perpendicular to the line that is equivalent to a half-turn, but there is no pair of reflections that will simultaneous do that to both of these lines, but yet a half-turn clearly preserves both lines. I do not see any reasonable way for that to have been included in any kind of formal system. So, if I had been insisting on formal systems, I would have missed out on the half-turn proof and not learned this bit of mathematics. I almost missed out anyhow and it was only because she was very persistent.

After that experience I started listening more to students and expecting that when they would say things that I didn't understand, that maybe they really did have something (and something that I could learn). I took the attitude that we are not working in a formal system, but that we are doing mathematics the same way that mathematicians mostly do mathematics. (In geometry, mathematicians do not stick inside any particular formal system, we use whatever tools might be appropriate: computers, linear algebra, analysis, symmetries. Mathematicians use symmetries a lot!) As I listen to students I have been learning more and more geometry from the students. I used to be surprised at that and thought it was just because I had not been teaching the course for very long. I thought that after I have taught it for a while then I will know it all and I will not see anything new. Well, what happened is that I have been teaching the course for 22 years now and now 30-40% of the students every semester show me some mathematics that I have never seen before! (These students are in different programs- some are mathematics majors, all the prospective secondary school teachers, and most of the mathematics education graduate students.) I would miss out on most of this new geometry if things were being done inside a formal system.

Another example: There are many properties of parallel lines in the plane (for example, any line which traverses two parallel lines will intersect those lines at the same angle) whose proofs depend on the parallel postulate. When we get to that point in the course I let the students come up with their *own* postulate – whatever it is that they think is most important to assume that will separate the plane from the sphere. There is a difference between the plane and the sphere and there is some difference that has to do with parallel lines. The students come up with all kinds of different postulates, many of which I think would be much more reasonable to assume than the usual parallel postulates. By the way, Euclid's parallel (fifth) postulate is true on the sphere - Euclid's parallel postulate is *not* what distinguishes spherical geometry from plane geometry, contrary to what many books say. I take that as evidence that people who have written such mistakes about spherical geometry have never really looked at a sphere – they have just been looking at the situation formally and thus made the mistake.

Let me give another example to show that I can think about something that is not just geometric. Here is the proof that is usually given to American secondary school students that 0.99 … = 1:

$$x = 0.99\ldots 9\ldots$$

$$10x = 9.9\ldots 9\ldots$$

now subtract both sides to get

$$9x = 9.0\ldots \text{ and thus } x = 1$$

This proof embodies a very useful technique for figuring out, when you have a repeating decimal, what fraction is equal to that repeating decimal. It is a very useful technique in that context. But I claim that it is not a proof in this context. I claim it is something that is masquerading as a formal proof: It looks like a formal proof, it has steps and *x's* and all that stuff. I started asking my calculus students at Cornell what they thought, and some of the best high school students in North America come to Cornell. They mostly know this proof, because they learned it; but only about half of them believe it, because they do not believe that 0.99 ⋯ 9 = 1. To show you why *I* think that this is masquerading as a proof and really isn't a proof, let us consider the following: Let us try to make this a little more precise as to just what it is we mean by 0.99⋯9⋯ (that is part of the problem here). Well, to most students what 0.99⋯9 means is, 0.9, then 0.99, then 0.999, ... – a limit of a sequence (at the time they are expressing this, they might not even know what a sequence is) – you keep putting on one more 9, you go on for ever-that is the way that they talk about it. It fits in nicely with calculus to do it that way and to think about 0.99⋯9⋯ as the limit of a sequence:

$$0.99\cdots 9\cdots \equiv \lim \{0.9, 0.99, 0.999, ...\}.$$

If you think of it as this limit and then follow the formal rules for subtracting sequences and multiplying sequences and so on, you come out with the amazing conclusion that:

$$x = 0.99\cdots 9\cdots \equiv \{0.9, 0.99, 0.999, ...\}$$

$$10x = 10 \times \lim \{0.9, 0.99, 0.999, ...\} = \lim \{9, 9.9, 9.99, ...\}$$

$$9x = \lim \{9, 9.9, 9.99, ...\} - \lim \{0.9, 0.99, 0.999, ...\} = \lim \{S.l, S.91, S.991, ...\}$$

$$x = (\lim \{8.l, 8.91, 8.991, ...\}) \div 9 = \lim \{0.9, 0.99, 0.999, \ldots\}$$

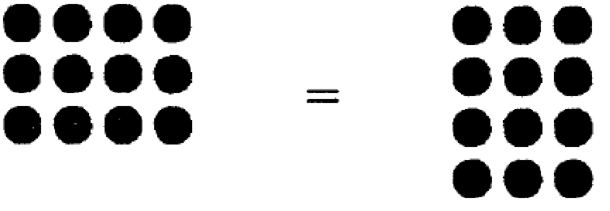
$$x = 0.99\cdots 9\cdots!$$

This is true – not very useful, but it is true. And it *has* to be that way, because there is an assumption being made here – the Archimedian Axiom. Way back, Archimedes knew that in talking about numbers it was possible to talk about ones which we now call infinitesimal, and then Archimedes had an axiom or principle which rules out these infinitesimals. The Archimedian Axiom (or Principle) gets stated in various different ways but is rarely mentioned these days in the North American undergraduate curricula-most textbooks (if they mention it at all) relegate it to a brief mention in a footnote or exercise. The usual approach these days is to subsume the Archimedian Axiom under Completeness in a hidden way so that you do not even notice that it is there. I think it is important for the students to know that this is an assumption. They can understand why it is convenient to assume that 0.99⋯9⋯ = 1 and understand that there are a lot of reasons for making that assumption. But we should tell them that it is an assumption – and it really is.

Another example – here is a theorem:

*For natural numbers.* $n \times m = m \times n$.

Now, the usual formal proof which I learned for this theorem is a complicated double mathematical induction. I dutifully learned this proof and was dutifully teaching it when I first started teaching. But here is the prop for another proof (I do not want to say 'another proof but only 'a *prop* for a proof):



Here we think of 3 x 4 as three 4's or four 3's (it seems that most mathematicians think of 3 x 4 as three 4's, but many of my students think of 3 x 4 as four 3's). I find a proof based on this schema as more convincing than the one with the double induction. And this proof can be visualized with having arbitrary numbers of dots, because the whole point is that you do not have to count the dots to know that this is true – there is symmetry. But it is hard to express in words and put down in a linear fashion on a piece of paper and all that kind of stuff.

- **Mathematics is being harmed.** Historically, most current-day mathematics was based on geometric explorations, geometric reasonings, and geometric understandings. The developers of our current deductive systems in algebra and analysis explicitly attempted to weed out all references and reliances on geometry and the geometric intuitions on which the algebra and analysis was originally based. When we confine mathematics to these formal systems we teach the students to distrust mathematics, not to value it, and not to use their intuitions in understanding mathematics. Many, many students who have a natural interest in mathematics are lost to mathematics by this process – I almost was.

## HOW SHOULD WE DESCRIBE WHAT IS MATHEMATICS?

David Hilbert is considered to be "the father of formalism" so I checked what he had to say. In 1932, late in his career he wrote in the Preface to *Geometry and the Imagination:*

> *In mathematics, as in any scientific research, we find two tendencies present. On the one hand, the tendency toward abstraction seeks to crystallize the logical relations inherent in the maze of material that is being studied, and to correlate the material in a systematic and orderly manner. On the other hand, the tendency toward intuitive understanding fosters a more immediate grasp of the objects one studies, a live rapport with them, so to speak, which stresses the concrete meaning of their relations.*
>
> *As to geometry, in particular, the abstract tendency has here led to the magnificent systematic theories of Algebraic Geometry, of Riemannian Geometry, and of Topology; these theories make extensive use of abstract reasoning and symbolic calculation in the sense of algebra. Notwithstanding this, it is still as true today as it ever was that intuitive understanding plays a major role in geometry. And such concrete intuition is of great value not only for the research worker, but also for anyone who wishes to study and appreciate the results of research in geometry. (Hilbert 1932)*

The last sentence in the first paragraph ("On the other hand ... ") is a very nice description of what a lot of us are trying to do and he goes on to say how important this is in mathematics. I even went back to his paper "On the Infinite" – he does not say that mathematics is formal systems or that all of mathematics should be formalized. He, in fact, says very explicitly there that mathematics is based on intuition and that intuition is an appropriate basis for what he calls "ordinary finite arithmetic." He wanted to introduce the formalization in order to take care of various paradoxes that were coming up in dealing with the infinite, because there seemed to be some problems with intuition around infinite things. He never claimed that mathematics was formal – that was his followers.

Here is a more recent view expressed by William Thurston, who is director of the Mathematical Sciences Research Institute at Berkeley and one of the most prominent American mathematicians. Thurston rejects the popular formal definition-theorem-proof model as an adequate description of mathematics and states that:

> *If what we are doing is constructing better ways of thinking, then psychological and social dimensions are essential to a good model for mathematical progress ...*
>
> *… The measure of our success is whether what we do enables people to understand and think more clearly and effectively about mathematics. (Thurston 1994)*

I will now give a description of mathematics that is what I think Hilbert and Thurston are talking about. I call it "alive mathematical reasoning" where I take the word "alive" from Hilbert's quote.

## WHAT IS ALIVE MATHEMATICAL REASONING?

Alive mathematical reasoning includes both abstraction and intuitive understanding as Hilbert says in the above quote.

**Alive mathematical reasoning is paying attention to meanings behind the formulas and words** – meanings based on intuition, imagination, and experiences of the world around us. It is not memorizing formulas, theorems, and proofs-this is again something that computers can do. We, as human beings, can do more. As Tenzin Gyatso, the fourteenth Dalai Lama has said:

> *"Do not just pay attention to the words;*
> *Instead pay attention to meanings behind the words.*
> *But, do not just pay attention to meanings behind the words;*
> *Instead pay attention to your deep experience of those meanings."*

**Alive mathematical reasoning includes "living proofs", that is, convincing communications that answers – Why?** It is not formal 2-column proofs-computers can now do formal proofs in geometry. If something does not communicate and convince and answer "why?" then I do not want to consider it a proof. What we need are *alive human proofs* which

- *communicate:* When we prove something to ourselves, we are not finished until we can communicate it to others. The nature of this communication depends on the community to which one is communicating and it is thus, in part, a social phenomenon.
- *convince:* A proof works when it convinces others. Proofs must convince not by coercion or trickery. The best proofs give the listener a way to experience the meanings involved. Of course some persons become convinced too easily, so we are more confident in the proof if it convinces someone who was originally a skeptic. Also, a proof that convinces me may not convince my students.
- *answer 'Why?':* The proof should explain, especially it should explain something that the listener wants to have explained. As an example, my shortest research paper [Henderson 1973] has a very concise simple proof that anyone who understands the

> terms involved can easily follow logically step-by-step. But, I have received more questions from other mathematicians about that paper than about any of my other research papers and most of the questions were of the kind: "Why is it true?" "Where did it come from?" "How did you see it?" "What does it mean?" They accepted the proof logically but were not satisfied-it was not alive for them.

One of my colleagues at Cornell was hired directly as a full professor based primarily on a series of papers that he had written even though at the time we knew that most of the theorems in the papers were wrong because of an error in the reasoning. We hired him because these papers contained a wealth of ideas and questions that had opened up a thriving area of mathematical research.

**Alive mathematical reasoning is knowing that mathematical definitions, assumptions, etc., vary with the context and with the point of view.** Alive reasoning does not contain definitions and assumptions that are fixed in a desire for consistency. It is an observable empirical fact that mathematicians and mathematics textbooks are not consistent with definitions and assumptions. We find this true even when the general context is the same. For example, I looked in the plane geometry textbooks in the Cornell library and found nine different definitions of the term "angle". Also, calculus textbooks do not agree on whether the function $y = f(x) = 1/x$ is continuous or not continuous; and analysis textbooks have many different axioms for the real numbers that have different intuitive connections and necessitate different proofs.

**Alive mathematical reasoning is using a variety of mathematical contexts**: 2- and 3-dimensional Euclidean geometry, geometry of surfaces (such as the sphere), transformation geometry, symmetries, graphs, analytic geometry, vector geometry, and so forth. It is not Euclidean geometry as a single formal system. When a mathematician is constructing a proof that needs a mathematical argument she/he is free to use whatever tools work best in the particular situation. Mathematicians do not limit themselves in this way. Also, those who use geometry in applications, do not feel restricted to a single formal system.

**Alive mathematical reasoning is combining** together all parts of mathematics: geometry, algebra, analysis, number systems, probability, calculus, and so forth.

**Alive mathematical reasoning is applying** mathematics to the world of experiences.

**Alive mathematical reasoning is using** physical models, drawings, images in the imagination.

**Alive mathematical reasoning is making conjectures**, searching for counterexamples, and developing connections.

**Alive mathematical reasoning is always asking "WHY?"**

## BUT WHAT ABOUT CONSISTENCY AND CERTAINTY?

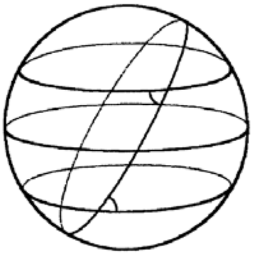
- Formal deductive systems do not gain consistency. For example, is the function $f(x) = 1/x$ continuous? Look in several calculus books. They give different answers! Differential geometry is another example where there is no consensus as to which formalism to use, but yet everyone thinks they are talking about the same ideas. Why?
- Formal deductive systems usually do not gain for us the certainty that we strive for. Formal deductive systems are useful and powerful in some circumstances, for example, in deciding which propositions can be logically deduced from other propositions and whether certain processes or algorithms will always produce the expected result. But, these deductive systems only give us certainty that certain steps (that can in principle be mechanized) can be carried out. They usually do not gain us

certainty for the human questions of "Why?" or the human desire for experiencing meanings.

## ALIVE MATHEMATICAL REASONING BRINGS BENEFITS TO MATHEMATICS

In my experiences, students with alive geometric reasoning are the most creative with mathematics. These are also the students who can step back from their individual courses and see the underlying ideas and strands that run between the different parts of mathematics. They are the ones who become the best mathematicians, teachers, and users of mathematics.

There is research evidence that successful learning takes place for many women and underrepresented students when instruction builds upon personal experiences and provides for a diversity of ideas and perspectives. See, for example Belenky et al. (1986), Cheek (1984), and Valverde (1984). Thus, alive mathematical reasoning in school classes may contribute to increasing the numbers of mathematicians who are women and persons from racial and cultural groups that are now underrepresented.

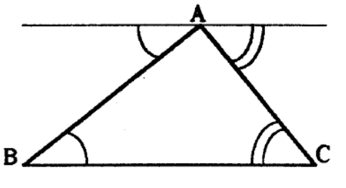


In my own teaching, I encourage students to use alive mathematical reasoning and observe how their thinking and creativity is freed and their participation is opened up. (See Lo et al.,1996.) As I listen to the alive mathematical reasonings of my students I find that 30-40% of the students show me mathematics that I have not seen before and that (percentage-wise) more of these students are women and persons of color than white men. (Henderson, 1996)

## CLOSING EXAMPLE

I will conclude with a proof that I learned from a student in a freshman course which is taught in the same style and using some of the same problems as the geometry course. The course was for "students who did not yet feel comfortable with mathematics" and who were social science and humanities majors. There, a student, Mariah Magargee, who was an English major, had been told all the way through high school that she was no good at mathematics and she believed it I want to share with you her proof that *the sum of the angles of a triangle on the sphere is more than 180 degrees.* We had previously, in class, been talking about the standard proof that on the plane the sum of the angles of a triangle *is* always 180 degrees:

> *Standard planar proof:* Given a plane triangle ABC, draw a line through A which is parallel to BC. The sides AB and AC are transversals of these parallel lines and therefore there are congruent angles as marked. We see now from the drawing that the sum of the angles is equal to 180 degrees.



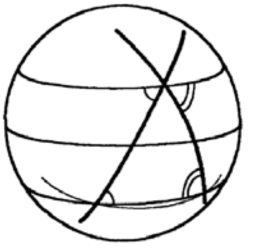

In class I stressed that the students should remember that latitudes circles (except for the equator) are not geodesics (straight on the sphere) and I urged them not to try to apply the notions of parallel to latitude circles. Mariah ignored my urgings and noted that two latitude circles which are symmetric about the equator of the sphere are parallel in two senses – first of all they are equidistant from each other and:

Note that: *Two latitude circles which are symmetric about the equator have the property that every (great circle) transversal has opposite interior angles congruent.*

> This follows because the two latitudes have half- tum symmetry about any point on the equator.

Now we can mimic the usual planar proof:

> We see that the sum of the angles of the "triangle" in the figure sum to a straight angle. This is not a true spherical triangle because the base is a segment of a latitude circle instead of a (geodesic) great circle. If we replace this latitude segment by a great circle segment then the base angles will increase. Clearly then the angles of the resulting spherical triangle sum to more than a straight angle.



> You can check that any small spherical triangle can be derived in this manner.

Nice proof! I like it. That is Mariah's proof. This is a student who believed that she was no good at mathematics and was told she was no good at mathematics, but she taught me a really nice proof.

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## QUESTIONS AND ANSWERS

**Question:** *How might we encourage our students to experience the passion that they have for mathematics-that passion can include joy, fear, excitement, however you want to interpret the word.*

**Answer:** I only know ways that I have tried and I do not think there is only one way. It seems to me that the most important thing is to expect it to happen and I do that and try to convey that I expect to see that from the students. The other thing that I started noticing when I started to have the students write is that a lot of the students' ideas and their passions are very fragile. There are a lot of students who do not dare to speak it in class, but they will write something. Maybe what they write isn't even directly what they really want to say but they will hint at it. So I have them write and then I respond to their writing and then they respond to my comments and so there is a dialogue that goes on in the writing. That I found to be the most powerful because then if someone just starts having an idea about something that is very tentative and very fragile, I can encourage it. I can encourage it easier in the written dialogue than it can happen in a class situation. That together with just expecting it and encouraging it whenever it happens and validating it is what I find that works for me.

**Question:** *What is formalism good for?*

**Answer:** One of the areas for which formalism is clearly good for is in computer science, in studying the algorithms and proofs in computer science. A huge area in computer science now is how to prove that a program does what you want it to do-it's a formal proof because that is what computers do, they are formal systems. The other place in which formalism is very powerful is in any situation like with groups. Studying groups is a good place to have axioms and build it up formally because there are a lot of different models for what a group is. So you can prove certain results that work for anything that satisfies these particular axioms and there are some examples and you can apply it across all the examples. There are a lot of areas like that in mathematics where that can happen. I think that Euclidean geometry is a particular bad place to apply formalism, because there is essentially only one model of Euclidean geometry and it is not a question of building these things up and then you can apply it somewhere else. Those areas where there actually are different models for a particular axiom system are areas where formal systems are powerful tools. I would also say that sometimes it is useful to use formal systems in areas where you actually have several different axiom systems-which we do not usually allow in courses as, we usually stick with only one. This happens in differential geometry. There are very complicated formalisms for differential geometry but there are a lot of different ones and to play the different ones off each other can be powerful. If you try to stick within one then you lose the geometric meaning, but if you go across them then the only thing which ties them together is the geometric meaning; and that is one way to get at what the geometric meaning is.

**Question:** *How is the way that you teach geometry dependent on whether you have preservice teachers or have mathematics majors?*

**Answer:** The main course that my book is based on has mathematics majors and preservice teachers (who are also mathematics majors) and mathematics education graduate students and then there are miscellaneous people (teachers, artists, or other members of the community who are interested in geometry) – I do the same thing with all of them. Because most of the feedback is based on the writing that they do, they can respond in different ways, so I can have a different dialogue going on for different students depending on where they are coming from. I have not been able to do this as well in other subjects, but geometry is particularly suited to this because most people do not have much background in geometry, and that evens them out. Also geometry is more accessible concretely and through the intuition. I should tell you about one workshop that I did that was very powerful, one of the most powerful workshops that I have lead. It was in South Africa and they had gotten together a group of about 50 people which included elementary school teachers (many of whom had not finished secondary school, so had very weak mathematics backgrounds and virtually nothing in geometry), secondary school teachers, mathematics education people, and research mathematicians (including the chair of the mathematics department) – the whole span. I had them work on the same problems in small homogeneous groups, the elementary school teachers worked with each other and the research mathematicians were working with each other. Of course, what they were doing in their small groups was very different, but I then had them report back to the whole group what they had found. Then the research mathematicians had to express it in a way that made sense for the elementary school teachers and the elementary school teachers were able to express what they had found and see that they had found things that the research mathematicians hadn't seen. It was very powerful – I try to have as much diversity as possible in my class, but I have never had that kind of diversity before or since.

**Question:** *Can or should geometry be taught separately from algebra?*

**Answer:** When I teach geometry, I encourage students to use whatever they find useful to use and, if that is algebra, then-great. Is that the kind of thing you were meaning by the question?

**Question:** *Should geometry be taught before algebra?*

**Answer:** My own feeling is that geometry comes before algebra-much of algebra developed out of geometry, historically, and that should not be lost. Mostly I would like to turn the question around: Should algebra be taught separate from geometry? And there my answer would definitely be NO. Whenever I teach linear algebra, geometry is there a lot. And that is true historically-much of linear algebra was developed to help with the description and study of the geometry of higher dimensional spaces.

**Question:** *Could you elaborate on your comments that formalist mathematics is harmful and destructive? Is this perhaps a function of generation, because Leslie Lee seemed to be really able to identify with your comments because she too wanted to build a cabin?*

**Answer:** Several people here came up to me afterwards and said they had had similar experiences, not just Leslie, and they were all about my age. Our generation was the generation before the baby boomers – I think the baby boom generation (the ones that went into mathematics and the ones that didn't) was just different and maybe that is part of the reason why my generation is affected. Also I think that mathematics was most formal when we were in school. There seems to be something about our generation; and we are now in charge so we are more visible and that puts more pressure on us. Also, this formalism is only a product of this century, so for a long while the leaders in mathematics who were doing the formalism also knew that it was not all of mathematics-like the quote I have from Hilbert and that was in the 1930's. But then somehow after the war in the late 1940's and 1950's, most of the mathematicians who had had direct personal contact with what was going before formalism came in had died off and so that may have affected our generation.

**Question:** *How has the harmfulness or destructive nature of formalist mathematics manifested itself beyond the urges to live in the woods?*

**Answer:** Well, I do not think that the urge to live in the woods is harmful! I think that it was mainly that when I was going through school I never grew up socially and I was not effectively encouraged to grow up socially. I was a 'brain' (or 'science whiz') and that particularly substituted for growing up socially. Being immersed in formalism sort of fit that and encouraged that. That's part of it and I think that with the baby boom generation there were lots of external forces that started drawing people out, that was just not around in my days. And people who were more alive and more socially active than I was, tended not to go into mathematics, or if they were in mathematics then they dropped out.

**Question:** *Could you picture a world where mathematics majors and graduate students could be extroverted and emotional and deal with people and still do formalism? Do think there is something inherent in formalism?*

**Answer:** I do not know for sure. I look at the graduate students now at Cornell and there are some of them who are like I was but there are also significant numbers of them now who are not like I was and who are alive in lots of ways. They are surviving at Cornell and Cornell is basically still a very formal place, but they are also interested in teaching and we have mathematics graduate students who are taking the initiative to do some educational reform. So, yes, I think it makes sense that it fits in with formalism. I should say more about formalism, there is the part of mathematics, the foundations of mathematics, which is specifically studying formal systems and now in lots of places, in particular at Cornell, there is almost no distinction between it and a part of theoretical computer science. That is an active area of research where there is a lot of exciting things going on. I am not talking about that, that is a part of mathematics. I don't know ... Let me tell you one observation that I had that is less true now but it used to be true 10 years ago or so. Almost all of the women graduate students did not go into geometry or geometry related areas, but instead went into very formal areas. I talked with some of these women trying to catch why this was true, because in most of these cases these women were very active outside of mathematics – they were involved in various social movements, political movements, feminist activities, and other such things going on 10-20 years ago. They expressed to me that they had to separate their lives – when they were doing mathematics they had to separate it off from the rest of their life and it was easier to do that when they were doing formal mathematics. That seems to fit in with what I am saying. I see that happening less now, for both the men and the women.

**Question:** *We are interested in the resurgence of geometry at Cornell. Can you give us a sense of the constellation of the geometry courses that are available at Cornell and the population that they are for?*

**Answer:** As far as I can tell, Cornell has more geometry courses than anywhere else in the world. The undergraduate geometry courses are:

1. Euclidean and spherical geometry (the course that the book is based on) that is required for prospective teachers but it is taken by lots of other majors also.
2. Hyperbolic and projective geometry.
3. Geometry and groups-tessellations, transformation groups, etc. Cornell has a strong geometric group theory research group and the course grew naturally out of that research group.
4. Differential geometry.
5. Geometric topology.
6. "From space to geometry" – A freshman course which is based on writing assignments.
7. "Mathematical explorations"-for first and second year students who are humanities and social science majors. I use a lot of the same problems that are in my book-Mariah's proof about the sum of the angles of a triangle on the sphere was from that course.
8. Applicable geometry-sometimes computational geometry, sometimes the geometry of operations research (such as convex polytopes), and other applied topics that vary from year to year.

**Question:** *Isn't the issue more one of HOW you run the course rather than the subject matter of geometry?*

**Answer:** First of all, only one of those eight geometry courses deals extensively with axioms and formalism and that is the hyperbolic and projective geometry course and it does not deal with axiom systems totally. But I agree with you that the important thing is, How? I think that as long as you can start with something that is a concrete contextual situation and use that to start building the area of mathematics, then it can be done with any subject; and I think all parts of mathematics have such grounding. Geometry is easier to get into because there is not a tradition of having a long string of prerequisites, this linear sequence of courses and so on, in geometry, so it is easier to jump in different places. But I have done it in an abstract algebra course where, because it was me doing it, I started with symmetries of polyhedra and ended up with Galois Theory. The main thing that I try to do is to have a concrete contextual situation where the students are able to experience the meaning of what is going on and so, in that way, it can be more constructive. The other part of it (what I mentioned in my answer to the first question) is eliciting from the students their ideas and their thinking, so that if there is some way that their imagination and intuition can latch on, then I just start pulling out of them what the ideas are and guiding them in the right ways and giving suggestions and writing challenging problems.

**Question:** *Are the other geometry courses at Cornell also not so prerequisite bound?*

**Answer:** None of the geometry courses have any of the other geometry courses as prerequisites.

**Question:** *It is perhaps true that the phenomena is not so much a growing interest in geometry but rather a growing interest in something mathematical that can engage the students in less formal ways?*

**Answer:** I think you may well be right and that geometry is just a particularly convenient or easier area to do that in. There is a debate going on in our department as to whether or not students who want to take my courses are wanting to take it because of me or is it something about the course. It is hard to gauge that but it seems to be what you say, that they are really looking for a different way in which to engage with mathematics – something that is less formal, that is not just lecture and exams.

**Question:** *Have you had any resistance to your approach from colleagues, other mathematicians, or from students?*

**Answer:** Yes, all of the above. The geometry courses, fortunately, are not required for mathematics majors except for prospective teachers, so students who do not want to do the course just do not take the course and I have had no complaints from the prospective teachers. But, I and some graduate students are trying to put some of these ideas into the calculus now. We have had a few cases of students getting up and stomping out of the room when we introduce small group work and other activities to engage the students. It seems that this happens because they are not there to learn calculus; they are there because they are required to take calculus and they want to do it with a minimum amount of effort to get their passing grade so they can go on to do whatever it is that it is required for. The students who want to learn calculus, they seem to love the new approaches. Sometimes I am teaching one section and there are other sections of the calculus course taught in traditional ways. I am required by the department for my students to have the same assignments and exams, so I am giving them stuff in addition to that. I tell them this right up front and I tell them why – I think they will learn it better and with more understanding, but they will have every week more assignments than the other sections. Typically, what happens when I announce that in the beginning, a few students drop out and other students hear about it and come in. At the end of the semester the students report that the extra stuff is the best part of the course.

# TOWARD A PRACTICE-BASED THEORY OF MATHEMATICAL KNOWLEDGE FOR TEACHING[4]

Deborah Loewenberg Ball and Hyman Bass[5]
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This plenary was chosen because of the unique combination of the two co-speakers that reflects CMESG's goal of creating a community of mathematics educators and mathematicians interested in mathematics education. The research collaboration of Debra Ball, a prominent mathematics education researcher, and Hyman Bass, a distinguished research mathematician who in his mid-sixties had started to put his creative energies into mathematics education, reminds us of the importance for the two groups they represent to work together. Their work opened up a significant chapter in the field in terms of the special kinds of mathematical and pedagogical knowledge needed by a mathematics teacher. This paper allows us to look back at the earlier stages of their work that eventually made this significant impact to the field of mathematics teacher education.

Cette conférence plénière a été choisie en raison de la combinaison des deux conférenciers, une combinaison qui reflète l'objectif du GCEDM de créer une communauté de didacticiens et de mathématiciens intéressés par l'éducation mathématique. La collaboration entre Deborah Ball et Hyman Bass, une chercheure importante en didactique des mathématiques et un chercheur de marque en mathématiques qui, dans la mi-soixantaine, a commencé à s'investir dans l'éducation mathématique, nous rappelle l'importance, pour les deux groupes qu'ils représentent, de travailler ensemble. Leur travail collaboratif a ouvert la voie à des travaux sur les types de connaissances mathématiques et pédagogiques nécessaires pour l'enseignement. Ce texte nous ramène aux premiers stades de leur programme de recherche, qui, éventuellement, a eu l'impact que l'on connait aujourd'hui sur le champ de la formation à l'enseignement des mathématiques.

> Mathematics Professor: *The situation is terrible: Only one of the students in my mathematics content course for teachers can correctly divide .0045789 by 3.45.*
>
> Fifth Grader: *Ms. Ball, I can't remember how to divide decimals. There's something my stepfather showed me about getting rid of the decimal point, but I can't remember what he said and, besides, I don't think that would work.*

With all the talk of teachers' weak mathematical knowledge, we begin with a reminder that the problem on the table is the quality of mathematics teaching and learning, not—in itself— the quality of teachers' knowledge. We seek in the end to improve *students'* learning of mathematics, not just produce *teachers* who know more mathematics.

Why, then, talk about teacher knowledge here? We focus on teacher knowledge based on the working assumption that how well teachers know their subjects affects how well they can teach. In other words, the goal of improving students' learning depends on improving teachers' knowledge. This premise—widely shared as it may be, however—is not well supported

[4] This work has been supported by grants from the National Science Foundation (REC # 0126237) and the Spencer Foundation (MG #199800202).
5 The authors acknowledge Heather Hill for her contributions to the ideas discussed in this paper.

empirically. We begin with a brief glimpse of the territory in which the problem on which we are working fits. Our purpose is to set the context for our proposal for reframing the problem.

## THE PROBLEM: WHAT MATHEMATICS DO TEACHERS NEED TO KNOW TO TEACH EFFECTIVELY?

The earliest attempts to investigate the relationship between teachers' mathematics knowledge and their students' achievement met with results that surprised many people. Perhaps the best known among these is Begle's (1979) analysis of the relationship between the number of courses teachers had taken past calculus and student performance. He found that taking advanced mathematics courses[3] produced positive main effects on students' achievement in only 10% of the cases, and, perhaps more unsettling, negative main effects in 8%. That taking courses could be negatively associated with teacher effects is interesting because the negative main effects are not easily explained by the criticism that advanced mathematics courses[6] are not relevant to teaching, or that course-taking is a poor proxy for teachers' actual mathematical knowledge. Such claims support finding *no* effects, but not *negative* effects.

So why might these variables be associated with negative effects? One explanation might rest with the compression of knowledge that accompanies increasingly advanced mathematical work, a compression that may interfere with the unpacking of content that teachers need to do (Ball & Bass, 2000a). Another explanation might be that more coursework in mathematics is accompanied by more experience with conventional approaches to teaching mathematics. Such experience may impress teachers with pedagogical images and habits that do not contribute to their effectiveness with young students (Ball, 1988).

Observational studies of beginning and experienced teachers reveal that teachers' understanding of and agility with the mathematical content does affect the quality of their teaching. For example, Eisenhart, Borko, Underhill, Brown, Jones, and Agard (1993) describe the case of a middle school student teacher, Ms. Daniels, who was asked by a child to explain why the invert-and-multiply algorithm for dividing fractions works. Ms. Daniels tried to create a word problem for three-quarters divided by one-half by saying that three quarters of a wall was unpainted. However, there was only enough paint to cover half of the unpainted area. As she drew a rectangle to represent the wall and began to illustrate the problem, she realized that something was not right. She aborted the problem and her explanation in favor of telling the children to "just use our rule for right now" (p. 198).

Despite having taken two years of calculus, a course in proof, a course in modern algebra, and four computer science courses, Ms. Daniels was unable to provide a correct representation for division of fractions or to explain why the invert-and-multiply algorithm works. In fact, she represented multiplication, rather than division, of fractions.

Many other studies reveal the difficulties teachers face when they are uncertain or unfamiliar with the content. In 1996, the National Commission on Teaching and America's Future (NCTAF) released its report which proposed a series of strong recommendations for improving the nation's schools that consisted of "a blueprint for recruiting, preparing, and supporting excellent teachers in all of America's schools" (p. vi). Asserting that what teachers know and can do is the most important influence on what students learn, the report argues that teachers' knowledge affects students' opportunities to learn and learning. Teachers must know the content "thoroughly" in order to be able to present it clearly, to make the ideas accessible to a wide variety of students, and to engage students in challenging work.

6 An "advanced course" was defined as a course past the calculus sequence.

The report's authors cite studies that show that teacher knowledge makes a substantial contribution to student achievement. They argue that "differences in teacher qualifications accounted for more than 90% of the variation in student achievement in reading and mathematics" (Armour-Thomas, Clay, et al., 1989, cited in National Commission on Teaching and America's Future, 1996, p. 8). Still, what constitutes necessary knowledge for teaching remains elusive.

An important contribution to the question of what it means to know content for teaching has been the concept of "pedagogical content knowledge" (Grossman, 1990; Shulman, 1986, 1987; Wilson, Shulman, & Richert, 1987). Pedagogical content knowledge, as Shulman and his colleagues conceived it, identifies the special kind of teacher knowledge that links content and pedagogy. In addition to general pedagogical knowledge and knowledge of the content, teachers need to know things like what topics children find interesting or difficult and the representations most useful for teaching a specific content idea. Pedagogical content knowledge is a unique kind of knowledge that intertwines content with aspects of teaching and learning.

The introduction of the notion of pedagogical content knowledge has brought to the fore questions about the content and nature of teachers' subject matter understanding in ways that the previous focus on teachers' course-taking did not. It suggests that even expert personal knowledge of mathematics often may be inadequate for teaching. Knowing mathematics for teaching requires a transcendence of the tacit understanding that characterizes much personal knowledge (Polanyi, 1958). It also requires a unique understanding that intertwines aspects of teaching and learning with content.

In 1999, Liping Ma's book, *Knowing and Teaching Elementary Mathematics* attracted still broader interest in this issue. In her study, Ma compared Chinese and U.S. elementary teachers' mathematical knowledge. Producing a portrait of dramatic differences between the two groups, Ma used her data to develop a notion of "profound understanding of fundamental mathematics", an argument for a kind of connected, curricularly-structured, and longitudinally coherent knowledge of core mathematical ideas.

What is revealed by the work in the years since Begle's (1979) famous analysis? Although his work failed to show expected connections between teachers' level of mathematics and their students' learning, it seems clearer now that mathematical knowledge for teaching has features that are rooted in the mathematical demands of teaching itself. These are not easily detected by how much mathematics someone has studied. We are poised to make new gains on an old and continuing question: What do teachers need to know to teach mathematics well? But we are poised to make those gains by approaching the question in new ways.

## REFRAMING THE PROBLEM: WHAT MATHEMATICAL WORK DO TEACHERS HAVE TO *DO* TO TEACH EFFECTIVELY?

The substantial efforts to trace the effects of teacher knowledge on student learning, and the problem of what constitutes important knowledge for teaching, led our research group[7] to the idea of working bottom up, beginning with practice. We were struck with the fact that the nature of the knowledge required for teaching is underspecified. On one hand, what teachers need to know seems obvious: They need to know *mathematics*. Who can imagine teachers being able to explain how to find equivalent fractions, answer student questions about primes or factors, or represent place value, without understanding the mathematical content? On the other hand, less obvious is what "understanding mathematical content" *for teaching* entails: *How* do

[7] Members of the Mathematics Teaching and Learning to Teach Project include Mark Hoover, Jennifer Lewis, Ed Wall, Rhonda Cohen, Laurie Sleep, and Andreas Stylianides.

teachers need to know such mathematics? What else do teachers need to know of and about mathematics? And how and where might teachers use such mathematical knowledge in practice?

Hence, instead of investigating what teachers need to know by looking at what they need to teach, or by examining the curricula they use, we decided to focus on their work. What do teachers *do*, and how does what they do demand mathematical reasoning, insight, understanding, and skill? We began to try to unearth the ways in which mathematics is entailed by its regular day-to-day, moment-to-moment demands. These analyses help to support the development of a *practice-based theory of mathematical knowledge for teaching*. We see this approach as a kind of "job analysis", similar to analyses done of other mathematically intensive occupations, from nursing to engineering and physics (Hoyles, Noss, & Pozzi, 2001; Noss, Healy, & Hoyles, 1997), to carpentry and waiting tables. In this case, we ask:

- *What* mathematical knowledge is entailed by the work of teaching mathematics?
- *Where* and *how* is mathematical knowledge used in teaching mathematics? How is mathematical knowledge intertwined with other knowledge and sensibilities in the course of that work?

## HOW WE DO OUR WORK

Central to our work is a large longitudinal NSF-funded database, documenting an entire year of the mathematics teaching in a third grade public school classroom during 1989–90[8]. The records collected across that year include videotapes and audiotapes of the classroom lessons, transcripts, copies of students' written class work, homework, and quizzes, as well as the teacher's plans, notes, and reflections. By analyzing these detailed records of practice, we seek to develop a theory of mathematical knowledge as it is entailed by and used in teaching. We look not only at specific episodes but also consider instruction over time, examining the work of developing both mathematics and students across the school year. What sort of larger picture of a mathematical topic and its associated practices is needed for teaching over time? How do students' ideas and practices develop and what does this imply about the mathematical work of teachers?

A database of the scale and completeness of this archive affords a kind of surrogate for the replicable experiment. More precisely, the close study of small segments of the data supports the making of provisional hypotheses (about teacher actions, about student thinking, about the pedagogical dynamics), and even theoretical constructs. These hypotheses or constructs can then be "tested" and, in principle, refuted, using other data with this archive itself. We can inspect what happened days (or weeks) later, or earlier, or look at a student's notebook, or at the teacher's journal for evidence that confirms or challenges an idea. Further, when theoretical ideas emerge from observations of patterns across the data, we can use them as a lens for viewing other records, of other teachers' practices, and either reinforce or modify or reject our theoretical ideas in line with their adaptability to the new data.

Structured data like those collected in this archive can constitute a kind of public "text" for the study of teaching and learning by a community of researchers. This would permit the discussion of theoretical ideas to be grounded in a publicly shared body of data, inherently connected to actual practice. As norms for such discourse are developed, so also would the expansion of such data sets to support such scholarly communication be encouraged. In our experience,

[8] These data were collected under a 1989 National Science Foundation grant to Ball and Magdalene Lampert, then at Michigan State University.

disciplined inquiry focused on such a practice-based "text" tends to dissipate ideologically based disputes, and to assure that theoretical constructs remain connected to practice.

Even with such records of practice in which much is available to be seen, casual observation will no more produce insight about teaching and learning than unsophisticated reading of a good mathematics text will produce mathematical insight. Teaching and learning are complex and dynamic phenomena in which, even with the best of records, much remains hidden and needing interpretation and analysis. Our approach to this work has been to mobilize an interdisciplinary group representing expertise in teaching practice, in disciplinary mathematics, in cognitive and social psychology, and in educational research. Over time we have collectively crafted well-honed skills for sensitive observation of records (particularly video) of teaching practice. One of our research aims is to articulate some of the demands, skills, and norms that this entails; in short, a kind of *methodology of interdisciplinary observation of teaching*.

Our work uses methods of mathematical and pedagogical analysis developed in previous research (see, for example, Ball & Bass, 2000a, b; 2003). Using a framework for examining practice, we focus on mathematics as it emerges within the core task domains of teachers' work. Examples of this work include representing and making mathematical ideas available to students; attending to, interpreting, and handling students' oral and written productions; giving and evaluating mathematical explanations and justifications; and establishing and managing the discourse and collectivity of the class for mathematics learning. As we analyze particular segments of teaching, we seek to identify the mathematical resources used and needed by the teacher. For example, when a student offers an unfamiliar solution, we will look for signs of whether and how the teacher understands the solution, and what he or she did, and what the mathematical moves and decisions are. Our coding scheme includes both mathematical content (topics, procedures, and the like) and practices (mathematical processes and skills, such as investigating equivalence, reconciling discrepancies, verifying solutions, proving claims). The goal of the analysis is twofold: First, to examine how and where mathematical issues arise in teaching, and how that impacts the course of the students' and teacher's work together; and second, to understand in more detail, and in new ways, what elements of mathematical content and practice are used—or might be used—and in what ways in teaching.

## WHAT MATHEMATICAL PROBLEMS DO TEACHERS HAVE TO SOLVE?

This approach has led us to a new perspective on the work of mathematics teaching. We see many things teachers do when teaching mathematics that teachers of any subject must do—keep the classroom orderly, keep track of students' progress, communicate with parents, and build relationships with students. Teachers select and modify instructional tasks, make up quizzes, manage discussions, interpret and use curriculum materials, pose questions, evaluate student answers, and decide what to take up and what to leave. At first, these may sound like generic pedagogical tasks. Closer examination, however, reveals that doing them requires substantial mathematical knowledge and reasoning. In some cases, the work requires teachers to think carefully about a particular mathematical idea together with something about learners or learning. In other cases, the work involves teachers in a kind of mathematical reasoning, unencumbered by considerations of students, but applied in a pedagogical context. Our analyses have helped us to see that teaching is a form of mathematical work. Teaching involves a steady stream of mathematical problems that teachers must solve.

Let us consider an example. Teachers often encounter students using methods and solutions different from the ones with which they are familiar. This can arise for a variety of reasons, but when teachers see methods they have not seen before, they must be able to ask and answer—for themselves—a crucial *mathematical* question: What, if any, is the method, and will it work for all cases? No pedagogical decision can be made prior to asking and answering this question. Consider, for example, three alternative methods for multiplying 35 x 25:

(A)
$$\begin{array}{r} 35 \\ \times\ 25 \\ \hline 125 \\ +\ 75 \\ \hline 875 \end{array}$$

(B)
$$\begin{array}{r} 35 \\ \times\ 25 \\ \hline 175 \\ +\ 700 \\ \hline 875 \end{array}$$

(C)
$$\begin{array}{r} 35 \\ \times\ 25 \\ \hline 25 \\ 150 \\ 100 \\ +\ 600 \\ \hline 875 \end{array}$$

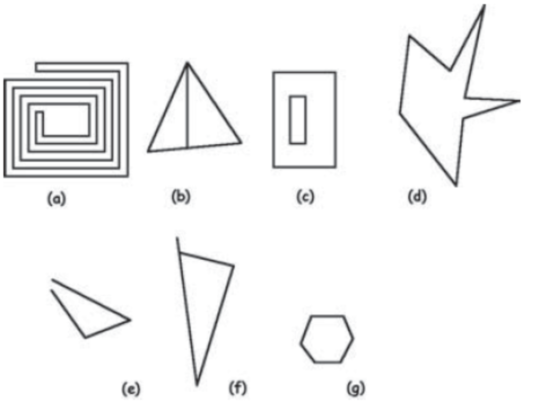
A teacher must be able to ask what is going on in each of these approaches, and to know which of these is a method that works for multiplying any two whole numbers. These are quintessential *mathematical* – not *pedagogical* – questions. Knowing to ask and how to answer such mathematical questions is essential to being able to make wise judgments in teaching. For instance, a decision about whether or not to examine such alternative methods with the students depends on first sizing up the mathematical issues involved in the particular approach, and whether they afford possibilities for worthwhile mathematics learning for these students at this point in time.

Being able to sort out the three examples above requires more of teachers than simply being able to multiply 35 x 25 themselves. Suppose, for example, a teacher knew the method used in (B). If a student produced this solution, the teacher would have little difficulty recognizing it, and could feel confident that the student was using a reliable and generalizable method. This knowledge would not, however, help that same teacher uncover what is going on in (A) or (C).

Take solution (A) for instance. Where do the numbers 125 and 75 come from? And how does 125 + 75 = 875? Sorting this out requires insight into place value (that 75 represents 750, for example) and commutativity (that 25 x 35 is equivalent to 35 x 25), just as solution (C) makes use of distributivity (that 35 x 25 = (30 x 20) + (5 x 20) + (30 x 5) + (5 x 5). Even once the solution methods are clarified, establishing whether or not each of these generalizes still requires justification.

Significant to this example is that a teacher's own ability to solve a mathematical problem of multiplication (35 x 25) is not sufficient to solve the mathematical problem of teaching – to inspect alternative methods, examine their mathematical structure and principles, and to judge whether or not they can be generalized.

Let us consider a second example. This example again helps to make visible the mathematical demands of simple, everyday tasks of teaching. Different from the first, however, it reveals that the mathematical demands are not always so closely aligned to the content outlines of the curriculum (in the example above, multiplication). Suppose that, in studying polygons, students produce or encounter some unusual figures and ask whether any of them is a polygon.




This is a natural mathematical question. Knowing how to answer it involves mathematical knowledge, skill, and appreciation. An essential mathematical move at this point is to consider the definition: What makes a figure a polygon? A teacher should know to consult the textbook's definition, but may well find an inadequate definition, such as this one, found in a current textbook:

> *A closed flat two-dimensional shape whose sides are formed by line segments.*

Knowing that it is inadequate requires appreciating what a mathematical definition needs to do. This one, for example, does not rule out (b) or (c) or (f), none of which is a polygon. But if the textbook definition is unusable, then teachers must know more than a formally correct mathematical definition, such as:

> *A simple closed plane curve formed by straight line segments.*

Teaching involves selecting definitions that are mathematically appropriate and also usable by students at a particular level. For example, fifth graders studying polygons would not know definitions for "simple" or "curve", and therefore would not be able to use this definition to sort out the aberrant figures from those we would call polygons.

To determine a mathematically appropriate and usable definition for "polygon", a teacher might try to develop a suitable definition, better than those found in the available textbooks. Consider this effort:

> *A sequence of three or more line segments in the plane, each one ending where the next one begins, and the last one ending where the first one begins. Except for these endpoints, shared only by two neighboring segments, the line segments have no other points in common.*

This definition, unlike the previous one in the textbook, is mathematically acceptable, as it does properly eliminate (b), (c), and (f), as well as (e). But a teacher would still need to consider whether or not her students can use it. Definitions must be based on elements that are themselves already defined and understood. Do these students already have defined knowledge of terms such as "line segments", "endpoints", and "plane", and do they know what "neighboring" and "in common" mean? In place of "neighboring", would either "adjacent" or "consecutive" be preferable? Knowing definitions for teaching, therefore, requires being able to understand and work with them sensibly, treating them in a way that is consistent with the centrality of definitions in doing and knowing mathematics. Knowing how definitions function, and what they are supposed to do, together with also knowing a well-accepted definition in the discipline, would equip a teacher for the task of developing a definition that has mathematical integrity and is also comprehensible to students. A definition of a mathematical object is useless, no matter how mathematically refined or elegant, if it includes terms that are beyond the prospective user's knowledge.

Teaching requires, then, a special sort of sensitivity to the need for precision in mathematics. Precision requires that language and ideas be meticulously specified so that mathematical problem solving is not unnecessarily impeded by ambiguities of meaning and interpretation. But the need for precision is relative to context and use. For example, a rigorous and precise definition for odd numbers as those numbers of the form $(2k + 1)$, or of even numbers as multiples of two, would not be precise for first graders first encountering the notion of "even number". Because they cannot decode the meaning of $(2k + 1)$ and do not have a definition of "multiple", the elements used to create a precise definition remain obscure and unusable to six-year-olds. Needed for teaching are definitions that are both correct and useful. Knowing what definitions are supposed to do, and how to make or select definitions that are appropriately and usefully precise for students at a certain point, demands a flexible and serious understanding of mathematical language and what it means for something to be *precise*. Taken together, these two examples show that knowing mathematics in and for teaching includes both elements of mathematics as found *in* the student curriculum – that is, standard computational algorithms, multiplication, and polygons – as well as aspects of knowing and doing mathematics that are less visible in the textbook's table of contents – sensitivity to definitions or inspecting the generality of a method, for example. These examples also provide a glimpse of how centrally mathematical reasoning and problem solving figure in the work of teaching.

## EXAMPLES OF MATHEMATICAL PROBLEMS OF TEACHING

To illustrate ways in which solving mathematical problems is a recurrent part of the work of teaching, we turn next to some examples. Each of our examples was chosen to show different aspects of the mathematical work of teaching, and to develop the portrait of the mathematics that teaching entails, and the ways in which mathematics is used to solve problems of teaching mathematics.

### 1. CHOOSING A TASK TO ASSESS STUDENT UNDERSTANDING: DECIMALS

One thing that teachers do is monitor whether or not students are learning. To do that, on an informal basis, they pose questions and tasks that can provide indicators of whether or not students are "getting it".

Suppose you wanted to find out if your students could put decimal numbers in order. Which of the following lists of numbers would give you best evidence of students' understanding?

| | | | | |
|---|---|---|---|---|
| a) | .5 | 7 | .01 | 11.4 |
| b) | .60 | 2.53 | 3.14 | .45 |
| c) | .6 | 4.25 | .565 | 2.5 |

Obviously, any of these lists of numbers can be ordered. One possible decision, then, is that the string makes no difference—that a correct ordering of any of the lists is as good as any other.

However, a closer look reveals differences among the lists. It is possible to order (a) and (b) correctly without paying any attention to the decimal point at all. Students who merely looked at the numbers, with no concern for decimal notation, would still put the numbers into the correct order. List (c), however, requires attention to the decimal places: If a student ignored the decimal point, and interpreted the list as a set of whole numbers, he would order the numbers as follows:

| | | | |
|---|---|---|---|
| .6 | 2.5 | 4.25 | .565 |

instead of:

.565 .6 2.5 4.25

So what sort of mathematical reasoning by the teacher is involved? More than being able to put the numbers in the correct order, required here is an analysis of what there is to understand about order, a central mathematical notion, when it is applied to decimals. And it also requires thinking about how ordering decimals is different from ordering whole numbers. For example, when ordering whole numbers, the number of digits is always associated with the size of the number: Numbers with more digits are larger than numbers with fewer. Not so with decimals. 135 is larger than 9, but .135 is not larger than 9. This mathematical perspective is one that matters for teaching, for, as students learn, their number universe expands, from whole numbers to rationals and integers. Hence, teaching requires considering how students' understanding must correspondingly expand and change.

## 2. INTERPRETING AND EVALUATING STUDENTS' NON-STANDARD MATHEMATICAL IDEAS: SUBTRACTION ALGORITHMS

Teachers regularly encounter approaches and methods with which they are not familiar. Sometimes students invent alternative methods and bring them to their teachers. In other cases, students have been taught different methods.

Suppose you had students who showed you these methods for multi-digit subtraction. First, you would need to figure out what is going on, and whether it makes sense mathematically. Second, you would want to know whether either of these methods works in general.

$$\begin{array}{r} 307 \\ -\ 168 \\ \hline 2\ {-6}\ {-1} \\ \hline 1\ \ 3\ \ 9 \end{array} \qquad \begin{array}{r} \overset{29}{\cancel{30}}{}^{1}7 \\ -\ 168 \\ \hline 139 \end{array}$$

The first method uses integers to avoid the standard, error-prone, method of regrouping. It surely works, for it reduces the algorithm to a simple procedure that relies on the composition of numbers, and does not require "borrowing". The second regroups 307 by regarding it, cleverly, as 30 tens plus 7 ones, to 29 tens and 17 ones. Asking mathematical questions, a teacher might ask himself: Even if the methods work, what would either one look like with a 10-digit number? Do both work as "nicely" with any numbers? Skills and habits for analyzing and evaluating the mathematical features and validity of alternative methods play an important role in this example. Note, once again, that this is different from merely being able to subtract 307 – 168 oneself.

## 3. MAKING AND EVALUATING EXPLANATIONS: MULTIPLICATION

Independent of any particular pedagogical approach, teachers are frequently engaged in the work of mathematical explanation. Teachers explain mathematics; they also judge the adequacy of explanations—in textbooks, from their students, or in mathematics resource books for teachers.

Take a very basic example. In multiplying decimals, say 1.3 X 2.7, one algorithm involves carrying out the multiplication much as if the problem were to multiply the whole numbers 13 and 27. One multiplies, ignoring the decimal points.

$$\begin{array}{r} 1.3 \\ \times\ 2.7 \\ \hline 91 \\ 26\phantom{0} \\ \hline 351 \end{array}$$

Then, because the numbers are decimals, the algorithm counts over two places from the right, yielding a product of 3.51.

But suppose one wants to explain why this execution of the algorithm is wrong:

$$\begin{array}{r} 1.3 \\ \times\ 2.7 \\ \hline 91 \\ 26 \\ \hline 11.7 \end{array}$$

and to explain why the standard algorithm works? In this typical instance, a student has not "moved over" the 26 on the second line, and has, in addition, simply placed the decimal point in the position consistent with the original problem.

Is it sufficient to explain by saying that the 26 has to be moved over to line up with the 6 under the 9? And to count the decimal places and insert the decimal point two places from the right?

These are not adequate mathematical explanations. Teaching involves explaining why the 26 should be slid over so that the 6 is under the 9: this involves knowing what the 26 actually represents. In whole number multiplication, if this were 13 x 27, then the 26 on the second line would represent the product of 13 and 20, or 260. In this case, the 26 represents the product of 1.3 and 2, 260 tenths, or 2.6. Developing sound explanations that justify the steps of the algorithm, and explain their meaning, involves knowing much more about the algorithm than simply being able to perform it. It also involves sensitivity to what constitutes an explanation in mathematics.

### What Does Examining the Work of Teaching Imply About Knowing Mathematics for Teaching?

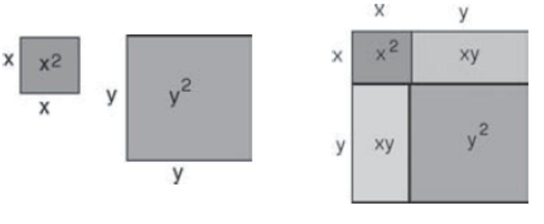
Standing back from our investigation thus far, we offer three observations. First, our examination of mathematics teaching shows that teaching can be seen as involving substantial mathematical work. Looking in this way can illuminate the mathematics that teachers have to do in the course of their work. Each of these involves mathematical problem solving:

- Design mathematically accurate explanations that are comprehensible and useful for students
- Use mathematically appropriate and comprehensible definitions;
- Represent ideas carefully, mapping between a physical or graphical model, the symbolic notation, and the operation or process;
- Interpret and make mathematical and pedagogical judgments about students' questions, solutions, problems, and insights (both predictable and unusual);
- Be able to respond productively to students' mathematical questions and curiosities;
- Make judgments about the mathematical quality of instructional materials and modify as necessary;

- Be able to pose good mathematical questions and problems that are productive for students' learning;
- Assess students' mathematics learning and take next steps.

Second, looking at teaching as mathematical work highlights some essential features of knowing mathematics *for teaching*. One such feature is that mathematical knowledge needs to be *unpacked*. This may be a distinctive feature of knowledge for teaching. Consider, in contrast, that a powerful characteristic of mathematics is its capacity to *compress* information into abstract and highly usable forms. When ideas are represented in compressed symbolic form, their structure becomes evident, and new ideas and actions are possible because of the simplification afforded by the compression and abstraction. Mathematicians rely on this compression in their work. However, teachers work with mathematics as it is being learned, which requires a kind of decompression, or "unpacking", of ideas. Consider the learning of fractions. When children learn about fractions they do not begin with the notion of a real number, nor even a rational number. They begin by encountering quantities that are parts of wholes, and by seeking to represent and then operate with those quantities[9]. They also encounter other situations that call for fractional notation: distances or points on the number line between the familiar whole numbers, the result of dividing quantities that do not come out "evenly" (e.g., $13 \div 4$, and later $3 \div 4$). Across different mathematical and everyday contexts, children work with the elements that come together to compose quantities represented conveniently with fraction notation. Meanwhile, their experiences with the expansion of place value notation to decimals develops another territory that they will later join with fractions to constitute an emergent concept of rational numbers. Teachers would not be able to manage the development of children's understanding with only a compressed conception of real numbers, or formal definition of a rational number. So, although such a conception has high utility for the work of mathematics, it is inadequate for the work of teaching mathematics.

Another important aspect of knowledge for teaching is its connectedness, both across mathematical domains at a given level, and across time as mathematical ideas develop and extend. Teaching requires teachers to help students connect ideas they are learning – geometry to arithmetic, for example. In learning to multiply, students often use grouping: $35 \times 25$ could be represented with 35 groups of 25 objects. But, for example, to show that $35 \times 25 = 25 \times 35$, or that multiplication is commutative, grouping is not illuminating. More useful is being able to represent $35 \times 25$ as a rectangular area, with lengths of 25 and 35 and an area of 875 square units. This representation makes it possible to prove commutativity, simply by rotating the rectangle, showing $a \times b = b \times a$. Or, later, helping students understand the meaning of $x^2 + y^2$ and how it is different from $(x + y)^2$, it is useful to be able to connect the algebraic notions to a geometric representation:




[9] Understanding the development of ideas was implied by Dewey in his distinction between the psychological and the logical aspects of subject matter in *The Child and the Curriculum* (1902). By "psychological", he did not mean the way in which a particular idea might be learned, but the epistemological composition of its growth.

Using these two diagrams helps to show that $x^2 + y^2$ is not the same as the $x^2 + 2xy + y^2$ produced by multiplying $(x + y)^2$.

Teaching involves making connections across mathematical domains, helping students build links and coherence in their knowledge. This can also involve seeing themes. For example, the regrouping of numbers that is part of the standard multi-digit subtraction algorithm is not unlike the renaming of fractions into equivalent forms. In each case, numbers are written in equivalent forms useful to the mathematical procedure at hand. To add two fractions with unlike denominators, it is useful to be able to rewrite them so that they have the same denominator. In subtraction, to subtract 82 – 38, it is useful to be able to rewrite 82 as "7 12" (7 tens and 12 ones) – also an equivalent form. Seeing this connection is useful in helping students appreciate that, to be strategic and clever in mathematics, quantities can be written in equivalent, useful forms.

Teaching also requires teachers to anticipate how mathematical ideas change and grow. Teachers need to have their eye on students' "mathematical horizons" even as they unpack the details of the ideas in focus at the moment (Ball, 1993). For example, second grade teachers may need to be aware of the fact that saying, "You can't subtract a larger number from a smaller one", is to say something that, although pragmatic when teaching whole number subtraction, is soon to be false. Are there mathematically honest things to say instead that more properly anticipate the expansion to integers, and the accompanying changes in what is true or permissible[10]?

One final observation about what we are finding by examining teaching as mathematical work: In our analyses, we discover that the critical mathematical issues at play in the lesson are not merely those of the curricular topic at hand. For example, in a lesson on subtraction with regrouping, we saw the students grappling with three different representations of subtraction and struggling with whether these were all valid, and, if so, whether and how they represented the *same* mathematical operation. They were examining correspondences among representations, investigating whether or not they were equivalent. Although the content was subtraction, the mathematical entailments of the lesson included notions of equivalence and mapping. In other instances, we have seen students struggling over language, where terms were incompletely or inconsistently defined, and we have seen discussions which run aground because mathematical reasoning is limited by a lack of established knowledge foundational to the point at hand. These lessons brought to the surface important aspects of mathematical reasoning, notation, use of terms and representation. Entailed for the teacher would be both the particular mathematical ideas under discussion as well as these other elements of knowing, learning, and doing mathematics. We have seen many moments where the teachers' attentions to one of these aspects of *mathematical practice* is crucial to the navigation of the lesson, and we have also seen opportunities missed because of teachers' lack of mathematical sensibilities and knowledge of fundamental mathematical practices.

Attending to *mathematical practices as a component of mathematical knowledge* makes sense. As children – or mathematicians, for that matter – do and learn mathematics, they are engaged in using and doing mathematics, as are their teachers. They are representing ideas, developing and using definitions, interpreting and introducing notation, figuring out whether a solution is valid, and noticing patterns. They are engaged in mathematical *practices* as they engage in learning mathematics. For example we often see students whose limited ability to interpret and use symbolic notation, or other forms of representation impedes their work and their learning. Similarly, being able to inspect, investigate, and determine whether two solutions, two representations, or two definitions are similar, or equivalent is fundamental to many arenas of

[10] A group of prospective teachers suggested saying, "We can't subtract larger numbers from smaller ones using the numbers we have right now".

school mathematics. Students and teachers are constantly engaged in situations in which mathematical practices are salient. Yet, to date, studies of mathematical knowledge for teaching have barely probed the surface of what of mathematical practices teachers would need to know and how they would use such knowledge.

## CONCLUSION: LEARNING MATHEMATICS FOR TEACHING

What we know about teachers' mathematical knowledge, learning mathematics for teaching, and the demands of teaching mathematics suggests the need to reframe the problem of preparing teachers to know mathematics for teaching. First, although many U.S. teachers lack adequate mathematical knowledge, most know some mathematics – especially some basic mathematics. Identifying what teachers know well and what they know less well is an important question for leveraging resources wisely toward the improvement of teachers' opportunities to learn mathematics. What many teachers lack is mathematical knowledge that is useful to and usable for teaching. Of course, some teachers do learn some mathematics in this way from their teaching, from using curriculum materials thoughtfully and by analyzing student work. However, many do not. Inadequate opportunities exist for teachers to learn mathematics in ways that prepare them for the work, and few curriculum materials effectively realize their potential to provide mathematical guidance and learning opportunities for teachers. Also important to realize is that professional mathematicians may often not know mathematics in these ways, either. This is not surprising, for the mathematics they use and the uses to which they put it are different from the mathematical work of teaching children mathematics. They, too, in helping teachers, will have mathematics to learn, and new problems to learn to solve, even as they also contribute resources. This summary suggests that reframing the problem and working on it productively is both promising and challenging.

Our analysis suggests that teachers' opportunities to learn mathematics should include experiences in unpacking familiar mathematical ideas, procedures, and principles. But, as the polygon example shows, learning mathematics for teaching must also afford opportunities to consider other aspects of proficiency with mathematics – such as understanding the role of definitions and choosing and using them skillfully, knowing what constitutes an adequate explanation or justification, and using representations with care. Knowing mathematics for teaching often entails making sense of methods and solutions different from one's own, and so learning to size up other methods, determine their adequacy, and compare them, is an essential mathematical skill for teaching, and opportunities to engage in such analytic and comparative work is likely to be useful for teachers. As we examine the work of teaching, we are struck repeatedly with how much mathematical problem solving is involved. It is mathematical problem solving both like and unlike the problem solving done by mathematicians or others who use mathematics in their work. Practice in solving the mathematical problems they will face in their work would help teachers learn to use mathematics in the ways they will do so in practice, and is likely also to strengthen and deepen their understanding of the ideas. For example, a group of teachers could analyze the three multiplication solutions presented here, determine their validity and generality, map them carefully onto one another. They could also represent them in a common representational context, such as a grid diagram or an area representation of the multiplication of 35 x 25 (see Ball, 2003).

Seeing teaching as mathematically-intensive work, involving significant and challenging mathematical reasoning and problem solving, can offer a perspective on the mathematical education of teachers, both preservice and across their careers. It opens the door to making professional education of teachers of mathematics both more intellectually and mathematically challenging, and, at the same time, more deeply useful and practical.

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# MATHEMATICS AS MEDICINE

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Dr. Ed Doolittle, mathematician and Associate Professor at First Nations University of Canada, gave his plenary address to CMESG in 2006. Doolittle, an Aboriginal mathematician and advocate for Indigenous people's mathematics and mathematics education for Indigenous people shared stories that reflected his view of mathematics as medicine. CMESG members were moved by his opening acknowledgement of his ancestors to his conclusion of the power of mathematics. Doolittle's story telling invited us to think about mathematics in the context of Aboriginal learners and culture. In considering this paper for inclusion in the volume we also reflected on how our structure of plenary address followed by small group discussions to generate questions for the plenary speaker and the subsequent question period is very effective in deepening the messages. We felt this was especially evident in Doolittle's plenary.

Dr. Ed Doolittle, mathématicien et professeur associé à l'Université des Premières Nations du Canada, a donné sa conférence plénière au GCEDM en 2006. Doolittle, un mathématicien amérindien défenseur des mathématiques des amérindiens et de l'éducation mathématique pour les amérindiens, a partagé avec nous quelques histoires reflétant sa vision des mathématiques en tant que médecine. Les membres du GCEDM ont été touchés du début jusqu'à la fin de sa conférence, de la reconnaissance exprimée envers ses ancêtres à sa conclusion sur la puissance des mathématiques. Les histoires racontées par Doolittle nous ont invités à penser aux mathématiques dans le contexte des apprenants amérindiens et de la culture amérindienne. Si nous avons choisi de publier ce texte dans le présent volume, c'est qu'il nous rappelle à quel point notre structure de conférence plénière suivie d'une période de préparation de questions en petits groupes, suivie à son tour d'une séance plénière de questions au conférencier ou à la conférencière, est très efficace pour approfondir le message. Il nous est apparu que ceci était particulièrement vrai dans le cas de cette conférence de Doolittle.

I remember well my first visit to the Navajo reservation in Arizona. I was traveling with a dear friend who had been a few times before. Driving through the desert in near total darkness, I spotted some strange coloured lights flashing on the horizon near the place where the town in the reservation should be, pulses which would stop and then start again in an irregular pattern. "What's that?" I asked. "I don't know," said my friend, clearly disturbed. "I've never seen that before." Thoughts of UFO abduction began to form in my mind. "You're pulling my leg," I said. "No really," she said, "I have no idea what those coloured lights are!" We continued to drive through the darkness, perplexed and staring in wonder at the coloured lights, for what seemed like ages. We were closer, but I still couldn't make sense of the experience. The lights disappeared behind a hill. As we climbed the hill I held my breath, knowing that the truth was on the other side. We reached the top and it finally unfolded clearly before me. The circus had come to the Navajo reservation.

I am an Indian. I am a mathematician. Those two aspects of my identity seem to be in constant opposition, yet I cannot let go of either.

My father, the late Edward Lorne Doolittle, was a Mohawk Indian from the Six Nations reservation in southern Ontario. My father's mother, Belda Brant, attended residential school where she lost her language and learned how to clean hotel rooms. My father's father, Clifford Doolittle, was killed in a railroad construction accident when my father was five years old. The settlement offered by the railroad company was $35 a month. My grandfather's spirit came to my grandmother to tell her that she should take the family off the reserve to find work, which she did. Although that helped to keep the family fed, it had the effect of further distancing them from their culture.

By the time I was a teenager, my father and mother, Eleanor Naylor, third generation English in Canada, had managed to pull together comfortable middle class existence. I grew up in the suburbs of Hamilton, Ontario, knowing almost nothing of my Indigenous culture. Aside from occasional weekend visits with my aunts and uncles, I had no idea what it meant to be Indian. I learned Latin instead of Mohawk, the Bible instead of *Kayanerekowa* and *Karihwiyo*, fairy tales instead of Coyote tales. I also enjoyed solving puzzles.

Despite all the advantages and privileges and resources available to me, I still felt there was something missing in my life. I tried to find what was missing in religion, but I stopped attending Baptist church at the age of 14. My Sunday school teacher had used some twine to tie my wrist to that of the other student in my age group, a sweet young woman whose name I can't recall, in order to illustrate some point about sin. I thought the problem might have been more with the teacher than with me or the girl. Instead I read about Zen Buddhism and searched for enlightenment. I studied religion instead of biology in high school, disappointing those who wanted to be able to call me Dr. Doolittle. My religion teacher invited a friend of his who we only knew as Krishna to come and talk about Hinduism. "Ask him whatever you want," said the teacher. So I asked Krishna, "Is it possible to convert to Hinduism?"

> "Yes," said Krishna reluctantly, "But I discourage it."
>
> What?
>
> "If you can't find what you're looking for in your own tradition, you won't be able to find it in ours."

Enlightenment. I don't even know what my tradition is, I wanted to say. How can I find anything within my own tradition if I don't even have a tradition? That was my problem, though, not Krishna's. He could not tell me where to find the something that was missing.

I was accepted to university, and my general intention was to study Artificial Intelligence at the University of Toronto. My parents had not planned for my university education. Instead they relied on the fact that status Indians received education funding from the federal government. While there was no denying that I was a status Indian—I had the card to prove it—I felt that the benefit was for real Indians, not privileged, pale, suburban half-Indians. At first I balked at the suggestion that I should accept Indian Affairs funding for my education, but that upset my parents greatly. I reconsidered, and (in a deal only with myself) accepted on the condition that I would do my best to deserve the benefit offered to me. I resolved to become an Indian.

At the University of Toronto I connected with the Indian Health Careers Program, a program designed to help to increase the representation of Aboriginal people in medicine and other health-related careers. Dianne Longboat, the director of the program, hosted gatherings with traditional teachers and elders, and invited me to attend. The experience of hearing elders speak turned me inside-out. For the first time I directly experienced a powerful tradition of thought and experience which stood completely outside of the Western tradition in which I had been educated. The power and wisdom of the words of the elders were like a streak of lightning shooting through my brain.

It felt like my whole life had been a preparation for those moments, when I understood that there really are different ways of thinking and being. Ways which were not only different, but truly powerful; a tradition that stood on its own, entirely independent from European thought, and had great gifts to offer. Even better, all this could somehow be mine. The search for that which was missing had ended. That is when I really started to become an Indigenous person. The work of bringing it in, filling myself with it had begun, continues to this day, and will not end until my life is over.

I am unbelievably fortunate. Something I could do to satisfy my obligations became something I could do for my own benefit. And not just a way to self-discovery and to fill a hole, but a way to power and strength, a way to change the world. The opportunity to become an Indigenous person is one of the greatest gifts I have ever been granted.

I am not wise, nor deeply knowledgeable about my culture, nor gifted in oratory like the elders. How can I, with my lack of gifts of expression, convince you that our way is a powerful way? I often think about three simple words spoken by Chief John Snow: "We have survived." Our ways must be powerful if they have helped people survive through one of the greatest holocausts in human history: one hundred million dead of disease, starvation, and warfare; loss of land, wealth, culture, and knowledge; injustice and wanton destruction all around. Through it all, we (the survivors) have survived.

Back in the regular world, I abandoned the study of Artificial Intelligence, which seemed to be reaching a dead end at the time, possibly because its mechanistic approach was just too simpleminded to approach something as complex as the human mind, and took up mathematics, which is what I seemed to do best and which always seemed fun and natural to me. I studied mathematics at the University of Toronto for twelve years, ultimately earning a PhD in pure mathematics under Peter Greiner; my dissertation was on the topic of hypoelliptic partial differential operators. Peter is like a father to me in a certain sense. He is also a great mathematician, and wise in his own way and in his own tradition. He is my mathematical father, connecting me to another strong lineage which includes Solomon Lefschetz and Carl Neumann. My two fathers have never met.

One of my major life goals is to resolve the apparent incompatibility between the two aspects of my identity, being a mathematician and being an Indigenous person.

To that end, I would like to explore various interfaces between mathematics and Indigenous thought. At this point I am more interested in searching for possibilities than organizing my thoughts in any particular way. I have tried to identify the main sources for my thinking, but I have neglected making exact references to the literature. I hope that you will forgive my poor scholarship, but the need for references is reduced because of the availability of such information in this modern age. In any case, I don't always remember things the way they were said or written, but I remember the impression they made on me.

Perhaps the most common, most straightforward, and simplest interface between mathematics and Indigenous people is the proposal that mathematics is a requirement for Indigenous people to succeed in the job market. The problem is often stated in terms of the desperate state of education of Aboriginal people in terms of math and science. Many researchers have attempted to quantify or otherwise justify that assessment and then conclude that we must find ways to improve outcomes and achievement indicators for the benefit of the students.

I am skeptical of that approach. For one thing, we have heard such talk before, in connection with residential schooling for example. I don't doubt the sincerity and desire to do good of those who take that point of view, but the concern that I have, partly from history, partly from personal experience, is that as something is gained, something might be lost too. We have some idea of the benefit, but do we know anything at all about the cost?

The complexity of the situation seems to expand endlessly the more it is examined. It is tempting to search for simple solutions to complex problems and to offer simple responses to complex situations; that is what Western thought (mathematics included) teaches us to do. However, such responses have not been adequate as we can see from the continuing nature of the problem (whatever that problem really is).

As examples of the surprising and complex nature of Indigenous mathematics education, I would like to offer some impressions taken from a paper by William Leap on the mathematics education of the Ute Indians.

Q: If he gets four dollars a day, how many is he going to have in two days?

A: Six.

Q: Let's imagine you have 72 pennies right here in a pile, and there's one boy sitting here, one boy there, one boy there, and one boy there. What would you do to make certain everybody got the same number of pennies?

A: Pass them out until they are all gone.

Q: If your brother took his truck to Salt Lake City, how much would he have to spend on gas?

A: My brother doesn't have a truck.

Another approach to the apparent incompatibility between Indigenous thought and mathematics is ethnomathematics. Roughly speaking, ethnomathematics expands the meaning of "mathematics" to include very general notions of counting, measuring, locating, designing, playing, and explaining. From the perspective of mathematics education, the task is to identify examples of such activities within a culture and use those examples to teach mathematics. Many different examples of Native American ethnomathematics have been discussed by authors such as Marcia Ascher, Michael Closs, and many others. For example, the peach pit bowl game of my people is discussed in Ascher's book *Ethnomathematics*.

Some of the most interesting examples of ethnomathematics in North America, in my opinion, involve the idea of mapping in an extended sense. The feeling I get from Native American maps and diagrams is that they are not static maps of locations and spatial relationships, but maps of processes, like how to get from one place to another, how to make a caribou dinner from scratch, how to give thanks and show respect to everything that's good, or how to mourn.

Ethnomathematics is far more reflective and respectful to Indigenous traditions of thought than the simpler reflex to help Indians succeed at improving their outcomes on standardized tests. However, the danger of oversimplification remains, perhaps more insidious because the motives are put forward as purer. An example of such oversimplification which I have encountered repeatedly in discussions with well-meaning people I call Cone on the Range. "The tipi is a cone," I have heard countless times. But that is surely wrong; the tipi is not a cone. Just look at a tipi with open eyes. It bulges here, sinks in there, has holes for people and smoke and bugs to pass, a floor made of dirt and grass, various smells and sounds and textures. There is a body of tradition and ceremony attached to the tipi which is completely different from and rivals that of the cone. Similarly, there is a ceremonial and spiritual tradition connected with the peach pit bowl game that is completely lost in Ascher's treatment.

Aside from being wrong, oversimplifications such as calling a tipi a cone or analyzing the peach pit bowl game only in terms of probabilities and odds may have other serious implications in an educational context. My feeling is that Indigenous students who are presented with such oversimplifications feel that their culture has been appropriated by a powerful force for the purpose of leading them away from the culture. The starting point (tipi, game) may be reasonable but the direction is away from the culture and toward some strange and uncomfortable place. Students may, implicitly or explicitly, come to question the motives of teachers who lead them away from the true complexities of their cultures.

There is a more pervasive and insidious example which I call Squaring the Circle. Of course, Squaring the Circle is one of the unsolvable mathematical problems of antiquity, but the term is also used by blues musicians such as Sterling "Satan" Magee for the process of reasoning too much about something that one should be feeling; I believe the term "square" is meant in a pejorative sense in that context.

In modern Indigenous thought, a tool called the medicine wheel is often used to divide complex situations into four simpler categories. Many Indigenous people will staunchly defend the process of dividing wholes into four aspects, such as the person into the physical, emotional, spiritual, and mental. However, I feel, based on personal experience, that such analyses square the circle; they are pale oversimplifications of complex and powerful traditions which have gone underground. One revelatory experience for me took place at a meeting with teachers, an elder, and a number of well-meaning researchers at the University of Saskatchewan. After the presentation of a rather complicated example of the use of the medicine wheel in the theory of science education, Elder Betty McKenna of Moose Jaw was asked what she thought about it. Betty responded: "I have worked on a real medicine wheel."

The implication, of course, is that a geometrical, abstract medicine wheel is not real. But what then, is a real medicine wheel? It is an approximately circular arrangement of stones on the ground, often with spokes radiating from a centre, sometimes with loops of stones occurring at irregular intervals around the perimeter. There are many pictures available on the Web of real medicine wheels such as the Bighorn Medicine Wheel. Note that they blend with the landscape as it rises and falls; they are not regular. The stones used to mark them are of different sizes and shapes and colours; the number of spokes is not necessarily a multiple of four and not clearly meaningful in any way at all. The purpose and meaning of such wheels is to some extent lost, or more likely has gone underground. My belief is that they were used not to divide and analyze, but as "maps" of processes of ceremony, thanksgiving, timekeeping, and communication. Or maybe not.

Notwithstanding the concerns I have about ethnomathematics in math education, I feel that ethnomathematics is a worthwhile pursuit. I would like to propose another example for the body knowledge of the ethnomathematics of Native North America. However, before I do so, I can't resist telling a joke which I first heard from Eber Hampton at a barbecue sponsored by Luther College on the occasion of the opening of First Nations University.

> *When the astronauts first landed on the moon, they saw a strange sight: a teepee sitting right there on the lunar surface some distance from the landing craft. The astronauts bounced over in their spacesuits to marvel at the sight. Finally, one of them got the nerve to knock on the hide covering the entrance. An old man parted the doorway and looked out, just as surprised to see the astronauts as they were to see him. They stared at each other for a few moments, and then the old man noticed the American flag planted some distance behind the astronauts. Seeing the flag, the old man exclaimed, "Oh no! Not you guys again!"*

The capture of Detroit is one of the highlights of Canadian military history. Near the beginning of the War of 1812, the government of Canada and its wartime leader, Isaac Brock, were concerned about its ability to fight a war on three fronts: the Detroit River, the Niagara River, and the St. Lawrence River. Brock decided to try to neutralize the threat in Detroit quickly by launching an immediate, overwhelming attack on the American forces stationed in Fort Lernoult, Detroit. Short of manpower, he gathered as many militia as possible and dressed them in red jackets to make them look like regulars, and recruited as many Indians as he could to the cause. Key to those recruitment efforts was the great chief Tecumseh, who was impressed with Brock and willing to support Brock's fight against the Americans.

In the decisive tactic of the attack on Fort Lernoult, Tecumseh had his Indians march past a point which the Americans could see, change their clothing somewhat, sneak back around to their starting point, and march again and again through the Americans' field of vision. "One little, two little, three little Indians … ." Several thousand non-existent Indians later the Americans thought they were severely outnumbered and surrendered without firing a shot.

That, I would say, is a fine example of the Native American use of mathematics. It is something which we own, something of which we can be proud. That is what is missing, from most of the examples of ethnomathematics used in education. In ethnomathematics, there is usually a sense that there is something larger behind the scene, let us call it "real mathematics", which is not ours. That perceived lack has the effect of making us feel ashamed rather than proud.

Passion was a major key to Tecumseh's success in the opinion of his biographer John Sugden. In the Indigenous world view, perhaps feelings like passion and pride are more valuable than the knowledge of facts, ideas, rules, regulations, and methods. We need to follow Tecumseh's example and instill a sense of pride and passion in our students, not shame and apathy.

Apropos are historian William Wood's words on the impact of the death of Brock at Niagara-on-the-Lake shortly after the capture of Detroit: "Genius is a thing apart from mere addition and subtraction." Brock was just one man, but his life and death changed the course of history. Arithmetic is not always the best tool to use. One good example notwithstanding, we are still left with the question of what we can do to resolve the apparent incompatibilities between Indigenous thought and mathematics. I would like to make two suggestions about how we might be able to proceed from here.

First, I would like to consider the question of how we might be able to pull mathematics into Indigenous culture rather than how mathematics might be pushed onto Indigenous people or how Indigenous culture might be pulled onto mathematics. What might be the difference between thought which is authentic to the culture rather than a simulacrum of an idea from elsewhere?

Let us consider how foreign words and concepts are introduced into the Mohawk. Some words are simply borrowed, in a process familiar to English speakers, from a European language as in *Kabatsya* = garbage, *Ti* = tea, and *Takós* = cat (probably from Dutch *de poes*, i.e., the puss). There are obvious signs that those words are not originally Mohawk words: the presence of strange sounds (the b sound in garbage), single syllable words, or words with stress on the wrong syllable. Some borrowed words have the overall style of Mohawk (e.g., begin with "*ra-*") but lack the internal structure of Mohawk words, as in *Rasanya* = lasagna which, if it were really a Mohawk word, would mean something like "he sanyaed", whatever sanyaing would be. A similar example is *Rasohs* = sauce, apparently from the French *la sauce*. All of those examples lack the nuance, complexity, and internal structure that Mohawk words typically have. If there is any connotation, it is ridiculous, as in "he sanyaed".

On the other hand, there are new Mohawk words to describe new concepts, words which developed within the Mohawk tradition. For example, we have *kaya'tarha* = television, literally "it has bodies on its surface"; *teyothyatatken* = banana, literally "the fruit that has bent itself"; *kawennokwas* = radio, literally "it throws out songs"; and *kawennarha*, literally "it has words on its surface", a word proposed, but not (yet?) generally accepted, for describing a computer. Those words really mean something and are not just dry tokens the way English nouns are. They are better because they ours, but it is not simply a matter of pride. Since they are ours, they are consistent and coherent with the rest of the language; they strengthen the language just as the language strengthens them; and they can be modified and built upon to add further complexity and sophistication to the language.

New words are coined constantly within the Mohawk tradition. The spirit of the language is inventive and playful, not acquisitive like the spirit of English. I myself have coined a few new words, for example *kahnekahontsi* = cola drink, literally "black water" or "black drink", and *Kwiskwis nikawahràsas* = bacon bits, literally "little pig meats". The latter made Kahnekotsyentha kenha laugh and is now regularly used by a small group in Six Nations. Some day it may come into general use.

Second, I would say we need to recognize that mathematics is an essentially simple (not complex, although often complicated) way of thinking. Mathematics is all about simplifying, clarifying, analyzing, and breaking down. On the other hand, Indigenous thought is all about developing and building up sophisticated, complex responses to complex phenomena such as the weather, animal migratory patterns, healing, and human behaviour. A colleague at First Nations House at the University of Toronto told me about one occasion on which her grandmother held a baby. "There's something wrong with this baby," said the grandmother. It turned out that the child had a serious illness, but the child's parents and doctor had all missed the problem until the grandmother felt that something was wrong. We can weigh and measure and test, but true complexity cannot be handled by simple means.

Time for another joke. This one I heard at the Sakewewak Storytelling Festival in Regina several years ago. I'm afraid I can't remember the name of the storyteller; if anyone out there knows, please tell me so I can credit him properly in the future.

> *In a town in a certain reserve in Saskatchewan, some young boys were breaking into houses. The RCMP investigated. They came into town and asked the first person they see, an old man sitting in front of his house, whether he knew anything about the break-ins. "Yup," said the old man. "Do you know who's been doing it?" asked the police. "Yup," said the old man, "those four boys." "Would you be prepared to testify in court?" asked the police. "Yup," said the old man. So the RCMP arrested the boys and charged them with break and enter.*
>
> *Court day arrived, and the old man took the stand. The prosecutor asked him, "Do you know who's being doing those break-ins?" "Yes," said the old man. The prosecutor asked, "Can you point to the individuals in question?" "Yes," said the old man, "it's those four boys sitting over there." "Thank you," said the prosecutor, "those are all the questions I have."*
>
> *Then the defence lawyer began his cross-examination. "Have you actually seen those boys breaking in to a house?" "No," said the old man, "I haven't actually seen it myself." "Then how do you know it's them?" asked the defence lawyer. "I have my ways of knowing," said the old man. "I'm sorry, your evidence is hearsay. We can't accept it," the defence lawyer said. The judge agreed, and dismissed the witness.*
>
> *Well, the old man was not too happy about being dismissed like that, so as he walked past the judge on the way back to his seat, he let out a fart. A long, loud one. A big one. The judge banged on his gavel and said, "I could have you charged with contempt of court for that!"*
>
> *The old man turned to face the judge and asked, "Did you see anything?"*

Given the apparent incompatibilities between Indigenous thought and mathematics, I suggest that instead of asking "What is Indigenous mathematics," it may be helpful to start with the following question instead: "What are the Indigenous analogues to mathematics?"

For example, we might ask what the role of mathematics is in non-Indigenous culture. I believe that one function mathematics plays is as a source of power, which is one reason people are so concerned about learning it or seeing that it is taught to their children. Power is also an important concept in my culture. In fact, the core message of the *Kayanerekowa*, the Great Good Way, is *Skennen, Kahsha'sten'tshera, Ka'nikonhriyo* = Peace, Power, and Good Mind. (The word "righteousness" is often seen in place of "good mind", but the latter is a better translation.) Power is central to our understanding of following a good way.

> *Seeing me in my patched-up, faded shirt, my down-at-heels cowboy boots, the hearing aid whistling in my ear, looking at the flimsy shack with its bad-smelling outhouse—it all doesn't add up to a white man's idea of a holy man. You've seen me drunk and broke. You've heard me curse and tell a sexy joke. You know I'm no better or wiser than other men. But I've been up on the hilltop, got my vision and my power, the rest is just trimmings. That vision never leaves me. - Lame Deer*

All this talk about power tends to make some people nervous. However, *kahsha'sten'tshera* in this context is not power in isolation, rather power within a strong ethical tradition, if "ethical" is the right word. Another aspect of the tradition in which power sits is humility. As Black Elk said,

> *I cured with the power that came through me. Of course, it was not I who cured, it was the power from the Outer World, the visions and the ceremonies had only made me like a hole through which the power could come to the two leggeds. If I thought that I was doing it myself, the hole would close up and no power could come through. Then everything I could do would be foolish.*

Black Elk's reference to power coming through him reminds me of Ramanujan, a great inspiration to me, one of the finest mathematical minds of the 20$^{th}$ century. Ramanujan could not describe the source of his mathematical insight, but believed it did not come from him personally; instead it came through him in dreams from his family goddess, Namakkal. Ramanujan had a morning ritual of writing down the thoughts that came to him in dreams shortly after awakening.

Indigenous spiritual traditions and mathematics are perhaps not really so far apart after all. Perhaps. Perhaps we can think of mathematics as a kind of medicine, a healing power. But can it make our lives better as a people, or are its benefits restricted to just a few fortunate individuals?

I would like to finish with the Blackfoot horse creation story. This version of the story is taken from Ted Chamberlin's most recent book, *Horse*.

> *A long time ago there was a poor boy who tried to obtain secret power so that he might be able to get some of the things he wanted but did not have. He went out from his camp and slept alone on the mountains, near great rocks, beside rivers. He wandered until he came to a large lake northeast of the Sweetgrass Hills. By the side of that lake he broke down and cried. The powerful water spirit—an old man—who lived in that lake heard him and told his son to go to the boy and find out why he was crying. The son went to the sorrowing boy and told him that his father wished to see him. 'But how can I go to him?' the lad asked. 'Hold onto my shoulders and close your eyes,' the son replied. 'Don't look until I tell you to do so.' They started into the water. As they moved along the son told the boy, 'My father will offer you your choice of animals in this lake. Be sure to choose the old mallard and its little ones.'*
>
> *When they reached his father's lodge on the bottom of the lake, the son told the boy to open his eyes. They entered the lodge, and the old man said, 'Come sit over here.' Then he asked, 'My boy, what did you come for?' The boy explained, 'I have been a very poor boy. I left my camp to look for secret power so that I may be able to start out for myself.' The old man then said, 'Now, son, you are going to become the leader of your tribe. You will have plenty of everything. Do you see all the animals in this lake? They are all mine.' The boy, remembering the son's advice, said, 'I should thank*

> *you for giving me as many of them as you can.' Then the old man offered him his choice. The boy asked for the mallard and it's young. The old man replied, 'Don't take that one. It is old and of no value.' But the boy insisted. Four times he asked for the mallard. Then the old man said, 'You are a wise boy. When you leave my lodge my son will take you to the edge of the lake, and there in the darkness, he will catch the mallard for you. When you leave the lake don't look back.'*
>
> *The boy did as he was told. At the edge of the lake the water spirit's son collected some marsh grass and braided it into a rope. With the rope he caught the old mallard and led it ashore. He placed the rope in the boy's hand and told him to walk on, but not to look back until daybreak. As the boy walked along he heard the duck's feathers flapping on the ground. Later he could no longer hear that sound. As he proceeded he heard the sound of heavy feet behind him, and a strange noise, the cry of an animal. The braided marsh grass turned into a rawhide rope in his hand. but he did not look back until dawn.*
>
> *At daybreak he turned and saw a strange animal at the end of the line—a horse. He mounted it and, using the rawhide rope as a bridle, rode back to camp. Then he found that many horses had followed him.*
>
> *The people of the camp were afraid of the strange animals. But the boy signed to them not to fear. He dismounted and tied a knot in the tail of his horse. Then he gave everybody horses; there were plenty for everyone and he had quite a herd left over for himself. Five of the older men in camp gave their daughters to him in return for the horses. They gave him a fine lodge also.*
>
> *Until that time the people had had only dogs. But the boy told them how to handle the strange horses. He showed them how to use them for packing, how to break them for riding and for the travois, and he gave the horse its name, elk dog. One day the men asked him, 'These elk dogs, would they be of any use in hunting buffalo?' 'They are fine for that,' the boy replied. 'Let me show you.' Whereupon he taught his people how to chase the buffalo on horseback. He also showed them how to make whips and other gear for their horses. Once when they came to a river the boy's friends asked him, 'These elk dogs, are they of any use to us in the water?' He replied, 'That is where they are best. I got them from the water.' So they learned how to use horses in crossing streams.*
>
> *The boy grew older and became a great chief, the leader of his people. Since that time every chief has owned a lot of horses.*

Given the frustrations and difficulties of the task facing us, it is reasonable to ask, "Do we really need this stuff anyway?" As a response I offer the completion of the earlier quotation by Chief John Snow: "We have survived, but survival by itself is not enough. A people must also grow and flourish."

# OLD AND NEW MATHEMATICAL IDEAS FROM AFRICA: CHALLENGES FOR REFLECTION

Paulus Gerdes
*ISTEG-University, Mozambique*

Dr. Paulus Gerdes, mathematician and mathematics teacher educator died just two years after he gave his keynote address to the Canadian Mathematics Education Study Group. He was 62 and at the height of his career, appreciated for his work to identify African (and Indigenous peoples) mathematics. His plenary was a mathematics talk. He shared the mathematics of basket weavers in Mozambique and the parallel mathematicization of the patterns found in the weavings. He shared the stories of mathematical people, those people who wove patterns and those people who coded patterns. His question period was a mathematics education discussion, offering images of curriculum and pedagogy through the stories of his life and the people he encountered.

Dr. Paulus Gerdes, mathématicien et formateur à l'enseignement des mathématiques, est décédé deux années seulement après avoir prononcé sa conférence plénière devant le Groupe Canadien d'Étude en Didactique des Mathématiques. Âgé de 62 ans et au sommet de sa carrière, il était reconnu internationalement pour son travail sur les mathématiques africaines et indigènes. Sa conférence plénière était une conférence mathématique. Il a présenté les mathématiques des tisserands de paniers au Mozambique et la mathématisation parallèle des motifs trouvés dans les tissages. Il a partagé avec nous quelques histoires de personnages mathématiques, ces personnages étant pour lui autant les créateurs de motifs que les codeurs de motifs. La période de questions a porté quant à elle sur l'éducation mathématique. Elle a permis au conférencier de présenter des exemples de contenus d'enseignement et d'approches pédagogiques à travers des histoires tirées de sa vie et de celle de personnes qu'il a croisées sur son chemin.

*Some challenges for reflection:*

- *How can culture(s) be a source of inspiration for mathematics education?*
- *Who does mathematics? Who invents mathematics? What is 'mathematical thinking'? Who or which culture defines it?*
- *Can African and other cultures be a source of inspiration for the development of new mathematical ideas?*

## AVANT PROPOS: SOME SIMILAR DESIGNS IN CANADA AND AFRICA

Figure 1. Detail of a wall decoration at the Université Laval, Québec.

Walking around at the Université Laval, I observed interesting geometric patterns and shapes. Yesterday night, when leaving the atrium of the Pavillon De Koninck after the sympathetic Wine & Cheese welcoming session, I was surprised to see outside the building a decorated wall with the inscription "Québec", with four replicas, in different colours, of a design well-known from Africa in various variations (figure 1). It appears, for instance, in the Cokwe culture from East Angola, drawn in the sand, to represent a tortoise (figure 2). The drawing consists of a reference frame of dots marked in the sand around which three closed lines are traced (figure 3); at the end the drawer adds the paws of the animal.

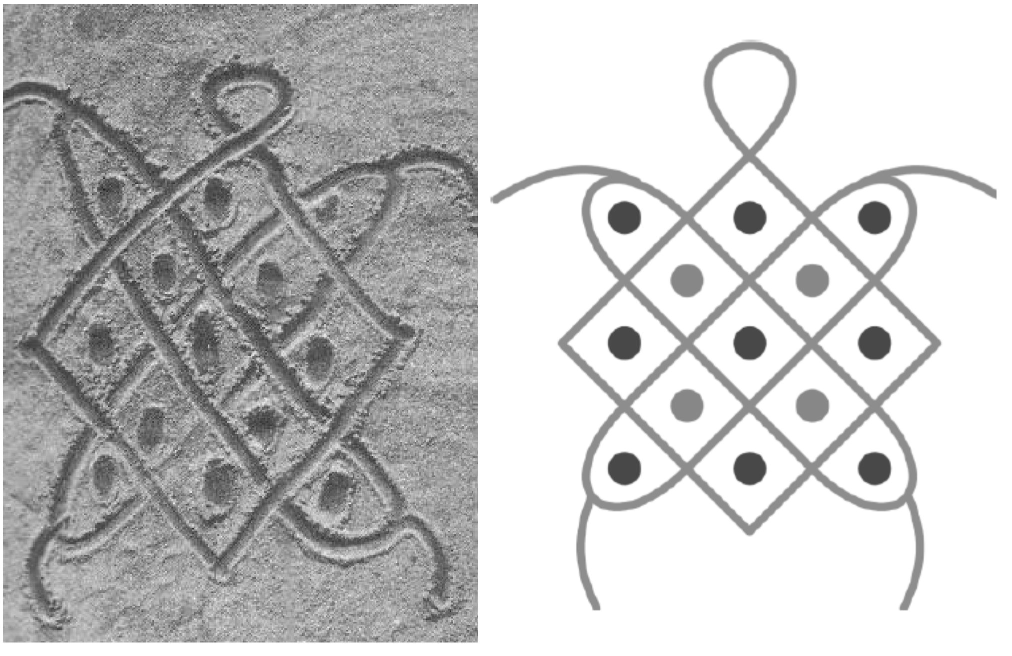


Figure 2. Cokwe representation of a tortoise.

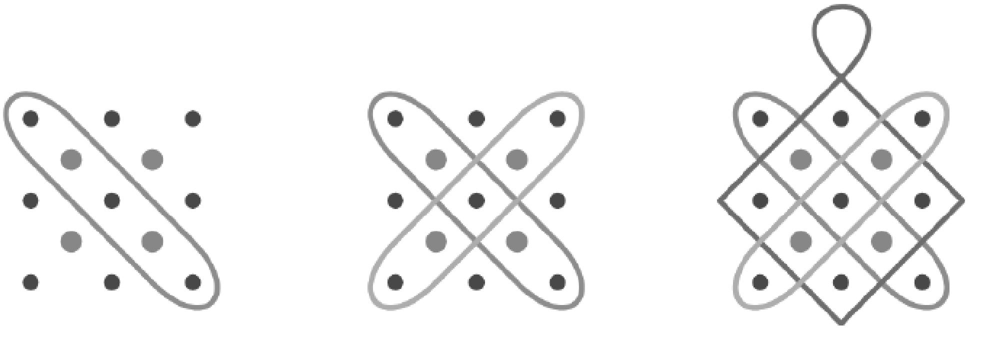


Figure 3. Execution of the drawing.

Before I continue, let me take the opportunity to thank the Canadian Mathematics Education Study Group for the invitation to give the 'mathematician' plenary at this year's conference at Laval University in Québec. My first contact with mathematics educators in Canada was when the late David Wheeler invited me to write papers for publication in *For the Learning of Mathematics*. I did so (Gerdes, 1985; 1986; 1988; 1990a; 1994a) and I am happy to see that the first paper was reprinted recently in Alan Bishop's (2010) book *Mathematics Education: Major Themes in Education*. It was also David Wheeler who, as a member of the international program committee, invited Alan Bishop, the late Peter Damerow, and me to organize the special 5th day of the 1988 International Congress of Mathematics Education in Budapest, dedicated to 'Mathematics, Education, and Society'. My 1988 paper in *FLM* was entitled "A widespread decorative motif and the Pythagorean Theorem" and dealt with the educational exploration of designs that appear both among Native Americans and in Africa. It included an infinite series of proofs of the theorem. My 1990 paper in *FLM* was on mathematical elements in the centuries old Cokwe *sona* sand-drawing tradition.

*Challenge for reflection: How can culture(s) be a source of inspiration for mathematics education? Example: Theorem of Pythagoras. Other examples?*

## 'SONA' GEOMETRY

The tortoise design in figure 2 is an example of a *lusona* (plural: *sona*). Formerly, Cokwe storytellers and educators used the *sona* as illustrations when teaching young boys. The colonial penetration and occupation contributed to the almost complete extinction of the knowledge about *sona*. I have been experimenting with the use of *sona* in mathematics education, both inside and outside the classroom. See, for instance, the children's book, *Drawings from Angola: Living Mathematics* (Gerdes, 2007a; 2012a), the book for high school pupils, *Lusona: Geometrical Recreations from Africa* (Gerdes, 1997; 2012b), the second volume on *Educational and Mathematical Explorations* (in English: Gerdes, 2013a; in French: Gerdes, 1995), and my trilogy *Sona Geometry from Angola* for use in mathematics teacher education and in the education of mathematicians. The first volume (Gerdes, 1995; 2006) of the trilogy deals with the reconstruction and analysis of mathematical ideas in the *sona* tradition and the third volume is a comparative study of *sona* with designs from Ancient Egypt, Ancient Mesopotamia, and India, and with Celtic knot patterns. As this is the 'mathematicians' plenary, I would like to present some new mathematical ideas that emerged in the attempt to analyze the mathematical potential of the (reconstructed) *sona* tradition.

*A small question: Is figure 1 positive or negative? In what sense? Why?*

## FROM A PARTICULAR CLASS OF 'SONA' TO THE CONCEPTUALIZATION OF MIRROR CURVES

Figure 4 shows the Cokwe *sona* that represent the stomach of a lion and the path followed by a chicken being chased by a hunter.

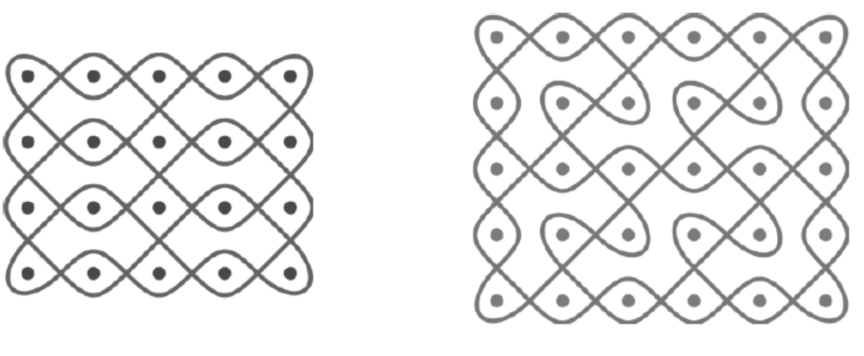


Figure 4. Lion's stomach and 'chased-chicken' path.

When I was analyzing *sona* like these two, I found that they can be generated in a particular way: Both are examples of what I call 'mirror curves', a concept I proposed for the first time in English in (Gerdes, 1990b). A mirror curve is

> *the smooth version of the polygonal path described by a light ray emitted from the starting place S at an angle of 45º to the rows of a grid (see figure 5); and as the ray travels through the grid it is reflected by the sides of the rectangle and by the 'double-sided mirrors' it encounters on its path. The mirrors are placed horizontally or vertically, midway, between two neighboring grid points, as in figure 6.*

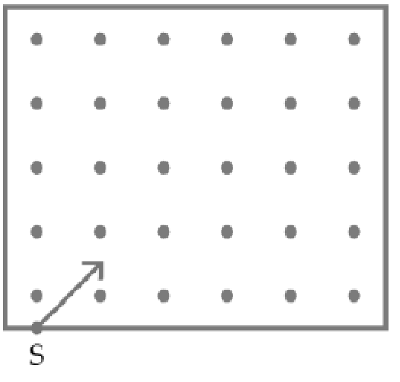



Figure 5. Light ray emitted from point S.

Figure 6. Possible positions of mirrors relative to neighboring grid points.

Figure 7 presents the position of the mirrors in the examples of the 'lion's stomach' and the 'chased-chicken' designs in figure 4.

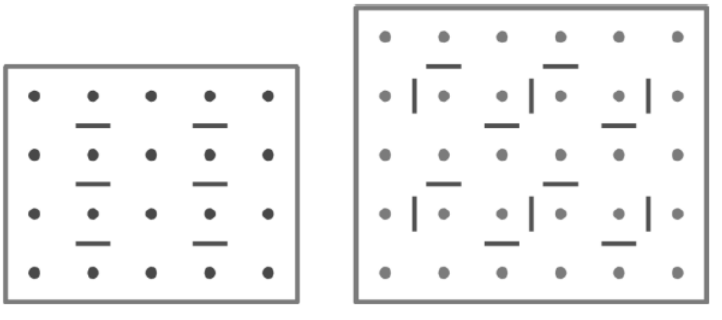


Figure 7. Position of mirrors in the case of the designs in figure 4.

Once I had defined the concept of mirror curve in general, I started to look for the properties of mirror curves. To facilitate the execution of mirror curves, I used to draw them on squared paper with a distance of two units between two successive grid points. In this way, a line drawing such as the 'chased-chicken' path passes exactly once through each of the unit squares inside the rectangle surrounding the grid (see figure 8).

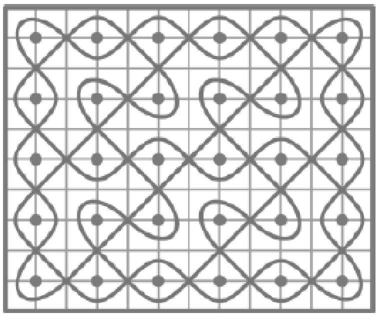


Figure 8. Line passing once through each of the unit squares.

This gives the possibility of enumerating the small squares 'modulo 2', with the number 1 being given to the unit square where one starts the line, and the number 0 to the second unit square through which the curve passes, and so on successively, 101010…, until the closed curve is complete. In this way a {0, 1}-matrix is produced. Colouring the unit squares numbered 1 black, and those numbered 0 white, a black-and-white design is obtained. As this type of black-and-white design generated by mirror curves was discovered in the context of analyzing *sona* from the Cokwe, who predominantly inhabit the Lunda region of Angola, I gave them the name of 'Lunda-designs'. Figure 9 presents two examples of Lunda-designs, using different colours.

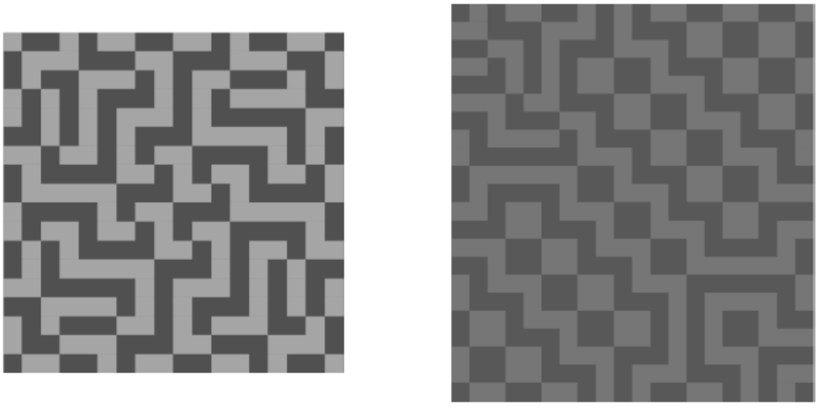


Figure 9. Two examples of Lunda-designs.

Lunda-designs have interesting symmetry properties, which often make them aesthetically attractive. For instance, in each row there are as many black unit squares as there are white unit squares. Also, in each column there are as many black unit squares as there are white unit squares. Furthermore, Lunda-designs have the following two characteristics:

1. Along the border each grid point always has exactly one black unit square associated with it (see figure 10);

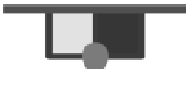


Figure 10. Symmetry situation along the border.

Of the four unit squares between two arbitrary (vertical or horizontal) neighboring grid points, two are always black (see figure 11).

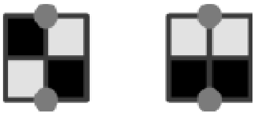


Figure 11. Symmetry situation inside the grid.

The concept of Lunda-design may be generalized in several ways. Circular and hexagonal Lunda-designs are some interesting possibilities (Gerdes, 1999; 2007b). The unit squares through which a mirror curve passes can be enumerated 'modulo *t*' instead of 'modulo 2', if *t* is a divisor of the total number of grid points. In this way *t*-valued matrices and *t*-Lunda-designs are created. Figure 12 gives two examples of 3-and 4-Lunda-designs.

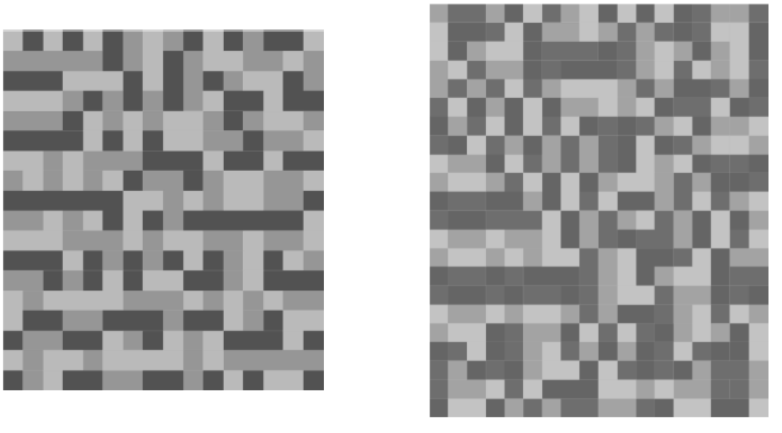


Figure 12. Examples of a 3-Lunda-design and a 4-Lunda-design.

## PATH OF DISCOVERY: FROM LUNDA-DESIGNS TO LIKI-DESIGNS AND SPECIAL MATRICES

In 2001, on the eve of the 4th anniversary of my daughter Likilisa, I started to analyze a particular class of 2-Lunda-designs. As these designs turned out to have some interesting properties, I called them Liki-designs. In the case of Liki-designs, the second property is substituted by the following stronger condition. Consider the four unit squares between two vertically or horizontally neighboring grid points. Two of them that belong to different rows and different columns always should have different colours (figure 13).

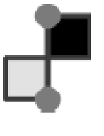
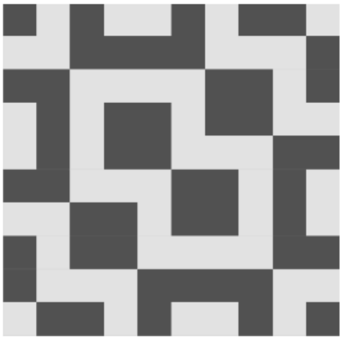


Figure 13. Situation inside the grid.

This property together with the border property (figure 10) implies that a square Liki-design and its associated Liki-matrix are composed of cycles of alternating black and white unit squares, or of cycles of alternating 1's and 0's, respectively. Figure 14 presents an example of a square Liki-design and its corresponding Liki-matrix L. The matrix has five {0,1}-cycles. A question that naturally emerges is what will happen with the powers of Liki-matrices.



| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 0 | 1 | 0 | 0 | 1 | 0 | 1 | 1 | 0 |
| 0 | 0 | 1 | 1 | 1 | 1 | 0 | 0 | 0 | 1 |
| 1 | 1 | 0 | 0 | 0 | 0 | 1 | 1 | 0 | 1 |
| 0 | 1 | 0 | 1 | 1 | 0 | 1 | 1 | 0 | 0 |
| 0 | 1 | 0 | 1 | 1 | 0 | 0 | 0 | 1 | 1 |
| 1 | 1 | 0 | 0 | 0 | 1 | 1 | 0 | 1 | 0 |
| 0 | 0 | 1 | 1 | 0 | 1 | 1 | 0 | 1 | 0 |
| 1 | 0 | 1 | 1 | 0 | 0 | 0 | 0 | 1 | 1 |
| 1 | 0 | 0 | 0 | 1 | 1 | 1 | 1 | 0 | 0 |
| 0 | 1 | 1 | 0 | 1 | 0 | 0 | 1 | 0 | 1 |

Figure 14. Example of a Liki-design (left) and its corresponding Liki-matrix (right).

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 5 | 2 | 2 | 1 | 1 | 3 | 3 | 3 | 3 | 2 |
| 2 | 5 | 1 | 2 | 3 | 1 | 3 | 3 | 2 | 3 |
| 2 | 1 | 5 | 3 | 2 | 3 | 1 | 2 | 3 | 3 |
| 1 | 2 | 3 | 5 | 3 | 2 | 2 | 1 | 3 | 3 |
| 1 | 3 | 2 | 3 | 5 | 2 | 2 | 3 | 1 | 3 |
| 3 | 1 | 3 | 2 | 2 | 5 | 3 | 2 | 3 | 1 |
| 3 | 3 | 1 | 2 | 2 | 3 | 5 | 3 | 2 | 1 |
| 3 | 3 | 2 | 1 | 3 | 2 | 3 | 5 | 1 | 2 |
| 3 | 2 | 3 | 3 | 1 | 3 | 2 | 1 | 5 | 2 |
| 2 | 3 | 3 | 3 | 3 | 1 | 1 | 2 | 2 | 5 |

$L^2$

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 16 | 9 | 15 | 10 | 9 | 16 | 12 | 13 | 15 | 10 |
| 9 | 10 | 16 | 16 | 15 | 13 | 9 | 10 | 12 | 15 |
| 15 | 16 | 9 | 9 | 12 | 10 | 15 | 16 | 10 | 13 |
| 10 | 16 | 9 | 13 | 16 | 10 | 15 | 15 | 9 | 12 |
| 9 | 15 | 12 | 16 | 15 | 9 | 10 | 10 | 13 | 16 |
| 16 | 13 | 10 | 10 | 9 | 15 | 16 | 12 | 15 | 9 |
| 12 | 9 | 15 | 15 | 10 | 16 | 13 | 9 | 16 | 10 |
| 13 | 10 | 16 | 15 | 10 | 12 | 9 | 9 | 16 | 15 |
| 15 | 12 | 10 | 9 | 13 | 15 | 16 | 16 | 10 | 9 |
| 10 | 15 | 13 | 12 | 16 | 9 | 10 | 15 | 9 | 16 |

$L^3$

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 75 | 60 | 60 | 53 | 53 | 68 | 68 | 66 | 66 | 56 |
| 60 | 75 | 53 | 60 | 68 | 53 | 66 | 68 | 56 | 66 |
| 60 | 53 | 75 | 68 | 60 | 66 | 53 | 56 | 68 | 66 |
| 53 | 60 | 68 | 75 | 66 | 60 | 56 | 53 | 66 | 68 |
| 53 | 68 | 60 | 66 | 75 | 56 | 60 | 66 | 53 | 68 |
| 68 | 53 | 66 | 60 | 56 | 75 | 66 | 60 | 68 | 53 |
| 68 | 66 | 53 | 56 | 60 | 66 | 75 | 68 | 60 | 53 |
| 66 | 68 | 56 | 53 | 66 | 60 | 68 | 75 | 53 | 60 |
| 66 | 56 | 68 | 66 | 53 | 68 | 60 | 53 | 75 | 60 |
| 56 | 66 | 66 | 68 | 68 | 53 | 53 | 60 | 60 | 75 |

$L^4$

Figure 15. Several powers of Liki-matrix L.

Figure 15 displays the first powers of Liki-matrix L. The third power has the same cycle structure as the first power: the first cycle of the third power is composed of alternating 16's and 9's, the second cycle of alternating 15's and 10's, etc. The even powers do not have the same cycle structure. Their diagonals are constant and they present other cycles, like the cycle of 2's in the second power. Figure 16 compares the cycle structures of the odd and even powers of the Liki-matrix L.

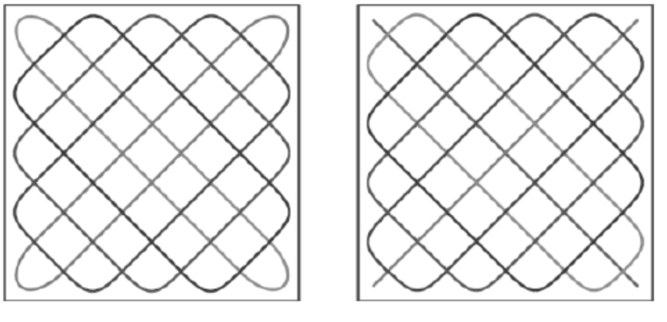


Figure 16. Cycle structures of odd and even powers.

The powers of a Liki-matrix, like the matrices $L_2$, $L_3$, etc., are themselves not Liki-matrices. Nevertheless, they display cycle structures. Let us call them cycle *matrices*. As the numbers on the cycles on the odd powers are alternating, we may say that these cycle matrices have period 2. As the numbers on the cycles on the even powers are constant, we say that these cycle matrices have period 1.

$$A = \begin{bmatrix} -1 & 3 & -4 & 6 & -2 & 0 \\ 3 & 6 & -1 & 0 & -4 & -2 \\ -4 & -1 & -2 & 3 & 0 & 6 \\ 6 & 0 & 3 & -2 & -1 & -4 \\ -2 & -4 & 0 & -1 & 6 & 3 \\ 0 & -2 & 6 & -4 & 3 & -1 \end{bmatrix} \qquad B = \begin{bmatrix} 4 & -2 & 3 & 5 & -3 & 2 \\ -2 & 5 & 4 & 2 & 3 & -3 \\ 3 & 4 & -3 & -2 & 2 & 5 \\ 5 & 2 & -2 & -3 & 4 & 3 \\ -3 & 3 & 2 & 4 & 5 & -2 \\ 2 & -3 & 5 & 3 & -2 & 4 \end{bmatrix}$$

$$AB = \begin{bmatrix} 14 & 7 & 5 & -17 & 18 & -9 \\ 5 & 14 & 18 & 7 & -9 & -17 \\ 7 & -17 & 14 & -9 & 5 & 18 \\ 18 & 5 & -9 & 14 & -17 & 7 \\ -17 & -9 & 7 & 18 & 14 & 5 \\ -9 & 18 & -17 & 5 & 7 & 14 \end{bmatrix} \qquad BA = \begin{bmatrix} 14 & 5 & 7 & 18 & -17 & -9 \\ 7 & 14 & -17 & 5 & -9 & 18 \\ 5 & 18 & 14 & -9 & 7 & -17 \\ -17 & 7 & -9 & 14 & 18 & 5 \\ 18 & -9 & 5 & -17 & 14 & 7 \\ -9 & -17 & 18 & 7 & 5 & 14 \end{bmatrix}$$

Figure 17. Cycle matrices A and B and their products.

Using the cycle structures, we may introduce the concept of a cycle matrix of period 2, independent of the context of Liki-designs in which I discovered the concept. Figure 17 displays two cycle matrices, A and B, of dimension 6x6, having period 2. Both A and B have the same cycle structure as the design at Laval University (figure 1) and the basic design for the Cokwe tortoise (figures 2 and 3). The products AB and BA have a different cycle structure, similar to the second cycle structure in figure 16. Compare matrices AB and BA. Do you note something remarkable?

Later in my lecture, I will return to cycle matrices in a very different context. At this moment, I would like to underscore the newness of mathematical ideas arising from the analysis of the old Cokwe *sona* tradition and the multiple relationships of these ideas with other areas of mathematics. This reflects the profoundness and the mathematical fertility of the ideas of the Cokwe master drawers.

Challenge for reflection: Can African and other cultures become a source of inspiration for the development of new mathematical ideas? Example: 'Sona' designs from Angola. Other examples?

After the elaborated example of the inspiration of *sona* geometry for developing new mathematical ideas, let me present a brief introductory overview of mathematics and mathematicians in African history and cultures, followed by some historic examples.

## MATHEMATICS AND MATHEMATICIANS FROM AFRICA

From the earliest times onwards, humans in Africa and elsewhere have created and developed mathematical ideas. Mathematical reflections from Ancient Egypt, from Hellenistic Egypt, from Islamic Egypt and from the Maghreb during the Middle Ages found their way to Europe and have been contributing to the development of 'international' mathematics (Djebbar, 2001; 2005). Hundreds of mathematical manuscripts—written in Arabic and in various African languages—from Timbuktu, in today's Mali, remain to be analyzed. These should lift the veil from some of the mathematical connections between Africa South of the Sahara and the north of the continent (Djebbar & Moyon, 2011). The astronomer-mathematician Muhammed ibn Muhammed (c. 1740) from Katsina, in today's Nigeria, was well-known in Egypt and the Middle East. Thomas Fuller (1710-1790), brought in 1724 from West Africa as a slave to North America, became famous in the 'New World' for his mental calculations (Fauvel & Gerdes, 1990). Some *sona* geometrical knowledge has survived until the beginnings of the 20th century in the Mississippi area among people of African descent. During the second half of the 20th century, the African continent produced thousands of PhDs, of whom several hundreds have been working as researchers in Europe and North America [see the catalogue (Gerdes, 2007c)].

For an introductory overview of mathematical ideas in the history of Africa South of the Sahara, the reader may consult (Gerdes, 1994b) or the classic book (Zaslavsky, 1973). The study (Gerdes & Djebbar, 2007a, b) presents an annotated bibliography of mathematics in African history and cultures, containing over two thousand entries and indices by region, ethnic or linguistic group, mathematician, and mathematical topic. This study is one of the outcomes of the activities of the AMU Commission on the History of Mathematics (AMUCHMA), created in 1986 by the African Mathematical Union (AMU), to do research and disseminate research findings through lectures, conferences, and publications. Most of the newsletters that AMUCHMA produced in English are available at the following webpage: www.math.buffalo.edu/mad/AMU/amuchma_online.html

Recently, the thirty-seven newsletters were reprinted in two book volumes (Gerdes & Djebbar, 2011).

## HISTORIC EXAMPLE: CRADLE OF MATHEMATICS

Very early on, humans living in Africa started to display an interest in symmetry, in constructing parallel lines, rectangles, and triangles, as attested by several objects made around 70,000-80,000 BC, and found during the last decade during excavations at the Blombos cave in the Eastern Cape region of South Africa. A counting rod, found in a cave in the Lebombo Mountains in the border area of South Africa, Swaziland, and Mozambique, dates from 33,000 BC. Better known are the bones found near Ishango in the East of today's Democratic Republic of Congo. The bones date from 20,000 BC. Figure 18 schematically displays one face of the first Ishango bone. The distribution of the number of engravings made into it give the impression that its marker was engaged, in one way or another, in duplication.

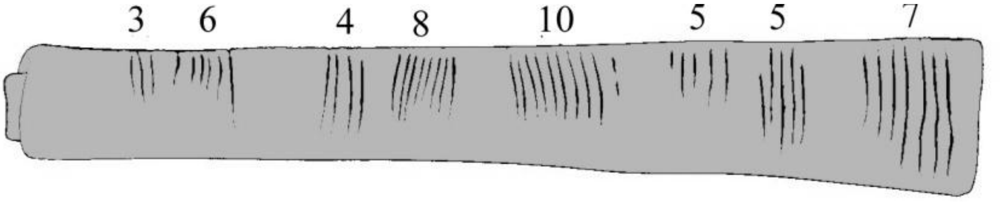



Figure 18. One face of the first Ishango bone.

Looking at the numbers of engravings in the first row of the second face (figure 19), we see four odd numbers between 10 and 20; 15 is left out. Do they represent only odd numbers, or also prime numbers?

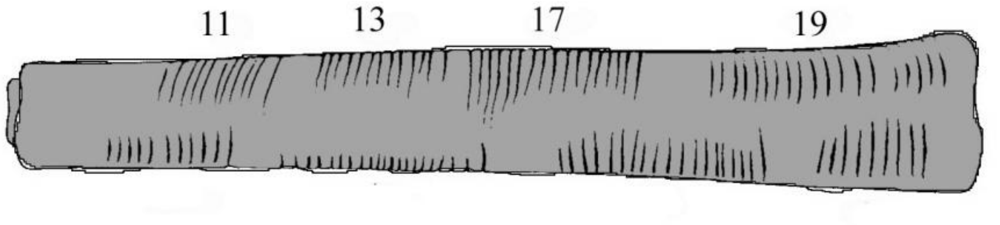



Figure 19. Quantities along the top side of the second face of the first Ishango bone.

Do the numbers at the lower row 11, 21, 19, and 9, that is, 10 + 1, 20 + 1, 20 – 1, and 10 – 1, reveal some special interest in multiples of 10 (figure 20)? Did the maker use a spoken numeration system with base ten?

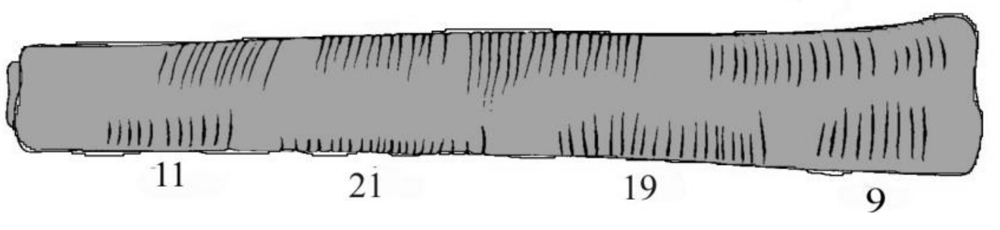



Figure 20. Quantities along the bottom side of the second face of the first Ishango bone.

Comparing the numbers in the two rows, one sees that the sums of both are equal:

$$11 + 13 + 17 + 19 = \mathbf{60} = 11 + 21 + 19 + 9$$

Does this reflect some early interest in the number 60? The Ishango bones have been the object of diverse attempts at interpretation ever since they were found in 1957. A special international conference dedicated to them took place in Brussels, entitled "*Ishango, 22000 and 50 Years Later: The Cradle of Mathematics?*" (February 28 – March 2, 2007) (cf. Huylebrouck, 2008). Considered a symbol of the birth of science in the world, a 7-meter high replica of the small, first Ishango bone was unveiled in 2010 as a statue in front of the Royal Theatre of Money in Brussels (figure 21).


Figure 21. Unveiling of the replica of the first Ishango bone in Brussels (Belgium).

## HISTORIC EXAMPLE: CONCEPTUALIZATION OF MATHEMATICS

The best-known mathematical text from Ancient Egypt is a papyrus written about 1,650 BC by the scribe Ahmes or Ahmose. It may be a copy of a text a couple of hundred years older. Unfortunately, the papyrus is often called the 'Rhind papyrus' after its 19th century buyer. In his book *Egyptian Geometry: Contribution of Ancient Africa to World Mathematics*, Théophile Obenga (1995) underlines that Ahmes' text is much more than a book of exercises with solutions. In particular, he draws our attention to the title of the papyrus: *Correct method of investigation of Nature in order to understand all that exists, each mystery, [and] all secrets*. Is this not a description or conceptualization of what mathematics is about?

Ahmes' title contains an early definition of mathematics. Even today it may stimulate a fruitful debate among mathematicians, philosophers, and mathematics educators about what is (the purpose of) mathematics (education).

Over the centuries, many other mathematicians in Egypt have contributed to the development of mathematics and have reflected about the nature of mathematics, like Euclid (4th century BC), Heron (1st – 2nd century), Diophant (3rd century), Theon and his daughter Hypathia (370-415), Abu Kamil (850-930), and Al-Haitham (965-1039), to name just a few mathematicians from the classic and medieval periods.

## HISTORIC EXAMPLE: INTRODUCTION OF SYMBOLS INTO MATHEMATICS

The Maghreb (North-West Africa) played an important role in the internationalization of mathematics in medieval times. One contribution with a lasting influence on mathematics and mathematics education is the invention and dissemination of diverse symbols since the 12th century. As an example, figure 22 presents a page of a manuscript of that time.

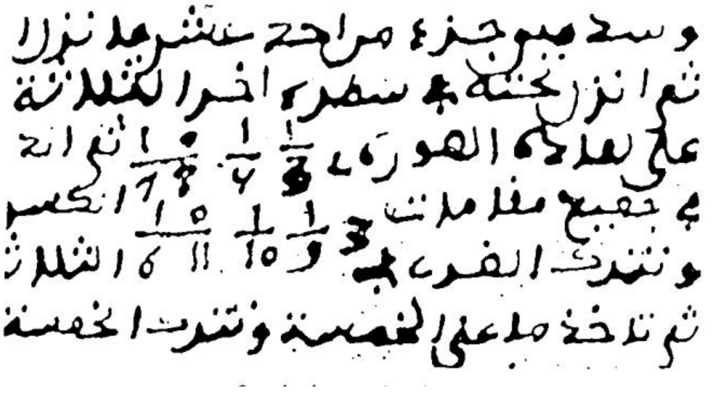


Figure 22. Page of a 12th century manuscript from the Maghreb.
(Reproduced with permission from Djebbar (2005), p. 93)

This text, written from the right to the left in Arabic, is at this moment the oldest text in which today's well-known notation for fractions appears that children all over the world learn to use. Symbols for arithmetical operations and extraction of roots were introduced about the same time. At present, it is not known for sure who the author of the text fragment in figure 22 was, or who introduced other symbols. It may have been the mathematician and poet Ibn al-Yasamin (d. 1204)—'son of the jasmine flower'. His mother was a black slave from south of the Sahara, freed in agreement with the legal customs of the day after having given birth, the father being a Berber. Also as a mathematics educator, Ibn al-Yasamin has had a long-lasting influence in the Maghreb: for centuries his mathematical poems were used to teach, learn and memorize the basics of arithmetic.

As in the next part of the lecture, the Rule of Signs

$$
\begin{aligned}
(-)(-) &= + \\
(-)(+) &= - \\
(+)(-) &= - \\
(+)(+) &= +
\end{aligned}
$$

will be referred to, it may be interesting to note here already that the Maghrebian geometer Ibn Al-Banna (13th C.) presented a proof of the Rule of Signs in one of his works.

*Challenges for reflection:*

- What is 'positive'? What is 'negative'?
- Is the design in Figures 1 and 2 positive or negative? Why?

## HISTORIC EXAMPLE: INTERWEAVING ART AND MATHEMATICS

In African cultures, mathematical and artistic ideas are frequently interwoven. I open the book, *Geometry from Africa* (Gerdes, 1999), with the following sentence: "The peoples of Africa South of the Sahara desert constitute a vibrant cultural mosaic, extremely rich in its diversity." Among the peoples of the sub-Saharan region, interest in imagining, creating and exploring forms and shapes has blossomed in diverse cultural and social contexts with such an intensity that with reason, to paraphrase Claudia Zaslavsky's *Africa Counts* (1973), it may be said that "Africa Geometrizes".

The books, Geometry from Africa (Gerdes, 1999), African Fractals, Modern Computing and Indigenous Design (Eglash, 1998), Women, Art and Geometry in Southern Africa (Gerdes, 1998), and African Basketry: A Gallery of Twill-Plaited Designs and Patterns (Gerdes, 2007d), present regional and thematic overviews of geometrical ideas and practices in African cultures. Case studies of geometrical exploration in specific African cultures are presented in the books, Sona Geometry from Angola: Mathematics of an African Tradition (Gerdes, 2006), Sipatsi: Basketry and Geometry in the Tonga Culture of Inhambane (Mozambique, Africa) (Gerdes, 2009), Otthava: Making Baskets and Doing Geometry in the Makhuwa Culture in the Northeast of Mozambique (Gerdes, 2010a; 2012c), and Tinhlèlò, Interweaving Art and Mathematics: Colourful Circular Basket Trays from the South of Mozambique (Gerdes, 2010b).

As a historic and current example, I will present the decorated mats woven by Makwe women in the extreme northeast of Mozambique, near the border with Tanzania (Gerdes, 2007d). Figure 23 presents the Makwe master weaver, Idaia Amade, with some mats and bags during an exhibition in the Mozambican capital, Maputo.

Figure 23. Master weaver Idaia Amade.

For centuries, Makwe women have been weaving their famous *luanvi* mats. In the 18th century these mats were among the most important products traded at Mozambique Island. The mats are made from brightly dyed palm fiber by sewing long plaited bands together. Mono-colour plain bands alternate with black-and-white ornamental bands. The (central parts of the) ornamental bands called *mpango* present all seven possible symmetry classes. Figure 24 presents one example of each class, with the international notation of each symmetry class indicated within brackets.

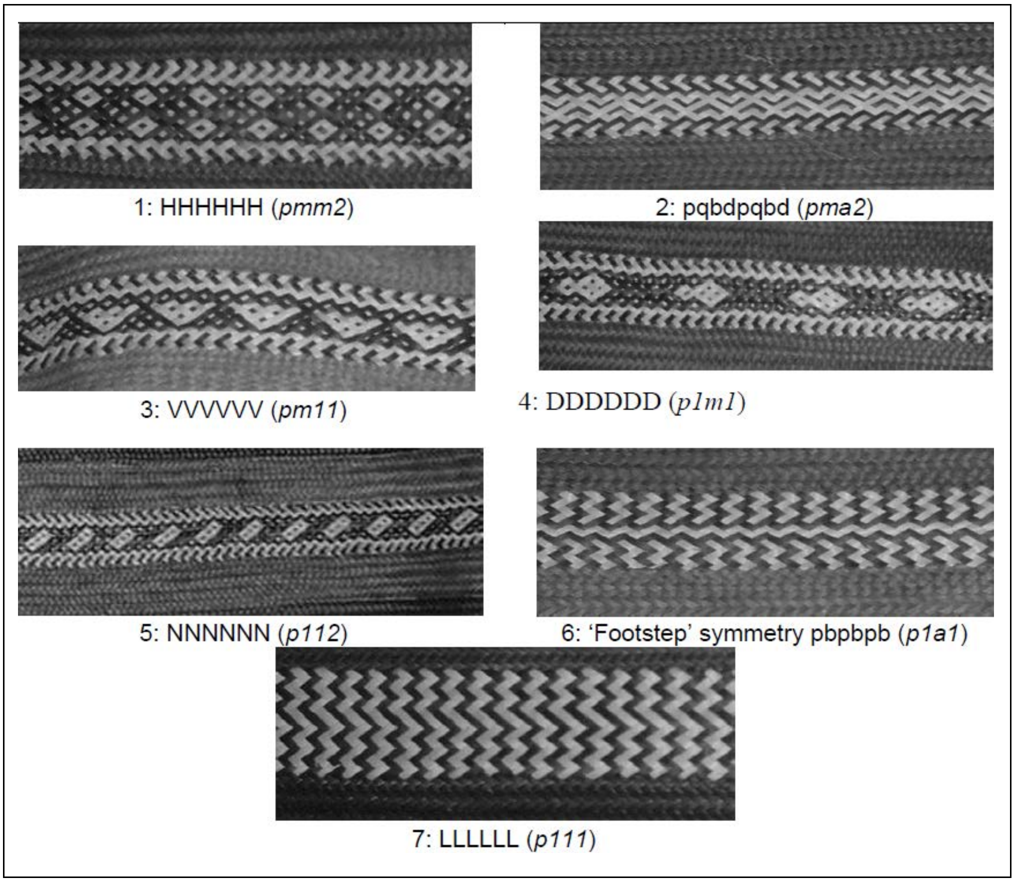



Figure 24. Examples of the seven symmetry classes.

In plaited and twilled basketry, front and backside of a woven band or mat display mostly the same image if they are made with black strips in one direction and with white strips in the other direction: only the colours are interchanged. In other words, one side is the 'photographic negative' of the other side. In Makwe weaving, the situation is different. Figure 25 presents an example: it displays both sides of a *mpango*.

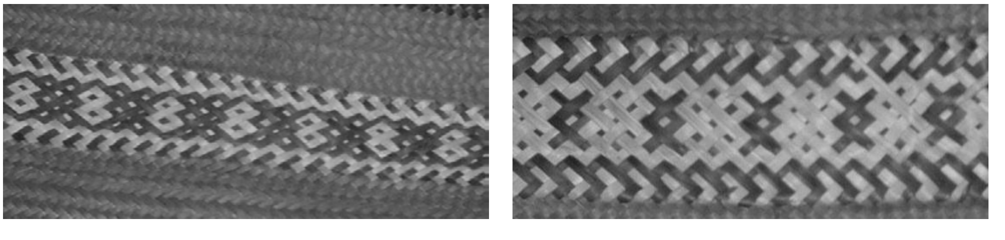


Figure 25. Front and back side of the same ornamental band.

The Makwe use a particular inversion of colour that is distinct from the photographic colour inversion. The black and white strands make angles of 45º with the borders of a decorative band. As, in both weaving directions, light coloured 'white' strands (0) and dark coloured 'black' strands (1) alternate (weaving code 01), we have the following situations:

1. Where a dark strand crosses with a dark strand, we see a dark unit square on both sides;
2. Where a natural strand crosses with a natural strand, we see a naturally coloured unit square on both sides;
3. Where a dark strand crosses with a natural strand, we have on one side a dark unit square but on the other side a natural unit square; the colours have been reversed.

As a consequence, under the Makwe colour inversion, half of the unit squares have the same colour on both sides of the mat (see the coloured unit squares in figure 26), whereas the other half of the unit squares (white in figure 26) have opposite colours on either side of the mat.

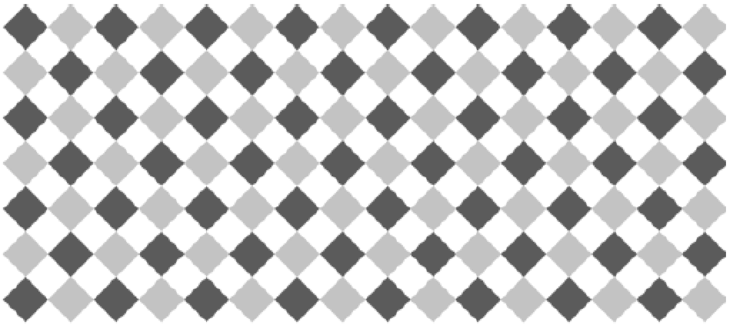


Figure 26. The coloured unit squares are invariant.

The design on a decorated band depends on the weaving algorithm used by the mat maker. Although the designs on both sides are normally distinct, they present the same symmetry. The weavers have invented various patterns with additional properties. For instance, both sides of the decorative band in figure 27 display the same design but the colours are interchanged.

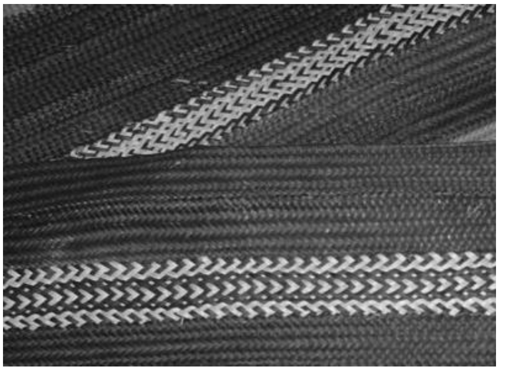


Figure 27. Special pattern with photographic colour inversion.

Makwe women have also explored weaving codes different from the 01-code, that is, they have explored other ways to alternate the colours in both weaving directions. For instance, they use the 011-code to produce the decorative band in figure 28: each time one white strand is followed by two black strands. Figure 29 presents an example of the use of the 00111-code.

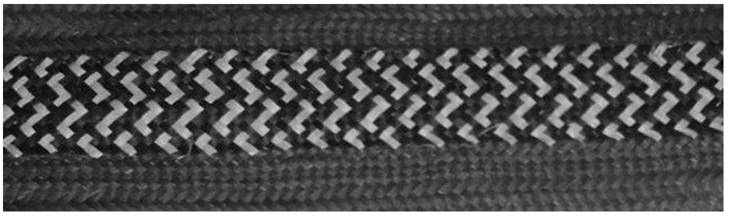


Figure 28. Example of the application of the 011-code.

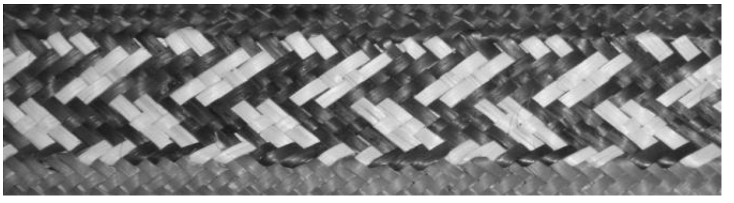
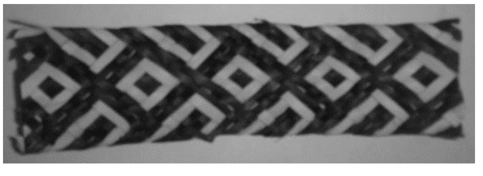


Figure 29. Example of the application of the 00111-code.

A very interesting case of the use of the 011-code is the chicken's eye pattern (Gerdes, 2013b). Figure 30 presents the front and back side of a piece of a band decorated with the chicken's eye design.



a: Front side

b: Back side

Figure 30. Band with the chicken's eye design.

The pattern has period six: 011011. On the backside the same pattern appears as on the front side, however its orientation is inversed and it is slightly displaced. How could the inventor have imagined such an exceptional design? It is surely not the result of experimentation, as there are too many possibilities. The inventor, several centuries ago, had consciously constructed the weaving texture using some kind of careful mathematical analysis. Calculations and geometry-symmetry considerations were involved. Figure 31 displays the underlying number frieze of the weaving texture: a place marked by a 1 means that the descending strand passes over the mounting strand; a place marked by a 0 is one where the descending strand passes under the mounting strand. The number frieze has vertical axes of symmetry and a horizontal anti-symmetry axis (inversion of 0's and 1's not belonging to the horizontal axis). This chicken's eye design may lead to the study of new types of number friezes (cf. Gerdes, 2013b).

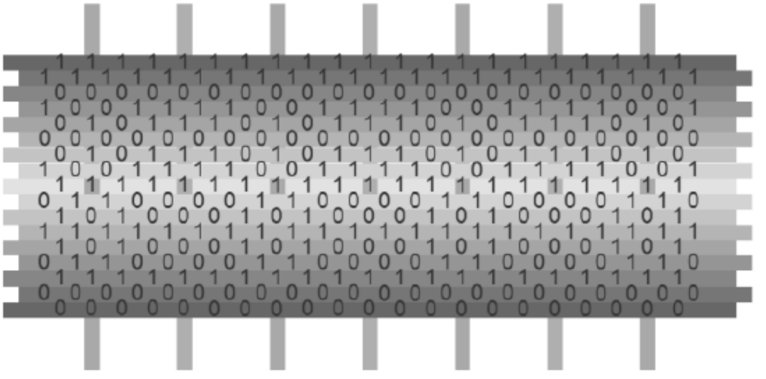
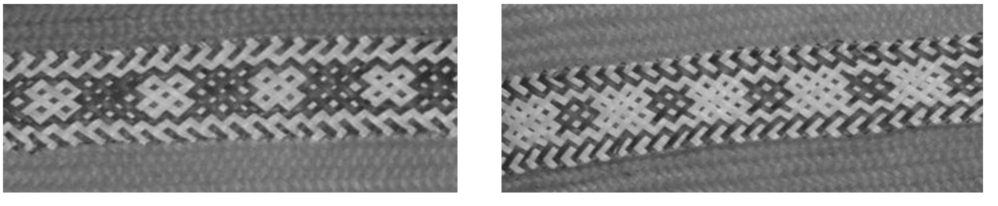


Figure 31. Underlying number frieze.

*Challenges for reflection:*

- Who does mathematics? Who invents mathematics? What is 'mathematical thinking'? Who or which culture defines it?
- Can African and other cultures serve as a source of inspiration for the development of new mathematical ideas?

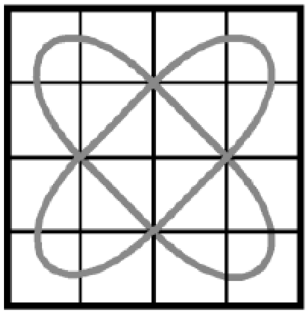
Let me now return to the code 01 and explore a particular case of it. Figure 32 presents the front and back side of a Makwe design called the 'footprints of a lion'.



a: Front side b: Back side

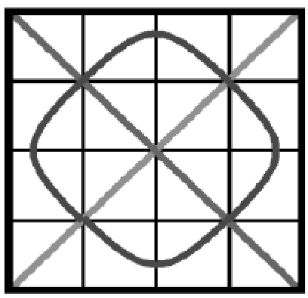
Figure 32. 'Footprints of a lion' design.

The designs on the front and back side are similar to the cycle structures we met earlier on (see figure 16). The design on the front side corresponds to the cycle structure in figure 33: we may attribute the number 1 to the unit squares through which the first cycle passes and the number 2 to those through which the second cycle passes. Analogously, the design on the back side corresponds to the cycle structure in figure 34, composed of two straight segments (numbered 3 and 5) and one cycle (numbered 4). In this way, we constructed two cycle matrices of dimensions 4x4 and of period 1.



| | | | |
|---|---|---|---|
| 1 | 1 | 2 | 2 |
| 1 | 2 | 1 | 2 |
| 2 | 1 | 2 | 1 |
| 2 | 2 | 1 | 1 |

Figure 33. Front side design and corresponding matrix.



| | | | |
|---|---|---|---|
| 3 | 4 | 4 | 5 |
| 4 | 3 | 5 | 4 |
| 4 | 5 | 3 | 4 |
| 5 | 4 | 4 | 3 |

Figure 34. Back side design and corresponding matrix.

Let us multiply these two cycle matrices and see if something interesting happens. Figure 35 presents an example of the multiplication of two matrices with the first cycle structure; the result is a matrix with the second structure. Figure 36 presents an example of the multiplication of two matrices with the second cycle structure; the result is once more a matrix with the second structure. Figure 37 presents an example of the multiplication of a matrix with the second cycle structure with one with the first cycle structure; this time, however, the result is a matrix with the first structure.

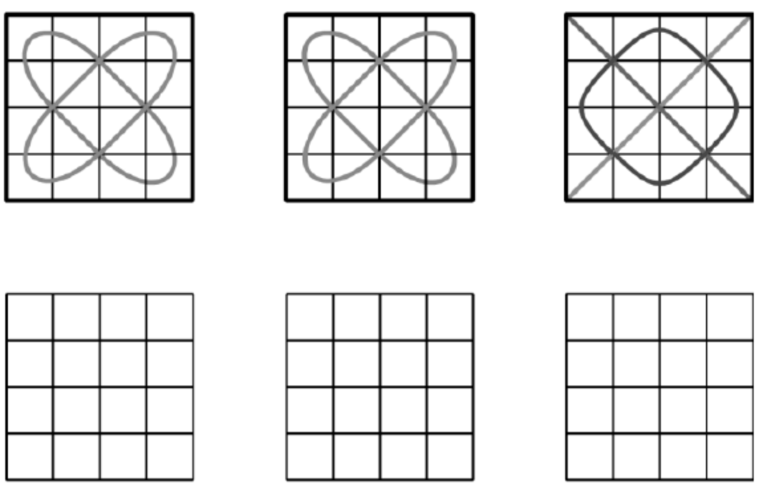


Figure 35. Example of a multiplication.

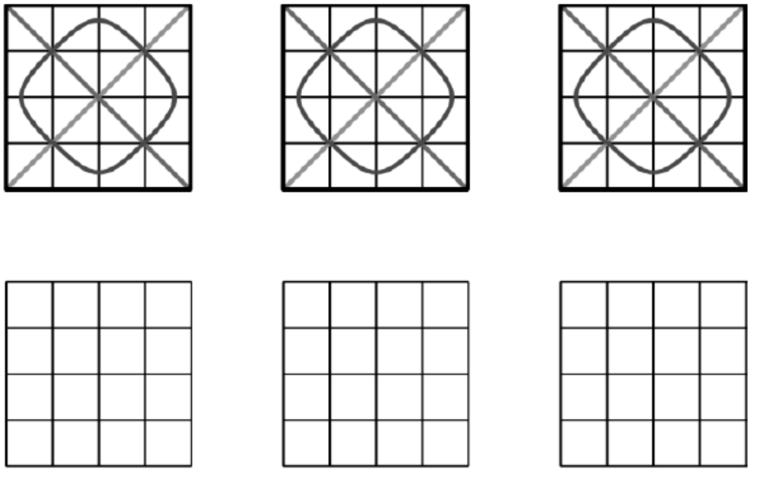


Figure 36. Example of a multiplication.

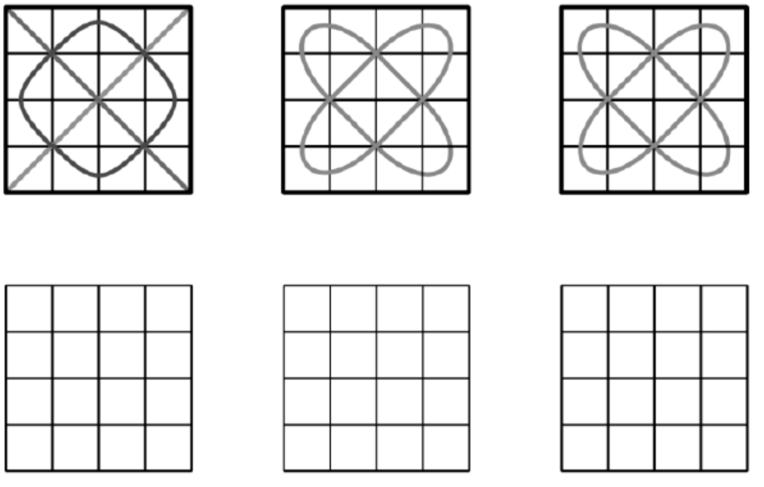


Figure 37. Example of a multiplication.

These results hold in general. For matrices of dimension 4x4 of period 1, having the first cycle structure (figure 33) or the second cycle structure (figure 34), we have the multiplication table shown in figure 38.

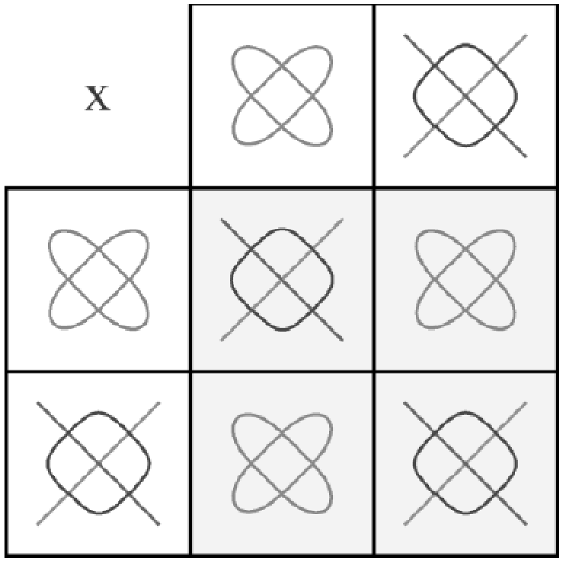



Figure 38. Multiplication table.

This multiplication table is similar to the aforementioned Rule of Signs for the multiplication of negative and positive numbers. The same holds for cycle matrices of any dimensions and any (admissible) period. Therefore, it is well justified to call designs and matrices with the first cycle structure 'negative', and designs and matrices with the second cycle structure 'positive'. The odd powers of Liki-matrix L (figures 14 and 15) are negative, whereas the even powers are positive cycle matrices of dimensions 10x10 and period 2. Matrices A and B in figure 17 are negative, while AB and BA are positive cycle matrices of dimensions 6x6 and period 2. The Laval University design in figure 1 and the Cokwe design in figure 2 are 'negative'!

Cycle matrices with their corresponding geometric designs have interesting properties, and may be applied in and outside mathematics. They are visually beautiful, like Lunda- and Liki-designs. They may be used as an attractive introduction to matrix theory, as I explain in my book, *Adventures in the World of Matrices* (Gerdes, 2008), written for high school students and undergraduates. Several proofs in the theory of cycle matrices may be given with geometric resources. Computer software may be explored to find properties of cycle matrices. From cycle matrices onwards it is possible to discover other types of matrices like helix and cylinder matrices (Gerdes, 2002a; 2002b).

## CONCLUDING REMARKS

There exists an immense variety of 'Old and New Mathematical Ideas from Africa'. In my address I presented only a small selection, influenced by my personal research experience. During millennia, Africans have been developing mathematical ideas in diverse cultural contexts. The contributions of African professional mathematicians and mathematical practitioners, like artists, musicians, drawing masters, storytellers, and mat weavers, may serve as a source of inspiration for new generations.

Mathematical ideas from Africa may be explored in mathematics education at all levels. Traditions with mathematical ingredients, like the *sona* of the Cokwe drawing masters-educators and the *mpango* of the Makwe mat weavers, may serve, as shown, as a source of inspiration for the invention of attractive new mathematical ideas and new educational explorations.

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# Working Groups

# Groupes de travail

# THE ROLE OF PROOF IN POST-SECONDARY EDUCATION

William Byers, *Concordia University*
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Ce groupe de travail illustre à merveille une partie de ce que l'on cherche à accomplir au GCEDM. Le groupe de travail était à la fois riche et diversifié, et David Henderson, conférencier plénier la même année, s'était joint au groupe. Il a su participer pleinement et respectueusement, et sa présence a largement contribué à notre travail. Nous avons abordé des questions touchant les preuves et leur place dans nos classes, à l'université. Je me souviens bien de nos discussions, qui ont marqué pour moi le début d'un questionnement profond sur ce qu'est l'enseignement, sur ce que constitue une preuve, et sur ce qui peut constituer la base d'un raisonnement correct ou être accepté sans preuve (axiomes).

This is a great example of a working group, illustrating some of what we try to achieve at CMESG. The working group was rich and diverse, and David Henderson, a plenary speaker that year, joined the working group. His presence added much to our discussion, and he took part respectfully. We discussed issues surrounding proofs and their place in our classrooms, at University. I still remember many things which were said, because they marked the beginning of a profound questioning of what it means to teach, of what constitutes a proof, of what constitutes acceptable or commonly accepted facts (axioms).

## PARTICIPANTS

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## INTRODUCTION

Before the Working Group met the following questions were distributed in order to facilitate some thoughts on the role of proof in post-secondary mathematics.

1. What is the point of rigour in mathematics?
2. What is the connection between proof and understanding?
3. Can you do mathematics without introducing the idea of proof?
4. Is it necessary to introduce *proof as* a distinct topic or course within the curriculum? If so, what should be the mathematical context: abstract or linear algebra, a first course in analysis, or a course on the structure of number systems?
5. Should mathematics departments have streams (or majors) in which mastery of *proof* is minimized or omitted?

[11] with the help of Joel Hillel, *Concordia University*.

## DISCUSSION

The session began with brief descriptions of two programs, one at Concordia University and the other at Simon Fraser University, which introduced the notion of proof in a separate course. A complete description of Concordia's program appears in Byers and Hillel (1994). The Working Group decided to divide into two subgroups - one concerning proofs and proving for pre-service high school teachers, and the other for students taking calculus, linear algebra, etc. The two groups met separately to discuss difficult questions and then shared their thoughts with each other at the end of the sessions.

We agreed that there should be a shift of emphasis from *proofs* to *the act of proving.* The act of proving refers to "habits of mind" (Cuoco, Goldenberg, and Mark, 1995) which involve questioning, anticipating, asking "what happens if', etc. More succinctly, proving has to do with the business of "inquiry with confirmation". Later, the notion of 'proving' was refined to include deduction. An act of proving is an inquiry with confirmation by means of deduction.

This line of discussion led to a flood of questions. Are such habits of mind unique to mathematics? Does writing a term paper in sociology or history not constitute an inquiry with confirmation? What is the difference between confirmation in mathematics and confirmation in, for example, history or sociology?

How can we tell whether a particular action of proving really involves deduction on the part of the person showing a proof? Could it just be a rote reproduction of an argument (even if logically correct)? Is the requirement of being able to communicate a result to another person a sufficient indicator of deductive reasoning?

Having shifted the focus from the originally proposed set of questions, the Working Group decided that they would rather consider the following questions.

1. What kind of results should we prove?
2. What would be considered as a convincing argument in a given context?
3. What is the difference between written proof and the students' understanding?
4. Are there technical differences between the terms validation, justification, confirmation, verification, proving, and convincing?

### WHAT KIND OF RESULTS SHOULD WE PROVE?

The first question needed clarification as to who is *we. We* could refer to university mathematics teachers and teacher trainers, or to their students (mathematics majors, or to pre- and in-service teachers) or to elementary and high school students. The group realized that at certain times in our discussions, we were shifting from one set of "we" to another. Clearly, what we should prove depended on the people involved.

The proof of the Chain Rule in elementary calculus proved to be an excellent example. Clearly the audience is calculus students. Why do we bother with such a proof and do we need to make it rigorous? What's wrong with the naive version of the proof using $\Delta y/\Delta x$? We spent time "unpacking" the proof and pointing to some of its features that are worth highlighting. What emerged was a consensus that the question is not whether the Chain Rule should be proved but how to go about it. The proof can be a focus of discussions, experimentation and group activities that can last several lectures.

## ARE THERE TECHNICAL DIFFERENCES BETWEEN THE TERMS VALIDATION, JUSTIFICATION, CONFIRMATION, VERIFICATION, PROVING, AND CONVINCING?

We didn't get very far with the fourth question - there was an attempt to distinguish the terms *conviction* and *justification* as they are used in philosophy of science, but this didn't help to clarify the issue. After some fruitless discussion, the question was dropped.

## WHAT WOULD BE CONSIDERED AS A CONVINCING ARGUMENT IN A GIVEN CONTEXT?

Question 2 was more central to the discussions. It was noticed that there is a social side to proving, which requires a shared repertory, and a personal side, which does not. Both of these aspects must be considered when discussing the act of proving for prospective teachers.

For high school teachers, one of the most important attitudes to foster is that of being curious and questioning. Therefore, high school teachers should have a strong background in mathematics so that they can *play* with the mathematical proofs in a *didactical* way. They should be able to give many different proofs of a result so that they are able to understand, justify, and explain high school material. They need to understand proofs, to be able to explain a result without betraying the proof.

Therefore, we should have prospective high school teachers approach proof in the manner of Lakatos (1976). Namely, have them attempt proofs of results and then enable them to gradually unveil the reasons why the initial attempts were incorrect. The aim of this approach is that teachers can detect where their proofs are not valid, can construct proofs on their own, can validate their reasoning or intuition, so that their students can validate or show as invalid, the results they obtain.

What, then, constitutes a convincing argument? The group decided that a convincing argument or proof is one such that the person should be able to produce a similar proof and explain it to others. As an example, consider the standard proof that $\sqrt{2}$ is irrational. Given a proof of this fact, the students should be able to produce a proof that $\sqrt{3}$ is irrational.

What if an algorithm is involved? One thought was that the person convinced should be able to explain it to someone else who would then be able to use the algorithm correctly. This view may still beg the question as to whether any deduction is involved in the explanation, or is it merely a repetition of something seen. We were left with the question as to whether this constituted an "inquiry with confirmation by means of deduction."

The discussion then moved to presenting "proofs by example" instead of giving very formal proofs. It was felt that in some cases, using a well-chosen example is more effective in convincing or in conveying a result or a technique than giving a formal, decontextualised proof. It was pointed out that some mathematicians who were known for their formal and pedantic writing of mathematics, behaved very differently in their lectures and when they supervised graduate students. Proofs by example raise the issue as to whether students can extract the generalized features of such proofs or whether they focus too much on the specific example.

We asked the question – What would be a convincing argument in the context of mathematics for teachers? The answer depends on the teacher's background. What is "obvious?" What is the "acquired knowledge?"

As noted earlier, the act of proving has both a social side and a personal side and the former relies on a shared repertory of things which are taken for granted which, in tum, determines if a proof is convincing or not. This raised the question of whether software tools such as Cabrie and Logo bring about a change of what constitutes the shared repertory. For example, consider a Logo procedure for generating three consecutive integers and then checking that their sum is divisible by 3. Is this a precursor to a formal proof? These thoughts pose more general questions. Are commonly accepted facts changing with Logo? With Cabrie? Is what we consider to be *obvious* changing?

The Working Group recognized that many people *understand* a concept or proof formally before the concept or proof is properly *understood* or internalized. The discussion then turned to *formal* proofs. It was felt that arguments that are *formal* may be easier to accept by students because of their perception that these proofs are given by someone in authority.

We closed our work by trying to decide what constituted a convincing argument. Two examples highlighted the discussion. The first was the illustrative proof that the sum of two odd numbers was even (see C. Hoyles, Figure 1, in this monograph). We all agreed that, given the appropriate audience, this would be a convincing argument.

There was a more heated discussion as to whether the following student's *proof* that the sum of the angles in a triangle is 180 degrees was acceptable as a convincing explanation. The proof relied on tessellation of the plane by triangles (Figure 1).

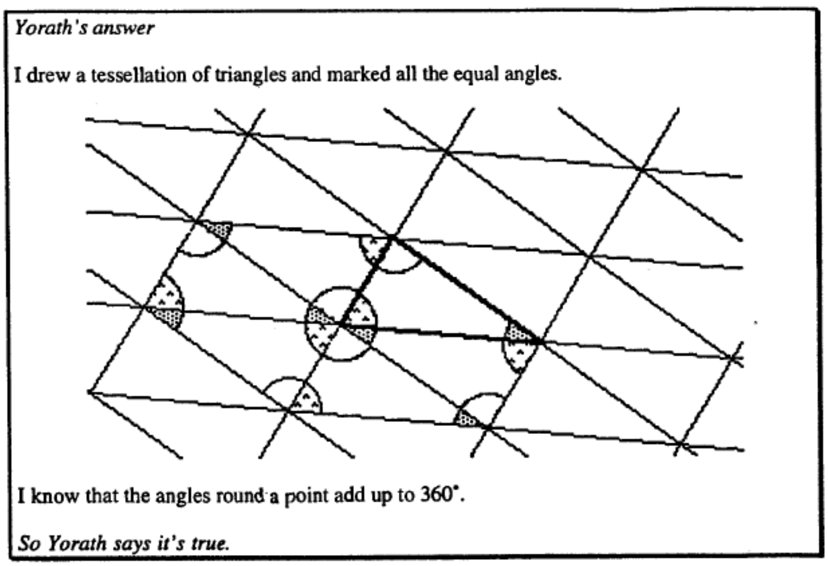


Figure 1: Yorath's answer.

Some found it novel and convincing, others pointed out the circularity of the argument (being able to tessellate by triangles presupposes the result) and others argued that we have to suspend judgment until we know what the student's starting points were.

WHAT IS THE DIFFERENCE BETWEEN WRITTEN PROOF AND THE STUDENTS' UNDERSTANDING?

Time didn't permit our discussing this question.

## CONCLUSION

The separation of the Working Group into two subgroups, one for proofs and proving for preservice high school teachers, and the other for students taking calculus, linear algebra, etc., seemed natural enough. However, many of us worked with both kinds of students and had some difficulty choosing one of the subgroups to the exclusion of the other. More interesting, the two groups were in agreement almost all of the time, and even chose some of the same illustrative examples when they met separately. Finally, it was noted that many of the issues about proofs and proving that were discussed in the Working Group are hardly ever discussed with mathematics students or student-teachers, thus leaving a serious gap in their mathematics education.

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# FROM THEORY TO OBSERVATIONAL DATA (AND BACK AGAIN)

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Voici un bon exemple de groupe de travail qui permet à des chercheurs en didactique des mathématiques, peu importe le domaine spécifique d'intérêt, de réfléchir à des questions de fond telles que : Qu'entendons-nous par théorie ? Qu'entendons-nous par données de recherche? ... Dès ses premières lignes, ce rapport met en évidence un phénomène souvent observé dans les groupes de travail au GCEDM/CMESG : tous les participants ont la possibilité de s'exprimer, de défendre leurs positions respectives, fussent-elles différentes, parfois même contradictoires, sans toutefois empêcher l'environnement ainsi créé d'être productif pour la réflexion ! Aussi, ce rapport met en évidence qu'au GCEDM les groupes de travail ne se déroulent pas toujours exactement comme prévu ... Dans ce cas-ci, les deux responsables, une chercheure établie et une étudiante au doctorat, racontent l'histoire de leur groupe de travail avec beaucoup de transparence et en donnant une grande place aux paroles des participants. C'est un peu comme si on y était ...

This is a good example of a working group that allowed researchers in mathematics education, regardless of the specific area of interest, to reflect on substantive issues such as: What do we mean by theory? What do we mean by research data? Etc. From its opening lines, the report highlights a phenomenon often observed in the working groups at CMESG/GCEDM: all participants have the opportunity to express themselves, to defend their positions, even if they are different, sometimes even contradictory, without preventing the environment created from being productive for thinking! Also, this report highlights that CMESG working groups do not always go exactly as planned. In this case, the two leaders, an established researcher and a PhD student, tell the story of their working group with transparency, and give place for the words of the participants. It's like being there ...

## PARTICIPANTS

| | | |
|---|---|---|
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## INTRODUCTION

*The phenomenon is the children working*
*The phenomenon is the videotape of the children working*
*The phenomenon is the story being told about the children working*
*Videotape is no good unless you were actually there*
*The videotape is the data*
*Data does not exist independently of the observer*
*Videotape is just videotape until we offer it to a community who decide to make it data*
*I have a relationship with the data*
*I am the data*
*I can see more clearly now how blurred the notion of data is*
*The best we can do is tell stories*
*What I learn from a researcher's analysis is what they are sensitive to*
*We can't see anything unless we have a structure to see it*
*I can only begin to work out my structure when I begin seeing*

How might statements such as these have emerged in a three-day conversation amongst peers? What might have led up to their expression? Those present at the reporting session on the final day of the 1998 Conference might recognize these statements as those read out by all the participants in this Working Group. Placed as they are here, with no preamble and removed from their context, the statements might suggest that our Working Group was a place of conflict, disagreement, and confrontation. As participants as well as leaders of the Group, though, we experienced a far more productive environment. We offer these statements as examples of the breadth of our conversations and hope to reveal in this report how the participants of the Group created an expansive and thought-provoking discussion which embraced these diverse points of view.

## A STORY ABOUT VIDEOTAPE

Let us step back, for a moment, and consider how this all began. Our story begins early in 1998 as we began to explore ideas for the Working Group we had been invited to lead At that time the Working Group had only a theme (suggested by the Executive) for structure – "From Observational Data to Theory." In one of our first e-mail exchanges, we decided that the suggested title didn't capture adequately the relationship between theorizing and data collection and analysis, and so began our conversations about the nature of theorizing and its relationship with data. Soon, the new title for the Working Group was born – "From Theory to Observational Data (And Back Again)." So far so good. However, the problem immediately became "So where do we begin in our Working Group—with theory or with data?" We began to explore possibilities for the structure of the three-day experience and it soon became clear that we needed to find just the right piece of videotape to share with the Group before we could plan how the three sessions might evolve. This became our priority, and we decided to try to obtain copies of the TIMSS (Third International Mathematics and Science Study) videotapes. When we finally tracked down a copy of the CD-ROM version, we were disappointed to discover that they contained only short excerpts of a number of lessons, and little context was provided to situate the clips. We believed that we needed to find a rich data source to support the exploration of a range of issues and to engage the members of the Working Group in nine hours of conversation and investigation. Our search began again. We finally settled on an excerpt featuring two Grade 7 students engaged in problem solving. We hoped that the excerpt, which was about thirty minutes in length and for which we also had copies of the students' written work and full transcripts of their conversation, would support investigation in a number of areas, such as the nature of the students' interaction patterns or the growth of their mathematical understanding. As we will recount later, though, all our agonizing over video excerpts was to prove unnecessary.

## A STORY ABOUT THEORIZING

Now that we had just the right piece of videotape, we turned to the task of planning the flow of activities for the Group. After lengthy e-mail exchanges we decided to begin the first day by having the participants introduce themselves and talk a little about the theoretical frames they use when analyzing their own data or conducting their own research, and then to broaden the discussion to consider the question "What do we all mean by 'theorizing'?" before introducing the chosen video excerpt sometime towards the end of the first session. Day One began just as we'd hoped, with an enthusiastic and diverse group sharing stories of their research interests. Many of the stories included passing reference to recognized philosophical frameworks such as social constructivism, enactivism, or phenomenology. As the conversation broadened to explore the issue of what we all meant by 'theorizing,' several participants explained their reluctance to affiliate themselves with a particular strand of thought such as, for instance, constructivism. One member articulated this in the following way:

> *I don't want to say that I belong to [that] club. I don't want to define myself in that way. And so, social constructivism? Yeah. OK. Fine. Radical constructivism? Fine. Behaviorism? Yeah, some of that as well. ... And actually there's quite a lot around that's quite helpful depending upon what it is that I'm looking at. ... But what I feel like saying, what I refuse to do is say 'I'm this'.*

This sentiment was echoed by another participant a few moments later:

> *The problem I have is not with identifying myself with some perspective of thinking, but it's the community wanting to keep me there. I would like to be a radical constructivist today, an enactivist tomorrow, a social constructivist the next day, because ... what it is that I'm interested in at that particular moment would dictate where I would like to be.*
>
> *As the discussion about what is meant by 'theorizing' broadened, one participant remarked on the need to take the discussion a step backwards:*
>
> *I think we need to talk about what we think theory is before I can talk about how I use it.*

Another participant responded by indicating that:

> *Some of us might have to [talk about] both together. In order to talk about theory [we need to] give it some context [by] describing how it's used.*

The discussion continued as Group members attempted to articulate their notions of theory and what it means to theorize. Comments included:

> *Maybe theory is just something that arrives in what you do, that it isn't 'out there', a framework.*

And:

> *I guess my theoretical framework is there for me to ignore it.*

Some participants described aspects of their own research work to explicate how they used theoretical frameworks in structuring their practice. We then offered a paraphrased excerpt from Cobb and Whitenack's (1996) paper on data analysis using videotape. The extract briefly outlined Glaser and Strauss's grounded theory method and emphasized Glaser and Strauss's view of the inextricability of the development of theoretical constructs and the process of data collection and analysis. One participant reflected on the extract by drawing a distinction between the 'grounded theory' discussed in the paragraph and a 'theory' like constructivism. He pointed out that the Group had been using the word 'theory' on (at least) two levels. Other participants responded by trying to differentiate between theories that structure the questions they pose, and theories that structure the ways in which they seek answers to those questions.

It was increasingly becoming clear to the Group that trying to talk about theory in the absence of data was problematic. One participant articulated this sentiment in the following way:

> *I think it's interesting how hard it is to talk about something like a theoretical framework or theory before having ... shared experiences that help foster the conversation. I'm waiting for the shared experience so that I can have some examples to use. I'm stuck without examples.*

This seemed to us to be the perfect moment to introduce the video extract we had chosen. As preparation, we first explained the context of the research, and handed out copies of the problems posed to the two Grade 7 students featured on the tape. Approximately fifteen minutes of the videotape was played to the Group, after which we asked the Group to share with us the aspects of the videotape to which their attention had been drawn. There was a variety of first impressions and features that had captured participants' interest, including aspects of school culture that one participant suggested were embedded in the students' problem-solving techniques, the influence that are-focusing of the camera lens during the episode had had on another Working Group participant's focus of attention, issues surrounding the interaction between the two students, particularly in terms of their collaboration processes, and the mathematical language used by the two students.

After initial discussion of these and other issues, the Group was split into two sub-groups, and one group moved to a separate room with a second copy of the videotape, to facilitate a more thorough investigation of a number of these issues.

## A STORY ABOUT THE MEANING OF DATA

When the Group met on the second day to continue discussion about what they had been seeing in the videotape, it did not take long before a rather substantial issue arose, that is, what do we mean by the word 'data'? Is the videotape data? If not, what is the data? (In this report, the word 'data' is used in its singular form-the form most often used by Working Group participants.)

A provocative entry into this conversation occurred with one participant's statement:

> *I have a theoretical position, which is that there is no event. The event consists of, for me, a collection of stories that people tell, which accumulate and accrete around this particular bit of videotape.*

If, as this participant argued, the event is the collection of stories, then how does such a position accommodate the notion of data and its existence? From that moment on, discussion moved in and out of what we thought we meant by 'data'. The perspective offered above was contrasted with others that included, for example, "seeing things in the videotape" or "referring to the data out there." Someone suggested that the polarity of the various positions could be crystallized with a question that is often asked by constructivists or by radical constructivists: "Where does reality reside?"

As the discussion continued, a participant stated that, in her own research, she found herself "returning to the videotapes to generate more data" and that, in so doing, former interpretations subsequently became data. She further suggested that, for her, it could be problematic to try to tell a story about another's videotapes because "you weren't there." The videotape and transcript that another provides are like notes of some event that occurred when you were not present. "It seems that you have to be there," she offered.

Another participant then put forward the idea that, whatever the phenomenon of interest, she could collect data on it by a variety of means. Thus, it was claimed that we could look at the same data obtained through different data sources. In other words, there was an object, called data, and rather than referring to the videotape as data, we might simply call it a data source. This led another participant to move in a somewhat different direction and voice how she felt that the relationship between a researcher and her data is a constantly changing one. Thus, the issue of whether the data remains unchanged-an implicit assumption of some of the earlier remarks-is a moot point, for it is never possible to return to it and to see it as it was previously. Our relationship to it has changed, as a result of interpretations we have brought to it. This opinion was echoed by some, but not by others who maintained that it was perhaps simply our sensitivity to the object called data that was changing rather than the data itself.

Considerable zigzagging continued to occur between the idea of researcher and data as one, and the separation of the two. For example, one participant advanced the question of how, if a researcher and her data are one, the various members of a given research team who look at the same piece of videotape can ever hope to negotiate commonality with respect to what is going on; in other words, how does the oneness become shared. [The same dilemma is one to which constructivism has been at odds to respond, that is, how do we come to have shared meanings if all sense-making is individual.] The question was, in fact, finessed by one participant's response that it is the collective resonance by a group of researchers with respect to what they are analyzing that constitutes the data (or that is equivalent to the data).

One participant returned again to the related claim that the data is one step removed from the videotape, which provoked once more the question of what she meant by the data. At that point, someone suggested that we might save ourselves some grief by eliminating the word data from our discussions and another followed with the reminder that, in the hermeneutic tradition, 'text' is used rather than 'data'. But the word data would not go away.

One participant then shared why she had problems defining the word data:

> *I'm not even sure that I want to specify what I think is data in my research. I'm not sure that it's helpful for me to say clearly whether the videotape is data or whether I'm the data or whether my relationship with the tape is the data. ... I'm studying my own practice, my own classroom. ... And then when I'm looking at that videotape later, I find it really hard to decide what it is that is the data. Is it the videotape that I'm seeing of this pair of students? But I know when I'm watching it I experience all kinds of things other than that tape, because I was in the room at the time. ... So there's all kinds of other influences on what becomes the data. I guess I'm asking how helpful it is that we actually do decide to call one thing the data.*

Another participant spoke of the entries that she makes in her journals when she does her research observations and pointed out that she uses these journal entries as her data. This statement led to an attempt by another participant to distinguish between what a linguistic analyst might do with those journals (i.e., use the journals themselves as the phenomenon of study-as the data) and what a mathematics education researcher might search for in those same journal entries. He continued with the following:

> *What data summons up for me is a world of significance-the world of significance is as a phenomenon which is in itself that which is being worked on. So, if the videotape is THE data, then the phenomenon is the sequence of phosphorescence that take place and we can look at that in all sorts of ways-from film techniques, etc., etc. But somehow we're trying to do more. We're trying to find meaning within the content of it, not as simply a phenomenon in itself. ...At a first level of analysis, we are trying to locate the phenomenon.*

Soon afterward, it seemed that a fatal blow to the slowly crumbling edifice of "videotape as data" was struck when one participant suggested that the same videotape and transcript could be given to, say, linguists, sociologists, psychologists, or mathematics education researchers. Each community of practice would focus on different aspects presented in the tape and transcript with the aim of arriving at consensual resonance within their own particular sphere of experience. The data for each group would be different. Thus, this example served to clarify the notion that the videotape is just a videotape until a community of practice looks at it and says that some particular thing is going to be data. It (whatever the it is) becomes data when we bring something to it; it becomes data at the moment when we make it data.

The story that has just been told did not proceed in a purely linear fashion. As one idea moved to the foreground, another moved into the shadows, only to re-emerge at a later time. However, just as the changed relationship between researcher and data makes it impossible to 'see' the data as it was seen initially-a notion advanced by one participant during our discussions—so it was equally problematic for these story-tellers to try to tell the story with the same voice in which it actually unfolded. Sensitivities that grew out of the three days of going in and out of the meaning of data make it quite unrealistic to attempt to go back and narrate events as they were prior to the dawning of new awarenesses. As well, there is no real ending to this story.

Even though certain issues related to the meaning of the word 'data' have moved out of the shade into the light, others remain a blur.

## A STORY ABOUT SURPRISES

Though the story we have just related began early on the second day, it continued throughout the remainder of the Group's time together. So intense was the discussion that the Group never returned to the videotape, a fact that surprised us as Group leaders very much. One viewing of fifteen minutes of tape, and a few minutes in which the sub-groups re-viewed parts of that tape, had been sufficient to generate two days of discussion. Though we (the Group leaders) had agonized for many weeks over the precise nature of the tape we should offer to the Group for analysis, our deliberations had been in vain. Or had they? During our preparation of this report, we have had cause to reflect on the value of the tape we used. In discussing the unfolding of the conversations that occurred in those last two days, we now believe that despite our earlier worries almost any tape would have sufficed. What seemed to be critical was that the Working Group had, as one participant put it, a "shared experience." Fifteen minutes of videotape had served not only as an 'example' to stimulate conversation, but also as a gathering place for ideas. Though the Group's contact with the videotape itself was brief, we believe it was necessary. What participants had seen and heard on the videotape, and their recollections and re-tellings of those events (if for a moment we can have permission to call them 'events'), became locations for the exploration of shared meanings. Participants referred frequently, at least initially, to specific aspects of what they had understood to be happening on the tape, and these explanations helped us all to understand the different positions that participants were taking with respect to both the nature of data and that of theory. In this way, strong statements of position, some of which are at the start of this report, had meaning for those of us in the room on those days. A reference to "the videotape" began to mean not only the specific videotape that we had all watched, but also other videotapes which we may have seen in the past, and ones we might see in the future, and by extension, other forms of recording material.

A second aspect of our work with this Group that surprised and excited us was the way in which the conversation, though wide in scope and great in depth, never strayed far from the orienting theme of the Working Group. We had anticipated that some participants might become so caught up in investigating a certain aspect of the happenings captured on the videotape that that sub-group might wander far from the theme of the Working Group, and be left with little time to think deeply about the issues we have described in this report. As we have already indicated, though, the participants' contact with the videotape was (by mutual agreement) brief, and, in fact, the sub-groups were never re-formed after the first day. Instead, we participated in the conversation as a whole group, feeling no need for sub-division, or further stimulus from the videotape. The theme that we traced, best described by our title "From Theory to Observational Data (And Back Again)," provoked a zigzagging of discussion from theory to data to theory to data. It seemed that each time we returned to one or the other of these two locations of conversation, the nature of the 'object', be it theory or data, had changed. Neither of these two elements of the conversation seemed static, and each was defined and re-defined, shaped and re-shaped, explored, examined, and investigated many times. Each position with respect to the nature of theorizing or the nature of data (and there were many more than we have been able to present here) was critiqued; each argument probed for insights. Participants' positions with respect to what is data, or what might a theory be, changed, sometimes subtly, sometimes more radically. And all this while, the theme of the Working Group was closely being traced.

We hope that the stories we have shared here reveal something of the complexity of the conversations that occurred, and the nature of the issues with which this Group grappled, over the three days. If the stories told here seem to have lost something in the telling for those who were not members of this Group, we nevertheless hope that this report will be revealing in another sense. As one participant suggested, "What I learn from a researcher's analysis is what they are sensitive to." Clearly, what is 48 Working Group B reported here are the statements that we heard, accentuated by our understandings, augmented by our interpretations, and shaped by our biases. We make no apologies for this fact, but acknowledge that what we have presented here is a story by two people representing a conversation among fourteen. It is, though, a story that we hope might provoke further reflection about data and theorizing, and the relationship between the two. After all, to echo the words of one participant, "The best we can do is tell stories."

## REFERENCE

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# WHERE IS THE MATHEMATICS?

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At a time when Mathematics Education was still young and there were still many stones still unturned, the CMESG Working Group was a place to turn over new stones – not through expert apprenticeship, but through bottom up organic activity, hypothesizing, and theorizing. This working group was chosen for inclusion in the volume, in part, because it represents such a process. Facilitated by talented and invested leaders this working group forged new ideas about the role of games, puzzles, and mathematical tasks.

Alors que la didactique des mathématiques était encore relativement jeune et qu'il demeurait de nombreux aspects à explorer, un groupe de travail du GCEDM représentait une opportunité pour explorer de nouvelles avenues – non pas à partir d'un mentorat expert, mais en travaillant ensemble, en émettant des hypothèses et en essayant de dégager une compréhension théorique. Le choix de ce groupe de travail s'explique en partie parce qu'il représente un tel travail. Guidé par des facilitateurs talentueux et pleinement engagés, ce groupe de travail a forgé de nouvelles idées quant au rôle des jeux, casse-têtes, et tâches mathématiques.

## PARTICIPANTS

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## SOME INITIAL INTENTIONS

As teachers, we give students tasks of various kinds. As educators, we expect novice teachers to develop skills in using and presenting tasks to students. If we construct tasks through didactic engineering starting from a knowledge of the situation, with emphasis on the 'mathematics' then there is an issue in acquainting others about the aims, intentions, and means so that potentials are actualized. As Dick Tahta observed (1981), in addition to an outer task (to perform what you are asked to do) there is an inner task (to make contact with mathematical ideas, to experience mathematical themes, to employ mathematical heuristics and powers, etc.). The aim of the Working Group was to approach a number of questions including: How does mathematics emerge from playing games, from using apparatus and from mathematical instruments? Is it possible to identify the properties of such instruments which motivate and facilitate the student's transition from the outer task to the inner task? How can this be planned for, enhanced, and exploited? What is the role of such instruments in the teaching and learning of mathematics?

## HOW THE TIME WAS SPENT

A summary of how the Group spent its time may provide some additional insight into this report. As the number of participants was fairly large, it took some time for everyone to introduce themselves, their interests and their experiences in the Working Group topic. As an ice-breaker, the game of SKUNK (see Appendix A) was played. In the first two sessions participants also worked with apparatus, which included the MIRA, hinged mirrors, leapfrog, Chinese jigsaw, and other games besides SKUNK that included Brock Bugs and Four- Bidden. The Group worked both in plenary, in small discussion groups and in pairs. The latter allowed for the sharing of reflections, team playing and the development of game strategies.

## REFLECTING ON EXPERIENCES

The first task proved very popular with most participants, to the extent that some wanted to carry on playing. An element of competitiveness entered, as well as interest in how an extreme strategy seemed to work rather better than a more considered one. Certainly it got the group off to an active and energetic start. The initial energy release may have made it difficult for more cerebral tasks later.

A large group of participants increases the possibility of divergent views, of experiencing shared situations in different ways and of not responding and reacting in unison. This was certainly the case for this group and tensions were generated that were not resolved. Within the context of the given situation individuals and groups reflected on the mathematics they used or developed, the links that facilitated their connection to the mathematics, the barriers that they perceived in accessing the mathematics, the role, timing of, and necessity of interventions, in order to move to the mathematics, and many other related points that arose during the activities. With manipulative-apparatus, including the MIRA, hinged mirrors, leapfrog and the Chinese jigsaw, points recorded were categorized as follows:

### GOALS

In the case of Leap Frogs and Chinese Jigsaw, goals were suggested at the beginning. For MIRA and hinged mirrors participants were free to set their own goals. Some teams felt that the goals got in the way and distracted from the process of doing mathematics. In the Chinese Jigsaw some pairs indicated that they would have benefited more if the goals had been set in broader terms. For example if it had been explicitly stated that all patterns and their relationships should be explored. Some groups expressed boredom with MIRA because no goals or mathematical problems had been set. Some teams changed the goals or set up new goals once they had reached the preset goals. Others were satisfied in reaching the goals and moved their attentions to other matters. Some discussions ensued about possible conflicts between the goals of the students in the activity and the goals of the teacher. How explicit should the teacher express his/her goals to the student? How and when should a teacher intervene when she/he realizes that the student is moving away from the intended mathematical goals?

### AWARENESS

Participants, as individuals or as teams, were asked to note whenever they became aware of positive or negative reactions or experiences, of changes in strategies or approach, and of mathematization. Participants recorded negative reactions to situations where they had used the manipulative-apparatus previously, and therefore had already gone through some of the mathematization. There was very little enthusiasm shown about extending previous experiences, preferring to engage in another activity. They noted positive reactions when they found the activity particularly interesting or when the mathematical modeling development was not obvious. Some of the groups did not even complete the activity but went straight to the mathematization, satisfied that they had a mental image of the activity. In the mathematization process participants used words such as abstraction, replacing the manipulative- apparatus by symbols, paper and pencil activities, looking for order, searching for patterns and trying to generalize. Others reflected that they had reached the goal with very little awareness of any mathematics.

The reactions reflect those of a group that brought together a very substantial set of experiences and mathematical power, a group that was disposed to approach the tasks, and individuals that were looking for the mathematical potential within those tasks. One can expect similar responses for activities that students enjoy but depending on the age and mathematical ability of students one can predict a need to motivate the mathematization of the situation.

The Group spent some time isolating properties of 'good' games and apparatus, in terms of providing a rich environment in which to mathematize. A good game or prompts reflection, is amenable to different approaches, motivates conjectures, is aesthetic and sensorial, is attention grabbing (it runs when you chase it), has the potential for generalization.

Is there a comparable list for students who bring a lower level of maturity, experience and mathematics background? We distinguish between the 'task' as introduced by the teacher (which may or may not be what they envisaged when planning nor what the author or other source had in mind), the 'task' as constructed by the participant, and the activity engaged in by the participant. The Group suggested that a good game for the classroom would require the following:

- the task as constructed by students needs a hook to engage them, and that the hook should be the mathematics itself and not the game;
- the task needs to provide enough impetus that students can break through the imposed structure and get hooked on the mathematics;
- the mathematics needs to be appropriate for that age group and not too deeply imbedded in the activity (i.e., it needs pointers to the fact that mathematics may be helpful), it needs some element of surprise in the mathematics and some unexpected results that challenge intuition (accessible but bothersome);
- the activity needs to provide various levels of mathematization allowing the better students to progress beyond the average.

Perhaps it is useful to extend the notion of 'the mathematics' to include the pleasure that comes from using one's mathematical powers. Thus, a hook can be that I find myself enjoying being called upon to imagine something, to express what I see or think in some way (through gesture and movement, through drawing and displaying, through words and symbols), to make conjectures but then find I want to modify them, to try to convince others to see what I see, to get a sense of what might be going on through trying some examples and then seeing through the particulars to a generality. A task which affords access to becoming more aware of such powers, and developing and honing them, contributes to mathematical development just as well as a task which prompts the student in rehearsing and refining a mathematical technique, or appreciation of a mathematical idea or topic.

A distinction was made between 'jeopardy' type games which provide an environment for practice—the game is more the motivator, and other games where the mathematics itself motivates the students to continue to examine it. Participants felt that the 'distance' between the game and the mathematics is shorter for Brock Bugs than it is for SKUNK. There is a notion that one can at the design/deployment stage modify this 'distance'. For example in Leap Frogs, adding a quest for minimality to the stated rules decrease the 'distance' (making game-play more directly engaged in problem solving); yet suppressing such a rule keeps the structure more flexible and open. In SKUNK the 'distance' was found to be huge, which does not diminish the value of the game but seriously changes the intervention model: where low-distance games like Brock Bugs and Leap Frogs are amenable to punctuated intervention. This may be due to the sophistication of the mathematical concepts required for analyzing SKUNK.

## APPENDIX A

### SKUNK

A description of this game can be found in pages 28 to 32, of the April 1994 issue of the journal "Mathematics Teaching in the Middle School". A short description follows. On the blackboard, the teacher draws five columns each headed with one of the letters of the word SKUNK, where each letter represents a different round of the game. Students play in teams of two or three and their aim is to accumulate as many points as possible over the five rounds of the game. For a team to acquire the points resulting from the sum of the values on the roll of two dice, all team members must be standing. Before each roll a team must decide whether to stand or to sit as a group. If at any stage a team decides to sit it has to remain seated until the end of that round. When standing a team gets the total of the dice unless a one comes up, then the play is over for that round and the standing teams loose the points accumulated for that round. If a double one comes up at any time all standing teams loose the points accumulated for all the rounds up to and including the present round. The game provides a blend of experiences in probability and decision making.

### MIRA

The MIRA is a plastic two-way mirror made for classroom use. It is available commercially together with various print materials developed for teacher and student use. Geometric properties of planar objects and constructions that involve translations, rotations, angle and line bisectors, etc. can be motivated by using this apparatus/manipulative.

### HINGED MIRRORS

The name of this apparatus describes it fully. Eric Muller brought number of pairs of mirrors hinged at one edge. No print or other materials were supplied to the Group.

### BROCK BUGS

The game of Brock Bugs was developed by Eric Muller for use in the teaching and learning of probability. The game aims to provide experiences in three specific concepts, namely relative frequency and probability distributions, expectations, and the use of the binomial probability distribution. Each of these concepts are explored in one of three levels of the game which allow students to progress through the game as they develop their understanding. Separate instructions are provided for the teacher and for students. The teacher 's notes include suggestions about classroom management, interaction with group of students, intervention to motivate understanding, etc. There is a game board with spots marked 1 to 14 and two teams are issued with different colour chips. To start the game the two teams take turn to place one of their chips on an unoccupied position of their choice. The positions of the chips stay fixed for 25 rolls of a pair of dice. For each roll the team whose chip is on the position corresponding to the sum on the faces of the two dice gets one point. The team with the most points after 25 rolls wins the game. Chips are now removed from the board and the game can be repeated. The objective is for students to first realise that strategies are in the appropriate placing of their chips and then to develop their optimum strategy to win the game. Level 2 is played on the same board but positions on the board carry different number of points. In other words a 7 may be worth two points while an 11 may be worth seven points. These points are shown in a square below each position. Level 3 asks students to explore reasons why the number of repetitions of the level 1 game were set at 25.

### LEAPFROGS

This is a classic Lucas problem that became the name of a group of mathematics teachers in the late 60s and early 70s. They met annually to work on mathematics and to design resources which required little or no introduction (more like stimuli perhaps, or phenomenon to attract attention and mathematization).

You are given a number of green frogs, and yellow frogs, lined up with greens together and the yellows together and a single space in between. Frogs can jump over another frog into a (the) vacant space, or slide into an (the) adjacent vacant space. The challenge is to interchange the green and yellow frogs and top predict the (minimum) number of moves required. Dudeney (I believe it was) extended this into two dimensions: you have two squares of frogs that overlap in just one square. This square is vacant. Again the challenge is to interchange the frogs and to predict the minimum number of moves.

### CHINESE JIGSAW

Nine coins are placed in a 3 by 3 array, with all but the centre coin showing a head. You are permitted to flip all the coins in a row, or all the coins in a column. The aim is to get all the coins facing the same way.

Eventually participants begin to conjecture that there are some difficulties. Perhaps if you are also permitted to flip all the coins in one or even both main diagonals? What if the two-state coins are replaced by $s$-state objects (so after $s$-flips they are back to their starting state)?

The name comes from a harder three-dimensional version, found in Chinese toy-stores. Nine cubes have been laid out in a 3 by 3 array and a picture has been pasted on the full set, then slit along the edges of the cubes to give one-ninth of the picture on one face of each cube. The cubes are rolled to display another face each and again a picture is pasted. When six pictures are pasted, so that each cube has one-ninth of each of six pictures, you have a six-fold jigsaw puzzle. Fortunately, the picture gluing is achieved by rolling all the cubes in each row in the same way (that permits four pictures) then in each column (for the remaining two pictures). In other words, once you have solved one picture, you can achieve the others by rolling all the rows, or all the columns, the same way and the same amount.

Now, suppose you displace just the centre cube by showing a different face. Can you roll the rows and or the columns and restore the picture? The non-commutativity makes analysis rather more difficult! (Comment: The coin version of this task generated considerable interest. It seemed to be at an appropriate level mathematically, and various ways of thinking afford access (groups, linear algebra, combinatorics), yet no really deep theorems or techniques are required. Mainly you have to reach the conjecture that it is not possible, and then to justify why.)

### FOUR-BIDDEN

Packs of cards (produced by ATM in UK) were offered. Each card has a technical term from secondary school, and four other terms which are 'forbidden'. Participants draw a card and give the others clues as to the technical term, without ever using the four forbidden terms.

Many different variants are possible such as only using the 'forbidden' terms, trying to work out the forbidden terms given the main term, using only diagrams or drawings to give clues, acting clues out in silence, team collaborations etc.

(Comments: This game generated considerable reaction. Some felt that students should not be restricted in how they try to express themselves, especially when the essence of the term is best expressed using some of the forbidden terms. In some cases it was not clear why certain terms were forbidden and not others.)

## REFERENCE

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# THE ARITHMETIC/ALGEBRA INTERFACE: IMPLICATIONS FOR PRIMARY AND SECONDARY MATHEMATICS • ARTICULATION ARITHMÉTIQUE/ALGÈBRE: IMPLICATIONS POUR L'ENSEIGNEMENT DES MATHÉMATIQUES AU PRIMAIRE ET AU SECONDAIRE

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Ce groupe de travail est un bon exemple de ce que le GCEDM/CMESG cherche à faire en regroupant des personnes d'horizons divers autour de thématiques liées à l'enseignement et à l'apprentissage des mathématiques. Guidés par deux chercheures d'expérience en didactique de l'algèbre, enseignants, mathématiciens et didacticiens des mathématiques se sont assis ensemble pour résoudre arithmétiquement des problèmes dits algébriques. Une belle entrée en matière pour ensuite s'attaquer ensemble à des questions du genre : Qu'est-ce que l'algèbre ? Qu'est-ce que l'arithmétique ? Comment faciliter la transition arithmétique/algèbre ? Comme le suggère le compte-rendu, rédigé de manière harmonieuse dans les deux langues, le groupe a fonctionné de manière toute naturelle en français et en anglais.

This working group is a good example of how CMESG/GCEDM seeks to gather together people of diverse backgrounds around themes related to teaching and learning mathematics. Guided by two researchers with experience in teaching algebra; teachers, mathematicians and mathematics educators sat together to arithmetically solve algebraic problems. His was a nice introduction to then jointly address questions like: What is algebra? What arithmetic? How to facilitate the transition between arithmetic and algebra? As suggested by the report, written harmoniously in both languages, the group worked very natural in both French and English.

## PARTICIPANTS

| | | |
|---|---|---|
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Les membres de notre groupe de travail proviennent de contextes différents, autant sur le plan géographique que sur le plan professionnel. Le groupe était en effet composé d'intervenants provenant de l'Ontario, du Québec, du Manitoba, de la Colombie-Britannique, de l'Alberta et des États-Unis, et donc de lieux où les programmes d'études, en ce qui concerne l'enseignement de l'arithmétique et de l'algèbre notamment, n'apparaissent pas nécessairement au même moment ni sous les mêmes formes. Sur le plan professionnel, ils ou elles interviennent comme enseignants ou enseignantes au niveau primaire ou secondaire, en formation initiale ou continue des enseignants au primaire et au secondaire, d'autres ont une expérience aux études avancées, et plusieurs des participants et participantes ont également une expérience d'intervention auprès d'adultes. Les contextes dans lesquels la question de « l'articulation arithmétique-algèbre » se pose sont donc multiples : ils touchent aux élèves de l'école secondaire appelés à faire cette transition (*middle school*), à la formation des enseignants, dans laquelle les étudiants sont appelés à se prononcer sur la compréhension éventuelle d'élèves, et sur leurs différentes stratégies de résolution face à des problèmes, au travail avec des adultes qui ont reçu un enseignement de l'algèbre mais ne l'ont pas réellement appris, et qui rencontrent des difficultés dans ce domaine. Cette variété de contextes, de positionnements vis à vis la thématique abordée dans le groupe de travail a contribué, comme nous le verrons dans ce compte rendu, à enrichir la discussion.

## THE FIRST SHARED EXPERIENCE

Two shared experiences on the second day produced rich questions and discussions, rich perhaps because of the diversity of perspectives in the group. The first shared experience involved working on the following set of "algebraic problems" arithmetically and considering the differences between the arithmetic and the algebraic reasoning involved.

1. Three tennis rackets and four badminton rackets cost $184. What is the price of a badminton racket if it costs $3 less than a tennis racket?

   *Trois raquettes de tennis et quatre raquettes de badminton coûtent 184$. Quel est le prix d'une raquette de badminton si celle-ci coûte 3$ de moins qu'une raquette de tennis?*

2. Luc has $3.50 less than Michel. Luc doubles his money. Meanwhile Michel increases his by 1/7. Now Luc has 40 cents less than Michel. How much did each have originally?

   *Luc a 3.50$ de moins que Michel. Luc double son montant d'argent. Pendant ce temps, Michel augmente le sien de 1/7. Maintenant Luc a 0.40$ de moins que Michel. Quel montant chacun avait-il au départ ?*

3. The dance troupe Petitpas is giving its annual recital tonight. Tickets were all sold ahead of time and the caretaker must now organize the hall. If he places 8 chairs in a row, 3 spectators will not have a chair. If he puts 9 in a row, there will be 27 empty chairs. How many people are expected to attend the recital?

   *La troupe de danse Petitpas donne son spectacle annuel ce soir. Les billets ont tous été vendus à l'avance et le concierge doit maintenant organiser la salle. S'il place 8 chaises par rangée, 3 spectateurs n'auront pas de chaises. S'il en met 9 par rangée, il restera 27 chaises disponibles. Combien de personnes attend-on à ce spectacle ?*

4. By increasing his speed to 5 km/h, a cyclist saves 37 minutes and 30 seconds. By diminishing his speed by 5 km/h, he loses 50 minutes. What is his speed and the length of the track?

   *En augmentant sa vitesse de 5 km/h, un cycliste gagne 37 minutes et 30 secondes. En diminuant sa vitesse de 5 km/h, il perd 50 minutes. Quelle est sa vitesse et la longueur du parcours ?*

5. A man takes five and a half hours to hike 32 km. He starts by walking on flat terrain and then climbs a slope at 4 km/h. He turns around at the top and returns on the same path to his starting point. We know he walked on the flat terrain for 4 hours (2 going and 2 returning) and that it took him twice as long to climb the slope as to descend it. Find the length of the flat part of his hike.

*Un homme met 5 heures et demie pour faire un trajet de 32 km. Il commence par marcher sur un terrain plat puis il monte une pente à la vitesse de 4 km/h. Il fait alors demi-tour et retourne au point de départ par le même chemin qu'à l'aller. Nous savons qu'il a marché pendant 4 heures (2 à l'aller et 2 au retour) sur le terrain plat et que la montée de la pente lui prend le double du temps que la descente. Calculer la longueur de la partie plate du trajet.*

6. It takes a man five and a half hours to complete a certain hike. He starts by walking on flat terrain at a speed of 6 km/h and then climbs a slope at 4 km/hr. He turns around at the top and returns on the same path to his starting point. We know that he descended the slope at 8 km/h and that the length of the slope is 2/7 of the total distance walked. Find the total distance he walked.

   *Il faut à un homme 5 heures 30 pour faire un certain trajet. Il commence par marcher sur une partie plate à la vitesse de 6 km/h et continue en montant une pente à la vitesse de 4 km/h. Il fait alors demi-tour et arrive au point de départ en faisant le même parcours qu'à l'aller. Nous savons que la vitesse de descente de la pente est de 8 km/h et que la longueur de la pente est les 2/7 du parcours total. Calculer la longueur du parcours.*

Solving the problem arithmetically was not necessarily an easy task for participants. Some of the problems appear to be quite complex. The hope was that it would contribute to our understanding of one facet of the interface between arithmetic and algebra in a particular context, that of problem solving. The two first problems were discussed at length.

## SOLUTIONS TO THE FIRST PROBLEM

For the first problem, the rackets problem, the group proposed a variety of solutions and some of these were the basis of considerable subsequent discussion.

**Solution A:** This trial procedure consisted of choosing a certain given number for the price of a tennis racket, then finding the corresponding price for the badminton racket and then the total amount for 3 tennis rackets and 4 badminton rackets. A new number was tried until the correct total price was reached

13 + 13 + 13
10 + 10 + 10 + 10

26 + 26 + 26
23 + 23 + 23 + 23

28 + 28 + 28
25 + 25 + 25 + 25

**Solution B:** The total number of rackets, 7, was divided into the total price, $184, in order to get a ballpark number for the price of a racket. Then a trial and adjustment procedure was undertaken.

| | | | | | | |
|---|---|---|---|---|---|---|
| 184 ÷ 7 = 26 … | First trial: 20 | 20 | 20 | | | |
| | | 17 | 17 | 17 | 17 | total: 128 |
| Then adjustment to a lesser amount: | | 16 | 16 | 16 | | |
| | | 13 | 13 | 13 | 13 | total: 93 |
| A new adjustment, raising the price … | | | | | | |

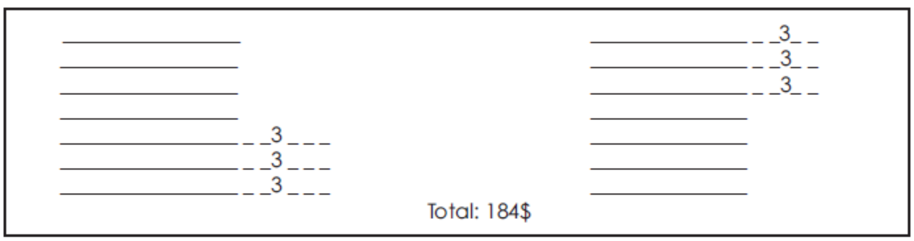
**Solution C:** This solution began with the fact that 4 of the rackets together cost $12 less. This was subtracted from $184 to get $172 (to have 7 rackets of the same price) and the latter amount was divided by 7 to get 24 and 4/7. The price of the more expensive racket was then fixed at three dollars more, 27 and 4/7. The price of the badminton racket was multiplied by 4 and that of the tennis racket by 3 with the result not coming out to $184.

Solution C led to some discussion on the difficulty of controlling the relationship "three dollars less than" and two corrected solutions (C1 & C2) by the group.

**Solution C1:** If they all cost the price of a tennis racket, then the bill would be $184 + $12. Dividing $196 by 7 gave the price of the tennis racket, $28 and thus $25 for the badminton racket.

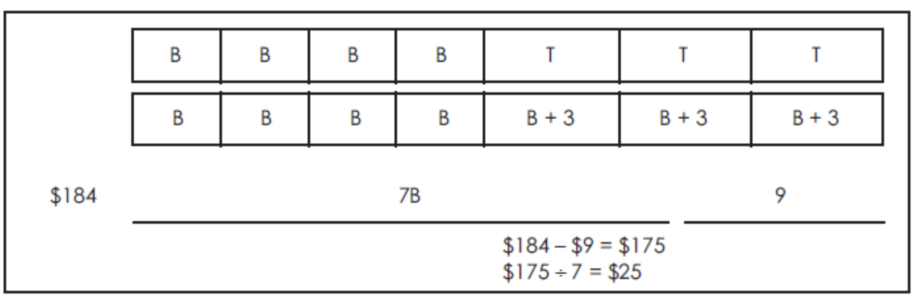
**Solution C2:** This solution began with the fact that a tennis racket costs $3 more than a badminton racket and so the three tennis rackets would cost $9 more altogether. If they all cost the price of a badminton racket, then the bill would be $184 – $9. Dividing this by 7 gave the price of a badminton racket.

It was noted that it is difficult to decide whether to add or subtract—controlling the relationship "$3 less than" and its influence on the total is difficult—and that sometimes a drawing is helpful in making the decision. Nadine offered her drawing of the problem situation and reminded us that at the turn of the 20th century this type of drawing could be found in the arithmetic problem solving sections of textbooks.




The solution that drew the most attention in the discussion later was the following:

**Solution D:**




SOLUTIONS TO THE SECOND PROBLEM

The second problem (Luc and Michel problem) also led to a variety of solutions, the most discussed of which were the two following solutions—reproduced here with an attempt to reflect the way the solvers explained them.

**Solution A:**

- Michel augmente son montant d'argent de 1/7, on va donc choisir au départ un nombre divisible par 7, disons 7$. Luc a alors 3,50$. Il double son montant d'argent, il a donc maintenant 7$, et Michel 8$ (7$ et 1/7 de 7$). La différence entre leurs deux montants est de 1$, ce qui ne convient pas puisque la différence devrait être de 0,40$.
- On va donc diminuer le montant de Michel choisi au départ pour pouvoir avoir une différence moindre à la fin. Prenons 6,30$ (nombre aussi divisible par 7). Luc a alors 2, 80$ (3,50$ de moins que Michel). Luc double son montant d'argent, il a maintenant 5,60$. Michel augmente son montant de 1/7, soit de 0,90$, il a donc maintenant 7,20$. La différence entre les deux montants est alors de 1,60$. La différence a augmenté et non diminué....
- Il faut donc que j'augmente le montant de Michel et non que je le diminue (plusieurs membres du groupe avaient fait cette erreur, mettant ici en évidence une des difficultés du problème, le contrôle ici de l'effet des transformations sur les grandeurs en présence). Prenons 7,70$ (nombre divisible par 7) pour le montant de Michel, Luc a alors 4,20$. Luc double son montant, il a maintenant 8,40$ et Michel a 8,80$ (7,70$ plus 1/7 de 7,70$). La différence est bien de 0,40$.

| Luc | | Michel |
|---|---|---|
| 3,50 | | **7** |
| 7 | | 8 |
| | différence = 1$ | |
| 2,80 | | **6, 30** |
| 5,60 | | 7, 20 |
| | différence = 1,60$ | |
| 4,20 | | **7, 70** |
| 8,40 | | 8, 80 |

**Solution B:**

- La différence entre les montants d'argent de Luc et Michel au départ était de 3,50$.
- Si les deux avaient doublé leurs montants d'argent, l'écart entre ceux-ci aurait alors été de 7$. Cependant cet écart n'est réellement que de 0,40$. On a donc réussi à regagner 6,60$.
- Luc a effectivement doublé son montant d'argent, mais Michel n'a pas réellement doublé son montant d'argent, il a juste augmenté celui-ci de 1/7. Il lui aurait fallu 6/7 de plus pour effectivement doubler son montant initial. Si on rajoutait 6/7 de la part de Michel, on aurait donc regagné 6,60$.
- Les 6/7 (de son montant de départ) correspondent donc à 6,60$.
- Michel avait donc 7,70$. Et Luc avait 4,20$ (3,50$ de moins).

(Montant de Luc au départ) _ _ _ _ _ (Montant de Michel au départ) ________

3,50$ (écart)

(Montant de Luc doublé) _ _ _ _ _ _ _ _ _ _ (Montant de Michel doublé)____________

7$ (nouvel écart)

0,40$ (écart réel)

6,60$ (regagné sur la différence)

## THE SECOND SHARED EXPERIENCE

The second shared experience involved watching a short video extract in which two future teachers were discussing their solutions to the following problem:

> *Another version of the Luc and Michel problem (distinct from that solved by the group)*
>
> Luc a 3.50$ de moins que Michel. Luc double son montant d'argent et Michel augmente le sien de 1.10$. Maintenant Luc a 0.40$ de moins que Michel. Trouve les montants que Luc et Michel avaient au départ.
>
> *Luc has $3.50 less than Michel. Luc doubles his money and Michel increases his by $1.10. Now Luc has 40 cents less than Michel. Find the amounts that Luc and Michel had originally.*

### EXCERPT FROM A DYADIC INTERVIEW

Éric (EC) «algebraic» problem solver, and Mireille (MI), «arithmetical» problem solver. (A partial translation of the verbatim[12] was provided by Nadine.)

| | |
|---|---|
| Notes that Mireille made as she explained how she solved the problem | L 3,50 M<br>? ,40 ?<br><br>3,50<br>− ,40<br>———<br>3,10<br>+ 1,10<br>———<br>4,20 |

MV: Okay. Luc has $3.50 less than Michel does (she writes down L,M and 3.50 as above). Now to start with, I suppose that ...

EC: *Michel has at least* $3.50.

MV: Well, let's say ... yeah, you could say that. Okay, Luc doubles his money. ... Well, when you get down to it, I go about it more using the difference between the two. I know that he, here there's 3.50 separating them. Uh, Luc doubles his money whereas Michel increases his money by $1.10. So I know that here there was an increase of $1.10. But I don't know the amount that they had (she writes down the two?)

EC: Okay.

MV: What I do know is that there was a difference and that afterwards, I've got Luc who's now got 40 cents less than Michel (she writes down .40). So I know that the difference between these two (she draws an arrow between $3.50 and $ .40) is $3.10.

EC: $3.10 you say ...

MV: *A difference of $3.10*, and I know already that...$1.10, here there was an increase of $1.10. So normally that would give the amount ... EC: ... *that Michel had*

MV: *Here, that Luc had.*

[12] Pour la recherche plus complète dont est tiré ce verbatim, voir S. Schmidt & N. Bednarz (2002), Arithmetical and algebraic types of reasoning used by preservice teachers in a problem-solving context, Canadian Journal of Science, Mathematics and Technology Education, 2(1), 67–91.

Because time was short, we did not observe the second video clip in which the same two students worked on the cafe croissant problem (see the problem with contradictory éléments below). We did, however, discuss the problem (the purpose here was to focus on the control of the process of solving problems in arithmetic and algebra) and Nadine provided a verbal description of the student interchange[13].

> *The cafe, croissant problem*
>
> Au restaurant, une tasse de café et trois croissants coûtent 2,70$. Deux tasses de café et deux croissants coûtent 3.00$. Trois tasses de café et un croissant coûtent 3,50$. Trouve le prix d'une tasse de café et d'un croissant.
>
> *At a restaurant, a coffee and three croissants cost $2.70. Two cups of coffee and two croissants cost $3.00. Three cups of coffee and one croissant cost $3.50. Find the price of one coffee and one croissant.*

One of the students (EC) immediately attempted an algebraic approach, writing three equations with two unknowns, solving two of them and then replacing the numbers found in the other one. When he observed that when he put them in the third equation, it didn't give him the right answer, he checked his method... He tried again with two other equations...He attempted three other algebraic solutions to the same problem ....he never returned to an analysis of the proposed relationships in the problem. On the other hand, the other student (MI) worked on the basis of the relations in the situation: Here I've got one coffee and three croissants; here I've got two coffees and two croissants, I've got one coffee more and one croissant less, and it costs 30 cents more. Then here, I have the same thing, I've got one coffee more and one croissant less. That costs me 50 cents more! That doesn't work!

Une certaine expérience partagée a ainsi constitué le point de départ de la discussion subséquente du groupe de travail. Elle portait d'une part sur la résolution arithmétique de problèmes usuellement présentés en algèbre, et l'explicitation de diverses solutions par les participants, et d'autre part sur le visionnement d'un extrait de vidéo dans lequel deux étudiants en formation des maîtres confrontaient leurs solutions (arithmétique et algébrique) à un même problème.

Les diverses solutions proposées par le groupe à certains problèmes, les discussions qu'ont provoquées certaines solutions plus spécifiquement, les réflexions issues de notre observation du vidéo par ailleurs, ont permis d'ouvrir sur un certain nombre de discussions. Nous reprendrons maintenant quelques uns des points les plus importants.

## QU'EST-CE QUE L'ARITHMÉTIQUE? QU'EST-CE QUE L'ALGÈBRE?

Le travail sur les différentes tâches, notamment la résolution arithmétique de problèmes, nous a amené à discuter longuement la différence entre arithmétique et algèbre: en quoi peut-on dire que cette solution est arithmétique ou algébrique? Où s'arrête l'arithmétique? Où commence l'algèbre? Par exemple certaines solutions arithmétiques présentées au problème des raquettes ont suscité une interrogation par certains participants: en quoi pouvait-on dire que cette solution était arithmétique et non algébrique? Ainsi, si une ligne (voir solution C, dessin proposé par Nadine) ou une boîte (voir solution D) est utilisée pour représenter les grandeurs en présence et leurs relations, ceci n'est-il pas une certaine façon de représenter l'inconnue et ne peut-on dire dans ce cas que le processus de résolution est algébrique? Le fait que certains élèves qui n'ont jamais reçu d'enseignement de l'algèbre produisent de telles solutions, ou encore que l'on

[13] For more details, see Schmidt & Bednarz (2002).

retrouve des illustrations semblables (solution C) dans de vieux manuels d'arithmétique, plaident toutefois en faveur de voir celles-ci comme des solutions arithmétiques. La question de savoir si une solution est arithmétique ou algébrique est vite apparue au groupe comme risquant de nous enfermer dans une discussion stérile, et celle-ci a été abandonnée au profit de l'intérêt qu'il pouvait y avoir à encourager certaines solutions particulières dans une perspective de transition à l'algèbre.

## WHAT IS ARITHMETIC, WHAT IS ALGEBRA?

Some of the arithmetic solutions produced by participants led to challenges by others as to whether or not they could also be classified as algebraic. For example, if a line or box is used to represent the unknown amount, is that just another way of representing the unknown and is the solution process essentially algebraic? The fact that some students who have never been exposed to algebra produce such solutions argued in favour of viewing these as arithmetic. Trying to reach a conclusion about whether such a solution was arithmetic or algebraic was eventually abandoned in favour of a discussion of the interest of encouraging this particular type of solution as a stepping-stone to algebra.

## POTENTIEL DE CERTAINES SOLUTIONS ARITHMÉTIQUES POUR UN PASSAGE À L'ALGÈBRE?

Le travail autour des solutions proposées par le groupe à quelques problèmes nous a amené à discuter très longuement du potentiel de certaines de ces solutions: en quoi ces solutions sont-elles porteuses de sens, riches pour un éventuel passage à l'algèbre? Comment favoriser la transition à partir de celles-ci à l'algèbre? Par exemple, les solutions arithmétiques mettent dans certains cas en évidence un contrôle très grand des relations en présence ou des transformations sur les grandeurs (voir les solutions C1 et C2, ou D au problème des raquettes, ou la solution B, au problème de Luc et Michel) ou s'appuient sur des propriétés des nombres (exemple de la solution A, au problème de Luc et Michel), elles constituent un atout important dans la mathématisation des problèmes en algèbre. Ces solutions rendent compte par ailleurs dans certains cas d'une notation ou représentation globale intéressante (exemple de la solution C, illustration proposée, ou de la notation utilisée en D au problème des raquettes). Cette dernière solution apparaissait même a priori pour plusieurs des participants très proche d'une résolution algébrique (faisant référence à une notation symbolique et semblant opérer sur ce symbolisme dans la notation même utilisée). Toutefois, la discussion a mis en évidence que la lettre ici ne joue pas vraiment le rôle d'inconnue. Elle sert juste à désigner les quantités en présence, c'est en quelque sorte une étiquette (badminton, raquette), ce qui constitue en fait un obstacle dans la résolution algébrique ultérieure (où la lettre représente un nombre et la symbolisation des relations exprime une relation entre des nombres qui n'est pas une traduction directe). La question de la transition à l'algèbre exige donc davantage. Elle peut sans doute s'appuyer avantageusement sur les habiletés développées en arithmétique et certaines de ces solutions, mais le passage n'est pas une simple transposition d'une procédure à l'autre.

## DO CERTAIN ARITHMETIC SOLUTIONS LEAD MORE EASILY INTO ALGEBRAIC SOLUTIONS?

Some arithmetic solutions produced in the group seemed to be more meaningful and offer a greater potential for an eventual passage to algebra. The question arose as to how to move from these particular solutions into algebraic ones. For example, arithmetic solutions exhibiting mastery of the relationships in the problem or of transformations of quantities (see solutions C1, C2 and D to the rackets problem, or solution B to the Luc and Michel problem), demonstrate skills that are important in mathematizing problems in algebra. These solutions also show interesting notation or global representations of the problem (for example, the illustration used in solution C and the notation in D). For many of the participants, solution D appeared to be very close to an algebraic solution in that it involved symbolic notation and operated on that notation. However the discussion brought out the view that the letter here did not really play the role of an unknown. Rather, the letter designated the quantities present and acted as a label (which is considered to be an obstacle in later algebraic work). Thus the passage to algebra requires additional insights and skills. It can certainly build on arithmetic skills and solutions but the passage is not a simple transposition from one to the other.

## TENSION DANS LA TRANSITION ARITHMÉTIQUE-ALGÈBRE ENTRE CONTEXTUALISATION ET DÉCONTEXTUALISATION

Un point important soulevé par le groupe et sur lequel nous nous sommes longtemps attardés est celui de la tension, dans la transition arithmétique-algèbre en résolution de problèmes, entre la nécessité de partir du contexte, pour construire notamment un sens à l'expression algébrique élaborée ou à toute autre représentation, et la nécessité de quitter le contexte pour aller plus loin dans la résolution. Dans nos solutions arithmétiques, nous nous appuyons en effet fortement sur le contexte, interprétant constamment les quantités et relations en présence pour pouvoir opérer. Chaque partie de la solution s'appuie sur le contexte, peut être vérifiée en regard du contexte. Nous reconnaissons que tel n'est pas le cas en algèbre, où le contexte sert seulement au début de la résolution du problème lors de la construction de l'équation ou des équations, et à la fin du processus dans l'interprétation de la solution trouvée. Plusieurs des participants du groupe pensent que cet abandon du contexte est un des gros obstacles dans la résolution de problèmes en algèbre. Le problème du café croissant, et sa résolution par les deux étudiants en formation à laquelle nous avons fait allusion précédemment, le montre bien et fournit plusieurs arguments en faveur du maintient d'un lien avec le contexte. Ceux qui en effet s'engagent dans une résolution algébrique semblent tourner en rond, essayant de résoudre à plusieurs reprises deux équations à deux inconnues puis de remplacer dans l'autre équation, sans nullement contrôler ce qui s'y passe. Ceux qui essaient de résoudre arithmétiquement le problème restent en contact avec le contexte, et rapidement réalisent que le problème ne fonctionne pas, qu'il n'y a aucune solution.

Nous nous sommes demandés s'il ne serait pas possible de prévoir des allers retours entre contexte et manipulations algébriques, au moins dans les premières étapes de l'apprentissage de l'algèbre. Bien que cette question soit restée ouverte, quelques oppositions à cette idée ont été exprimées. Tout d'abord, l'algèbre est un outil pour résoudre une classe plus générale de problèmes et sa puissance réside justement ici dans l'abandon du contexte, d'autre part, il est extrêmement difficile, voire impossible, de donner à chaque étape de la manipulation une signification dans le contexte.

## TENSION BETWEEN KEEPING THE CONTEXT AND WORKING IN ABSTRACT

In our arithmetic solutions we were constantly reading and interpreting the problem situation. Every line of the solution was or could be checked for sense in the context of the problem. We recognized that this is not the case in algebra where the context serves only at the beginning and end of the solution process. Several participants thought that this abandonment of context was one of the big obstacles to problem solving with algebra. We wondered whether it would be possible to move back and forth between the algebraic manipulations and the context, at least in the early stages of algebra. Although this question remains open, there was some opposition to the idea: firstly, algebra is a tool for solving general problems and its power lies in the abandonment of context and secondly, it is extremely difficult to do—perhaps more difficult than the manipulations themselves. The café/croissant problem above provided some arguments in favour of maintaining a connection with context. Those who leaped into an algebraic solution ended up going around in circles trying to solve 3 equations in two unknowns. Those who looked at the problem arithmetically or stayed in touch with the context, quickly realized the impossibility of a solution.

## SURVALORISATION DE L'ALGÈBRE ET DÉVALORISATION DE L'ARITHMÉTIQUE EN RÉSOLUTION DE PROBLÈMES

Dans l'extrait vidéo que nous avons visionné, la difficulté du futur enseignant de mathématiques au secondaire (ER) à comprendre la solution arithmétique produite par l'autre étudiante (MI), et son absence de volonté apparente à vouloir comprendre celle-ci, ont questionné les participants. À l'opposé, bien que MI ait eu de la difficulté à suivre le raisonnement algébrique de son coéquipier, elle a fait l'effort de comprendre celui-ci et a été tout à fait capable à la fin d'expliquer et de refaire ce raisonnement. Derrière l'indifférence du solutionneur «algébrique» envers la solution arithmétique de l'autre (qu'il perçoit comme de la magie), il est possible d'y lire une certaine supériorité de l'algèbre sur l'arithmétique, ce que le groupe a nommé «une certaine arrogance de l'algèbre». Venant de travailler nous-mêmes sur des solutions arithmétiques à des problèmes, nous étions naturellement impressionnés par la solution arithmétique de MI et par le raisonnement sous-jacent mis en jeu. D'où notre étonnement à voir l'inhabileté du solutionneur algébrique à apprécier, lui de son côté, cette solution arithmétique. Les conséquences d'une telle attitude selon nous dans la classe sont importantes. Elle questionne en effet la capacité du futur enseignant à comprendre les stratégies des élèves et amène à penser que lorsque l'algèbre est introduite, tout raisonnement arithmétique est de fait évacué. Ceci peut nous laisser penser que le raisonnement arithmétique est de fait, au moment de l'introduction à l'algèbre et après, négligé, voire même qu'il régresse. Ceux qui ne rentrent jamais dans l'algèbre courent ainsi le risque d'être laissés de côté avec aucun outil pour résoudre les problèmes, et aucune confiance dans leur capacité à résoudre des problèmes. Du point de vue de la formation des maîtres, un travail important est à faire, en valorisant entre autres les enseignants qui veulent comprendre les stratégies premières des élèves.

## TO WHAT EXTENT IS ALGEBRA OVER VALUED AND ARITHMETIC UNDER VALUED IN PROBLEM SOLVING?

In the video extract, we were all slightly appalled by the future high school teacher's inability and unwillingness to understand the arithmetic solution produced by the future special education teacher. Although the latter found her partner's algebraic solution difficult to follow, she did make the effort to do so and in the end was able to follow it. Behind the indifference of the algebraic solver towards the arithmetic solution, we read a sense of superiority attributed by both students to the algebraic solution and coined the term "the arrogance of algebra". Having just worked on arithmetic solutions, we were naturally quite impressed with the arithmetic solution and the mathematical reasoning involved. It was worrisome to recognize in the algebraic solver the inability to appreciate an arithmetic solution. The consequences of this in the classroom assure that once algebra is introduced, all arithmetic reasoning is outlawed. Hence arithmetic reasoning atrophies and those who never quite "get" algebra are left with no tools and no confidence to solve mathematics problems. In teacher training, it also seems important to value teachers who want to understand students' primitive strategies.

## DEVRIONS-NOUS INTRODUIRE DES PROBLÈMES «D'ALGÈBRE» EN ARITHMÉTIQUE?

Nous avons aussi discuté la pertinence qu'il pouvait y avoir à introduire des problèmes classiques d'algèbre, comme ceux que nous avions examinés, pas nécessairement les derniers considérés comme complexes pour les élèves, mais d'autres plus simples, avant toute introduction à l'algèbre. Beaucoup d'arguments en faveur d'une telle introduction ont été mis en évidence par le groupe:

- Il y a plusieurs stratégies de résolution possibles comme nous l'avons vu, dont le potentiel est riche pour le développement d'habiletés en résolution de problèmes: essais erreurs raisonnés s'appuyant sur certaines propriétés des nombres; fausse position: on fait semblant que...en se donnant un nombre et on réajuste; travail sur les relations et comparaison. ...
- Le recours à plusieurs méthodes de résolution fait partie du curriculum (est requis par celui-ci)
- Il semble toutefois important dans ce travail d'aller au delà de la simple procédure d'essais erreurs pour forcer une réflexion sur les relations. L'arithmétique, si elle est un appui important pour le passage ultérieur à l'algèbre, doit être une arithmétique relationnelle.
- Les notations utilisées, la manière dont nous rendons compte de ces stratégies, dont nous les explicitons, est aussi un appui important pour le travail ultérieur en algèbre: notations séquentielles versus notations globales (rendant compte de l'ensemble des relations en présence), recours à des illustrations aidant à contrôler les relations, présence possible d'une riche variété de notations, représentations (diverses représentations explicitant l'ordre de grandeur des notations ou leurs relations, notations symboliques ...)
- Le travail sur différents types de problèmes est possible: travail sur des régularités (exemple trouver la somme des 45 premiers nombres entiers rapidement...), problèmes mettant en jeu des relations de comparaison, développant une flexibilité à jouer avec ces relations de comparaison (se les représenter, les formuler de différentes façons ...)
- Il est possible de discuter certains critères avec les élèves dans le retour sur les stratégies (clarté à des fins de communication de celles-ci à quelqu'un d'autre, efficacité, certaine stratégies sont-elles plus efficaces que d'autres?: qu'arrive t-il si l'on change certaines données du problème, la solution fonctionne t-elle encore? ...)

La question de savoir si nous devrions enseigner certaines de ces stratégies , si nous devrions parfois insister sur la mise en évidence de certaines stratégies plutôt que d'autres a été posée.

## SHOULD WE INTRODUCE "ALGEBRA" PROBLEMS IN ARITHMETIC?

A number of arguments were made for introducing classic algebra problems such as those examined in the working group before any introduction of algebra.

- As we saw, a number of solution strategies emerge that are potentially rich for developing problem solving abilities: trial and error strategies grounded in number sense, trial and adjustment, work on relationships and comparisons. (It is important though to go beyond trial and error strategies and move to reflection on the relationships between the quantities in the problem.)
- The use of a variety of solution strategies is required by the curriculum.
- The notation used (sequential as opposed to global notation, recourse to illustrations, multiple notations and representations) and the ensuing discussions are important for future work in algebra.

- Work on a wide variety of problems is possible: on patterns or regularities (for example, find the sum of the first 45 whole numbers rapidly), on comparisons (expressing and representing them in different ways), …
- In discussions of strategies with the students, criteria can be established for clarity in presentation, efficiency in solutions (Are some solutions more elegant than others? What happens if we change some of the givens in the problem?)

The question arose as to whether or not we should teach certain of these strategies.

## DEVRIONS-NOUS CONTINUER LE TRAVAIL EN ARITHMÉTIQUE APRÈS L'INTRODUCTION DE L'ALGÈBRE?

Notre travail sur les problèmes nous a amené à la conclusion que l'arithmétique ne devrait pas être mise de côté dans le travail en résolution de problèmes, après que l'algèbre ait été introduit. Nous avons trouvé quelques unes des solutions arithmétiques que nous avons partagées riches sur le plan du raisonnement mis en jeu. Toutefois, encourager des solutions arithmétiques et continuer le développement de l'arithmétique chez les étudiants tout au long de l'école secondaire ne fait pas vraiment partie des expériences des participants. Le curriculum arrête en effet l'enseignement de l'arithmétique en général lorsque l'algèbre est introduit. Dans le programme récemment introduit en France, on réintroduit cependant l'arithmétique dans les dernières années du secondaire et même au niveau postsecondaire. Il reste à voir ce qu'on appelle ici arithmétique. Il y a un intérêt à considérer les raisons qui ont conduit les responsables de ce curriculum à vouloir y réintroduire l'arithmétique, ce dernier contribuant selon eux au développement d'une certaine rationalité mathématique. Les participants du groupe ont mentionné l'intérêt que pourrait avoir un tel travail en arithmétique, articulé par exemple autour de la théorie des nombres, pour le développement du concept de variable. La réflexion est ici à poursuivre.

## WHAT WOULD BE THE BENEFITS OF CONTINUING THE STUDY OF ARITHMETIC THROUGHOUT SECONDARY SCHOOL?

Our experiences in the group work led us to the conclusion that arithmetic should not be set aside in problem solving work after algebra is introduced. We found some of the arithmetic solutions to the shared problems both simple and mathematically exciting. However, encouraging arithmetic solutions and continuing the development of students' arithmetic throughout high school had not been part of the experience or expectations of any of the participants. Nadine spoke about the new programs in France where arithmetic has been re-introduced in the last years of secondary school. There was considerable interest in that curriculum and conjectures that students would be much better prepared for tertiary mathematics particularly in the area of number theory. One learning shared by all was that the meeting of arithmetic and algebra does not just occur in a year or two somewhere in middle school. It impacts on all of us wherever we are intervening in the school system.

# IMAGES OF UNDERGRADUATE MATHEMATICS

Miroslav Lovric, *McMaster University*
David Poole, *Trent University*

Animé par deux mathématiciens passionnés par l'éducation mathématique, le groupe avait une participation équilibrée de mathématiciens, didacticiens et éducateurs mathématiques. En cela, le groupe reflète bien une des caractéristiques du GCEDM. Les discussions sur ce vaste sujet en ont ainsi été enrichies de perspectives diverses, et le groupe a manifestement permis d'explorer plusieurs images différentes. Il en résulte un rapport intéressant, parfois intriguant, et des images qu'il faut voir dans l'appendice.

Led by two mathematicians with a strong interest in mathematics education, this working group brought together a well-balanced group of mathematicians, researchers in mathematics education and mathematics educators. This feature of the group reflects one on CMESG's characteristics. The discussions on this vast topic were enriched by these varied perspectives, and the group was obviously able to consider many different images. This gave and interesting and somewhat puzzling report, and images worth looking at in the Appendix.

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The theme that the working group was given to discuss, named "Images of Undergraduate Mathematics," is certainly very broad and far-reaching. Thinking about it, we published the following abstract:

> "I hate math!", "What is Fermat's Last Theorem about?", "I really liked your lecture on infinity.", "Fractals are cool but I hated those area and perimeter calculations.", "Do I have to teach that calculus course again?", "Do all mathematicians look like that guy *in Good Will Hunting*?", "I have always liked math, and was good at it.", "This textbook is useless.", "Why do we need all that geometric stuff?", "Who is Sophie Germain?", "Chinese students are expected to do well in mathematics.", "What is all this theory good for?", "The problem is that students don't learn that stuff in high school anymore.", "Why are you bothering me with questions, just give me the damn answer!"
>
> Images, opinions, and views of mathematics are uncountable ... so are emotions and stereotypes. The comments above come from students' comments in course evaluations and journal entries, faculty comments over coffee, and comments in the media. Can you tell which is which?
>
> This working group plans to look at this large and complex space of undergraduate mathematics—to discuss, investigate, and analyze, in an attempt to describe what it looks like. We will not restrict our attention to courses for mathematics majors only:

> "service" courses will also be considered and we will explore what each type of course can learn from the others.
>
> There are many approaches (historical, cultural, ethnomathematical, teaching/learning, epsilon-delta/no epsilon-delta, etc.), and viewpoints (undergraduate students, university lecturers, high school students and teachers, mathematics education researchers, media and popular culture, etc.) What attracts students to mathematics? What repels them? How can we keep interested students from "turning off" to mathematics? How can attract those able students who are already turned off? How to perceptions and misconceptions play into these issues?
>
> We'll be searching for interesting facts, fresh ideas, and creative insights. What can we learn? Can it help us appreciate mathematics more? Can it lead towards improvements in the way we teach and learn mathematics? What other questions can we possibly ask (and answer?)

Moreover, to provoke, motivate, and suggest possible directions of discussion, we produced a four-page handout (see Appendix).

Two-and-a-half days of (somewhat unfocussed) discussion resulted in the following list of statements and issues.

1. Mathematics is difficult. Instructors and teachers need to be honest and straightforward about it.
2. Mathematics is useful. Good reasons to learn mathematics include gaining valuable skills and increased chances of finding a good job.
3. Mathematics is exciting. This excitement, enthusiasm, motivation for mathematics needs to be 'transferred' to our students.
4. Mathematics is not calculus. However, at most universities calculus is the only course that a large majority of students take. Although calculus is essential to understanding certain areas of mathematics and applications, there are other mathematics disciplines that are of equal importance, but are not taught to non-mathematics majors. New courses at the entry-level are needed.
5. Mathematics education is communication. We need to facilitate communication between high school teachers, college/university mathematics instructors, and mathematics educators.
6. List of knowledge and skills. Is it possible, and on what level (local, provincial) to create a list of knowledge and skills that high school students possess (i.e., that are expected of them).
7. Mathematics instruction needs small classes.
8. Mathematics education needs the support of NSERC, CMS, and others.

Although we have touched upon numerous subjects, we will focus our presentation on the above statements.

## AD 1

Mathematics—on any level, from elementary school to graduate courses—is, for most of our students, a difficult subject to learn. They know it, and we know it. We should be honest about it, and tell our students that they are about to learn something difficult. With our support and help, and lots of work on their own, our students should be able to learn the material.

Issues to consider include:

- How to teach mathematics to the 'lower end'—i.e., to the students who are not adequately prepared for a university course in mathematics;
- 'Upper end' students usually get neglected; need to create challenging contexts for them;
- Drill has a place in learning mathematics; it can also serve as a motivational tool;
- Reading a mathematics textbook is not easy—we need to teach our students how to read mathematics.

## AD 2

We know that mathematics is useful. But what about our students? How can they conclude, reading a calendar description of a course (which is usually very short, and more often than not, quite vague), why it's important for them to take it? How will they benefit from it? How does the course fit, more globally, into their program? What skills are they going to gain, and why are those skills important and relevant to them?

Course syllabus is an ideal medium to address these questions. Of course, it cannot answer all students' questions and concerns, but could certainly be a good start.

Consider, as an example, the list below, given in the syllabus for the first-year science calculus course (Math 1A3) at McMaster University[14]. It is expected that the Math 1A3 course will:

- give you a detailed discussion of basic concepts of calculus of functions of one variable;
- give you some experience in relating mathematical results obtained using calculus to solutions of problems in other disciplines and to "real-world" problems;
- give you experience in constructing and interpreting graphs of functions, so that you will be able to interpret pictorial data obtained from various sources (computers, reference manuals, instruments, various reports, etc.);
- give you experience in reading and writing mathematics, so that you will be able to communicate your mathematical and technical ideas to others and use various reference sources;
- teach you how to use computer software to enhance understanding of the material and to solve various problems.

This is just a start. In lectures, when the opportunity arises, students' attention is brought to the above items. This way, they can see how the promises given in the syllabus 'materialize' in context of the course. The list of course objectives, that accompanies the above list gives further information on what will be happening in the course:

- to learn about basic concepts of calculus (function, limit, continuity, derivative, integral);
- to learn how to think logically (mathematically);
- to learn how to communicate mathematics ideas in writing;
- to learn about mathematics as a discipline (what is a definition? theorem? why do we need to prove statements in mathematics? why does mathematics insist on precision and clarity?).

Some universities use so-called 'mission statements' or 'rationale for a course' statements to precisely describe their courses in terms of knowledge an expectations. It was suggested that,

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[14] Parts are taken from a similar course taught at University of Waterloo that one of the authors (M. Lovric) taught some time ago.

in some format, these statements should enter university's official document ('course calendar'). Statements about the course should also include the following:

- 'location' of the course in terms of other courses (what are prerequisites); what courses are sequels to it; how does the course fit into a 'general philosophy' of a particular program;
- detailed list of material that students are expected to know or be familiar with;
- suggestions on how to review the background material, possibly with a good reference.

Given the reasons, and assuming that our students are convinced that learning mathematics is useful, how do they learn mathematics?

The above-mentioned syllabus document for the McMaster calculus course includes the following:

- Learning mathematics (physics, chemistry, philosophy, etc.) requires dedication, discipline, concentration, significant amount of your time, and hard work.
- To learn mathematics means to understand and to memorize.
- To understand something means to be able to correctly and effectively communicate it to somebody else, in writing and orally; to be able to answer questions about it, and to be able to relate it to known mathematics material. Understanding is a result of a thinking process. It is not a mere transfer from the one who understands (your lecturer) to the one who is supposed to understand (you).
- How do you make yourself understand math? Ask questions about the material and answer them (either by yourself, or with the help of your colleague, teaching assistant, or lecturer). Approach material from various perspectives, study solved problems and work on your own on problems and exercises. Make connections with previously taught material and apply what you just learnt to new situations.
- It is necessary to memorize certain mathematics facts, formulas, and algorithms. Memorizing is accomplished by exposure: by doing drill exercises, by using formulas and algorithms to solve exercises, by using mathematics facts in solving problems.
- The only way to master basic technical and computational skills is to solve a large number of exercises. You need to drill, i.e., solve (literally) hundreds of problems.
- It is not really possible to understand new mathematics unless one has mastered (to a certain extent) the required background material.

These are not statements to be taken for granted. Rather, they are supposed to start a discussion on the topic, motivate students to think about hows and whys of their learning.

## AD 3

For most of us, various aspects of doing mathematics are quite exciting. It could be working on a research problem, or trying to develop a new approach to teaching certain topic, or creating a good problem-solving set for our students (or all of the above). We need to 'transfer' this energy to our students. If we teach with enthusiasm and are excited about the material we are discussing, our students will be better motivated and will learn mathematics better. Although learning style is a matter of personality, there are certain attitudes that we all can 'learn' and 'act' when we lecture/teach. If we show interest in what we teach, so will our students.

## AD 4

At most universities calculus is the only course that most of the students take. It is the 'ultimate' mathematics service course, taught (quite often) to large audiences, made up of majors in almost every field: from political science and business, to business and health sciences. Often, calculus also serves as a 'gatekeeper'—a course through which all mathematics majors must (first) pass—and a 'filter' for other disciplines (e.g., engineering) looking for a quick enrolment-management tool.

Although calculus is essential to understanding certain areas of mathematics and its applications, there are other mathematics disciplines that are of equal importance, but are not taught to non-mathematics majors. New courses at the entry level are needed. For instance, students could profit from an entry-level course in linear algebra, probability, number theory, or discrete mathematics.

Why not have a course that discusses mathematics used in the human genome project? Or viruses? These could be very rich courses, spanning across areas as diverse as geometry, combinatorics, and probability.

Further suggestions for courses offered at entry-level:

- Inquiry-type course (e.g., McMaster has 'Inquiry in Mathematics' in its offering of first-year courses);
- Problem-based courses;
- Problem-solving courses;
- Foundations course (e.g., University of Waterloo);
- Geometry (e.g., SFU has a course on Euclidean geometry);
- Philosophy of math course (e.g., University of Glasgow);
- Interdisciplinary courses (e.g., Queen's University has a math and poetry course at the upper-year level; could there be an entry-level analogue?).

Among the issues to consider and useful information are the following:

- Courses developed for math teachers at Brock University became mainstream math courses;
- Some students are not allowed to take first year courses, even remedial, because they do not qualify. (Should we have courses that allow these students to take math at university level?);
- Danger of early streaming: students who change their minds are forced to retake courses
- High failure rates in first-year calculus courses;
- Review prerequisites scheme to broaden the base of students;
- Role of calculus as a filter/gatekeeper should be re-examined.

## AD 5

Communication amongst university/college mathematics instructors, mathematics educators in faculties of education, and high school teachers is very important. Opportunities abound for interaction: math contests in the schools, summer math camps, professional development workshops, and local/regional mathematics associations are some examples. We need to look for ways to foster dialogue and interaction. CMESG and CMS can play a lead role here in making the mathematics community aware of the possibilities that exist and providing easy, online access to lists of organizations and key contacts.

The aforementioned dialogue would be greatly facilitated if mathematics departments would consider hiring in the area of mathematics education, or making cross-appointments with faculties of education. Joint projects between mathematicians and mathematics educators are another possibility.

## AD 6

The question of identifying a list of essential mathematical skills that students should acquire in high school is a sensitive one. Any attempts in this direction will need to be phrased positively ('These are skills that will help you succeed.') rather than as an admonishment ('If you don't master this list of skills, you will struggle with university mathematics.') There is also the risk that such a list will be seen as "bashing" high school teachers.

One thing seems clear: the high school curriculum is over-packed. It is difficult for students (and teachers) to know what is essential when there are so many topics to be covered in so little time. The same is true for university mathematics courses. We need to take a critical look at the mathematics curriculum at all levels in an effort to provide our students with the time to explore, savour and become familiar with the truly essential ideas in mathematics. Can we teach fewer topics? What is essential and what is not?

This notion was the focus of Working Group D and we will allow them to elaborate on it.

## AD 7

Humanities departments have been more successful at arguing that small class size is integral to the way their subject is taught. The argument applies equally to mathematics but we have for too long been willing to accept large classes, especially at the introductory level.

Our best teachers are needed in first-year classes. This is the 'make or break' time for many students. A bad experience can cost us mathematics majors and ill-serve students in other disciplines.

We need to involve younger faculty too but tenure and promotion considerations often militate against their involvement in teaching issues and curriculum development. (If NSERC were to support research in mathematics education, some junior faculty would take advantage of it. It was noted in our working group that the situation is quite different in the U.S., where the NSF supports both 'traditional' mathematics research and research/projects of an educational nature; many highly funded NSF projects have focussed on the first-year experience in mathematics.)

## AD 8

Mathematics and research in mathematics education need full recognition by bodies/organizations that financially (and otherwise) support other areas of mathematics and science.

Levels of recognition include:

- Local: tenure based on strong teaching record and record of research in mathematics education ('scholarship of teaching');
- CMS needs to support research in mathematics education more strongly (supporting projects that promote mathematics is a very positive sign, but the support should not stop there); stronger ties between mathematics and mathematics education are needed;
- Funding agencies, such as NSERC, should support research in mathematics education.

# A MATHEMATICS CURRICULUM MANIFESTO

Walter Whiteley, *York University*
Brent Davis, *University of Alberta*

Breaking with CMESG traditions, this working group was conceived, proposed, and approved during the week before the annual meeting actually began. It arose from a recommendation of a meeting of the Canadian Math Education Forum (CMEF) held in Montréal just a few weeks earlier, that university-based mathematicians prepare and publish a statement on the technical and conceptual preparation that was needed for success in university mathematics courses, the point being that the "overstuffed closet" of the Canadian high school mathematics curriculum was a result of a perceived high level of knowledge required by the universities. Our deliberations on this question had a sense of urgency and purpose—a sense that folks were waiting eagerly to hear what we had to say.

En rupture avec la tradition, ce groupe de travail avait été conçu, proposé et adopté la semaine précédant notre rencontre annuelle. Il émergeait d'une recommandation issue du Forum canadien sur l'enseignement mathématique (FCEM) qui s'était tenue à Montréal quelques semaines auparavant : que des mathématiciens universitaires publient un énoncé portant sur la préparation technique et conceptuelle requise pour réussir dans les cours de mathématiques universitaires. Cela se présentait en réaction au carcan dans lequel les curricula dans les écoles secondaires canadiennes étaient pris, dû à une perception qu'une abondance de connaissances étaient requises par les universités. Nos discussions revêtaient donc un sentiment d'urgence et avaient un objectif précis, animées par le sentiment que l'on attendait avec impatience ce que nous aurions à dire.

## PARTICIPANTS

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## BACKGROUND

Breaking with CMESG traditions and protocols, this working group was conceived, proposed, and approved during the week before the annual meeting actually began. We appreciate the cooperation of the CMESG in responding to this unusual initiative. The main impetus for this unorthodox action was a recommendation that was made at a forum on school mathematics that had been hosted by the Canadian Mathematics Society (CMS) in Montréal in mid-May. A purpose for the forum was to examine ways in which the CMS might participate meaningfully in efforts to reform school mathematics in Canada.

One prominent topic of discussion at the forum was the "overstuffed closet" of Canadian high school mathematics curriculum. It seemed that a near-universal theme in teachers' and researchers' responses to proposed pedagogical innovations was that an overprescribed curriculum, with long, detailed specific lists of topics, militates against any sort of meaningful change to their teaching. The perception is that an unwieldy syllabus compels lockstep, fragmented progress and does not allow time for, let alone encourage, lingering engagements. Coupled to this perception is a widespread belief that current curricula are 'held in place' by university-based mathematicians. All-too-often, the rationale for maintaining (or even adding to) a curriculum topic is that mastery of that topic is needed to ensure that university-bound students are prepared to survive undergraduate mathematics.

With these issues in mind, a proposal went forward at the CMS Forum that university-based mathematicians prepare and publish a statement on the sorts of competencies and other preparations that are understood as necessary for success in university mathematics courses. The goal of this proposal, and of our work, is not to have university mathematicians prescribe, or even describe the necessary mathematical curriculum or the pedagogy. The goal is that university mathematicians describe what mathematics is, and what background of experiences is appropriate preparation for doing mathematics.

We, as prospective working group leaders, felt that the CMESG conference could be a good place to continue discussion of the topic, given both the timing of the meeting (immediately after the CMS forum) and the mix of teachers, educational researchers, and mathematicians that is present at every CMESG gathering. The response within the working group, and from the CMESG executive, confirmed this judgment.

## THE TASK

The May 2003 CMS Mathematics Education Forum was actually the second of three (the first was held in Québec City in May, 1995), with the third tentatively scheduled for late spring, 2005. (See Dubiel, pp. 87–89 of this volume for further details.) The principal intentions for the second forum were to present participants with opportunities to identify important issues and to propose courses of action that would—hopefully—help to frame the third forum.

One of the working groups at the Montréal forum recommended that, by the 2005 gathering, there should be a clear statement from the CMS about what students need from school and university mathematical experiences, worded to lift this perceived burden and to open up more engaging possibilities for teaching and curriculum. It was agreed that such a statement would not consist of a detailed list of topics, but would address the broader goals of such a curriculum. The end-in-view was a curriculum that offered rich coherent mathematical experiences over many concepts, in contrast to what is often described in terms of (as) superficial and fragmented encounters with a great many disconnected topics.

As group leaders, we elected to frame the collective's efforts with the task of making recommendations to the CMS for the content and wording of a 'manifesto'. More specifically, we proposed that a goal of the working group would be to draft a statement that might be useful to members of the CMS as a starting place for discussions—to focus, to mine, to hone, to problematize, and to elaborate.

## THE DISCUSSIONS

As is often the case, our discussions revolved in large part around the meanings of key terms. It quickly became apparent that at least part of our task involved the interrogation and redefinition of such seemingly transparent notions as 'curriculum' and 'mathematics'.

The word *curriculum*, in particular, seemed subject to two incompatible interpretations. On the one hand, curriculum was used in reference to formal 'must-do' lists of topics and expected levels of learner competence. On the other hand, curriculum was also used to refer to the obligation to engage in mathematically rich tasks that engender encounters with particular topics. The two conceptions are not necessarily incompatible: A well-conceived list can help to frame rich engagements. However, an over-engineered list—one that reduces broad topics into micro-competencies and rigid sequences—can have quite the opposite effect as it compels a fragmented approach to instruction.

Unfortunately, it would seem that the 'must-do' conception of curriculum prevails, and it has contributed to a popular conception of mathematics as an amalgam of discrete, mechanical procedures that are so often dismissed as irrelevant in the oft-heard statement, "I was never very good at math". This proved to be a critical point in our discussions. For many, and perhaps most adult Canadians, 'mathematics' is understood in terms of their fragmented, algorithmic high school experiences with the subject matter. Our opinions on the nature of mathematics and mathematical engagement—as mathematicians, mathematics education researchers, and mathematics teachers—would seem to have little impact on popular and dominant beliefs about and attitudes toward the subject matter.

Given this backdrop of popular opinion, it seemed to us reasonable to conceive of our task in terms of reverse-engineering a curriculum from a conception of mathematics as more integrated and imaginative. The following were among the qualities that we hoped might be embodied in a mathematics curriculum:

- Mathematics arises when engaged in extended investigations and tasks.
- Students prepare to do mathematics by doing mathematics.
- A conception of curriculum as a prespecified, micro-detailed list of topics, algorithms and competencies prevents the extension of activities; a more flexible understanding of curriculum can promote the development of critical mathematical abilities.
- Mathematics is evolving with new experimental approaches, new topics, and new tools.

Such reframings of 'mathematics' and 'curriculum' prompted our discussions toward several other issues, including matters of prerequisites, uniformity of student experience, and the simultaneous need to rethink university-level mathematics. Regarding prerequisites— a notion that underpins contemporary linearized curricula, widespread 'readiness' testing, and endless debates over 'basics'—it was agreed that current emphases on preparing learners for future studies were at the expense of immediate and deep engagements with mathematics. An emphasis on prerequisites, for example, can support a conception of a singular route through mathematics and eclipse the important realization that mathematics has multiple pathways. It was also agreed that in many cases the claimed prerequisites were an illusion—material covered but not material learned and available as grounding for the next course. It was during discussion of this issue that we agreed that an important element in a CMS manifesto would be a statement to the effect that rich extended activities on broad topics can be expected to draw in most of the topics that might be identified in a curriculum as important—and that those topics that are not addressed in a particular activity can likely be set aside to emerge in a different activity or to recur when needed.

On the issue of uniformity of student experience, it was noted that major curriculum revision efforts over the past decade, provincial and interprovincial, have been in large part framed by a perceived need for learners to follow similar curricula. This 'need' is prompted by a recognition of high mobility of Canadians. The most common argument for a unified and highly regulated curriculum is that students, quite literally, should be on the same page so that they can move without interruption between schools and jurisdictions.

We agreed that the CMS manifesto would have to address this prominent concern. A consensus among us was that the argument of mobility, in and of itself, was just as applicable to the conception of mathematics curriculum that our group had begun to articulate: In brief, students whose mathematics experiences have been framed by flexible engagement with meaningful inquiries would be well prepared to a move among similar settings. In fact, it was argued, a revisioned curriculum might help to ease the problems that are often associated with students moving from one location to another. Learners who have been involved in activities that foreground interconnections of concepts, inventions of new possibilities, and so on might be expected to be able to adapt well to new environments.

All of this being said, an issue that university-based mathematicians must themselves address is the fact that many undergraduate courses can be described in very much the same terms that are criticized here: over-engineered, focused on mastery of disconnected topics, and so on. It was thus acknowledged that the manifesto would also have to include some sort of commitment to the transformation of university mathematics courses.

With these considerations foregrounded, the working group undertook the task of a draft manifesto on our final day together. The statement at the end of this report is a synthesis of pieces that were developed in small and whole group discussion.

An important aspect of the words chosen for the statement, and the words omitted, were the tactical understanding of context and appropriate voice. In proposing this statement, we agreed that this proposal is a gamble on the deepest priorities of university mathematicians. Although most teachers have some concerns about the missing skills of their students, we concluded that a deeper discussion of the larger conceptual background which would prepare students to effectively learn and apply mathematics at the university level. There were some concerns about the expressions from university faculty in other disciplines, whose priorities do not center on the rich experiences of doing mathematics. The proposed statement would come from university mathematicians, and the discussion must be thoughtfully focused on the deeper issues, not fall into the trap of a quick survey of 'topics you want covered' as has sometimes happened in the past.

It was also recognized that this discussion is also linked to the discussions of Working Group C on the undergraduate curriculum and pedagogy. We foresee further discussions of these connections both within university departments, and at future CMESG conferences (see next steps below).

It is important to note that the text of the manifesto is written in the voice of the CMS. This mode of expression was adopted for purely pragmatic reasons. Framing the text in this way enabled us to speak more directly and to avoid endless qualifications. It is not in any way our intention to 'put words in the mouth' of the Canadian Mathematics Society—merely to offer, in as concise terms as we are able, suggestions that we hope are useful in their own discussions.

## NEXT STEPS

Despite having actually achieved the task that had been set, the group did not in any way see its work as completed. In addition to the circulation of a version of the above draft to members of the CMS for discussion, the following tasks presented themselves in our closing discussions:

- the collection of background research to support discussions within the CMS;
- participation with the CMS in the identification of commissions (likely CMS-based) for background reports, surveys, and so on, to be available as support of CMS statements;
- potentially, the preparation of a parallel document by CMESG on pedagogical issues to complement the mathematical issues emphasis of the CMS statement;
- potentially, a follow-up Working Group at the 2004 CMESG meeting, this one perhaps concerned with the topics of Working Groups C and D at the 2003 meeting;

- participation in the mathematics education sessions at the CMS meetings in December 2003 (Vancouver) and June 2004 (Halifax);
- engagement with other regional groups, including Fields (Mathematics Education Forum), PIMS (Changing the Culture, 2004), and AMQ;
- possible submission of discussion documents to regional newsletters, *CMS Notes*, and/or the CMESG newsletter;
- sharing (and perhaps collaboration) with other subject areas (including physics, biology, and statistics);
- preparation of a report as follow-up to the CMS Forum group.

At the time of this writing (November 2003), we have followed up on several of these steps. A draft version of the manifesto was presented for discussion at the June 2003 meeting of the CMS Education Committee and the draft Manifesto was a focus of discussion at the June meeting of the Mathematics Education Forum of the Fields Institute. It has also been circulated on some lists of teachers in Ontario, and to the person heading up the review of the K–12 Ontario Mathematics curriculum. It will now be the focus of discussion in a Mathematics Education Session at the CMS Winter Meeting in Vancouver, December 2003.

## A MANIFESTO

*(Drafted by a Canadian Mathematics Education Study Group Working group, for discussion with the Canadian Mathematics Society)*

The CMS endorses the general aims of the current K–12 commonly found at the beginning of mathematics curriculum documents across Canada. However, we believe that the structure of these curricula is an obstacle to student learning of mathematics. Over-specified and fragmented lists of expectations misrepresent what mathematics is and militate against deep and authentic engagement with the subject—which, in turn, reduces recruitment and retention of people into the mathematical sciences.

The aim of this document is to describe the necessary preparation for the student who intends to study mathematics in university. We are aware that this statement also implies that change is necessary in undergraduate mathematics programs, including the mathematics programs offered to pre-service teachers.

The practice of mathematics is constantly evolving. Important new approaches include modeling, and numerical and symbolic work with computers. Student needs in such a changing environment cannot be met by adding more topics (or substituting new content for old) within an already overstuffed curriculum. They must be addressed in a more fundamental way.

We find that:

- students coming out of high school mathematics must be able to engage effectively with complex problems; they require the ability to 'think mathematically'—that is, to investigate the mathematics in a situation, to refine, to expand, and to generalize;
- students' mathematics concepts must be woven into a connected set of relationships;
- students must be able to independently encounter and make sense out of new mathematics.

These aims should have priority over any specific selection of content; and it is our judgment that it is impossible to achieve these objectives if teachers are required to cover each item on a curriculum list.

In support of our view, we point out that:

- the need for detailed lists of prerequisites in mathematics has been exaggerated. While there is some hierarchy of concepts, a more appropriate image of mathematics centers on the rich problems themselves with their relationships among concepts and that highlights both multiple entrance points into topics and multiple directions for expanding one's practice.
- a mathematical topic that appears isolated to the students and the teacher reveals a problem of placement and/or selection. Choose topics that offer opportunities to generalize and to connect.
- there are diverse modes of mathematical practice, ranging from established paths and practices of logical reasoning to modeling, investigation, and technology-supported experimentation.

Although a de-emphasis on checklists would result in variations between schools, we believe that the approach to mathematics described herein would not increase problems connected to student movement among schools and educational jurisdictions because it focuses on a central goal of mathematics education—namely, teaching students to think mathematically about a broad range of situations.

While we have not yet made explicit recommendations, we hope that, in the list of this statement, ministries and boards of education will re-examine the following:

- the structures of curriculum documents and the designs of resource materials;
- support for teachers' initial and on-going development of professional knowledge;
- assessment and reporting of students' abilities to engage with mathematically rich problems, to think mathematically, and to make sense of mathematics.

The CMS is committed to supporting teachers and curriculum developers in these difficult and important tasks.

# LEARNER-GENERATED EXAMPLES

Nathalie Sinclair, *Michigan State University*
Anne Watson, *Oxford University*
Rina Zazkis, *Simon Fraser University*

This Working Group illustrates the rich opportunity that these sessions can provide for productive activity, collaboration and learning. Participants were actively engaged in a sequence of tasks and reflective discussion, working to build the theory of Learner-Generated Examples while gaining insight into their own teaching practice.

Ce groupe de travail illustre les possibilités offertes par un tel cadre de travail, alors qu'activités, collaboration et apprentissage peuvent y être des plus productifs. Les participants ont pu s'engager dans des tâches suivies de discussions, et ont travaillé ensemble afin de dégager une théorie des « exemples créés par les apprenants » tout en portant un regard sur leurs propres pratiques enseignantes.

## PARTICIPANTS

| | | |
|---|---|---|
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| Jean-Lou De Carufel | John Mason | Vicki Zack |

## PROLOGUE

*Give an example of an irrational number between 0 and 1. And another, and another, and 10 more. How were these examples found/constructed/generated?*

In this workshop participants were offered a sequence of mathematical tasks to work on in small groups, or by themselves. All the tasks involved the construction of mathematical objects in some sense, or exploration of what it means to work with 'learner-generated examples' (LGEs). Some explanation of terms is necessary here. By 'mathematical object' we mean something we can make sense of mathematically, an expression, a question, a class, a number, an equation, a diagram, a proof—anything which in mathematics can be the focus of attention. By 'construction' we mean building, creating, generating such an object rather than being given one by an authority such as a teacher or textbook. We are using the phrase 'learner-generated example' to show that when a learner creates an object it is an example of some class of such objects, and for this purpose we sometimes refer to the workshop participants as 'learners' because they were often in the position of having to construct such objects.

The purpose of the workshop was for all the participants, including the three leaders, to explore the power, purposes and pedagogic implications of asking learners to construct their own examples. Some of the tasks, and the thinking behind them, arose during the writing of a forthcoming book by Anne Watson and John Mason. Rina Zazkis and Nathalie Sinclair both had draft copies of this book, and Nathalie had been using this with practicing teachers. However, the workshop was far from being a presentation of fully-formed ideas, rather it was an arena in which all of us could work authentically to gain further insight into the issues from our own perspectives and starting points, for our own working contexts. In this sense, the workshop contributed to praxis and every participant's story will be different. This is an attempt to summarise what was spoken in the public domain.

It is widely acknowledged that mathematics is learned by engagement in mathematical activity that invites the construction of meanings which are mediated, in classrooms and textbooks, by conventional distinctions and discourse. In usual teaching, the objects of attention are given by authorities and learners may or may not see in them what the teacher intends. Thus mathematics lessons might appear to involve observing a sequence of presented things whose genesis is a mystery and which may seem to be unconnected and incoherent. As a student once said to Bateson 'we know you are offering us examples, by we don't know what they are examples of' (Bateson, 1973). We are wondering if asking learners to construct their own examples of objects on which to work mathematically is possible and useful. Certainly as teachers we have found it to be a powerful way to engage learners, and teachers with whom we have worked (experienced and new) react as if this approach makes sense to them also. The literature supports this idea as well (e.g., Dahlberg and Housman 1997). There is a substantial literature on students posing their own problems or test questions, generating their own examples for inductive reasoning, and contributing 'owned' examples for motivational purposes (such as using their own heights for statistical analysis). In this sense, the focus of this workshop was not new but was part of an attempt to bring these kinds of activity under one umbrella and see what the world looks like if we ask 'can all types of mathematical object be learner generated?'

## TASK 1:

*Each participant was asked to say their name and give a number between 99 and 100. Then they were asked to discuss in small groups how they chose their number.*

This task was intended to be introductory on several levels. It proved to be unexpectedly rich socially, pedagogically and mathematically. Socially, of course, it got people talking but, in addition, it illustrated that the business of choosing a number also contained an element of self-presentation. Some people were aware that their choices would give an impression to others of their mathematics and this influenced the way they chose numbers, as well as their emotional response to the task. For some, the element of competition was mathematically fruitful in pushing them to expand their example space, while for others it was inhibiting. In addition, for some there was an attempt to 'guess what the teacher is thinking' and to try to anticipate what the workshop leaders might be looking for. In fact, we were looking for discussion of these reactions! Mathematically, the richness of the task came from the fact that people were working on 'number' in creative and imaginative ways— the property of 'richness' was in the ways people worked, not in the task itself which also had the potential to become superficial. One pedagogic observation was that it was the discussion which made the task rich, but several people reported that the way they had chosen their number was not at all superficial, hence showing that individuals can enrich a task by the way they impose personal constraints and private rules. For example, some 'learners' used their knowledgeable reactions to numbers involving 9s and 1s to offer numbers which expressed a sense of 100 as a limit; one gave a number which had already been said 'ninety-nine and a half', but in a different form ('ninety-nine point five') to focus on representation; one used a number he had used earlier in the day for something else and offered '100 minus one over root 2'. This last offering was hotly debated since some people thought it was an operation rather than a number. One table engaged in an exploration about the likelihood of a number randomly selected between 99 and 100 being rational or irrational; another table offered research evidence to claim that people tend to choose numbers towards the upper end of an interval.

The responses to this task offer fragments towards a theory of LGEs:

1. Learners select from a personal, task-specific example space influenced as much by the situation and their expectations of it as by their prior knowledge. This space is likely to be a proper subset of what they know.
2. A 'good' example has to be seen to be an example of something.
3. Examples are examples of something. A mathematical object may be a member of multiple classes; what is seen by one learner to be an example of class G may be seen by another to be an example of class H.
4. Sharing examples exposes a range of possibilities which may not have occurred to others, but one person cannot necessarily make use of the examples given by another. New examples are more likely to be taken up by other learners if there is proximal relationship to someone's existing example space, and if they are seen to be useful or interesting in some way.

## TASK 2:

*Write down a pairs of numbers which differ by 2.*

> *… now write down another pair of numbers which differ by 2.*
>
> *… now write down another pair of numbers which differ by 2.*

This task was done individually and then discussed in groups. The difference between writing privately and giving publicly was deliberate, since the pressures of speaking out had been raised earlier. It was interesting that for some people this was an entirely new task, but for others the extended example spaces developed in the previous task were used for this one. One participant said that she found herself using numbers for task 2 she would never have chosen if she had not just heard the discussion of task 1. Some reported trying to find interesting pairs, but the interpretation of 'interesting' is personal. For some, (1, -1) was interesting since it straddles zero which gave it a singular status; others developed generating sequences which would produce as many new pairs as were required; others had personal, social or emotional reasons for choosing certain numbers. For most, the decisions about how to create second and third pairs were mathematical in some way, such as creating a pattern, or going to some extreme, or stating a generality of some subclass of possibilities. This led to further fragments:

5. Learners can employ personal constraints and goals which can make a task more interesting for them. Constraints and goals may have the same effect, although differently expressed.
6. The effect of having a generalisation expressed can be to energise or to de-energise. Sometimes it is more useful to hold the generalisation until a particular is needed; other times exploration of particulars is interesting enough to postpone, or put aside, generalisations.

## TASK 3:

*Write down a time at which the hands of a clock are at 90 degrees.*

> *… now write down another time when the hands of a clock are at 90 degrees.*
>
> *… now write down another.*

This task is, for many, intrinsically more mathematically interesting than the previous task, but is of the same generic structure. We did not offer a plenary discussion of the mathematical findings, although plenty of time was spent on the maths. Instead we asked participants to compare the two 'similar' tasks, since both ostensibly ask for LGEs and push learners beyond their immediate offerings.

Theoretical fragments:

7. Asking for more than one example can force exploration of a class of objects, and more particularly of mathematical structures as learners find out what is possible.
8. It makes a difference how familiar the class is; how many dimensions of possible variation it has; how easily it can be explored; whether there are finitely or infinitely many possibilities …
9. If an example is not easy to find, learners may be pushed towards generalisation which suddenly gives access to multiple examples.
10. Limitations and possibilities may be perceived differently by teachers and learners.

## TASK 4:

*Teacher: 7 squared take away 1 is … ?*

*Learner: 48*

*Teacher: 6 times 'what' is 48?*

*Learner: 8*

*Teacher: now you make up one like that for me*

*Learner: 10 squared take away 1 is … ?*

*Teacher: 99*

*Learner: 9 times 'what' is 99?*

*Teacher: 11*

This task continued to be passed between participants around the room until someone said 'n squared take away 1'. This task was offered as an example of the kind of repetition which can help learners participate in mathematical structures through speech patterns. In the literature on problem posing, researchers have shown that learners usually copy the kinds of questions asked by their teachers, and this is often taken to be a negative result. In this task, we try to show that such copying can be a positive feature of classrooms when used to demonstrate generalities through LGEs. However, learners may not see the same generality which the teacher intends, and there was discussion of how to handle this in the classroom without being insensitive to learners. An example arose in which the 'teacher' tried to start another 'round the room' sequence.

*Teacher: what number must I take away from 4 to get -4?*

*Learner: 8*

*Learner: what number must I take away from 5 to get -2?*

The teacher had intended to work on 'n – (-n) = 2n' but the learner had apparently taken it to be an example of more general subtraction of negative numbers. Discussion of this seemed to converge towards:

11. Learners may not sense the same generality that the teacher intends; being 'wrong' can be uncomfortable unless a classroom ethos has been strongly established that there are always other possible generalities and these may have to be put to one side.

## TASK 5:

Find a *data set of seven items which have a mode of 5, median of 6 and mean of 7.*

*… and another*

*… and another*

*Alter your set to make the mode 10, the median 12, and the mean 14.*

*Alter your set to make the mode 8, the median 9 and the mode 10.*

Discussion of this task led to the observations that, for many, it had not felt like an example-generating situation until a decision had to be made about whether and how to vary the set to form new ones. The scope of possible variation, and the ranges of numbers it was possible to have, and their relationships, were all significant. It was generally agreed that this had led to significant engagement with the structures of these averages—though, as with all tasks, not for everyone. Supplementary questions about the smallest possible data set for which these averages can have any pre-assigned values, and how much choice there is for data-sets, were provided but also arose from the mathematical activity. One participant said that 7 data points had not provided him with enough challenge, so he had decided to limit himself to 5. It was important that the participants used a range of different images and methods, such as 'balancing' or 'algebraic', to develop their examples, because this gave rise to the first of the following fragments:

12. Learners' example spaces are structured in individual ways, and these structures depend also on the initial entry into the space, different images possibly giving access to different relationships.
13. 'To find' questions are asking for LGEs, but the number of members of the example space may be limited to one. Thus 'finding' can be seen as imposing a sequence of constraints on possibilities until only one example fits all the constraints.
14. Creating an LGE can lead learners to engage deeply with the meaning of concepts; varying constraints can expose assumptions.
15. Creating examples can involve searching through dusty corners of knowledge, or creating new things from known things.

## TASK 6:

We watched a video of a teacher, a student of Nathalie's, who has decided to include LGEs as a central feature of his teaching. He was working with a class who was by then used to participating in a range of ways. He asked them to choose numbers between 75 and 100 and to design prisms which would have that number as their volume. After some discussion about what was meant by 'prism' students started work. Using the 'another and another' prompt he encouraged students to generate more than one idea, and some chose to do this by adapting or transforming their first ideas. Participants spent some time exploring the task themselves. We then watched on video three students showing the rest of the class their ideas at the end of the lesson, including one who decided to assign an area of 1 to his cross-section, and discussed what had happened. Some issues which emerged were:

- Choice of total volume was a good idea to engage learners and ensure a variety of possibilities, but the choice of number significantly affects the nature of the task.
- The student who chose a cross-section of 1 may have done it as a joke, but it showed the power of using extreme examples.

- Had the teacher thought about why only boys were demonstrating their answers? Had he thought about the purpose of sharing results?
- Indeed, there was extended discussion of whether sharing results was ever useful unless learners had a purpose for listening to each other, such as comparing methods, deciding which is most powerful, seeing if there were generalisations to be made, and so on.

The teacher's view is that these new-to-him strategies he is using have led to significant improvement in the participation of students and his own enjoyment of teaching.

Pedagogic theoretical fragments arising from this video and discussion included:

16. Listening to learners includes giving up detailed advance planning.
17. Listening to learners involves giving up authority to learners for the direction of the lesson, and depending on the intrinsic warrants for validity within mathematics, expressed through discussion and mathematical activity, such as exemplification and counter-exemplification
18. There may be an optimal level of constraint; too much constraint may stifle creativity or willingness to engage.

Giving up authority creates ethical implications for whether and how the teacher, or the whole class, handles 'wrong' answers (Chazan & Ball, 1999).

## TASK 7:

*Give[15] an example of an arithmetic sequence, and another, and another.*

Rina has for some time been asking learners to give examples of concepts, or classes of objects subject to certain constraints, as a regular feature of her teaching (Hazzan & Zazkis, 1997, 1999; Zazkis & Liljedahl, 2002). She offered us some transcripts arising from clinical research interviews with students who were learning about arithmetic sequences. The aim was to ask students to give numbers which would appear in a given sequence, and to ask them also if particular numbers would or would not appear, and why. Thus students' understanding of the generality expressed by the sequence could both be explored and would also develop within the interview. Below are the specific excerpts presented in the working group.

EXCERPT #1

| | |
|---|---|
| Interviewer: | Okay. I would like you to look at a different sequence, and it is 17, 34, 51, 68 and so on. And I would like to ask you about the number 204. Is it an element of this sequence? |
| Dave: | If it's a multiple of 17 it is . . . |
| Interviewer: | And if it is not a multiple of 17? |
| Dave: | Then it shouldn't be. |
| Interviewer: | So this will guide your decision. |
| Dave: | Um hm. |
| Interviewer: | So 204 is indeed 17 x 12. . . |

[15] Not all LGE tasks need to be of this form—but we have found that it can be particularly fruitful. Look in Watson & Mason (forthcoming) and Bills, Bills, Watson and Mason (2004) for many more.

| | |
|---|---|
| Dave: | Then it's in. . . |
| Interviewer: | It's in. Can you please give me an example of a big number which is in this sequence? |
| Dave: | 17,000. |
| Interviewer: | Another one. |
| Dave: | 17,051. |
| Interviewer: | Okay. What makes you believe that 17,051 is an element of this sequence? |
| Dave: | (pause) 17,000 is 1,000 x 17 and that's a multiple of 17. . . |
| Interviewer: | Um hm. . . |
| Dave: | I also know that 51 is a multiple of 17, and so it's the 3 x 17, so I add 1,003 17's, I've still got a multiple of 17, it's still going to be in there. |

EXCERPT #2 — CONSIDERING SEQUENCE 8, 15, 22, 29…

| | |
|---|---|
| Interviewer: | Okay, so can you give me an example of a number that you believe is not in the sequence and example of a number that you believe is, or could be in the sequence? |
| Leah: | Um, I don't think 714 would be in the sequence, um, a number that could be, I would just pick a number that hasn't a factor of 7, so like 511 possibly, or something |
| Interviewer: | And you are saying possibly because. . . |
| Leah: | Just, I just picked a number that wasn't, didn't have 7 as a factor. |

EXCERPT #3 — CONSIDERING SEQUENCE 17, 34, 51, 68, …

| | |
|---|---|
| Interviewer: | How would you decide whether 204 is an element in this sequence? |
| Eve: | Okay, (pause) okay I guess I would use 204 and divide by the first number in here, because it looks like, when I'm looking at this sequence it looks like um all these numbers are multiples of 17, so if 204 is a multiple of 17 which means that it will also occur in this sequence, so in order to be a multiple of 17, 204 divided by 17 must give us a result of a whole number and no decimal places. |
| Interviewer: | Okay. . . |
| Eve: | So 204 divided by 17, that gives us 12, okay it's 12, this whole number, so it's a number in this sequence. |
| Interviewer: | Okay. Can you please give me an example of any 4-digit number in this sequence? |
| Eve: | I could just randomly pick any, okay. . . |
| Interviewer: | Yes, please pick any, but convince me that it is in the sequence. |
| Eve: | Okay. 17, um, (pause) I just keep on adding 17 to get um this sequence up 85, 102, 119, 136 and 153. . . |
| Interviewer: | Yeah, this is a pretty hard work. . . |
| Eve: | Yeah. . . |
| Interviewer: | If I want a 4-digit number, it will take you quite a while to get that. . . |
| Eve: | Oh, you want a 4-digit number. . . |
| Interviewer: | Yeah. . . Eve: Umm, (pause) I don't know how to do this. |

EXCERPT #4 — CONSIDERING SEQUENCE 8, 15, 22, 29, …

| | |
|---|---|
| Interviewer: | So 704 is not divisible by 7, none of these elements in this sequence you believe will be divisible by 7, so can you draw conclusions from what you have now? |
| Sally: | It's, it's um very possibly in this set. |
| Interviewer: | Um hm. What, what will convince you? |
| Sally: | (laugh) Well just because it's not divisible by 7, doesn't mean it's in the set, right? |
| Interviewer: | Can you give me an example of a number that you know for sure that is not in this arithmetic sequence? |
| Sally: | Um hm, um 700. . . |
| Interviewer: | Another one. . . |
| Sally: | Um, 77. |
| Interviewer: | Okay. And how about 78? |
| Sally: | It may be in the set, but it's not divisible by 7. . . |
| Interviewer: | (laugh) So 77 you're sure is not, 78 you're not sure. |
| Sally: | Right. |
| Interviewer: | 79? |
| Sally: | Could be. . . |
| Interviewer: | Could be. 80? |
| Sally: | Could be. . . |

EXCERPT #5 — CONSIDERING SEQUENCE 2 ,5, 8, 11, …

| | |
|---|---|
| Interviewer: | Could you please give me an example of a number, and I would like a relatively big number, like 3-digit number or 4-digit number, that you're sure will be listed in this sequence [2,5,8, ...] ? |
| Sue: | Mmm, okay, I'm guess it has to be a multiple of 3, because it's common difference, so um 333, maybe? |
| Interviewer: | So you think that 333 will be listed in this sequence? |
| Sue: | I think so. […] Sue: Hmm, wait a minute, 360 is a multiple of 3, yet I just said that it didn't go in, right... |
| Interviewer: | You did. . . |
| Sue: | So then this might not go in there, I don't know, um, (pause) I'm not sure (laugh). I think I'll have to guess a couple, I'll have to do trial and error to figure it out. |
| Interviewer: | And what do you mean by trial and error here? |
| Sue: | Like um, I'm going to start with pick a couple of numbers that I think would work and then put it back into this formula. . . |
| Interviewer: | Okay. . . |
| Sue: | To see if I get a whole number. . . |
| Interviewer: | For? |
| Sue: | For N. (The interview excerpts are from Zazkis & Liljedahl, 2002.) |

After reading excerpts from several transcripts, participants were invited to continue the transcript by imagining the next few interchanges. They were then asked to provide another continuation, and then another continuation, thus using the LGE strategy on our own learning about the use of examples as a way to engage with mathematical structures. During our discussion of these tasks the following arose:

19. The concept of 'sufficiency' relates to whether the properties the learner is using are those the teacher imagines are operating. Asking about exemplification both ways, ('give me an example' and 'is this an example?') gives a structure to explore this further.

## TASK 8:

Participants formed groups to produce teaching sequences which used LGEs, either using a strategy employed already in the workshop or inventing 'new' ones. Thus everyone had experience of thinking through the pedagogic issues. Teaching sequences could be to teach us something new, or to look at some old knowledge in new ways. Tasks presented are left for readers to contemplate:

- Make up a problem whose answer is ${}_7C_3$ ('7 choose 3')—these were then accepted or rejected by the 'teacher' saying 'I like it' or 'I don't like it'.
- Choose a number between 9 and 19. Make triangles with integer sides, the lengths of which add up to that number, and another, and another ….
- Make up a number system which is based on some system of (cultural?) values.
- Ask learners for an example of a quadrilateral, then ask for one with a constraint that excludes the quadrilateral just drawn, then another, and so on. What is the longest string of examples which you can create using this structure and any class of objects?
- Give me a number bigger than 1; bigger than 2; … bigger than 17 000, …
- Construct 'greater than' and 'less than' statements about the numbers of pens each pair of people at a table of four have. Pass the statements to an adjacent table and they have to interpret the statements to put people in order of their possession of pens.

Finally, each participant constructed a statement about what had struck them about the content of the workshop. Many of these were restatements of the theoretical fragments reported above, so only a few further reflective thoughts will be summarised here:

> *The learners' choice of examples tells the teacher how they are classifying and what they have as prototypes (Participants referred to Women, Fire and Dangerous Things by George Lakoff).*
>
> *Errors provide opportunity for growth in a safe environment.*
>
> *LGEs are only useful if the teacher is working on strategies for reacting to and valuing what is generated, and how to handle surprises.*
>
> *Yes-examples and no-examples are both of value, cf. concept-attainment.*
>
> *LGE tasks give a way to move students 'out of the box', to be inventive, to feel safe while on the edge of not-knowing.*
>
> *Asking for LGEs to be constrained in some way forces awareness (Gattegno).*
>
> *Shifts from algebra to number, or number to algebra, change the nature of the exploration.*
>
> *Sometimes LGE tasks cause one to look for particulars, sometimes for patterns and generalities.*
>
> *LGE tasks are cognitive and constructive.*

## EPILOGUE

The workshop concluded with offering participants several additional tasks for exploration. These tasks are from Watson and Mason (2005, hopefully). We included some of them here for the readers' perusal.

INTER-ROOTAL DISTANCE

We have decided to call the horizontal distance between neighbouring roots of a polynomial function the inter-rootal distance. Imagine a quadratic equation with two real roots. What families of quadratic curves have the same inter-rootal distance?

INTER-ROOTAL DISTANCE CONSTRAINED

Find three different examples of quadratics whose roots are 1 and 2.

FINDING PRIMES

Find a prime number which cannot be expressed as 4k + 1 for any positive integer k. Find a prime number which cannot be expressed as 6m + 1 for any positive integer m. Find a prime number which cannot be expressed as 8n + 1 for any positive integer n.

SELF-PERPENDICULAR HEXAGONS

Draw a hexagon with each pair of opposite edges perpendicular to each other.

## ACKNOWLEDGMENT

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# ÉTUDE DES PRATIQUES D'ENSEIGNEMENT

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This Working Group was chosen as an example of a bilingual session at its best. The report reflects the way the working group was run, moving smoothly between French and English. Participants spoke their preferred language slowly and clearly; translation was provided when needed. The discussion was lively, engaging and thought provoking—an excellent experience for all.

Un parfait exemple d'un groupe de travail bilingue à son meilleur. Le rapport illustre le fonctionnement du groupe, qui passait harmonieuse du français à l'anglais, et vice-versa. Les participants s'exprimaient dans la langue de leur choix, lentement et clairement, et la traduction était offerte au besoin. La discussion a été vivante, engageante et riche – une excellente expérience pour toutes et tous.

## PARTICIPANTS

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## INITIAL DESCRIPTION OF OUR WORKING GROUP

The study of teacher practice is increasingly taken into account in research in mathematics education. There is increasing evidence of influences on teacher practice—such as context and situation, subject matter knowledge, beliefs, and pedagogical knowledge. From pre-service, to in-service, to graduate studies, to research, the effect of teaching practice on learning is known to be essential—such as the impact on teacher behaviour and possibly student achievement (although to date the direct relationship between teaching and learning has not been shown). The concept of teaching practice is central to research with an interest on the impact of teachers' work.

In this working group, we wanted to discuss various questions about the study of teacher practice around three dimensions: 1) the foundations of the notion of practice, 2) the reasons for studying teacher practice, and 3) the possible methodologies to do so.

1. The concept of teacher practice relies on the various perspectives, beliefs, and conceptualizations of "practice". What are the underlying (implicit) principles of the concept of practice? What are some of the different meanings the notion of practice can have in various epistemological perspectives? In what sense is teaching a practice? How is the concept of practice useful for mathematics education?
2. Why should the concept of practice be studied? What can we learn from the study of teacher practice? From the point of view of students' learning? From the point of view of in-service teacher training? From the point of view of pre-service teacher training? From the point of view of research in mathematics education?
3. Using particular elementary school level and secondary school level examples of teacher practice to explore the methods of analysing mathematics teaching practice,

what dimensions should be taken into account? How can we describe teaching practice? Is it possible to refer to teacher practice without referring to a particular teacher's practice? Is it possible to refer to teacher practice without taking into account the content that is taught?

## BACKGROUND

The initial discussion of the working group illuminated a number of common and shared conceptions of teacher practice. Common themes emerged, for example: i) the importance of beliefs and confidence in oneself as a teacher; ii) concerns for the selection and management of materials, resources, and mathematics content; iii) the importance of the knowledge of students and of self—used in the processes of interpretation and understanding of classroom experiences, and in making pedagogical decisions with respect to particular tasks for mathematical learning opportunities; and iv) social and cultural factors that influence and impact the teachers' classroom practices and the learners' mathematical experiences.

These emergent themes began to coalesce into three overarching conceptions: "institutional/logistics", "personal/professional/theoretical", and "classroom students" or alternatively conceptualized as "a priori knowledge", "responsiveness in the classroom", and "post-reflective turns", respectively. These categories and category titles segued into an articulation that as researchers or classroom teachers, we carry various frameworks of conceptions of mathematics teaching and learning, and that these frameworks are often evident theoretically in our thinking as well as practically in our actions.

### DIFFICULTÉ À CERNER LA NOTION DE PRATIQUE D'ENSEIGNEMENT

Au cours de la première séance du groupe de travail, nous avons tenté de définir en quoi consiste la pratique d'enseignement. Dans la littérature scientifique, il se dégage consensus à l'idée que le travail de l'enseignant ne s'effectue pas uniquement pendant le temps de classe. L'enseignant travaille aussi en dehors de la classe. Étudier les pratiques d'enseignement exige alors de s'intéresser au travail de l'enseignant en classe et en dehors de la classe. Pour Wenger (1998), la pratique relève du "faire", dans ses dimensions à la fois historiques et sociales, et dans sa capacité à produire de la structure et une signification aux actions. Il en découle plusieurs caractéristiques d'une pratique :

1. La pratique relève du faire mais n'est pas réductible aux actions posées par l'enseignant et/ou observées par le chercheur. Elle doit inclure le système qui les oriente. Dans ce sens, le chercheur ne doit pas s'arrêter aux actions de l'enseignant mais tenter de remonter aux logiques de ces actions et leurs déterminants.
2. Une pratique s'inscrit dans la durée et dans une historicité. Une activité isolée, sans précédent historique, sans référent historique, sans fondement historique ne relèverait pas de la pratique.
3. Cela ne signifie pas que la pratique n'évolue pas. Au contraire, une pratique est dynamique, elle évolue dans le temps. L'étude de la pratique vise, entre autres, à « déceler le permanent dans le changement (…)» (Weil, 1996).
4. En tant qu'activité humaine, une pratique est ancrée dans un contexte social. Une pratique est socialement partagée, au moins partiellement. L'enseignant n'est pas un être solitaire. Il appartient à des collectifs de travail, qui influencent ses choix.
5. Toujours selon Wenger (1998), le concept de pratique inclut à la fois le champ de l'explicite (le langage, les outils, les documents, les symboles, les procédures, les règles que les différentes pratiques rendent explicites), et le registre du tacite (relations implicites, conventions, hypothèses, représentations sur le monde).

Dans l'étude des pratiques, le chercheur est pris par un certain nombre de tensions qui ont fait l'objet de discussions au sein du groupe.

1. Singularité / régularité
2. L'analyse du travail de l'enseignant est située et conditionnée par un ensemble de facteurs. Le chercheur doit distinguer ce qui dans l'action de l'enseignant est singulier, de ce qui témoigne d'une certaine régularité. Il doit pour cela observer l'enseignant durant une longue période et faire varier certaines variables selon l'objet de son étude (différentes classes de même niveau ou de niveaux différents; différents objets d'enseignement, etc.). Le risque est de prendre pour générique une action spécifique de l'enseignant.
3. Particulier / général
4. Les pratiques d'enseignement sont avant tout des pratiques d'enseignants. Ce sont ces derniers qui pratiquent l'enseignement. Une pratique est donc toujours une pratique de quelqu'un. Comment alors la pratique d'un enseignant rend-elle compte d'une pratique d'enseignement au sens générique du terme. En outre, il n'y a pas d'enseignement sans un objet d'enseignement; il n'y a pas de pratique sans un objet de pratique. Comment alors la pratique d'un enseignant relativement à un objet d'enseignement rend-elle compte de la pratique d'enseignement de cet enseignant ?
5. Microscopique / macroscopique
6. Le niveau de grain d'analyse est un enjeu important dans les choix méthodologiques d'analyse de la pratique d'enseignement. La centration sur l'identification de régularités amène à utiliser un grain d'analyse assez gros. En revanche un grain microscopique rend difficile de dépasser le caractère singulier des observables.
7. Point de vue du chercheur/point de vue de l'enseignant
8. Le point de vue du chercheur seul n'est pas suffisant pour rendre compte de la pratique d'enseignement. La prise en compte du point de vue de l'acteur est primordiale.

## TEACHER EFFICACY - AIMS

The aim of this section of the working group was to articulate a possible framework to encompass these expressions of the factors that reflect our own teaching practices. Encouragingly, the three conceptual frameworks selected prior to the working group meeting appeared to encapsulate the working group's expressions, thoughts, and feelings. The three conceptual frameworks are: teacher efficacy (Bandura, 1997; Tschannen-Moran & Woolfolk Hoy, 2001; Tschannen-Moran, Woolfolk Hoy & Hoy, 1998), teacher orientation (Feimen-Nemser, 1990), and teacher concern (Borich & Tombai, 1997; Fuller & Bown, 1975). The following few paragraphs will describe these conceptual frameworks in relatively general terms.

> *We teach who we are. So* ***who*** *is the Self that teaches? From the perspective grounded in teacher formation, that Self is the* ***who*** *we are 'disposed' to be, not the* ***who*** *external forces maintain we are 'supposed' to be (Hare, 2007, p. 143, bold in original).*

Specific beliefs that teachers carry regarding their confidence in their teaching ability, within the context of teaching, are called teacher efficacy. These beliefs also pertain to the teacher's beliefs of his or her capacity to affect student performance. Teacher efficacy has been found to be a powerful construct that appears to explain and/or predict many aspects of teaching and learning. For example, teacher efficacy is related to student achievement (Tschannen-Moran & Woolfolk Hoy, 2001, 2002), pre-service teacher behaviours and pre-service teacher preparation (Ashton, 1984; Bruce, 2005; Gordon & Debus, 2002; Watters & Ginns, 1995), in-service professional development effects (Ross & Bruce, 2007), attitudes towards children and control (Woolfolk & Hoy, 1990), and mathematics reform efforts (Smith, 1996; Wheatley, 2002). This is not an exhaustive list, however it appears research into teaching and learning has benefited from the teacher efficacy perspective, and the working group's thoughts and reflections on teacher practice appear to align with this framework.

Feimen-Nemser (1990) suggests that the 'orientation', or complex mix of orientations, held by the pre-service program faculty influences the development of, and subsequent classroom practice orientation of pre-service teachers. Feimen-Nemser (1990) conceptualized teacher orientation with five 'extreme' cases: Academic, Technological, Practical, Personal, and Critical/Social orientations. The diversity of orientations reflected as attitudes and perspectives held by pre-service education faculty was understood to have influence, to impact, to align with, and/or to conflict with, the developing pre-service teachers' sense of self and classroom instructional behaviour. Feimen-Nemser (1990) and Cotti and Schiro (2004) noted pre-service teachers' orientations may reflect the faculty orientation to beliefs about the purpose and means of teaching mathematics. In practice, an institution or an individual teacher would express a mix of these orientations. The combination and integration of these orientations would be apparent in the focus of a pre-service education course, and subsequently may become apparent in pre-service teachers' articulations and expressions of their classroom practice, e.g., classroom management or instructional strategies.

The following is the prompt for the working group, and the descriptive paragraphs of each teacher orientation used in the working group discussion:

*"Which of these most appeals to your sense of yourself as a mathematics teacher?"*

ACADEMIC ORIENTATION

> *The teacher's academic preparation is vital. The knowledge of the structure, concepts, skills, and processes of mathematics is the fundamental basis for successful teaching in a secondary school math classroom. I know the mathematics and my professional treatment of mathematics determines the quality of my teaching.*

TECHNICAL (TECHNOLOGICAL) ORIENTATION

> *There are tried and true skills, processes, and steps to follow in order to be an effective classroom teacher. There are basic principles and procedures to be used by teachers to achieve specified goals. In order to be a successful teacher, it is necessary to develop proficiency in the skills of teaching.*

PRACTICAL ORIENTATION

> *The greatest source of knowledge about teaching mathematics is the experience of teaching mathematics. To become a successful classroom mathematics teacher, one must be immersed in the classroom environment as the teacher. Daily practical dilemmas and situations in the classroom provide the opportunities to develop and hone the teacher's ability to learn to teach and develop the practical wisdom of expert teachers.*

PERSONAL ORIENTATION

> *To be a good teacher one must know students as individuals. In order to select appropriate materials and tasks for student learning, the teacher must know the student's individual interests, needs, and abilities. In addition teachers must know themselves and work towards personal fulfillment and meaning as a classroom teacher. These dual goals intersect and the attention to the personal development of the students and the teacher creates the opportunities for quality learning and successful teaching.*

CRITICAL/SOCIAL ORIENTATION

> *The critique of schooling in combination with a progressive social vision provides the optimal classroom environment for quality teaching and learning. A teacher's social justice focus empowers students to connect the relevance of mathematics to their personal identity and find influential experiences in the larger local and/or global community.*

Teacher concerns are the perceived problems or worries of teachers (Fuller, 1969). Teaching behaviours and classroom practices for teaching and learning mathematics arising from teachers' beliefs may be related to teachers' concerns. Fuller and Bown (1975) identify a set of concerns experienced by teachers and suggest that these concerns are somewhat linear, in that teachers progress through stages of concerns. Initially, teachers experience concerns for 'self' in a focus on survival and "*one's adequacy … as a teacher, about class control, about being liked by pupils, about supervisors' opinions, about being observed, evaluated, praised, and failed*" (p. 37). Then, stretching into the first years of teaching, teachers' concerns turn to 'task', the knowing and presenting of the mathematics content, lesson timing issues, and other instructional duties, and then, to 'impact' and being aware of the learner and his or her needs, and the evaluation of learning fairness.

These stages are also articulated within the context of power in the classroom (Staton, 1992). This sense of power in the classroom is pre-service and in-service teachers' sense of control and being controlled, their sense of power exerted by them in the classroom and exerted on them from outside the classroom. The teachers' sense of control is also related to their sense of teacher efficacy, for example, their beliefs in their ability to provide effective classroom behaviour management. Erikson (1993) suggests that there exists a continuum on which beliefs and classroom practice change at different rates, and it is possible that varying degrees of all three concerns will be evident in teachers' expressions of beliefs and classroom practice (in Muis, 2004).

The following is the prompt for the working group members, and the descriptive paragraphs of the three teacher concerns used in the working group's discussion:

*"Which of these most accurately represents the concerns you have in your mathematics teaching at this point?"*

SELF

> *I feel stress about class control—that is, classroom management and discipline and dealing with student behaviours in my class. I think about having to master all the content of courses I am to teach, and not knowing the answers to the mathematics questions my students will ask. I am concerned about being evaluated by supervisors, like my Associate Teacher during practicum, and soon my Principal when I start teaching in my own classroom. I feel in survival mode and sometimes I wonder how I will ever learn to teach at all.*

TASK

> *I worry about finding the appropriate teaching method. It is vitally important to find the right materials too, like the right activity, task, computer applet, and using the smartboard or graphing calculators. For each topic I worry about how I am going to teach it. I know the math I am to teach, I am just unsure how I should teach it.*

IMPACT

> *Assessment is the first thing I think of and I worry about finding and then using the right assessment tool (for example: rubrics, achievement categories, checklists, performance criteria, marking schemes). I look for the right task and materials to adequately measure student achievement. Fairness and equitable assessment is*

*important because I work to continually recognize the social and emotional needs of my students as well as their intellectual development.*

## DISCUSSION PROCESS IN THE WORKING GROUP

During the working group session, the participants were asked to read the paragraphs of teacher orientation from the perspective of a classroom teacher, and to reflect upon and draw a picture of their own individual teacher orientation, using intersecting circles where size, position on the paper, and overlap with the other circles, would indicate the interactions and connections of the five orientations. Participants then shared their diagrams with each other and the group as a whole. The paragraphs for teacher concern were then read, from the perspective of the classroom teacher. A short response was written to the prompt, "*What is your teacher concern? Give an example to support your claim.*" This was then shared with the working group. Teacher efficacy was explored through the Teachers' Sense of Efficacy Survey (Tschannen-Moran & Woolfolk Hoy, 2001). The working group as a whole discussed the implications of our own teacher efficacy with regard to the 'study of teacher practice'.

The purpose of the presentation of these three theoretical constructs was not to provide a rigid structure on which to fit all future discussion, but to provide a possible conceptual framework to deepen an inquiry and understanding of teacher practice, which might align with the themes and relationships expressed by the working group members about 'the study of teacher practice'.

## PARTICIPANTS' PERCEPTIONS OF CHALLENGES IN THE STUDY OF TEACHING PRACTICE

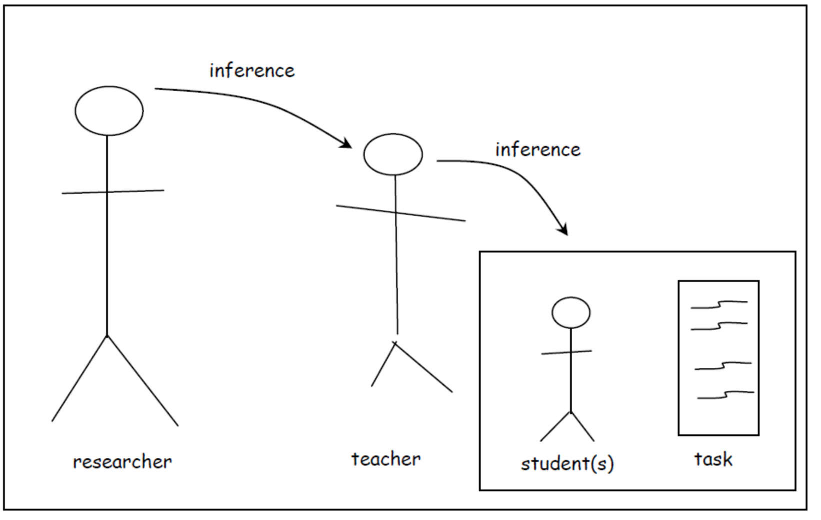
At the end of the working group, we asked the participants to write down the challenges they believe the study of teacher practice needs to face. In the following paragraphs, we will present some excerpts of the participants' answers.

Several participants mention the complexity of teacher practice as one of the challenges of its study. For one participant, the number of variables involved constitutes a problem: "*The complexity of the factors involved in teaching practice makes the study of teacher practice challenging*". This complexity translates on one hand into the necessity of using a large quantity of data, and on the other hand, the challenge of "*making sense of abundant data*". The concept of practice itself also seems to lack clarity, as one of the participants identified "*defining what is (constitutes) practice*" as one of the challenges.

On a methodological level, one participant points out that the study of teacher practice needs to deal with different levels of inferences. "*For example, the teacher makes inferences on how students are "dealing" with the task and the researcher makes inferences on how the teacher is "dealing" with the students "dealing" with the task.*"

In the same way, the researcher needs to put aside his own point of view in order to get into the teacher's logic. « *L'effort de mettre de côté temporairement* la *posture du chercheur pour pouvoir rentrer dans la logique de l'acteur*. » Furthermore, teachers' and researchers' theoretical points of view are not necessarily the same, which generates a "*potential […] of synergies or disconnects between teachers' and researchers' theoretical perspectives*."

Still on a methodological level, the tension between the individual practice of a particular teacher that has been observed and the practice of teachers in general is also seen as a challenge by several participants. On one hand, a practice necessarily has an individual part, which is difficult to generalize: « *Une difficulté est le côté propre individuel d'une pratique et donc, le peu de généralisation possible.* » For another participant, the generalization is problematic, because practice is necessarily related to a particular context: "the context is complex and an essential factor". The same participant also wonders what the researcher is seeing while analyzing a teacher's practice: "*A teacher's practice is organic, making it difficult to be able to capture any more than a snapshot of the practice (or a series of snapshots…?)*".




The tension between what is particular to a situation of a teacher and what can be considered as general appears, for one participant, both in what the researcher sees and what his aims are. Therefore, he needs to decide whether "*we are dealing with the particular or the general"* and whether *"we want to say something particular or general*". Another participant wonders how the presence of the researcher influences the practice of the teacher: "*Are we seeing a "real" picture, or a "special occasion" for the purpose of the research?*" The tension between the particular and the general is also mentioned in relation to the research methodology: if the researcher chooses a fine-grained analysis, the results will be difficult to generalize, and if the analysis is too general, results will be vague.

Finally, one participant sees one of the main challenges of the study of teacher practice in its use during teacher training: « *Un des défis de l'étude des pratiques est de contribuer à l'amélioration en alimentant la formation des enseignants et de la recherche.* » However, this use is also associated with dangers, mentioned by two participants: the "*prescription of* researcher *practice*" and the identification of what constitutes "*good, best or effective teaching*". "*The dangers being that a province/government can take this and then say that research shows this is effective practice and so we will insist that every teacher should do this. This leads to teachers carrying out this surface level behaviour (because they are told they have to) but without a connection with their own set of beliefs, etc. The result of this can be very ineffective teaching or teachers are just replicating surface level behaviour. It also sends a signal where teachers are not being encouraged to develop their own pedagogical beliefs.*"

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# MATHEMATICS OF PLANET EARTH 2013: EDUCATION AND COMMUNICATION • MATHÉMATIQUES DE LA PLANÈTE TERRE 2013 : FORMATION ET COMMUNICATION

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2013 was designated by the Canadian Mathematics Society (and in fact, world-wide) as the year of the Mathematics of Planet Earth. This was an interesting workshop as it combined questions of curriculum and pedagogy with the challenge of modeling the biosphere, an area that mathematics has significantly interacted with over the past 50 years. To emphasize the significance of this these for CMESG, the WG had been preceded two years earlier by one led by Barwell, Craven, and Lidstone on climate change and mathematics teaching, and was followed in 2014 by one led by Caron, Lidstone and Lovric on Complex dynamical systems, in which we used a "hands-on" approach to simulate predator-prey dynamics. Any of these three WGs could have been chosen to represent this theme. This 2013 study was of notable interest in that it featured the use of appropriate graphical display, for example the extraordinary *Gapminder* program, to bring the data to life.

Notons tout d'abord que 2013 avait été désignée année mondiale des Mathématiques de la Planète Terre par l'UNESCO. L'intérêt de ce groupe de travail venait de la combinaison de questions touchant le curriculum et la pédagogie, avec le défi présenté par la modélisation de la biosphère, laquelle est fortement liée aux mathématiques depuis plus de 50 ans. L'importance de ces questions pour le GCEDM est mise en évidence lorsque l'on se rappelle que ce groupe de travail avait été précédé, deux ans auparavant, par celui animé par Barwell, Craven et Lidstone portant sur les changements climatiques et l'enseignement des mathématiques, et suivi en 2015 par celui de Caron, Lidstone et Lovric sur la dynamique des systèmes complexes, dans lequel avait été utilisée une approche expérimentale pour simuler la dynamique prédateur-proie. Ces trois groupes de travail auraient pu être choisis pour illustrer ce thème. Le groupe de travail de 2013 se distinguait en particulier par l'intérêt porté sur l'importance d'une représentation qui permet de donner vie aux données, tel que cela est permis par exemple par l'extraordinaire programme *Gapminder.*

## PARTICIPANTS

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## INTRODUCTION: A NOTE FROM THE CO-LEADERS

The first challenge we had after being invited to lead a Working Group on Mathematics for the Planet Earth 2013 was picking a meaningful topic that would be narrow enough to be manageable over the course of three days—9 hours—and yet not so narrow that we took the stuffing out of this potentially rich and very relevant subject. The MPE2013 website (http://mpe2013.org/) identifies four major themes:

- A Planet to Discover
- A Planet Supporting Life
- A Planet Organized by Humans
- A Planet at Risk

And three major mission statements:

- Encourage research in identifying and solving fundamental questions about planet earth.
- Encourage educators at all levels to communicate the issues related to planet earth.
- Inform the public about the essential role of the mathematical sciences in facing the challenges to our planet.

There was much here from which to choose—too much to simply leave wide open.

In 2011, Barwell, Craven, and Lidstone led a CMESG working group on climate change and mathematics teaching, and although we felt that climate change was a critical issue, we did not want simply to duplicate the experience of two years before. We chose to focus on mathematics education and communication, examining the challenges of effectively communicating mathematics-related issues related to planet earth. This frequently took us to topics in *A Planet at Risk*, including climate change, with frequent links also to the other three themes, especially *A Planet Organized by Humans*. In preparing, we drew heavily on the graphical communication work of Howard Wainer (1997, 2009), Edward Tufte (1983, 1997), and Hans Rosling, creator of *Gapminder*.

Our Working Group Abstract captured our planned focus:

> *Variability, uncertainty, modeling and risk are central mathematical concepts at the core of the investigations. How these are presented has a major impact on what is communicated and what decisions are made. Examining both the scientific literature and what appears in blogs and public discussion, graphic displays and visually presented simulations are how people choose to present their 'information'. One theme for the working group will be probing such displays, to ask 'where's the math' and 'what's the math' in different choices of graphic presentations. These types of questions are a central issue of mathematics education. Given the importance of 'rhetorical communication' on the vital debates involving Planet Earth, we will consider 'graphical rhetoric'. How do we put mathematical arguments into these displays and how do people extract mathematical reasoning from such graphic displays?*

We began the discussion on Day 1 by reviewing the general theme we selected, and sharing with the group what we personally saw as questions on the visual representation issues that came to mind for each of us. This then led to an introduction to powerful historical examples of the early use of graphic representations of quantitative data intended not only to represent but to advocate, as well as other graphic resources the group was invited to explore.

Day 2 started with an introduction by Walter to the work of Tversky, particularly her seminal paper, *Cognitive Principles of Graphic Display* (1997), and sharing the principles of graphic representation offered by Tufte. This was followed by an extended discussion of items raised in Day 1. We concluded Day 2 with an exploration of data using Rosling's *Gapminder World* program, led by Kathleen.

Day 3 focused on education for the Mathematics for Planet Earth, the graphic representation of quantitative MPE data, and "Where's the math, what's the math?" Doug introduced courses by two US-based mathematicians that attempted to address mathematics issues in climate change at an early undergraduate level, as well as some University of Cambridge resources on modeling risk, epidemics, etc. He also presented a schematic of what he saw as the shape of the issues and the focus of the first two days. Working group members then split into two groups to discuss topics that were of particular interest: one we might call "context, good data, and good mathematics" and the other, "the mathematics (education) of risk and impact".

This report will now present the events and discussions of the three days in more detail.

## DAY 1: GRAPHIC REPRESENTATION—CONTEXT, CONTENT, AND STORY

Because our working group was so 'visual' and depended on everyone being able to access, show, and discuss graphic data, whether from websites or *PowerPoint* slide shows, as well as to be able to add new resources over the three days, we shared information using memory keys and established a *Dropbox* location. For our first day, Walter and Kathleen had provided *PowerPoint* presentations of "FAQs" and "Issues KP" respectively.

### WALTER'S PRESENTATION

Walter's slides featured climate change-related visual representations taken from various websites. He focused on seven themes related to climate change issues. The content of Walter's "Some Climate Change FAQs" presentation is given here. The specific links for the various themes are provided in Appendix 1.

Theme I: Increased Carbon Dioxide

- Keeling Curve (Carbon dioxide concentration at Mauna Loa Observatory)
- Questions Walter asked based on the Increase in $CO_2$ evident in this graphic representation:
    - Is this man-made?
    - Is it higher than anything in the last 800,000 years?
    - What are the impacts on ocean, atmosphere?
    - How long will it last?
- Sources: Walter's recommendation to explore the Keeling Curve at the given site on different time scales. What one notices in these graphic representations of the data is that each has a different message of chaos and pattern.

Theme II: Ocean Acidification

- Slide is from the IPCC report (2007). It is a double graph showing: (a) Ocean $CO_2$ levels over a 20 year period (1985-2005); and (b) Ocean water acidity over the same period. Placed side by side (using scales appropriate, respectively, to $CO_2$ levels and acidity), the graphs are roughly a reflection of each other. That is the power in the display—it seems clear that as $CO_2$ levels have increased, the global ocean pH level has decreased (and thus become more acid). The eye in this case is helped by the presentation of a drawn (non-linear) line of best fit to the data.
- "CO2 enters Oceans (makes acid)." Walter's comments and questions—
    - More acid oceans change key parts of the ecology (coral, shells)...
    - Changes in species in particular environments (extinction)
    - How long does the new equilibrium last?

Theme III: Continuing Carbon Emissions? [Global Warming]

- Slide of the 'hockey stick' graph: "Carbon pollution set to end era of stable climate" (period: 10 000 BCE to 2000 CE)
    - Reveals the start of an upward change around the year 2000 in "temperature change relative to 1961-1990 mean" and a projected very large and rapid upward change subsequent to that. Scale is used to dramatically illustrate relative stability over a very long period of time, and strength of projected change.
- Walter's comments and questions—
    - Do carbon emissions (and other emissions—methane, etc.) warm the planet?
    - Are there other sources that can compensate/dominate these human causes?
    - There are differences (more change in the north, less at the equators).
    - What are the risks if we are still uncertain?
    - Is temperature alone (including ocean temperature) the key problem?
- Walter then asked: Do continuing carbon emissions imply Global Warming?
    - How sensitive is the average temperature to CO2 levels?
        - Skeptics say atmospheric warming has slowed.
        - Scientists say same total heat but more went into the ocean.

Theme IV: Global Warming and Extreme Weather

- A series of slides of graphic displays—
    - "5 year average precipitation categories" relative to 20th Century in 2085.
        - Reveals wetter polar and northern/southern temperate regions, roughly stable equatorial region, and dryer subtropical regions. Display is based on colour change.
        - Source: NOAA
    - View of global northern hemisphere showing colder/warmer than average regions for November 2010 (Polar area shown to be 4 to 10º C higher). Warmer indicated by increasing deeper shades of red.
    - Source: NASA
- Walter' s comments (drawing upon the implications of precipitation and temperature changes visually displayed)—
    - More humidity stronger events.
    - Amplification (bigger waves) and slower movement.
    - Dry gets dryer, wet gets wetter.
    - Systems can stall—many snow storms, flooding.
    - ... then hot drought in summer …

Theme V: Sea Level Rise

- Slide showing global change in "Sea Level Trend 1993-01/2012-12 (mm/Year)"
    - Regional trends illustrated by change in colour—darker (thus more emphasis) indicates greater change (drop or rise in SL).

- Walter's concerns—
    - Melting ice sheets, glaciers.
    - Expansion of water due to warming (1 m?)
    - Risk of extreme storm surges.

Theme VI: A Budget for Carbon Emissions?

- Slide: Graphic of "Oilsands vs. Global $CO_2$ Budget"
    - A graphic of inset (but not concentric) circles offering a number of $CO_2$ emissions comparisons, and questioning the claims of Federal Government Natural Resources Minister Joe Oliver's claims about the limited impact of the Oilsands, in contrast to the "Game Over" claims of scientist Jim Hansen.

- Walter's comments and questions—
    - When I was younger—concern was over peak oil (scarcity). Now concern about too much oil, coal …
    - Is there a maximum safe limit for carbon sources we can burn to create $CO_2$?
    - How would this be determined?
    - If there is a budget—whose resources will be left 'in the ground'?
    - If there is a budget, who will have the right to use the associated energy?
    - Will your pension plan go bust investing in oil/coal?

Theme VII: Communication Barriers

- Slide from *thinkprogress.org*—histogram of "Public Perception of [Climate Scientist] Consensus"
    - Graph suggests that only 30-50% of climate scientists agree on human-caused global warming, while in reality (based on review of peer-reviewed literature) there is 97% consensus.
    - Indicates that there is confusion, or unwillingness to accept that there is an issue, on the part of the public.

- Slide illustrating that scientists need to change the way they communicate with the public about climate change significantly from the way they communicate with each other. A graphic display indicating an "inverted" approach to communicating with the public compared to that with each other.
- Walter's comments and questions—
- Rhetorical devices: repetition.
    - Story (narrative) or formulae / graphics.
    - Metaphors to live by: e.g. "Climate is like body temperature: there is a safe range beyond which there is major risk."
    - Graphics, sequences of graphics, animation.
    - Mathematics is a barrier to public communication: numerically and visually.
    - Does this illustrate a fundamental failure of Math Ed?
    - What does research show works/does not work?
    - Hope works better than fear.

Walter focused on major climate change issues in his presentation. Among the features of his presentation that raised thoughtful questions were: the nature (form and structure) of visual displays themselves; the relationship between context, mathematics, and interpretation; the potential story or narrative embedded in the graphic display—what story does the display tell (indeed, does it tell a story?), to whom, and are the stories 'read' by the creator and the reader/viewer consistent with each other; has mathematics education (and science education) failed in its goal of supporting the development of an informed citizenry—or, does it even fully understand that as a goal it ought to have?

### KATHLEEN'S PRESENTATION

Kathleen then presented her issues-based set of slides [My issues/Mes préoccupations].

She started by stating that she felt as though she was drowning in a sea of information. She posed the comment and questions: "*I am wary of media coverage; I sense that there is an underlying agenda. What's good? What's not? How do I differentiate?*" She then gave an example of what she meant by 'good' and 'bad' presentations. Again, the specific links are provided in Appendix 1.

> *The Financial Post, on April 15, 2013, stated that Lawrence Solomon—"one of Canada's leading environmentalists" according to his biography—claimed that "Arctic sea ice was back to 1989 levels, and now exceed the previous decade."*

Kathleen asked: "*Vraiment?*" She presented the graphic representation of the NSIDC data that Solomon reportedly used to make his claim (approximate time period, 1978 to 2013). It became clear immediately that Solomon was being extremely selective in the choice of data points in order to make this claim.

This example again raised the question of how the 'story' that the data tells, in its visual representation, is highly dependent on the perspective and intentions of the story teller (in this case, Solomon). Kathleen again asked, "*What's good? What's not? What can I do to counter reporting à la Solomon?*"

Implicit in both Kathleen's and Walter's presentations was the issue of advocacy. Visual representations have been used to support arguments covering a range of views and competing agendas. How does one counter what is just 'bad mathematics', or evidently misleading interpretations of 'good mathematics'?

Kathleen took this opportunity to introduce the "Wall of Advocacy / *Le mur de l'action réfléchie*". She invited working group members to:

> *Take a look at what is posted and add to it when the mood strikes. Jetez-y un coup d'oeil et mettez-y du vôtre quand l'envie vous prend!*

She noted the expression: "*Une image vaut 1000 mots* – A picture is worth 1000 words." She concluded by adding to her earlier expression of concern as a citizen, her concerns as a teacher: "*How do I integrate MPE themes in my teaching? I'm not an expert…where do I get the data? What can I do with it without tainting it?*"

She offered the following suggestion: "*A good start is looking at graphic displays*."

### DOUG'S PRESENTATION

Doug took a somewhat different opening approach, seeing a significant issue in the mere presence of tens, often hundreds, even thousands, of graphic images on a topic, available on the resource that most turn to now, the internet. On many MPE-related topics, one might consider one can find visual representations of a wide variety of related data. They vary by time of creation, nature of the data collected, period over which the data were collected, geographic region in which the data were collected, choice of scale, choice of graph or other graphic, table parameters, colour scheme, and other factors, often particular to the intentions and interests of the person(s) presenting the display. He showed the example of the rather esoteric but critical to the planet topic of *nitrogen fertilizer*—its overuse around the world. A quick *Google* search of "nitrogen fertilizer overuse – images" produced approximately 900 images—ranging from photos of people, crops, pollution caused by fertilizer overuse, manufacturing plants and the like, to scientific data tables and the graphic displays of such data, to graphic cartoons warning of the dangers.

How does one make sense of such a collection? What does it take to sift through a collection of images like these to identify what might be of particular interest (and perhaps more important, of greater social and ecological value), to identify particularly significant contexts, and separate 'bad' graphics from 'good' graphics, for example? As a grandfather of young children, Doug said he was concerned for their future, but noted that it took thoughtfulness and (mathematical) understanding to make sense of what the images portray, let alone move people to take action. Mathematics education, he thought, needs to pay attention to this weave of contextual and mathematical sense-making.

### DISCUSSION

During the latter part of Day 1, working group members were invited to explore the data, graphic representations, and contexts already shown in Walter's and Kathleen's slides, the mathematics (including the graphical displays) of the MPE issues that were of particular interest to themselves and not yet discussed, or to investigate the "Graphical Resources" set of slides that were also located in the *Dropbox*. Briefly, this set of slides included images of five early ($19^{th}$ C) graphic displays of quantitative data (William Playfair, "Price of Wheat," 1821; John Snow, "London cholera map," 1854; Florence Nightingale's "rose" graph of British military deaths due to disease in the East, 1858; E. J. Marey's graphical French train schedule, 1885; and C. J. Minard's graphic of Napoleon's invasion of, and retreat from, Russia, 1869). Tufte, of whom more will be said shortly, has described Minard's chart as "probably the best statistical graphic ever drawn". These displays are especially marked by their quality, their contextual particularity, and, for several of them, their explicit political or advocacy nature. Also among the slides was Andy Lee Robinson's "Canary in the Coalmine" image of the decline of Arctic ice, and an animated GIF offering a view of the difference between how 'skeptics' and 'realists' view global warming (see Appendix 1 for links to both these websites.)

## DAY 2: PRINCIPLES OF GRAPHIC DISPLAY, A DISCUSSION OF ISSUES, AND AN EXPLORATION OF DATA WITH *GAPMINDER*

This day began with a presentation of "Some Principles of Graphic Design" we compiled from the works of Tufte (1983, 1997) and Tversky (1997). It was noted that these guidelines were written to refer to static and individual displays, not the animated, sometimes-interactive displays that one often finds now on the internet. Nevertheless, they represent important principles by which to judge visual displays of quantitative data, wherever one finds them.

What follows is a brief summary, with attributions where possible, of that full group discussion, first on the issues raised by day 1 topics, and second, by the Tversky and Tufte functions and principles.

### THOUGHTS ON DAY 1 TOPICS

- Nenad suggested we need to differentiate between mathematics, science, and social science.

- France raised the topic of *risk*—how might we visualize *risk*? She suggested the work of economists and mathematicians such as Graciela Chichilnisky (*risk*) and Doyne Farmer (*complex systems*).
- Miroslav asked what mathematics is needed for risk studies.
- Dave suggested that we face challenges when working from a corporate model that places profit at the top of the list. These are based on a carbon economy. We need to have an alternative model—what is available that we might consider?
- Richard remarked that mathematics has limitations—it cannot do all things, such as model human experience in an ecosystem, or model Peter's concern for his grandchildren. Some important parameters cannot be modeled.
- Miroslav acknowledged this while observing that mathematics is distinct from reality, offering the quote, "All models are wrong, just some are useful." What is important to understand are the assumptions on which the model is based.
- Richard offered the view that it was dangerous and egocentric thinking to assume that through mathematics we can control and predict, and thus change conditions to suit us.
- Walter offered the example around the launch of the space shuttle Challenger where the use of a flawed graphic (with the irrelevant independent variable—date of launch—rather than the critical independent variable—temperature at launch) did not support deciding not to launch, even though those creating the visual wanted to communicate the engineers' concerns.
- Context and the issues associated with context were important: Jennifer commented that there had to be much more to both context and mathematics than, for example, simply displaying a graph in class on Arctic ice changes.
- Richard saw a need to rethink the teaching of mathematics—for example, not starting with the mathematics.
- Peter felt that teaching a 'named' course (e.g., "Linear Algebra") could be too confining, and forced one into a lock-step programmatic approach, rather than a more inclusive approach which he would prefer.
- Frédéric offered something of a reality check, noting that there often were real difficulties with trying to do something different in mathematics class, something that would make a difference. The (varied) level of students' mastery of mathematics was a problem, a barrier.
- Stewart raised the point that classroom contextual discussions need to be meaningful to the students: they have to see themselves in the class, and be engaged. "Why care if you don't see yourself?" On the other hand, as a good story teller, Stewart could convince the students that studying environmental issues in math was appropriate, but the issue was, "Where's the math?"
- France observed that the discussion had made clear for her the tension (and challenge) for mathematics educators between (mathematics) content and context (e.g., social, ecological) when it came to the classroom.

## THOUGHTS ON DESIGN FUNCTIONS AND PRINCIPLES

- Frédéric questioned why *understanding* was not one of the functions listed by Tversky regarding the functions of graphic design. Graphic representations ought to deepen the understanding of the context for both the designer and the viewer. [Tversky's list of functions included: *attract attention and interest*, *serve as models of actual and theoretical worlds*, *serve as a record of information*, *facilitate memory*, and *facilitate communication*.]
- Some took issue with Tufte's claim that a good information display should be "causal", while Dianne asked where *relationship* was in the list. [Tufte claimed that information

displays should be *documentary*, *comparative*, *causal and explanatory*, *quantified*, *multivariate*, *exploratory*, and *skeptical*.]

- The topic shifted somewhat to a need to understand who the intended viewers of a display were. Richard noted that we have been thinking in terms of the public being the audience for the graphs we have looked at, while in fact (for example), the IPCC 2007 graphs were intended for scientists, not the public.
- Steven referred to the distinction between 'sensitized' and 'non-sensitized' viewers, and used the expression "visual connoisseurship".
- Richard noted that *rhetoric* was missing from Tufte's list of intentions for a good display: presenting an argument to a particular audience.
- France concluded that it was necessary to think of graphs as "living things", evolving and subject to change.

### A NOTE ON ETHICS AND *THE WALL OF ADVOCACY*

As an attempt to give a ubiquitous shape to advocacy as an underlying issue to be addressed by our group, the *Wall of Advocacy* was set up on Day 1 and was accessible throughout. The wall served to post articles, website addresses, and documents testifying to different forms of advocacy. As a subset of a quite large resource file that was provided to participants on Day 1, posted items included advocacy websites, letters to journalists denouncing flawed graphic displays—showing how they could be designed to better reflect the data, as well as interesting graphics and telling images gleaned off the internet.

Markers, post-its and sticky gum were available for anyone of the group to share ideas. Though many read and took note of what was posted on the *Wall* on Day 1, little was added to it over the course of the three days. The *Wall* did, however, bring about a short but interesting discussion on what form advocacy should or should not take in the classroom. In their professional role as mathematicians and mathematics educators, participants felt the need to be cautious and not distort the information contained in data. This perspective can be seen in the comments shown above made on Day 2, as well as on Day 3. Summing up, we all come to teaching with our personal set of biases and must thread a mighty fine line.

### L'EXPLORATION DE DONNÉES AVEC *GAPMINDER*

Pour poursuivre la réflexion, nous avons introduit les graphiques interactifs. En particulier, nous avons exploré des données à l'aide du gratuiciel *Gapminder*, un outil de visualisation Internet pour l'étude de données statistiques.

Comme courte introduction au gratuiciel *Gapminder*, nous avons présenté la vidéo 200 Years that Changed the World, http://www.gapminder.org/videos/200-years-that-changed-the-world-bbc/, dans laquelle Hans Rosling commente un graphique qui lie l'espérance de vie au revenu par personne selon le pays.

After the video, participants were provided with the following two *Gapminder* graphs:

- www.bit.ly/b9p3dA—linking $CO_2$ emissions per person with Income per person (GDP/capita, PPP$ inflation adjusted) where the size of the bubbles shows total emissions/year for the country.
- www.bit.ly/13UnIhm—linking Water withdrawal (cubic meters per person) and Income per person (GDP/capita, PPP$ inflation adjusted) where the size of the bubble is the total water withdrawal/year for the country.

They were invited to address the *Gapminder* questions: "The USA or China, who emits the most $CO_2$?" and "Does income matter?"; and to play with variables and scales, create their own graphs, and explore the available data, noting as they went what works, what doesn't, where they were lead to, etc.

Some of the comments posted:

- 5 dimensions is very rich
    - Trailing one country to notice patterns of growth & relating to history generates conjectures.
- Great for teaching critical thinking and generating questions.
- Can be used in secondary curriculum. What students' projects would it support?
- Produces effective graphics: Leads to exploration & development of critical questioning & critical thinking (must source background socio-political & economic activities to provide explanation).
- Use of animated graphs always requires knowledge of historical events & content.
- Lends itself to the need for definitions (e.g. How do we compare 'incomes' over time? What does inflation-adjusted mean? Are tonnes in metric or imperial?)
- Caveats.
- Can be challenging to master all parameters.
- Data predetermined but abundant.
- Predetermined data limits the scope of the questions.

Riche, le gratuiciel *Gapminder World*, disponible sous l'onglet *Gapminder World* du site http://www.gapminder.org/, permet de jouer avec cinq variables, est facile d'approche et les données disponibles sont abondantes. L'interaction a amené un questionnement plus audacieux et profond que ce qu'a l'habitude de provoquer les graphiques statiques. De plus, ayant accès à Internet, les données aberrantes, souvent reléguées aux oubliettes, ont donné lieu à davantage d'exploration afin d'en comprendre la signification d'un point de vue économique et sociopolitique, mariant ainsi le contenu mathématique au contexte social.

One item of discontent did arise from the fact that only the indicators available in *Gapminder World* can be displayed. However, anyone interested in displaying their own data sets à la *Gapminder* can do so with a Google Docs spreadsheet (previously known as Motion Chart Gadget).

## DAY 3: A VISUAL INTERPRETATION, AND MATHEMATICS AND MATHEMATICS EDUCATION

At the start of the day, Doug presented a rough sketch of his interpretation of the discussion themes the first two days: context and content figured significantly. A slightly refined version of the diagram is shown in Appendix B.

A short discussion of the previous day followed, and then Doug presented some educational resources:

- descriptions from two US mathematicians of their courses developed on climate change and mathematics (one calculus-based, the other data-based);
- Cambridge University's educational resources for middle and high school students on modeling health and risk (*Motivate Maths*);
- the *Carbon Mitigation Initiative* at Princeton University; and
- *My World 2015*.

Following this, the working group formed two smaller groups to pursue mathematics topics of particular educational interest to them, essentially, context and mathematics, and risk. An outline of these discussions is presented below.

## GROUP 1: DATA, GOOD MATHEMATICS, AND MEANING

Six people were in this group (Egan, Frédéric, Jennifer, Margaret, Richard, and Yasmine), and their general focus was on the resources necessary to develop the good mathematics to support an informed citizenry, and the challenges that represented. The following description is based on notes taken at the time. Because only some comments noted at the time were attributed to specific speakers, the decision here is to avoid any attribution.

The resources considered included data sources, as well as software such as *Gapminder*. As was noted, if an instructor and students are to engage with these contexts—such as climate and economic change, for example—in a mathematically significant way, then good data are needed. Data now are to a high degree coming from online sources. Although this in turn suggests a mathematics that is increasingly computer-based, questions related to the reliability of the data, their general availability, how and when they were collected—all connected to the consistency, integrity, and context of the data—are critical.

In a teaching environment, to start with, as one member mentioned, the mathematics teacher must become familiar with the data and any software that he or she will use for instructional purposes. When using *Gapminder*, for example, it is important that the teacher first gain comfort and comprehension of the program and how it analyzes the data and presents its very graphic representations of the results of that analysis. After having spent some time exploring data with *Gapminder*, one member noted how the program offered opportunities to develop proportional reasoning skills, concepts and skills related to analytic geometry, and, to some extent, transcendental functions (because *Gapminder* makes extensive use of the logarithmic scale).

A pre-service teacher context was offered by another member as an example, with the intention of supporting these candidates in the process of developing plausible and substantial questions based on *Gapminder*. One progression-oriented classroom strategy might look like the following:

- Have teacher candidates begin with an initial exploration (i.e. play with *Gapminder*).
- The instructor (as the more experienced person) then selects questions for teacher candidates to further investigate.
- This in turn leads to more in-depth statistical investigations and understanding of the meaning of the data.
- And finally, the conclusion with an investigation or project with the intention of producing graphic representations, with a particular audience in mind. Here one might consider, for example, both a more sophisticated scientific audience, and an audience composed largely of the parents of the students that one might be teaching. What might be the similarities and differences in how the data are represented to these two audiences?

This then represents a linkage between the mathematics, the context, and the audience. The context serves as a reason for undertaking the mathematics, while the mathematics helps inform the particular audience about the contextual circumstances.

There was in fact substantial discussion of context among many of the group's members—for example, climate change and the visual representation of climate-change-related data. One person noted that a contextually based discussion allowed for the intention that teachers—and students—have the opportunity to learn to ask good questions. Visual representations of data in context need to be created and interpreted at deeper levels than they often are, in order to take greater advantage of the learning opportunities they present. Having noted that, however, it was observed that visual representations are very audience dependent. Creating visual representations is audience dependent; unpacking existing visual representations is audience dependent. When given displays—and one finds many thousands when exploring the internet—it is important to know who they were created for, and who created them. 'Insider' audiences, for example scientists, may be able to make assumptions about the data, and consequently their graphic representations, that the 'public' community or audience are not able to make (and thus may misinterpret representations intended for an 'insider' audience).

The relationships among context, the mathematics (including the creation and presentation of graphic representations), and audience meaning are in fact complex, with an inherent tension, as suggested by the preceding comments. Data and their representations are not inherently imbued with meaning. Data points on a visual display are in fact quite abstract objects, potentially leaving 'viewers' cold and uncertain how to react. Finally it was also noted that in the evolving contexts being discussed (those to do with 'planet earth'), uncertainty was an unavoidable fact of life, so to speak, and teachers, students, and audiences generally must accept that their mathematical analysis (modeling) will not produce a final solution.

### GROUP 2: RISK, PROBABILITY, AND IMPACT

The second, larger group of Working Group members (Dave, Diane, Dianne, France, Krista, Minnie, Miroslav, Nanad, Peter, Steven, and Stewart) focused on the question of *risk*, which had arisen as a thread of discussion in talking about 'change' when it is relatively rapid and substantial—climate change (e.g., global warming), health change (e.g., epidemics), economic change (e.g., financial collapse), etc.—and the impact of that change on the planet. How do we measure the risk associated with these events and changes—what is the impact? And, how do we support the development of an educated citizenry that understands these issues? What follows is a summary of rough notes made while the group members shared their thoughts. The focus shifted somewhat throughout as different group members interjected with what was on their mind at that moment. Again, there is no attribution of comments to particular persons.

'Black swans'—that is, one-off catastrophic events—represented the extreme of these changes, but may in fact be most impactful. Probability concepts, although important to develop and understand, do not do a good job of accounting for 'black swans', and certainly not of understanding the risk associated with them in terms of their impact. One discussion thread looked at winning the lottery. There is a very low probability of winning (which could be calculated), but purchasers do not care about the low likelihood of winning because the impact of not winning is not going to affect their lives (in most cases)—ticket prices are too low. So there is little or no risk involved in purchasing a lottery ticket. [This discussion considered the risk associated with losing; the impact of winning the lottery apparently was not discussed.] "Lotteries are Pink Swans," one person observed.

It was noted that students often have difficulty grasping probability concepts, so some time needs to be devoted to their development; but to discuss risk and impact, students need to go beyond conceptual development, and engage with real data. Questions of risk associated with past and potential events—even catastrophic—represented by the data, and the impact, need to be examined. This would add relevance and serve as motivation. Some examples:

- 'False positives' – e.g., in health-related issues:
    - risks and impacts
    - side effects

- effect on living

- Coal versus nuclear—which is safer?
- Swain vs. US court case:
  - Brief background:
    - 1964—black man convicted of rape, sentenced to death, Alabama
    - Appealed to US Supreme Court on basis that there were no black jurors and that potential black jurors were 'struck' from jury duty by the prosecution strictly on the basis of race. Supreme Court turned down the appeal.
  - A group member asked: What is the probability of never having a black juror in 20 years?
    - North Sea rising and Holland's dikes.
    - Actuarial tables and insurance rate changes.

The educational intention was not to create actuaries, but to support the development of informed citizens who could understand the risk associated with a potential event by being able and prepared to sift through information and make an informed decision about the risk to them. Currently, it was noted, this is not in the [mathematics] curriculum. An outline of the mathematical processes involved was suggested:

- Generate the questions to be examined or answered
- Decide on the data
- Analyze the data
- Summarize the results of the analysis
- Revisit the questions

Modelling was a theme that weaved through the discussion, in part because of the challenging relationships between probability, risk, and impact. One suggestion was to address only impact and not talk about probability at all, in such situations. Another suggested way of thinking of risk was that sometimes it's about probability, while at other times it is impact that must be considered. As an example of 'a way in', perhaps one could picture risk as change in insurance rates. But modelling the impact function was difficult—and we do also need the probability function. The *STELLA* dynamic systems modelling software was mentioned—which raised the question, should *dynamic systems* be taught earlier than it is currently? The hope was that students would come to university and college already with the ability to calculate and interpret numerical results [the implication being that presently they often do not arrive with these capabilities]. The comment was made that it was also important to be aware that the probabilities of events [that take place on or happen to Earth] change over time, and therefore we cannot rely exclusively on 'old' data.

Modelling based on data also raised the important question of the need to understand the assumptions embedded in the tables of secondary data that one might use with students. It was also critical that as an instructor one needed to be clear on the mathematics that students were to learn from the experience. In a reference to the impact of the North Sea rising on Dutch dikes, the question was asked, "How would one model the impact?" For example, would it be stepwise? Exponential? Modelling, it was noted, was about making decisions and "going with it"—and then discussing and refining the model. [One might ask: On what basis?]

Finally, this mathematics, it was also noted, was being affected or influenced by science (biology, and physics, for example), with a focus on functions. But discrete mathematics is addressed in a significant way in high school, and notwithstanding the question of how to handle missing values in a table of values, complex systems are accessible through discrete methods, and iterative processes are highly suitable to computer analysis.

The discussion was lively, varied, and complex, reflecting closely the nature of the topic itself.

## CONCLUSION

Mathematics of planet earth is rich in data, modeling and in questions that generate lively debates. This was true for the participants in the working group, and we predict it would be true for classrooms. Assessing risk and communicating to support decision-making highlight the importance of statistical work with large data sets, of stochastic modeling with both uncertainty and enough certainty to act. They also highlight visual displays as essential tools of communication—and supports for debate. Both the reading of information from graphical displays and the development of effective honest graphical displays are important, and learnable. These tools and these discussions have an important place in mathematics and statistics classrooms.

## APPENDIX A: LINKS TO WEBSITES DESCRIBED IN THE REPORT

### WALTER'S DAY 1 "FAQS" PRESENTATION

Theme 1: Increased Carbon Dioxide

- Keeling Curve: http://keelingcurve.ucsd.edu/

Theme 2: Ocean Acidification

- A link to investigate: http://en.wikipedia.org/wiki/Ocean_acidification
- Interactive resources:
    - http://i2i.stanford.edu
    - http://i2i.loven.gu.se/AcidOcean/AcidOcean.htm

Theme 3: Continuing Carbon Emissions?

- "Hockey stick" graph: http://thinkprogress.org/climate/2013/04/23/1903001/the-hockey-stick-lives-new-study-confirms-unprecedented-recent-warming-reverses-2000-years-of-cooling/?mobile=nc
- Scientists and global warming: More heat has gone into the ocean: http://www.realclimate.org/index.php/archives/2013/04/the-answer-is-blowing-in-the-wind-the-warming-went-into-the-deep-end/
- Skeptics and global warming: atmospheric warming has slowed: http://www.economist.com/news/science-and-technology/21574461-climate-may-be-heating-up-less-response-greenhouse-gas-emissions

Theme 4: Global Warming and Extreme Weather

- See NOAA and NASA sites.

Theme 5: Sea Level Rise

- Sea level trend: http://en.wikipedia.org/wiki/Current_sea_level_rise
- Risk of storm surges (extremes): http://oceanservice.noaa.gov/facts/sealevel.html

- A student's guide to global warming:
    - http://www.epa.gov/climatestudents/impacts/signs/sea-level.html
    - This EPA slide reveals the upward, roughly linear trend in sea level rise for the period 1870-2010—a change of approximately 9 inches. [Units of measure—e.g., inches, millimetres, Fahrenheit, Celsius—as well as scale choice are factors to consider in these graphic representations of scientific data.]

Theme 6: A Budget for Carbon Emissions?

- Oilsands vs. Global $CO_2$ Budget: http://www.vancouverobserver.com/blogs/climatesnapshot/do-oilsands-threaten-our-safe-climate-hansens-game-over-vs-olivers-minuscule
- Do the Math: http://math.350.org

Theme 7: Communication Barriers

- A need for scientists to change the way they communicate—suggested links:
    - http://www.physicstoday.org/resource/1/phtoad/v64/i10/p48_s1?bypassSSO=1
    - http://www.wilsoncenter.org/dialogue-program/severe-weather-and-climate-change-there-connection
    - http://www.climatechangecommunication.org/
    - http://www.climatecentral.org/
    - http://environment.yale.edu/climate-communication/article/sixAmericasMay2011

- What does research show works / does not work?
    - http://climateandcapitalism.com/2009/11/23/naomi-klein-on-climate-debt/

- Hope works better than fear:
    - http://www.nature.com/nclimate/journal/v2/n8/full/nclimate1610.html

- Adapting or mitigating … (prepare for it or prevent it)
    - http://en.wikipedia.org/wiki/Adaptation_to_global_warming
    - http://www.global-greenhouse-warming.com/climate-mitigation-and-adaptation.html

KATHLEEN'S DAY 1 "ISSUES KP" PRESENTATION

Lawrence Solomon's Arctic sea ice-related links:

- Financial Post: http://opinion.financialpost.com/2013/04/15/lawrence-solomon-arctic-sea-ice-back-to-1989-levels-now-exceeds-previous-decade/
- Challenging the claim: http://tamino.wordpress.com/2013/04/16/worth-more-than-a-thousand-words/

"GRAPHIC RESOURCES" LINKS

- Andy Lee Robinson's "Canary in the Coalmine" Arctic sea ice: http://thinkprogress.org/climate/2013/02/14/1594211/death-spiral-bombshell-cryosat-2-confirms-arctic-sea-ice-volume-has-collapsed/
- Animated GIF on Global Warming (skeptics and realists):

http://thinkprogress.org/climate/2013/03/28/1785461/as-scientists-predicted-global-warming-continues/

ADDITIONAL ONLINE RESOURCES IDENTIFIED BY THE WORKING GROUP

- Cambridge University: "Motivate Maths": http://motivate.maths.org/content/
- "Carbon Visuals" illustration (from Stewart): http://artthreat.net/2012/11/carbon-visuals/
- David McCandless: "The Beauty of Data Visualisation" http://www.ted.com/talks/david_mccandless_the_beauty_of_data_visualization.html
- Graciela Chichilnisky: "Catastrophic Risks" (from France) http://chichilnisky.com/pdfs/catastrophic-risks.pdf
- J. Doyne Farmer: "Economics Needs to Treat the Economy as a Complex System" http://ineteconomics.org/sites/inet.civicactions.net/files/farmer_berlinpaper.pdf
- Hans Rosling & *Gapminder*:
    - http://www.gapminder.org/videos/
    - 200 Countries, 200 Years, 4 Minutes: http://www.gapminder.org/videos/200-years-that-changed-the-world-bbc/
    - Let my data set change your mindset: http://www.gapminder.org/videos/ted-us-state-department/
- Princeton University: "Carbon Mitigation Initiative"(CMI): http://cmi.princeton.edu/
- Thomas Goetz: It's time to redesign medical data: http://www.ted.com/talks/thomas_goetz_it_s_time_to_redesign_medical_data.html [Blood work, CRP and others tests are 'rewritten', inspired by the nutritional value info on cereal boxes.]
- Visual Learning for Science and Engineering http://old.siggraph.org/education/vl/vl.htm

## APPENDIX B: ADDITIONAL INDIVIDUAL AND SMALL GROUP COMMENTARY

France, Krista, and Minnie's comments on the *Ocean Acidification* graphic, comparing $CO_2$ level graph with ocean pH level graph:

- Challenging the necessity of including zero. Telling a story is drawing attention to something. Thin line with manipulation.
- The variability in the trend line: What's the story? The meaning? The purpose? To make it look more authentic? Respectful of the data? Scientific? Less naïve?
- Choosing which variables to display. Only two at a time? Playing with size, colour or dynamism… What would you gain if you added a graph of pH as a function of $CO_2$? Or a dynamic version of that with respect to time?
- The mirror image with different variables:
    - Tells the story well
    - Strong aesthetic appeal
    - Seems too perfect to be true
- The hidden information:
    - Where do the points come from? All over the earth? Are they collected on a regular basis?

    - Requires expert knowledge to know that pH is a logarithmic indicator (pH).

- What does this linear trend, on a logarithmic indicator mean in terms of the relation?

France, Krista, and Minnie comments on *Oil Sands vs. Global $CO_2$ Budget* graphic:

- What's the story? What's the message?
- The choice of form is misleading: Diameter or areas? Could they be spheres?
- Canada fossil fuels ever burned vs. world global $CO_2$ budget: too much going on…

# INDIGENOUS WAYS OF KNOWING IN MATHEMATICS

Lisa Lunney Borden, *St. Francis Xavier University*
Florence Glanfield, *University of Alberta*

This working group was chosen because it represents an example of the diversity of the topics and profoundness of the issues explored at CMESG. The participants experienced alternate ways of communicating about, and within, mathematics as they discussed key ideas, fundamental to the future of Indigenous and Western mathematics education.

Ce groupe de travail a été choisi parce qu'il représente un exemple de la diversité des thèmes et la profondeur des sujets explorés au GCEDM. Les participants ont pu éprouver des moyens divers de communiquer les mathématiques en discutant des idées clés qui sont fondamentales pour l'avenir de l'enseignement des mathématiques autochtones et occidentales.

## PARTICIPANTS

| | | |
|---|---|---|
| Yasmine Abtahi | Roman Chukalovskgy | Limin Jao |
| Annette Braconne-Michoux | Stewart Craven | Kate Mackrell |
| Élysée Robert Cadet | Osnat Fellus | Cynthia Nicol |
| Bev Caswell | Frédéric Gourdeau | Jamie Pyper |
| | David Guillemette | Annie Savard |

## INTRODUCTION

As we planned for this group we acknowledged that there are many ways that this working group might 'work' on these ideas. Over the three days, this group engaged in a variety of experiences to explore what is meant by *Indigenous knowledges* and how we hold onto these ideas in relationship with what we've come to know as 'mathematics' or 'mathematics education' or 'mathematics teacher education'.

## STARTING EACH DAY IN A GOOD WAY

As we planned, we imagined that we would be able to have an Elder alongside. However, this did not occur. However, Florence brought along a bundle of sage from the Indigenous Teaching and Learning Gardens at the University of Alberta. The sage was a reminder of the sacred medicines and the importance of the conversations that we were about to embark upon. The sage was gift from the place we now call Alberta. The sage bundle provided us with a reminder of our connectedness to the land each morning as we started with a sharing circle. The sharing circle began each of the three days so that we could each acknowledge our heart and mind.

The teachings that Florence has been given around the importance of the circle as a place of beginning is that individuals acknowledge their own heart and mind as they enter into the conversation. In the circle, each person is invited to describe "*what is on the heart*" and "*what is in the mind*" as they hold the sage bundle. Once this is shared, it is now a part of the 'collective' or the 'community' and we can now 'hear each other'.

## DOING BIRCH BARK BITING

The second part of the first meeting was dedicated to experiencing birch bark biting first hand. Lisa shared her story of coming to birch bark biting following a conversation with a Mi'kmaw elder who described it as a common pastime when she was young and encouraged Lisa to learn more about it. Lisa shared that while following up on the conversation she came across an article that demonstrated that birch bark biting was indeed a historical part of the Mi'kmaw community:

> *That she was "the last one that can do it" was the same phrase echoed in 1993 by Margaret Johnson, an Eskasoni Micmac elder from Cape Breton. Continuing research has revealed that two other Micmac women – including Johnson's sister on another reserve – can also do it. (Oberholtzer & Smith, 1995, p. 307)*

Lisa had known both Margaret, who was affectionately known as Dr. Granny, and her sister, Caroline Gould, who had resided in the community where Lisa had taught and often visited the school. Unfortunately both women had already passed away by the time Lisa began researching this practice.

Birch bark biting involves folding thin pieces of bark and biting shapes into the bark to create designs. The act of folding the bark presents an opportunity to think about fractions, angles, shape, and symmetry. Creating the designs draws in geometric reasoning and visualization of geometric shapes. However, on this day we began in much the same way Lisa has explained she begins with students—showing photos of birch bark biting and asking the question, "*How do you make this?*"

After some quick tips on how to fold the bark, we first practiced on waxed paper which enabled people to get a sense of folding and biting before moving on to using real birch bark. Fold lines are easily seen on the waxed paper (see Figure 1). After a few tries with waxed paper, we were ready to work with the real birch bark. The bark had to be peeled into single layers which were very thin and often very delicate.

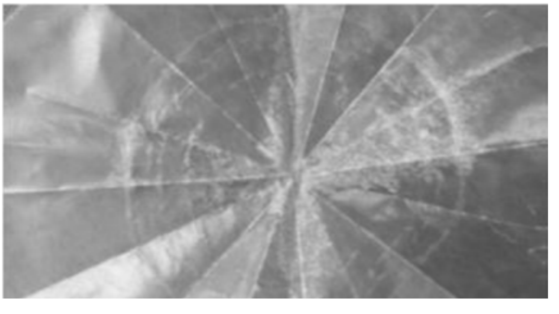

Figure 1. Wax Paper Practice.

Through doing the birch bark biting many questions and observations emerged. The pictures tell the story of the birch bark biting. See Figures 2 and 3.

Figure 2. Working on our creations

Figure 3. Some of our work.

## EXPLORING ALTERNATE WAYS OF DOING MATHEMATICS

On day 2, we reconnected in circle, sharing some thoughts from the previous day and were excited to continue learning. As a way of considering how mathematics learning can emerge through centring Indigenous knowledge, Lisa shared videos from two projects that were part of the Show Me Your Math (SMYM) program in Nova Scotia public and Mi'kmaw schools. These SMYM projects were inspired by Doolittle's (2006) idea of pulling in mathematics by beginning in aspects of community culture where the already present, inherent ways of reasoning within the culture can help students to make sense of the 'school-based' concepts of mathematics in the curriculum. One goal of this work is to have teachers and students learning alongside one another as they explore practices that are relevant to the community. As such, these projects have been called *Mawkinamasultinej!* Let's Learn Together! as a way of emphasising this focus on learning together. As a group we watched the videos from the eel project done by a grade 4/5 class in a public school called #KataqProject using the Mi'kmaw word *kataq* meaning eels (see Figure 4). We also watched *Sap to Syrup*, a project about maple syrup making done by a grade 5/6 class in a Mi'kmaw school. The links to these videos are given below.

#KataqProject Link:
https://www.youtube.com/playlist?list=PLJzemBBS0KqjfxhC8sdlwImxjgOAdi8xZ

Sap to Syrup Link:
https://drive.google.com/a/pictoulandingschool.ca/file/d/0B9xZFTcbPGMienpJekVNZU1Bb3c/view?usp=sharing

These videos provided examples of culturally-based inquiry that can be connected to the school-based mathematics curriculum. Teachers involved in these projects have shared that the outcome connections easily emerged as they engaged in these projects.

Others in the group shared videos and presentations about their work in communities. Cynthia described her work in British Columbia, Annie in Northern Québec, Bev in Northern Ontario, Elysée in Southern Québec, and Florence in Alberta.


Figure 4. The #KataqProject.

Our discussions of how mathematics might emerge in different contexts led us to discuss how we do mathematics differently in different contexts, including using different algorithms. Our diverse group had diverse approaches to doing a standard mathematics problem, which helped to prompt discussions about these varied approaches and the implication for teaching and learning.

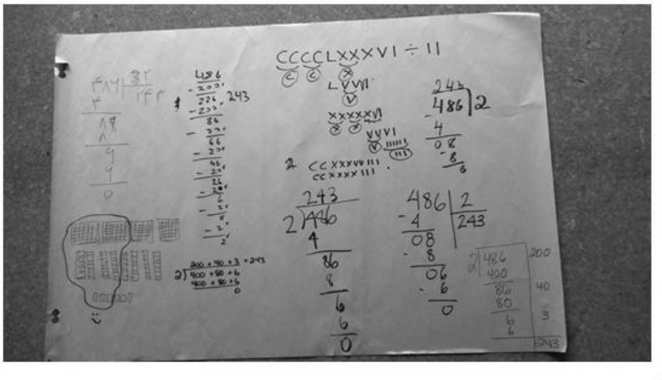

Figure 5. Working Group recording of varied approaches to division.

### SHARING OUR WORDS

Those of us who had extensive experience working in Indigenous contexts were able to share some words we had been given that we felt helped us to conceptualize some of the ideas we were discussing. Lisa shared *mawikinutimatimk* meaning "coming together to work together", a word use to describe a way of working together to solve a problem or discuss an issue, and *mawkina'masultinej* meaning "let's learn together", a word that has been used to describe the inquiry projects that have become part of the Show Me Your Math program. In a similar vein, Florence shared the Cree word *Miyowichitowin* meaning "coming together to learn and live 'in a good way'" From her experience working in Fort Francis, Bev share the word *Gaa-maamawi-asigagindaasoyang*' which was developed by Ojibwe language teacher, Jason Jones, in February of 2015 to describe mathematics. It translates as "We are the ones doing the math together" and has a sense of the ongoing nature of this work. The components of the word are identified below:

- *maamawi* – together
- *asig* – gathering
- *agind* – read or count/put it through thought
- *aaso* – performing useful action
- *Gaa* and *yang* are used to put the word into a noun

These words were used to guide our thinking and inform our discussions.

## KEY LEARNINGS

On day 3, our focus turned to discussions again in circle and in small groups to begin to synthesize our learning. Some key learnings are described below.

### RELATIONSHIPS/RELATIONALITY

We discussed the idea that 'being together with' or 'building with' or 'learning together with' community is essential in this work. Trusting relationships are at the heart of such decolonizing endeavours. In particular, if we are not members of an Indigenous community, we must acknowledge that we have as much to learn as we have to teach; we are not the experts. We must work to build community and develop trust. We need to be aware of who we are working with and honour what the community identifies as needs. We must honour elder knowledge and value the contributions of these knowledge holders. This requires that we listen with sincerity and openness, in a true spirit of learning.

### PLACE

Connecting to place and learning from place were key ideas that emerged in our discussions as well. We discussed questions of how we learn to connect to place, especially if we are outsiders to that place, and in so doing begin to learn from place. We discussed how connections to place might help us to generate different ways of knowing and different ways of relating to one another and to the place itself.

### RECLAIMING / RECONNECTING / REVITALIZING

The SMYM program, the Indigenous teaching and learning gardens at the University of Alberta, Cynthia's work in Haida Gwaii, Bev's work in Fort Francis, Elysée's work in Québec, and Annie's work in Northern Québec, all provided examples of how mathematical inquiries can provide opportunities to centre Indigenous knowledge as a place of learning. Such centring often generates an opportunity for communities to reconnect with knowledge that has been nearly forgotten, taken away by forced assimilation and colonization. When such reclaiming or reconnecting happens, this often inspires community members to share stories almost forgotten or recall memories not yet shared with the next generation. This allows students to see that mathematical thinking has always been a part of community knowledge though it may not be articulated in the same way as it is in mathematics textbooks.

#### MATHEMATICS IN RELATION

Our discussions also focused on the nature of mathematics itself. We considered how we might bring Western and Indigenous mathematics and knowledge systems into conversation with one another.

## FINAL THOUGHTS: MATHEMATICS EDUCATION FOR RECONCILIATION

As we reflected on our discussions and the ideas that had emerged in these conversations, we turned our attention to the recently released report of the Truth and Reconciliation Commission of Canada that identified 94 *Calls to Action* (www.trc.ca).

#### FROM THE CALLS TO ACTION OF THE TRC:

62. We call upon the federal, provincial, and territorial governments, in consultation and collaboration with Survivors, Aboriginal peoples and educators to:
    i. Make age-appropriate curriculum on residential schools, Treaties, and Aboriginal peoples' historical and contemporary contributions to Canada a mandatory education requirement for Kindergarten to Grade Twelve students.
    ii. Provide the necessary funding to post-secondary institutions to educate teachers on how to integrate Indigenous knowledge and teaching methods into classrooms.
    iii. Provide the necessary funding to Aboriginal schools to utilize Indigenous knowledge and teaching methods in classrooms.
    iv. Establish senior-level positions in government at the assistant deputy minister level or higher dedicated to Aboriginal content in education.
63. We call upon the Council of Ministers of Education, Canada to maintain an annual commitment to Aboriginal education issues including:
    i. Developing and implementing Kindergarten to Grade Twelve curriculum and learning resources on Aboriginal peoples in Canadian history, and the history and legacy of residential schools.
    ii. Sharing information and best practices on teaching curriculum related to residential schools and Aboriginal history.
    iii. Building student capacity for intercultural understanding, empathy, and mutual respect.
    iv. Identifying teacher-training needs relating to the above. (TRC, 2015, p. 7)

We asked ourselves, what is the role of mathematics education, mathematics educators, and mathematics in responding to the TRC *Calls to Action*? We left our working group with this enduring question as we move towards teaching mathematics in a good way and developing research with communities.

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# Topic Sessions

# Séances thématiques

# THE COMMONSENSE OF TEACHING

David Wheeler
*Concordia University*

David Wheeler is one of the forefathers of Mathematics Education in Canada in general, and CMESG in particular. As such, it is only fitting that, from time to time, he calls the field to attention and holds it responsible for the knowledge it has produced (or failed to produce). This Topic Session is just such a call to attention. Taking a "common sense" approach he challenges the notion of teaching and the theories that abound. When considering Topic Sessions for inclusion this one came to the fore not only for who David Wheeler was to CMESG, but also what he continued to give to the community well past his retirement.

David Wheeler est l'un des pères fondateurs de la didactique des mathématiques au Canada, et en particulier au GCEDM. En tant que tel, il est normal que, de temps à autres, il nous rappelle que nous sommes responsables des connaissances que nous avons produites (ou que nous n'avons pas produites). Cette séance thématique est un tel rappel. Approchant le tout avec un "bon sens" commun, il remet en question la notion d'enseignement et les abondantes théories. Parmi les séances thématiques considérées pour inclusion dans ce recueil, celle-ci s'est distinguée non seulement à cause de l'importance de David Wheeler pour le GCEDM, mais aussi en vertu de ce qu'il a continué à donner à la communauté longtemps après sa prise de retraite.

> *This talk was in Cassandra mode. You know the sort of thing: society is crumbling, war and pestilence are around the corner, the situation is hopeless, and similar animadversions. Those of a nervous disposition would be well advised to stop reading now and take a calming stroll.*

When I first thought about giving this talk, I intended to give most of the time to technical aspects of teaching, especially the teaching of mathematics, a topic which doesn't seem to me to get the detailed attention and study it deserves. In starting to work on the talk, however, I found a lot of more general issues about teaching came into my mind that couldn't be pushed aside. What follows is mostly this other "general stuff', so the paper should now be read as a prelude to the one I had originally hoped to present.

A second "caution to the reader" may be in order. I open with some "negative thoughts" that appear to be dismissive of a great deal of very dedicated work by many teachers, teacher trainers, innovators, and researchers. I start with these because they indicate the position of the "frame" through which I am looking at questions in the field. The frame is deliberately placed to emphasize how much further remains to be travelled than the distance we have come. It insists on reminding us that we have hardly begun to articulate and communicate the skills that underlie good teaching. This viewpoint was important for my purpose in giving the talk, which was to provoke my listeners into thinking as much about teaching as they do about learning. At other times I have other purposes, or I am talking to other audiences, and then I adjust the frame accordingly.

## NEGATIVE THOUGHTS

I took up my first teaching appointment, in a high school, in 1947 and last year I gave up the editorship of *For the Learning of Mathematics,* so I've been involved in one way or another with the teaching of mathematics for 50 years. What do I see now when I look at teaching from the perspective of this long haul?

- Almost all teaching is amateurish.

Amateurs may love their work, as the etymology of the word suggests, but society expects professional expertise from its teachers. I see some very effective teachers, but I also see many who don't seem to have the resources of skill and know-how needed to teach effectively in the difficult circumstances that many schools present. I have little sense that there is agreement among teacher trainers about the technical and/or professional equipment a teacher needs, and some of the people training others to teach seem to doubt whether teaching involves the application of any techniques at all.

- Almost all that is said about teaching is banal.

This thought reinforces the first. The banality seems to arise from uncertainty about the basic requirements for effective teaching. Because we haven't resolved the matter of what comprises the basic equipment, conversation about teaching never goes much beyond discussing "starting points" and we hardly ever get to work on the more searching and sophisticated questions that classroom practice throws up. The claim that "teaching is an art" can too easily become an evasion of responsibility.

- Though teachers review and reflect on their actions, they almost never reflect on their beliefs.

This is a tricky point. Western societies, in general, permit people to believe what they want-it's one of the pillars of a free society. The downside to this freedom is that people begin to think that beliefs don't have to be checked out, that evidence for or against their validity doesn't have to be considered, that the only authority a belief requires is that enough people hold it. Teachers, whose beliefs affect what they do, and whose beliefs may not be entirely compatible with actions the educational system tells them to take, need to be particularly alert to both the overt and covert effects of the beliefs that touch on their work in their classrooms.

- Almost all educational trends are essentially concerned with reinventing the wheel.

Re-inventing the wheel, in spite of the old jibe, isn't altogether a bad thing to do. Each generation or two of teachers meets fresh educational and social problems, or old problems with a new twist, and a re-inspection of the ways they are being handled can prove useful. But of course this point highlights the lack of a well-founded *tradition* of good teaching practices and of a suitable machinery for inducting new teachers into it. (Our wheels remain square, one could say, and we continue to find them unsatisfactory, rolling them first this way, then that, to no appreciable advantage.)

*N. B. The above statements are not, of course, for general circulation! Should any parent or politician accuse me of uttering them I shall immediately deny that I made any such observations.*

The following four sections offer my choice of themes connected to teaching that I put forward as worth thinking about.

## PARADIGMS

Here I insert a point about the irreducible components of a model of teaching, and to suggest other paradigms of asymmetric social interaction that can be usefully compared with and differentiated from teaching. (In all that follows I am chiefly thinking of the teaching that takes place in institutional settings-"classroom teaching.")

French didacticians have helpfully focused attention on the centrality of the triad: *teacher/student/subject matter* and have framed many of their empirical studies to clarify the interactions among its components. In this form, however, the triad makes no explicit acknowledgment of the culture (in all the large and the small senses of that word) within which the triad is situated. The effect of this culture (or of these cultures, because the classroom is the meeting ground of a number of independent and sometimes incompatible cultures) is easily overlooked, yet it seems plausible to me that in institutional teaching the cultural factors embedded in the teaching environment-the customs, values, expectations, etc., particular to the systems involved-have effects that always influence, and sometimes dominate, the interactions among the elements of the triad. Taking account explicitly of this "fourth party" can safeguard us from painting an unrealistic picture of practical pedagogical possibilities.

The irreducible components of a model of teaching, I suggest, are: *teacher/ student/topic/context*, where "context" covers all of the physical, linguistic, social, and cultural attributes of the place where the teaching happens. The French model assumes, metaphorically speaking, two actors with a text, and I am suggesting it is important to integrate these into the "setting" – the theatre itself, the type of stage, the scenery, the audience's expectations.

Are there insights to be obtained by comparing the "teaching paradigm" to others? Here are a few pairings to consider.

- Teacher / student
- Craftsman / apprentice
- Mother / infant
- Guru / disciple
- Coach / ball player
- Counsellor / client
- Abuser / victim

In many ways the last six can be regarded as variants of the teaching paradigm. I include the very last one as a hint to you to entertain the idea that teaching doesn't always have positive and liberating effects. Other pairings will probably occur to you.

Having noted some similarities between the paradigms, we can then try to identify what, if anything, is special to the first. This may help us become clearer about what actions properly belong to teaching and what prevents it from slipping or sliding into one of the other related but different activities.

Perhaps, too, it is worth considering the ways people learn how to play their parts in these various activities.

## THEORIES AND SUCH

The following is the introduction to a paper in the Fall 1995 issue of the journal *Daedalus.*

> *"Two challenges face American education today: I) raising overall achievement levels and 2) making opportunities for achievement more equitable. The importance of both derives from the same basic condition-our changing economy. Never before has the pool of developed skill and capability mattered more in our prospects for general economic health. And never before have skill and knowledge mattered as much in the economic prospects for individuals. There is no longer a welcoming place in low-skill, high-wage jobs for individuals who have not cultivated talents appropriate to an*

> *information economy. The country, indeed each state and region, must press for an overall higher level of such cultivated talents. Otherwise, we can expect a continuation of the pattern of falling personal incomes and declining public services that has characterized the past twenty years.*
>
> *The only way to achieve this higher level of skill and ability in the population at large is to make sure that all students, not just a privileged and select few, learn the high-level, embedded, symbolic thinking skills that our society requires. Equity and excellence, classically viewed as competing goals, must now be treated as a single aspiration." (Resnick 1995)*

The author's main argument in the paper is that the belief that *aptitude* is the chief determinant of educational success has unfortunate consequences; in particular, it discourages students from attempting "to break through the barrier of low expectations." She recommends a shift to an emphasis on *effort*, based on the assumption that "effort generates aptitude."

In this paper Professor Resnick makes the telling point that many of the practices in American schools enshrine the claim that aptitude is the most important factor in educational success, and that the effects of these practices covertly reinforce the belief even after public or professional opinion has (overtly) moved away from it. Probably the most striking example of such a practice is the use of SAT scores, which aim to be "knowledge-free," as important indicators for college admission. Could anything be more absurd than failing to consider the knowledge that 18 year old students have already acquired when deciding whether to accept them for further education?

The status of the proposition "aptitude determines success" appears to be that of a *theory* in the field of education, but we can remark that the concepts it deals with are not entirely clear, and the assertion doesn't seem to have clearly articulated connections to other theoretical statements. Does the proposition have empirical support? Can it defeat arguments attempting to disprove it? Perhaps it's an item of *folklore*, not of theory. Either way, Resnick reminds us, too many institutionalised practices which may at some time in the past have been derived from the "theory" now serve as at least a partial substitute for it, extending its life invisibly.

We need much more than this single instance to establish a significant generalisation, but it nevertheless triggers in me two small "lemmas":

- Educational "theory" doesn't govern educational "practice" in a straightforward way.
- What educators say they believe about teaching doesn't necessarily match the beliefs embedded in their practices.

Before quitting this example, I must express my extreme disquiet with Resnick's strategy. She expresses the basic options in simplistic either/or terms – "aptitude or effort," "equity or excellence" – as if her readers would be unable to appreciate a more nuanced account of the complexities she is dealing with, and her introduction is as shocking in its use of crude generalisations based on unexamined assumptions as anything in the beliefs and practices she criticises.

For my second example I go back over two hundred years to Britain. In 1749 David Hartley presented a systematic formulation of his psycho-philosophical theory of *associationism.* The philosophies of Locke and Hume were among its influences, J. S. Mill and Spencer among its later adherents[16]. The theory (which I simplify and abbreviate considerably) held that:

- Ideas and sensations are reflections of external objects.

[16] I am indebted to Brian Simon's paper (1985), *Samuel Taylor Coleridge: The education of the intellect,* for triggering my thoughts about this example and for some of the detail in my account of it.

- Particles transmit vibrations from the objects through the senses to the nerves and the brain. (This description of the material connection between object and sensation was later modified and eventually abandoned.)
- "Complex" ideas are associations of simpler ones, which connect according to laws of simultaneity, contrast, contiguity, etc.

The first thing this theory does is bring the study of thought, reason, and other mental processes, into line with the cosmological and scientific beliefs current in its time: that (physical) phenomena are governed by fixed (mechanical) principles whose laws can be discovered through a combination of observation and insight. In the second place, it holds out the possibility that particular ideas can be generated in the mind by an appropriate manipulation of those immediate objects belonging to the material world. This latter implication was seized with enthusiasm, suited as it was to an intellectual climate much concerned with exploring the possibilities of secularising and democratising society. Among those influenced by the theory were the early utopian socialists, exemplified by Robert Owen and his slogan, "Circumstances make man!"

Associationism is essentially a theory of learning and when interpreted within a determinist frame of reference it proposes the possibility of making education truly scientific, producing guaranteed effects in learners by controlling in detail their educational environment. The theory generated a quite extraordinary mood of pedagogical optimism.

Coleridge and Marx, both originally highly sympathetic to the theory, came to acknowledge its drawbacks. Coleridge allows that associationism explains very clearly why certain items of knowledge are easy for us to retain and recall, but beyond that it fails by treating the learner as properly passive, "a lazy looker-on." He points out that by an act of will anyone can arbitrarily give distinction to any item of knowledge whatsoever. Marx asks in what way the theory can explain how the *educator* comes to know what learning is desirable and hence what sort of learning environments to construct, and adds that, rather than being the creature of circumstances, man is someone who changes circumstances and in the course of that action changes himself.

This example draws our attention to the fact that an educational theory doesn't have to have a proven track record to be adopted with enthusiasm. Together with the previous example it should encourage us to be sceptical about the development of sound educational theories, and especially about the too hasty attempt to deduce practical consequences from them.

One is very tempted to say, "a plague on all your theories."

## NEW FACTORS

I consider myself very lucky to have begun teaching in the immediate post-World War 2 years. I was energetic and enthusiastic, and so was the general mood in society at the time. Teaching looked a worthwhile thing to be doing, its importance wasn't questioned, and everyone was optimistic about what institutional education would eventually bring to all students in society, not just those who, because of their social class, or because they were deemed unusually "bright", had already been led to expect it. How different the mood seems now, fifty years later! Yet, I don't side with those who say there has been a substantial falling-off in what teachers and students achieve: the hard evidence is scanty and somewhat ambiguous. But I can't deny that the optimism of society has been replaced by pessimism and that the schoolteacher's job at the end of the millennium appears to be immeasurably more difficult than it was 50 years ago.

Without claiming to be able to give all the reasons for this shift, I draw attention to three major factors affecting the environment within which teachers now have to work.

A. Education started out as the education of an elite. This is still the only education that is done well. In elite education the teachers and students share a common culture-a culture with a common language, common values, common expectations. Teachers confronted with the demands of "education for all" are faced with the genuine difficulty of working effectively in and around distinct and often incompatible subcultures.
B. Teaching in classrooms used to be teaching "behind walls," physically and metaphorically. Classrooms have traditionally been spaces where teachers practised their instructional expertise unquestioned by their peers and unobserved by all except their students. To an extent the walls have now been toppled. Teachers today are evaluated by their students, held accountable by their employers, and generally subjected to intense public scrutiny and pressure.
C. Computers and the Internet pose threats, real and imagined, to traditional teaching methods and to the traditional instructional vehicle, viz., books.

No wonder classroom teaching now seems so much more difficult and so exhausting!

It's highly unlikely the challenges now facing teachers can be met by trying to go back, to reassert the methods and values of the recent past. But the past is enormously powerful. Large educational systems have an intrinsic inertia now reinforced by the new openness of the system to outside criticism and influence. Parents and politicians even more than teachers may seek security in the familiar and shrink from radical change. How else can one account for the widespread popular rejection of the electronic calculator as a tool for teaching arithmetic in elementary schools? The rejection seems the expression of a fervent wish that the calculator didn't exist, had never been invented, so everyone could go on teaching as they did before it arrived. But the calculator *does* exist, the calculator *has* been invented, and there is no way the teaching of elementary number operations can or should go on as before.

## THE PRACTICE OF TEACHING

Maybe the need now is for more creative and radical kinds of pedagogy. First, though, in this new "open" climate, we have to try to establish that pedagogy is *important*, that it's not just an academic word for something trivial, like knowing something and then telling or showing it to someone else. Teaching mathematics to all students is difficult, not because mathematics is particularly difficult, nor because students are, but because we now have to be able to teach it in such a way that *anyone* can access it if they need it and if they want to, and this is a requirement that can't possibly be met without the aid of a skilful and quite sophisticated pedagogy. We need to stop talking as if *teaching is an art*, which is only a sly way of saying "a few can teach, most can't," and as if *teaching is a science*, which would require a consensus about basic theories which won't be achieved for a long while, if ever, and settle for teaching as essentially a technical matter-not in the sense of a full-fledged technology but as a set of know-hows, a sort of kit bag for dealing with the practical demands of the classroom, a kind of *bricolage.*

Because I have spent so much time on this occasion talking about other issues, I can't go on here and now to give genuine substance to the previous sentence, though that is what I hope to begin to do another time. It's also my hope that some among my listeners and readers will also want to work in detail on this question of the technical nature of teaching.

## POSTSCRIPT

In spite of the low esteem it suffers, and the institutional restrictions that confine it, teaching remains a wonderfully worthwhile activity. Just occasionally it yields the reward that is above all other-the 98 Topic Session 3 awareness that for a particular student *intelligence has been revealed to itself*[17]. To make "revealing intelligence to itself' an explicit target may be unwise since we can so rarely be sure whether, how, or when the target has been attained, but as a general orientation, a vision we store in the back of our mind, it may ready us to seize any opportunity the classroom does offer to bring this gift to our students.

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[17] I have lifted the phrase from the sentence, "The problem is to reveal an intelligence to itself," on page 28 of Jacques Rancieres (1991) *The Ignorant Schoolmaster: Five Lessons in Intellectual Emancipation*.

# A HISTORICAL PERSPECTIVE ON MATHEMATICS EDUCATION RESEARCH IN CANADA: THE EMERGENCE OF A COMMUNITY[18]

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This special session paper was selected because it connects with both parts of the theme of CMESG 2016 meeting: "celebrating the past, inspiring the future." It reminds us of the emergence and growth of the Canadian mathematics education research committee from the early 1920s to 1995 and the role of CMESG/GCEDM in it. The hope is that it will inspire others to update this important historical account of our community from 1996 to the present.

Cette séance thématique spéciale a été choisie parce qu'elle peut être reliée aux deux parties du thème de la rencontre du GCEDM de 2016 : «célébrons le passé, inspirons le futur». Carolyn Kieran y raconte l'émergence et la croissance de la communauté canadienne de recherche en éducation mathématique du début des années 1920 jusqu'à 1995, en faisant ressortir le rôle joué par le GCEDM dans l'émergence et la croissance de cette communauté. Notre souhait est que ce texte inspire d'autres chercheurs à poursuivre cette importante analyse historique de notre communauté, de 1996 à aujourd'hui.

This paper describes the Canadian mathematics education research community—from its preemergence in the early 1920s up until 1995. Because data collection ended in 1995, the more recent scholars of the community are not included in this story. A future update will hopefully rectify this situation. In preparing the chapter on which this paper is based, I first asked myself, "What defines a research community? What are its characteristics? How does a research community develop? Are there events one can point to that could be said to have contributed to its emergence?" I finally decided to use Etienne Wenger's (1998) concept of *communities of practice* as a unifying thread to describe those happenings that I felt were pertinent to the formation of the Canadian community of mathematics education researchers. In my analysis of the events that fostered the emergence of our community, I weave together the concurrent growth of research communities both abroad and within the provinces and the roles that these played. The milestones I present are intended to illustrate the spirit of its overall evolution. In taking a particular focus, I have regrettably missed certain individuals or events that merit inclusion; for this, I offer my sincerest apologies.

## WHAT IS A COMMUNITY OF PRACTICE?

According to Wenger:

> *We all belong to communities of practice. At home, at work, at school, in our hobbies—we belong to several communities of practice at any given time. And the communities of practice to which we belong change over the course of our lives. In fact, communities of practice are everywhere. ... In laboratories, scientists correspond with colleagues, near and far, in order to advance their inquiries. Across a worldwide web of computers, people congregate in virtual spaces and develop shared ways of*

[18] This paper is an abridged version of a chapter, "The Twentieth-Century Emergence of the Canadian Mathematics Education Research Community", to appear in *A History of School Mathematics*, edited by George Stanic and Jeremy Kilpatrick, which will be published by the National Council of Teachers of Mathematics in 2003.

> *pursuing their common interests. ... We can all construct a fairly good picture of the communities of practice we belong to now, those we belonged to in the past, and those we would like to belong to in the future. We also have a fairly good idea of who belongs to our communities of practice and why, even though membership is rarely made explicit on a roster or a checklist of qualifying criteria. Furthermore, we can probably distinguish a few communities of practice in which we are core members from a larger number of communities in which we have a more peripheral kind of membership. (pp. 6–7)*

However, not all groupings and associations are communities of practice. The focus is on practice. For example, the neighborhood in which one lives may be called a community, but it is not a community of practice. More precisely, according to Wenger, communities of practice have the following three dimensions that associate practice and community, all of which are required for communities of practice: mutual engagement, joint enterprise, and shared repertoire.

### MUTUAL ENGAGEMENT

Mutual engagement involves, according to Wenger, "taking part in meaningful activities and interactions, in the production of sharable artifacts, in community-building conversations, and in the negotiation of new situations" (p. 184). He adds:

> *[communities of practice] come together, they develop, they evolve, they disperse. ... Thus, unlike more formal types of organizational structures, it is not so clear where they begin and end. They do not have launching and dismissal dates. In this sense, a community of practice is a different kind of entity than, say, a task force or a team. ... A community of practice takes a while to come into being. (p. 96)*

### JOINT ENTERPRISE

Joint enterprise concerns the way in which members do what they do. Wenger emphasized that "because members produce a practice to deal with what they understand to be their enterprise, their practice as it unfolds belongs to their community in a fundamental sense" (p. 80). Practice is, in fact, shaped by the community's way of responding to conditions, resources, and demands.

### SHARED REPERTOIRE

Wenger described the shared repertoire of a community of practice as follows:

> *The repertoire of a community of practice includes routines, words, tools, ... concepts that the community has produced or adopted in the course of its existence, and which have become part of its practice. The repertoire combines both reificative and participative aspects [reificative: documents, instruments, forms, etc.; participative: acting, interacting, mutuality, etc.]. It includes the discourse by which members create meaningful statements about the world, as well as the styles by which they express their forms of membership and their identities as members. (p. 83)*

The above description would appear to focus more on the repertoire of an existing community. But, as Wenger pointed out: "A community of practice need not be reified as such to be a community: it enters into the experience of participants through their very engagement" (p. 84). Indeed, the reificative aspects of a community are of two types. There are those that are the products of the enterprise, such as—for a community of researchers—research reports, publications, and so on. There are also those that are more related to the processes engaged in by the community. The latter might include the taking on of a more formal, organizational structure, but this is not essential for a community to exist. In either case, as Wenger emphasized: "Reification is not a mere articulation of something that already exists. ... [It involves] not merely giving expression to existing meanings, but in fact creating the conditions for new meanings" (p. 68).

As reificative aspects can yield evidence related to a community's coming together, emerging, and developing, I first present two rather broad examples of reificative aspects of the shared repertoire: doctoral dissertation production and government-funded research (note that master's theses would have been included as an example in this category, were it not for the challenge of obtaining reliable information regarding their production across the country over the past century). The overview that is presented in the following section not only signals the growth that occurred during approximately three-quarters of a century but also helps situate the later discussion of the communities of practice that emerged at both the local and national levels and the interactions between them.

## AN OVERVIEW OF THE GROWTH OF A COMMUNITY

When Jeremy Kilpatrick (1992) wrote a history of research in mathematics education in the 1992 *Handbook of Research on Mathematics Teaching and Learning*, he argued that disciplined inquiry into the teaching and learning of mathematics in the United States and elsewhere in the world had its beginnings in the universities. Therefore, signs of the beginnings of a Canadian community of mathematics education researchers were sought in the universities.

### THE EARLY CANADIAN RESEARCH RELATED TO MATHEMATICS EDUCATION

The first doctorate from a Canadian university for research that was related to school mathematics was awarded in 1924, a Doctor of Pedagogy from the University of Toronto (U of T) (*Dissertation Abstracts* 1861–1996). The dissertation had the title *Practice in Arithmetic or the Arithmetic Scale for Ontario Public Schools*. It was followed by three more in 1929, 1943, and 1945 at the same university. These first dissertations centered on surveys of the teaching of arithmetic, the development of arithmetic evaluation instruments, and the diagnosis and remediation of arithmetic learning problems. The subject matter of these dissertations suggests that, as early as 1920 at U of T, there were individuals, perhaps even a group, whose main research interest was mathematics education.

In fact, there was very definitely an interest in school arithmetic at U of T, and this preoccupation preceded by several years the 1924 dissertation just mentioned. It seems that in the late 1880s John Dewey had been contacted by James McLellan, who was Director of Normal Schools for Ontario and a professor of pedagogy at U of T (see Dykhuizen 1973, p. 60), to write a psychological introduction to a book that McLellan was authoring on educational theory and practice. The outcome of that collaborative effort was published in 1889 (McLellan 1889), but more interestingly it led to a second book on the study and teaching of arithmetic, *The Psychology of Number and its Applications to Methods of Teaching Arithmetic* by McLellan and Dewey in 1895. However, McLellan passed away in 1907, at the age of 75, and thus had no direct role in the supervision of the first dissertations.

The doctoral research related to school mathematics that was carried out at U of T from the 1920s to the 1940s was succeeded by similar work in the late 1940s and early 1950s at Université de Montréal (one dissertation in 1947) and Université Laval (one dissertation in 1951). In 1948, U of T added another to its set of dissertations related to school mathematics. Thus, the total number of dissertations completed at Canadian universities on research related to the mathematics curriculum or the teaching and learning of school mathematics during the 1924–1951 period was seven. Even though a national community of practice was still far from being a reality, it was clear that small groups had begun to be involved in mathematics education research in Ontario and Québec by the middle of the twentieth century. But progress during these years was very slow. According to the U.S. scholar Ellen Lagemann (1997), mathematics education as a research discipline in its own right did not exist in many countries prior to the late 1950s and early 1960s, at which time educational research began to be more discipline-based (see Kilpatrick 1992 for an extensive discussion of this process both in the United States and abroad). Nevertheless, the topics of interest in these early Canadian dissertations related to mathematics education reflected themes that were equally of interest south of the border.

It took the post-World War II population boom to provide a jump-start to the mathematics education research enterprise in Canada. The population, which in the 1920s stood at 9 million, rose to 14 million in 1950 and then 18 million in 1960, registering in that latter decade the highest percent of increase since the years 1900–1910. To accommodate the growing numbers of students in the 1960s, the already existing universities had to expand, and new ones were created. The number of graduate programs increased, too, which meant that more research would now be done than ever before.

## A PERIOD OF GROWTH FOR THE UNIVERSITIES

Of the several universities created in Canada in the post-World War II years, the two that were formed in Montréal—Concordia University in 1964, and Université du Québec à Montréal (UQAM) in 1969—made an innovative decision regarding the intersection of mathematics and education. Scholars who were interested in the teaching and learning of school mathematics were affiliated with mathematics departments rather than education departments. This was a period of intense educational reform in the province of Québec. The traditional eight-year classical colleges of the French-language educational system of the province were disbanded in the 1960s and replaced by high schools, *CEGEPs* (from Collèges d'enseignement général et professionnel [in English, Colleges of General and Professional Training]; i.e., colleges that dispensed both preuniversity and technical or vocational courses), and universities. Those who had taught at the upper levels in these classical colleges were integrated into the new CEGEPs and universities of the province. The same reform that closed the classical colleges also brought an end to the French- and English language normal schools and transferred instructors and students alike to both the new and existing universities. Similar events with respect to the transfer of normal school teachers to the universities occurred in other provinces as well, but at different times; for example, in Alberta, this changeover took place in 1945.

It might have been expected that once the teacher-training faculty had been incorporated into universities across the nation, they would soon enough get involved in research. But it took time because many of the freshly appointed education professors had to work at obtaining doctoral degrees themselves and learning about the research process. At some universities across the nation, a tension arose between the role of teacher trainer and that of researcher, and was not resolved for several decades in many education faculties. As of the early 1990s, there continued to be faculties of education in several universities where the emphasis was clearly on teacher training. Research was simply not part of the culture of these faculties of education, as was indicated by the lack of graduate programs with a research component. At such universities, students might have been able to obtain a master's degree, but had to go elsewhere if they wished to continue on to the doctoral level in education. In 1990, there were merely seventeen Canadian universities where it was possible to obtain a doctoral degree involving research related to mathematics education (see Kieran and Dawson 1992).

## GROWTH IN DISSERTATION RESEARCH RELATED TO MATHEMATICS EDUCATION

The years from 1955 to about 1969 were years of continuing gradual growth in Canada for dissertation-based research related to school mathematics. During this period of population increase, of reform in the educational systems of various provinces, and of the beginnings of mathematics education as an identifiable field of research study in many countries of the world, the number of school-mathematics-related dissertations in Canadian universities showed a modest increase. As well, the production moved beyond the universities of Ontario and Québec. The Universities of Alberta and British Columbia had also begun to develop research groups interested in school mathematics.

But the period of most intense growth in dissertation research related to mathematics education in Canadian universities occurred from the late 1960s onward. See table 1 for the number of dissertations related to school mathematics for which doctoral degrees were awarded in Canadian universities from 1924 up to 1995. The data from table 1 are re-presented in graphical form in figure 1 so as to see at a glance the periods that were peaks with respect to Canadian math education dissertation production. Note the rise in the mid- to late 1970s (of the 43 doctorates awarded during the years 1974–1979 for research related to mathematics education, 17 were from the U of A and the remainder from six other universities across the country). This rise during the 1970s was followed by a period of slower growth, until the 1990s when the sharpest increase took place—resulting in the highest peak in 1994 with twenty-one dissertations.

Over the 72-year period from 1924 through 1995, the lone entrant of the early years— U of T— was joined during the latter part of this period by several other universities. Nevertheless, the majority of the 80 doctorates (63 of them, or 79 percent) for research related to mathematics education that were awarded from 1990 to 1995 came from 6 universities (U of T, including OISE; U of A; Université Laval; Université de Montréal; UQAM; and UBC). The remaining 17 doctorates awarded during this period were from 11 other universities across the country. By the end of the 1990s, there had been a 60 percent increase in the number of universities offering doctoral programs in education over that of 1990. It had become possible to earn a doctorate for research in mathematics education in all provinces of the country except for New Brunswick, Prince Edward Island, and Newfoundland (see table 2 for province-by-province totals of dissertations produced during the 1990–1995 period for research related to mathematics education; data reflect the recency of the doctoral programs in Saskatchewan, Manitoba, and Nova Scotia). The increase in mathematics education dissertation production over the 72-year period suggested the presence of communities of mathematics education researchers at certain universities and, along with other events to be discussed, reflected as well the growth in the Canadian mathematics education research community at large.

Table 1. The yearly number of Canadian-university doctoral dissertations related to mathematics education for which a degree was awarded during the period 1924–1995 (A skipped year indicates that no dissertations were produced.)

| | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1924 | 1 | 1951 | 1 | 1964 | 1 | 1971 | 5 | 1977 | 4 | 1983 | 1 | 1989 | 8 |
| 1929 | 1 | 1955 | 3 | 1966 | 1 | 1972 | 3 | 1978 | 6 | 1984 | 4 | 1990 | 10 |
| 1943 | 1 | 1956 | 1 | 1967 | 3 | 1973 | 4 | 1979 | 9 | 1985 | 5 | 1991 | 7 |
| 1945 | 1 | 1957 | 1 | 1968 | 1 | 1974 | 8 | 1980 | 6 | 1986 | 4 | 1992 | 18 |
| 1947 | 1 | 1959 | 3 | 1969 | 3 | 1975 | 8 | 1981 | 6 | 1987 | 8 | 1993 | 13 |
| 1948 | 1 | 1962 | 1 | 1970 | 3 | 1976 | 8 | 1982 | 4 | 1988 | 3 | 1994 | 21 |
| | | | | | | | | | | | | 1995 | 11 |
| | | | | | | | | | | | | Total: | 212 |

Note: This compilation is extracted from the following data bases: *Bibliothèque Nationale du Canada* 1947–1981; *Bibliothèque Nationale du Canada* 1981–1984; *Canadian Education Index* 1976–1996; *Dissertation Abstracts* 1861–1996; and specialized university listings such as that of *Université du Québec à Montréal* 1996–1997. For work carried out during the early part of the period and for which the data bases provided no descriptors or abstracts, only the title of the dissertation could be used for deciding whether the content was related to school mathematics.

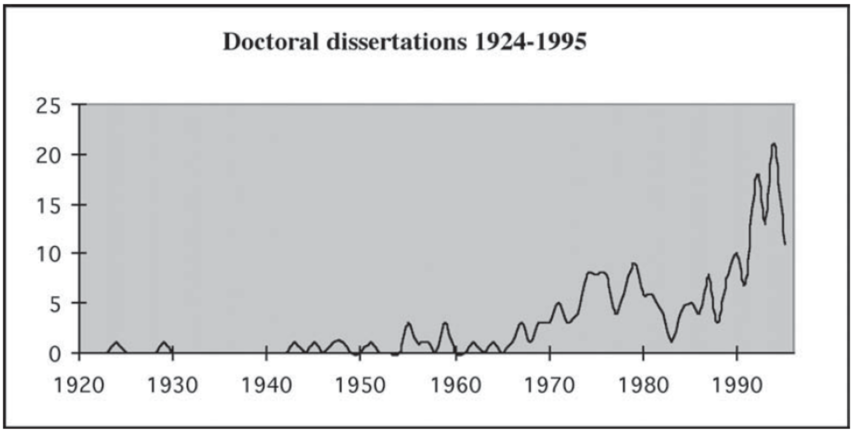


Figure 1. A graphical representation of the growth in the number of doctoral dissertations related to mathematics education that were produced in Canada between 1924 and 1995.

Table 2. Province-by-province distribution (in a west-to-east order) of mathematics education related dissertations for which a doctorate was awarded during the 1990–1995 period in Canada

| Province | Number of dissertations |
|---|---|
| British Columbia (BC) | 10 |
| Alberta (AB) | 13 |
| Saskatchewan (SK) | 2 |
| Manitoba (MB) | 1 |
| Ontario (ON) | 23 |
| Québec (QC) | 30 |
| New Brunswick (NB) | 0 |
| Nova Scotia (NS) | 1 |
| Prince Edward Island (PE) | 0 |
| Newfoundland (NF) | 0 |

## FEDERAL GOVERNMENT FUNDING OF MATHEMATICS EDUCATION RESEARCH

The continued increase in doctoral dissertation research in mathematics education was but one of the global indicators of the growth of the Canadian community. Another indicator, one that is also linked with reificative aspects of the shared repertoire, was government funded research. Nearly 50 years elapsed between the production of the first dissertation related to school mathematics in Canada and the initial awarding of federal funds in 1970 for mathematics education research carried out by university faculty (or independent scholars associated with a university). However, as table 3 illustrates, the 1970s were not especially productive for federally funded research in mathematics education. It was not until 1983 that such projects became more significant in number. (Note that federal funding for mathematics education research was under the control of the Canada Council, which was set up in 1957, and then under the Social Sciences and Humanities Research Council which replaced it in 1978.)

The low figures in table 3 for certain provinces reflect the fact that universities in some of those provinces had not, as of 1990, developed doctoral programs where one could obtain a degree for research related to mathematics education. Thus, academics in those universities had not, in general, applied for federal research funds. In contrast, the high figures for the province of Québec reflect the emergence of communities of researchers in the 1970s and 1980s who were strongly encouraged and supported at both the university and provincial government levels (I will say more about this in a later section when I treat the communities of practice in various provinces).

By graphically overlaying the data on doctoral dissertation production with those on federally-funded research projects (see figure 2), one obtains an overview that suggests three phases of growth over the years 1924–1995. The years up to approximately 1967 can be considered the years of *preemergence* of the community—the number of doctoral dissertations had not increased dramatically and no research projects related to mathematics education had yet been funded. The years from 1967 to approximately 1983 can be considered the years of *emergence*—there was significant growth with respect to doctoral dissertation production and mathematics education research by university faculty had begun to be funded, even if somewhat sporadically. The years from 1983 onward can be considered the years of *continued development* that followed the middle phase of emergence—doctoral dissertation production had gone on to reach new highs, after a brief slowdown period, and federally funded research had come into its own. In fact, during the third phase, both dissertation production and federally funded research greatly increased together—a sign that the community had already emerged.

Table 3. Number of new research projects in mathematics education in Canada funded by the Canada Council (1957–1977) and the Social Sciences and Humanities Research Council (1978–1995)

| Year | BC | AB | SK | MN | ON | QC | NB | NS | PE | NF | Total # projects funded |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | Province (arranged in a west to east order) | | | | | | | | | | |
| Up to 1970 | ... | ... | ... | ... | ... | ... | ... | ... | ... | ... | **0** |
| 1970 | ... | ... | ... | ... | ... | 1 | ... | ... | ... | ... | **1** |
| 1971 | ... | ... | ... | ... | ... | 1 | ... | ... | ... | ... | **1** |
| 1972 | ... | ... | ... | ... | ... | ... | ... | ... | ... | ... | **0** |
| 1973 | ... | 1 | .... | ... | ... | 1 | ... | ... | ... | ... | **2** |
| 1974 | ... | ... | ... | ... | ... | 1 | ... | ... | ... | ... | **1** |
| 1975 | ... | ... | ... | ... | ... | ... | ... | ... | ... | ... | **0** |
| 1976 | ... | ... | ... | ... | ... | ... | ... | ... | ... | ... | **0** |
| 1977 | ... | ... | ... | ... | ... | ... | ... | ... | ... | ... | **0** |
| 1978 | ... | 1 | ... | ... | ... | ... | ... | ... | ... | ... | **1** |
| 1979 | ... | ... | ... | ... | ... | ... | ... | ... | ... | ... | **0** |
| 1980 | ... | ... | ... | ... | ... | ... | ... | ... | ... | ... | **0** |
| 1981 | ... | ... | ... | ... | ... | ... | ... | ... | ... | ... | **0** |
| 1982 | ... | ... | ... | ... | ... | ... | ... | ... | ... | ... | **0** |
| 1983 | 1 | ... | ... | ... | 1 | 2 | ... | ... | ... | ... | **4** |
| 1984 | ... | ... | ... | ... | ... | 1 | ... | 1 | ... | ... | **2** |
| 1985 | ... | 1 | ... | ... | 1 | 2 | ... | ... | ... | ... | **4** |
| 1986 | 1 | ... | ... | ... | ... | 3 | ... | ... | ... | ... | **4** |
| 1987 | ... | ... | ... | ... | 2 | 5 | ... | ... | ... | ... | **7** |
| 1988 | 1 | 1 | ... | ... | 2 | 2 | ... | ... | ... | ... | **6** |
| 1989 | ... | ... | 1 | ... | ... | 6 | ... | ... | ... | ... | **7** |
| 1990 | 1 | 1 | ... | ... | ... | 4 | ... | ... | ... | ... | **6** |
| 1991 | 1 | 1 | ... | ... | ... | 1 | ... | ... | ... | ... | **3** |
| 1992 | ... | ... | ... | ... | 1 | 2 | ... | ... | ... | ... | **3** |
| 1993 | 1 | 1 | ... | ... | 1 | 4 | ... | ... | ... | ... | **7** |
| 1994 | ... | 1 | ... | ... | ... | 5 | ... | ... | ... | ... | **6** |
| 1995 | ... | ... | ... | ... | 3 | 3 | ... | ... | ... | ... | **6** |
| Totals/ province | 6 | 8 | 1 | 0 | 11 | 44 | 0 | 1 | 0 | 0 | **71** |

Note: The first year in which a given project was funded is the year used, as is the province of the principal investigator. Sources: *Annual Reports* of Canada Council (1958–1978) and *Annual Reports* of Social Sciences and Humanities Research Council of Canada (1978–1996).

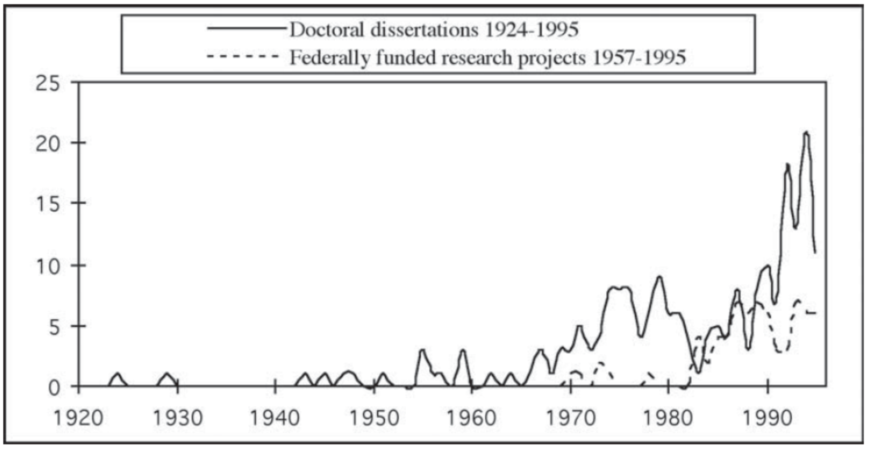


Figure 2. The number of doctoral dissertations related to mathematics education during the period 1924–1995 compared with the number of federally funded research projects related to mathematics education (Note that the federal funding agency was not established until 1957.)

## INTERNATIONAL INFLUENCES AND INTERACTIONS RELATED TO THE EMERGING CANADIAN COMMUNITY OF MATHEMATICS EDUCATION RESEARCHERS

Few countries develop their research communities in isolation. The influences of, and interactions with, mathematics education researchers in other countries, along with research related events that took place outside Canada, all served to shape the Canadian community. In this part of the paper, the focus switches to the activities of individual Canadians and the roles they played both at home and abroad, particularly during the 1960s and 1970s—activities that had an effect on the evolution of the Canadian community of mathematics education researchers. As will be seen, the impact of several of these individuals was felt in both the local and the national communities of practice that they worked to develop.

### EARLY INTERACTIONS WITH THE UNITED STATES MATHEMATICS EDUCATION RESEARCH COMMUNITY

The United States was a source of influential ideas with respect to the growth of research in mathematics education in Canada not only during the initial three decades of the 1924–1995 period, when the first doctoral dissertations were produced at U of T, but also in the decades that followed, especially from the mid-1950s through the 1960s. Douglas Crawford in the 1970 *History of Mathematics Education in the United States and Canada* has described some of the interactions that occurred between Canadians and Americans during the 1950s and early 1960s. But the most obvious source of influence of the U.S. mathematics education research scene on the developing Canadian community during these years was its journals. The U.S. research journals played an important dual role, especially in the late 1960s and 1970s, in that they not only enabled the growing number of Canadian mathematics education researchers to read about the kinds of research that were being conducted in the United States but also published the work of members of the newly emerging Canadian research community. These journals thus provide a window on the research activities of many Canadians during that period.

For example, in 1969, the *Review of Educational Research* (*RER*) published a special issue on mathematics education research. In that special issue was a paper by Tom Kieren on "Activity Learning," which contained several references to the mathematics education research activities being engaged in by fellow Canadians. The *Journal for Research in Mathematics Education (JRME),* which published its first issue in January 1970, also featured the work of Canadian researchers during its early years, for example, Tom Kieren, Daiyo Sawada, Walter Szetela, W. George Cathcart, David Robitaille, Frank Riggs, Doyal Nelson, Lars Jansson, Jim Sherrill, and D. Kaufmann. Equally important to note is that these early contributors to *RER* and *JRME* were primarily from Canada's western provinces—in particular, from U of A and UBC—reflecting the ties that had developed between the English speaking mathematics educators of western Canada and their National Council of Teachers of Mathematics (NCTM) counterparts in the United States. This western Canada-United States connection parallels the pull felt by many of the French-speaking researchers of the country, along with some of their Anglophone colleagues, toward Europe (e.g., to the International Commission on Mathematical Instruction and its quadrennial International Congresses on Mathematical Education, the soon-to-be-formed International Group for the Psychology of Mathematics Education, and the Commission Internationale pour l'Étude et l'Amélioration de l'Enseignement des Mathématiques).

## INTERNATIONAL INTERACTIONS

Developments in mathematics education research at an international level in the late 1960s and 1970s attracted the attention of Canadian researchers. Canadians played a role in the growth of international associations of researchers, while at the same time coming into contact with fellow researchers from other parts of Canada and thereby forging the relationships needed for a Canadian community of practice.

### THE EARLY INTERNATIONAL CONGRESSES ON MATHEMATICAL EDUCATION IN THE 1960S AND 1970S

The first International Congress on Mathematical Education (ICME), which was held in Lyon, France, in August 1969, drew twenty-three Canadians—sixteen from Québec, six from Ontario, and one from Manitoba. It exposed them to the work and ideas of several invited speakers, including the research-related plenary presentations of Ed Begle from the United States, Efraim Fischbein from Israel, and Zoltan Dienes from Canada (see ICME-1 1969). Dienes was a researcher, born in Hungary, who had spent several years in Sherbrooke, Québec, directing the research center he founded there in the 1960s. The next ICME, held three years later in Exeter, England, attracted even more Canadians than the previous one had; this time, fifty-two Canadians attended—twenty-three from Ontario, fourteen from Québec, seven from British Columbia, three from Nova Scotia, three from Alberta, one from New Brunswick, and one from Manitoba. The program allotted more time to research than ICME-1 had, with two of the thirty-nine working groups targeted explicitly towards discussion of research on learning and teaching (ICME-2 1972), one chaired by Efraim Fischbein on the psychology of learning mathematics and another chaired by Bent Christiansen on research in the teaching of mathematics. At this ICME, among the Canadians attending were Marshall Bye, Douglas Crawford, Claude Gaulin, Bill Higginson, Claude Janvier, Raynald Lacasse, and Richard Pallascio.

The increasing worldwide interest in mathematics education, and in mathematics education research in particular, was reflected in Canadian participation at the third ICME in Karlsruhe, Germany, in 1976. Fifty-one Canadians attended: twenty-nine from Québec, thirteen from Ontario, five from Alberta, one from Manitoba, one from British Columbia, one from Saskatchewan, and one from Prince Edward Island. Participants had the opportunity to congregate with, and talk to, many active mathematics education researchers from around the world, and to hear about current international research activities, such as those reported by Heinrich Bauersfeld and by Jeremy Kilpatrick. But, by far, the most important research -related event that occurred at ICME-3 was the formation of the International Group for the Psychology of Mathematics Education (originally IGPME, then changed to PME), a group that was to become the largest association of mathematics education researchers in the world. Not only did

Canadians play an important role in the genesis of this international group, but their participation in its creation served also to increase the interest in the research enterprise at home and to contribute a particular identity to the emerging Canadian community.

## THE FORMATION OF THE INTERNATIONAL GROUP FOR THE PSYCHOLOGY OF MATHEMATICS EDUCATION

At the founding meeting of PME in Karlsruhe, 100 or so persons were present, including six Canadians: Claude Dubé, Nicolas Herscovics, Joel Hillel, Claude Janvier, Dieter Lunkenbein, and David Wheeler—all of whom were from Québec. The Canadians provided a considerable amount of leadership and support in the setting up of PME. The contributions of Nicolas Herscovics to the founding of PME were remembered by Efraim Fischbein at the opening session of a much later PME conference, in 1995, when he delivered a eulogy in memory of Herscovics, who had passed away the previous year:

> *Nicolas was instrumental in setting up a Committee, in suggesting the election of a president and in contributing to the project of organizing, as quickly as possible, the first international conference of the organization. The activity of Nicolas was a decisive factor in the creation of the new body. Nicolas understood from the beginning that one had to create an organizational body which would facilitate interaction in that area [the psychology of mathematics education], would promote common research efforts, would contribute to new ideas, new research methods, and would confer on mathematics education, a theoretical and investigative dimension which it was lacking before. (Fischbein 1995, p. 1)*

Canadians continued to play a major role in PME and remained active through the late 1990s. They assisted in the direction of PME in the following capacities: Eight served on the international committee (Jacques Bergeron, Claude Gaulin, Gila Hanna, Nicolas Herscovics, Claude Janvier, Carolyn Kieran, Gerald Noelting, and Vicki Zack); one was elected to the presidency for a three-year term from 1992 to 1995 (Carolyn Kieran); and three Montréalers (Jacques Bergeron, Nicolas Herscovics, and Carolyn Kieran), supported by many others from the Québec community of mathematics education researchers, hosted the eleventh annual PME conference in 1987. By the end of 1995, PME had 684 members, of whom 26 were Canadian—from British Columbia on the west coast to Newfoundland on the east coast**.**

Since PME's beginnings, Canadians have both drawn on the research that was presented by international colleagues at the annual conferences and also contributed their own work. One example that suggests how the research of Canadians may have been an influence on the larger PME community is reflected in the book published by PME in 1992, *Mathematics and Cognition* (Nesher and Kilpatrick 1992). The major authors in that book were all either Canadian, French, or Israeli, with the Canadian chapters focusing especially on the learning of early arithmetic and algebra. Additionally, Canadians served as contributing authors to the chapters on the epistemology and psychology of mathematics education, language and mathematics, and advanced mathematical thinking.

## THE ROLE OF THE COMMISSION INTERNATIONALE POUR L'ÉTUDE ET L'AMÉLIORATION DE L'ENSEIGNEMENT DES MATHÉMATIQUES IN THE DEVELOPMENT OF FRENCH-SPEAKING CANADIAN MATHEMATICS EDUCATION RESEARCHERS

An important international association involving French-speaking Canadians during the early years of their developing practice as mathematics education researchers was the Commission Internationale pour l'Étude et l'Amélioration de l'Enseignement des Mathématiques (CIEAEM), which had been set up in 1950. According to Claude Janvier (personal communication, September 1997), "Québecers were not researchers in the late 1960s and early 1970s; they were university teachers looking for the best curricula and the best approaches for teaching that curricula." Janvier emphasized that those were the years when the "new math" movement was being experienced. Teachers had to be retrained. In their search for answers to questions regarding new approaches to teaching mathematics, Québec mathematics educators came across examples of European research. So they decided to attend CIEAEM conferences where much of this research was being generated and discussed. The style of research appealed to Québec participants: critical questioning of what was being promoted in many countries under the name of new math. According to Janvier, the Québecers were also attracted to the voice being given in CIEAEM conference discussions to teachers and their experience.

## LOCAL COMMUNITIES OF PRACTICE

The images presented thus far suggest that communities of mathematics education researchers had been established here and there throughout the country. Indeed, local communities of mathematics education researchers had either emerged or were in the process of emerging during the 1960s and 1970s in several provinces. Not all of these communities were at the same point of development at the same time. Nor were they all of the same size or strength. But there was no question as to their existence. Space constraints do not allow me to cover all of the provinces, such as for example, Manitoba and the work of Lars Jansson, or Nova Scotia where Yvonne Pothier and Mary Crowley were instrumental in developing a local community of practice, or the efforts of Lionel Pereira-Mendoza in Newfoundland, or the more recent work of Vi Maeers in Saskatchewan.

As will be seen, the local communities were the roots of the national community, a community whose period of emergence included a reificative act of a more formal nature in 1978. However, these communities were more than roots. They continued to develop, in interaction with the national community, during the 1980s through the 1990s. As will be seen from the examples of local communities of practice to be presented, it was not always the case that communities of a provincial nature emerged. In some provinces, structures—either of a formal or informal nature—had simply not been put into place that would encourage the development of a single provincial community. In contrast, in other provinces, the local communities of practice had become quite interconnected.

### THE COMMUNITIES OF PRACTICE IN ALBERTA

From about 1965, the University of Alberta (U of A) was an active center of mathematics education research, with the work of Doyal Nelson and Sol Sigurdson. Tom Kieren's arrival at U of A in 1967 signaled a bustling period devoted to theorizing in this local community of mathematics education research. In the 1970s, the research by Kieren, his colleagues, and their many graduate students focused on such themes as models of the use of concrete materials, rational number construct theory, rational number mechanisms, and neo-Piagetianism. The group's strong interest during the 1970s in constructivism developed further when, in the 1980s, Maturana and Varela's theories were brought into play.

Another local community of mathematics education researchers in the province of Alberta was situated at the University of Calgary where Bruce Harrison became a faculty member after being the first graduate of the new mathematics education doctoral program at U of A in 1968. One of Sigurdson's students, Harrison had focused his doctoral research on elaborating and testing the work of Richard Skemp on reflective thinking. This community continued its evolution throughout the decades that followed, a period that included the arrival of Olive Chapman during the 1990s. In another area of Alberta, at the University of Lethbridge, the mathematics education group benefited from interactions during the 1970s with Sigurdson from U of A on "direct meaning" teaching and curriculum.

The Alberta community of mathematics education researchers, from its early days in the 1960s and 1970s up to the late 1990s, was characterized by a strong theoretical leaning toward a nonrepresentationist, constructivist, enactivist perspective. It could be said that theory building was a main feature of their joint enterprise. And this interest was reflected in the dissertation topics of several of the mathematics education doctoral students trained in Alberta, including the early graduates such as Bruce Harrison, Sandy Dawson, and Bill Higginson. Tom Kieren advised many of the U of A students—in fact, he guided to completion more mathematics education Ph.D. students than any other Canadian academic during the last three decades of the twentieth century. As the Ph.D. graduates from U of A took up posts at other Canadian universities, characteristics of that community of practice took root at these other sites. And because many of these researchers continued to interact with fellow Canadians in the development of a joint enterprise, the Canadian community of mathematics education researchers reflected some of the features of the extended Alberta community.

## THE COMMUNITIES OF PRACTICE IN BRITISH COLUMBIA

In the 1960s, concomitant with the arrival of the new math era, the mathematics education department of the University of British Columbia (UBC) mushroomed to become what was then the largest in Canada. As the new math period drew to a close, David Robitaille assumed the headship of the department, and he and Jim Sherrill in particular redirected the department to contemporary concerns.

Shortly after his arrival, David Robitaille was asked to take over the clinic for school children having trouble with mathematics, a project with which the department had been involved for some time. The diagnostic research focus gradually evolved during the late 1970s and 1980s into more qualitative work on children's misconceptions and on constructivist approaches to learning. Examples include studies on problem solving and metacognition (e.g., the work of Walter Szetela, Jim Sherrill, and Tom Schroeder), and decimal- fraction learning within a Piagetian perspective (the research of Doug Owens). As well, there was a growing interest in large-scale evaluation. This interest was exemplified in the role that UBC mathematics education researchers played in both the second and third international mathematics studies, in particular, as the international coordinating center (directed by David Robitaille) for the entire TIMSS study from its outset and until the end of 1993.

The story of the development of the UBC community of mathematics education researchers illustrates the multifaceted nature of the joint enterprise of that community, which had produced a shared repertoire covering many different themes of research, from diagnostic work to problem solving to large-scale evaluation. This diversification continued through the 1990s when new faculty were hired: Susan Pirie, with her interest in models of students' understanding; Ann Anderson, who focused on mathematics learning in young children; and Cynthia Nicol, whose research centered on the education of mathematics teachers.

During these same years, another local community of mathematics education researchers developed in British Columbia—at Simon Fraser University (SFU). The SFU community was characterized by a consistent theme that ran through its approach to mathematics education research over the years—that of studying the implementation of innovative teaching practices that arose from an investigation of the nature of mathematics. That theme was originated by John Trivett, who joined the SFU faculty of education in 1967; was strengthened in the early 1970s when Sandy Dawson was hired; and became fully developed with the addition of Tom O'Shea in the early 1980s. The theme arose out of, and was based on, a close collaboration between the mathematics department at SFU (Len Berggren, Harvey Gerber, and others) and the above mathematics educators of the faculty of education.

The same theme that characterized the mathematics education research at SFU from the 1960s through the 1980s was reinforced and elaborated when Rina Zazkis joined the faculty in the early 1990s. When Rina arrived, not only did she work closely with members of the mathematics department, continuing the tradition established by her colleagues, but she also taught courses in that department, in particular the mathematics-for-teachers course that the mathematics department had offered for a number of years. Rina also used this opportunity to begin an investigation of the understandings that preservice teachers have of number theoretic concepts. The close connection between mathematicians and mathematics educators, that was one of the distinguishing features of the local community of mathematics education researchers at SFU, will be seen to be a characteristic as well of the national community of mathematics education researchers from the late 1970s onward.

## THE COMMUNITIES OF PRACTICE IN ONTARIO

In 1965, U of T's departments of Graduate Studies and Research were transferred to the newly created Ontario Institute for Studies in Education (OISE), an autonomous institution with an affiliation agreement with U of T. Despite OISE's newly acquired prominence on the graduate education scene of Ontario in the 1960s and the past history of U of T in doctoral research related to mathematics education prior to the 1960s, events in the 1960s and 1970s seemed to work against the growth of a unified community of mathematics education researchers centered at OISE. The new math movement brought several actors onto the Ontario mathematics education stage, but they belonged to different communities, each with its own forms of engagement, enterprise, and shared repertoire. In fact, these communities held quite opposite views of what was important in mathematics education and of what it might mean to do research in mathematics education.

In 1985, Gila Hanna joined the OISE faculty, after having been a research associate there since 1978. Her main research interests focused on gender studies and the role of proof in mathematics. She also advised most of the doctoral candidates in mathematics education in Ontario from 1985 through the late 1990s. These years, which were important ones in the growth of the mathematics education research community at OISE, signaled a period of intense activity that was due in no small measure to her leadership. In 1999, just a year before her retirement from OISE-U of T, Gila established a new bilingual Canadian journal, which she would coedit with two colleagues. These years also witnessed important work being carried out by the educational psychologists at OISE, for example, Robbie Case's research on rational number and Rina Cohen's on the learning of mathematics in Logo environments.

While a local community of practice in Ontario was developing at OISE in the late 1980s, other communities of mathematics education researchers that were emerging in Ontario included one at the University of Western Ontario, where Doug Edge, Eric Wood, Barry Onslow, and Allan Pitman were active, and another at the University of Windsor, which centered on the work of Erika Kuendiger and her colleagues and students.

The last local community of practice in Ontario to be discussed, but by no means the smallest or the most recent, is the community of mathematics education researchers that emerged at Queen's University in Kingston during the 1970s. A faculty of education had been established there in the late 1960s, during the period when teachers colleges and normal schools in Ontario had begun to be affiliated with the universities. The corps of active researchers in mathematics education at Queen's included Hugh Allen, Douglas Crawford, and Bill Higginson. Under Bill's leadership, the group established Queen's as an important center of mathematics education and mathematics education research, in particular in the area of technology applied to the teaching and learning of school mathematics.

A significant feature of this local community of mathematics education researchers was the involvement of some of the faculty from the mathematics department. Working relationships were developed between the mathematics educators and mathematicians such as John Coleman. Peter Taylor was another Queen's mathematician who actively collaborated with the mathematics educators. Taylor, known for his research on the teaching of calculus, was honored in the 1990s for his contributions to mathematics education by a 3M Teaching Fellowship awarded by the Society for Teaching and Learning in Higher Education.

Other Ontario mathematicians who conducted research in mathematics education included Pat Rogers, then of York University, whose research focused on increasing the participation of women in university mathematics courses, research for which she, too, was awarded a 3M Fellowship in the 1990s. Similarly, Eric Muller from Brock University and Ed Barbeau from U of T (as well as Bernard Hodgson from Québec) were recipients of the Adrien Pouliot award for sustained contributions to mathematics education in Canada, an award given each year since 1995 by the Canadian Mathematical Society. Other awardees included the group from the University of Waterloo for their work on Canadian mathematics competitions at the secondary school level.

The productive interactions between mathematicians and mathematics educators that had been fostered at Queen's from the late 1970s through the 1990s were reflected in other such collaborative work occurring in the province. For example, in the 1990s, Eric Muller and Ed Barbeau, along with Gila Hanna and Bill Higginson, served on the Forum of the Fields Institute that played a major role in the revision of the Ontario secondary school mathematics courses. Interactions such as these between mathematics educators and mathematicians set in place the mechanisms for the creation of a center for mathematics education at the Fields Institute.

## THE COMMUNITIES OF PRACTICE IN QUÉBEC

The 1960s were exciting years in Québec. Zoltan Dienes, a recent arrival to Canada, had just set up his research institute, Centre de Recherche en Psycho-Mathématiques, at Université de Sherbrooke. His research center attracted visitors from all around the world, thereby exposing Québecers to international mathematics education research. Other signs of research activity in Québec in the 1960s and 1970s included the research centers such as Institut National de la Recherche Scientifique. However, an event that was among the most significant with respect to the emergence of a community of mathematics education researchers in Québec was the creation of a government funding agency that paralleled the federal SSHRC. In 1970, the Programme de Formation des Chercheurs et d'Actions Concertées (FCAC) was set up by the newly formed ministry of education of Québec. One of the key elements of FCAC funding, as well as that of the Fonds pour la Formation de Chercheurs et l'Aide à la Recherche (FCAR) that replaced it in 1984, was the encouragement of teams of researchers, including teams drawn from various universities. Table 4 gives the number of FCAC/FCARfunded projects in mathematics education research between 1972 and 1995.

In comparison with the data presented earlier in table 3, which showed that mathematics education research was hardly present on the federal funding scene prior to 1983, the data of table 4 reveal that mathematics education research was in fact being funded in Québec during the years 1972–1983 and that 1981 was an especially productive year for the emerging community. Thus, as of 1970, Québec researchers had access to two governmental funding agencies in contrast with researchers from the rest of the country who could only submit research proposals to the federal funding agency SSHRC.

The first recipient of an FCAC grant for mathematics education research, in 1972, was Claude Gaulin (with Hector Gravel as co-investigator). Gaulin was one of Québec's pioneers in mathematics education research, having carried out studies on the teaching of fractions from 1966 to 1971 with colleagues from Collège Ste-Marie, a college that was incorporated into Université du Québec à Montréal (UQAM) in 1969. Another pioneer from the 1970s was Dieter Lunkenbein from Université de Sherbrooke, who conducted research on the teaching of geometry in the early grades. And at Université Laval was a mathematician named Fernand

Lemay, whose theoretical reflections were a great influence on the conceptual and epistemological thinking of some of the Québec mathematics education researchers of that time.

An additional development of significance for the growing Québec community of mathematics education researchers during these years was related to a series of retraining courses for mathematics teachers provided by the ministry of education from 1965 to 1970. These courses evolved into the highly successful in-service distance education program for the

Table 4. Number of new research projects in mathematics education in Québec funded by FCAC (1970–84) and FCAR (1984–95)

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1972 | 1 | 1976 | 2 | 1980 | 0 | 1984 | 3 | 1988 | 5 | 1992 | 1 |
| 1973 | 0 | 1977 | 1 | 1981 | 9 | 1985 | 4 | 1989 | 2 | 1993 | 1 |
| 1974 | 0 | 1978 | 2 | 1982 | 3 | 1986 | 1 | 1990 | 1 | 1994 | 2 |
| 1975 | 0 | 1979 | 4 | 1983 | 4 | 1987 | 2 | 1991 | 2 | 1995 | 2 |

Note: The first year in which a given project was awarded is the year used. Sources: *Subventions accordées* of FCAC (1971–73), *Crédits alloués* of FCAC (1973–77), *Répertoire des subventions allouées* of FCAC (1977–79), *Crédits alloués: équipes et séminaires* of FCAC (1979–84), *Rapports annuels* of FCAR (1984–91), and *Répertoire des subventions octroyées: soutien aux équipes de recherche* of FCAR (1990–96).

retraining of mathematics teachers throughout the province (known as the PERMAMA program), a program that not only involved many of the province's mathematics educators but also served as a basis for elaborating some of their research orientations. Another important event of the 1970s was the formation of the Groupe des Didacticiens en Mathématiques (GDM), an association of Québec mathematics educators interested in research.

During the late 1970s, as well, FCAC became more structured and began to give both more and larger grants. The excitement that had been generated with the creation of the international PME group at Karlsruhe in 1976 led to the formation of new teams of funded researchers in Québec, for example, the group of Joel Hillel and David Wheeler, followed by the collaboration of David Wheeler and Lesley Lee, at Concordia University, and the team situated at U de M of Nicolas Herscovics from Concordia University and Jacques Bergeron from U de M. The latter researchers' work on the learning and teaching of early number in the late 1970s and 1980s attracted several doctoral students to the team (e.g., Jean Dionne, Nicole Nantais, Bernard Héraud), who in turn became advisors to later graduate students of mathematics education at other universities in Québec. The work of Roberta Mura at Université Laval in the 1980s on women in mathematics brought certain socio-cultural issues to the fore in Québec mathematics education research. And Anna Sierpinska's arrival at Concordia in 1990 from her native Poland injected new dimensions into the research being carried out on understanding and epistemological obstacles.

Another event that aided the growth of the Québec mathematics education research community was the creation of CIRADE in 1980, a center whose initial roots were located in the Centre de Recherche en Didactique that was set up when UQAM was established in 1969. Mathematics education research flourished there during the 1980s and 1990s. Several international seminars and colloquia were held; these involved not only the members of CIRADE and their international visitors but also many other mathematics education researchers of Québec. Such colloquia focused, for example, on representation and the teaching and learning of mathematics organized by Claude Janvier, on epistemological obstacles and sociocognitive conflict, and on approaches to algebra—perspectives for research and teaching, organized by Nadine Bednarz, Carolyn Kieran, and Lesley Lee.

The 1980s were also the years in Québec when the potential of the computer programming language, Logo, as a mathematical exploration tool, sparked the interest of many researchers. Just about all Québec universities and colleges had their Logo groups of mathematics education researchers in the 1980s, such as the UQAM collaboration of Benoît Côté, Hélène Kayler, and Tamara Lemerise, as well as the inter-university team of Joel Hillel, Stanley Erlwanger, and Carolyn Kieran. During that same decade, the 17 full-time faculty members of the mathematics education section of the UQAM Mathematics Department made that group the largest ever contingent of mathematics education researchers across the country.

## A MAJOR REIFICATIVE EVENT FOR THE CANADIAN COMMUNITY OF PRACTICE: THE FORMATION OF THE CANADIAN MATHEMATICS EDUCATION STUDY GROUP / GROUPE CANADIEN D'ÉTUDE EN DIDACTIQUE DES MATHÉMATIQUES

Let us return for a moment to the late 1960s and early 1970s. Local communities of practice had been evolving in various provinces since then. Members from some of these communities had come together at the early ICMEs, where connections among them had been formed. Then, in 1976, the third ICME had led to the establishment of the international PME group of researchers. The momentum created by these events beyond Canada's borders sparked not only an increased interest in mathematics education research at home but also a need for a structure that would permit members of the various local communities of Canada to get together. An occasion would soon present itself, even if it was planned with a somewhat different purpose in mind.

In 1977, John Coleman, Bill Higginson, and David Wheeler invited thirty mathematicians and mathematics educators from across Canada to join them at a mathematics education conference at Queen's University, Kingston, Ontario (sponsored by the Science Council of Canada) to discuss the general theme "Educating Teachers of Mathematics: The Universities' Responsibility." The conference had been convened primarily as part of the follow-up to the Science Council's Background Study No. 37 (Beltzner, Coleman, and Edwards 1976) to consider the place and responsibility of Canadian universities in the education of teachers of mathematics. Wheeler (1992) wrote that "one purpose of the conference was served by the mere fact of bringing participants together and the consequent pooling of ideas and information by those who have overlapping interests but seldom meet" (p. 2). Despite the intended "teacher education" agenda of the meeting, Coleman wrote in a letter accompanying the proceedings of the 1977 conference, "The meeting was noteworthy for the fact that, as far as we were aware, there had never been a comparable gathering of [Canadian] university staff whose prime concern was *research* in mathematics education" (Coleman, Higginson, and Wheeler 1978, p. 1 of accompanying letter, emphasis added). Wheeler (1992) pointed out that "the encounter generated a demand from many of the participants for further opportunities to meet and talk" (p. 1). The Science Council supported a second invitational meeting in June 1978 at which the decision was taken to establish a continuing group to be called the Canadian Mathematics Education Study Group (CMESG)/ Groupe Canadien d'Étude en Didactique des Mathématiques (GCEDM). This reificative act was a very important one for the emerging Canadian community of mathematics education researchers. The formation of this group would further enable the connectedness that had already been developing among individual researchers across the country, as well as extend the joint enterprise over a broader base. At the close of the 1978 meeting, the participants voted for an acting executive committee; a formal constitution was approved at the 1979 meeting; and the first elections under the terms of the constitution took place in 1980.

Tom Kieren, in his plenary address at the 1977 meeting entitled "Mathematics Education Research in Canada: A Prospective View," emphasized the "need for much more interrelated mathematics education research to tackle the problems [of mathematics education]" and suggested that "perhaps our small numbers in Canada and our personal interrelationships will allow us to engage in such interrelated research" (Kieren 1978, p. 19). He then offered a few recommendations to effect the cooperation needed in Canadian mathematics education research, among which was the regular meeting of groups of researchers and teachers to discuss problems of mathematics education in Canada.

CMESG/GCEDM tried to find some balance between focusing on teacher training and on research. Wheeler (1992), in a historical retrospective of CMESG/GCEDM written in 1992, described the concerns of CMESG/GCEDM as follows: "The two main interests of CMESG/GCEDM have been teacher education and mathematics education research, with subsidiary interests in the teaching of mathematics at the undergraduate level and in what might be called the psycho-philosophical facets of mathematics education (mathematization, imagery, the connection between mathematics and language, for instance)" (p. 5). However, because many Canadian mathematics education researchers were also responsible for the training of mathematics teachers and did in fact focus their research on teacher training, the two main spheres of interest were intertwined.

CMESG/GCEDM has attempted to provide a forum where research could be discussed— and even where new research partnerships could be formed—as well as set up an encouraging atmosphere where novice researchers could find out how to begin. For individuals coming from universities or provinces where no local community of mathematics education research practice had yet emerged, this latter provision was extremely important. Through its activities, CMESG/GCEDM gave some mathematics educators a taste for research. Wheeler (1992) wrote that CMESG/GCEDM "has shown them that their puzzlement about some aspects of mathematics is shared by many mathematicians; and it has shown some mathematicians that learning can be studied and that teaching might be made into something more than flying by the seat of the pants" (p. 8). The fact that the study group included among its active members both mathematicians and mathematics educators gave a particular flavor to the nature of the research enterprise as engaged in by its participants. One of the aspects of this particularity was a fairly wide vision of what it means to do research in mathematics education, as suggested by the following: "The Study Group takes as its essential position that the teaching of mathematics and all the human activities that are connected to it can, and should, be *studied*, whether the study has the form of an individual's reflections, the reasoned argument of professional colleagues, or the more formal questioning of empirical or scholarly research" (Wheeler 1992, p. 8).

From the beginning, the format of the four-day CMESG/GCEDM meetings fostered a unique form of mutual engagement of its participants. Three half-days were spent within one of the working groups. Designed to be the core activity of the meetings, these working groups were based on themes related to research, teacher development, and mathematical topics. During the 1990s, a novel feature was added to the annual meeting programs: the reporting by new mathematics education doctoral graduates of their dissertation research. This feature became a standard component of the program and had the effect of encouraging younger mathematics education scholars to join the community. But it succeeded in doing more than that. It made provision for the community of practice that came together at CMESG/GCEDM meetings to be a community of learners in which new practices and new identities were formed for both the existing members and the new members. Wenger (1998) argued that "engagement is not just a matter of activity, but of community building and ... emergent knowledgeability" (p. 237) and that "practice must be understood as a learning process, ... learning by which newcomers can join the community and thus further its practice" (p. 49); "from this perspective, communities of practice can be thought of as shared histories of learning" (p. 86). From its beginnings in 1978, CMESG/GCEDM succeeded in creating both the accumulation of a history of shared experiences and the development of interpersonal relationships—processes that, according to Wenger, are characteristically entailed in the work of engagement of a community.

In describing the dimension of a community of practice that is the shared repertoire, Wenger emphasized the ways of doing and talking about things, as well as the reified written forms of its work. CMESG/GCEDM remained rather steady in size—about sixty people attending the annual meetings, with a core of regulars present every year—so the participative aspects of the community stayed quite constant over time. The only written trace of the annual meetings is the proceedings, but these do not always manage to convey the spirit of the annual get-togethers. One has to look further to obtain a sense of the reified repertoire of this national community, for example, to the publications of its members or to the journal *For the Learning of Mathematics* (*FLM*). This journal, which was established by David Wheeler in 1980, often published the texts of various contributions made at the annual CMESG/ GCEDM meetings. When David retired in the mid-1990s, the administration of the journal was handed over to CMESG/GCEDM. David (1997) emphasized, however, that the journal would not become the source of "Canadian news and views," but would continue to retain its international character. Nevertheless, the "Canadianness" of the journal was articulated by Bill Higginson in a special 1997 "retirement of the founding editor" issue of *FLM*:

> *Let me point to two other aspects of FLM that have loomed large for this reader. The first is the extent to which it has been for me a quintessentially Canadian publication in the best possible sense of that term. Partly that has been because of geography. The journal was born in the bilingual richness of Montréal (subliminally, I suspect that the real meaning of FLM is Front for the Liberation of Mathematics) and then, like many other institutions and individuals, succumbed to the siren call of mellower British Columbia. More importantly, however, are its close links with one of David Wheeler's other legacies, the Canadian Mathematics Education Study Group (Groupe Canadien d'Étude en Didactique des Mathématiques) the small but vital organization which he was instrumental in creating in the late 1970s. ... The other unique feature of FLM for me has been the extent to which it exemplifies what I would like mathematics education to be. I have always found the inside front cover proclamation of the journal's aims ("... to stimulate reflection on and study of ...") to be a succinct and graceful statement. (Higginson 1997, p. 18)*

Because of David Wheeler's influence on the Canadian community of mathematics education researchers from the 1970s to his passing in 2000, the above remarks of Higginson can be said to be related not merely to *FLM*; they relate as well to the spirit and to the "ways of doing and talking about things" of a national community whose emergence and development were stimulated by the founding of CMESG/GCEDM.

The national community of mathematics education researchers and CMESG/GCEDM were not one and the same, even if it was difficult at times to disentangle them. But CMESG/ GCEDM encouraged community building and it was this community building that was so vital to the growth of a national community of mathematics education researchers. This is not to say that, when CMESG/GCEDM meetings were over, members did not return to their local communities and work at the continued development of these communities. But they also participated in a national community by, for example, collaborating in joint research teams with other Canadians, consulting on their research projects, coadvising doctoral students from other universities across the country, and organizing research colloquia and conferences with fellow Canadian researchers.

The annual meetings of CMESG/GCEDM continued to contribute to the emergence and later development of the national community of practice; however, there were additional events that played a role as well. One of these was the formation of the North American chapter of the International Group for the Psychology of Mathematics Education (PMENA), in which Canadians participated both as founders and regular contributing members. Another was the preparation for and participation in ICME-7, held in Québec City in 1992, an event that entailed the involvement in one form or another of all Canadian mathematics education researchers.

### The Canadian Mathematics Education Research Community at the End of the Twentieth Century

The main focus of this paper has been a description of the events related to the emergence of the Canadian community of mathematics education researchers, an emergence which could be said to have occurred in the block of years from the mid-1960s to the mid-1980s. The discussion of those events also touched upon the period of preemergence prior to the mid- 1960s, as well as the years of continued development from the mid-1980s onward. What remains is to take a final look at the community in the 1990s.

In 1993, Roberta Mura of Université Laval conducted a survey of all mathematics educators who were faculty members of Canadian universities in order to learn more about the community that they constituted. Mura (1998) stated that "since the vast majority of universities do not have mathematics education departments, 'mathematics educator' is a label that individual members of various departments may or may not choose to apply to themselves" (p. 106). She therefore sent questionnaires to all those whose names appeared in the CMESG/GCEDM mailing list or in the CMESG/GCEDM research monograph produced by Kieran and Dawson in 1992, as well as to any other university-based Canadian mathematics educators known to these recipients. Of the 158 questionnaires sent out, 106 were returned; of these, 63 were retained as they were considered to belong to the target population. To be retained, one had to have answered positively to both of the following questions: (a) Do you hold a tenured or tenure-track position at a Canadian university? and (b) Is mathematics education your primary field of research and teaching? (Mura estimated that the total number of Canadians satisfying these two conditions was about 100, coming from approximately twenty-eight universities across the country.)

Mura reported that 44 of the 63 were men (70 percent). The mean age of the 63 respondents was fifty years, with a range from thirty to sixty-four. Forty-one of the respondents (65 percent) spoke English at work and 22 (35 percent), French. Of the 63 who did acknowledge mathematics education as their primary field, 47 (75 percent) worked in education departments, 13 (21 percent) in mathematics departments, and three had joint appointments. Eleven of the 13 employed in mathematics departments worked at two Québec universities, Concordia University and Université du Québec à Montréal, where mathematics education was a section of the mathematics departments. Concerning their education, 56 of the 63 respondents (89 percent) held doctoral degrees—46 in education, eight in mathematics, and two in psychology. For 57 percent of the survey participants, their highest degree was from a Canadian university, while for 33 percent it was from a U.S. university (the remaining 10 percent were from various other countries). Regarding the supervision of doctoral students, 29 percent had directed the research of at least one doctoral student.

Mura asked, "How do you define mathematics education?" In responding to this open-ended question, many referred to the goals of their field—some in theoretical terms, others in practical terms. Twenty-two respondents identified the aim of mathematics education in a way classified as "analyzing, understanding and explaining the phenomena of the teaching and learning of mathematics" (Mura 1998, p. 110). Twenty-one respondents assigned to mathematics education the goal of improving the teaching of mathematics and the facilitation of its learning. But Mura pointed out that these two identified goals are not mutually exclusive:

> *In fact, four respondents integrated elements of both tendencies in their definitions of mathematics education. Contrary to what one might expect, even withdrawing these four individuals, the group who expressed a theoretical orientation and the group who expressed a practical orientation do not differ substantially from each other in their involvement in research as measured by the number of publications and communications, the number of theses supervised and manuscripts reviewed, membership in editorial boards, participation in joint research projects, co-authored publications and exchange of information with colleagues in Canada and abroad. (p. 110)*

Despite some intersection of goals, the main characteristic of the community as uncovered by Mura was its diversity: *"Le portrait dessiné par les résultats de l'enquête est celui d'une communauté professionnelle diversifiée"* ["The portrait drawn by the survey's results is that of a diversified professional community"] (Mura 1994, p. 112). This diversity was based partially on the fact that the Canadian community consisted of both Anglophones and francophones, each group having a different history that was clearly related to the school systems in which many of them taught before becoming university academics. But diversity also existed within the strictly Anglophone communities of practice where various perspectives on what was important in mathematics education existed. Another facet of the community's heterogeneity was related to the fact that it included persons who were trained as mathematicians but who considered mathematics education to be their primary field of research and teaching. Many of these individuals tended to focus their research on the learning of mathematics by undergraduate students (e.g., Muller 1991; Taylor 1985). Consequently, they often reported their research at meetings of mathematicians, such as the Canadian Mathematical Society or the Mathematical Association of America. These researchers also published their work more often in the journals and monographs of those mathematical societies than in the usual mathematics education research periodicals. Wenger (1998) argued that it is the community that creates its own practice. In this regard, the community of mathematics education researchers that was created in Canada was one whose practice was markedly characterized by diversity.

However, the multifaceted nature of the Canadian community was not attributable solely to linguistic factors or to the discipline of initial training. There were also differences among Canadian mathematics education researchers that were related to the theoretical tools they used for framing research questions and for analyzing data. Some of these differences were evident from the 1980s in, for example, the theoretical perspectives held by U of A researchers. But, the variety of theoretical perspectives increased even more across the Canadian mathematics education landscape during the 1990s when a widespread shift toward theorizing occurred. One of the indicators of this shift was the mix of espoused theoretical positions that were discussed within the 1994 CMESG/GCEDM working group on "Theories and Theorizing in Mathematics Education" (led by Tom Kieren and Olive Chapman). Much of the previous research of the Canadian community had tended by and large to be Piagetian in spirit, often focusing on constructivism, cognitive conflict, and epistemological obstacles, and usually paying less attention to the role of cultural and social factors. But in the 1990s, theoretical frameworks broadened considerably to include, for example, Vygotsky's socio-cultural psychology, Brousseau's theory of didactical situations, and the interactionist perspective of Bauersfeld—a shift that could also been seen on the international scene.

As the 1990s ended, there could not be said to be one perspective that characterized the Canadian community of mathematics education researchers. Even though it was possible to speak in the 1990s of a French or Italian or German school of thought in mathematics education research, there was no such single view in the Canadian community. The Canadian community of mathematics education researchers was basically quite eclectic with respect to theories and theorizing. However, theoretical sameness, even though it may exist in a community of research practice, is not required, for, as Wenger (1998) argued, "If what makes a community of practice is mutual engagement, then it is a kind of community that does not entail homogeneity; indeed, what makes engagement in practice possible and productive is as much a matter of diversity as it is a matter of homogeneity."

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# LESSONS FROM THE PAST, QUESTIONS FOR THE FUTURE: MÉDITATION SUR THÈME IMPOSÉ

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Cette présentation thématique a été retenue pour la profondeur de la réflexion qu'elle suscite chez le lecteur, mais aussi parce que son thème peut être relié à celui de la rencontre du GCEDM 2016. En 2002, à Queen's, à l'occasion de la 25e rencontre du GCEDM, Roberta Mura, mathématicienne de formation et chercheure établie en didactique des mathématiques au Canada, est invitée par le comité exécutif à faire une présentation sur le thème «Leçons du passé, questions pour l'avenir». Elle donne rapidement le ton à sa présentation, tant à l'oral qu'à l'écrit : «Lessons from the past ? We don't learn any (...) Nous répétons toujours les mêmes erreurs» ! Le texte, provocateur, s'adresse à toute personne intéressée par l'enseignement des mathématiques et par les divers développements dans ce domaine.

This presentation was selected for the depth of the reflection it arouses in the reader, as well as because its theme can be connected to the 2016 CMESG meeting. In 2002, on the occasion of the 25th meeting of CMESG, Roberta Mura, mathematician, researcher, and prominent mathematics educator, was invited by the Executive Committee to make a presentation on "Lessons from the past, questions for the future". With her statement, "Lessons from the past? We do not learn any (...) We always repeat the same mistakes", she quickly sets the tone for both the presentation and the text that follows. The text is provocative and is for anyone interested in the teaching of mathematics and its various developments.

## INTRODUCTION

*Perchè — quando si è sbagliato — si dice « un'altra volta saprò comee far » quando si dovrebbe dire: « un'altra volta so già come farò? »*

– Cesare Pavese, *Il mestiere di vivere*[19]

Ma première réaction au titre proposé a été : « Lesson from the past? We don't learn any! » Nous n'apprenons rien de nos expériences passées, encore moins de celles des autres, nous répétons toujours les mêmes erreurs. (Je ne suis pas d'un naturel optimiste, Pavese ne l'était pas non plus.)

Après réflexion, pourtant, j'ai dû admettre que cette conclusion ne pouvait être tout à fait acceptable, puisqu'elle était elle-même le fruit d'expériences passées et constituait donc bel et bien une leçon du passé, fût-elle la seule!

« Apprendre une leçon » peut signifier simplement « savoir que les choses sont ainsi » et pas nécessairement « modifier son comportement ». Il fallait distinguer les deux sens de cette expression.

L'échec de ma première leçon m'en suggérait immédiatement une seconde — c'est-àdire une première, puisque la précédente n'en était plus une : méfions-nous des déclarations trop

[19] Cesare Pavese, *Il mestiere di vivere (Diario 1935-1950)*, Torino, Giulio Einaudi editore, 1964, p. 99. Il s'agit de l'entrée datée du 25 avril 1938 : « Poquroi — quand on s'est tormpé — diton « une auter fois, je saurai comment faire », quand on devrait dier : « une auet rfois, je sais déjà comment je ferai »? » Cesear Pavese, *Le métier de vivre*, traduit de l'italien par Michel Arnaud, Paris, Gallimard, 1987, p. 120.

catégoriques! Ce n'est pas une leçon spécialement agréable; il est bien moins fatigant de penser et de s'exprimer sans nuances. Les démagogues le savent bien : la nuance confond le public et entrave l'action. Toutefois, justement, nous avons affaire à l'éducation et non à la démagogie. Si ma leçon permet de distinguer les deux, c'est bon signe.

J'ai bien senti le danger que cette nouvelle leçon ne se retourne contre elle-même, comme la première : n'était-elle pas justement une déclaration un peu tr op nette? Sans doute que si, mais j'ai voulu résister à la tentation de m'enliser dans ce terrain.

On ne peut donc pas nier l'existence de leçons du passé. Reprenons alors avec plus de calme et un brin d'optimisme.

**INTRODUCTION, PRISE DEUX**

If I have seen *farther than others, it is because*
*I have stood on the shoulders of giants.*

– Isaac Newton

Que nous enseigne le passé? D'abord, qu'il est passé. Nous pouvons le regretter ou nous en réjouir, c'est selon. Toutefois, dans bien des cas, avec le passage du temps, nos jugements sur les événements et sur les courants de pensée se modifient, des certitudes se lézardent et s'effritent, des théories sur l'éducation sont supplantées par d'autres, parfois davantage à cause d'un changement de valeurs que de nouvelles découvertes. Il est prudent alors de ne pas trop s'attacher aux idées du présent, de s'abstenir d'opinions trop tranchées et de croyances trop fermes, de ne pas se lier, corps, coeur et âme, à une conception particulière de l'enseignement des mathématiques.

La vision cumulative de la science évoquée par la citation de Newton n'est pas au goût du jour. On a critiqué la vision absolutiste des mathématiques; la même critique doit s'adresser à une vision absolutiste de la philosophie et de la didactique des mathématiques. Si les mathématiques ne peuvent livrer la vérité et ne peuvent aspirer à la permanence, la philosophie et la didactique des mathématiques le peuvent encore moins (y compris lorsqu'elles rejettent une vision absolutiste..., mais résistons à la tentation de nous enliser!). Ce qui, aujourd'hui, nous apparaît comme la vérité et le bon sens en matière d'enseignement pourra fort bien être perçu comme erroné et ridicule dans quelques années. Je vous propose une expérience : songez aux théories passées de mode (positivisme, modernisme, behaviorisme, structuralisme, etc.) et au jugement que vous portez sur elles. Essayez ensuite de vous placer d'un point de vue futur et de regarder de ce point de vue les théories en vogue actuellement, essayez d'imaginer ce que l'on en dira dans 10 ans, dans 30 ans ou dans un siècle... Toutes les générations ont cru être sur la bonne piste et ont estimé que leurs prédécesseurs s'étaient fourvoyés. Pouvons-nous penser sérieusement que nous sommes l'exception, que nous ne faisons pas fausse route, que notre piste est réellement la bonne? La contemplation du sort des théories passées nous incite à la prudence, à l'humilité et au nonattachement (avec modération!). Voilà donc ma leçon du passé : il nous faut regarder nos croyances avec du recul et éviter de les prendre trop au sérieux, de nous prendre trop au sérieux.

D'une part, donc, il semble difficile, en didactique, de se tenir « debout sur les épaules de géants » et d'imaginer les générations futur es debout sur les nôtres. D'autre part, comment y renoncer? Quelle serait la valeur d'une discipline où nous rejetterions systématiquement la vision de nos prédécesseurs pour voir ensuite la nôtre mise de coté à son tour par nos successeurs? Ne finirions-nous pas par tourner en rond? N'est-ce pas, du moins un peu, ce que nous faisons? Peut-être, en didactique — et plus généralement en éducation — avons-nous un peu trop soif de changement et souvent tendance à procéder par réaction, à nous définir par opposition à une théorie précédente, ou concurrente, que nous érigeons en rivale et que nous « démonisons » pour nous justifier de la condamner en bloc. Cette attitude est au coeur du drame cyclique de la réforme des programmes, dont le scénario pourrait se résumer comme suit :

1. On juge la situation catastrophique : on prétend qu'au sortir de l'école les élèves ne savent pas grand-chose et comprennent encore moins, que leurs connaissances sont désuètes, inutiles et inapplicables, que l'école ne fournit de préparation convenable ni à la vie ni aux études supérieures. On crie au scandale. Il est toujours possible de dresser un constat d'échec, peu importe la situation : tout est dans la façon de s'y prendre pour observer et mesurer le phénomène;
2. On cherche un coupable. On impute le désastre au curriculum (on pourrait aussi mettre en cause la compétence du personnel enseignant, ce qui renverrait aux faiblesses des programmes de formation des maîtres);
3. On produit un nouveau curriculum en réaction au précédent : on le définit par contraste, on raisonne par dichotomies et on propose des ruptures radicales. On fait table rase. On risque alors de donner aux nouvelles idées, bonnes en soi, une application d'une étendue exagérée, qui les pervertit et les voue à l'échec à leur tour. Comme on dirait en didactique, on les pousse au-delà de leur « domaine de validité »;
4. On implante le nouveau curriculum, souvent rapidement, et on l'évalue, fréquemment de façon prématurée, sans tenir compte du temps nécessaire à un changement en profondeur. (La patience est une autre leçon qu'il nous faudrait apprendre du passé.) Il s'en suit un nouveau constat d'échec et une reprise du cycle[20].

Peut-être qu'une certaine humilité et un peu de détachement à l'égard de nos propres idées nous prédisposeraient à percevoir dans les théories concurrentes autre chose que des défauts et à profiter de tout élément utile qu'elles pourraient contenir; cela nous aiderait également à ne pas vouloir pousser nos idées au-delà de leurs limites et à prévenir ainsi de nouveaux dégâts ou, du moins, à les circonscrire.

L'enseignement, nous disait David Wheeler il y a quatre ans, est une sorte de bricolage, ce n'est pas une science, car cela suppose un consensus sur des théories de base qui est loin d'être atteint, si jamais il devait l'être[21]. J'avais lu une mise en garde similaire dans un petit livre que m'avait offert Fernand Lemay, un de mes collègues qui se sont chargés de ma formation en didactique. Dans cet ouvrage[22], Krishnamurti soutenait qu'aucune méthode ni système ne peut fournir la bonne sorte d'éducation (p. 23). Il recommandait de ne pas penser selon des principes et de ne pas suivre de méthode, car, selon lui, cela conduit à accorder plus d'importance à la méthode qu'à la réalité des élèves (p. 25).

Dans ce qui suit, je développerai cette méditation sur les vicissitudes de l'enseignement des mathématiques à la lumière de la leçon que j'ai dégagée du passé, en prenant comme point de départ mon histoire personnelle.

## L'ENSEIGNEMENT DES MATHÉMATIQUES : LES MATHS MODERNES, LA RÉFORME DES PROGRAMMES ET LA TRADITION

[20] Parfois, comme l'a rappelé Lesley Lee lors de la période de discussion, ce scénario n'est pas suivi et on lance une réforme sans fournir de justification. Une culture qui valorise le changement et la nouveauté fait que cette pratique rencontre peu d'opposition.

[21] David Wheeler, « The commonsense of teaching », danYsv onne M. Pothier (dir.), Proceedings of the Annual Meeting of the Canadian Mathematics Education Study Group, University of British Columbia, May 29–June 2, 1998, Mount Saint Vincent University Press, 1998, p. 98.

[22] J. Krishnamurti, *Education & the Significance of Life*, San Francisco, Harper & Row, 1981/1953, p. 125.

*Le traité prend les mathématiques à leur début, et donne des démonstrations complètes. Sa lecture ne suppose donc, en principe, aucune connaissance mathématique particulière, mais seulement une certaine habitude du raisonnement mathématique et un certain pouvoir d'abstraction.*

– N. Bourbaki, *Éléments de mathématique, Mode d'emploi de ce traité*

Mon tout premier contact avec la didactique des mathématiques, à part mes expériences comme élève, a eu lieu dans le contexte de la réforme des maths modernes, vers la fin des années 60. J'étudiais les mathématiques à l'université, en Italie, et l'on m'avait invitée à donner quelques heures de cours dans le contexte d'une activité de perfectionnement d'enseignantes et d'enseignants. Je devais leur parler d'ensembles, de relations d'équivalence et d'autres notions de ce genre.

J'étais pleine d'enthousiasme pour ces idées que je venais de découvrir, j'étais heureuse de les partager, et le projet de les introduire à l'école me paraissait bon, car il me semblait répondre à un besoin réel. Je me souvenais, par exemple, d'avoir été frustrée, au secondaire, par l'absence d'une définition du mot « fonction ». Il y avait les polynômes, les fonctions trigonométriques, les logarithmes et les exponentielles. En existait-il d'autres? On me disait que oui. Cependant, qu'est-ce que c'était une fonction au juste? Mystère! C'est seulement à l'université que l'on m'avait enfin révélé qu'une fonction était une correspondance univoque entre deux ensembles. (À l'apogée des maths modernes, on envisagera d'enseigner cela au préscolaire!) Cette définition m'avait libérée d'un long malaise. Un peu plus tard j'ai pris connaissance d'une variante de cette définition, à savoir qu'une fonction est un sous-ensemble du produit de deux ensembles respectant certaines conditions, et cette autre formulation m'avait plu également, non seulement parce qu'elle faisait ressortir le lien avec l'idée familière de graphique, mais aussi parce qu'elle livrait d'emblée la fonction tout entière et qu'elle ne contenait aucune suggestion de mouvement (comme ce va-et-vient entre les deux ensembles suggéré par le mot « correspondance » dans la première formulation). Je trouvais cela satisfaisant et apaisant.

En fait, la teneur en maths modernes de mon éducation a été très faible : nulle à l'école, relativement modeste à l'université. Si j'en ai appris un peu plus, c'est en raison de mon initiative personnelle de tenter de lire, je dis bien « tenter », ce fameux traité qui ne supposait, en principe, aucune connaissance mathématique particulière. Cela a été frustrant, bien sûr, mais je n'ai pas été rebutée par la chose comme d'autres qui en ont été gavés en bas âge. Au contraire, je me souviens de moments de réel plaisir, comme lorsque j'ai lu qu'un couple (a,b) pouvait se définir comme l'ensemble {a,{a,b}}. Jusque-là, la notion de couple m'avait agacée, puisque je ne voyais pas comment distinguer (a,b) de (b,a) sans importer en mathématique des notions physiques telles que « droite » et « gauche » ou « avant » et « après » (j'ai toujours aimé mes mathématiques très peurs)[23].

Aujourd'hui, il est rare que l'on ne qualifie pas de « faillite » le mouvement des maths modernes et le discours structuraliste qui le sous-tendait. Les exposés systématiques, genre « axiomes, définitions, théorèmes, démonstrations », qu'il s'agisse des Éléments d'Euclide ou de ceux de Bourbaki, sont donnés comme exemple d'une mauvaise approche didactique. Pourtant, la motivation, du moins la motivation initiale, derrière ces traités était d'ordre didactique[24]. Leur

[23] Voilà une autre leçon du passé, du moins de mon passé : l'appr entissage qui fait le plus plaisir est habituellement celui qui résout un malaise préexistant, ou satisfait une curiosité préexistante.

[24] Frédéric Patras, *La pensée mathématique contemporaine*, Paris, Presses universitaires de France, 2001, p. 16; Maurice Mashaal, « Bourbaki. Une société secrète de mathématiciens. Un groupe se forme »*P, our la science*, fév.-mai 2000, p. 6–9*L*; *e matin des mathématiciens, entretiens sur l'histoire des mathématiques présentés par* Émile Noël, Paris, Éditions Belin-Radio France, 1985, p. 43.

but était de présenter les concepts et les résultats de base d'une discipline, de façon cohérente, claire et ordonnée, avec un degré optimal (pas nécessairement maximal) de généralité.

N'est-ce pas là des intentions louables? Où est l'erreur? Je ne m'y attarderai pas longtemps, car les critiques sont bien connues : les maths modernes ont donné lieu à de nombreux excès et dérives, mais, même sans cela, cette approche, qui aspirait pourtant à introduire à l'école des « vraies » mathématiques, des mathématiques alignées sur celles qui se pratiquaient à l'université, avait le défaut d'offrir aux élèves un savoir achevé, sans leur montrer par quels chemins on en arrivait à s'intéresser à telle question, à la circonscrire au moyen de tels concepts, définis de telle manière. Elle occultait les raisons du choix des définitions et des axiomes, qui pouvaient paraître alors purement gratuits. Les problèmes auxquels répondaient les théorèmes, les cas particuliers à partir desquels on avait bâti abstractions et généralisations, les stratégies qui avaient permis d'obtenir les résultats, bref tout le processus de création, demeuraient cachés.

Il fallait donc (re)donner aux élèves la chance de se familiariser avec ce processus, de faire appel à leur intuition, d'élaborer les concepts de façon graduelle et de tenter de résoudre des problèmes. Il fallait leur permettre de passer par toutes les étapes qu'une présentation axiomatique les forçait à sauter. Cependant, cet excellent programme risque lui aussi de provoquer des effets pervers, notamment en conduisant à dévaloriser, voire éliminer, l'étape finale de systématisation et d'organisation du savoir.

L'indignation contre l'erreur des maths modernes et l'engouement pour la résolution de problèmes ont entraîné une certaine indifférence, presque de la méfiance, à l'égard de tout ce qui est abstraction, théorie mathématique, système ordonné de résultats. Si avant on négligeait l'activité mathématique, le processus, maintenant on risque d'en oublier le produit. Voilà donc ma première question pour l'avenir : comment trouver et maintenir un juste équilibre entre les deux[25]? Dans la conjoncture actuelle, le défi me semble être d'éviter que les concepts demeurent au stade d'intuitions, emprisonnés dans des représentations qui ne devaient jouer qu'un rôle d'échafaudage[26], et d'éviter que les résultats (les solutions des problèmes) s'accumulent sans que l'on se soucie de les organiser en structures.

« Le mode d'exposition suivi est axiomatique et abstrait; il procède le plus souvent du général au particulier. »[27] Aujourd'hui, cette façon de procéder et l'idée même d'« exposition » sont frappées d'anathème. L'orientation actuelle en didactique veut que l'on procède du particulier au général. Cela paraît avalisé autant par le bon sens que par la recherche. J'ai tout de même à ce sujet des préoccupations de deux ordres : 1) que la nouvelle façon de procéder soit appliquée correctement, que l'on prenne réellement le temps, que l'on fasse vraiment l'effort, de se rendre au « général », que l'on ne se perde pas dans la multitude des « particuliers »; 2) que l'on n'érige pas en dogme une façon unique de opérer. Même si pour l'instant elle semble être la meilleure, elle ne l'est sans doute pas pour tout le monde et en toute occasion.

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[25] D'après Frederick Leung, une plus grande attention accordée au produit, plutôt qu'au processus, est une des caractéristiques qui distinguent l'enseignement des mathématiques en Asie de l'Est (les pays de culture confucianiste) de celui qui se pratique en Occident (les pays anglo-saxons). Cette plus grande attention au produit, ainsi que tous les autres traits qui, selon Leung, définissent l'identité asiatique dans l'enseignement des mathématiques, me semble pourtant se retrouver aussi, avec autant de relief, dans la tradition « occidentale » : Frederick K.S. Leung, « In search of an East Asian identity in mathematics education », *Educational Studies in Mathematics*, vol. 47, no 1, 2001, p. 35–51. Par ailleurs, j'ai appris de mon collègue Christian Laville que, dans l'enseignement de l'histoire, il existe une tension similaire entre curriculums modernes centrés sur la pensée historique (le processus) et curriculums anciens centrés sur le récit historique (le produit).

[26] Roberta Mura, « L'épaisseur d'un décimètre carré »,*La Revue canadienne de l'enseignement des sciences, des mathématiques et des technologies*, vol. 1, no 3, 2001, p. 291–303.

[27] N. Bourbaki, Éléments de mathématique, Mode d'emploi de ce traité.

L'histoire devrait nous inciter à faire preuve de retenue. Nous reconnaissons aisément l'intérêt d'un principe de précaution en repensant à l'aventure des maths modernes et en nous disant : « Ils auraient dû... », mais le même principe vaut tout autant pour les tendances dominantes actuelles! Il est facile, maintenant, d'exhiber des horreurs tirées des manuels scolaires d'époques révolues[28], mais mettons-nous à la place de nos collègues du futur et essayons de regarder le matériel didactique contemporain à travers leurs yeux. Peut-être pourrons-nous déjà entrevoir ce qui leur paraîtra risible. (Je pense, par exemple, à certaines « situations-problèmes significatives »...) Si nous avons de la difficulté à apprendre des leçons du passé, peut-être que voyager mentalement dans l'avenir pour regarder en arrière vers le présent pourra nous aider, pour ainsi dire, à apprendre des leçons du futur!

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J'ai contrasté l'esprit des nouveaux programmes et celui des maths modernes, alors qu'il est plus courant de l'opposer à celui de l'enseignement dit « traditionnel ». Les maths modernes sont un épisode bien délimité dans l'histoire de la didactique, un adversaire déjà vaincu. L'ennemi par rapport auquel se définissent les nouvelles orientations est l'enseignement « traditionnel ». On le dépeint comme la transmission de faits et de techniques au niveau intellectuel le plus bas qui soit. Sans doute cela a-t-il existé, et malheureusement cela existé- il encore, mais supposer une uniformité dans la tradition plurimillénaire de l'enseignement des mathématiques est une simplification inacceptable. En fait, cette tradition contient déjà les principales idées qui animent la réforme actuelle.

Il y a un siècle, par exemple, Mary Boole, épouse de George Boole, prônait et pratiquait déjà une approche par découverte[29] :

> *For mathematical purposes, all influence from without, which induces the pupils to admit a principle as valid before his own unbiased reason recognises its truth, come under the same condemnation (p. 9).*
>
> *Qualities of a teacher [...] Great reserve on the part of the teacher in even stating to pupils the special conclusions to which he has been led, lest he should arrest the normal exercise of their investigating faculties (p. 11).*
>
> *[The teacher's] object should be to efface himself, his books, and his systems; to draw aside a curtain from between the child and the process of discovery, and to leave the young soul alone with pure Truth (p. 14).*

L'auteure de ces propos n'était pas une visionnaire isolée. Au contraire, Boole fait allusion à des théories éducatives de son époque plus radicales que les siennes, condamnant tout apprentissage mécanique, théories dont elle se démarque en adoptant une position plus modérée, en reconnaissant, à côté de moments privilégiés d'apprentissage pleinement conscient, l'utilité de périodes d'entraînement (p. 15).

Le caractère récurrent des préoccupations à propos de l'enseignement des mathématiques ressort bien d'un autre passage du même texte. Boole y rapporte l'indignation d'un professeur devant l'incapacité des étudiants à se servir de leurs connaissances mathématiques dans leurs études de génie ou de physique. Cela, poursuit-elle, a rallumé l'intérêt pour une question qui avait été négligée pendant une ou deux générations, mais qui avait retenu l'attention de savants 60 ans plus tôt, à savoir (p. 20) : « What are the conditions which favour a vital knowledge of

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[28] Voir, par exemple, Maurice Mashaal, « Bourbaki. Une société secrète de mathématiciens. Les « maths modernes » à l'école » *Pour la science*, fév.-mai 2000, p. 83.
[29] D.G. Tahta, *A Boolean Anthology. Selected Writings of Mary Boole on Mathematical Education*, Derby, U.K., The Association of Teachers of Mathematics, 1972. Merci à David Pimm d'avoir cité ce texte dans un de ses articles et de m'avoir ainsi permis de le découvrir.

mathematics? ». L'idée de connaissances vivantes me semble très proche du discours contemporain sur la compétence à se servir des mathématiques dans des contextes variés.

Un autre aspect des nouveaux programmes que l'on peut retracer dans la tradition concerne justement l'accent mis sur l'utilité des mathématiques. La part importante du « temps d'antenne » réservé aux mathématiques à l'école se justifie par le fait qu'elles « sont partout » et qu'elles sont devenues indispensables à la vie en société axer l'enseignement des mathématiques sur leur utilité n'était certainement pas le souci des promoteurs des maths modernes, mais ce n'est pas non plus une idée nouvelle. À ce propos, la tradition oscille entre deux pôles. D'une part, l'étude des mathématiques est conçue comme une poursuite intellectuelle gratuite, « pour l'honneur de l'esprit humain », comme l'écrivait Jacobi et le répétait Dieudonné[30]. Une anecdote célèbre illustre bien cette vision. On raconte qu'un élève, après avoir appris un théorème, a demandé à Euclide à quoi cela lui servirait. Euclide aurait ordonné alors à son esclave de donner une pièce de monnaie au garçon, puisque ce dernier avait besoin de tirer un avantage de ce qu'il apprenait[31]. L'objet de l'histoire est de promouvoir une attitude désintéressée et idéaliste envers le savoir. On peut y voir aussi, et dénoncer, une attitude arrogante et élitiste d'hommes privilégiés[32], mais il reste que tout le monde est en droit d'aspirer à une part de loisir à consacrer, éventuellement, à la spéculation gratuite.

D'autre part, la tradition comprend aussi, et depuis longtemps, une vision plus utilitaire des mathématiques. Sans parler des mathématiques babyloniennes qui s'exprimaient essentiellement par des problèmes de nature économique, même dans la Grèce classique, il semble que l'éducation mathématique des jeunes, jusqu'à 14 ans, avait une orientation surtout pratique[33]. Par la suite, pour les élèves de 14 à 18 ans, l'inclusion dans le curriculum de matières plus abstraites, comme l'astronomie et la géométrie, et les dangers d'en pousser l'étude trop loin faisaient l'objet de discussions. Platon, qui souhaitait enrichir le contenu mathématique des programmes d'études, citait la meilleure éducation des enfants égyptiens, ce qui n'est pas sans rappeler le rôle des comparaisons internationales dans les débats contemporains...

Qu'est-ce donc que la tradition? Un héritage culturel précieux à chérir ou une tyrannie étouffante contre laquelle on doit se révolter si l'on veut progresser?

Pour améliorer quoi que ce soit, nos conditions de vie ou l'enseignement des mathématiques, il faut innover (l'inverse n'est pas vrai), et pour innover, il faut s'écarter de la tradition, c'est évident. D'où la connotation négative du terme « traditionnel[34] ». Cependant, une tradition aussi ancienne, riche et variée que celle de l'enseignement des mathématiques, une tradition qui contient tout et son contraire, n'est sans doute pas à mettre au rancart en bloc! Et, ne l'oublions

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[30] C.G.J. Jacobi, *Gesammelte Werke*, vol. 1, Berlin, 1881, cité par Frédéric Patras, *op. cit.*, p. 4, note 4. Dieudonné a intitulé un de ses ouvrages *Pour l'honneur de l'esprit humain* : JeanA lexandre Dieudonné, *Pour l'honneur de l'esprit humain : les mathématiques aujourd'hui* , Paris, Hachette, 1987.

[31] Thomas L. Heath, *Greek Mathematics*, New York, Dover Publications, 1963, p. 10.

[32] On répète souvent et avec un peu trop de désinvolture que, si les mathématiques telles que nous les connaissons ont été façonnées dans la Grèce classique, c'est parce que les penseurs grecs étaient « des hommes libers » : Denis Gued*Lje, théorème du perroquet*, Paris, Éditions duSeuil, 1998, p. 179, cité par Bernard Hodgson, « Pourquoi enseigner les mathématiques à tous ? », dans Elaine Simmt, Benrt Davis et John Grant McLoughlin (dir.), *Proceedings of the Annual Meeting of the Canadian Mathematics Education Study Group, Université du Québec à Montréal, May 26–30, 2000*, p. 164. Lorsqu'on tient des pr opos de ce genre, il faudrait prendre soin de ne pas contribuer à perpétuer une image idéalisée de cette société antique et rappeler que ces « *hommes* libres » devaient leur liberté non pas à une supposée démocratie, mais au travail des femmes et des esclaves.

[33] Thomas L. Heath, *op. cit.*, p. 7–10.

[34] Dans d'autres contextes, pourtant — dans l'artisanat par exemple — ce terme n'a pas la même connotation.

pas, même ce qui nous semble dépassé pourra être revalorisé plus tard. Pensons au domaine artistique, où les styles sont constamment réévalués.

Il en va ainsi, en éducation, des valeurs et des méthodes d'enseignement. Il ne faudrait alors peut-être pas oublier entièrement certaines valeurs démodées, telles la clarté et la vérité. La clarté était jadis parmi les qualités considérées comme les plus désirables pour l'enseignement. Quant à la vérité, elle a été, traditionnellement, la qualité idéale de la connaissance attendue par les élèves et dispensée par les maîtres. C'était ce que j'attendais et recherchais quand j'étais élève. Cela n'exclut pas l'esprit critique; au contraire, celui-ci doit être bien éveillé pour tester la vérité des connaissances proposées. Si l'école renonce à dispenser la vérité, elle laissera un vide, un désir insatisfait qu'il faudra combler par d'autres moyens. D'ailleurs, que cherchons-nous lorsque nous faisons de la recherche, si ce ne sont des connaissances vraies? Bien sûr, dans le domaine intellectuel, la vérité n'est pas absolue, elle varie selon les points de vue, mais cette affirmation aussi est une vérité. L'existence de plusieurs niveaux de vérité et son caractère relatif n'impliquent pas que ce concept soit dépourvu d'intérêt.

Il ne faudrait pas écarter non plus la possibilité de façons d'apprendre autres que celles qui tiennent actuellement la vedette, soit la résolution de problèmes, la recherche personnelle ou collective, l'exploration, la découverte et la discussion. Il n'est pas impossible d'apprendre aussi par l'écoute, l'observation, l'imitation, la lecture, l'entraînement, la pratique, voire la mémorisation[35]. N'avons-nous pas appris nous-mêmes par un mélange de ces approches? Pourquoi rejeter ce qui a fonctionné pour nous? Personnellement, je crois que j'ai beaucoup appris des livres. À l'occasion, même de livres qui se situaient au-delà de ma « zone de développement proximale ». Je ne comprenais pas, mais je voulais comprendre, je me disais qu'un jour je comprendrais. Ces livres constituaient pour moi un but, un horizon vers lequel marcher.

En somme, pour ce qui est de la tradition, il me semble qu'un regard sur l'histoire, même un regard sur quelques fragments seulement, nous incite à faire preuve de prudence, d'une part, afin de ne pas créer une image stéréotypée de ce qu'est la tradition de l'enseignement des mathématiques et, d'autre part, afin de ne pas rejeter entièrement et définitivement des éléments de cette tradition qui pourraient s'avérer encore profitables.

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Revenons maintenant à la réforme actuelle des programmes. J'ai parlé de l'importance de rechercher un équilibre entre processus et produit, entre créativité et systématisation du savoir. Une autre problématique où prudence et modération me semblent nécessaires concerne l'utilisation des mathématiques. La question que je voudrais poser à l'avenir à ce sujet est la suivante : comment éviter l'abus des mathématiques, la mathématisation à outrance, à tort et à travers? Y a-t-il moyen de contrer cette tendance par l'éducation? La nouvelle insistance sur les situations-problèmes favorisera-t-elle une attitude critique à ce propos ou, au contraire, contribuera-t-elle à empirer la situation?

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[35] Selon Leung (voir la note 7), le rôle accordé à la mémorisation, même avant qu'une pleine compréhension soit atteinte, est le deuxième aspect qui caractérise l'enseignement des mathématiques en Asie de l'Est par rapport à ce qui se fait en Occident. Le troisième élément est la vision de l'étude comme un travail sérieux, difficile et pénible. Leung contraste cela et la recherche, en Occident, d'une manière d'apprendre qui soit agréable, voire amusante. Encore une fois, je noterais que les caractéristiques que Leung attribue à la vision occidentale de l'enseignement des mathématiques n'en résument pas toute la tradition, celle-ci comprenant également des courants qui accordent autant d'importance aux aspects que cet auteur considère comme typiques de l'approche asiatique. Il suffit de penser, par exemple, à la célèbre réponse de Ménechme à Alexandre (ou d'Euclide à Ptolomée) à savoir qu'il n'existe pas de chemin royal en géométrie : Thomas L. Heath,*o p. cit.*, p. 158.

Je ne veux pas parler ici de la fabrication d'armes, du clonage de monstres ou de la survente de billets d'avion. Une éducation critique peut sensibiliser au rôle des mathématiques dans tout cela, mais je doute que la didactique offre des moyens d'endiguer le mal. Le problème que je veux soulever est moins grave, mais quand même irritant et davantage de notre ressort. Il s'agit de l'emploi de formules, d'images ou de termes mathématiques à mauvais escient, là où ils n'apportent rien à la compréhension d'une situation et deviennent même une source de confusion. Cette pratique revient, encore une fois, à pousser trop loin une bonne idée et nuit au projet de montrer aux élèves l'utilité des mathématiques et de leur apprendre à s'en servir.

La prolifération de schémas inspirés des maths modernes en constitue un exemple anodin, mais typique. Désormais, il ne reste plus rien, à l'école primaire (et bien peu au secondaire), de ces notions de théorie des ensembles qui ont été la marque de commerce des maths modernes et la cible des railleries de leurs détracteurs[36]. Entre-temps, par contre, certains éléments du langage graphique qui les accompagnait, comme les diagrammes de Venn, sont passés, dénués de leur sens, dans l'usage courant. Ironiquement, on en trouve de nombreux exemples parmi les illustrations du nouveau programme pour le préscolaire et le primaire au Québec[37], programme qui, justement, a évacué du contenu d'étude les dernières traces de ce langage! Loin d'éclairer quoi que ce soit, la plupart de ces ovales et de ces flèches jouent, au mieux, un rôle purement décoratif. Souvent ils trahissent et encouragent un flou intellectuel qui se traduit par des schémas dans lesquels des flèches colorées remplacent des connexions logiques que l'on aurait du mal à expliciter.

Pourquoi s'inquiéter de ces pratiques maintenant, à l'heure de l'implantation d'un curriculum qui met l'accent justement sur l'utilisation des mathématiques? Celui-ci ne devrait-il pas éduquer à en faire un usage judicieux? En principe, oui. Cependant, les contextes qui se prêtent à des activités adaptées aux élèves et dans lesquels les mathématiques jouent un rôle véritablement significatif ne sont pas si faciles à trouver. Je crains — à tort, je l'espère — que devant cette pénurie on ne se rabatte sur des situations artificielles où l'on plaque des éléments mathématiques sans trop se soucier de la pertinence de l'opération, comme l'on met des schémas inspirés des diagrammes de Venn en guise d'illustration d'un texte.

Ne nous faisons pas d'illusions, l'introduction à l'école des mathématiques appliquées, pour ne parler que de cet aspect de la réforme actuelle, demande au personnel enseignant un effort majeur de perfectionnement, un effort comparable à celui qui a été exigé à l'époque par les maths modernes. Je me souviens encore très bien du sentiment d'incompétence que j'ai éprouvé au début de ma carrière lorsque j'enseignais la géométrie projective et que des étudiants d'architecture m'ont posé une question pratique, portant sur la couverture d'un toit. J'ai oublié la question, mais je me rappelle que je me suis sentie paralysée, entièrement dépourvue de moyens pour aborder une question de ce genre. Je crois que beaucoup d'enseignants et d'enseignantes n'ont pas plus de préparation à cet égard, aujourd'hui, que je n'en avais alors, après quatre ans de spécialisation en mathématiques, ou que leurs collègues d'antan n'en avaient sur la théorie des ensembles, l'algèbre linéaire et l'algèbre abstraite.

C'est le moment de tirer profit des leçons de l'histoire. Le perfectionnement est indispensable, mais il n'est pas suffisant pour le déploiement optimal d'un curriculum radicalement nouveau. S'il est vrai, comme l'a rappelé Vicki Zack dans un commentaire écrit remis à la fin de la séance, que l'on peut apprendre, changer et se développer tout au long de sa vie, je pense que ce ne seront pas tous les enseignants et les enseignantes qui le feront, et qu'il est parfois difficile de

[36] Voir par exemple la caricature intitulée « Intersection d'un bébé et d'une pomme de terre », dans Didier Nordon, *Les mathématiques pures n'existent pas!*, Paris, Actes Sud, 1981, p. 6.

[37] Ministère de l'Éducation du Québec, *Programme de formation de l'école québécoise. Éducation préscolaire. Enseignement primaire*, 2001. Voir tous les schémas, notamment ceux des pages 8, 43, 99, 125, 197, 253 et 257.

se débarrasser d'habitudes acquises dans sa jeunesse. Ce n'est que lorsqu'une génération entière d'enseignantes et d'enseignants aura reçu une formation initiale en accord avec le nouveau curriculum que celui-ci pourra donner sa pleine mesure. D'ici là, il faudra traverser quelques décennies de transition pendant lesquelles la plupart des maîtres n'en sauront pas beaucoup plus que ce qui se trouve dans les manuels de leurs élèves et garderont une vision des mathématiques et de leur enseignement plus ou moins décalée par rapport aux nouvelles orientations. Dans le passé, nous n'avons pas eu la patience d'attendre une si longue période. L'aurons-nous cette fois-ci? Choisirons-nous d'endurer les ratés inévitables et de les corriger graduellement ou en ferons-nous un argument pour condamner la réforme et changer de cap une fois de plus?

## CONCLUSION

> *A popular misconception is that we can't change the past— everyone is constantly changing their own past, recalling it, revising it. What really happened? A meaningless question. But one I keep trying to answer, knowing there is no answer.*
>
> – Margaret Laurence, *The Diviners*[38]

Tirer des leçons de l'histoire est une démarche éminemment subjective, qui consiste à interpréter des souvenirs. Encore faut-il que ces derniers soient disponibles. Malheureusement, la mémoire humaine est tout sauf fiable et les documents se révèlent souvent ambigus ou incomplets, certains ayant été perdus ou détruits, d'autres étant devenus indéchiffrables, sans compter toutes les pensées et tous les événements qui n'ont pas été enregistrés et dont la mémoire ne s'est pas transmise. Nous ne saurons jamais ce qui s'est passé dans toutes les classes de mathématiques et quels en ont été les effets. Les souvenirs que nous interprétons posent eux-mêmes problème. L'incertitude n'est pas l'apanage de l'avenir.

La prudence est donc souhaitable, aussi, lorsque nous reconstruisons le passé. Difficile de savoir ce qui est arrivé « en réalité », si tant est que la question ait un sens. Nous suivons notre tendance à créer des récits cohérents, et cela nous permet d'y lire des leçons. Je vous en ai proposé une, à vous maintenant de fouiller dans vos souvenirs pour en trouver d'autres.

[38] Margaret Laurence, *The Diviners*, Toronto, McClelland & Stewart, 1988/1974, p. 70.

# BUILDING THINKING CLASSROOMS: CONDITIONS FOR PROBLEM SOLVING[39]

Peter Liljedahl
*Simon Fraser University*

This topic session was chosen because it is representative of the many levels and overreaching applicability of the topics discussed at CMESG. The response and composition of the audience, from practicing secondary teachers to grad students to university professors, speaks to the relevance and resonance of the findings. Conversations were sparked that continued beyond the temporal boundaries of the conference. This topic session has significance for all, not only those who attended.

Cette séance thématique a été choisie parce qu'elle convient aux nombreux niveaux scolaires ainsi qu'aux variétés de sujets abordés au GCEDM. La réaction et la composition du public, qui incluait des enseignants de mathématiques du secondaire, des étudiants de doctorat, ainsi que des professeurs universitaires, démontrent la pertinence de cette approche. Certains échanges amorcés en cours de séance se sont poursuivis au-delà des limites temporelles de la rencontre annuelle. Cette session thématique est pertinente pour tous, et non uniquement pour ceux qui étaient présents.

*In this session, I first introduce the notion of a thinking classroom and then present the results of over ten years of research done on the development and maintenance of thinking classrooms. Using a narrative style, I tell the story of how a series of failed experiences in promoting problem solving in the classroom led first to the notion of a thinking classroom and then to a research project designed to find ways to help teachers build such a classroom. Results indicate that there are a number of relatively easy-to-implement teaching practices that can bypass the normative behaviours of many classrooms and begin the process of developing a thinking classroom.*

## MOTIVATION

My work on this paper began over 10 years ago when I was observing a grade 7/8 teacher introducing problem solving into her teaching for the first time. Problem solving was something that, at the time, was becoming more and more prominent in the BC curriculum, and Ms. Ahn was interested in incorporating it into her classroom. Despite her best intentions the results were abysmal. The students gave up almost as soon as the problem was presented to them and they resisted her efforts and encouragement to persist. After three days of constant struggle, Ms. Ahn and I both agreed that it was time to abandon these efforts. Wanting to better understand why our well-intentioned efforts had failed, I decided to observe Ms. Ahn teach her class using her regular style of instruction.

That the students were lacking in effort was immediately obvious, but what took time to manifest was the realization that what was missing in this classroom was that the students were not being asked to think. More alarming was that Ms. Ahn's teaching was predicated on an assumption that the students either could not, or would not, think. The classroom norms that

[39] An extended version of this article, including research on teacher uptake, can be found in Liljedahl (in press).

had been established in Ms. Ahn's class had resulted in, what I now refer to as, a *non-thinking classroom*. Once I realized this, I proceeded to visit other mathematics classes—first in the same school and then in other schools. In each class I saw the same basic behaviour—an assumption, implicit in the teacher's practice, that the students either could not, or would not think. Under such conditions it was unreasonable to expect that students were going to spontaneously persist through a problem-solving encounter.

What was missing for these students, and their teachers, was a central focus in mathematics on thinking. The realization that this was absent in so many classrooms that I visited motivated me to find a way to build, within these same classrooms, a culture of thinking, both for the student and the teachers. I wanted to build, what I now call, a *thinking classroom*—a classroom that is not only conducive to thinking but also occasions thinking, a space that is inhabited by thinking individuals as well as individuals thinking collectively, learning together, and constructing knowledge and understanding through activity and discussion. A classroom where thinking was assumed to be possible and was expected in every activity. Such a classroom will intersect with research on mathematical thinking (Mason, Burton, & Stacey, 1982) and classroom norms (Yackel & Rasmussen, 2002). It will also intersect with notions of a *didactic contract* (Brousseau, 1984), the emerging understandings of *studenting* (Fenstermacher, 1986, 1994; Liljedahl & Allan, 2013a, 2013b), *knowledge for teaching* (Hill, Ball, & Schilling, 2008; Shulman, 1986), and *activity theory* (Engeström, Miettinen, & Punamäki, 1999).

My early efforts to do so involved a series of three workshops designed to help teachers implement problem solving in their classroom. The results of these workshops were mixed. Some teachers reported that they saw great enthusiasm in their students, while others reported experiences similar to those I had observed in Ms. Ahn's class. Further probing revealed that teachers who reported that their students loved what I was offering tended to have practices that already involved some level of problem solving. It also revealed that those teachers who reported that their student gave up easily or resisted their efforts had practices mostly devoid of problem solving. The experiences that that the teachers were having implementing problem solving in the classroom were being filtered through their already existing classroom norms (Yackel & Rasmussen, 2002). If there was already a culture of thinking and problem solving in the classroom then the teachers were reporting success. If the culture was one of direct instruction and individual work then, although some students were able to rise to the task, the majority of the class was unable to do much with the problems.

These latter classroom norms are a difficult thing to bypass (Yackel & Rasmussen, 2002), even when a teacher is motivated to do so. The teachers that attended these workshops wanted to change their practice, at least to some degree, but their initial efforts to do so were not rewarded by comparable changes in their students' problem-solving behaviour. Quite the opposite, many of the teachers I was working with were met with resistance and complaints when they tried to make changes to their practice.

From these experiences I realized that if I wanted to build thinking classrooms—to help teachers to change their classrooms into *thinking* classrooms—I needed a set of tools that would allow me, and participating teachers, to bypass any existing classroom norms. These tools needed to be easy to adopt and have the ability to provide the space for students to engage in problem solving unencumbered by their rehearsed tendencies and approaches when in their mathematics classroom.

This realization moved me to begin a program of research that would explore the elements of thinking classrooms. I wanted to find a collection of teacher practices that had the ability to break students out of their classroom normative behaviour—practices that could be used not only by myself as a visiting teacher, but also by the classroom teacher that had previously entrenched the classroom norms that now needed to be broken.

## GENERAL METHODOLOGY

The research to find the elements and teaching practices that foster, sustain, and impede thinking classrooms has been going on for over ten years. Using a framework of noticing (Mason, 2002)[40], I initially explored my own teaching, as well as the practices of more than forty classroom mathematics teachers. From this emerged a set of nine elements that permeate mathematics classroom practice—elements that account for most of whether or not a classroom is a thinking or a non-thinking classroom. These nine elements of mathematics teaching became the focus of my research. They are:

1. the type of tasks used, and when and how they are used;
2. the way in which tasks are given to students;
3. how groups are formed, both in general and when students work on tasks;
4. student work space while they work on tasks;
5. room organization, both in general and when students work on tasks;
6. how questions are answered when students are working on tasks;
7. the ways in which hints and extensions are used while students work on tasks;
8. when and how a teacher *levels*[41] their classroom during or after tasks;
9. and assessment, both in general and when students work on tasks.

Research into each of these was done using design-based methods (Cobb, Confrey, diSessa, Lehrer, & Schauble, 2003; Design-Based Research Collective, 2003) within both my own teaching practice as well as the practices of a number of teachers participating in a variety of professional development opportunities. This approach allowed me to vary the teaching around each of the elements, either independently or jointly, and to measure the effectiveness of that method for building and/or maintaining a thinking classroom. Results fed recursively back into teaching practice, each time leading either to refining or abandoning what was done in the previous iteration.

This method, although fruitful in the end, presented two challenges. The first had to do with the measurement of effectiveness. To do this I used what I came to call *proxies for engagement*—observable and measurable (either qualitatively or quantitatively) student behaviours. At first this included only behaviours that fit the *a priori* definition of a thinking classroom. As the research progressed, however, the list of these proxies grew and changed depending on the element being studied and teaching method being used.

The second challenge had to do with the shift in practice needed when it was determined that a particular teaching method needed to be abandoned. Early results indicated that small shifts in practice, in these circumstances, did little to shift the behaviours of the class as a whole. Larger, more substantial shifts were needed. These were sometimes difficult to conceptualize. In the end, a contrarian approach was adopted. That is, when a teaching method around a specific element needed to be abandoned, the new approach to be adopted was, as much as possible, the exact opposite to the practice that had shown to be ineffective for building or maintaining a thinking classroom. When sitting showed to be ineffective, we tried making the students stand. When leveling to the top failed we tried levelling to the bottom. When answering questions proved to be ineffective we stopped answering questions. Each of these approaches needed further refinement through the iterative design-based research approach, but it gave good starting points for this process.

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[40] At the time I was only informed by Mason (2002); since then I have been informed by an increasing body of literature on noticing.

[41] *Levelling* (Schoenfeld, 1985) is a term given to the act of closing of, or interrupting, students' work on tasks for the purposes of bringing the whole of the class (usually) up to certain level of understanding. It is most commonly seen when a teacher ends students work on a task by showing how to solve the task.

## FINDINGS

In what follows I present, in brief, the results of the research done on each of the nine elements and discuss how all nine elements hold together as a framework to build and maintain thinking classrooms. All of this research is informed dually by data and analysis that looks both on the effect on students and the uptake by teachers.

### 1. THE TYPE OF TASKS USED, AND WHEN AND HOW THEY ARE USED

Lessons need to begin with good problem solving tasks. At the early stages of building a thinking classroom these tasks need to be highly engaging collaborative tasks, usually non-curricular, that drive students to want to talk with each other as they try to solve them (Liljedahl, 2008). Once a thinking classroom is established the need for problems to be inherently engaging diminishes. As a result, the problems shift to towards curricular mathematics (Schoenfeld, 1985) that can be linked to the curriculum content to be 'taught' that day and permeate the entirety of the lesson.

### 2. THE WAY IN WHICH TASKS ARE GIVEN TO STUDENTS

Tasks need to be given orally. If there are data or diagrams needed, these can be provided on paper, but the instructions pertaining to the activity of the task need to be given orally. This very quickly drives the groups to discuss what is being asked rather than trying to decode instructions on a page.

### 3. HOW GROUPS ARE FORMED, BOTH IN GENERAL AND WHEN STUDENTS WORK ON TASKS

Grouping needs to be done frequently through visible randomizations (Liljedahl, 2014). Ideally, at the beginning of every class a visibly random method is used to assign students to a group of 2-4 for the duration of that class. These groups will work together on any assigned problem solving tasks, sit together or stand together during any group or whole class discussions.

### 4. STUDENT WORK SPACE WHILE THEY WORK ON TASKS

Groups of students need to work on vertical non-permanent surfaces such as whiteboards, blackboards, or windows. This will make visible all work being done, not just to the teacher, but to the groups doing the work. To facilitate discussion there should be only one felt pen or piece of chalk per group.

### 5. ROOM ORGANIZATION, BOTH IN GENERAL AND WHEN STUDENTS WORK ON TASKS

The classroom needs to be de-fronted. The teacher must let go of one wall of the classroom as being the designated teaching space that all desks are oriented towards. The teacher needs to address the class from a variety of locations within the room and, as much as possible, use all four walls of the classroom. It is best if desks are placed in a random configuration around the room.

### 6. HOW QUESTIONS ARE ANSWERED WHEN STUDENTS ARE WORKING ON TASKS

Students only ask three types of questions: (1) proximity questions—asked when the teacher is close; (2) stop thinking questions—most often of the form "*Is this right?*"; and (3) keep thinking questions—questions that students ask so they can get back to work. Only the third of these types should be answered. The first two need to be acknowledged, but not answered.

### 7. THE WAYS IN WHICH HINTS AND EXTENSIONS ARE USED WHILE STUDENTS WORK ON TASKS

Once a thinking classroom is established, it needs to be nurtured. This is done primarily through how hints and extensions are given to groups as they work on tasks. *Flow* (Csíkszentmihályi, 1990) is a good framework for thinking about this. Hints and extensions need to be given so as to keep students in a perfect balance between the challenge of the current task and their abilities in working on it. If their ability is too high the risk is they get bored. If the challenge is too great the risk is they become frustrated (Liljedahl, 2016).

### 8. WHEN AND HOW A TEACHER LEVELS THEIR CLASSROOM DURING OR AFTER TASKS

Levelling needs be done at the bottom. When every group has passed a minimum threshold, the teacher needs to engage in discussion about the experience and understanding the whole class now shares. This should involve a reification and formalization of the work done by the groups and often constitutes the 'lesson' for that particular class.

### 9. ASSESSMENT, BOTH IN GENERAL AND WHEN STUDENTS WORK ON TASKS

Assessment in a thinking classroom needs to be mostly about the involvement of students in the learning process through efforts to communicate with them where they are and where they are going in their learning. It needs to honour the activities of a thinking classroom through a focus on the processes of learning more so than the products, and it needs to include both group work and individual work.

## DISCUSSION

However, this research also showed that these are not all equally impactful or purposeful in the building and maintenance of a thinking classroom. Some of these are blunt instruments capable of leveraging significant changes while others are more refined, used for the fine-tuning and maintenance of a thinking classrooms. Some are necessary precursors to others. Some are easier to implement by teachers than others, while others are more nuanced, requiring great attention and more practice as a teacher. And some are better received by students than others. From the whole of these results emerged a three-tier hierarchy that represents not only the bluntness and ease of implementation, but also an ideal chronology of implementation (see Table 1).

| STAGE ONE | STAGE TWO | STAGE THREE |
|---|---|---|
| • begin lessons with problem-solving tasks<br>• vertical non-permanent surfaces<br>• visibly random groups | • oral instructions<br>• defronting the room<br>• answering questions | • levelling<br>• assessment<br>• managing flow |
| BLUNTNESS | | |
| DIFFICULTY OF IMPLEMENTATION | | |

Table 1. Nine elements as chronologically implemented.

These results are not definitive, exhaustive, or unique. The teaching methods that emerged as effective for each of these elements emerged as a result of an *a priori* commitment to make change in a contrarian fashion. This continued until positive effects began to emerge, at which point refinements were recursively explored. It is possible that a different approach to the research would have yielded different methods. Different methods could, likewise, emerge a different set of stages optimal for the development of thinking classrooms.

## CONCLUSIONS

The main goal of this research is about finding ways to build thinking classrooms. One of the sub-goals of this work on building thinking classrooms was to develop methods that not only fostered thinking and collaboration, but also bypassed any classroom norms that would potentially inhibit this from happening. Using the methods in stage one while solving problems, either together or separately, was almost universally successful. They worked for any grade, in any class, and for any teacher. As such, it can be said that these methods succeeded in bypassing whatever norms existed in the over six hundred classrooms in which these methods were tried. Further, they not only bypassed the norms for the students, but also the norms of the teachers implementing them. So different were these methods from the existing practices of the teachers participating in the research that they were left with what I have come to call *first person vicarious experiences*. They are first person because they are living the lesson and observing the results created by their own hands. But the methods are not their own. There has been no time to assimilate them into their own repertoire of practice or into the schema of how they construct meaningful practice. As such, they experienced a different way in which their classroom could look and how their students could behave. They experienced, thorough these other*ly* methods an other*ly* classroom—a thinking classroom.

The results of this research sound extra-ordinary. In many ways they are. It would be tempting to try to attribute these to some special quality of the professional development setting or skill of the facilitator. But these are not the source of these remarkable results. The results, I believe, lie not in what is new, but what is not old. The classroom norms that permeate classrooms in North America, and around the world, are so robust, so entrenched, that they transcend the particular classrooms and have become institutional norms (Liu & Liljedahl, 2012). What the methods presented here offer is a violent break from these institutional norms.

By constructing a thinking classroom, problem solving becomes not only a means, but also an end. A thinking classroom is shot through with rich problems. Implementation of each of the aforementioned methods associated with the nine elements and three stages relies on the ubiquitous use of problem solving. But at the same time, it also creates a classroom conducive to the collaborative solving of problems.

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# Ad Hoc Sessions

## Ad hoc Session

# WHERE DO I WANT STUDENTS' ATTENTION? AND WHAT CAN I DO TO AFFECT THEIR ATTENTION?

Dave Hewitt
*University of Birmingham*

This session was chosen a nice example of what Ad Hocs are meant to be—an opportunity for deep conversations on topics of interest or issues arising during the conference. In this Ad Hoc, Dave Hewitt picks up on an idea raised in John Mason's Plenary talk at that meeting, *Structure of Attention in Teaching Mathematics*, to delve more deeply into what particular actions teachers can do to affect students' attention.

Cette séance a été choisie car elle représente bien ce que les séances ad hoc sont censées être – l'occasion de discuter en profondeur de sujets ou d'idées qui émergent pendant la conférence. Dans cette séance, David Hewitt part d'une idée émise dans la conférence plénière de John Mason, La structure de l'attention dans l'enseignement des mathématiques, pour explorer davantage les actes qu'un enseignant peut poser pour guider l'attention des étudiants.

Following John Mason's lecture, we explored further the issue of where attention is placed and reasons why a teacher might want to affect where a student places his/her attention. Furthermore, we considered articulating techniques which might be employed to help affect students' attention.

We began with the following written on the board: 5 + 5 =

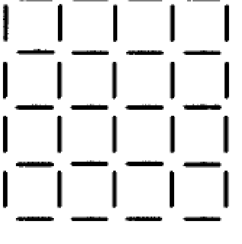
We discussed different consequences of what is stressed and what is ignored. For example (underlining indicates stressing):

$5 + 5 = \underline{5} + 4 + 1 = \underline{5} + 3 + 2 =$

$5 \underline{+} 5 = 6 \underline{+} 4 = 7 \underline{+} 3 =$

$\underline{5} + \underline{5} = 2 \times 5$ (2 lots of 5). A shift of attention can then result in $2 \times 5 = 2 + 2 + 2 + 2 + 2$

We looked at the different ways of stressing parts of the following drawing when given the task of counting the numbers of 'matches' involved:



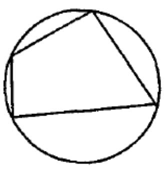
This resulted in statements such as the following:

$$2n(n+1) = \qquad n(2n+1)+n \qquad 2n^2 + 2n \qquad 4n + 2n(n-1)$$

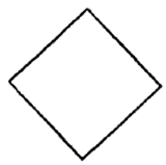
These resulted from stressing what calculations each person was carrying out to fmd the number of 'matches', rather than actually carrying out any calculations. The stressing of the process rather than the answer shifts attention from arithmetic to an algebraic structure, which can then be expressed as an algebraic statement.

The following was then put on the board and people were asked to consider, if they were teaching something using these images/symbols, where they would wish students' attention to be:

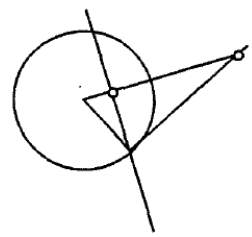
A



B



C



D $200 + 300$ E $x$

This was followed by considering what might be done to try to focus a student's attention onto the aspect chosen in each case. I will offer two examples of what was offered:

For B: rotating the figure so that one of the sides became horizontal, and then rotating it back to its original position (attempt to shift attention onto the property of squareness when such a figure is often not considered to be square by students).

For D: stressing aspects of the number names: two *hundred* plus three *hundred* (attempt to place attention on the value aspect of the number names in an attempt to help students use an existing number 'fact', 2 + 3 + = 5, in this new situation).

I leave the reader to consider how they would approach other images.

# MEASURING THE IMPACT OF A MENTOR PROGRAM

A. J. (Sandy) Dawson
*University of Hawaii*

This ad hoc was selected in memory of Sandy Dawson, a friend, mentor, and leader in CMESG. Sandy was president of CMESG from 1993 to 1997 and provided ongoing support to the CMESG executive by always advising on matters he thought were important for the organization. The ad hoc represents what he dedicated the last decade of his life to after retiring from Simon Fraser University and moving to the University of Hawaii. It is a testament of his passion to make a difference to teacher education in Pacific Island communities.

Cette présentation ad hoc a été choisie en mémoire de Sandy Dawson, un ami, un mentor et un leader au sein du GCEDM. Sandy a été président de 1993 à 1997 et il a toujours offert un support constant au comité exécutif en le conseillant sur des aspects qu'il considérait fondamentaux pour l'organisation. Cette présentation ad hoc représente ce à quoi il s'est consacré pendant la dernière décennie de sa vie, après avoir pris sa retraite de l'Université Simon Fraser et être déménagé pour travailler à l'Université d'Hawaii. Ce texte témoigne de sa passion à faire la différence en formation des enseignants des communautés des îles du Pacifique.

In 2002 Pacific Resources for Education and Learning (PREL) received funding for five years from the National Science Foundation (NSF) teacher enhancement program to implement Project MENTOR (Mathematics Education for Novice Teachers: Opportunities for Reflection). Project MENTOR staff work with 4-member teams of mentors drawn from departments and ministries of education and institutions of higher education in the 10 U.S.-affiliated Pacific island communities of American Samoa, the Commonwealth of the Northern Mariana Islands, the Federated States of Micronesia (FSM, which includes Chuuk, Kosrae, Pohnpei, and Yap), Guam, Hawai'i, the Republic of the Marshall Islands, and the Republic of Palau. Project MENTOR established a mentoring program for novice teachers aimed at developing in novice teachers the knowledge, skills, and dispositions necessary to become effective teachers of mathematics.

The Project has a number of areas of desired impact:

- Increase novice teacher (0–3 years experience) ability to plan, implement and assess instructional sequences that reflect a standards-based approach;
- Develop novice teachers' and mentors' abilities to reflect critically on their practices;
- Develop mentors' abilities to effectively mentor novice teachers;
- Develop mentors' abilities to provide effective professional development that contributes to the professional growth of novice teacher;
- Increase mentor and novice teacher mathematical knowledge;
- Increase leader and novice teacher collaboration.

The Project created a number of strategies in an attempt to accomplish the impact desired. These strategies included at least the following:

- Annual week-long institute provided by Project staff for mentors that focuses on (1) mathematical content and pedagogical knowledge, (2) mentoring skills and techniques, and (3) assessment strategies;

- Project staff undertake bi-monthly video and/or telephone conferences with mentors, and where feasible, deliver local workshops and demonstration lessons for mentors and novice teachers;
- Annual week-long institute provided by mentors for novice teachers that focuses on (1) mathematical content, (2) pedagogical knowledge, and (3) and classroom assessment strategies;
- Monthly seminars provided by mentors for novice teachers that focus on the classroom experiences of the novice teachers;
- Monthly observations of the novice teachers by the mentors.

The tools for measuring impact selected by the Project are the following:

- Mathematics Content Test administered to novice teachers each year for the three years they are involved with the Project;
- Novice Teacher Questionnaire administered each year (questionnaire focuses on attitudes and dispositions towards the teaching of mathematics);
- Novice Teacher Survey administered after the yearly institutes that focuses on the support provided to novice teachers by the mentors;
- Mathematics Content Test administered to mentors at the beginning and end of the Project;
- Mentor Questionnaire administered at the beginning and end of the Project;
- Institute assessment instrument completed by mentors at the conclusion of the yearly Project staff-led institutes;
- Group interviews of mentors by island community regarding the observations made by mentors of the novice teachers.

The presentation sought input regarding the match among the focus of impact, strategies chosen, and the tools designed to measure impact. Participants made several suggestions for alternate ways to measure impact that may be better suited to the setting of the Project.

# Appendices

# Annexes

## APPENDIX A / ANNEXE A

## PLENARY LECTURES AT EACH ANNUAL MEETING / CONFÉRENCES PLÉNIÈRES DES RENCONTRES ANNUELLES

| | | |
|---|---|---|
| 1977 | A.J. COLEMAN | The objectives of mathematics education |
| | C. GAULIN | Innovations in teacher education programmes |
| | T.E. KIEREN | The state of research in mathematics education |
| 1978 | G.R. RISING | The mathematician's contribution to curriculum development |
| | A.I. WEINZWEIG | The mathematician's contribution to pedagogy |
| 1979 | J. AGASSI | The Lakatosian revolution |
| | J.A. EASLEY | Formal and informal research methods and the cultural status of school mathematics |
| 1980 | C. GATTEGNO | Reflections on forty years of thinking about the teaching of mathematics |
| | D. HAWKINS | Understanding understanding mathematics |
| 1981 | K. IVERSON | Mathematics and computers |
| | J. KILPATRICK | The reasonable effectiveness of research in mathematics education |
| 1982 | P.J. DAVIS | Towards a philosophy of computation |
| | G. VERGNAUD | Cognitive and developmental psychology and research in mathematics education |
| 1983 | S.I. BROWN | The nature of problem generation and the mathematics curriculum |
| | P.J. HILTON | The nature of mathematics today and implications for mathematics teaching |
| 1984 | A.J. BISHOP | The social construction of meaning: A significant development for mathematics education? |
| | L. HENKIN | Linguistic aspects of mathematics and mathematics instruction |
| 1985 | H. BAUERSFELD | Contributions to a fundamental theory of mathematics learning and teaching |
| | H.O. POLLAK | On the relation between the applications of mathematics and the teaching of mathematics |
| 1986 | R. FINNEY | Professional applications of undergraduate mathematics |
| | A.H. SCHOENFELD | Confessions of an accidental theorist |
| 1987 | P. NESHER | Formulating instructional theory: the role of students' |

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|---|---|---|
| | | misconceptions |
| | H.S. WILF | The calculator with a college education |
| 1988 | C. KEITEL | Mathematics education and technology |
| | L.A. STEEN | All one system |
| 1989 | N. BALACHEFF | Teaching mathematical proof: The relevance and complexity of a social approach |
| | D. SCHATTSNEIDER | Geometry is alive and well |
| 1990 | U. D'AMBROSIO | Values in mathematics education A. SIERPINSKA On understanding mathematics |
| 1991 | J .J. KAPUT | Mathematics and technology: Multiple visions of multiple futures |
| | C. LABORDE | Approches théoriques et méthodologiques des recherches françaises en didactique des mathématiques |
| 1992 | ICME-7 | |
| 1993 | G.G. JOSEPH | What is a square root? A study of geometrical representation in different mathematical traditions |
| | J CONFREY | Forging a revised theory of intellectual development: Piaget, Vygotsky and beyond |
| 1994 | A. SFARD | Understanding = Doing + Seeing ? |
| | K. DEVLIN | Mathematics for the twenty-first century |
| 1995 | M. ARTIGUE | The role of epistemological analysis in a didactic approach to the phenomenon of mathematics learning and teaching |
| | K. MILLETT | Teaching and making certain it counts |
| 1996 | C. HOYLES | Beyond the classroom: The curriculum as a key factor in students' approaches to proof |
| | D. HENDERSON | Alive mathematical reasoning |
| 1997 | R. BORASSI | What does it really mean to teach mathematics through inquiry? |
| | P. TAYLOR | The high school math curriculum |
| | T. KIEREN | Triple embodiment: Studies of mathematical understanding-in-interaction in my work and in the work of CMESG/GCEDM |
| 1998 | J. MASON | Structure of attention in teaching mathematics |
| | K. HEINRICH | Communicating mathematics or mathematics storytelling |
| 1999 | J. BORWEIN | The impact of technology on the doing of mathematics |
| | W. WHITELEY | The decline and rise of geometry in 20th century North America |
| | W. LANGFORD | Industrial mathematics for the 21st century |
| | J. ADLER | Learning to understand mathematics teacher development and change: Researching resource availability and use in the context of formalised INSET in South Africa |

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| | B. BARTON | An archaeology of mathematical concepts: Sifting languages for mathematical meanings |
| 2000 | G. LABELLE | Manipulating combinatorial structures |
| | M. B. BUSSI | The theoretical dimension of mathematics: A challenge for didacticians |
| 2001 | O. SKOVSMOSE | Mathematics in action: A challenge for social theorising |
| | C. ROUSSEAU | Mathematics, a living discipline within science and technology |
| 2002 | D. BALL & H. BASS | Toward a practice-based theory of mathematical knowledge for teaching |
| | J. BORWEIN | The experimental mathematician: The pleasure of discovery and the role of proof |
| 2003 | T. ARCHIBALD | Using history of mathematics in the classroom: Prospects and problems |
| | A. SIERPINSKA | Research in mathematics education through a keyhole |
| 2004 | C. MARGOLINAS | La situation du professeur et les connaissances en jeu au cours de l'activité mathématique en classe |
| | N. BOULEAU | La personnalité d'Evariste Galois: le contexte psychologique d'un goût prononcé pour les mathématique abstraites |
| 2005 | S. LERMAN | Learning as developing identity in the mathematics classroom |
| | J. TAYLOR | Soap bubbles and crystals |
| 2006 | B. JAWORSKI | Developmental research in mathematics teaching and learning: Developing learning communities based on inquiry and design |
| | E. DOOLITTLE | Mathematics as medicine |
| 2007 | R. NÚÑEZ | Understanding abstraction in mathematics education: Meaning, language, gesture, and the human brain |
| | T. C. STEVENS | Mathematics departments, new faculty, and the future of collegiate mathematics |
| 2008 | A. DJEBBAR | Art, culture et mathématiques en pays d'Islam (IX$^{e}$-XV$^{e}$ s.) |
| | A. WATSON | Adolescent learning and secondary mathematics |
| 2009 | M. BORBA | Humans-with-media and the production of mathematical knowledge in online environments |
| | G. de VRIES | Mathematical biology: A case study in interdisciplinarity |

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|---|---|---|
| 2010 | W. BYERS | Ambiguity and mathematical thinking |
| | M. CIVIL | Learning from and with parents: Resources for equity in mathematics education |
| | B. HODGSON | Collaboration et échanges internationaux en éduction mathématique dans le cadre de la CIEM : regards selon une perspective canadienne • ICMI as a space for international collaboration and exchange in mathematics education: Some views from a Canadian perspective |
| | S. DAWSON | My journey across, through, over, and around academia: "...a path laid while walking..." |
| 2011 | C. K. PALMER | Pattern composition: Beyond the basics |
| | P. TSAMIR & D. TIROSH | The Pair-Dialogue approach in mathematics teacher education |
| 2012 | P. GERDES | Old and new mathematical ideas from Africa: Challenges for reflection |
| | M. WALSHAW | Towards an understanding of ethical practical action in mathematics education: Insights from contemporary inquiries |
| | W. HIGGINSON | Cooda, wooda, didda, shooda: Time series reflections on CMESG/GCEDM |
| 2013 | R. LEIKIN | On the relationships between mathematical creativity, excellence and giftedness |
| | B. RALPH | Are we teaching Roman numerals in a digital age? |
| | E. MULLER | Through a CMESG looking glass |
| 2014 | D. HEWITT | The Economic Use of Time and Effort in the Teaching and Learning of Mathematics |
| | N. NIGAM | Mathematics in Industry, Mathematics in the Classroom: Analogy and Metaphor |
| 2015 | E. RODITI | Diversité, variabilité et convergence des pratiques enseignantes • Diversity, Variability, and Commonalities Among Teaching Practices |
| | D. HUGES HALLET | Connections: Mathematical, Interdisciplinary, Electronic, and Personal |

# APPENDIX B / ANNEXE B

## WORKING GROUPS AT EACH ANNUAL MEETING / GROUPES DE TRAVAIL DES RENCONTRES ANNUELLES

1977 Queen's University, Kingston, Ontario

- Teacher education programmes
- Undergraduate mathematics programmes and prospective teachers
- Research and mathematics education
- Learning and teaching mathematics

1978 Queen's University, Kingston, Ontario

- Mathematics courses for prospective elementary teachers
- Mathematization
- Research in mathematics education

1979 Queen's University, Kingston, Ontario

- Ratio and proportion: a study of a mathematical concept
- Minicalculators in the mathematics classroom
- Is there a mathematical method?
- Topics suitable for mathematics courses for elementary teachers

1980 Université Laval, Québec, Québec

- The teaching of calculus and analysis
- Applications of mathematics for high school students
- Geometry in the elementary and junior high school curriculum
- The diagnosis and remediation of common mathematical errors

1981 University of Alberta, Edmonton, Alberta

- Research and the classroom
- Computer education for teachers
- Issues in the teaching of calculus
- Revitalising mathematics in teacher education courses

1982 Queen's University, Kingston, Ontario

- The influence of computer science on undergraduate mathematics education
- Applications of research in mathematics education to teacher training programmes
- Problem solving in the curriculum

1983 University of British Columbia, Vancouver, British Columbia

- Developing statistical thinking
- Training in diagnosis and remediation of teachers
- Mathematics and language
- The influence of computer science on the mathematics curriculum

1984 University of Waterloo, Waterloo, Ontario

- Logo and the mathematics curriculum
- The impact of research and technology on school algebra
- Epistemology and mathematics
- Visual thinking in mathematics

1985 Université Laval, Québec, Québec

- Lessons from research about students' errors
- Logo activities for the high school
- Impact of symbolic manipulation software on the teaching of calculus

1986 Memorial University of Newfoundland, St. John's, Newfoundland

- The role of feelings in mathematics
- The problem of rigour in mathematics teaching
- Microcomputers in teacher education
- The role of microcomputers in developing statistical thinking

1987 Queen's University, Kingston, Ontario

- Methods courses for secondary teacher education
- The problem of formal reasoning in undergraduate programmes
- Small group work in the mathematics classroom

1988 University of Manitoba, Winnipeg, Manitoba

- Teacher education: what could it be?
- Natural learning and mathematics
- Using software for geometrical investigations
- A study of the remedial teaching of mathematics

1989 Brock University, St. Catharines, Ontario

- Using computers to investigate work with teachers
- Computers in the undergraduate mathematics curriculum
- Natural language and mathematical language
- Research strategies for pupils' conceptions in mathematics

1990 Simon Fraser University, Vancouver, British Columbia

- Reading and writing in the mathematics classroom
- The NCTM "Standards" and Canadian reality
- Explanatory models of children's mathematics
- Chaos and fractal geometry for high school students

1991 University of New Brunswick, Fredericton, New Brunswick

- Fractal geometry in the curriculum
- Socio-cultural aspects of mathematics
- Technology and understanding mathematics
- Constructivism: implications for teacher education in mathematics

1992 ICME–7, Université Laval, Québec, Québec

1993 York University, Toronto, Ontario

- Research in undergraduate teaching and learning of mathematics
- New ideas in assessment
- Computers in the classroom: mathematical and social implications
- Gender and mathematics
- Training pre-service teachers for creating mathematical communities in the classroom

1994 University of Regina, Regina, Saskatchewan

- Theories of mathematics education
- Pre-service mathematics teachers as purposeful learners: issues of enculturation
- Popularizing mathematics

1995 University of Western Ontario, London, Ontario

- Autonomy and authority in the design and conduct of learning activity
- Expanding the conversation: trying to talk about what our theories don't talk about
- Factors affecting the transition from high school to university mathematics
- Geometric proofs and knowledge without axioms

1996 Mount Saint Vincent University, Halifax, Nova Scotia

- Teacher education: challenges, opportunities and innovations
- Formation à l'enseignement des mathématiques au secondaire: nouvelles perspectives et défis
- What is dynamic algebra?
- The role of proof in post-secondary education

1997 Lakehead University, Thunder Bay, Ontario

- Awareness and expression of generality in teaching mathematics
- Communicating mathematics
- The crisis in school mathematics content

1998 University of British Columbia, Vancouver, British Columbia

- Assessing mathematical thinking
- From theory to observational data (and back again)
- Bringing Ethnomathematics into the classroom in a meaningful way
- Mathematical software for the undergraduate curriculum

1999 Brock University, St. Catharines, Ontario

- Information technology and mathematics education: What's out there and how can we use it?
- Applied mathematics in the secondary school curriculum
- Elementary mathematics
- Teaching practices and teacher education

2000 Université du Québec à Montréal, Montréal, Québec

- Des cours de mathématiques pour les futurs enseignants et enseignantes du primaire • Mathematics courses for prospective elementary teachers
- Crafting an algebraic mind: Intersections from history and the contemporary mathematics classroom
- Mathematics education et didactique des mathématiques : y a-t-il une raison pour vivre des vies séparées? • Mathematics education et didactique des mathématiques: Is there a reason for living separate lives?
- Teachers, technologies, and productive pedagogy

2001 University of Alberta, Edmonton, Alberta

- Considering how linear algebra is taught and learned
- Children's proving
- Inservice mathematics teacher education
- Where is the mathematics?

2002 Queen's University, Kingston, Ontario

- Mathematics and the arts
- Philosophy for children on mathematics
- The arithmetic/algebra interface: Implications for primary and secondary mathematics • Articulation arithmétique/algèbre: Implications pour l'enseignement des mathématiques au primaire et au secondaire
- Mathematics, the written and the drawn
- Des cours de mathématiques pour les futurs (et actuels) maîtres au secondaire • Types and characteristics desired of courses in mathematics programs for future (and in-service) teachers

2003 Acadia University, Wolfville, Nova Scotia

- L'histoire des mathématiques en tant que levier pédagogique au primaire et au secondaire • The history of mathematics as a pedagogic tool in Grades K–12
- Teacher research: An empowering practice?
- Images of undergraduate mathematics
- A mathematics curriculum manifesto

2004 Université Laval, Québec, Québec

- Learner generated examples as space for mathematical learning
- Transition to university mathematics
- Integrating applications and modeling in secondary and post secondary mathematics
- Elementary teacher education – Defining the crucial experiences
- A critical look at the language and practice of mathematics education technology

2005 University of Ottawa, Ottawa, Ontario

- Mathematics, education, society, and peace
- Learning mathematics in the early years (pre-K – 3)
- Discrete mathematics in secondary school curriculum
- Socio-cultural dimensions of mathematics learning

2006 University of Calgary, Calgary, Alberta

- Secondary mathematics teacher development
- Developing links between statistical and probabilistic thinking in school mathematics education
- Developing trust and respect when working with teachers of mathematics
- The body, the sense, and mathematics learning

2007 University of New Brunswick, New Brunswick

- Outreach in mathematics – Activities, engagement, & reflection
- Geometry, space, and technology: challenges for teachers and students
- The design and implementation of learning situations
- The multifaceted role of feedback in the teaching and learning of mathematics

2008 Université de Sherbrooke, Sherbrooke, Québec

- Mathematical reasoning of young children
- Mathematics-in-and-for-teaching (MifT): the case of algebra
- Mathematics and human alienation
- Communication and mathematical technology use throughout the post-secondary curriculum • Utilisation de technologies dans l'enseignement mathématique postsecondaire
- Cultures of generality and their associated pedagogies

2009 York University, Toronto, Ontario

- Mathematically gifted students • Les élèves doués et talentueux en mathématiques
- Mathematics and the life sciences
- Les méthodologies de recherches actuelles et émergentes en didactique des mathématiques • Contemporary and emergent research methodologies in mathematics education
- Reframing learning (mathematics) as collective action
- Étude des pratiques d'enseignement
- Mathematics as social (in)justice • Mathématiques citoyennes face à l'(in)justice sociale

2010 Simon Fraser University, Burnaby, British Columbia

- Teaching mathematics to special needs students: Who is at-risk?
- Attending to data analysis and visualizing data
- Recruitment, attrition, and retention in post-secondary mathematics
- Can we be thankful for mathematics? Mathematical thinking and aboriginal peoples
- Beauty in applied mathematics
- Noticing and engaging the mathematicians in our classrooms

2011 Memorial University of Newfoundland, St. John's, Newfoundland

- Mathematics teaching and climate change
- Meaningful procedural knowledge in mathematics learning
- Emergent methods for mathematics education research: Using data to develop theory • Méthodes émergentes pour les recherches en didactique des mathématiques: partir des données pour développer des théories

- Using simulation to develop students' mathematical competencies – Post secondary and teacher education
- Making art, doing mathematics • Créer de l'art; faire des maths
- Selecting tasks for future teachers in mathematics education

2012 Université Laval, Québec City, Québec

- Numeracy: Goals, affordances, and challenges
- Diversities in mathematics and their relation to equity
- Technology and mathematics teachers (K-16) • La technologie et l'enseignant mathématique (K-16)
- La preuve en mathématiques et en classe • Proof in mathematics and in schools
- The role of text/books in the mathematics classroom • Le rôle des manuels scolaires dans la classe de mathématiques
- Preparing teachers for the development of algebraic thinking at elementary and secondary levels • Préparer les enseignants au développement de la pensée algébrique au primaire et au secondaire

2013 Brock University, St. Catharines, Ontario

- MOOCs and online mathematics teaching and learning
- Exploring creativity: From the mathematics classroom to the mathematicians' mind • Explorer la créativité : de la classe de mathématiques à l'esprit des mathématiciens
- Mathematics of Planet Earth 2013: Education and communication • Mathématiques de la planète Terre 2013 : formation et communication (K-16)
- What does it mean to understand multiplicative ideas and processes? Designing strategies for teaching and learning
- Mathematics curriculum re-conceptualisation

2014 University of Alberta, Edmonton, Alberta

- Mathematical habits
- Mathematical Habits of Mind • Modes de pensée mathématiques
- Formative Assessment in Mathematics: Developing Understandings, Sharing Practice, and Confronting Dilemmas
- Texter mathématique • Texting Mathematics
- Complex Dynamical Systems
- Role-Playing and Script-Writing in Mathematics Education: Practice and Research

2015 Université de Moncton, Moncton, New Brunswick

- Task Design and Problem Posing
- Indigenous Ways of Knowing in Mathematics
- Theoretical Frameworks in Mathematics Education Research • Les cadres théoriques dans la recherche en didactique des mathématiques
- Early Years Teaching, Learning and Research: Tensions in Adult-Child Interactions Around Mathematics
- Innovations in Tertiary Mathematics Teaching, Learning and Research • Innovations au post-secondaire pour l'enseignement, l'apprentissage et la recherche

# APPENDIX C / ANNEXE C

## PROCEEDINGS OF ANNUAL MEETINGS / ACTES DES RENCONTRES ANNUELLES

Past proceedings of CMESG/GCEDM annual meetings have been deposited in the ERIC documentation system with call numbers as follows:

| | |
|---|---|
| *Proceedings of the 1980 Annual Meeting* . . . . . . . . . . . . . . . . . . . | ED 204120 |
| *Proceedings of the 1981 Annual Meeting* . . . . . . . . . . . . . . . . . . . | ED 234988 |
| *Proceedings of the 1982 Annual Meeting* . . . . . . . . . . . . . . . . . . . | ED 234989 |
| *Proceedings of the 1983 Annual Meeting* . . . . . . . . . . . . . . . . . . . | ED 243653 |
| *Proceedings of the 1984 Annual Meeting* . . . . . . . . . . . . . . . . . . . | ED 257640 |
| *Proceedings of the 1985 Annual Meeting* . . . . . . . . . . . . . . . . . . . | ED 277573 |
| *Proceedings of the 1986 Annual Meeting* . . . . . . . . . . . . . . . . . . . | ED 297966 |
| *Proceedings of the 1987 Annual Meeting* . . . . . . . . . . . . . . . . . . . | ED 295842 |
| *Proceedings of the 1988 Annual Meeting* . . . . . . . . . . . . . . . . . . . | ED 306259 |
| *Proceedings of the 1989 Annual Meeting* . . . . . . . . . . . . . . . . . . . | ED 319606 |
| *Proceedings of the 1990 Annual Meeting* . . . . . . . . . . . . . . . . . . . | ED 344746 |
| *Proceedings of the 1991 Annual Meeting* . . . . . . . . . . . . . . . . . . . | ED 350161 |
| *Proceedings of the 1993 Annual Meeting* . . . . . . . . . . . . . . . . . . . | ED 407243 |
| *Proceedings of the 1994 Annual Meeting* . . . . . . . . . . . . . . . . . . . | ED 407242 |
| *Proceedings of the 1995 Annual Meeting* . . . . . . . . . . . . . . . . . . . | ED 407241 |
| *Proceedings of the 1996 Annual Meeting* . . . . . . . . . . . . . . . . . . . | ED 425054 |
| *Proceedings of the 1997 Annual Meeting* . . . . . . . . . . . . . . . . . . . | ED 423116 |
| *Proceedings of the 1998 Annual Meeting* . . . . . . . . . . . . . . . . . . . | ED 431624 |
| *Proceedings of the 1999 Annual Meeting* . . . . . . . . . . . . . . . . . . . | ED 445894 |
| *Proceedings of the 2000 Annual Meeting* . . . . . . . . . . . . . . . . . . . | ED 472094 |
| *Proceedings of the 2001 Annual Meeting* . . . . . . . . . . . . . . . . . . . | ED 472091 |
| *Proceedings of the 2002 Annual Meeting* . . . . . . . . . . . . . . . . . . . | ED 529557 |

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| *Proceedings of the 2003 Annual Meeting* . . . . . . . . . . . . . . . . . . . | ED 529558 |
| *Proceedings of the 2004 Annual Meeting* . . . . . . . . . . . . . . . . . . . | ED 529563 |
| *Proceedings of the 2005 Annual Meeting* . . . . . . . . . . . . . . . . . . . | ED 529560 |
| *Proceedings of the 2006 Annual Meeting* . . . . . . . . . . . . . . . . . . . | ED 529562 |
| *Proceedings of the 2007 Annual Meeting* . . . . . . . . . . . . . . . . . . . | ED 529556 |
| *Proceedings of the 2008 Annual Meeting* . . . . . . . . . . . . . . . . . . . | ED 529561 |
| *Proceedings of the 2009 Annual Meeting* . . . . . . . . . . . . . . . . . . . | ED 529559 |
| *Proceedings of the 2010 Annual Meeting* . . . . . . . . . . . . . . . . . . . | ED 529564 |
| *Proceedings of the 2011 Annual Meeting* . . . . . . . . . . . . . . . . . . . | ED 547245 |
| *Proceedings of the 2012 Annual Meeting* . . . . . . . . . . . . . . . . . . . | ED 547246 |
| *Proceedings of the 2013 Annual Meeting* . . . . . . . . . . . . . . . . . . . | ED 547247 |
| *Proceedings of the 2014 Annual Meeting* . . . . . . . . . . . . . . . . . . . | *submitted* |
| *Proceedings of the 2015 Annual Meeting* . . . . . . . . . . . . . . . . . . . | *submitted* |

## NOTE

There was no Annual Meeting in 1992 because Canada hosted the Seventh International Conference on Mathematical Education that year.

## Appendix D / Annexe D

# LIST OF PARTICIPANTS / LISTE DES PARTICIPANTS

Aboagye, Samuel
Abtahi, Yasmine
Abu-Bakare, Veda
Adihou, Adolphe
Adler, Jill
Adler, L.S.
Afodjo, Nabil
Agassi, Joseph
Alalouf, Eva
Alexander, David
Allaire, Richard
Allan, Amanda
Allan, Darien
Allard, Hughette
Allen, Hugh
Alsalim, Lyla
Alvarez, Melania
Anderson, Ann
Antropov, Alexander
Anwandter Cuellar, Nathalie
Appelbaum, Peter
Archibald, Tom
Arden, Ann
Arias de Sanchez, Gabriela
Armstrong, Alayne
Arsenaull, Isabelle
Artigue, Michele
Aruliah, Dhavide
Avoseh, Jimmy
Badee, Jenny T.
Bailey, James
Baird, Ben
Bakos, Sandy
Balacheff, Nicolas
Bale, David
Ball, Deborah
Ballheim, Cynthia
Bany, B.
Barabé, Geneviève
Barbeau, Edward
Barclini, Caroline
Barnes, Judy
Barnson, Pat
Baron, Lorraine
Barrera, Raquel
Barry, Souleymane
Bartolini Bussi, Mariolina
Barton, Bill
Barton, Shelley
Barwell, Richard
Bass, Hyman
Bates, Thomas
Bauersfeld, Heinrich
Beamer, James
Beatty, Ruth
Beattys, Candice
Beaumont, C.F.A.
Bednarz, Nadine
Begg, Andy
Beirnes, Norman
Beisiegel, Mary
Bélanger, Jean-Philippe
Bélanger, Maurice
Belostotski, Gregory
Bemmouna, Benyounès
Bengo, Priscilla
Benoit, David
Ben-Rherbal, Abderrahmane
Benton, Nicola
Berezovski, Tetyana
Bergé, Analia
Bergen, Katharine
Bergeron, Jacques
Berggren, Tasoula
Berglind, Bob
Berneche, Christian
Berry, John
Bertrand, Richard
Beswick, Kim
Betts, Paul
Bilinski, Robert
Bishop, Alan
Bisson, Caroline
Blake, Rick
Blum, Carolyn
Bobos, Georgeana
Bochonko, Helen
Bogomolny, Marianna
Boileau, André
Boileau, Nicolas
Boisclair, Noëlange
Boissett, Annick
Boissy, Leo
Bolduc, Yvonne
Borasi, Raffaella
Borba, Marcelo
Borgen, Katharine
Borwein, Jonathan
Boswall, Alberta
Boucher, Claude
Boulanger, Cindy
Bouleau, Nicolas
Bourgeois, Roger
Boutilier, Debbie
Bowers, Craig
Brace, Alec
Braconne, Annette
Braconne-Michoux, Annette
Braich, Pavneet
Brandau, Linda
Branker, Maritza
Breen, Chris
Brettler, Elias
Brew, Christine
Brian, Mary
Brochmann, Harold
Brodie, Iain
Brody, Josef
Brody, Zofie
Broley, Laura
Brouwer, Peter
Brown, Allan
Brown, Burke
Brown, Laurinda
Brown, Stephen
Bruce, Catherine
Bryan, Robert
Buerk, Dorothy
Burnett, Dale
Businskas, Aldona
Buteau, Chantal
Byers, Trish
Byers, Victor
Byers, William
Cadet, Élysée Robert
Calin, Lucas
Calvert, Brian
Cameron, Mary
Campbell, Catherine
Campbell, Stephen
Capling, Lynn
Carbno, Clifford
Caron, France
Caron, Renee
Carr, Donna
Carrodus, Jon
Carruthers, Carol
Carson, Rosalind
Cassidy, Charles
Caswell, Beverly
Cathcart, George
Cattari, Dave
Cerullj, Michele
Challita, Dalia
Chandrasekhar, Vandana
Chapman, Olive
Charbonneau, Louis
Chavoshi Jolfaee, Simin
Chazan, Daniel
Chernoff, Egan
Chorney, Sean
Chow, Valeen
Christian, Robert
Cimen, Arda
Civil, Marta
Clark, John
CoacheLessard, Michele
Cohen, Rina
Coleman, John
Colgan, Lynda
Confrey, Jere
Conne, Francois
Connelly, Ralph
Conrigan, Arlene
Cooper, Alan
Copes, Larry
Cordy, Michelle
Cormier, Janelle
Correa, Priscila
Corriveau, Annie
Corriveau, Claudia
Coté, Benoit
Craven, Stewart
Crawford, Douglas
Crawford, Sandra
Cremo, John William (Tiny)
Crespo, Sandra
Cristall, Eleanor
Crowley, Mary
Cudmore, Donald
Currie, Gina
Cyr, Gilles

Cyr, Stéphane
D'Ambrosio, Beatriz
D'Ambrosio, Ubiratan
d'Entremont, Yvette
Dale, Meghan
Dalrymple, Shirley
Damiano, Toni
D'Amour, Lissa
Dance, Rosalie
Davenport, Linda
Davidson, Michelle P.
Davidson, Natasha
Davis, Brent
Davis, George
Davis, Philip
Dawes, Mike
Dawson, Sandy
de Cortet, Sophie Rene
De Santos, Lydia
de Vries, Gerda
DeBlois, Lucie
DeCarufel, Jean-Lou
Defence, Astrid
de Flandre, Charles
deFreitas, Elizabeth
Deguire, Paul
Del Gobbo, Assuntina
Demana, Frank
Demers, Luc
d'Entremont, Yvette
Desjardins, Sylvie
Devitt, J.S.
Devlin, Keith
Dhombres, J.
Dianne, Kenton
Dick, Catherine
Dickmann, Evelyn
Dietiker, Leslie
Dionne, Jean
Dirks, Michael
Djebbar, Ahmed
DoCarmoVila, Maria
Doctorow, Gord
Doctorow, Roz
Doolittle, Edward
Douamba, Kirsi Jean-Pierre
Drakes, Chiaka
Dreyfus, Tommy
Drost, Dale
Dube, Michael
Dubiel, Malgorzata
Ducharme-Rivard, Alexandre
Dumouchel, Mélanie
Duret, Doris
Dutfield, Anna
Easley, Jack
Eastman, Don
Eddy, Ronald
Edge, Douglas
Edwards, Laurie
Egodawatte, Gunawardena
Egsgard, J.C.
Ekol, George
El Bouazzaoul, Uabiha
Elliot, Michelle
Ellis, David
Erlwanger, Stanley
Ertman, Trina
Esmonde, Indigo
Etchecopar, Philippe
Evans, Norma
Facey, Sharon
Falkenberg, Thomas
Fast, Gerald
Fellus, Osnat
Fenwick-Sehl, Laura
Fillmore, Peter
Finney, Ross
Fioroni, Marcella
Fisher, JChris
Flewelling, Gary
Flynn, Tara
Fortier, Marielle
Francis, Krista
Frankenstein, Marilyn
Franks, Doug
Fraser, Grace
Fraser, Holly
Freiman, Viktor
Frempong, George
Frid, Sandra
Fullerton, Olive
Furey, Rosalita
Furlong, Caitlin
Gadanidis, George
Gaddis, Kelly
Gagnon, Luc
Gagnon, Ludovick
Garanson, Maurice
Garashuck, Kseniya
Garekwe, Kgomotso
Gattegno, Caleb
Gattuso, Linda
Gaulin, Claude
Gauthier, Pierne
Geddes, Keith
Georgescu, Elena
Gerber, Harvey
Gerdes, Paulus
Gerofsky, Susan
Gervais, Nicole
Gholamazad, Soheila
Giles, Carmen
Gillis, Allan
Girard, Jeanne-d'Arc
Giroux, Jacinthe
Glanfield, Florence
Gol Tabaghi, Shiva
González-Martin, Alejandro
Goolann, Faroon
Gordon, Lynn
Gould, Holly
Goupille, Cecile
Gourdeau, Frédéric
Gouthro, Cathy
Grant McLoughlin, John
Grasse, Cindy
Graves, Barbara
Griffith, John
Grignon, Jean
Gruen, Frank
Guentuer, Grant
Guillemette, David
Gula, Taras
Gunthrope, Gerrenne
Gupta, H.N.
Guzman, Iosé
Hackenberg, Amy
Hagen, Pamela
Haguel, Marie-Iane
Haines, Jim
Hall, Jennifer
Hall, Joe
Halmaghi, Elena
Hamilton, Gordon
Handscomb, Kerry
HangmoSubba, Bijoy
Hanna, Evelyn
Hanna, Gila
Hanson, Denis
Harburn, Keith
Hardy, Nadia
Hare, Andrew
Harrison, Bruce
Harrison, Peter
Hawkins, David
Headley, Velmer
Hébert, Michelle
Heffernan, Jane
Heinrich, Kathy
Henderson, David
Herbel-Eisenmann, Beth
Herbst, Behnaz
Herscovics, Nicolas
Hershkovitz, Rina
Hewitt, Dave
Higginson, William
Hillel, Joel
Hillman, Thomas
Hilton, Peter
Hitt, Fernando
Hoang, Chinh
Hodgson, Bernard
Hoechsmann, Klaus
Hoffman, Martin
Holm, Jennifer
Hooper, Jeff
Hope, Jack
Horrobin, Michelle
Hoshino, Richard
Howitz, Thomas
Hoyles, Celia
Hudon, Wendy
Hughes Hallett, Deborah
Humell, Marilyn
Hunkln, Leon
Hunter, Kim
Hunter, Shelley
Hussin, Véronique
Iordi, Isabelle
Irvine, Jeff A.
Iverson, K.
Jackiw, Nicholas
Jackson, Renee
Jamshid Nejad, Masomeh
Janes, Rita
Jansson, Lars
Janvier, Claude
Janvier, Bernadette
Jao, Limin
Jarvis, Daniel
Jaworski, Barbara
Jayakody, Gaya
Jayasundera, Kanthi
Jaykody, Gaya
Jeannotte, Doris
Jeanotte, Doris
Jeffery, Gordon
Jesso, Andrew
Johns, B.
Johnson, Carl
Johnson, Richard
Jones, Adrian
Jonker, Leo
Jordi, Isabelle
Joseph, George
Judah, Mary-Lee
Kajander, Ann
Kaller, C.L.
Kam, Mimi
Kantowski, M.G.
Kapoor, Don
Kaput, Jim
Karadag, Zekeriya
Karrass, Margaret
Karsten, Donna
Karuku, Simon
Kastner, Bernice

Kaul, Saroop
Kaur, Harpreet
Kavousian, Shabnam
Kaye, Brian
Kaye, Ella
Kayler, Héléne
KayPohl, Margaret
Keitel, Christine
Kelecsenyi, Klara
Kelleher, Heather
Kelly, Valerie
Kennedy, Shannon
Kestell, Mary Lou
Khan, Steven
Kibedi, Francisco
Kieran, Carolyn
Kieren, Thomas
Kilpatrick, J.
Kinelovsky, Dasha
Kirshner, David
Klamkin, M.S.
Klasa, Jacqueline
Klawe, Maria
Kleiner, Israel
Kletskin, Ilona
Klompmaker, Matt
Kloster, Aldona
Knipping, Christine
Knoll, Eva
Koch, Martha
Koe, Carryl
Kondor, George
Kondratieva, Margo
Korkmaz, Ozlem
Korytowski, W.
Kotsopoulos, Donna
Krause, Jason
Kubota-Zarivnij, Kathy
Kuendiger, Erika
Kuenzang
Kuntz, Gerard
Kutas, Cecilia
Kwan, Kerry
Labelle, Gilbert
Laborde, Collette
LaCroix, Lionel
Lacroix, Norbert
Ladouceur, Andre
Lafleur, Geri Ann
Lagu, Indy
Lai, Frank
Lajoie, Caroline
Lakramtl, Ahmed
Lalli, B.S.
Landry, Laurie
Langen, Kimberley
Langford, Bill
Langwenya, Sibongile
Larsen, Judy
Lavoie, Mario
Lavoie, Paul
Lawrence, Lloyd
Lawson, Alex
Lawson, G.
Lazarus, Jill
LeBlanc, Manon
Lee, Lesley
Lee, Peter
Leikin, Rosa
Lemay, Fernand
Lemieux, Collette
Lemoyne, Gisele
Lepage, Ernestine
Lerman, Stephen
Leron, Uri
LeSage, Ann
Lessard, Gary
Lessard, Genevieve
Lester, June
Lewis, Matt
Lewis, Ron
Lidstone, Dave
Liljedahl, Peter
Lin, Terry Wan Jung
Linchevski, Liora
Liu, Minnie
Lo, Jane-Jane
Loewen, Craig
Lomas, Dermis
London-McNab, Susan
Long, Julie
Lorway, Geri
Losasso, Anita
Lovric, Miroslav
Lucas, Calin
Lunkenbein, Dieter
Lunney Borden, Lisa
Luo, Lixin
Luthuli, Dexter
Lynch, Grace
Lytle, Pat
MacGregor, Stephen
Maciejewski, Wes
Mackay, Jack
MacKinnon, Richard
Mackrell, Kate
Maclean, Angus
MacNeill, I.E.
MacPherson, Eric
MacRae, LaDonna
Maeers, Vi
Maher, Carolyn
Maheux, Jean-François
Mai Huy, Khoi
Makischuk, Chester
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Malik, MohdAbdul
Mamolo, Ami
Mantyka, Sherry
Manu, Stan
Manuel, Dominic
Manuel, Peter
Marchand, Patricia
Margerm, Pat
Margolinas, Claire
Marshall, Neil A.
Marshman, Bev
Martin, Lyndon
Martin, Vincent
Martinovic, Dragana
Mary, Claudine
Marynowski, Richelle
Mason, John
Mason, Ralph
Mathieu-Soucy, Sarah
Matthews, Asia
Matthews, Wayne
Maurice, Louise
Mavor, Martha
Mayberry, John
Mazerolle, Steve
Mazur, Dorota
McAskill, Bruce
McCarthy, John
McCarthy, Peter
McCaul, Marjorie
McCurdy, Sarah
McDonald, Billie-Dawn
McDonald, Ian
McDonald, Joanne
McDonald, Judith
McDougall, Doug
McFeetors, Janelle
McGarvey, Lynn
McGee, Robert
McLaughlin, David
Mclntosh, Alistair
McLoughlin, John
McNab, Susan
McNicol, Shirley
McPherson, Murray
McQuillan, Anna
Meglis, Irene
Mendoza, Lionel
Mennie, Jim
Menz, Petra
Mercer, Mark
Mestinsek, Jason
Mgombelo, Joyce
Miesen, Jeanne
Millett, Kenneth
Millroy, Wendy
Miranda, Helena
Modafferi, Dominic
Moisan, Patrick
Monagan, Michael
Monette, Georges
Monk, Steve
Moore, Karen
Morin, Marie-Pier
Morris, Dawn
Moschkovich, Judit
Moshe, Lily
Moss, Joan
Mowat, Elizabeth
Muldoon, Martin
Muller, Alice
Muller, Eric
Mura, Roberta
Murray, Erin
Myers, Laura
Nadeau, Déborah
Namukasa, Immaculate
Naud, LaJune
Nelson, Doyal
Nesher, Pearla
Newell, Craig
Nichol, Kathleen
Nicol, Cynthia
Nicolson, Cynthia
Nigam, Nilima
Nikfarjam, Parinaz
Nnadozie, Alfred
Noble, Tanya
Nolan, Kathy
Norrie, Alexander
Northcott, Katherine
Nosrati, Nadia
Noss, Richard
Nouh, Mohamed
Novak, Dan
Núñez, Rafael
Oates, Robert
O'Connor, Barbara
Odunuga, Ayopo
Oesterle, Susan
Oh, Hyunsuk
Oishi, Tony
Oktac, Asuman
Oladosu, Lydia
Oliveira, Izabella
Olson, Alton
Openshaw, Sheila
Orsten, Sandy
Ortigosa, Amaya
Orzech, Grace
Orzech, Morris
Osana, Helena
O'Shea, Tom
Otto, William
Owens, Douglas
Owston, Ronald
Padamadan, Marina
Pagnucco, Lyle
Pallascio, Richard
Palmer, Chris K

Pante, Joao
Paradis, Hélène
Parent, Deborah
Pascuzzo, Tony
Passaro, Valériane
Pateman, Neil
Payerl, Leah
Pearlman, Miriam
Pearson, T.L.
Peled, Irit
Penton, Kelly M.
Percival, Irene
Pfeiffer, Benoite
Philips, Eileen
Pileggi, Steven
Pimm, David
Pinard, Sophie
Pineau, Kathleen
Pirie, Susan
Pitici, Mircea
Pitman, Allan
Poirier, Louise
Poland, John
Pollak, Henry
Polotskaia, Elena
Poole, David
Poscente, Krista
Pothier, Yvonne
Powell, Arthur
Powell, Jayne
Preciado-Babb, Armando Paulino
Proppe, Hal
Proulx, Jérôme
Puchalska, Ewa
Pulford, Stephen
Pyper, Jamie
Quigley, Martyn
Rachar, Brock
Rachlin, Sid
Radakovic, Nenad
Radford, Luis
Radu, Oana
Rahim, Medhat
Ralph, Bill
Raman, Marie
Rankin, Graham
Rapke, Tina
Ratti, Susan
Rawlyk, Teresa
Raymond, lean-Luc
Redekopp, Reynold
Reid, David
Reid, Kris
René de Cotret, Sophie
Reopolos, Bruce
Reshmin, Afsana
Reynolds, Patrick
Rhéaume, Stéphanie
Rhodes, Mary
Richard, Philippe R.
Richer, Danielle
Ricketts, Theresa
Riggs, F.
Rioux, Miranda
Ripley, Ron
Rivard, Alexandre
Robertson, Monica
Robitaille, David
Roditi, Eric
Rogers, Pat
Roodal Persad, Veda
Rosamond, Frances
Rose, Barbara
Rosenfield, Steven
Rosenthal, Helen
Rouleau, Annette
Roulet, Geoff
Rousseau, Christiane
Rousseau, L.A.
Routledge, Joan
Roux, Yvan
Roy, Anne
Roy, Ghislain
Roy, Keith
Royea, Diana
Rozins, Carly
Rubin, Andee
Russell, Gale
Russell, Lise Sara
Ruttenberg-Rozen, Robyn
Saarimaki, Peter
Sabbagh, Bouchra
Sabbaghan, Soroush
Saboya, Mireille
Saint-Aubin, Yvan
Saint-Pierre, Yves
Saldanha, Luis
Saloojee, Riaz
Sangar, Robert
Santos, Manuel
Savard, Annie
Savoie, Mylène
Sawada, Daiyo
Scarfe, Donna
Schattschneider, Doris
Schoenfeld, Alan
Schroeder, Thomas
Scoins, Ronald
Scott, Patrick
Seager, Suzanne
Seaman, Rick
Sebena, Cristina
Sedaghat Jou, Vajiheh (Mina)
Sedghatjou, Mina
Senteni, Alain
Sewell, Alice
SeyedMousavi, SeyedHossein
Shaw, Janice
Shenltzer, Abe
Sherrill, James
Shipulina, Olga
Shiu, Christine
Shockey, Tod
Sibbald, Tim
Siddo, Radcliffe
Sierpinska, Anna
Sigurdson, Conrad
Sigurdson, Sol
Simmt, Elaine
Sinclair, Margaret
Sinclair, Nathalie
Sirotic, Natasa
Skovsmose, Ole
Small, Marian
Smart, Angela
Smith, Constance
Smith, Geoffrey
Snider, Wesley
Solomon, Avery
Sookochoff, Shannon
Soucy, Jérôme
Sousa, Bernardo
Spreen, Marina
Springer, Jean
Squalli, Hassane
Sriraman, Bharath
St.Pierre, Jocelyn
Staal, Ralph
Stanley, Darren
Steen, LynnArthur
Steinke, Tom
Sterenberg, Gladys
Stevens, Brett
Stevens, Christine
Stewart, Christine
Stewart, Pat
Stienstra, Wendy
Stocker, David
Stordy, Mary
Stuart, Susan
Stuckless, Tara
Subedi, Krishna
Sul, Gay
Sullivan, Jillian
Sutherland, Rosamund
Suurtamm, Christine
Swanson, Dalene
Swift, Jim
Szetela, Walter
Tahta, Dick
Tanguay, Denis
Taplin, Anna
Tarala, Erin
Tariq, Nessim
Taylor, Alan
Taylor, Jean
Taylor, Keith
Taylor, Peter
Taylor, Ruth
Taylor, Tara
Tepylo, Diane
Terenta, Tania
Theis, Laurent
Therien, Loic
Thibault, Mathieu
Thom, Jennifer
Thomas, G.H.M.
Thomas, Kevin
Thomas, Robert
Thompson, Tony
Thornton, Steve
Thurston, Hugh
Tingley, Maureen
Tirosh, Dina
Toft, Bjarne
Toor, Amanjot
Touma, Georges
Tourigny, JeanMarc
Towers, Jo
Traore, Kalifa
Traub, Ross
Travis, Sonja
Tremblay, Mélanie
Tremblay, Gervais
Trgalova, Jana
Trlvett, John
Trudel, Josianne
Truman, Jeffrey
Tsamir, Pessia
Turcotte, Olivier
Turgman, Abraham
Vaillancourt, R.
Vakharia, Vanessa
Valiquette, Adelina
Vanbrugghe, Bernard
Velfin, Emilia
Venturini, Marta
Vergnaud, Gerard
Verhille, Charles
Vermette, Sylvain
Vervoort, G.
Vickers, Anne
Vik, Jarmila
Villeneuve, Jean-Philippe
Vincent, Martin
Voigt, Jorg
Voyer, Dominic
Voytsekhovska, Svitlana
Vukets, Barb
Vukets, Barbara

Wagner, David
Walshaw, Margaret
Wamboldt, Kristina
Wan, F.
Wang, Bingjie
Wang, Zhaoyun (Helen)
Wangdi, Kinley
Watson, Anne
Wei, Sun
Weinstein, Pauline
Weinzweig, A.I.
Wells, Kevin
Wells, Kevin J.
Weston, Harley
Westrom, M.
Wheeler, David
White, Harry
Whitehead, Janet
Whiteley, Walter
Whitney, Steve
Wiebe Bouchanan, Christine
Wiest, John
Wiggan, Lorna
Wijeratne, Chanakya
Wijeweera, Ananda
Williams, Edgar
Wiseman, Dawn
Wolf, Valerie
Wolodko, Brenda
Woo, Ronald
Wood, Elizabeth
Wood, Eric
Worth, Joan
Wright, Don
Xiaoheng, Yan
Yan, Joel
Yeshey, Karma
Yoshioka, Alan
Zack, Vicki
Zazkis, Dov
Zazkis, Rina
Zerpa, Carlos
Zevenbergen, Robyn
Zhou, Julie
Zhu, Hongmei
Zilliox, Joseph