UAlberta

CANADIAN MATHEMATICS EDUCATION STUDY GROUP (CMESG)

(1978 c)

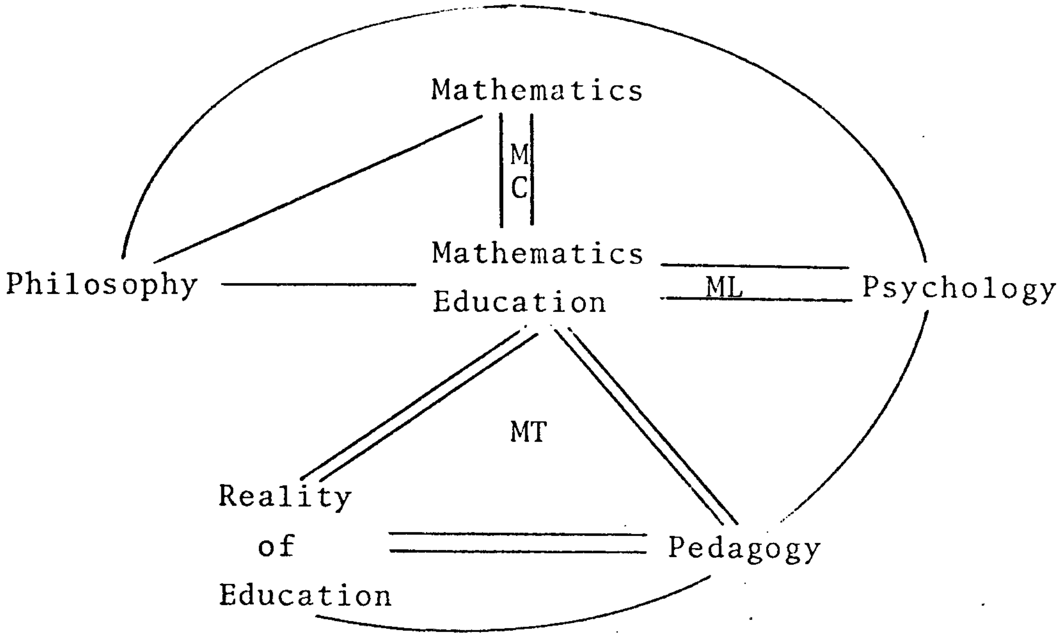
The Kingston Meeting 1978 Research Group

What follows in no way is a chronological report of the sessions of the research group. Our basic style was to interact at length on a few pieces of work in which the group was interested and which seemed to embody important characteristics of what might come to typify mathematics education research in Canada. (Three of these pieces by Noelting, Nelson and Lunkenbein are attached as an appendix to the document.).

1. Didactique de la mathématique

In attempting to define a mathematics education research task it seemed that a characterization of mathematics education was both useful and necessary. Although the French term, la didactique de la mathématique, for the field seems expressive, the English translation has such a limited meaning, the research group chose to continue the use of the more static term mathematics education.

Mathematics education simply put lies in the domain of ideas related to many fields, among them mathematics, philosophy, psychology, pedagogy as well as in relation to the realities of the educational enterprise normally typified by schooling in the broad sense. It is difficult to choose a "language", in the Rising sense, to describe the relationship network but Figure 1 below attempts such a characterization.



MT : Mathematics Teaching
MC : Curriculum Analysis and Development
ML : Mathematics Learning

Figure 1: Mathematic Education: a Characterization.

Mathematics Education is not central in the Ptolemaic sense, but its ideas are to an extent bounded by, although not fully covered by, the other fields mentioned. Three particular activities, represented by bonds which are central to them (MC, MT, ML), are mathematics curriculum, mathematics teaching, and mathematics learning. These general activities are the super sets containing the 3 key notions in Bauersfeld's (1976) triangle matter meant, matter taught, matter learnt.

2. Canadian perspective of a Mathematics Education Research Network

Given this sketch of mathematics education, what is seen to be a reasonable related research enterprise in Canada? Central to this entire enterprise and indeed its goal is the improvement of the Mathematical Learning Experience of the Person. Figure 2 below shows 5 kinds of research and one activity aimed at the above goal.

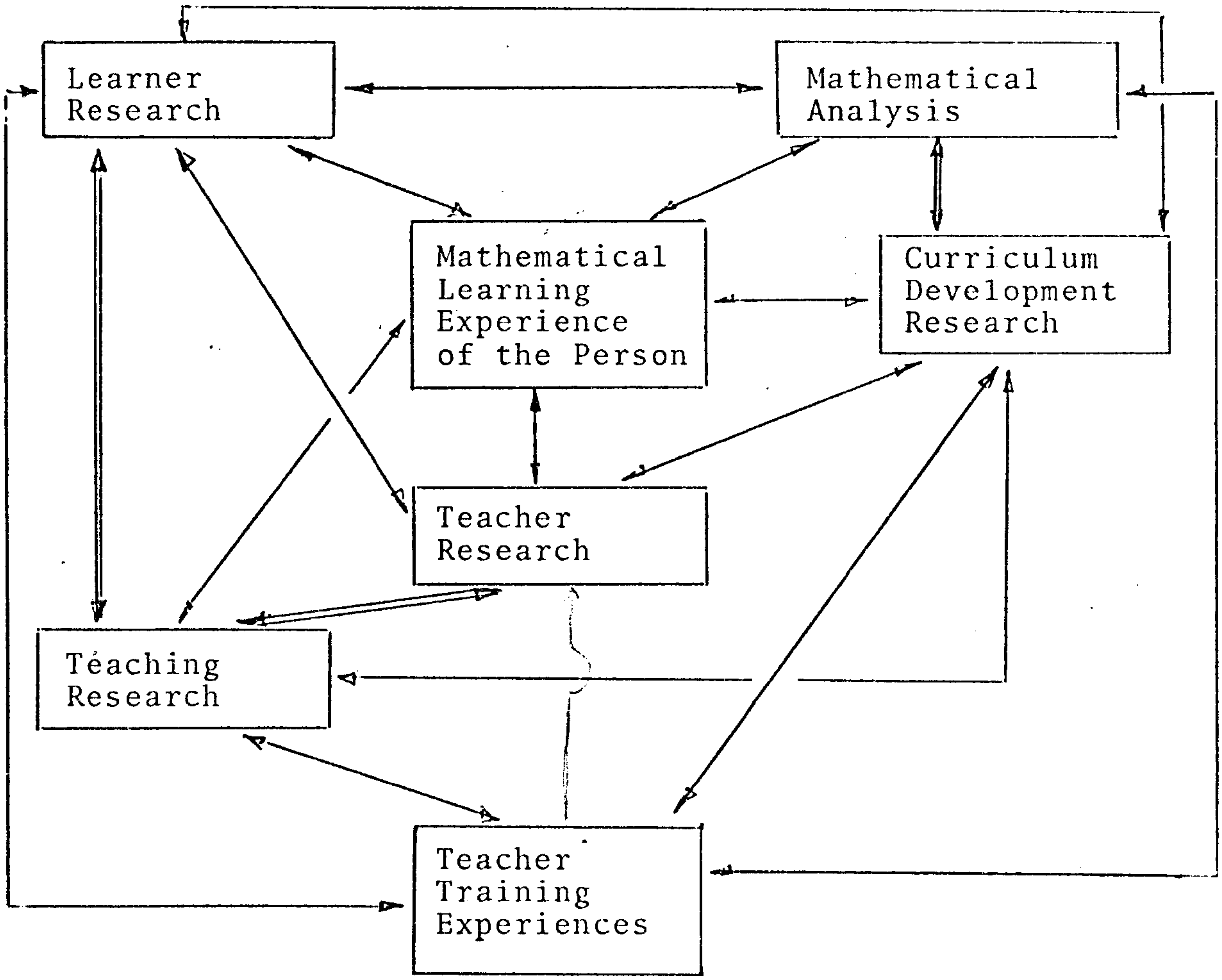


Figure 2: Research Network.

There are many things to note about the network pictured. One important aspect are the connecting arrows. These connections show that relationships are intended. Further, they show that germs of research ideas in any cell can come from any other cell. This means that although there may be strings of research done (eg. mathematical analysis of fractions - learner studies on fraction learning - teaching studies on fraction units and then curriculum development and teacher education), such a sequence is not necessary and certainly not always feasible. Finally, but of key importance, the connections represent important research activities in their own right. (For example, hew are the results of a learner study on fractions useful in the classroom or in teaching research?).

What follows is a brief description of the contents of the boxes. The attachements (Lunkenbein, Nelson, Noelting, Lunkenbein and Kieren) further elaborate on research activities.

a. Teacher Research

Teacher research captures the daily planning and reflecting efforts of teachers as well as longer range planning and action research carried out by many conscientious teachers. This effort is and must remain a central mathematics education research activity.

Teacher education must strengthen the ability of teachers to do this important work. (Although research in teacher education is important it is not detailed further in this discussion).

b. Mathematical Analysis

Research in this area concerns the advanced study of mathematical topics and processes in order to better illuminate them for *educational purposes*. This work has long standing with the work of Dedekind and Felix Klein coming easily to mind. The work of the Mathematization Group and future activity in this area is another example fo this work from a process point of view (as is the work of Polya).

Other current work in this field is exemplified by Lunkenbein (1977) -groupings, Kieren (1976) - rational numbers, Nelson and Kirkpatrick (1975) -problem solving for young children and Weinzweig (1977) -geometry.

c. Learner Research (see the Nelson and Noelting attachments)

Research in this area focuses on the learner doing a mathematical task. The central methodology important here is that of dynamic structural protocol (Easley, 1977). The work is descriptive and normative.

Some of its qualities are its ordering and categorizing mathematical behaviour and thinking, as well as searching for mechanisms which allow the learner to function (eg. counting, partitioning). Such research can generate rich protocol date on video and/or audio tapes and transcripts and these "facts" are useful to the broader community.

d. Teaching Research (see the Lunkenbein attachment)

Teaching research focusses on the careful development of a teaching unit. This unit is then used with children under carefully documented circumstances. This research sees the teaching-learning event as an open cybernetic system and also sees research in this light (Easley, 1977 - model 3). Once again detailed protocols are a central data base. Relations among matter meant, taught and learnt are a theoretical goal and experience packages and teacher information a practical one. This research area is seen as a high need field, with only a very limited amount of current information available.

e. Curriculum Development Research

Curriculum development research is not writing the common commercial textbook. It might be typified by the work of Risings "clever creative person". This work demands designing, implementing and evaluating learning experience in new, imaginative and useful ways and can make use of and stimulate all other kinds of research.

f. Work in the Network

One can make several comments on research needs and parametres with respect to the network. There is a history of work (though not consciously done as such) in the curriculum development area in Canada which can be seen in the work of La Zerte and Sawyer to name but two. There is a high need for more and more reported work in this area today. There is considerable current work in Mathematical Analysis and an amount of promising Learner Research being done in Canada. Perhaps the area of greatest need is well done Teaching Research.

In conducting this research two almost opposite things need be rated. First there is a great need for related and coordinated research. To the extend that researchers can and will be map out related studies there will be quicker and perhaps higher pay offs in better learning experiences. However, it should be noted that these researchs can be parallel. One research effort need not wait for another and in fact it may be unpractical to wait for another.

Relating studies is a high need both in coordinated and parallel studies. Such relationships will be based on the communication not only of results, but of details of procedures, protocols, analyses of these protocals and organized data. These "facts" stimulate useful generalizations and generate research, curriculum and hypotheses in other studies. Easley (1977) has suggested seven lines of enquiry in education. While not neglecting others, the research suggested above will legitimately use systems approaches, language analysis inquiry and dynamic structural protocal methodologies.

Such related studies cannot be done in a mathematical vacuum. The "sausage link" image of Rising seems very useful in applying the results of current research, sponsoring the input of process concerns and supporting and demanding broad rather than narrow research topics. Mathematical ideas which seem particularly fruitful in prospect are rational numbers, transformations of all kinds, algebraic ideas, mathematical languages, aspects of mathematizing as they develop, problem solving and algorithm development.

What might be unique about Canadian mathematics education research? It would be provincial to think that it will occur without consideration of other research efforts in the world. However, the network focus which sponsors a broad definition in the doing and reporting of research (and we hope funding) is unique. The willingness to see the need for extensive controlled protocols and related data sharing will be a key feature. Finally, a broad support from the mathematics community as well as roots in it is important.

3. Current Matters and Details

The mathematics education research community has a responsability to the Canadian Public to provide useful information and guidance on mathematics education matters. In addition, there are demands for information and the opportunity to inform both ourselves and the world mathematics education community of our activities. Attached are various groups efforts to address the following issues.

a) Collection and review of assessement data.

b) Bibliography of Canadian mathematics education research.

c) Reports to various scholarly periodicals.

d) Communication with Council of Ministers.

e) Review of mathematics education research in various Canadian centres.

It is hoped that this report fairly summarizes our activity, presents a practical but visionary scheme of action and suggests attention to short range problems.

Research Group	Thomas Kieren for Dale Burnett Dale Drost Dieter Lunkenbein Shirley McNichol Doyal Nelson and Gerald Noelting

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5. D. Nelson and J. Kirkpatrick (1975), "Problem Solving" in Payne, J.: *Mathematics Learning in Early Childhood, 37th Yearbook*: National Council of Teachers of Mathematics, Reston, VA, 1975.

6. A.I. Weinzweig (1977), "The Erlanger Program and the Child's Conception of Space", in R.A. Lesh: *Recent Research Concerning Development of Spatial and Geometric Concepts*, Columbus, Ohio: ERIC/SMEAC, 1978.

THE KINGSTON MEETING: RESEARCH FOCUS*

The purpose of this paper is to provide a partial skeleton and a stimulus for our work. The ideas represent a point of departure. While looking toward goals of the sessions, they are not an outline of products.

What might be the "products" of our meeting? Three tasks face us. The first is rather concrete. Do we have anything to say to the world mathematics education community about mathematics education research in Canada? If so, how do we present and explain these ideas? The second task in a sense builds on the first. Given the current research enterprises and Canadian conditions, what are profitable avenues to pursue? (I will give some ideas in the research perspectives below and Dieter has several in his paper. Please bring your own notions with you.)

The third task again relates to the first two. Should we organize a community of persons doing mathematics education research in Canada? If so who should we contact? What should we do?

The sessions allotted to us should devote themselves to providing answers to some of the above questions and others. Since some written product is desireable much of our time will probably be spent in very small groups working on particular questions. Lengthy input will probably best be given in writing. We will have a session, probably late on Tuesday, given over to summarizing our progress.

MATHEMATICS LEARNING AND TEACHING IN CANADA

As we try to present a picture of Canadian mathematics education the following status studies come to mind. We are in the process of doing a large number of provincial assessments. What kind of images of achievement do they present? We have much unique and interesting curriculum building in Canada. How can we summarize this? If one analyzed Canadian mathematics curricula, what are its unique features? The Mathematical Sciences report (Coleman et al. 1975) is an important mathematics educational document. What is the 1980 perspective on its findings and recommendations?

There is an interesting study of 11 schools in the United States and their science programs (Easley, 1978). There is much interest in Canada in ethnographic research. What would an in-depth study of urban and non-urban schools across Canada reveal?

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*The second paper attached is by Dieter Lunkenbein. Like this one, it is designed to stimulate, not pre-or proscribe our thinking.

RESEARCH PERSPECTIVE 1: THE LEARNER AND MATHEMATICS

There are two lines of inquiry which seem to be important in Canadian research in this domain. The first studies children and young adults to try to trace the growth of their mathematical ideas, what mechanisms they use in this development and the relationship of various instruction practices to this development.

The second line of research is more philosophical in nature. This work analyzes the content and processes of mathematical structures for their educational implications. The connections between mathematics and cognition and mathematics and learning experience are sought and/or exploited.

RESEARCH PERSPECTIVE 2: PROBLEM SOLVING

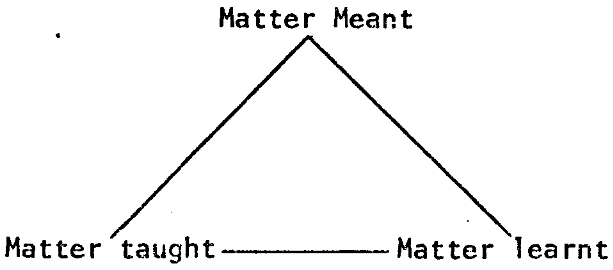
As is a world-wide trend, problem solving is a research focus in Canada. One line of such research uses clinical methodology to study children's reaction to problems at varying age levels. The purpose of this research is to build up a background of information upon which to generate hypotheses about mathematical problem solving instruction.

A second line of research looks at the performance of persons on various problem solving tasks. Here the attempt is to study the heuristics of the person and the effects of particular teaching on performance.

RESEARCH PERSPECTIVE 3: MATHEMATICS INSTRUCTION

A major Canadian research concern has focussed on instructional patterns under which the learner's processes of learning mathematics are developed. Questions of sequence of experiences, appropriate mathematical development, teacher activity as well as a variety of outcomes are studied.

A second thrust in this domain is in its embryonic stage. Because of interest and structures on research on teaching in various provinces, the following "triangle" is being studied. Bauersfeld (1976) posed the following model for discussing mathematics instruction.



Research in this area usually involves researchers and teacher teams. They carefully analyze mathematics and specify instructional acts for themselves. Classroom behaviour of teachers and students is then studied using technologies, clinical methods and various instruments. The strength and importance of relationships in the "triangle" can be explicated.

It should not be thought that these or any limited list of perspectives cover Canadian research. Because such research should have applicational goals, much of it is topical (for example, various studies on calculator use). This research needs to continue but its impact may be greater in some coordinated, cooperative or at least cross-informed scheme. The existence of such schemes may well be the reason for an organization of researchers in Canada.

Thomas Kieren
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Research in Mathematics Education: Suggestions for discussion.

I Mathematics Education and Research in Mathematics Education

It is possible, that we have different conceptions of what mathematics education (didactique de la mathématique) is or should be and how research in mathematics education is to be carried out. It should be interesting to have a brief discussion on such a general topic in order to outline globally the field of mathematics education and to indicate roughly some domains of research in mathematics education. Some reflexions on methods used with respect to goals persued would have to be included in such a discussion if we wanted to establish to what degree mathematics education is an autonomous science or what field of science it is a part of.

II Documentation of current research (or research interest) in mathematics education (in Canada)

It seems to me, that there is a lack of information on research carried out across Canada, on research interests manifested in different institutions as well as on competencies (in research in mathematics education) in the different Canadian Universities. It should be very useful to establish a short documentation of research being carried out at what place, by whom, in what area and to what advancement. If we want to get an overview of the development and tendencies of research in mathematics education in Canada and if we want to encourage collaboration across Canada, such a documentation (to be revised periodically) should be of crucial importance.

III Goals of meetings of Canadian researchers in mathematics education

I imagine, that the main goals of such meetings are

- information on research in mathematics education;
- clarification of research domains and goals through discussions amongst researchers of similar interests; and
- coordination of related projects and collaboration of Canadian researchers.

The means by which such goals (or others) are to be achieved ought to be outlined or at least discussed at the Kingston meeting in June. Is it possible and realistic to establish a "Canadian Association of Researchers in Mathematics Education", given all the provincial, American and international associations we are already members of? Is there a Canadian perspective of or approach to mathematics education which would justify a Canadian Association? Should such an association be autonomous or affiliated with already existing Canadian associations (CMC)? Could we achieve those goals by simply joining forces with interest groups like the "Georgia Center for the Study of Learning and Teaching of Mathematics"? Is it thinkable that more or less informal yearly meetings (like the one we are attending this year), with reports and discussions on particular research projects across Canada, are (for the time being) sufficient means to achieve those given goals?

As I write those notes, I can't help thinking of the danger that we might lose much time discussing ways of organising a Canadian Association instead of doing some constructive work in the field of research in mathematics education while we are together. So I hope, that organisational questions and problems (as interesting and necessary they may be) won't prevail over actual research questions and problems we all have and which we would like to communicate to and to discuss with our collegues.

Dieter Lunkenbein

Research in Mathematics Education - A Teacher Trainer's Approach*

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The Training of Teachers (of Mathematics) and Research

The domain of competency of the University Teacher who is responsible for the training of Mathematics Educators or Mathematics Teachers is what is usually called Mathematics Education (didactique de la mathématique) or Mathematics Teaching (enseignement de la mathématique). This field of research activity is rather young and its description will have different nuances according to the main preoccupations of the researcher. From the point of view of the Teacher Trainer, Mathematics Education describes a field of study which includes the domain of Mathematics Teaching, but which includes still other contributions which are neutral towards the teaching of mathematics. Amongst these latter contributions, one finds those that could be classified in the epistemology of ideas, the growth (genèse) of mathematical notions in relation to the mental development of the learner and others. These contributions could certainly find a place (at least a peripheral one) in one or the other of the resource sciences (like Mathematics, Pedagogy, Psychology, Sociology, etc.) and they share with these resource sciences a descriptive and normative character with respect to Mathematics Teaching.

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*) Prepared as a contribution to the discussions of the research group: Mathematics Education Study Group, Kingston, 1978.

The domain of Mathematics Teaching distinguishes itself by its prescriptive and constructive character: its contributions study systematically the practice of the teaching of Mathematics from the point of view of the teacher in order to develop an optimal planification and efficient instructional material. WITTMANN (1975) compares this part of Mathematics Education with the engineering sciences, particularly with operations research, where the system of the teaching of Mathematics is analyzed and studied systematically.

We tried to picture some relations between Mathematics Education and its resource sciences on the one hand and between Mathematics Education and the practice of the teaching of Mathematics on the other in a schema like this:

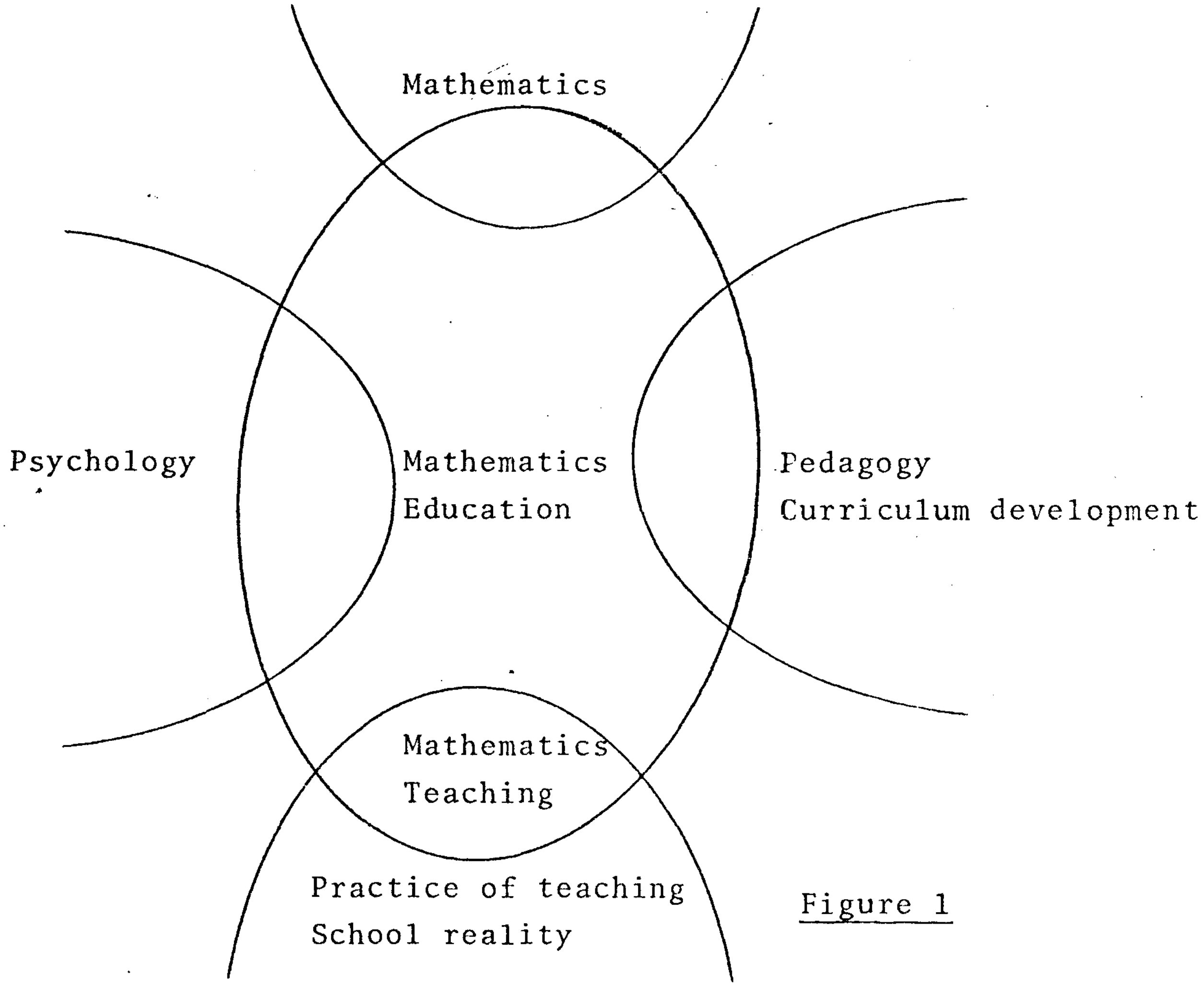

Figure 1

The main purpose of this picture is to communicate the conception of the domain of Mathematics Teaching as the part of Mathematics Education which has a special concern for school reality, without excluding from Mathematics Education those contributions that do not take in account school reality explicitely.

Another particularly important aspect of the domain of Mathematics Teaching is its multidisciplinarity. In effect, it has to take in account a multitude of theoretical reflections (concerning parts of Mathematics, Psychology, Sociology, Educational Philosophy, etc.), to consider their relevance for a special purpose at a given time and to integrate them in such a manner, that they will form a well balanced, continuous and realistic teaching unit. In other parts of Mathematics Education, one can consider just some special aspects of the whole domain, but in Mathematics Teaching one has to face the global and synthetic character of the process of teaching (and learning) in a normal classroom setting.

This domain of Mathematics Teaching seems to be the natural field of study for the trainer of teachers of Mathematics. Research in this field has not yet been well established but is of greatest need since it provides the teacher with suggestions of applications of theoretical or particular findings to normal classroom situations.

Research in Mathematics Teaching

The careful development of teaching units, its applications and evaluations and, subsequently, its modifications are the characteristics of this kind of research. It seems to involve a process of systematic refinement and adjustment, which leads gradually to teaching units or learning sequences, the foundations of which are more and more explicit and the effects of which are better and better known or predictable. Such a process has been described by LUNKENBEIN (1977) as a working model which, certainly, will have to be detailed and modified according to relevant experience.

First experiences confirm the complexity of the enterprise resulting from the simultaneous consideration of a great variety of factors involved. At the same time they indicate particular fields of investigations necessary for the satisfactory solution of partial problems. Amongst others, we need here:

a. Investigations of mathematical nature: how can particular mathematical topics be "structured" in order to be accessible to a given group of learners (without blocking further development at a later stage)?

b. Investigations of psychological nature: what do results of learning experiments mean for the classroom teaching situation? Also, what is the relevance of the notion of grouping in the context of mathematics teaching?

c. Investigations of evaluation methods: how does one evaluate the efficiency of teaching units according to the aims and processes involved.

Series of teaching units are then to be organized and related in order to be integrated into and organic program. Thus, considerations of curricular nature must not be neglected in this kind of research.

By its global and synthetic character, research in Mathematics Teachings seems to be, if not a central, at least an essential part of research in Mathematics Education. It applies findings of theoretical and particular kinds to the classroom situation and, in turn, motivates and stimulates investigations of more specific character.

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A HOPEFUL TREND IN
PROBLEM SOLVING RESEARCH

by

Doyal Nelson

The failure of mathematics instruction to develop problem solving skills is well enough documented but the form it often takes can perhaps best be illustrated by describing an incident that happened a few years ago. It concerns Bruce, a neighbor boy, who at the time was a ninth grade student. What Bruce lacked in general mathematical ability he more than made up for in his enthusiasm for engaging in problems of a practical nature. Whenever I entered my combination garage and workshop some sixth sense informed Bruce that an opportunity for exercising his favorite problem solving ability was in the making. He always appeared in less than five minutes.

On this particular day I was completing a workbench and Bruce was my willing helper. I had put aside a good piece of 1 by 4 lumber which I intended to cut into three strips of equal width to trim along the front of the bench. The bench was 22 feet long and the 8 foot board would provide just enough for the trim and the fitting. When Bruce understood what was to be done he offered to mark the board for sawing.

His first step was to divide 4 (the width of the board in inches, so he thought) by three. He got $\frac{4}{3}$ then $1\frac{1}{3}$. The trouble was that the units in his calculations did not jibe with any of the units on the square he was using. He finally decided to estimate $1\frac{1}{3}$ inches and did indeed measure quite precisely but the last mark was obviously much closer to the edge of the board than it should be. Anyone familiar with lumber knows that the width of boards is usually the width before planing. Milling of a 4 inch board reduces its width to about $3\frac{5}{8}$ inches. That seemed to explain to Bruce's satisfaction why the second mark was always closer to the edge than it should be.

When I suggested he measure the board and found it to be only $3\frac{5}{8}$ inches wide he looked a bit confused but went on with a revised calculation. He divided $3\frac{5}{8}$ (the measure in inches he had obtained) by 3 and though the computation gave him some trouble he finally got $1\frac{5}{24}$. Twenty-fourths were not marked on the square of course and he didn't even attempt to estimate. He just gave up.

Carpenters have a neat way of solving a problem like this. To mark the board into three pieces they would take a whole number greater than the width of the board to be divided but also a number divisible by 3. In the case of a 4 inch board 6 would be a good choice. The square is then laid on the board obliquely as shown in Figure 1.

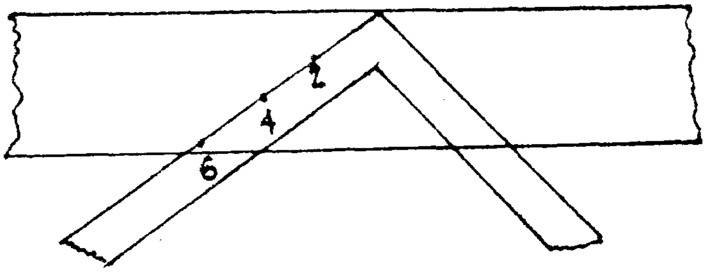

FIGURE 1

Note that the positioning of the square puts the vertex of the right angle on one edge of the board and the 6 on the other. Then marks are made at 2 and 4. The square is moved along the board and the process repeated. The marks are joined as shown in Figure 2.

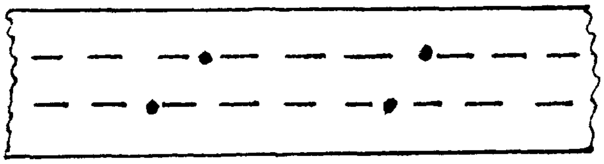

FIGURE 2

Sawing along the lines will provide three strips of equal width (providing marking and sawing is done with care).

The underlying elementary geometric principle is usually taught in connection with the study of similar triangles but is sometimes introduced earlier in connection with parallel lines and transversals. Bruce had recently made a rather thorough study of similar triangles in his ninth grade mathematics course so I preferred to relate it to that. The more I tried the more obvious it became that Bruce was not buying that similar triangle thing. His question was: How could similar triangles be involved when you didn't even have to draw a triangle? After a few more half-hearted attempts it was my turn to give up; which I did. What procedures would have to be developed in mathematics instruction to make the parallel structure of physical situations and the related mathematical notions more apparent to the learner? What research methods and procedures show the best potential for answering these questions?

The object of this paper is to suggest a general procedure for doing research in problem solving and to describe some studies in which an attempt was (or is being) made to apply the procedure.

The paper does not contain a review of the literature on problem solving as it relates to school mathematics. The position taken here is that we need to know a great deal more about how children learn to structure their world and how their real experiences interact in instruction to result in learning basic mathematical structures. It is assumed that investigations involving problems of real or practical significance to children is the best way to begin. The discussion will be mainly concerned with elementary school age children and younger.

If we look in at any class of elementary school children there would be general agreement that the basic reason for giving them instruction in mathematics would be to help them solve problems which they are likely to meet in their daily living. Yet the methods usually adopted in teaching mathematics at this level tends to foster the growth of a skein of mathematical ideas, process and skills which seem to have little or no connection with the real world experiences or problems faced by the child. Analysis of the results of the National Assessment of Educational Progress in mathematics (M. T., October 1975) revealed that while elementary school children had developed considerable computational skill yet they lacked even such fundamental problem solving process as "checking the correctness or reasonableness of a result, or making an estimate..."

In the very early stages of mathematics learning there is general recognition that all basic mathematical ideas have their source in real world experiences so methods of teaching rely to a greater or lesser extent on real problems or at least on manipulation of concrete materials. After the first year or two of elementary school mathematics, teaching tends less and less to be concerned with real world problems and their solutions and more and more with computation and with the symbolic or abstract aspect of mathematics.

Let me hasten to point out this is certainly not an indictment of teachers presently offering instruction in elementary school mathematics. In the first place research has very little to say to them about the precise role problem applications have in mathematics learning nor how mathematical structures, once attained by the child, find easy applications in the solution of everyday problems. In the second place some so-called experts in early mathematics learning recommend that the processes and skills of mathematics should be learned and that applications can and should be found later. Their argument hinges around the real or imagined difficulty in finding any but trivial applications in the early stages. Finally, collections of applications and application ideas available to teachers are apt to include very few which would be appropriate at the elementary school level.

In the face of all these difficulties let us take the case of a hypothetical teacher who decides to provide a problem solving base of a practical nature in teaching mathematical processes. As an example let us assume that the lesson is an attempt to construct a symbolic form for simple division using real problems. To keep it simple the teacher decides to be concerned only with measurement division; that is the form of division which specifies the number in each group and requires that the number of groups be found.

The teacher carefully constructs a layout which, let us say, consists of seven joined enclosures to represent stalls and fifteen horses which are to be placed in the stalls, three per stall. A protocol is then carefully worked out which specifies exactly how the problem is to be presented. This protocol, in short, tells the child how many horses are to occupy each stall and also asks the question, "In how many stalls will there be three horses?"

Most children would have little difficulty in placing the horses as required in the stalls, counting the occupied stalls and supplying the answer. One should note that the manipulation can be done as easily whether the child knows the total number of horses to begin with or not. In fact, there is no reason apparent to the child for having the information. Yet if this is to be related to the symbolic or computational form, most teachers I know (and I have no alternate recommendation to give them) would write down the number fifteen and proceed as follows:

15

Then $\overline{)15}$ to indicate the division process.

Then $3\overline{)15}$ to indicate the number in each stall.

Then $\begin{array}{r} 5 \\ 3\overline{)15} \end{array}$ to indicate that there will be horses in five of the stalls.

How can the process expressed in symbols be related to the actual manipulation or problem solution when the very first number to be written down in symbols is not needed at all to do the real problem? No research I know provides any definitive help to the teachers in this situation.

Suppose the teacher had decided instead of horses and stalls to set up a problem in which items of cargo had to be loaded onto a truck. The items might be fifteen blocks and the problem to load a toy truck from a loader which carries three blocks at a time and to find how many loads the loader will have to take to get all the blocks onto the truck. Changing the substance of the situation has not made the need to know the total of number of blocks any more apparent. The real problem can be solved as well whether the total number to begin with is known or not.

But this particular situation has embedded in it a further peculiarity as well. There is no guarantee once the truck has been loaded that it would be apparent how many loads the loader had taken. The child would have had to know in advance that some means would have to be found which would preserve the integrity of each group so they could be counted in the end or some mental record would have to be kept of the number of loads. There is no such requirement in the horse and stall example.

No doubt children would as likely encounter one of these situations in reality as the other. But no research exists to indicate which might be more efficient in helping the child understand the process of division or seeing sense in its symbolic representation. One might hypothesize that there is enough difference in the two solutions that learning the symbolic form would actually interfere with solving the problem with objects and vice versa.

Learning the process of division is further complicated because sometimes remainders are involved and the child has to exercise keen judgement in order to give a sensible result. Then there is the partitive form of division whose symbolic representation is not altered but in which the requirement is to find the number in each equal group when the number of groups is known. Here again the real world problems could be solved readily by manipulation without ever knowing the total number of objects while the symbolic form requires it as a start. Research has revealed that few children up to the age of eight or nine have an entirely systematic way of partitioning objects (Bourgeois and Nelson) but there is no research that directs the teacher in the best way to teach partitioning so it becomes a natural, systematic process for the child, readily related to a symbolic representation.

This example serves to illustrate some of the complications which must be faced when a decision is made to use practical problems and their solutions as the basis for even a fundamental mathematical operation. If those outlined were the only complications encountered the course of research could be fairly specifically mapped out. Some other complications are to be discussed later but are outlined in Bourgeois and Nelson (1977).

In spite of the difficulties associated with using real problems to help build up mathematics few would recommend trying to do it without the use of such problems.

In fact if there had to be a choice the child would probably be better served learning the process in the real situation than to have to learn some vaguely understood symbolic form. Future research may reveal, in fact, that children need to have a very clear idea of how to solve various problems in real situations with real objects over an extended period of time before any attempt at all is made to render such solutions in symbolic form. That research would have to depend on careful observations of childish responses and extensive descriptions of variations in procedures.

If a basic aim of elementary school instruction in mathematics is to assist children in solving various practical problems that occur in their daily lives; if practical problems and experiences, at least in the early stages, provide the actual basis for mathematical learning and understanding and if we are to understand the complex interactions which occur on the interface of developing mathematical structures and related experiences in the real world then it is clear that a great deal of emphasis must be placed on problem solving research. Exhortation, testimonial and speculation must be replaced by empirical data which will provide more definitive guidance for planning learning experiences in elementary school mathematics.

There is considerable difference of opinion about what constitutes a problem and simply calling it a practical problem won't do much to clarify the meaning or resolve the differences. If it is a practical problem the implication is that it has to make sense and be useful to the person who is expected to solve it. If it is ultimately to be a part of mathematics instruction or mathematics learning it would certainly have to have some readily identifiable mathematics significanace. That significance obviously would have to be related to the maturity, experience and interests of those expected to solve the problem.

Anyone preparing to do research around practical problems would be advised to adopt something like these or other criteria or guidelines in creating, selecting and constructing problem situations to be used in such research. At any rate this appears to be a more sensible approach than to try to work out some kind of definition of practical problems or of problem solving. Sets of criteria for problems and problem solving would probably differ from investigator to investigator and would no doubt reflect more precisely the individual's own concept of what a problem is. Provided criteria or guidelines for construction of problem situations and related problems are clearly stated and faithfully applied then the effects of any particular criterion could readily be tested as part of the analyses of the responses children make to the problems. Such evaluation would at least provide the means of constant refinement of the criteria. The important point to be made here is that differences in criteria or guidelines used by different investigators are of little concern providing sets of criteria are carefully formulated and provision made in the research to evaluate them.

If practical problems, constructed according to identifiable guidelines or criteria, are to form the basis for empirical research there are a number of further conditions which have to be met. In the first place the problem situation and associated physical apparatus would have to be carefully constructed and their specifications clearly described. The examples in the previous section indicate how two practical situations, both seemingly involving identical aspects of measurement division and which could meet similar construction criteria, might stimulate two quite different sets of responses on the part of children to the process of measurement division.

As long as it is clear under which set of conditions empirical data are to be collected no great difficulty in their interpretation is likely to occur. Indeed, subtle changes in conditions and their effects on children's responses are precisely what this type of research is best designed to clarify. Distortions in interpretation would be almost certain to occur in the absence of precise information on the nature of these two situations.

Related to this requirement and of equal importance is the necessity of working out very precise protocols for presenting the problems to children. The context in which a problem is presented may have a profound influence on the way children respond to it. Whether the problem is presented to a group of children or to an individual child has to be clearly stated. Differences in language used to describe the problem and suggestions or instructions about the form responses will take are likely to be most critical. Other considerations are whether one problem precedes another or whether specific instruction was given sometime prior to the child's response. Whatever verbalizations or directions are given to the child should be accurately reported along with an accurate report of the child's response.

Once such protocols are established there should be no substantial deviation from them. Deviations from protocol from child to child will make any interpretation of responses extremely difficult if not altogether impossible. When working with pre-school children, especially three and four year olds, investigators will normally be tempted to modify procedures if a child shows any fear or reluctance to respond. Such temptations should be overcome. It is more appropriate to use some clearly described means as part of the protocol to alleviate any fear or reticence prior to presenting the problem.

The ultimate purpose of problem solving research at this level is to obtain information which can be used to improve instruction in mathematics. The first step, however, is far removed from this ultimate goal. A great deal of information about how children respond to specific problems selected on the basis of specific criteria must be amassed before they have any application in day to day instruction. The state of the art at this time would suggest that a clinical situation in which children respond individually should dominate the methodology. If time has to be spent developing this methodology it should be considered part of the task we are facing. It has to be admitted that clinical research has not been developed to a great extent in North American mathematics education and it is important that part of reports of clinical studies should be devoted to a discussion of the specific methodology used.

In the absence of detailed information about how children will respond to practical, concrete problems in a clinical situation it is difficult to set up in advance an adequate response framework or schedule. To overcome the difficulty a number of investigators have used audio or video tape which does provide a reliable and faithful means of collecting such data. No information need be lost and analysis can proceed as time permits using any number of schemes which show some promise of providing insight into the meaning of childish responses to the problem. Each scheme can be applied simply by running the tape through again.

Introducing (into the clinical situation) the technological devices necessary for such recording, however, produces its own peculiar problems. The situation which contains two or three video cameras with recorders along with the technical personnel required to operate them can be disturbing to children and may substantially alter their responses from those which might be obtained in a less busy atmosphere. Measures taken to simplify the technical set-up and to ease the situation are as likely as not to result in an inferior record or incomplete data. In spite of these disadvantages some form of recording all of the responses of children seems to be mandatory at least in the early stages of such exploratory research.

One difficulty that cannot be overemphasized is the expense connected with collecting and analysing audio and video records of children's responses. Even the most well endowed investigator will become discouraged when analysing taped data. The reason for taping in the first place is that no adequate encoding scheme now exists which will select out all the important responses as they occur. But the flexibility and richness of taped data is at once a serious source of concern. A scheme or set of schemes, as often as not, have to emerge from the data themselves which involves viewing taped segments over and over again.

Not only is it difficult to devise and select an encoding scheme which will permit convenient analysis but the sheer logistics of finding wanted tape segments in reel after reel of similar segments can be overwhelming. There are few investigators who have the tolerance required to encode taped material or to devise encoding schemes for more than two or three hours at a time with the level of alertness required by the task. Where research has to be conducted in the face of budget limitations it is important that the investigator make an accurate assessment of the time that analysis will take and to adjust the amount of data collected accordingly. It is better to collect only those data which can be analysed with available resources than to collect large masses of data and hope that funds can be found eventually to analyse them. It is my guess that there are many hours of carefully collected taped data lying around right now waiting for analysis which will never be done because funds will never be available for the analysis. It is also my guess that funding agencies are turning down potentially good research involving taped data because their advisors or referees are ignorant of the power (and expense) of using the medium.

What has been said here about methodology in problem solving research may suggest a rather narrow view of the scope of such research. It is admittedly a narrow view but one which is taken to emphasize the need to make accurate observations of children's responses and behaviors when confronted with real, concrete, significant problems. We need to get a clearer picture of how children construct their own reality, what problem solving abilities they possess at various levels and how these abilities develop with age and experience, what part spontaneous language plays in their constructions, how they interact with various visual and verbal stimuli in solving problems, and how their real experiences are used to build the mental structures we call mathematics. These and related questions have to be answerd before we can confidently address the intricate instructional and curricular questions which is our ultimate task.

Up to this point research in all aspects of mathematics education has emphasized experimentation and the need to find a theory to account for learning phenomena.

It has essentially skipped the phase which is purely discipline and which depends on the ability of the investigator to make careful observations of children in learning situations. In their attempts to apply the methods of the physical sciences researchers forget that development in disciplines such as physics, chemistry and biology were preceeded by years of simple observation and description. If we are going to make significant progress in research involving practical problems it is essential that the phase which is charaterized by observation and description precede serious attempts to experiment or to develop a theory.

Some Outcomes of Research Involving Real or Practical Problems

This section will be devoted to describing certain aspects of two lines of research involving practical problems and currently being conducted at the University of Alberta. The main procedures and methodology of both emphasize observation and description. While they do not define the scope of such research they do provide samples of a kind of methodology that shows process.

The first project was designed by Nelson and Sawada and is concerned with responses of children in the age range three to nine years as they are presented with a selection of six practical problems paired with six others. The pairing of problems was arranged so that the physical situations in which each pair occur were dissimilar while the mathematical structure on which each pair was based was the same. Problems involved the following mathematical processes or notions: division (measurement and partitive); locating positions in two and three dimensional space; sequences, geometric constructions, predicting movement in a plane and factoring. Criteria for the construction of these problems appears elsewhere. (Nelson & Sawada, 1975)

Sampling of responses was arranged to account for development of responses across the age range with longitudinal verification after one year. Sampling procedures also took into account the effect of order of presentation of a problem and its equivalent. ~~Responses~~ of the children at each age level as they interacted with six problems at each the cross sectional and longitudinal aspects of the study were recorded on video tape. For the cross sectional sampling there were ten children who did the problem in each case and five of them also did its equivalent. In the longitudinal aspect there was some attrition but the plan was for ten children to do the equivalent and five to do the related problems./

[illegible]

Data were subsequently analysed by viewing the taped segments of each child who responded to a particular problem or its equivalent. No pre-arranged coding scheme was designed in advance but schemes were allowed to grow out of the observations. Discipline accounts of the results have appeared in several publications (Nelson; Nelson and Sawada; Bourgeois and Nelson; Nelson and Kieren). Analyses for some of the problems have not yet been completed.

The other project to be considered was designed by Kieren and was preceded by a careful analysis of possible interpretations of the rational number constructs. Kieren identified seven interpretations for fractional and rational numbers:

- fractions
- decimals
- ordered pairs (equivalent classes)
- measures
- quotients
- operators
- ratios

The **cognitive and instructional structures re**quired for building a rational number construct as suggested by Kieren are: part-whole relationships, ratios, quotients, measures and operators. For each of these can be devised a set of tasks or problems appropriate for children learning the construct.

The same criteria for constructing problems used by Nelson and Sawada were used by Kieren to gather information on the child's notion of rational numbers as operators. The operator notion is based on mechanisms which map a set (or region) multiplicatively onto another set. (A "3 for 4" operator would map a domain element 16 onto a range element 12 while a $\frac{3}{4}$ operator maps a region onto a similar region reduced in size.)

The practical problem consisted of a card stacking machine whose input could be compared with an output to define the nature of the operator. Observations of forty-five children in the age range 8 years 11 months to 14 years 7 months and descriptions of their responses to these kinds of situations have been described. (Kieren and Nelson)

Here again no coding scheme was developed in advance but grew out of the observations of children as they solved problems in the situations as outlined. However, [illegible]
([illegible])

With no hypotheses to reject what is the form that reports of such observations take? Can the results be used as a basis for more formal experimental research? Are new insights into childish behavior possible with these methods? To help answer these and related questions some results which have so far been obtained will be reported here.

In both the studies outlined problems were embedded in physical layouts to which children could respond in a physical way. Symbolic or written responses were not essential or required. The protocols developed for the Nelson and Sawada study provided for support in case children did not respond but the precautions were found to be unnecessary and were not included in the longitudinal sampling. In most cases children were not only prepared but were eager to respond in a physical way. This phenomena was no more apparent in younger than in older children even though the older ones may have been able to respond symbolically. Anyone doing research involving real, practical problems need have no fear of any reluctance on the part of children -- even the very young -- to respond eagerly (for the most part) in a readily interpretable manner.

~~Although neither of the studies were designed primarily to evoke verbal responses,~~ Spontaneous language used by the children was of considerable interest. There was, for example, in the Nelson and Sawada study a noticeable change in language function across the age range.

Whether language was being used to help solve the problem or whether the problem proved to be a useful source of language generation could not be determined but younger children used language extensively to monitor their actions. In fact with three and four year olds particularly, the language often defined some problem other than the intended one. Older children on the other hand, used language to pose questions in order to clarify more precisely what problem they were expected to solve. Five children older than five altered the problem to suit themselves.

Provoked language, as in the exploratory study reported by Kieren and Nelson (1978) can provide rather clear insights into children's modes of thought in dealing with problems. When asked to describe how they thought the fraction machine functioned, for example, it was clear that many children thought subtractively and not multiplicatively. For example, in looking at the $\frac{2}{3}$ operator such children would say it's subtracting 4 (12 8), it's subtracting 10 (30 20) and thus never focussed on the constant multiplier involved.

Interpretation of language function may have been improved in some instances if an expert in the language development of children had been part of the investigating teams. Those proposing to do research in the problem solving area would do well to recruit such a person in the early planning stages.

Kieren is in the process of exploring in greater depth the role of the operator in the development of the rational number construct in children. The main thrust of this research will be to investigate more thoroughly the tendency of children to think subtractively rather than multiplicatively when working with operators and [illegible] ~~the important role one half plays in the early stages~~. It should be noted that in the exploratory study nearly all children mastered the $\frac{1}{2}$ task but when faced with the $\frac{3}{4}$ task on the machine the vast majority of children under 12 would give 12 as an output for an input of 24. Questioning revealed that they knew it was not a $\frac{1}{2}$ machine but when confused would respond as if it were. The global role of $\frac{1}{2}$ in early thinking obviously needs careful investigation.

A third function of Kieren's exploration will be to trace the developing ability of children to move from functioning with unit operators to functioning with all forms of operators. The method will be clinical and will emphasize careful observations and descriptions of how children respond to protocol problems involving operators.

In the Nelson and Sawada study there were twelve problem situations (or more precisely, six pairs of problem situations). In general, these took the form of layouts or materials which children could manipulate in order to solve associated problems. As pointed out before there was no reluctance on the part of children to respond but there were a number of other observations that applied to more than one problem situation. For example, children were often distracted from making appropriate responses to problems because of various spatial and physical characteristics of the situation. [illegible] tendency of children to focus on the color [illegible] prevented them from making [illegible] in the measurement division p[illegible]

[illegible] division problem it was nece[illegible] [illegible] cars in front of simulated [illegible] be the same number of cars [illegible] each house. Some children would not park all the cars because that would be "too many" cars for each house. Or they refused to park cars on the "grass" near the house. A three year old was so interested in the make and model of plastic cars used in the parking lot problem (locating positions in two dimensions) that he forgot the rules given for parking.

The vulnerability of children to such distractions is not new. It is evident in the now conservation behavior of children. Children who cannot conserve (whether it be number, length, area, volume or whatever else) cannot do so because of some irrelevant or distracting element in the situation to which these children respond. The point is that if we are going to provide practical problems to children we have to be able to predict with some confidence (as in the conservation phenomena) what may be distracting in the problem situation and thus interfere with the child's ability to cope with the problem.

The information from the Nelson and Sawada study indicating the tendency of children to be distracted or, more precisely, to respond to distracting elements of the situation has led to more searching clinical study with this phenomena (Bana and Nelson, 1977; Bana and Nelson, 1978). Although the work is far from complete these studies have revealed some interesting results. ✕/ Some [illegible] [illegible] ~~For example, children seem to have a greater tendency~~ to be distracted if when the distracting element is brought into play it provides a plausible alternative problem for the child to solve. There is also some evidence to support the contention that the way a problem is posed can determine whether a child will be distracted or not. Whether these two observations can be verified and if so whether they are in fact, part of the same difficulty depends on further carefully designed clinical research with appropriate problem settings.

Whenever children in the Nelson and Sawada study were required to predict an outcome there was a distinct reluctance on the part of many to attempt to do so. In fact, nearly half of the children across the age range refused to predict without considerable urging. The proportion of those who were reluctant to predict showed little change from three to eight years. The same phenomena shows up in the Kieren study as these older children also appeared more happy to say nothing than to be wrong. It is not clear at this point what the true dimensions of this phenomena are. If it were school induced it is not likely it would manifest itself so strongly in pre-school children.

There are some specific outcomes which warrant mention here as examples of the kind of information research involving real problems is likely to reveal.

It is generally conceded that partitive division is a more difficult process for young children than measurement division. In any case, making groups of a specific size can be more easily systematized than partitioning a given number of objects into smaller groups. Children in this study had no completely systematic way of partitioning and generally found these partitioning problems more difficult except when in measurement division no provision was made in the problem to preserve the integrity of the equal groups. Thus when animals were placed in cages, children had no difficulty saying how many cages were needed. But when the ferry boat had finished hauling cars across three at a time, children had trouble remembering how many trips the ferry took. (Bourgeouis and Nelson, 1977) This example should serve to illustrate the necessity of making careful and detailed descriptions of the real problem, exactly as it is presented to the child. Some apparently small differences in situations can lead to profound differences in children's responses to them. — The [illegible] [illegible] — [illegible]. (See [illegible]

Even very young children were successful in constructing complicated three dimensional figures when provided with a number of two dimensional elements (Nelson and Kieren, 1977). Although they appeared in many cases to be solutions strictly on the physical level providing little or no mathematics-logical experience such problems seemed to be appropriate for the whole age range three to nine. What effect such early experience has on the subsequent development of spatial abilities in children is yet to be determined. Their skill in making structures and their eagerness to do so suggest that the effect on these abilities may be considerable.

A pair of problems were designed to determine if any children in the age range three to nine related numbers and their factors. One problem was called the factor platform. This was an upright structure slightly sloping backward with thirteen slots and blocks which could be piled in the slots. Children were presented first with twelve blocks in four of the slots arranged so there was not the same number of blocks in any two slots. They were asked if the blocks could be rearranged in the four slots so there were the same number in each slot. This proved to be easy to verify for almost all the children (some three year olds piled all the blocks in one slot) but few if any thought of twelve blocks being arranged in four groups of three. When one block was removed so that there were now a total of eleven and they were distributed in four slots again so no two slots contained the same number of blocks, most children persisted in trying to arrange them in equal piles. Their failure to do so did not in any case, suggest to them a difference in factorability of eleven and twelve. This was expected to provide only physical experience for the three, four, five and six year olds but it was expected that at least some of the older ones would suspect what was going on. Experience with the factor board which had spaces for blocks to fit in twos, threes, and fours did not make it any easier for children to see in advance that eleven blocks could not be made to fit exactly in any of them. The inappropriateness of this set of problems to reveal anything of importance is in sharp contrast to the other problems included in the study. Nine year olds in the longitudinal sample who had been in school as much as four full years could have been expected to respond more appropriately to these situations if any instruction at all had been provided in school to partition the set of counting numbers. Either that or the notion of partitioning according to factorability of numbers is too complicated for eight and nine year olds to cope with. The examples given above serve to illustrate the kind of outcomes that can be expected in clinical methods involving real problems. While most of the observations need further clarification and more rigorous verification they do form the basis of a methodology which promises more profound insights into the way children go about solving problems.

[illegible] back to critical example

(see additional note)

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PAGE 20 --

Replace line 5 in second paragraph beginning, "Response of the children....related problems." (9 lines) with the following:

> "The cross sectional data were made up of responses of each child to six different problems. These responses were all recorded on video tape in one setting. Sampling of the problem was so arranged that ten different children at each age level did each basic problem while 5 of these also did its equivalent. The same pattern was to have applied in collecting the longitudinal data a year later except that now ten children at each age level were to do the equivalent problems while five of these were to do the basic one. Normal attrition reduced these numbers slightly but not enough to do serious harm."

PAGE 22 --

Paragraph 5 starting, "Here again no coding..." Add to that one sentence paragraph:

> "However, it should be pointed out that the conditions and experiences were carefully designed so that responses to them could be readily observed."

PAGE 23 --

Third paragraph, sentence starting, "Although neither of the..." Omit entire phrase at the beginning and start "Spontaneous language..."

PAGE 25 --

Replace in line 8 the phrase, "the important role one half plays in the early stages" by:

> "to look at the partitioning act as a vehicle in problem performance."

PAGE 27 --

Second paragraph after setence ending, "some interesting results."

> "Kieren is finding, for example, in the machine problem that children preserved their own answer by using completely inconsistent explanations. The necessity of justifying their answers appears to be so distracting that logic and consistency is overpowered. Bana and Nelson have found that children seem to have a greater tendency ... (line 8)."

(Pick up original from end of line 8.)

PAGE 29 --

After first paragraph are sentences:

> "The relative success of very young children in some of these tasks were the result no doubt of more or less favorable modes of presenting the problems."

(Running on from last line on page 31.)

Despite the crudities in methodology the studies cited in previous sections lend support to the following general conclusions.

1. Distraction appears to be a key element for children dealing with practical problems. It is manifested in the form of responses young children make to various irrelevant physical, spatial and numerical aspects of the problem situation. It also occurs in a somewhat altered form in older children who are so attracted to justifying their own answers that they cannot give logical explanations for the mathematical procedures involved.

2. Children can get involved in more elaborate mathematical processes when they are embedded in relevant, practical problems than when the same processes are presented in their more formal, abstract or symbolic forms. Thus 3 and 4 year olds, though not necessarily in a perfectly systematic way, find solutions to partitive division problems with real objects while 10 year olds perform the complicated partitioning required in handling compound fractional operators (multiplication of fractions) provided the process is embedded in the card stacking machine. What this means in terms of instruction is not altogether clear but children seem to be able to "act out" mathematical processes in real problems long before the same processes make any sense at all in the symbolic forms.

3. Children involved in solving real problems are more apt to engage in a genuine search for solutions. This stands in sharp contrast to their responses in solving verbal problems where there is a search of sorts but that search is for a formula or a procedure which can be applied to produce desired answers.

4. Careful observations of children solving real problems provides a brighter picture of the interface between their development and their experience. Distraction, for example, seems to occur more as a function of being able to formulate a plausible alternative problem to the one intended than of how complex the problem is. In fact, complexity does not appear to be an important factor in whether or not a child will be distracted.

There may be others which could be drawn but these four are illustrative of how rich the field is or can become.

CONSTRUCTIVISM AS A MODEL FOR COGNITIVE DEVELOPMENT AND (EVENTUALLY) LEARNING[1]

The development of proportional reasoning in the child and adolescent

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July 1978

INTRODUCTION

The main emphasis of constructivism is on the dual aspect of subject and object intervening in a process of interaction and reciprocal construction.

At equilibrium, the subject grasps the object exhaustively and bears a judgment which is adequate to the whole object.

However, this same object, which is autonomous in the environment, can become more complex. The consequence will be that one part of the object will be grasped by the subject, while the other part is either ignored or interpreted erroneously. This leads to *centration* or *confusion*.

An example is a subject, familiar with natural numbers, who is placed in front of a rational number. He will interpret it in the light of what he "knows". The number 4/9 will be considered "large", while 2/3 is "small". A certain number of "modifications" have to be made to the concept of natural number, in order to fit it (or equilibrate it) to the new "object" which is the rational number.

Equilibration theory is based on the process of adjusting existing schemes to fit more complex objects in the outer world. It is the outer world which unbalances an existing scheme and forces it to evolve. But the process of change and the reconstruction of a new, "magnified" scheme, is the subject's business, and has to do with what is ordinarily called "understanding". When a subject says: *"I do not understand"*, he means that the

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new object in front of him is too complex for him to adjust, or is presented to him in such a manner as not to enable him to proceed easily to an adjustment. While the expression: *"Now, I understand"* means that the appropriate modifications have been made and the subject is able to integrate the unfamiliar object and play with it adequately.

Piaget (1975)[1] has called "équilibration majorante" the process through which an increased or magnified scheme is constructed, adapted to the new object in the environment.

THE PHASES OF MAGNIFYING EQUILIBRATION

"Magnifying equilibration"comes about in a step-wise process. Let us examine its premises and consider these phases.

1. - Data in the environment are interpreted by the subject, or "assimilated" through identifiable patterns of behavior or "schemes".
2. - Unfamiliar data in the environment cause a "disturbance" in the functioning of the scheme.
3. - The subject reacts to a disturbance in the environment through a process of "compensation".
4. - The mechanism of compensation is not a one-step process, but consists of a succession of identifiable "phases".
5. - Three phases can be described, which are the following:
    - i) At the α-phase, the subject "neglects" the disturbance or simply "avoids" it.
    - ii) At the β-phase, the subject "modifies" his scheme in order to "assimilate" the new datum.
    - iii) At the γ-phase, the subject integrates the new datum in a hierarchical system.
6. - Finally each "period" of development, i.e. sensori-motor, preoperational and operational, is the seat of a complete process of"magnifying equilibration",each period beginning with a phase of nonbalance and terminating with the construction of a hierarchical system.

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[1] PIAGET, J., (1975). *L'équilibration des structures cognitives, problème central du développement.* Paris: Presses Universitaires de France, translated as: *The Development of Thought.* New York: Viking Press, 1977.

CONSTRUCTION OF PROPORTIONAL REASONING

We shall examine a series of data, obtained on the development of concepts in the child and adolescent, to test:

(1) whether equilibration theory holds, and, if so,
(2) what is the nature of its "phases",
(3) whether we find these at each of the "periods" of development.

The development of proportional reasoning will here be studied. Two distinct findings are made:

a) Development of the ratio concept occurs in stages which can be both chronologically and structurally differentiated.
b) These stages can be seen as resulting from equilibration processes which can be reorganized into two distinct "periods".

The first preoperational period bears on "terms" : a natural number is equilibrated with an inverse generating, in a four-phase process, the concept of 1:1 ratio varying inside its equivalence class.

The second, operational period, bears on "ordered pairs" : the 1:1 ratio is differentiated in an *a:b* ratio, where terms are independent in magnitude, both in their state and their transformations, generating in a four-phase process, the Common Denominator and Common Factor algorithms.

The "phases"of equilibration, described by Piaget (1975), are seen here to be distinct "stages", structurally defined and imbedded in one another.

Equilibration, inside a "period", takes place in four phases, Piaget's β-phase being subdivised in two, giving the following:

α- phase: centration on known part of a situation, ignoring unknown part.
β_1- phase: assimilation of new part as complement (state-differentiation).
β_2- phase: relation between complementary parts (differentiation of relation or operation).
γ- phase: hierarchical integration.

THE EXPERIMENT

The development of rational number was studied, under its aspect of ratio, in the Orange Juice Test. This was devised in Quebec, and experimented both in individual and group forms. Through a number of Doctor and Master theses, the methodological aspects of developmental research were set down, with methods for differentiating stages, comparing them chronologically, verifying their integrative character, and determining problem-solving strategies at each level (Noelting, Cloutier and Cardinal, 1975)[1]. The results of a group experimentation will be given here. A publication is in preparation.

Instrument: Orange Juice Test.

A test was devised comprising 23 items, where each consisted in comparing the relative orange taste of a mixture, made up of a certain number of glasses of orange juice and a certain number of glasses of water (see Table 1A).

The items were the outcome of a certain number of previous experiments. Items 24 and 25 were later added to make up the final stage IIIB.

Procedure.

Two items are first discussed with the whole group, with explanations given (items I and II, see Figure 1).

Then each child or adolescent is asked to answer the experimental items 1 to 23, first choosing among three possible choices, then explaining why he made his choice.

Correction

Items 1 to 15 are corrected as given by the subject. Items 16 to 25 needed the examination of explanations, in order to eliminate accidental correct answers due to sole centration effects and no operations being put into use.

Sample.

A sample of 321 subjects were tested, from 6 to 16 years of age (see Table 1C). This corresponded to one class per grade level of Elementary

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[1] NOELTING, G., CLOUTIER, R. et CARDINAL, G., *Stades et mécanismes dans le développement de la notion de proportion chez l'enfant et l'adolescent.* Rapport de recherche, Université Laval, Québec, 1975.

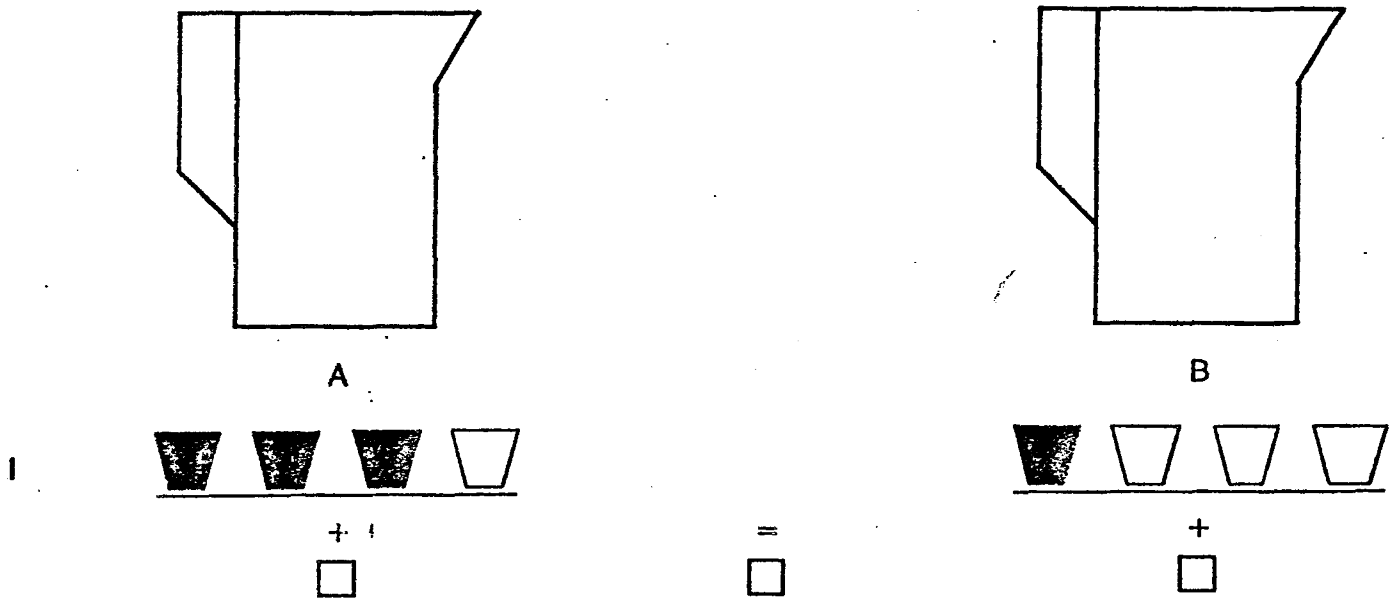
ORANGE JUICE (FORM A)

Date ______________________

Name ______________________

Age ______________ Date of birth ______________________

School ______________________ Class ______________________



Why ? ____

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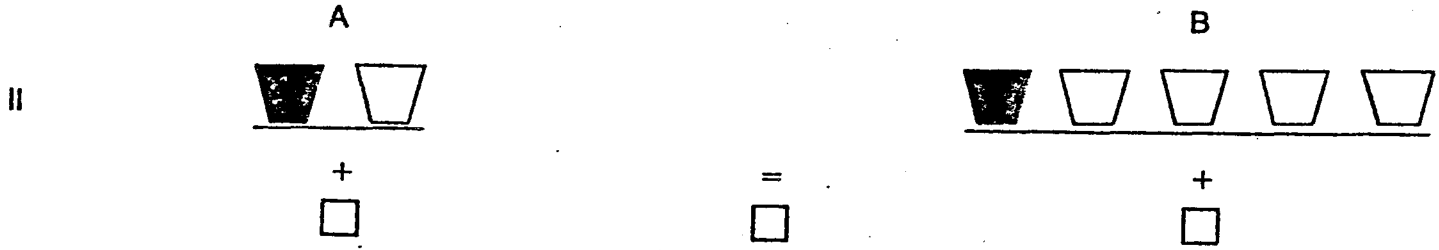

Why ? ____

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FIG. 1. — First page of Orange Juice Test (Group Form A).

TABLE 1A
ITEMS OF ORANGE JUICE TEST (GROUP FORM A)
WITH CORRECT ANSWER AND STAGE

Items	Composition	Correct answer	Stage
I	(3,1) vs. (1,3)	A	IA
II	(1,1) vs. (1,4)	A	IB
1	(1,0) vs. (1,1)	A	IB
2	(4,1) vs. (1,4)	A	IA
3	(1,2) vs. (1,5)	A	IB
4	(1,2) vs. (2,1)	B	IA
5	(1,1) vs. (1,2)	A	IB
6	(3,1) vs. (2,2)	A	IA
7	(1,1) vs. (2,2)	E	IIA
8	(2,3) vs. (1,1)	B	IC
9	(2,2) vs. (3,3)	E	IIA
10	(2,2) vs. (3,4)	A	IC
11	(1,1) vs. (3,3)	E	IIA
12	(1,2) vs. (2,4)	E	IIB
13	(2,1) vs. (3,3)	A	IC
14	(2,3) vs. (1,2)	A	IIIA
15	(4,2) vs. (2,1)	E	IIB
16	(2,1) vs. (4,3)	A	IIIA1
17	(1,3) vs. (2,5)	B	IIIA1
18	(2,1) vs. (3,2)	A	IIIA1
19	(2,3) vs. (3,4)	B	IIIA2
20	(6,3) vs. (5,2)	B	IIIA2
21	(3,2) vs. (4,3)	A	IIIA2
22	(4,2) vs. (5,3)	A	IIIA2
23	(5,2) vs. (7,3)	A	IIIB
24	(3,5) vs. (5,8)	B	IIIB
25	(5,7) vs. (3,5)	A	IIIB

NOTE. — Items 24 and 25, of Stage IIIB, have been added after further experimentation, in order to complete the stage of Higher Formal Operations.

and Secondary Schools. Mathematically advanced classes were chosen at each level, from the same socio-economic level (upper-middle class) of a suburb of Quebec City.

Results.

Items are ordered according to difficulty, then submitted to a Guttman-type scalogram analysis with the help of a computer program (Dixon, 1971)[1]. Satisfactory coefficients were obtained for CR, MMR and PPR, showing that items formed a "perfect" hierarchical scale.

Adjacent items on the scale were then grouped through a process of categorization (Table 1B). Subjects succeeding items of one level, but failing at the next, were considered to make up a "stage". These stages were compared, as to the age distribution of subjects, with the Kolmogorov-Smirnov Test (Siegel, 1956). This is a non-parametric test, as the scale involved is ordinal. Adjacent-stage comparison gave significant differences for the last five stages (IC to IIIB). Earlier stages had to be differentiated in an individual experiment (Table IC).

Examination of problems involved at each stage, and strategies used to solve them, led to assign operational levels to these stages, following the Piagetian chronology of development. Typical items of each stage are given in Table 1D.

Explanations subjects give at each stage, for solving the particular problem of the stage, were set in mathematical form (Table 1E). Symbols used are described in the section titled Symbolism.

Finally, the succession of stages was analyzed in terms of equilibration process. Two "periods" of equilibration were found, one corresponding to pre-operational processes leading to the construction of the concept of ratio (Table 1F), the other to construction of the Common Denominator and Common Factor algorithms (Table 1G).

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[1] DIXON, W.G., Ed. *Biomedical Computer Programs.* Berkeley: University of California Press, 1971.

TABLE 1B

ITEMS OF ORANGE JUICE TEST (GROUP FORM A)

ORDERED ACCORDING TO DEGREE OF SUCCESS, THEN CATEGORIZED TO FORM STAGES

Stage	Item	Composition	Frequency of success	Characteristics
0	0	(1,0)vs.(0,1)	-	Differentiation of terms.
IA	2	(4,1)vs.(1,4)	319	Difference between first terms of ordered pairs.
	6	(3,1)vs.(2,2)	319	
	4	(1,2)vs.(2,1)	319	
IB	1	(1,0)vs.(1,1)	311	Like first term, difference between second terms of ordered pairs.
	3	(1,2)vs.(1,5)	307	
	5	(1,1)vs.(1,2)	305	
IC	8	(2,3)vs.(1,1)	295	Equality vs. difference between terms of ordered pairs.
	13	(2,1)vs.(3,3)	291	
	10	(2,2)vs.(3,4)	297	
IIA	9	(2,2)vs.(3,3)	251	(1,1) equivalence class.
	11	(1,1)vs.(3,3)	244	
	7	(1,1)vs.(2,2)	231	
IIB	12	(1,2)vs.(2,4)	186	Any equivalence class.
	15	(4,2)vs.(2,1)	156	
IIIA1	16	(2,1)vs.(4,3)	141	Ordered pairs with two corresponding terms multiple of one another.
	17	(1,3)vs.(2,5)	131	
	14	(2,3)vs.(1,2)	107	
	18	(2,1)vs.(3,2)	88	
IIIA2	20	(6,3)vs.(5,2)	87	Same after simplyfing one pair or extracting (1,1) unit.
	22	(4,2)vs.(5,3)	71	
	19	(2,3)vs.(3,4)	65	
	21	(3,2)vs.(4,3)	59	
IIIB	23	(5,2)vs.(7,3)	51	Any fraction.
	24	(3,5)vs.(5,8)	-	
	25	(5,7)vs.(3,5)	-	

TABLE 1c

COMPARISON OF AGE DISTRIBUTION AT EACH STAGE OF ORANGE JUICE TEST (GROUP FORM A)

Age	N	Stage							
		0	IA	IB	IC	IIA	IIB	IIIA	IIIB
6	14	0	1	2	8	3	0	0	0
7	26	1	1	7	14	2	1	0	0
8	35	1	0	4	12	10	6	2	0
9	43	0	1	2	9	12	13	6	0
10	32	0	0	1	3	13	8	6	1
11	38	0	0	1	5	12	7	9	4
12	34	0	3	1	0	9	5	14	2
13	31	0	2	0	0	2	9	17	1
14	20	0	0	0	1	1	2	10	6
15	29	0	0	0	0	0	8	16	5
16	19	0	0	0	0	1	2	8	8
Total	321	2	8	18	52	65	61	88	27
p[a]		–	–	–	–	<.01	<.01	<.01	<.01
Age of accession[b]		–	–	–	–	8;1	10;5	12;2	(17;0)

NOTES. — [a] Probability level of difference between age distribution of the stage, compared with preceding one, assessed by Kolmogorov-Smirnov Test.

[b] Age of accession to a stage is the age where 50% of Ss solve at least one item of the stage.

TABLE 1D

STAGES IN THE DEVELOPMENT OF THE CONCEPT OF RATIO
(ORANGE JUICE TEST, GROUP FORM A)

Stage	Name	Age of accession (50% Ss)	Typical item	Characteristics of stage
0	Symbolic	(2;0)	■ □ (1,0) vs. (0,1)	Identification of elements.
IA	Lower Intuitive	(3;6)	■■■■□ ■□□□□ (4,1) vs. (1,4)	Comparison of first terms only.
IB	Middle Intuitive	6;4	■□□ ■□□□□□ (1,2) vs. (1,5)	Like first terms, comparison of second terms.
IC	Higher Intuitive	7;0	■■■□□□□ ■■□ (3,4) vs. (2,1)	Inverse relation between terms of both ordered pairs.
IIA	Lower Concrete Operational	8;1	■□ ■■□□ (1,1) vs. (2,2)	Equivalence class of (1,1) ratio.
IIB	Higher Concrete Operational	10;5	■■□□□ ■■■■□□□□□□ (2,3) vs. (4,6)	Equivalence class of any ratio.
IIIA	Lower Formal Operational	12;2	■□□□ ■■□□□□□ (1,3) vs. (2,5)	Ratios with two corresponding terms multiples of one another.
IIIB	Higher Formal Operational	15;10	■■■□□□□□ ■■■■■□□□□□□□□ (3,5) vs. (5,8)	Any ratio.

TABLE 1E

PROBLEM-SOLVING STRATEGIES AT DIFFERENT STAGES IN THE DEVELOPMENT OF PROPORTIONAL REASONING

Stage	Name	Age of accession (50% Ss)	Typical item (a,b) vs. (c,d)	Strategy
0	Symbolic	(2;0)	(1,0) vs. (0,1)	$a_1 \in A$, $d_1 \in D$
IA	Lower Intuitive	(3;6)	(1,4) vs. (4,1)	$c > a$ Therefore $(c,d) > (a,b)$
IB	Middle Intuitive	6;4	(1,5) vs. (1,2)	$a = c$, $b > d$ Therefore $(c,d) > (a,b)$
IC	Higher Intuitive	7;0	(2,1) vs. (3,4)	$a > b$, $c < d$ Therefore $(a,b) > (c,d)$ even though $a < c$
IIA	Lower Concrete Operation	8;1	(1,1) vs. (2,2)	$m(1,1) = (m,m)$
IIB	Higher Concrete Operation	10;5	(2,3) vs. (4,6)	$m(a,b) = (ma,mb)$ with $a \neq b$
IIIA	Lower Formal Operation	12;2	(1,3) vs. (2,5)	$ma = c$ $m(a,b) = (ma,mb)$ $ma = c$, $mb > d$ $(c,d) > (ma,mb)$ Therefore $(c,d) > (a,b)$
IIIB	Higher Formal Operation	15;10	(3,5) vs. (5,8)	$a+b = g$ $(a,b)g = (a/g,b/g)$ $c+d = h$ $(c,d)h = (c/h,d/h)$ $hg = gh$ $h(a,g) = (ha,hg)$ $g(c,h) = (gc,gh)$ $(gc,gh) > (ha,hg)$ Therefore $(c,d) > (a,b)$

TABLE 1F

THE FOUR STAGES OF EQUILIBRATION OR ADAPTATIVE RECONSTRUCTION IN THE GENESIS OF EQUIVALENCE CLASSES OF RATIOS

PROBLEM: Working out the relation between terms of a ratio takes place here. The difficulty in understanding the equivalence class of a ratio is differentiating between complementation of terms in the state and covariation of terms in the transformation.
Math. object involved: ordered pair.- Part known: 1st term. - New part: 2nd term.

Equilibration stage	Typical item with strategy	Operatory mechanism	Resulting behavior
α-stage Centration by scheme on familiar part of object (1st term).	IA. - (1,4)vs.(4,1) $a < c$ therefore $(a,b) < (c,d)$	Centration on familiar part of object (1st term). New part rejected or treated by same scheme (confusion).	Direct relation between familiar part of object (1st term) and whole (ratio) in all items passed with success.
β_1-stage New part seen as having inverse effect on whole.	IB. - (1,5)vs.(1,2) $a = c,\ b > d$ therefore $(a,b) < (c,d)$	Assimilation of unknown part of object through inversion of scheme.	Oscillation between centration on either 1st or 2nd terms of ratios.
β_2-stage Internal compensation of terms.	IC. - (2,1)vs.(3,4) $a > b,\ c < d$ therefore $(c,d) < (a,b)$ even if $c > a$	Compensation of parts in each object with internal comparison, then conclusion.	Comparison between 1st and 2nd terms in each pair, and conclusion when possible.
γ-stage Hierarchical organization of states in transformation.	IIA.- (1,1)vs.(2,2) Mode C $m(1,1) = (m,m)$ Mode D $1/1 = m/m$	Differentiation between complementation of parts and inversion of scheme. Parts compensate each other (reciprocals) but covary in same direction (complexifying or simplifying ratio).	Complementary parts covary both in direct and inverse directions. Equivalence class of 1:1 ratio.

TABLE 1G

THE FOUR STAGES OF EQUILIBRATION OR ADAPTATIVE RECONSTRUCTION IN THE GENESIS OF THE COMMON DENOMINATOR OR PERCENT ALGORITHM

PROBLEM: At the end of last period, the child has grasped to inverse relationship between terms in the 1:1 ratio.
At this period, he must understand their independence as to size (state) and variation (transformation).

Equilibration stage	Typical item with strategy	Operatory mechanism	Resulting behavior
α-stage Fragmentation between multiplicative and additive parts of (a,b) ratio.	IIA.- i) (1,1)vs.(2,2) $m(1,1)=(2,2)$ (success) ii) (2,3)vs.(4,6) → 2(1,1)+(0,1) vs. 4(1,1)+(0,2)(failure)	Application of (1,1) scheme and centration on excess.	Adequate treatment of 1:1 ratio.
β_1-stage Terms of ratio seen to be independent in state.	IIB.- i) (2,3)vs.(4,6) $m(a,b)=(ma,mb)$ (success) ii) (3,1)vs.(5,2) → (2,1)+(1,0) vs. 2(2,1)+(1,0) (failure)	Differentiation of terms in the state. Application of (a,b) scheme with centration on excess.	Equivalence class of any ratio is grasped. But only conjunctive variation possible.
β_2-stage First coordination between conjunction and disjunction.	IIIA. - (3,1)vs.(5,2) $mb = d$ $m(a,b) = (ma,d)$ $ma > c, \quad mb = d$ therefore $(ma,mb) > (c,d)$ whence $(a,b) > (c,d)$	Differentiation of terms in the transformation. Multiplicative conjunction combined with additive disjunction.	Only ratios where corresponding terms are multiple one of another are treated.
γ-stage Hierarchical organization of logical connectives by algebraic operations.	IIIB. - (3,5)vs.(5,8) i) Algebraic addition $a+b = g$ $(a,b) \rightarrow (a,g)$ $c+d = h$ $(c,d) \rightarrow (c,h)$ ii) Algebraic multiplication $hg = gh$ iii) Logical comultiplication $h(a,g) = (ha,hg)$ $g(c,h) = (gc,gh)$ iv) Logical disaddition $ha < gc, \; hg = gh$ $(ha,hg) < (gc,gh)$	Differentiation between logical and algebraic aspects of system. Combinational system along two dichotomies, logical: conjunction vs. disjunction; algebraic: multiplicative vs. additive.	Equivalence class of each ratio integrated by common denominator or common factor. Then additive treatment of numerators. Algorithm of rational number addition is established.

SYMBOLISM

Symbolism of items in algebraic form was introduced to express the strategy common to a stage. A uniform method of placing the two ordered pairs in each item was found necessary, in order to make items comparable. The following rule was applied: the ordered pair with the smaller first term is put first, e.g. (1,4) vs. (4,1). When first terms are equal, the ordered pair with the greater second term is put first, e.g. (1,5) vs. (1,2). With equivalence classes, the pair with the lowest terms thus comes first, allowing multiplicative covariation: $3(1,1) = (3,3)$. This rearrangement when symbolizing an item will be called the *standardized form*. In the test, the order of pairs in an item is put at random.

The following symbols were adapted to express the sets and subsets of each item. The sets, when placed in the standardized order are called G and H, with g and h expressing their number. The respective subsets of orange juice and water of set G are A and B, with a and b their number. The subsets of set H are C and D, with c and d their number. Thus an item in standardized form remains (a,b) vs. (c,d). Individual elements of a subset such as A are called a_1, a_2, etc. Various operators are introduced by subjects modifying each term of the ordered pairs, i.e. a, b, c and d. The symbols f, j, m and n will be used to denote natural numbers (excluding zero).

QUALITATIVE DESCRIPTION OF STAGES - In choices: A means first pair, B second, E equality. Space does not allow to give examples of each stages.

Examples of behavior of some characteristic stages are given.

Stage IA: Lower intuitive. Centration on the first terms of the ordered pairs.

Success at items such as (1,4)vs.(4,1) and (1,2)vs.(2,1).- The child compares the number of glasses of orange juice in both pairs, or opposes predominance of juice in one pair and water in the other.

Examples of success:

Diane, 4;0	Item B4: (1,2)vs.(2,1) Chooses B (success).	*"Here there is more orange".*
Nathalie, 5;0	Item B3: (4,1)vs.(1,4) Chooses A (success).	*"Because there is a lot of orange juice and only one glass of water".*
Gilles, 4;0	Item B4: (1,2)vs.(2,1) Chooses B (success).	*"There is a lot of orange juice (B). There is a lot of water (A)".*

Examples of failure:

i) Globalism

France, 5;0	Item B6: (1,0)vs.(1,1) Chooses B (failure).	*"Because there are many".*

ii) Centration

Louis, 4;7	Item D6: (1,1)vs.(1,0) Chooses E (failure).	*"It will taste the same because there is one glass of orangeade there (B) and one glass of orangeade there (A)".*

Stage IIA: Lower concrete operation. Equivalence class of ratio (1,1).

Success at items such as: (1,1)vs.(2,2) and (2,2)vs.(3,3).

Examples of success:

i) Covariation (Mode C)

Johanne, 11;0	Item A7: (1,1)vs.(2,2) Chooses E (success).	*"Each glass dilutes one glass. So A has one glass of juice and B has two; A has one glass of water and B two. They are equal only there is more liquid mixed in B".*

ii) Division (Mode D)

Martine, 8;0	Item B12: (2,2)vs.(3,3) Chooses E (success).	*"Two for two, here (A); three for three, here (B)".*

Subjects differentiate between state and transformation. The relation between complementary terms in the pair is stabilized as an "invariant". The relation between corresponding terms between pairs is mobilized as a transformation (either co-multiplication or co-division). This yields the simplest equivalence class, the 1:1 ratio. Strategy for the divisive mode is $m/m = n/n$, corresponding to transposition of a ratio. Strategy for the multiplicative mode is $m(1,1) = (m,m)$; $(m,m)/m = (1,1)$, corresponding to complexifying or simplifying a ratio.

However ratios, where terms are not equal, are failed.

Two modes of behavior are distinguished and will be found at each stage:

Mode C or covariation - Mode D or internal division.

Examples of failure:

i) Centration on the residual after (1,1) covariation (thus using strategy of the stage)

Louise, 11;0	Item A12: (1,2)vs.(2,4) Chooses A (failure).	*"Because the left side has one glass of water more, while the right side has two of them more".*

ii) Centration on either juice or water (repression to earlier strategy)

Diane, 8;0	Item A12: (1,2)vs.(2,4) Chooses A (failure).	*"It is that there are less glasses of water".*

Stage IIIB: Higher formal operation. Common Denominator and Percentage Algorithms.

Examples of success:

i) Common Denominator (Mode C)

Sylvie, 14;0	Item A19: (2,3)vs.(3,4) Chooses B (success).	*"At the right, there is 3/7 of juice for 4/7 of water, that is 15/35 of juice; at the left there is only 14/35".*

ii) Percentage (Mode D)

Réjean, 13;0	Item A23: (5,2)vs.(7,3) Chooses A (success).	*"A = 71 3/7 % because 5/7 orange juice.* *B = 70% because 7/10 orange juice".*

Characteristics of stage IIIB: Differentiation between logical and algebraic transformations, with hierarchical integration.

At stage IIIB, a combinatorial system is formed, where algebraic and logical transformations are differentiated and integrated. These are defined as follows:

Algebraic:

A *binary operation on elements* is an operation in set which combines two elements of the set into a third element of the set.

The addition of two terms a and b of a ratio to find their sum g is a binary addition. The terms of a ratio are considered here as natural numbers, with their sum a natural number.

The multiplication of two denominators to find their product is a binary operation. The operation of join or meet on two denominators to find their LCM or HCF is also a binary operation.

Logical:

A *binary operation on propositions* is a connective introduced on two propositions. We shall consider a binary operation on elements as a proposition. Thus the coordinated multiplication or division of both terms of an ordered pair, to find an equivalent, will be considered a conjunctive operation. The isolated multiplication of one term of a ratio or fraction will be considered a disjunctive operation.

Binary operation on terms : $a+b = g$
$g \cdot h = gh$

Binary operation on propositions:

Two propositions: $a \rightarrow ma$, $b \rightarrow mb$
Conjunction: $(a,b) \rightarrow (ma,mb)$ equivalence class
Disjunction: $(a,b) \rightarrow (ma,b))$ operation on rational

This distinction is best summarized in the following table :

	Algebraic (terms)	Logical (pairs)
Additive	I $a + b = g$	III $(ha,hg) \gtrless (gc,gh)$
Multiplicative	II $h \times g = hg$	IV $(a,b)/g = (a/g,b/g)$ $h(a,g) = (ha.hg)$

PEDAGOGICAL CONCLUSIONS

I. - *Specific to the proportion concept.*

(1) The concepts of ratio between quantities (e.g. 2 glasses of orange juice for 3 glasses of water), fraction of a set (e.g. 2 glasses of orange juice for 5 glasses of liquid) and fraction of a unit (e.g. 2/5 juice in each glass) should be carefully distinguished.

(2) Proper (< 1) and improper (< 1) fractions should be worked upon simultaneously, e.g. 1/2, 2/4 ..., 2/1, 4/2 ...
Improper fractions should be considered as rational numbers without immediate retrieval of the unit.

(3) From level IIA onwards, any fraction should be envisaged under both its internal division aspect (mode D) and covariation aspect (mode C).

(4) The passage from equivalence of unit fractions, to equivalence of any fraction, should be considered a difficult step, and taking up many years. The child must here differentiate between independence of terms in the state and covariation in transformation. This is the proper problem of the elementary school. Equivalences of 3/5 and 5/7, for instance, in concrete situations, are still considered difficult at the end of the elementary school.

(5) A lot of time should be devoted to problems like 1/3 + 2/9. The multiplicative relation between denominators should be discovered by the pre-adolescents themselves. We find here a combination of the covariation of terms in the equivalence class and "disvariation" of terms in the addition of fractions with like denominators. This is a coordination of conjunction and disjunction applied to the same content. It is characteristic of the formal level of thinking. It is abstract thinking, an operation (additive operation) on an operation (equivalence).

(6) Common denominator (mode C), and reduction to unit (mode D), should be seen as inverse strategies used to liken different denominators. Percentage should be seen as a way of better expressing the ratio to a unit.

II. *Relative to the concepts of periods and phases of equilibration.*

(1) Respect of periods of equilibration i.e. concrete and formal modes of thinking.

A sharp distinction must be made between concrete operations and formal operations. It is a distinction between operation on terms and operation on operations. In period I, operation is performed on data themselves (i.e. a, b, c, d). In period II, operation is performed on data constructed from the data (e.g. ma, a/b, etc.). This differentiation is especially important to make for teachers in Grade 6, when children are at the frontier of concrete and formal thinking. Some problems, though intricate, are easy because they involve only concrete data. Others, though apparently much simpler, are difficult for the pre-adolescent, because they involve simplification, equivalence, which seems automatic to an adult, but involve retention of constructed data, then operation on these.

(2) Equilibration should be made of what the child knows, to the unknown which is brought to him. In particular the new variable introduced at each period should be clearly identified by the teacher and related to existing schemes in the child.

III. *Conclusions of a general nature bearing on mathematics.*

(1) Emphasis should be put on laying strong foundations rather than rapid but evanescent techniques. Notions should be constructed in their hierarchical order. Motivation is kept up by the process of discovery and construction, and by varying the content for a same structure.

(2) Equilibration - The novel aspect introduced by equilibration theory is the constant interplay between interaction and construction. The dialectical process of uncovering new data is constantly related to the process of structuring the data inside a coherent whole where the new and the old are interrelated. Equilibration to novelty is

related to reorganization of internal structure. Growth consists in being open to the world, but proceeding with system. This opens up a new field of study: the psychology of mathematical construction.

(3) At all levels concrete problems should be worked upon in parallel to symbolic representation of problems.

(4) Stage IIIB, in proportional reasoning, is characterized by the combination of logical reasoning and algebraic operations. The difference between the nil transformation, characteristic of logic (e.g. $p\bar{q}$) and the inverse transformation, characteristic of operations (e.g. $2 + 3 = 5$; $5 - 3 = 2$) should be made much earlier. Usually a nil transformation is introduced in a combinatorial setting at the formal level: e.g. $pq \vee p\bar{q} \vee \bar{p}q \vee \bar{p}\bar{q}$. This should be prepared at the concrete level (elementary school) by introducing the difference between constancy and variation. Their combination in the constancy-variation scheme is already put into use at stage IB (middle intuitive: 5-6 years of age). The very important role of the "agreement and difference" principle (basis of scientific reasoning as set forth by Roger Bacon and John Stuart Mill) render the early introduction of the constancy-variation scheme imperative. The "all other things being equal" principle is the basis of organized thought at all levels.

(5) Axiomatics vs. constructivism. - Disinterest for mathematics on the part of the layman and the child is due to the obsessive character of axiomatics in contrast with the creative quality of constructivism. Mathematics should become constructive, with emphasis on process, instead of obsessive, with emphasis on structure. Time is ready for a change in attitude in mathematics.

(6) "Intuition" plays a certain part in mathematics, but is never formalized. Cognitive-developmental theory rejets "intuition" and replaces it by "actions" of subjects upon the environment. These actions are reversible and are formalized as "operations". Rules should arise from the nature of mathematical objects upon which activity is performed and their constraints and not from the "axioms" which are elaborated as end products.