Canadian Mathematics Education Study Group
Groupe canadien d'étude en didactique des mathématiques

Proceedings of the 1979 Meeting

Queen's University at Kingston, Ontario
June 1 - 5

CMESG/GCEDM

November 1979

CMESG/GCEDM
1979 Meeting
Proceedings

Note

The four Working Groups met at the same times during the Meeting for a total of 9 hours each. Each person attending the Meeting was expected to participate fully in one of these groups. Review Groups were scheduled to meet for $2\frac{1}{2}$ hours each. Special Groups were *ad hoc* groups added to the programme at the suggestion of the respective leaders.

ON MATHEMATICS EDUCATION: THE LAKATOSIAN REVOLUTION*

Joseph Agassi

Boston University and Tel-Aviv University

When a philosopher like me is invited to address a professional group like the present august audience, it is unreasonable to expect that he show expertise in their specialization of the level required from one addressing a conference of his peers in his own specialization. Hence, such an invitation must be based on a different expectation. Possibly the philosopher is expected to perform a ritual function, akin to that of a priest. I have been invited for ritual purposes many times in the past, but no more. One advantage of a reputation is that it prevents such understandable cross-purposes. Whatever the reason for my being invited, it is no longer to offer platitudes or homilies. I will neither soothe nor preach, and by now this is known. An invitation to an outsider may also be an invitation to impart to one field the fruits of another. For example, mathematicians were invited to physics conferences to teach professional physicists some Lie algebra. I do have some specialized results to tell you about, yet hardly from the field of philosophy. But there is one good reason for inviting a philosopher to any specialized conference: he may be able to make quite a lot of trouble in a short time.

I am telling you this in advance because of my past experiences. There is no need to accept my offer to make big

---

* Paper written while I was a senior fellow of the Alexander von Humbolt Stiftung resident at the Zentrum fur Interdisziplinare Forschung, Universitat Bielefeld, and read to the Canadian Mathematics Education Study Group Meeting at Queen's University at Kingston, Ontario, June 1979.

trouble, and like everyone else I can be dismissed on any one of many sorts of pretext. I often meet indignation, and I have one thing to say to the indignant: his indignation is but an excuse for dismissing me, and dismiss me he may anyway, so that his indignation is redundant. And, as Spinoza noticed, it is expensive. So it is better to dismiss me with no indignation if dismiss me you will.

The indignant, if I may pursue the matter for one more paragraph, will now vehemently protest, though vehemence is, like indignation, an expensive redundancy. He will say that he is indignant not because of his own personal reasons but because I am dangerous to the system. I appreciate the compliment and for a fleeting megalomaniac moment I am even tempted to accept it. But now is my lucky break and perhaps you and I together may become a little dangerous. As I say, I have come here to try and stir up a little trouble.

We are all working within a system, both in a broad sense of the word and in a narrow sense. And my first thesis to you, which is my first import -- from philosophy this time -- is that one's efficiency much depends on the question, in which framework is one operating? To give you a few examples. In mathematics the same theorem may be trivial in one system and deep in another. At times, it is worthwhile to embed systems within systems so as to make an easy kill and import some results. Abraham Robinson, for example, proved that all theorems of analysis in a non-standard interpretation are valid in the standard interpretation and that some theorems which are deep in the standard reading are trivial in the non-standard one. A corollary derived from this is for mathematics education. If, instead of moving along in a lecture course in mathematics as fast as possible, proving all the necessary theorems, introducing all the necessary tools, etc., if instead of this one takes it slowly and explains to one's audience what one is doing and one's rationale for it, then one's audience will find the course both more fun and more rewarding. I will return to this later.

Another example. The problem of induction is easily soluble in one system and utterly insoluble in another. I will not enter this vast topic now but only mention results of other studies of mine. When one assumes Daltonian chemistry, one can employ induction when performing complex tasks of chemical analysis. The system is known to be false, since some atoms are unstable and disintegrate into new atoms, contrary to the basic assumptions of Daltonian chemical analysis, yet in many cases this is irrelevant. If, however, one clings to the classical view of science as empirically demonstrable and hence error-free, then the solution here outlined must be rejected as faulty or at least as question-begging. Similarly in technology. An airline which follows legal requirements is not necessarily to be condemned when a vessel which belongs to it is disastrously destroyed. It can claim that it has complied with the law and thus has adequately discharged its responsibility. The government agency operates within a framework too, and this may justify its having unknowingly permitted the faulty vessel to fly. Of course, both the airline company and the government agency, perhaps also other parties, may be found reprehensible, but the catastrophe alone is not proof of neglect. The solution to the problem of induction, if perfect, would allow one to preclude all the imperfections that permit faulty vessels to be used, and would make such use, then, automatically reprehensible.

My final example is Marxism. Not the one officially endorsed by communist governments at home, which most of us are quite unfamiliar with, but in the West. Of all the vaguenesses of western Marxist thinking, I pick up the one about the framework: which framework do Marxists wish to destroy? We can see the vagueness of most Marxists by observing the clarity of those who are still clear. For example, Ivan Illich wants the whole social system destroyed: he thinks society will be better off for that; perhaps he also thinks that once the current system be destroyed, a better one will emerge by itself. If so, he does not tell us how. But he quite rightly observes that the school system is a pillar of the social system -- this is true of all social systems -- and so he wants it destroyed. How, then, will the education of the members of society proceed after the revolution? Will a new school system emerge all by itself? No, says Ivan Illich. We do not need any, he avers. Following Paul Goodman he claims that children in the park can learn more from the elderly folk who sit on the benches to warm their bones than from professional teachers. I do not wish to deny that this vision holds for some cases. Nor am I totally insensitive to its beauty and allure. Yet I do think that as a substitute for the present system it is simply disastrous. It will destroy the present system, for sure; but it will destroy everything else as well.

Hence most Marxist thinkers are muddled because they do not want to share Illich's framework but cannot provide an alternative that is at all convincing. Other people respond to Illich either by endorsing his vision or by going ultra conservative. Both responses are conservative in practice since Illich's alternative is plainly a non-starter.

I do not mean to condemn Illich for his extremism but to congratulate him on his relative clarity. As to extremism, his is by no means the limit. For example, the idea of the discovery method, though in one framework highly conservative - i.e. in that it keeps the classroom structure as it is - in another framework is more extremist and radical than Goodman or Illich. It says that by sheer prompting and coaxing a child can discover for himself what the greatest minds in the history of mathematics did over twenty-five centuries. This is a megalomaniac view either of prompting and coaxing *per se* or of the prompting and coaxing teacher.

So let me describe now the exact framework or system that I wish to see destroyed, the one I wish to see replace it, and the way I propose to effect such a revolution, the Lakatosian revolution, as I have elsewhere called it. And I begin with what I have already covered in this brief introduction. First, I do not wish to see schools closed down or teachers dismissed. Second, I oppose prompting, coaxing, motivation, and any other forms of leadership. Instead I recommend that teachers, like any other adult citizens, explain to their audience their purposes and invite them to partake in exciting intellectual and other ventures.

Let me start, then, with the last point, with motivation versus declaring one's purpose. There are two systems within which the matter just introduced may be discussed. First, educational coaxing and motivation is a form of deception. To take an example from Moses Maimonides, a teacher offers fruits and nuts to children as a reward for their acquisition of literacy. The children, says Maimonides, may think they work for peanuts. And, as the teacher has promised these, he must indeed reward his pupils with peanuts. But the true reward lies elsewhere. The parent says to his child, the teacher says to his pupil, when you are adult you will comprehend my purpose and see its benevolence towards you and then you will thank me. Motivation, then, is making the child do what he has no inclination to do. One can achieve this by offering him peanuts, by offering him the absence of flogging (that is to say by promising him flogging unless he is motivated; and then, is one still obliged to flog the lazy pupil?), and, worst of all, one can achieve this by promising love to the good pupil and no love for the lazy one. Replacing the rationing of love for the rationing of non-harassment is one of the most famous, most highly praised advance of the modern age, of the age of enlightened education. One of the clearest results from the field of criminology and of psychiatry is that the rationing of love, regardless of any other factor that may or may not accompany it, the sheer system of rationing of love, is more damaging and has more lasting effects than flogging or threats to flog. I need not mention the obvious fact that the discovery method has been tried chiefly in experimental and/or progressive schools where enlightened teachers use every subtle psychological means, except brute physical violence, to make their pupils work their hardest to achieve the impossible. My horror of this knows no bounds: I find physical violence merciful in comparison to sophisticated psychological manipulation.

The claim that stands behind this manipulation, may I repeat, is that a child is unable to comprehend the true purpose of the exercise. And this enables one to switch from the educational framework or system to the political framework or system. We can ask, what is the purpose of education? It may be to conserve; it may be to raise the cadres for the revolution; it may be to help our successors do better than we have done. To cut a long story short, I endorse the third purpose. Education, then, is for independent, free, able citizens. Hence I am for the true discovery method, not for the method of coaxing and cajoling and manipulating children, but of talking to them as to equals and of discussing with them the past faults of the system, the joys of discovery and of improvements and of innovation, and their desire to evolve their necessary faculties.

The claim of the establishment still precludes this. The establishment, to repeat yet again, denies that young ones know or can know the purpose of the exercise. We thus have to shift our framework yet again, from the psychology of education, through the politics of education, to the intellectual framework that the educator himself endorses, and not only as an educator, but also as a citizen in the broader sense. He claims authority over his charge on the pretext that his charge does not know. Before entering on a discussion of this claim, let us notice the implied claim that he does. Does he?

I am told that some of you are not familiar with the great classic work of Imre Lakatos, Proofs and Refutations. I much recommend it as the most revolutionary work in the philosophy of mathematics in the post-World-War-II period, no less profound and wider in scope than Paul Cohen's discovery or the discovery of category theory or anything else. What Lakatos has succeeded in showing is that all mathematical presentations in standard mathematics textbooks are ill conceived as to their purpose, and that mathematicians, even some of the greatest, are surprisingly vague about the rationale of mathematics, and about the aims and methods of mathematical proofs in particular.

The framework within which these people operate is what handicaps them, and to that extent it makes their success as great mathematicians all the more miraculous. I cannot discuss this framework here: I have devoted a whole book to it. But I can describe it here briefly. The framework, devised by Parmenides and developed by Plato, is of particular significance for the understanding of the history of Greek mathematics, and thus of the history of mathematics at large, as was ably argued by Arpad Szabo, the person who was Lakatos' teacher in their native Hungary.

Parmenides divided our whole cultural and intellectual and moral world into two extreme poles, truth and falsity. Truth is demonstrable and gives us the nature of things. Falsity is mere appearance and convention. To be more specific at the cost of adding to the original text, Parmenides and Plato were not interested in any old falsehood, in blunt mathematical error, or in cock-and-bull stories and fairytales. Rather, they were disturbed by prevalent error, what is at times called truth by convention or by agreement or by popular support. There is truth by convention which is mere _doxa_, i.e., opinion, i.e., popular opinion, which is mere appearance, i.e., seemingly convincing. And there is truth by reason, i.e., by proof, i.e., _logon piston_ or _episteme_, i.e., knowledge proper. In the whole field of western thought this dichotomy runs through. In bullish mood thinkers opted for knowledge, in bearish mood they settled for convention. Even in bullish mood, the status of truth by convention was granted as a consolation prize. Thus, Maxwell gave his own theory first prize and that of Lorenz (the Dane) the consolation prize. Some political philosophers, Edmund Burke and Hegel in particular, tried to defend their reactionary philosophy by appeal in both nature and convention. Yet generally most naturalists are radicalists -- like Ivan Illich -- and most conventionalists are conservative -- like James Bryant Conant who defended conventionalism in science: scientific truth, he said, in his celebrated Harvard Case Studies and elsewhere, is truth by convention.

When we come to any branch of learning, but particularly to logic and mathematics, the dichotomy of nature and convention is so dreadful because it cuts out purpose: nature leaves no room for my desires and convention makes them arbitrary. This is why in logic and in mathematics most philosophers are either naturalists, logicists, ideal language theorists, etc., or formalists who deem any axiom system as good as any other. Both are in error: systems are man-made but not arbitrary: they are designed to answer certain desiderata, and these desiderata are themselves subject to debate.

I now come to a major fork. The structure of my discussion moves here. My desideratum, I have told you, is to overturn the educational system. I have delineated my framework: we want to decide what are the desiderata of mathematics education -- or education in general -- and show the system wanting with respect to them, devise a better system, and transact the change. In particular, I have said, math teachers do not quite know the aim of math ed, as Lakatos has proven, and the philosophies of both mathematics and education are cast in a Parmenidean-Platonic framework that tones down desiderata, so we have a big task on our hands which we have hardly begun. We can now move (1) to the purpose of education, with special reference to mathematics; or (2) to the purpose of mathematical instruction with special reference to the education of mathematics teachers, of mathematics researchers, applied mathematicians, and amateurs; and we can discuss (3) the aim of mathematical research.

I do not know the aim of mathematical research. I could say "to discover the nature of mathematics", and I could say "to devise the mathematical system most useful for our studies of nature or for our conquest of nature". Doing so casts mathematics well within the Parmenidean-Platonic framework. It is the enormous merit of Lakatos that he has trussed this framework's seams. Even Szabo and Polya did not touch on this issue. Karl Popper, Lakatos' other teacher, even if not a Hungarian, has devised a philosophy of science and a social philosophy that do not fit the Parmenidean-Platonic dichotomy between nature and convention; yet in logic and mathematics he was a conventionalist until he met Lakatos. Lakatos himself was no less shaken by Popper than Popper by Lakatos. Popper said of science, and Lakatos said of mathematics, that each is full of errors to be corrected and hence is neither nature, i.e., not only demonstrable final truths, nor convention, i.e., not truths arbitrarily nailed down and defended against criticism. They agreed that we hope to progress towards the truth, or nature. At times, though, Lakatos seems to suggest that we progress towards meeting conceptual desiderata. Lakatos wanted to say the same of logic. But he was persuaded that, as both Popper and Quine argue, the law of contradiction has a special status: we cannot hope to criticize it; for to criticize it is to discover a true contradiction; which is absurd. Lakatos, then, was swayed by Popper to become a conventionalist in logic and Popper was swayed by Lakatos to reject conventionalism in mathematics. The result is that neither of them offered a comprehensive view.

This is not a critique. It is clear that mathematics is more deeply linked with language than physics. I have no need to mention the contribution of formalization to mathematics, from Hilbert to Cohen. No attempt to formalize physics has done anything for physics, as yet -- to philosophy and to mathematics, but not to physics. Moreover, whereas with Hilbert and Godel and von Neumann it is not hard to declare each of their papers as belonging essentially to logic <u>or</u> to mathematics, model theory, at least since Abraham Robinson, is decidedly both a central contribution to logic <u>and</u> to mathematics, as Godel noticed in his comment on Robinson.

In view of all this, I hope I am allowed to conclude that the field is in a fluid state and be forgiven for my ignorance. We once knew at least what objects mathematics handled, whether numbers and figures, or groups and fields. With the stalemate in the foundation of mathematics, one can conclude that even the question of what is mathematics about is very much an open question.

This leads me to training. Two major evils are specific to training, if we leave to the end what is wrong with education. Ever since Euclid became a standard text, and at least since the days when Archimedes wrote his breathtaking masterpieces, it has been customary, more so in mathematics than in any other field of study, to train through the teaching of textbooks. Textbooks have characteristics that have evolved, but which started as the characteristics of Euclid's <u>Elements</u>. Anyone in doubt should look at any geometry textbook and see how much it is indebted to either Euclid or Hilbert, not to mention Hilbert's conscious debt to Euclid. Historians are now arguing about Euclid's aims: why did he write his classic book? I do not know. I cited a famous sentence of Proclus, in which he says Euclid was executing Plato's program, in the presence of some leading historians of mathematics, and one of them, a very clever fellow, said to me: I like Proclus' statement since I am biased in favour of metaphysics, otherwise I could argue against it just as well. He said that Proclus was not the most reliable author in antiquity. I conceded. This exchange took place after a dramatic public exchange between Szabo and Jaakko Hintikka about the question: what was Euclid's aim? Why did he not tell us? Clearly he did not know we would be so interested.

The Lakatos revolution is the end of the textbook. Hilbert told us why he wrote his book on geometry, and was generally excellently clear about his aims. There is still a debate about his program, but at least he had one. The study of mathematics in the future will be frankly programmatic: programs will be put on the agenda, everyone will belong to the steering committee that will decide the agenda, simply because groups with different agenda will do different things and they will then pool information and rediscuss the agenda. I am predicting the future on the basis of a hypothesis that learning by agenda is ever so much more powerful than learning by textbook, that by natural or competitive selection the one to introduce it will be the winner.

And the increased efficiency is two-fold. First, specificity. The mathematics required by (1) the amateur, (2) the applied mathematician, (3) the math teacher, and (4) the researcher mathematician, are so very different that each needs a different agenda. Even when all four want to know what an axiom is, each of them will doubtless approach matters differently. And agenda-making is active student-participation, and educational psychology is unequivocal about certain matters: there is no better training than by active participation, trial and error, etc. Here a few writers in different fields collude: Wiener in cybernetics, Piaget in developmental psychology, Chomsky in psycholinguistics, Popper in scientific method: they all favour active participation. So I can move on without driving home this point any further.

I wish to speak of education proper now. The worst thing about motivation theory, to return to my old <u>bete noire</u>, is not that it is an advocacy of lies, though it is; not that it is based on contempt for the one to be educated, an unjust expression of superiority of the educator, though it is that too; the worst of it is that it is a system of training for independence. The teacher makes a simple mistake, has a simple optical illusion. He wants his student to listen to him, and for that he demands obedience to himself. When he does not get it, and children are hardly ever fully obedient by the clock, he feels justified in breaking their backbones. He then breeds well-read, well-trained professionals lacking any backbones.

I wish to state my view, if I may, that even the most intellectual, most abstract achievement, is impossible without a measure of moral independence, since without it the Church would still dominate the universities today as when it founded them. I also wish to state, if I may, that independence is a way of life, and a constant struggle. Allow me to illustrate this last point. I have always deemed myself fairly independent, but then I noticed that once, after I had read a paper and received an adverse comment, I did not handle the comment as maturely as I should have done. It was a few years ago; both Professor Grunbaum and I were guests together in some German university, and he did me the honour of attending my lecture. He said it was ill-prepared and half-digested and advised me to put it aside. I did so. After a couple of years I read it and decided I had been browbeaten. I do not wish to blame friendly Professor Grunbaum, nor to express a conviction that my paper was good and his judgment incorrect: for all I know he was right. Rather, I want to confess I was easily browbeaten, and though he only advised me I took it as a browbeating and capitulated instead of attempting to form a judgment of my own -- whether in agreement with his or not. There was no harm done, and the paper will soon be printed, I hope. But I wanted to indicate how easy it is to be browbeaten. How much easier it was to be browbeaten when I was an inexperienced, bewildered pupil who could not even effectively rebel!

I think the following question always arises at this junction: how soon do you take the child to be a fully responsible student, a real researcher? My answer is, from the very start. We can always present agendas to children, both in intellectual terms and in occupational terms. And if their assessments are erroneous, at least they are theirs. We learn from both ethology and evelopmental psychology that it is useless to work on a stage prior to the student's ability to be in that stage. And he will arrive quickest at that stage by proofs and refutations, where he can be aided by his teacher -- so it is not the discovery method -- where he can see where he is going -- so he needs no phony motivation -- and where his progress is the correction of his error. The Lakatos method has the merit of taking the student from where he stands and using his interruptions of the lecture as the chief vehicle of his progress, rather than worrying about the teacher's progress.

But there is more to it. We still introduce geometry using the axiomatic method -- without knowing why, without being able to explain why. And often enough we use a variant of Euclid. This is wrong. The axioms only bewilder innocent students. We still teach the multiplication table as if there are no pocket computers. But children know there are and view standard arithmetic instruction as sheer punishment. I put all this as empirical findings.

There is still more to this. We all know of _idiot savants_. These are people able to do arithmetic like computers. They don't have to be idiots -- for example, Von Neumann was an _idiot savant_ -- but it helps. Why? This is an intriguing question. We can try to find the answer in another and similar phenomenon, the infant musical prodigy. In the last century it was taken for granted that only a _Wunderkind_ can become a concert pianist. Today we know otherwise. In the last century, however, all concert pianists had been _Wunderkinder_. Why? Because others had piano teachers who insisted on their keeping their elbows to their waists. In other words, _Wunderkinder_ escaped instruction. Perhaps the same holds for _idiots savants_. If we assume that physically all brains are similar enough, we may assume all brains are pocket computers, and school usually destroys them. If so, we can help kids learn arithmetic without pressure and see whether every child can be made an _idiot_ savant. After all, there is the Japanese school that by this method makes children into muscial prodigies! We can play with them, counting up and down and in all sorts of series and see how well they learn the names of numbers and of operations without learning them as operations; and if they get the results of the operations right we may soon have grandchildren who will not purchase pocket computers for want of any need for them.

This is frankly a speculation, and one that may well be refuted. Still, I hope I am allowed to conclude with an empirical observation of how arithmetic is taught in vast portions of the western world. It is a fact easy to assess by seeing what teachers' training colleges advise their students to do and how this affects their conduct with children. The schoolma'am -- the sexism is not mine but of the system -- begins with counting concrete objects and adding concrete objects as a preparation for the abstraction from objects to numbers. I interrupt the story for a comment. It is not clear whether concrete cases such as two apples are easier to comprehend than abstract cases such as the number two. By Frege and Russell this is so, by Zermelo amd Fraenkel it is not, and as to Peano I cannot say. Staying within the system of cardinals, though, is the Frege-Russell view better than the Zermelo-Fraenkel? Both Fraenkel-Bar-Hillel and Quine abstain from judgment. But our schoolma'am does not know. She is convinced that the concrete counting is simpler and should be abstracted to abstract counting. In this idea of abstraction she is very advanced and unknowingly follows Dedekind Was sind und was sollen die Zahlen? Except that no mathematician today agrees. Our ma'am is a bit apprehensive since she was told in a didactics class to prepare the ground well for the jump. After many boring repetitions of two plus two this make four this, and two plus two that make four that, she asks the brightest kid in class what is two plus two. She is nervous and therefore so is he; he fumbles; she gets impatient; he gets lost; she takes a grip on herself and goes a step back: she asks him: how many are two this plus two this? Four this, he mumbles. She encourages him. And two that plus two that? she triumphs. Four that, he answers, encouraged. And two plus two? He fumbles again.

I interrupt again for a comment. The mistake was to choose a bright kid; he smells a rat. He cannot articulate his trouble, but being troubled makes him an excellent Russellian, perhaps even a potential Lakatosian. But the schoolma'am loses her nerve. She knows the right moves, makes them, and fails; her self-confidence is shattered. She blames her pupil for treason and threatens to withhold her love. He breaks down and says, four. Yes, he knew the expected answer all along, but hated it. He is in the right and is told he is in the wrong. His only way to maintain his independence is to say, I am bright but am no good at math; I will move to poetry. Except that he may have the same experience there too.

This little drama is no creation of my frustrated theatrical talent. I have seen this happen in schools in several countries, in arithmetic and in algebra and in geometry. I have no time to report the case of solving equations of the second order and analysing that case -- I have done so in another paper of mine. My point is, this ought to be stopped.

I have thus arrived at the end of my first part, my critique. There remains the design of the new program and the way to effect the transition. I will now conclude this presentation with one brief point on these two items. Any of you who has tenure and feels very brave and experimental can try the following recipe. First, let him treat his students as respectfully as possible and not motivate them, and explain frankly to them his conduct and position and situation as best he can. Second, let him discuss with them the shortcomings of the system as best he can. Third, let them discuss the design of the new system, and the possible ways of effecting the change as equals with him as moderator. I cannot do so now. I have a few papers and two books on the topic. I cannot get them published. I was amazed and delighted with the fact that this group has invited me to express my very uncomplimentary view of the state of the art to which you are devoted, fully or in part, but committed to it; and I can only express my ardent wish to see the members here go home and start to implement the Lakatosian revolution. Whether it goes this way or that way, it is bound to do good and be most exciting.

ALTERNATIVE RESEARCH METAPHORS AND THE SOCIAL CONTEXT OF MATHEMATICS TEACHING AND LEARNING*

Jack Easley

Committee on Culture and Cognition

University of Illinois

My point of view toward mathematics and mathematics teaching is undoubtedly colored by my background. I hated arithmetic, but I liked algebra because I could do it better than most of my fellow students, and I loved geometry for itself.

I took my first degree in physics because I was a natural born tinkerer, and mathematics had no lab, or computers, in those days, to provide an external source of insight. Besides, what math was taught in the small liberal arts college I attended was clearly subservient to physics; namely, calculus, advanced calculus, and differential equations. There was no reason for me to suspect it was the queen of the sciences. I took my graduate degrees in education, having abandoned physics in the middle of quantum mechanics whose differential equations I couldn't solve -- perhaps, as I thought, because I couldn't visualize the physical system. They seemed to work for others like magic and that I didn't like. But education was, and still is, searching for foundations: philosophy (but philosophy of science was where the action was), psychology (but cognitive psychology was where the action was), sociology (but linguistics and ethnography were where the action

*Based on a talk given to the Canadian Mathematics Education Study Group, Kingston, Ontario, June, 1979.

was -- and still is). Clearly, I wanted action -- something to compete with physics which I had abandoned - or it me? I still wonder if it will return to Einstein's view that being able to compute psi square is not enough.

I joined mathematics education formally with my second academic post, in 1962, as Research Coordinator for Max Beberman's second UICSM math project -- the one that produced "Stretchers and Shrinkers," "Motion Geometry " and "Vector Geometry." (The third one, on elementary school math, died abortively when he died in England some ten years ago.)

What I admired in Max was that he wanted children to think, to discover mathematical ideas and patterns, and to learn how to think mathematically. He was a master craftsman! Given a complete mathematical analysis of the curriculum which Herb Vaughan, Bernie Friedman (whose version of applied math turned out to be as pure as Vaughan's) and other mathematicians taught him, he could bring it back to life in the classroom. I became acquainted with the two upper-right boxes in the diagram reported last year from the CMESG. (Figure 1)

Max Beberman had a few apprentices. I had been an apprentice in '59-'60 during the first UICSM project. Alice Hart, Eleanor McCoy and a half-dozen others were much better apprentices -- totally dedicated to absorbing his classroom creations for years or even decades. I wasn't dedicated to mastering his craftsmanship. I had found out what he could do, in 9th grade algebra and learned how to do it myself. I wanted to know why it was possible, and I wanted to master teacher training in a more direct way than Max himself could.

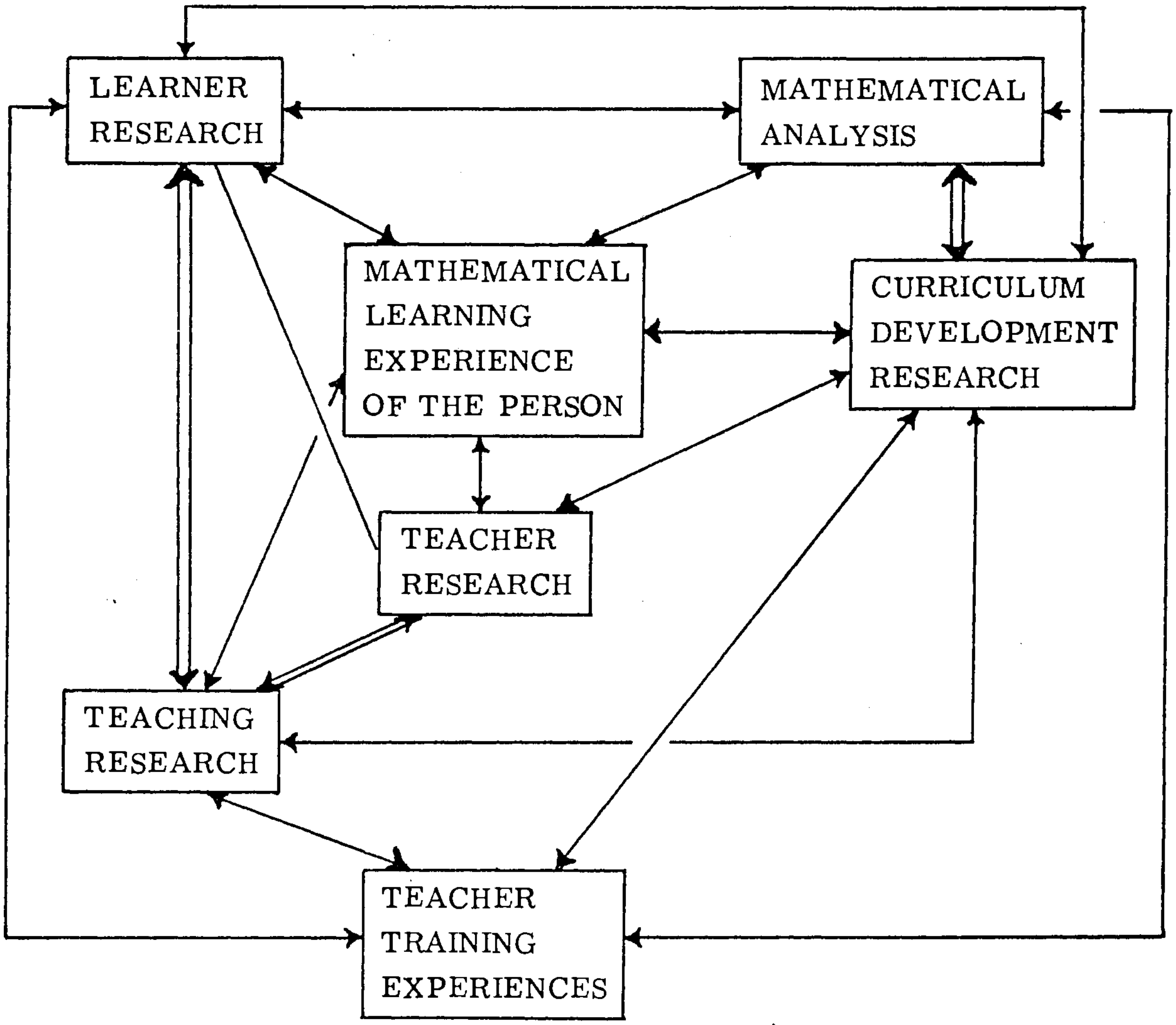

LEARNER RESEARCH
MATHEMATICAL ANALYSIS
MATHEMATICAL LEARNING EXPERIENCE OF THE PERSON
CURRICULUM DEVELOPMENT RESEARCH
TEACHER RESEARCH
TEACHING RESEARCH
TEACHER TRAINING EXPERIENCES

Figure 1

I was fascinated by his critics, whom Max regularly invited to visit the project: Morris Kline (who never came), Alexander Wittenberg, George Polya, and Imre Lakatos (whom I invited). Their arguments (and demonstration classes in Wittenberg's case) were very moving. But I believed Max was right -- you couldn't construct a formal pedagogy out of their non-mechanical intuitions! (It took a mechanical [i.e., dead] analysis built on sets. Gentzen's natural deduction (1969) or the like, to produce a clear set of classroom principles.) If only most mathematic teachers were amateur mathematicians, reading Martin Gardner's column in Scientific American regularly, trying out his puzzles, being taken in by his April Fool jokes and sheepishly spending the next few months working out how they got taken in!!! Then, they'd know about mathematical intuition and recognize it when they saw it. (Or would they?)

I became convinced that the way Max's teaching worked could be better understood by means of Piaget's research. Piaget, it seemed, had the only psychology of mathematics and physics going -- but I was later persuaded I was wrong. Piaget wasn't dealing with psychology at all, but philosophy, anti-positivist, anti-empiricist, anti-realist, above all anti-phenomenologist epistemology. Psychology was only a sort of weapon with which to fight Carnap, Popper, Russell, Husserl and Merleau-Ponty (see especially Piaget, 1969). He flaunted his genetic fallacy (genetic epistemology) in the face of the criterion philosophers, and undermined the experience philosophers with cognitive structures. I shifted my attention to the two blocks in the upper left of Figure 1.

Piaget appealed to mathematicians and logicians to help him discover how mathematics was biologically possible and was intrigued with the first, and the so-called "second", Erlangen program because Kline saw geometry and MacLane saw algebra as emerging from the underlying structures of transformation groups and categories, respectively.

The paper Piaget circulated at Exeter (1975) endorsed the "new math" as a content field within which teacher and pupil should work, but he objected to formal methods of mathematical work. This didn't seem to help very much. What kind of theory was needed on which to found a pedagogy of mathematics? No answer was forthcoming, but the narrowness of most research in math and science education appalled me, and I tried to prepare my doctoral students for a "no-holds barred" attack on this problem. We needed to be able to understand why things went right and wrong, explain to teachers what puzzled them, and provide a conceptual framework against which they could work on their own problems, pedagogical ones. The first part of what follows illustrates my quest for such positive foundations and the second part questions it all. (Dan Knifong, George Shirk, Stan Erlwanger produced notable math education studies. Ted Nicholson, Bernadine Stake, and Shirly Johnson have dissertations in progress. Linda Brandau and James Kan are doing preliminary studies.)

In 1977 I published a paper, from which the table of Figure 2 comes. I don't want to discuss that paper except to say that I didn't find it very helpful in guiding my research. I wrote it to call the attention of my students (in several content fields) to metaphors implicit in most educational research. I couldn't solve the problem of how to choose better ones.

I do want to say something about that table that's not in the paper. There is an order -- perhaps a partial order -- in the table concerning the proportion of formal and informal research being done. (See Figure 3.) By formal, I mean that algorithms or mathematical theorems can be used; by informal, I mean they cannot be -- at least, until they are developed. So the seventh perspective has the least use of mathematics, but still it has some. Other metaphors are needed, and informality is to be expected with new ones. (I would like to add two more metaphors: process and intuition, which have as yet been used mainly without mathematics. I'll say more about them later.)

Mainly, I want to say that with the informal uses of metaphors, the choice of metaphor problem can be worked on; but formal methods tend to stop the search for better metaphors.

KEY METAPHORS	Combinations ① Mixtures	Sampling ② Dice	Feedback ③ Codes Thermostat	Games ④ Poker	Operations Criteria ⑤ Consistency	Syntax ⑥ Language Games	Organic Structures ⑦ "Experts"
Typical Questions:	What % variance is accounted for by these variables?	What is probability of this result?	What is the means-ends program?	What are the rules the strategies?	What are the criteria?	What is the "logic" of "teaching," "learning," etc.?	What are the structures, how do they work?
Typical Solutions to:							
1. What makes t good at teaching x to y?	Interaction of t variables with y variables	Arranged or chance matching of t variables with y variables	Efficient coding system	Creative "plays"	Consistent modelling of precise language	y agrees on language game with t	Good match between t and y structures
2. Why can't p learn to do q?	p ignores q variables	p's sample of q too small	p uses wrong codes	p is playing wrong game	Use of inadequate language	Confusion of multiple meanings	p assimilates lesson to wrong structures
3. Why can't t teach x to y better?	t ignores y variables	Arranged or chance mis-match of t with y variables	t has poor information on p	Poor sportsmanship, etc.	Non-critical thinking	Thinks words carry their meanings	Forced patterns of behavior, against p's structures
Uses of Knowledge	policy to control variables	avoid decisions based on poor samples	adjust system for efficient information processing	follow optimum strategy	use consistent language	use folk wisdom	look for opportunities to teach

Figure 2

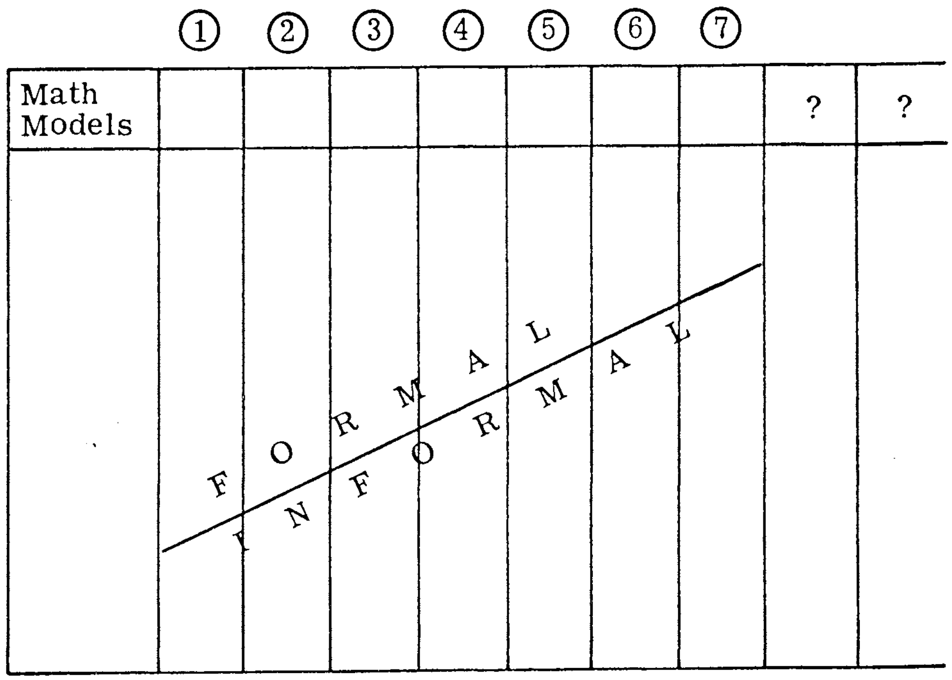


Figure 3

Perhaps the most important creative work in research is done between successive studies; that's where metaphors can change. That's what I want to try to share with you tonight - the thoughts and feelings between some of the studies I've been involved in -- with or without my students. So I won't give you the research details you might expect.

The following problems have concerned me for a long time:

What accounts for successful teaching of mathematics by the few?

What accounts for unsuccessful teaching and learning of mathematics by the many?

What could be done to remedy unsuccessful teaching and learning?

While I want to explore with you alternatives to those seven (widely recognized and utilized) modeling perspectives about which I wrote in 1977, I should say more about my disappointment in them.

There is a powerful cognitive effect in using one or more of the established perspectives: You believe you are open to every possible question and every possible answer, yet you tend to ask and answer only those questions that come most naturally within that perspective. You use the power of the models within the perspective but you don't choose the perspective heuristically. "Gestalt switches" between some of the seven key metaphors are very difficult. I find myself unhappy settling on any one of them, because each one seems impotent to improve mathematics and science teaching. However, what people want to know is why you can't modify or adopt some of these standard approaches to solve the problems of practice - and the only answer I've come up with is that I have really tried doing this, and I'd like to show you how, as a practitioner (teacher, tutor, teacher educator), I've tried to use a variety of modified standard approaches.

This is autobiographical, but that's not the point of it. The items are selected to make the above point and not to say who I am. Many other workers could select research experiences of their own to make the same point. In fact, only in that way can each researcher check the meaning and validity of my point for him or her.

Starting with the first perspective, the combinatorial or "mixture" conception, with Robert Comley (1965), I tried item-by-item comparison of student populations from different curricula as suggested by Cronbach (1964). Item profiles and matrix sampling don't tell you much -- it's too easy to make up hypotheses to explain the differences observed between the two populations (see Figure 4a.) Of course, this approach is more sensible than using test scores to compare students taking different curricula, when the test is, at best, related to just one of them. At any rate, we found few differences in the profiles from UICSM and traditional algebra classes -- certainly not enough to justify the investment that had been made in developing the new curriculum.

Here the conception imposed by a profile is that of a mixture of different ingredients in different proportions. However, good math learning does not appear to be a particular mixture of certain "ingredients", it's more like a structure -- more like a flower arrangement or a bonsai than a martini. At least, that's the way it feels to me.

With populations of students and teachers, factor analysis by Hiroshi Ikeda (1965) helped to make sense out of the observed mixtures. The hypothesis of Figure 4b, was that teacher-held objectives make a difference in what the pupils learn. What we found out was that teacher preferences for our test items could be represented by the two bi-polar factors shown. However, teacher's preferences did not correlate significantly with class performance on the same items. So how <u>do</u> you improve math education? We tried to use the psychometric mixture metaphor creatively but made no progress in a useful understanding. It is too formal and too shallow!

Of course, a structure should have many profiles projected onto different spaces. The choice of a space should be precisely determined by the inner form of the structuring, as a chemist determines it. However, without that guidance, the dimensions available make too rich an assortment to be searched through by trial and error until the structure is found.

The inference from test items to what concepts or attitudes are learned is much too "stretched out" to be made

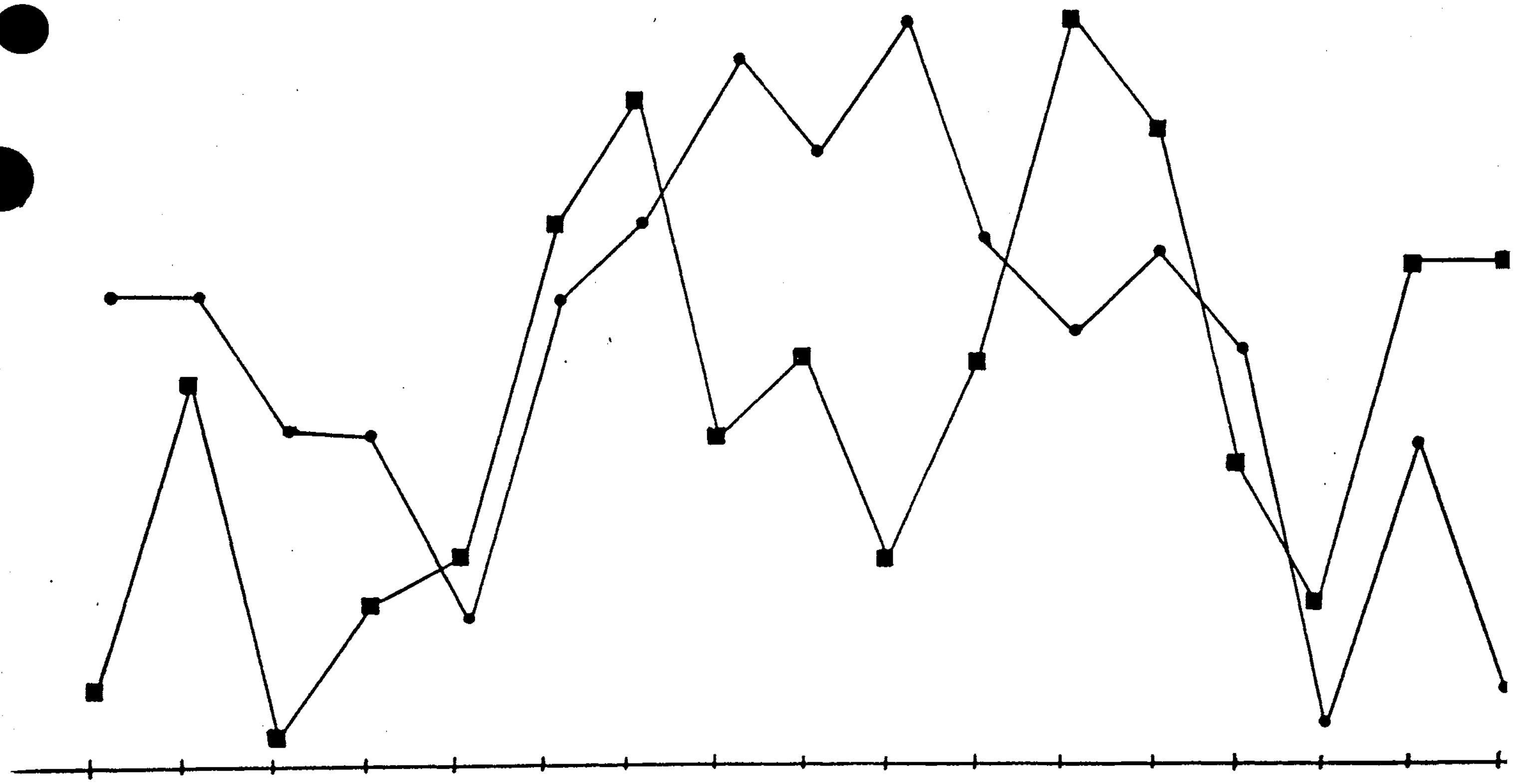


Figure 4a

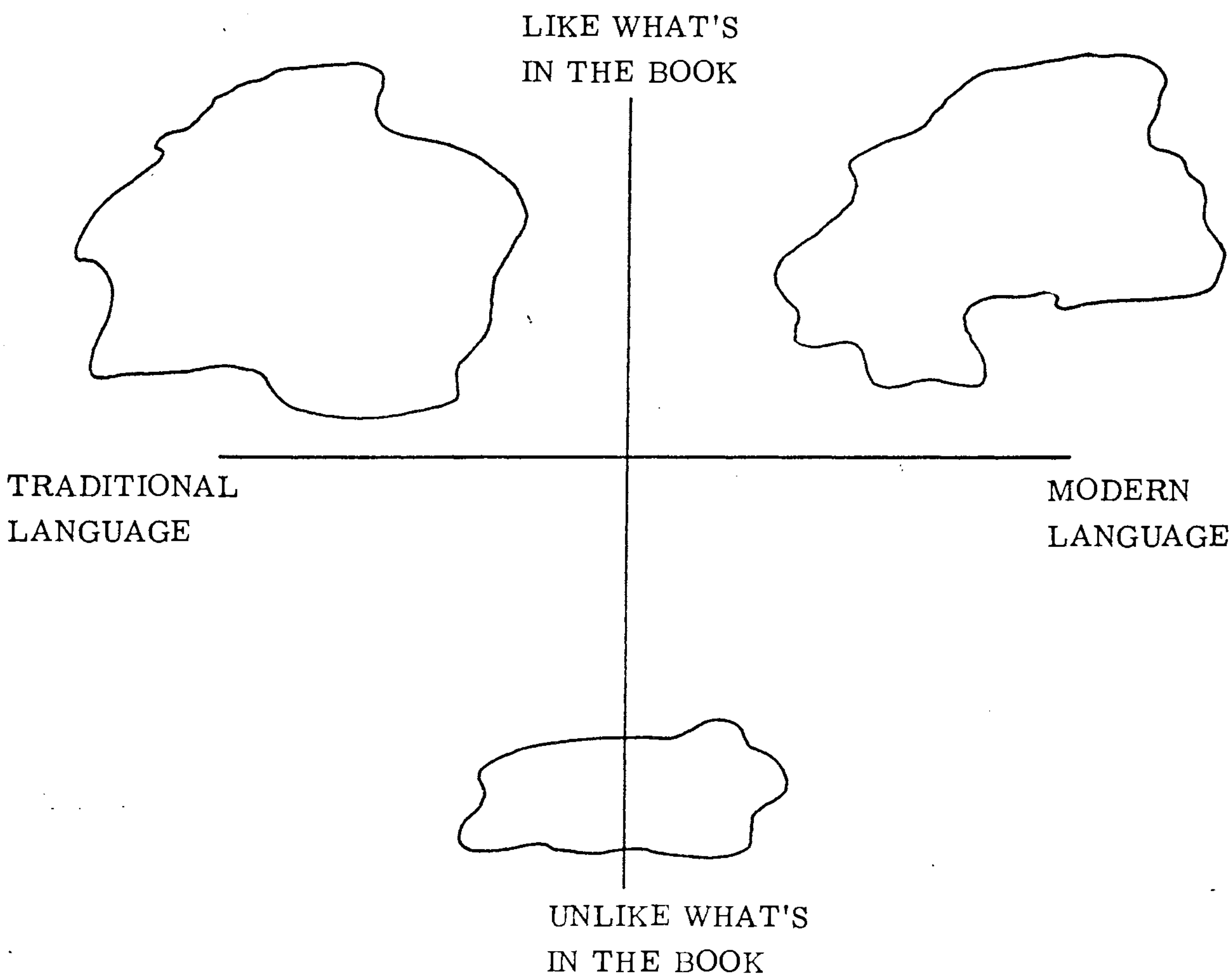


Figure 4b

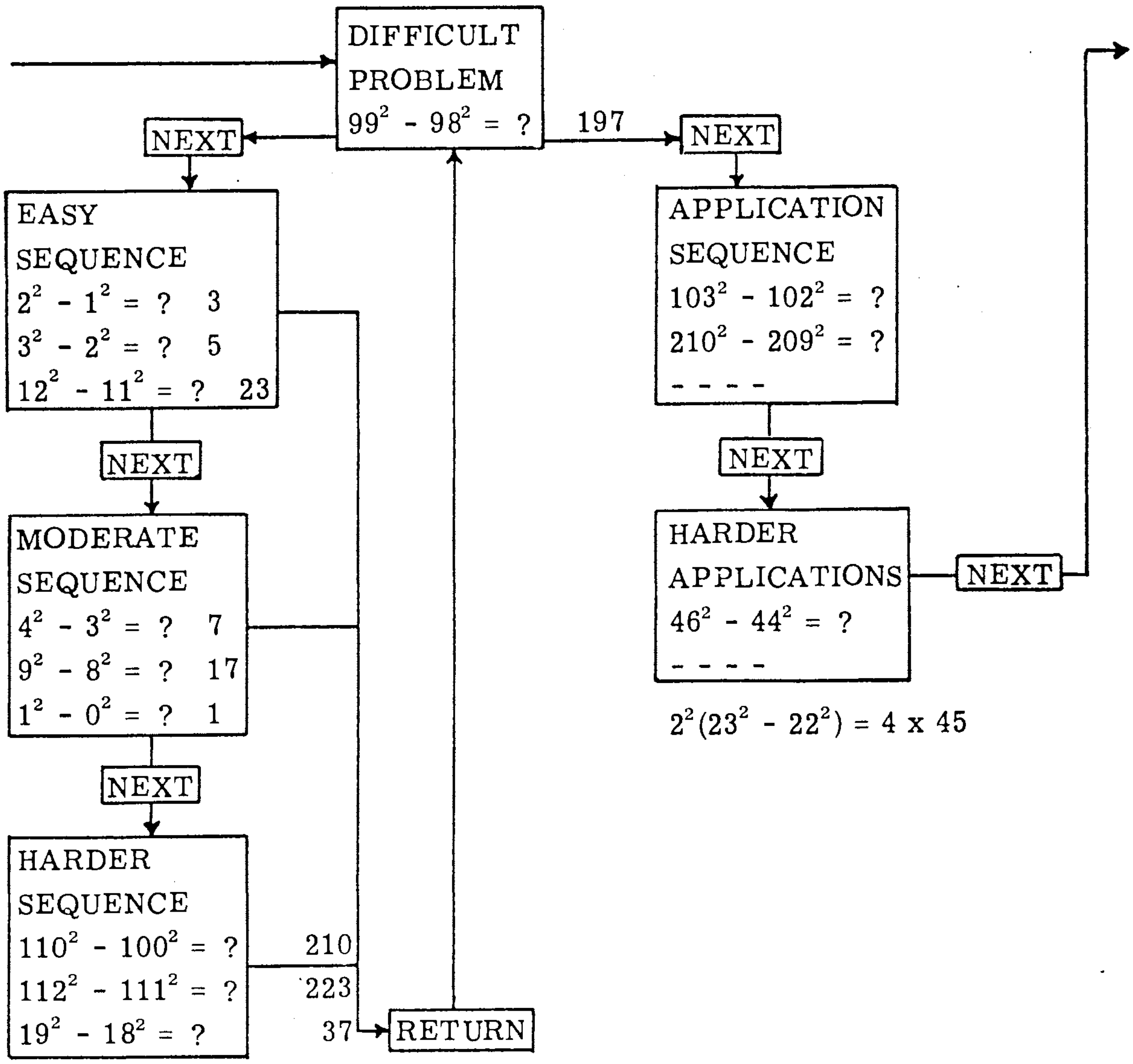
with confidence. This method, like other psychometric and statistical methods, is primarily a method of demonstration, i.e., demonstrating whatever variance is accounted for by the items used. So one needs a prior conception in order to generate the items. Today, with hidden curricula and alternative preconceptions better known, factor analysis seems very unconvincing.

Sequencing of Problems

Teaching is linear in time -- so there should at least be a partial order in the problems used in successful teaching that would be beneficial to other teachers. Such an order could be defined in terms of the properties of the problems used. Teaching appears to be much more than sequencing problems in terms of their formal properties. Of course, there are well-known formal snags in most disciplines that can be avoided by proper preparation. However, on the whole, learning does not appear to be linear and incremental; rather, it seems to involve unpredictable insights, forgetting, and much relearning.

Pattern recognition is a new metaphor, related to some others, that is emphasized by Polya. Like many metaphors, it seems to work when promoted by a skillful teacher. Could we understand it as a process in individual students?

There's been talk of "teachable moments". The idea occurred to me to try to capture them in terms of heart-rate increases, measured while students worked math problems on PLATO. I decided to try to find out if I could produce moments of insight in terms of the formal structure of problem solving using response times. (Of course, without paying attention to response times, it is possible to give correct answers to all the problems in such a sequence without recognizing the pattern.) Herb Wills, Al Avner, and Pat Cutler helped me devise problems (Wills, 1967), set up this program, and connect electrodes to the students' chests and analyze the data. Problems were arranged on PLATO as illustrated in Figure 5. Mathematical insight was indicated by successful return to the difficult problem after guided progress through easier



DIFFICULT PROBLEM
99² - 98² = ?
197
NEXT
EASY SEQUENCE
2² - 1² = ? 3
3² - 2² = ? 5
12² - 11² = ? 23
NEXT
MODERATE SEQUENCE
4² - 3² = ? 7
9² - 8² = ? 17
1² - 0² = ? 1
NEXT
HARDER SEQUENCE
110² - 100² = ? 210
112² - 111² = ? 223
19² - 18² = ? 37
RETURN
NEXT
APPLICATION SEQUENCE
103² - 102² = ?
210² - 209² = ?
- - - -
NEXT
HARDER APPLICATIONS
46² - 44² = ?
- - - -
NEXT
2²(23² - 22²) = 4 x 45

Figure 5

problems. Such insights were sometimes "hot" and sometimes "cool", as indicated by an increase or constancy of heart-rate. Moreover, there were some "hot" moments that were not related to any mathematical event. Clearly, this wasn't it! Something more than pattern recognition is going on in a class that becomes excited when they discover a pattern. What else is going on besides individual mathematics? Something social must be going on when teachers skillful at getting a class to make pattern discoveries exploit the emotional response of insightful students. So, it doesn't appear that pattern recognition is regularly exciting or motivating, but sometimes it clearly is.

In Figure 6a, the kinetic energy of a pupil thinking out loud while working a math problem is depicted as sound and movement energy might be recorded. Figure 6b represents the kinetic space constrictions and expansions during the same time period (Witz and Easley, 1976.) This opens new doors conceptually, but does not help solve the problem of math teaching.

In Figure 6c, a teacher-pupil dialogue is interpreted figuratively in terms of classroom competition and social control. Here's a new metaphor -- mathematics as social control (see Stake and Easley, 1978, Ch. 16.) Experienced teachers undoubtedly read the figurative meaning of dialogue and pay a lot of attention to it. Studying it helps researchers find out what else is going on in teacher-pupil interaction. Theoretically, it affects mathematical learning. But once anthropologists have learned to interpret the symbolic meanings of the classroom, can they be of any help to the improvement of learning mathematics? Some say not. I observe that:

1. It is hard to train observers to record this kind of meaning. It took me many years to get the feel of it and, even now, I have to concentrate very hard or I slip back into the so-called literal code.

2. It's not yet clear what teachers get out of this kind of reporting, i.e. there are only two or three teachers who have reported positive benefits, certainly not a representative group.

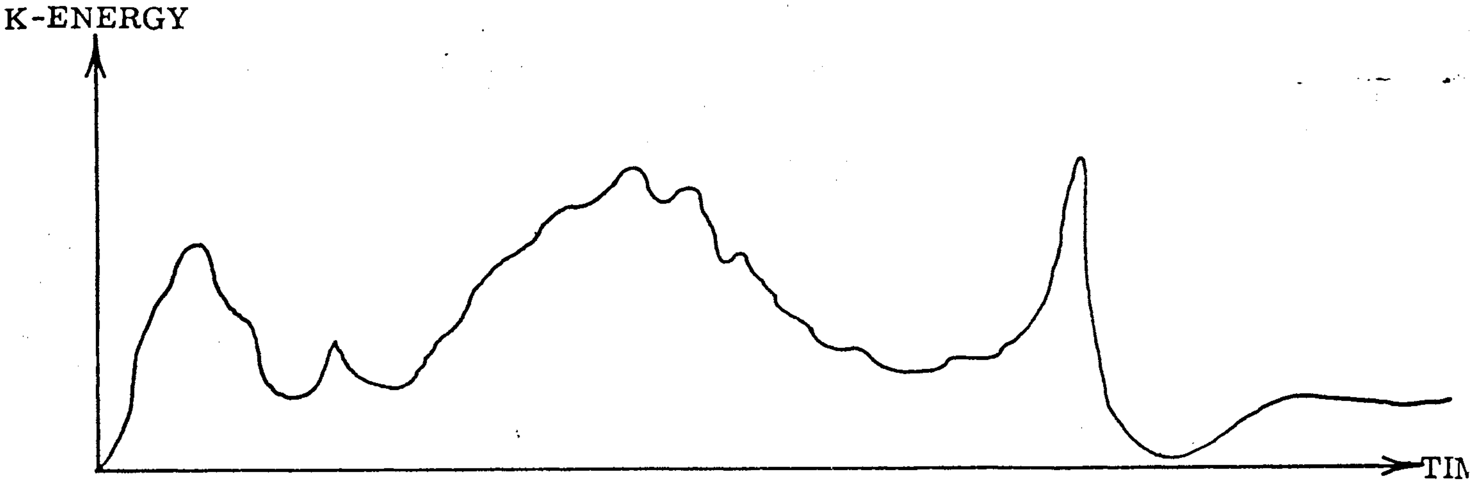


Figure 6a

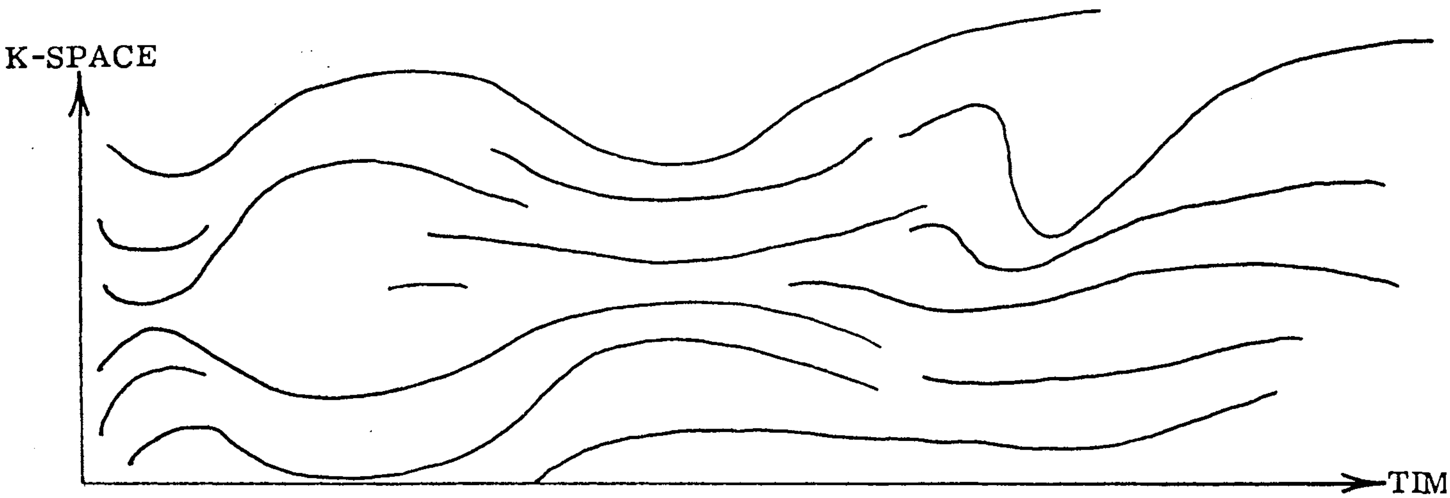


Figure 6b

Literal meanings	Figurative meanings
T: What's negative four, squared? (watches hands waving) Bill?	
Bill: That's easy, sixteen.	[You've got to make points when you can.]
John: Positive sixteen.	[Why'd you call on him? I'm smarter.]
T: We agreed to that.	[Cut it out, you guys. Everyone gets a turn.]
What's negative four, cubed? Mary?	

Figure 6c

3. It is far from clear what pupils will get out of this research although there are some promising signs. Certainly, we have them in mind.

Back into the literal code! Can we increase information processing and crowd out symbolic meaning?

Working with PLATO had inspired the notion, an elaboration of the third perspective, that its data-retrieval and editing potential could be used to develop a man-machine system that would be self-improving (Figure 7a) and adaptive to the learning characteristics of students. Starting with a curriculum system, it would augment the human intelligence that was involved with a computer. This is McLuhan-esque extension of our brains and sense organs. Don Bitzer (see Easley, 1966), Kikumi Tatsuoka, Betty Kendzior, Al Avner and others helped. It didn't work.

Teachers and authors were much too busy and too self-directed to hook into a man-computer system. Although I had mastered the PLATO computer operations for searching for error patterns and detecting possible pupil misconceptions, I could only discover the most trivial misconceptions. I couldn't use the system to improve it. All improvements I made came from reflecting on the natural teacher-pupil interaction phenomena. (Reports on this project, Easley, 1968, SIRA, were filed with the U.S. Office of Education.)

This remains a problem today with PLATO and similar systems. It was also a problem in the UICSM vector geometry curriculum project. Formative evaluation of UICSM texts was spoiled by the slow process of data collection and analysis. Ideas for improvement come from other sources. The big difficulty that took me years to recognize, is that learning how to teach mathematics is not the inductive, accumulative process all these models assume it is. It is some kind of constructive process. The constructive metaphor is another reason to look to Piaget and the seventh perspective. Furthermore, learning about mathematics teachers should be looked at similarly, as I intimated at the end of my 1979 paper. Using Papert's metaphor for the mind of the child as a "room-full of experts" (1975) who respond independently to problems in terms

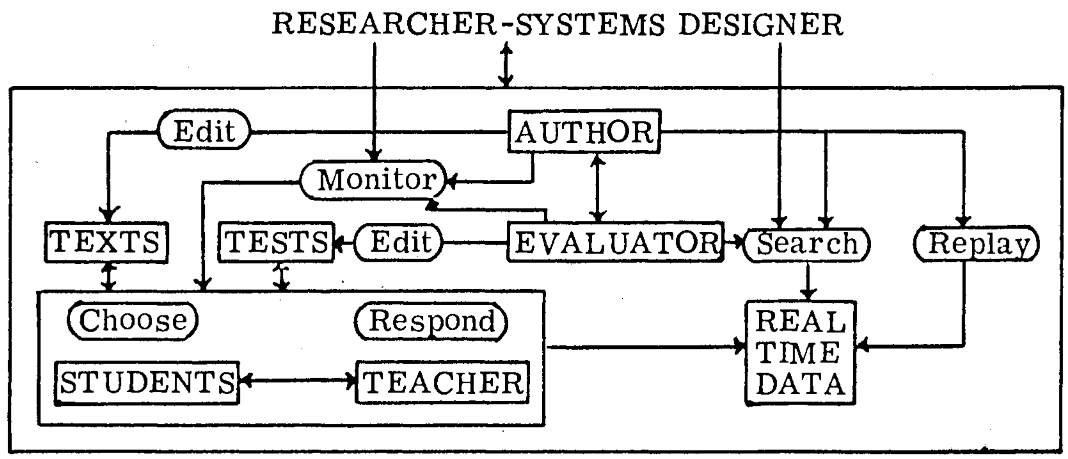


Figure 7a

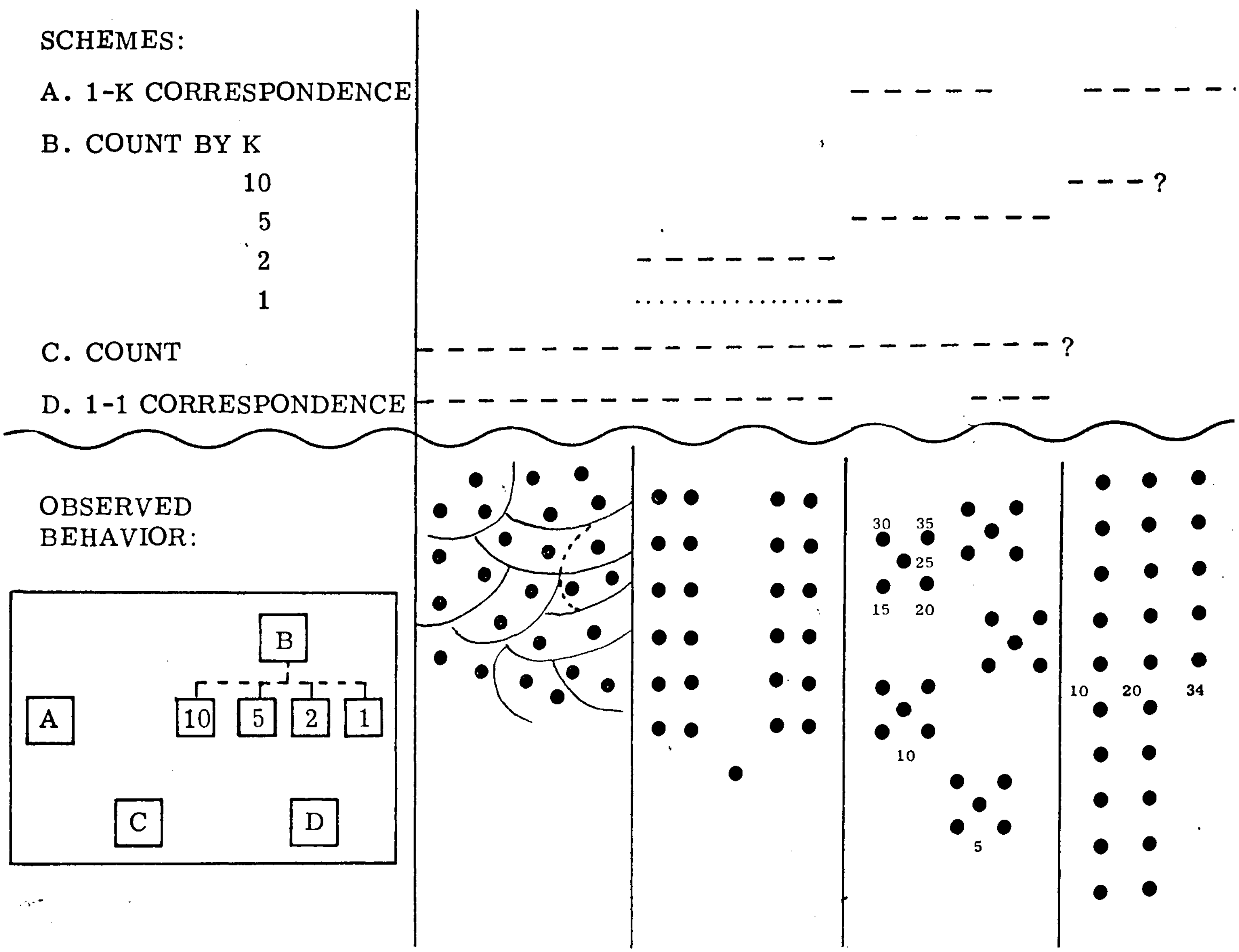


Figure 7b

of their expertise (but without any executive to decide which one responds to which problem), Bernardine Stake (1980) and I have been studying counting by primary grade children. We have found that the process of counting is more important to most K-2 children than the result of counting. Process is a key metaphor. Here's where the dynamic processes of cognitive structures need development. Several schemes can be distinguished as being involved in different aspects of counting (Figure 7b). For example, for the first one depicted, we developed a theory of certain counting errors. Figure 7b depicts the problem of controlling the partition or envelope that separates what has been counted from what hasn't yet been. The confirmation of a child's counting is illustrated as a small-scale process in which a cluster of objects is subitized as containing 1, 2, 3 or 4 and enumerated with a matching cluster of numbers from the counting sequence. The problem of how to control the spreading envelope that separates what has been counted from what hasn't been is not often addressed, in our experience, by primary grade children. The confirmation or checking process is located at the "micro" level rather than the "macro" level where the result occurs. This theory facilitates posing counting problems for K-1 children to work on, if teachers could be persuaded that skill in counting disorderly collections is <u>not</u> an objective for which all children should be taught a solution.

Some answers to some problems are so clearly not produced by pedagogical procedures that it is necessary by inference to bring in intuition as another metaphor. In another metaphor, "teaching by listening," Easley and Zwoyer (1975) placed a lot of hope. Just listening closely to a pupils' erroneous explanation often seems to stimulate incubation or richer, better understandings. Several points need to be made:

1. Probing, in a clinical interview, to expose students' ideas is a necessary art.
2. Listening is difficult when you are concentrating on what you want to teach.

3. Offering counterexamples can help clarify students' ideas, if one has listened well.

4. The student often straightens himself out a day or so after he/she has tried hard to explain things to a good listener.

When Bernadine Stake and I interviewed primarily grade students on mathematical concepts twice on successive days, they had always improved on the second. Figure 8 illustrates a confusion which a 5th grade student straightened out by himself within a week of this interview.

Working in the seventh perspective, Klaus Witz and I solved the problem of the identification of cognitive structures (Witz, 1973; Easley, 1974.) We showed what problems are created for test item analysis and profile interpretations by the alternative structures a child would likely have available to use in responding to a single item. But our interview methods can't be used by many teachers in the classroom -- no probing is usually possible there.

Bernadine Stake and I worked on teachers' problems. For example:

Why aren't children in primary grades better able to remember numbers they have just counted or computed? We looked at long-term and short-term memory. What are children doing? Why? There's one partial answer in our theory above.

Teachers do a lot of piloting of calculational processes. Why is so little significant struggling with conceptual frameworks going on in classrooms? Teachers try to make math easier and wind up making it harder later on. Piloting is Lundgren's metaphor (1979). I want to study piloting systematically. The structural reconstruction of children's ideas really doesn't help many teachers (even very good ones) because they have their own "agenda". Consider the dialogue in Figure 9 (abstracted from several times this much material in Rosalind Driver's (1973) dissertation). The teacher's intervention clearly interrupts the students' discussion. The teacher and the students are operating in two incompatible frameworks. The teacher automatically reduces everything to measurable variables and the students employ

S. 1.06 > 1.50 > .79 > .4 > .002

I. Explain it. What about the first two?

S. That's just the way it is. 1.06 is bigger than 1.50.

I. What if you had 1.60?

S. That's the same thing as 1.06.

You see, $.06 = \frac{6}{10}$ and $.60 = \frac{60}{100}$

and $\frac{60}{100} = \frac{30}{50} = \frac{6}{10}$. So, .06 = .60 and 1.06 = 1.60 .

I. What if you had $1.60 and I had $1.06, who'd have more?

S. I think you're trying to kill me with money.

Let's see, with money it's different. It's the way you say it, you'd have a dollar and six cents and I'd have a dollar and 60 cents. But with gallons or liters, you'd have a fraction, and they're the same.

I. So what does .002 mean?

S. $\frac{2}{10}$

I. .4?

S. $\frac{4}{10}$

I. .79?

S. $\frac{79}{100}$

Figure 8

S_1: Well, I say that when you put more weight on the table there's no more resistance. It just sits there.

S_2: But, you see, it's got to resist more to hold up the larger weight.

S_1: Don't be silly, the table's just there in the way.

T: (dropping by) What variables have you been measuring?

S_2: Well, we haven't actually measured the weights yet, we're waiting for a scale.

T: What else are you going to measure?

S_1: How high up the spring holds it.

T: Oh, the length of the spring. Anything else?

S_2: Oh, the diameter of the wire.

T: Here, you can use <u>that</u> scale. (leaves)

S_1: The table's not doing <u>anything</u>.

S_2: If you make the weight big enough, it'll do something. It'll smash.

S_1: Well, if you let an elephant stand on it, sure! Then it doesn't resist anymore.

S_2: Look (places a meter stick so half of it extends off the table), when you put a weight on there, it will bend--so you know how much it pushes back.

Figure 9

concepts of resistance in which whether resistance is a variable quality for a given table and whether it is measurable or not are precisely the points of their argument. They tolerate this pluralism with mutual respect and get along fine.

Stake and Easley (1978) published ethnographic portrayals of what math teachers are teaching (see also Easley, 1979.) We found they teach:

a) personality and personal values by exhortation and example

b) group countrol (without either grouping by ability or isolating the slow, difficult and destructive)

c) socialization: trying hard
following directions
following rules
admitting errors
neatness
submitting to drill, hard work, memorizing

d) values: "Do your own work."

We also found that teachers:

a) have problems understanding forgetting and learning.
b) cry for help in adapting math materials to children of different backgrounds from their own.
c) believe that math supervisors, teacher educators, and administrators can't help because they're not even considering how to use math to control and socialize children.

In Figure 10a is depicted the closing moves of a game of "guess my rule" as played by a 7th grade pre-algebra class. The teacher watches while the class guesses the rule of one pupil, in the process generating the table on the left and the equation underneath. When the second table is set up by the winner of the previous game, the teacher intervenes, calling first one pupil and then another to the board to guess entries

GUESS MY RULE: PRE-ALGEBRA, 7th GRADE

$\triangle$	$\square$
1	0
2	3
3	8
4	15

$$(\triangle \times \triangle) - 1 = \square$$

$\triangle$	$\square$
4	1
9	2
5̸	

$$\sqrt{\triangle} - 1 = \square$$

$$\triangle = \bigcirc \times \bigcirc$$

$$\bigcirc - 1 = \square$$

T: They have to keep trying!

Figure 10a

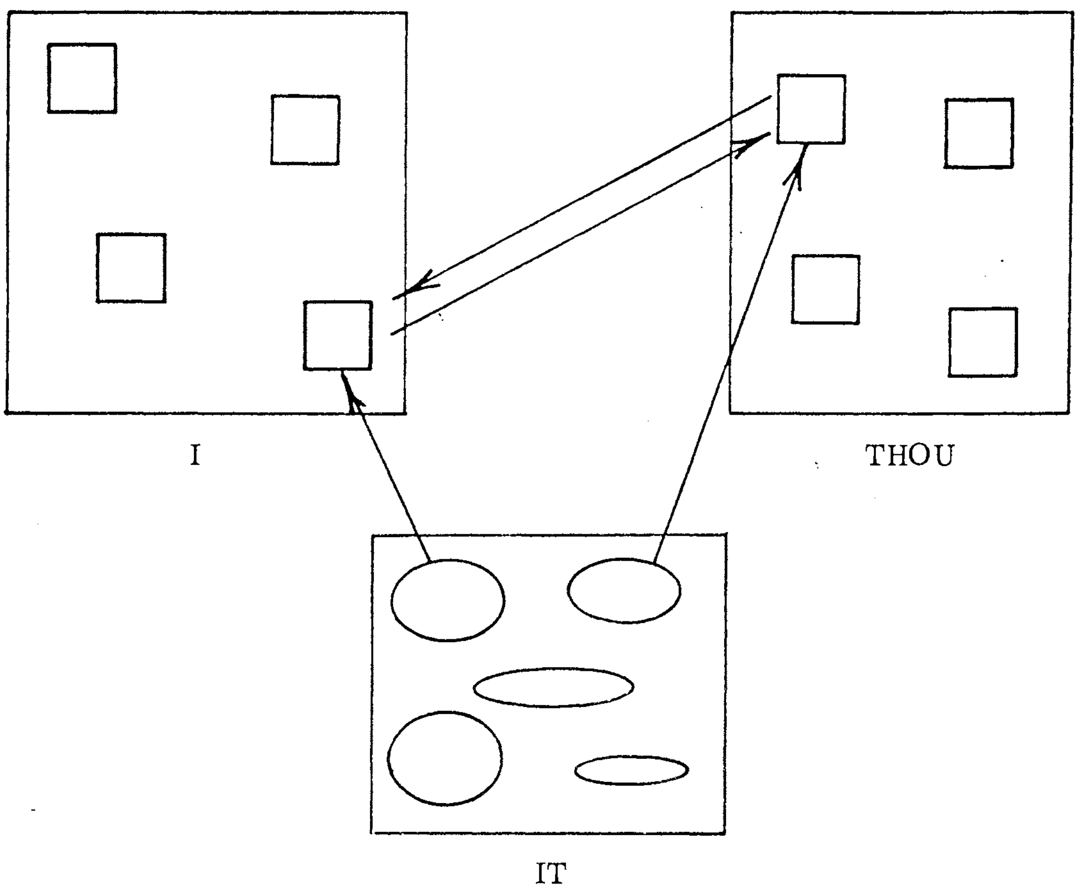


Figure 10b

for the left column. Two are stumped because their entries are not accepted (they are not perfect squares) and they refuse to continue guessing -- to the teacher's great consternation. Later, she explains: "They have to keep trying!" The teacher's dramatic take-over of a student game is somewhat softened when she tries to write the question without using the radical sign -- since it has not yet been introduced. The observer helps her out with the suggestion that she write two equations - which she does at the right. The items on her prime agenda are clear! She wants to shape all of her pupils into persons who'll cope well with algebra in terms of their personal style or manifest attitude.

How does this picture help? To many it is a picture of despair. No one group of people (teachers, math specialists, administrators) can do anything much without the others getting involved. The whole "eco-system" has to be examined by its own elements (we have to discover the enemy and recognize it as us.) Old pedagogical metaphors get buried in mathematical research methods -- but that's the wrong way to do applied math. Fresh conceptions are needed that are sufficiently close to the target practitioners: teacher educators, teachers themselves, or students, so that they can recognize how to express their problems and can (or could) participate in shaping the research. Then they won't see research as a monstrous distortion, casting their problems into a foreign metaphor -- or if they do, and they have the courage to say so, they will recognize, as some researchers don't, that no particular metaphor is a necessary one. It is truly an "I, Thou, It" situation (Hawkins, 1968). In figure 10b I attempt to represent that different aspects of the "It" are interpreted in terms of different metaphors in the "I" and "Thou".

Whether multiple metaphors can help is a good question. Perhaps, it may be argued the real problem cannot be fully represented in any one metaphor. It is difficult for me to presuppose a real world in which the real problem exists objectively. An alternative we can consider is that the problem exists in the constructed world of the practitioner's experience, and it is that experience that holds together the multiple images of several metaphors. Lakatos (1976) has implicitly shown that within the Eulerian research program of Proofs & Refutations, unlike in a stable Kuhnian period of normal science, inquiry moves across several different metaphors of polyhedra. Should we expect less of mathematics practitioners and researchers?

Considered strategically, the first main point is that such informal methods as clinical interviews and participant observations have convinced me that children and teachers alike have ideas I wouldn't have suspected, which if ignored may lead them to mistrust their own responses to teacher education programs. As a teacher educator, I risk putting teachers down if I ignore their ideas but demonstrate to them how they can use their students' ideas. Can I work in a way that shows the same respect for teachers' ideas as I have for students' ideas? My concern for the damage they may be doing to students' own self-confidence may so intrude in my relation with teachers that I spoil their self-confidence and the opportunity for real communication with them.

The second point is: the more I study math teachers, at any level, the more convinced I become that mathematics is so embedded in social interaction -- where they have their own sense of how the game is to be played, what standards of justice they are to maintain -- that I am forced to enter into a dialogue, with them about ethics, personal values and culture, and help them struggle with issues they are even less prepared to face. Where do either of us get support for rethinking the ethics and sociology of math teaching and learning? Is equity the only issue? Is social stratification of society in terms of mathematical ability wrong? Shouldn't those in responsible technical positions: doctors, lawyers, engineers, etc. be required to have superior abstract reasoning ability?

The third point is that a very special design is needed to open up free communication between groups of math educators taking different perspectives.

1. CAMP MEETINGS (Communication Across Multiple Perspectives)

i.e. identification of viewpoints to study mastery to criterion by all.

Video		Teachers
	X	Students
Tapes		Specialists

2. LONGITUDINAL CASE STUDIES of teachers and students representing different perspectives

3. Put downs are so highly likely that an advocacy system may be needed.

Methodologically, the need to represent to researchers and curriculum specialists the problems that really concern teachers is primary. This means representing problems in their own frameworks, for the frameworks that researchers are accustomed to use really distort their problems and make the results useless for most teachers. (See Figure 11.) To do this, informal methods are required, by definition (except possibly for those few teachers with standard research training).

Clinical interviews with teachers and participant-observer case studies are required to capture their view of their problems, and some integration during investigations of such problems is needed to lead toward solutions they will recognize as worthy of their attention. It seems to be an act of faith that a teacher's problem solved will represent a step toward real improvements for students' mathematical learning. Of course, solving one problem often exposes another more serious one. But failure to solve genuine problems serves only to leave teachers who have those problems stranded and unable to reach the breakthroughs.

Conclusion

Even in the most educationally valuable classrooms, potential communication is missed to an extent that would shock the teachers if it were demonstrated to them. That is, as teachers we always think we understand a good deal more

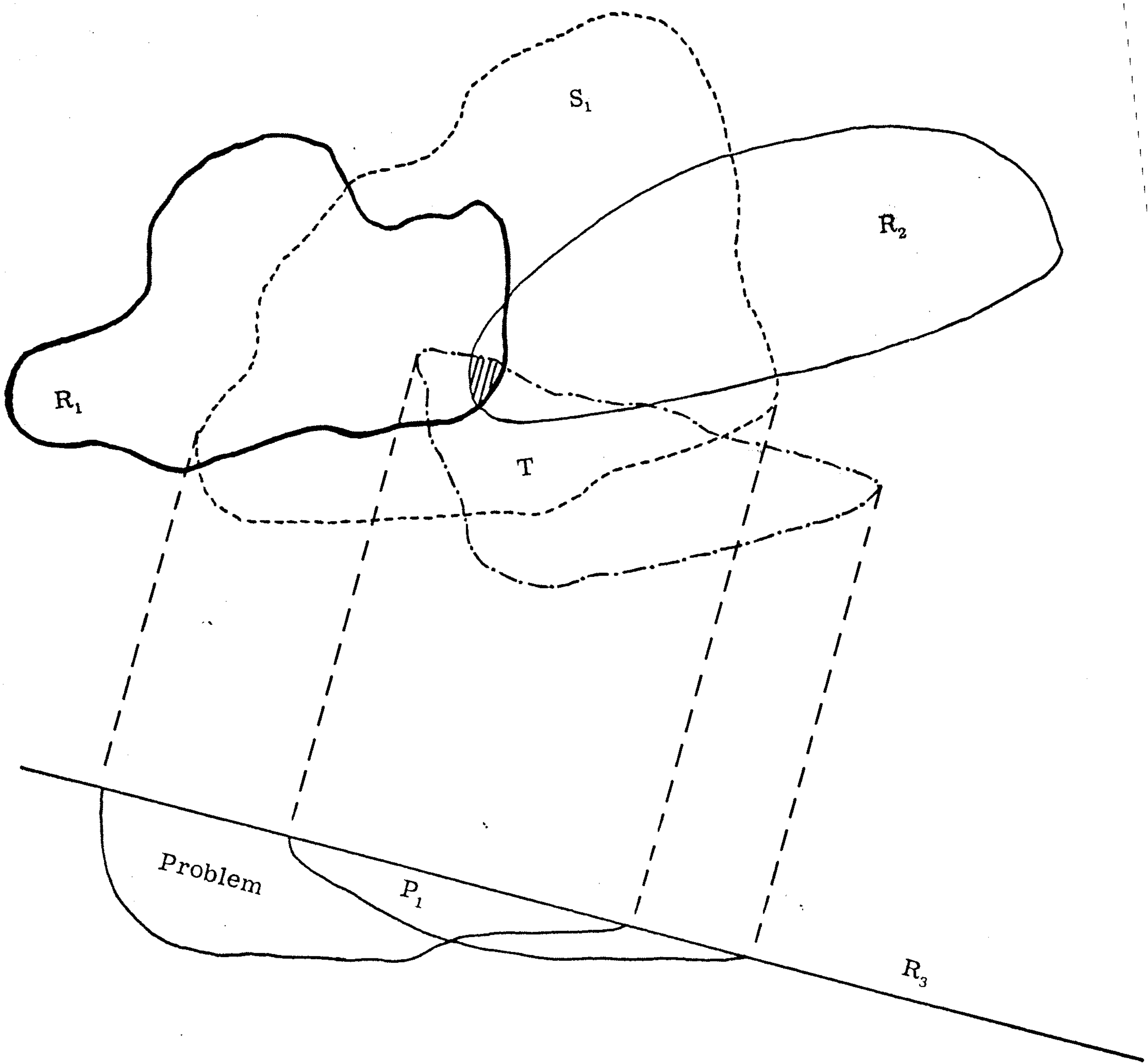


Figure 11

REFERENCES

Bourbaki, N. (1939) Eléments de mathématiques. Paris: Herman.

Comley, Robert E. (1965) "A follow-up study of UICSM students who started course 1 in 1958 or 1959." UICSM Research Report No. 9, 1965. Also Comley's Ph.D. dissertation, University of Illinois, Urbana/Champaign.

Cronbach, L. J. (1964) "Evaluation for course improvement." Reprinted in R. W. Heath (Ed.) New Curricula. New York: Harper and Row.

Driver, Rosalind P. (1973) "The Representation of Conceptual Frameworks in Young Adolescent Science Students". Ph.D. Dissertation (UIUC).

Easley, J. A. J. (1966) "The natural sciences and educational research - a comparison." The High School Journal, 50: 39-50. Also in H. G. Broudy, R. H. Ennis, L. F. Krimerman (Eds.) Philosophy of Educational Research, New York: John Wiley and Sons (1973).

Easley, J. A. Jr., (1968) " A project to develop and evaluate a computerized system for instructional response analysis." U.S.O.E. Final Report Contract N. OE610184.

Easley, J. A., Jr. (1974) "The structural paradigm in protocol analysis." Journal of Research in Science Teaching, 11:281-290.

Easley, J. A., Jr. (1977) "Seven modelling perspectives on teaching and learning." Instructional Science 6: 319-367.

Easley, J. A. Jr. (1979) "A portrayal of traditional teachers of mathematics in American Schools." <u>Critical Review in Mathematics Education</u>. Materialism and Studien, Band 9: 84-108.

Easley, J. A., Jr. and Zwoyer, R. E. (1975) "Teaching by listening - toward a new day in math classes." Contemporary Education. 47: 19-25.

Erlwanger, Stanley H. (1974) "Case studies of children's conceptions of mathematics." Ph.D. dissertation, University of Illinois, Urbana/Champaign.

Gentzen, G. (1969) "Investigations into logical deduction" in M. E. Szabo (Ed.) <u>The Collected Papers of Gerhard Gentzen</u>. Amsterdam: North Holland Publishing Co.

Hawkins, David (1968) "I, Thou, It." Reprinted in <u>The ESS Reader</u>.

Ikeda, Hiroski (1965) "A factorial study of the relationships between teacher-held objectives and student performance in UICSM high school mathematics." Ph.D. Disseration, University of Illinois at Urbana/Champaign.

Klein, Felix (1939) <u>Elementary Mathematics from An Advanced Standpoint</u> (2 vols.). New York: Macmillan.

Knifong, James Dan (1971) "The Representation of Cognitive Studies of Four-and-a-half-year old children." Ph.D. dissertation, University of Illinois, Urbana/Champaign.

Lundgren, U. P. (1979). Background: The conceptual framework. In Lundgren and Pettersson (Eds.) <u>Code, Context and Curriculum Processes</u>. Studies in Education and Psychology, 6. Stockholm Institute of Education.

Papert, Seymour (1975) Remarks made in a lecture given at the University of Illinois, at Urbana/Champaign.

Piaget, Jean (1969) Sagesse et illusions de la philosophie. Paris: Presses Universitaires de France.

Piaget, Jean (1975) "Comments on Mathematics Education." In A.G. Howson (Ed.) Developments in Mathematical Education. Proceedings of the Second International Congress in Mathematical Education. Cambridge: Cambridge University Press. Also in Contemporary Education 47: 510.

Shirk, G.B. (1973) "An Examination of Conceptual Frameworks of Beginning Mathematics Teachers" Ph.D. Dissertation, University of Illinois, Urbana/Champaign.

Stake, Bernadine (1979) "Counting as Problem Solving", in Part II of Harold Lerch and Jack Easley, Children's Conceptions of Number and Numeral. Final Report, NIE 400-77-0075.

Stake, R.E. and Easley, J.A., Jr. (1978) Case Studies in Science Education, Vols I & II. Washington: Government Printing Office. Stock Nos. 038-000-00377-1 and 038-000-00376-3.

Wills, Herbert (1967) "Transfer of Problem Solving Ability Gained through Learning by Discovery." Ph.D. dissertation, University of Illinois, Urbana.

UICSM Stretchers and Shrinkers, Books 1, 2, 3, & 4. Harper and Row, Publishers.

UICSM Motion Geometry, Books 1, 2, 3, & 4. Scranton, Pa.: Harper and Row, Publishers.

UICSM Vector Geometry, Volumes 1 & 2. New York: The Macmillan Company.

Witz, K.G. (1973) "Analysis of frameworks in young children." Journal of Children's Mathematical Behavior. 1(2): 44-66.

Witz, K.G. and Easley, J.A., Jr. (1976) "New approaches to cognition." (mimeographed) Committee on Culture and Cognition, University of Illinois at Urbana/Champaign.

RATIO AND PROPORTION: A STUDY OF A MATHEMATICAL CONCEPT

Group leaders:

Thomas Kieren
Faculty of Education
University of Alberta

David Wheeler
Department of Mathematics
Concordia University

Report - D. Alexander and D. Wheeler

Report of Working Group A

The approach proposed to the group was to examine ratio and proportion from four different perspectives: historical, epistemological, experimental and experiential. The historical and epistemological views would study the concept "from above", as it were: the experimental and experiential views "from below". In historical and epistemological studies one is concerned to analyse what one already knows with the aim of arriving at a better understanding of what it constitutes to "know", say, ratio and proportion. On the other hand one may study the process by which this knowledge is acquired, either by devising experiments which elicit responses from children and others that can be noted, or by looking introspectively at one's own experience of using the concept.

It was hoped that the four views - which overlap considerably - could be combined to give a richer and truer picture than any one of them could do on its own. Indeed, it was hoped that the procedure might be a paradigm for the study of mathematical concepts with education in mind. In the event, the time was too short, the concept too complex, and the group members too unpractised in the method. Nevertheless, the group members made some progress in the clarification of their own understanding even if there was not much of this that could be communicated concretely for the benefit of all.

1. During the first session the discussion centred on determining the meanings of ratio and proportion and whether these concepts were different from others, such as fraction, function, etc. The group looked at the usual high school curriculum and considered the occurrences of the concept - e.g. in rate, scale, direct and inverse proportion, conversion, similarity, etc.

It was suggested that an exclusive focus on two-termed ratios is misleading since $x:y = 2:3$ can be interpreted as $x/y = 2/3$ while $x:y:z = 2:3:5$ cannot be interpreted, unambiguously, as $x/y/z = 2/3/5$. Perhaps a more general formulation is that $x:y:z = 2:3:5$ implies that there is a real number k such that $x = 2k$, $y = 3k$, $z = 5k$. k is called the scale factor.

It was noted that e.g. 2:3:0 could be given a meaning with this definition.

The above reminded the group of similar triangles. For example, AB:BC:CA = DE:EF:FD (where ABC and DEF are two triangles) imples that AB/DE = BC/EF = CA/FD. It was suggested that inserting the scale factor could be enlightening.

e.g. AB:BC:CA:k = DE:EF:FD:1 imples that AB/DE = BC/EF = CA/FD = k . But some felt uneasy because it did not seem clear whether k was supposed to be a length or a number.

Freudenthal (in Weeding and sowing) distinguishes "internal" and "external" ratios. The former are between elements of the same set, or magnitudes of the same type, whereas the latter are between elements of different sets or magnitudes of different types. Although Freudenthal appears to limit his discussion to two-term ratios (arising from direct proportionality examples), a parallel can be seen between differentiating the fraction approach from the scale factor approach and Freudenthal's distinction between internal and external ratios.

The group made a list of examples of ratios in real situations and these seemed to form natural groups:

Maps	Microscopes	Recipes	Rate
Drawings	Telescopes	Dosages	
Models		Nutrition	

The examples indicated the close connection between ratio, rate, scale, fraction, function. Throughout the three days of discussion, keeping these concepts distinct proved to be a continuing difficulty.

Difficulties in the initial exposure of children to the concept are evident in such statements as "An ant is stronger than a man", or "The sides of a triangle are 4:5:3 (in yards) or 12:15:9 (in feet)". But perhaps the most significant learning difficulty for the student may be in deciding whether a particular comparison should be measured by a difference of the quantities involved or by a quotient - and school textbooks are unanimously silent on this point!

2. A number of experimental researches into the conceptual understanding of ratio and proportion have been carried out, mainly with a view to showing that the learning develops over time. An experiment devised by Karplus concerns "Mr. Short and Mr. Tall": the heights of two similar (human) figures are measured with two different sizes of paper clips and students are asked, essentially, to complete a fourth proportional. As well as some expected "misunderstandings" about the additive or multiplicative nature of the comparison, some students show that the notion of growth interferes with their performance on the task. Related experiments used by Karplus include one involving students in predicting where to place masses on a beam balance, and another where the students have to consider building walls of differing heights with bricks of differing sizes.

One of Noelting's experiments asks children to compare two mixtures of glasses of water and glasses of orange juice and say which would taste more "orangey". A sequence of questions displays the development of the children's ability, from being able to deal with "obvious" cases to a complete operational mastery whatever the proportions of the mixtures. Another experiment presents two schematic pictures of some cookies and some children's faces, the pictured cookies to be shared, the children are told, among the pictured group of children, and each is then asked which group s/he would prefer to belong to.

Kieren has devised some experiments concerning an imaginary "packing machine" which converts, say, an input of six similar objects into an output of 3 objects. Children are asked to predict outputs from inputs, or infer inputs from outputs, when the machines are used singly or sequentially.

Although interested in the accounts of these researches, some of the group felt that it was not certain that the same concept was being tracked in each case. Didn't the "noise" of the chosen setting or model often intrude? This is not a characteristic of these researches alone, but a difficulty.

confronted by most researches where some concept is embodied in a concrete problem situation.

3. The group looked at a draft of questions for a survey of methods of introducing and teaching ratio and proportion and found itself having to attack many of the proposed questions for being confused and muddled. Textbook examples are often unsatisfactory too. Direct and inverse proportion questions, for example, can only be solved by assuming that certain crucial rates are held constant, but in the situations described (men ploughing fields, say) there is no way of knowing whether the rates would or should remain constant in practice. This lack of concern for basic plausibility can only reinforce the tendency of students to neglect the reasonableness of their answers to word problems.

Are there any significant differences between fractions and ratios? Historically they have been used interchangeably, and the colon notation (not to mention the double colon ::) is of relatively recent introduction. At first sight it appears as if they must be discriminated because fractions can be added whereas ratios generally cannot. But the distinction begins to disappear for good when it is realized that the sum of two fractions can only be given a meaning when both operate on the same quantity.

There seems to be nothing of mathematical significance in the area of ratio and proportion that could not be equally well handled by the use of fractions and linear functions.

APPENDIX A (Working Group A)

FRACTION TASKS

Fraction Task 1: Measurement & Partitioning

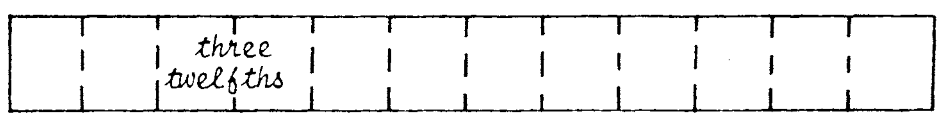
1. Take a piece of calculator tape and "work it" until it lays flat rather than curling up. Cut the ends so that they are perpendicular to its length.

2. Consider your piece of tape as a unit. Use your unit to measure the following objects:

Table	________	units
Book	________	units
Your partner's height	________	units
Your waist	________	units

Because your unit will not usually fit "evenly", you must sub-divide your unit into 2, 3, 4, 6, 8, 12, 16 etc. parts. You can do this by folding your tape appropriately. (e.g. How can you fold "thirds"?) Write the names of the division lines on your tape.

e.g.



Make the measurements using your divided tape.

3. What do you do if your divisions don't give you an even measure?

   Why can you always find numbers to represent your repeated partitions?

4. This activity is done to answer the following questions.

   a) Are fractional numbers always less than one?

   b) Counting is a useful mechanism in understanding whole numbers. What mechanism appears useful in understanding fractions?

Fraction Task 2a

8. List the fractions on the "1/2" fold.

   1/2, 2/4, _____, _____, _____, _____, _____, _____.

   What can we say about these fractions?

   Why are there no "thirds" in this list?

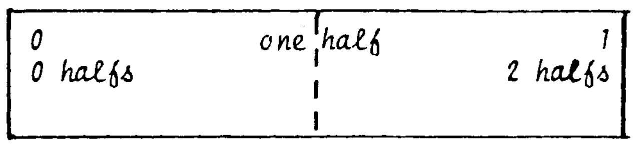
   Give other sets of equivalent fractions from your tape.

9. How could you generate other fractions to go on the "7/12" fold?

Fraction Task 2a: Measurement, Order and Equivalence

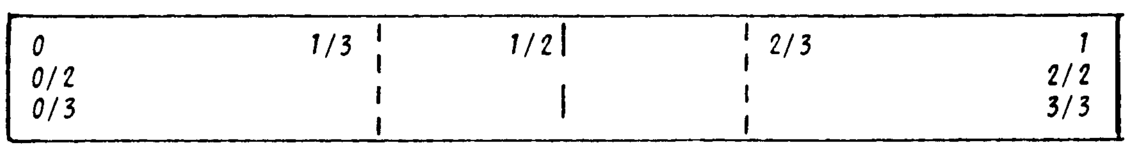
1. Take a piece of calculator tape about 1 metre long and work it until it lays flat. Cut the ends perpendicular to the length and make them straight. Label the ends 0 and 1 right at the top of the tape.

2. Fold the tape length-wise in two equal parts. Label as follows.



Because of space limitations you will want to use the formal forms 0/2, 1/2, 2/2, but remember as children learn fractions start with word names first and only later use ordered pairs of numbers.

3. Fold the tape length-wise in three equal parts. Think before you act and do it carefully. Label the folds on the tape as follows:



4. Fold the tape into 6 equal parts, label the ends and the "sixths" folds appropriately (remember to add the label "2/6" to the "1/3" fold, etc.)

5. Fold the tape in 12 equal parts. Label the ends and the "twelfths" folds. (remember to label the "2/3" fold with "8/12", etc.)

6. Fold the tape in 4 equal parts. Label as above.
   Fold the tape in 8 equal parts. Label as above.

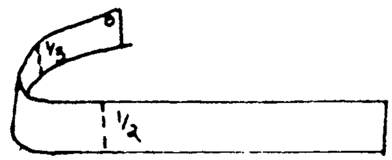
7. Is 5/8 greater than 7/12? How can you tell?

   Make up a half a dozen ordering tasks using your tape.

Fraction Task 2b: Meaning of Addition and Measurement

1. Take your tape from task 2a. Hold the "1/3" fold directly on the "1/2" fold.

   Where does the "zero" end lie?

   Why?

   A mathematical sentence to describe this is:

   1/3 + 1/2 = 5/6

2. Repeat 1 but fold 1/3 on 7/12. Write the appropriate mathematical sentence.

   1/3 + _____ = _____.

   Do other "additions" using your tape.

   Can you "add" fractions without like denominators?

3. What happens if you lay the "2/3" fold on the "5/6" fold?

   Can you figure out how much beyond 1 the tape extends?

   Complete this mathematical sentence

   2/3 + 5/6 = 1 _____

4. Using the tape do other additions whose sum is greater than 1. Write the related mathematical sentences.

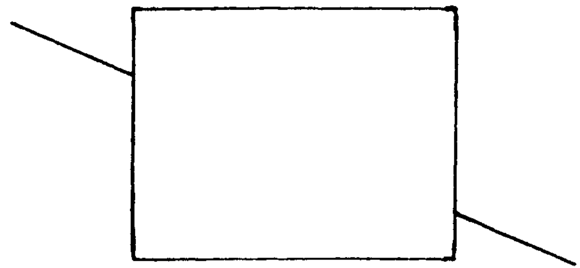
5. Using your tape (and imagination) solve the following.

   1/3 + _____ = 5/6   _____ + 1/12 = 11/12   5/8 + 1/2 = _____

6. Think up a way to show subtraction using your tape.

Fraction Task 3: Operators and Machines

1. Complete the table below



Input	Output
10	6
20	12
50	30
100	60
15	____
5	____
75	____
5000	____
3000	____

The name of this machine is a ________ for ________ machine. For every ___5___ that go in ________ come out.

2. For the above machine complete this table

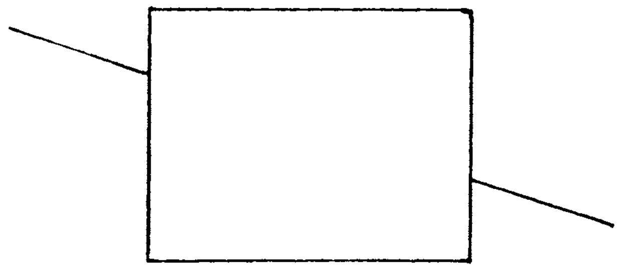
Input	Output
____	9
____	300
____	600

How did you know these results?

You were using the notion of "inverse". A machine which would do the reverse of the 3 for 5 machine above would be a 5 for 3 machine.

Fraction Task 3

3. Find a partner. Each of you make up a machine with a mixed list of 8 inputs and outputs. Make sure you give 3 complete pairs. Exchange lists and see who can give the most correct answers. Here is a sample game machine.



Input	Output
15	10
9	6
60	40
6	_____
_____	2
24	_____
18	_____
_____	20

This machine is a _________ for _________ machine. Its "inverse" machine would be a _________ for _________ machine.

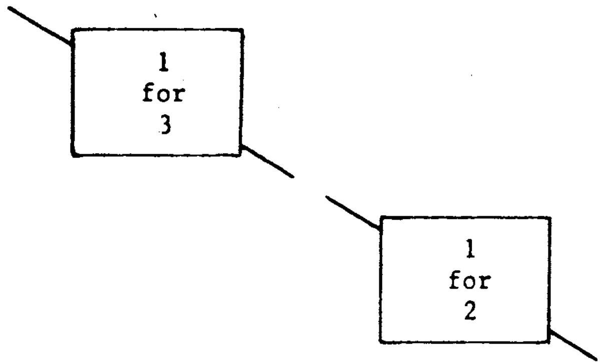
4. Here is a mysterious machine's input and output list. Can you complete it?

Input	Output
10	10
3	3
727	727
_____	46
29	_____
11	_____
7777	_____
_____	21

Can you name this machine. _____ for _____. A formal mathematical name for this machine is the identity machine. Its inverse machine would be a _____ for _____ machine.

Fraction Task 3

5. Here are 2 machines.



The output from the first machine is the input for the second machine. Can you complete the table below?

Machine 1			Machine 2	
I	O		I	O
30	10	----------	10	5
12	4	----------	4	2
24	8	----------	_____	_____
60	20	----------	_____	10
90	_____	----------	_____	_____
120	_____	----------	_____	_____
_____	2	----------	_____	1
_____	_____	----------	_____	4
_____	_____	----------	_____	2000

Look at these machines carefully. What fraction could you use to automatically get the final result if 300 were put into the first machine?

We can write this result 1 for 3 followed by 1 for 2 is the same as 1 for 6.

In more mathematical symbols 1/3 _____ 1/2 = 1/6.

6. Use the machine idea to solve the following:

1 for 2 followed by 1 for 2 is _____ for _____.
3 for 4 followed by 1 for 2 is _____ for _____.
1 for 1 followed by 3 for 7 is _____ for _____.
3 for 5 followed by 5 for 3 is _____ for _____.
*4 for 7 followed by _____ for _____ is 1 for _____.

7. What mathematical operations and what ideas are related to this approach to fractions?

Fraction Task 4: Part-Whole and Equivalence

1. Using the set of 72 objects in front of you complete the following list of all the ways you can divide 72 objects into subsets of the same size.

   1) 36 sets of 2
   2) _____ sets of 36
   3) 24 sets of _____
   4)
   5)
   6)
   *
   *
   *

   How many ways of partitioning the 72 object set did you get?

   Why are there such an abundance of ways? (remember Kennedy, pp. 268-278)

2. Looking at one's list of partitions helps one see ways in which fractions can be expressed. For example because there are 4 sets of 18 in 72, 18/72 can be expressed as 1/4.

   Complete the following lists of ways that the partitioning of 72 suggests for expressing various fractions.

   a) 18/72, 1/4, _____, _____, _____

   b) 24/72, 2/6, _____, _____, _____

   *c) 10/72, _____, _____, _____, _____

   *d) 4/72, _____, _____, _____, _____

   *e) 17/72, _____, _____, _____, _____

   *Can you fill all the blanks? Why or why not?

   What can you say about the fractions in set "a" above?

   About all the sets of fractions above?

Fraction Task 5: <u>Measurement and Addition</u> (another look)

1. Look back to Fraction Task 1 and use your tape from that task or make a new tape.

2. Measure the following objects as precisely as you can and complete the following table.

OBJECT	A SIDE 1	B ADJACENT SIDE	C BOTH SIDES IN A SINGLE MEASURE	D SUM OF A + B
a) Book				
b) Table				
c) Room or Part of Room				

3. What appears to be the relationship between columns C and D in the table?

4. If the two sides of the room measure 6 1/2 and 4 3/8 tapes, we can relate these to the total length 10 7/8 with the following sentence:

$$6\ 1/2 + 4\ 3/8 \doteq 10\ 7/8$$

Using your data write 3 sentences which describe the relationship between the side lengths and the total.

Why do we use $\doteq$ instead of $=$?

Why in theory or in elementary school texts can we write:

$$6\ 1/2 + 4\ 3/8 = 10\ 7/8?$$

5. Use the two pieces of mayfair board which contain units marked off in eighths. Label the points starting at 1/8 with appropriate fractional and whole number names (e.g., 3/4, 11/8, 3/2, etc.)

6. Use the two rulers to add 1/2 and 1/4. Result __________. Write a set of directions for Grade 6 or 7 children which would tell them how to add numbers using these rulers.

Fraction Task 5

7. Use your ruler to add the following numbers:

a) 3/8 + 3/4 = _____

b) 5/8 + 3/2 = _____

c) 7/4 + 1/8 = _____

d) 17/8 + 1/2 = _____

8. Re-label your rulers using mixed numerals (e.g. 1 1/2, 2 1/4, etc.) or at least think of the partitions in those terms. Complete the following:

a) 3/8 + 3/4 = __________

b) 1 3/8 + 1 1/2 = __________

c) 1 1/4 + 7/8 = __________

*d) 1 3/4 + 1 5/8 = __________

9. Use your ruler to answer the following questions:

a) 3/4 = /8?

b) 5/2 = /4?

c) 5/4 = /8?

d) 5/4 + 7/8 = __________

This is the same question as /8 + 7/8 = __________

e) 3/2 + 5/4 = __________

is the same as /4 + 5/4 = __________

10. The purpose of this task sheet has been to show two things.

A. Fractions or rational numbers can be added! There is no question of common denominators!

B. When you are making up a quick algorithm which uses symbols only, equivalence allows you to make use of the common denominator notion to do so.

Fraction Task 6: Units

1. Take a set of 10 different rods. Choose some rod to be your unit. Write fractional number names for all the other 9 rods in terms of your unit.

   Have your partner choose a longer rod as a unit and do the same task.

   Compare your green rods. Do they have the same name in both systems?

   Why or why not?

2. Could the following sentences ever be true? Explain.

   3/4 < 1/2

   5/4 = 1

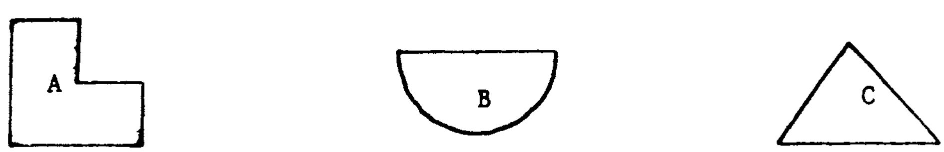
   2/3 > 3/4

3. What kinds of learning problems are posed by the aspect of fractions described above?

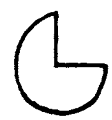
4. Find the piece of yarn at your table. If that piece is represented by 4/5, cut a piece of yarn from the ball which would represent 1. Describe how you did this.

5. Make up similar problems for children of age 10 or 11 which would help them focus on the notion of unit.

6. On your table find shapes labelled A, B and C. Below draw the shapes of the figures represented by the given fractions if A, B or C were considered as units.



Fraction Task 6

UNIT	NUMBER	SHAPE
C	1	
A	2/3	
B	1/2	
*C	3/4	
B	____	

7. Make up exercises in a more interesting and motivating style, which would be like those in 6 but appropriate for Division II students. (e.g. use humour or fantasy)

Fraction Task 7: Teaching

1. Make up fraction representation problems and fraction addition problems using:

    a) Cuisenaire rods

    b) Graduated beakers or cans
    (What is the problem here?)

Fraction Task 8

Ratio Numbers*

1. From the box of rods, select a set of rods, one of each color and order them. Associate a number from 1 to 9 with each rod.

2. Take a red and a light green rod. Describe the relationship between them in as many ways as you can.

red	=	____	light-green	(a)
light-green	=	____	red	(b)
____ red	=	____	green	(c)

The ratio of red to light green is ____ to ____. (d)

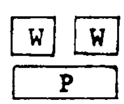
The ratio of light green to red is ____ to ____. (e)

How are the numbers used to describe the relationships in a and b and d and e related?

3. Select two other pairs of rods and write mathematical sentences which describe the relationship between them.

4. i) How are the red and white rods related?

We can picture this relationship as follows.



Find all the other rod pairs which have the same relationship.

List the set of ratio numbers used to describe these rod pairs:

{white to red, ___, ___, . . .}

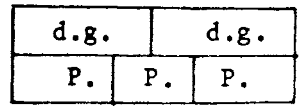
{½, ___, ___}

How do you know physically that the rod pairs are in the relationship?

What can you say about the ratio numbers used to describe these rod pairs?

ii) Test two ratio numbers used to describe the relationships between pink and dark green rods.

Write an equation which describes this picture.



Find all the other rod pairs which share the relationship pictured above.

What can you say about the ratio numbers which describe these rod pairs?

iii) Suppose that there was a silver rod and a violet rod which were related in the following way.

$$3\ S = 7\ V$$

Write two sets of ratio numbers which would describe rod pairs which would have the same relationship.

{3/7, ___, ___, ___, ___, ___}

{7/3, ___, ___, ___, ___, ___}

How many ratio numbers would fall in such sets, if one included all possible ones?

iv) How would you know physically that rod pairs would be represented by equivalent ratio numbers?

(extra for experts) Suppose we have a rod pair of colors x and y such that

$$ax = by$$

Give a ratio number which relates x and y and give five equivalent ratio numbers.

*The idea for this task sheet was taken from the work of Alan Bell and the South Nattinghom Project in England.

Decimal Task Set 1. Tenths, hundredths, thousandths

On your table you should have 2 flats divided into one hundred congruent smaller squares, a number of longs, and a number of smaller cubes. You should also have one large cube.

1. If a long is considered as 1 unit, what fractional name would describe one of the small squares.

   Why?

2. Using longs and small cubes illustrate the following:

   a) 7/10

   b) 3 6/10

3. Using longs and cubes find the following:

   a) 3 6/10 + 4 1/10

   b) 2 5/10 + 1 7/10

   Explain your result

   c) 4 5/10 - 2 1/10

   d) 1 7/10 - 9/10

4. Write a set of directions for children to have them answer the following using longs and squares.

   a) 2 7/10 = /10

   b) 13 4/10 = /10

   c) 43/10 = 4 /10 = 3 /10

Decimal Task 1

*5. Sketch a cork board display which shows how to find the following results using longs and cubes.

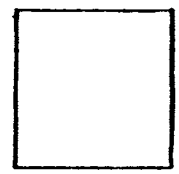
$$11 \div 10$$

6. Change gears! Suppose a <u>flat</u> is now considered as a <u>unit</u>. What fractional name is now assigned to:

   a) longs __________?

   b) small cubes __________?

   c) large cubes __________?



7. Using the materials illustrate each of the following at least 2 ways:

a)

	flats (1)	longs (1/10)	cubes (1/100)
25/100	________	________	________
	________	________	________

b)

	flats (1)	longs (1/10)	cubes (1/100)
1 13/100	1	1	3
	________	________	________
	________	________	________

8. Develop a short <u>demonstration</u> for a child to show how to use the blocks to solve the following:

   a) 1 3/10 + 27/100 =

   b) 36/100 + 29/100 =

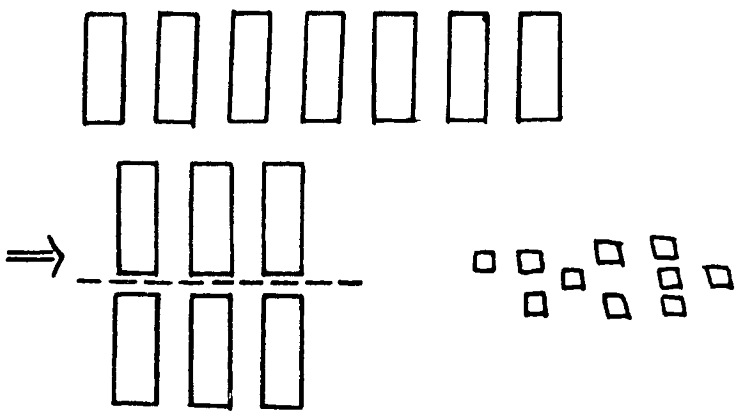
   c) 4/10 + 3/100 + 49/100 =

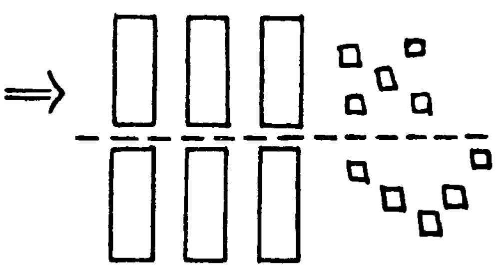
   *d) 1/2 + 1/5 + 3/20 =

9. Using the materials show two fractions equivalent to the following.

a)	1/2	5 longs (5/10)	50 cubes (50/100)
b)	1/4		
c)	2/5		
d)	3/4		
e)	7/20		

10. The diagram below shows 7/10 ÷ 2.





∴ 7/10 ÷ 2 = 35/100

Sketch diagrams for the following.

a) 8/10 ÷ 5

b) 1 8/10 ÷ 5

*c) 25/100 ÷ 5/10
(Can you think of an equivalent division question?)

Decimal Task 1

11. Change gears one more time. Suppose the large cube is a unit. What is a sequence of activities for children which will lead up to their being able to do the following:

    a) 37/100 + 25/1000 + 5/10

    *b) $2 \div 3$

The purposes of this task set has been to show a way of providing meaning to fractions which relate to decimals, to show physically the simplicity of the decimal operation of adding, and to show experiences relating decimal fractions to other fractions.

Decimal Task Set 2. Decimal numeration and fractions

1. 3 3 3
   a b c

The value of the digits indicated by a, b and c above are:

a) 300

b)

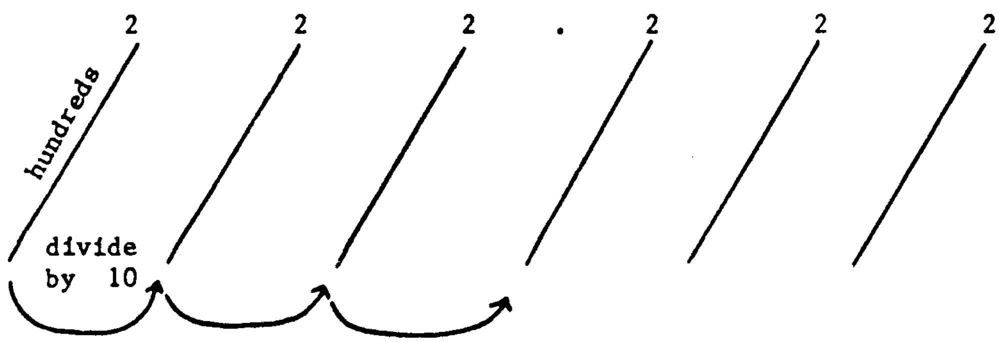
c)

Why?

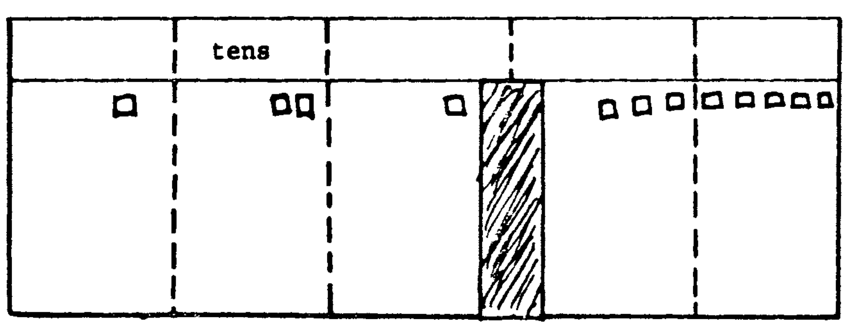
To get the value of b from a one can ______________________

To get the value of c from b one can ______________________

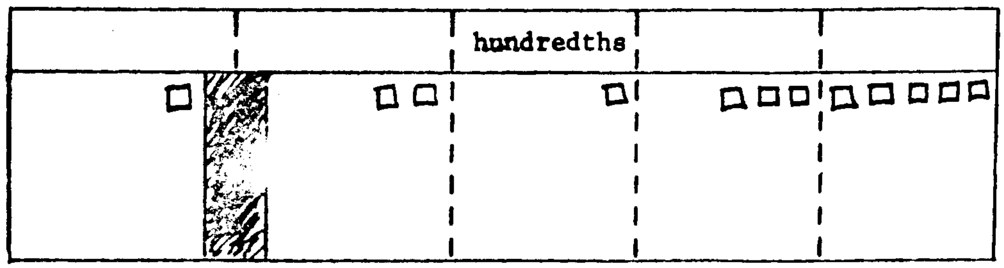
2. Complete the following demonstration for children:



3. a) If the bar represents the "decimal point" give the number represented by the chart in decimal form.



Decimal Task 2



b) Represent 2.3012

c) Represent (2 x 10) + (0 x 1) + 3/10 + 0/100 + 5/1000

4. Sketch a place value pocket chart you would use with your class (What is the value of a "moveable" decimal point?)

   a) Write up a set of 6 exercises for children using the chart.

   b) Explain how the chart could be used for addition.

   c) Explain how the chart could be used for division.

Decimal Task Set 3. Addition, meaning and equivalence

Your table should have at least 2 metre sticks divided into decimetres, centimetres and millimetres. It should also have a long piece of string and calculator tape.

1. Complete the following table.

OBJECT	A length of side 1	B length of side 2	C measure of string combining side 1 & 2	D A + B
Book				
Table				
Bookcase				

*Give length in decimal fractions of a metre. That is use the metre as your

______________________________.

2. Carefully cut a piece of calculator tape 1 metre long with ends cut perpendicular to the length. Label the ends 0 and 1.

   a) Fold the tape in two. Label the fold and ends in 1/2's.

   b) Fold the tape into 4 congruent parts. Label the folds and ends in 1/4's.

   c) Repeat b for 1/3's, 1/6's, 1/8's, 1/12's.

   d) (Key exercise!) Use a metre stick to add a decimal fraction to the list of equivalent fractions on each fold.

Decimal Task 4. Homework

During the next week collect as many different observed uses of decimals as you can find. Make a display which you could use to motivate the study of decimals in your classroom.

T.E. Kieren

APPENDIX B (Working Group A)

NOTES ON DIFFICULTIES IN THE INTERPRETATION OF RATIOS AND PROPORTIONS

Many of the problems which students encounter in the use of fractions, ratio, proportion, percentages, rates, etc. appear to stem from too rapid abstraction to a mathematical entity and the neglect of the fact that this entity has no meaning except in the original context; eg. one fourth of a 20cm. pizza is not equal to one fourth of a 30cm. pizza. The context in which the fourths arise becomes just as important as the arithmetic that can be performed with the 1/4. It has been suggested that the reason why the fraction is called one fifty rather than one over five is that one fifth begs the question "one fifty of what?"

In some sense the two term ratio is a generalization of the fraction concept and is usually used when the total composition of the set is of interest.

For example, if we say 2/3rd of the fruit in this basket is bananas, nothing is known about the composition of the remaining fruit except that they are not bananas. However, by saying the ratio of bananas to apples to pears is 3:2:1 (short hand notation?) the total composition of the basket is specified. These ratios can be interpreted in a number of ways. It is of concern to us that most teachers give only one of the many possible interpretations. Isolating one particular interpretation may have serious implications later on when another would have been more appropriate to tackle the problem at hand.

Whatever the interpretation we wish to give to ratios, errors will be minimized if a conscious effort is made to retain the units. e.g we are told that the ratio of girls to boys in a particular class is 3 to 4 (or 3:4). By this statement we mean

(i) for every three girls there is exactly (no more no less) 4 boys or for every girl there is exactly 4/3* (four thirds) boys

- units (boy)/(girl) or (boys per girl).

* The equivalent form 1.33 leads to difficulties since it doesn't beg the question "of what?"

or (ii) for every four boys there is exactly 3 girls or for every boy there is 3/4 (three fourths) girls
- units (girl)/(boy) or (girls per boy).

or (iii) for every seven children (collective noun available in this example) there is exactly 3 girls <u>and</u> 4 boys
- the composition of the set is completely specified, which in this case is trivial since if the children are not girls they must be boys

or 3/7 (three sevenths) of the children are girls <u>and</u> 4/7 (four sevenths) of the children are boys.

The aforementioned three interpretations become important depending on whether we wish to estimate

(a) the number of boys in the school given the number of girls, from

(i) units (boy X girl)/girl boy

(b) the number of girls in the school given the number of boys from

(ii) units (girl X boy)/boy girl

(c) the number of children in the school given either number of boys or number of girls in the school.

It is clear to us that the last interpretation is hardly even mentioned by teachers (it doesn't even arise as a possibility in the proposed questionnaire on Ratio, Proportion and Percent prepared for the IEA). Yet this interpretation is important, especially in areas like chemistry where concentrations are relative to total volumes and not to volumes of component parts. To some of us the examples discussed at the meetings (dilution of orange juice and children with cookies) can appear very different. In the first, one tends to look at it from the point of view of how much orange is present relative to the total, while in the second, it is clear that a child will see it as "how much cookie do

I (the child) get?"

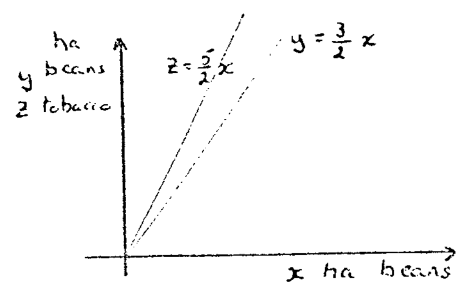
Although all of the above interpretations could be seen as constant rates, the meaning is rather artificial. The meaning of <u>constant</u> rate is more clearly visible when elements are of different types, e.g. candies per boy, hits per at bat, etc., and when non integral values of the variables are permitted, eg. freight rates (dollars per kilogram), gas rates (dollars per litre), etc. although the latter are rarely expressed as ratios and the examples in texts are artificially contrived.

The unqualified use of the equals sign between two equivalent fractions or ratios can cause many difficulties - this procedure is usually called a proportion. The use of the four dots for equivalent ratios or a proportion does have the merit of alerting the student that equality does not automatically hold.* This is made evidently clear when multiple (more than two terms) ratios are introduced, where the statement: the hectares of corn (x) to beans (y) to tobacco (z) is in the ratio of 2:3:5. This is usually summarized as x:y:z = 2:3:5 which is to be interpreted in any one of the following ways:

(I) Comparison with corn as a basis

$$y = \frac{3}{2} x \text{ (ha beans)}$$

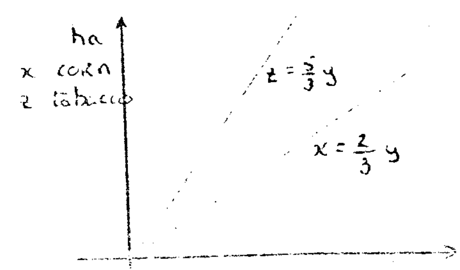
and $$z = \frac{5}{2} x \text{ (ha tobacco)}$$



(II) Comparison with beans as a basis

$$x = \frac{2}{3} y \text{ (ha corn)}$$

and $$z = \frac{5}{3} y \text{ (ha tobacco)}$$

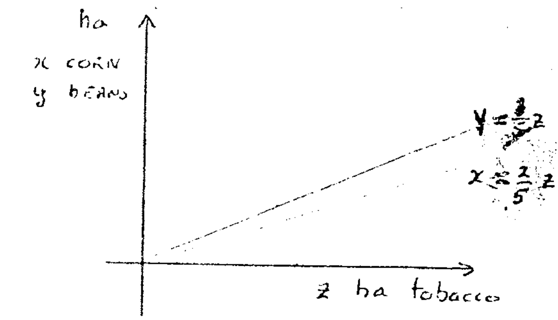


* We are not advocating the return of this notation.

(III) Comparison with tobacco as a basis

$$x = \frac{2}{5} z \text{ (ha corn)}$$

and $$y = \frac{3}{5} z \text{ (ha beans)}$$

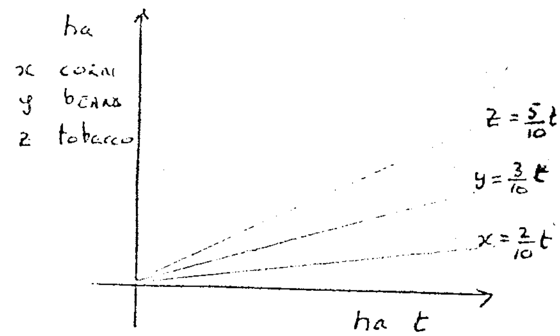


(IV) Comparison with the total ha (t)

$$x = \frac{2}{10} t \text{ (ha corn)}$$

$$y = \frac{3}{10} t \text{ (ha beans)}$$

$$z = \frac{5}{10} t \text{ (ha tobacco)}$$

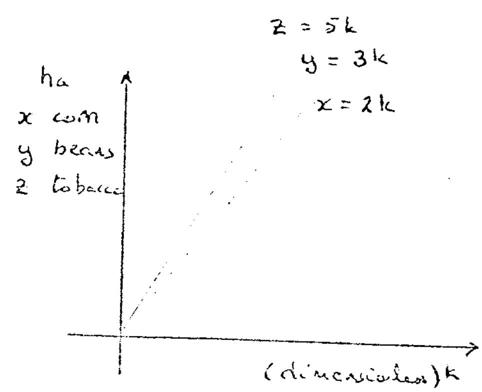


(V) Comparison with a (dimensionless) constant k

$$x = 2k \text{ (ha corn)}$$

$$y = 3k \text{ (ha beans)}$$

$$z = 5k \text{ (ha tobacco)}$$



Martin Hoffman, Mathematics Department,
Queen's College,
CUNY.

Eric Muller, Mathematics Department,
Brock University.

APPENDIX C (Working Group A)

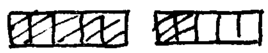
RATIO AND PROPORTION: SOME MATHEMATICAL, HISTORICAL AND EPISTEMOLOGICAL ASPECTS.

1. $a/b = c/d$ iff $ad = bc$. If this means anything, this is a definition of /. It has nothing to do with the reason why two fractions may be equal, or two ratios equivalent, or four quantities in proportion.

2. History has saddled us with the words ratio, proportion, fraction. A case could be made out that each is so loaded with unfortunate associations that we'd be better off without them.

Fraction - fracture, broken into parts. Hence, inevitably, "improper" fractions. For, indeed, if a fraction is a part, then it is not possible for a fraction to be greater than the whole.

Williams, quoted by Olson, says that students label the diagram



as 7/10 rather than 7/5. Faced with a silly question, the students appear to do an eminently sensible thing - adjust the nature of the whole so that the question is no longer silly.

Textbook writers get into great difficulties with area representations of fractions because they don't respect the conventions of the representation they have chosen. Because they know that improper fractions are around the corner and will sooner or later have to be mentioned, they attempt to stretch the diagram convention to cover improper fractions, forgetting that the area representation only works for proper fractions. Oddly enough, the same writers would never dream of using more than one circle in a statistical pie chart.

Proportion. "Portion" equals part, share (and here we go again). But usage is not consistent, for we have "well-proportioned", which has nothing to do with parts. Indeed, it is hard to see what talk about aesthetic or musical proportions has to do with mathematical proportion.

And then the Greeks were fascinated with "mean proportionals", which are special cases. They defined at least ten different varieties of "mean" - but this included the arithmetic mean which we would not be inclined to consider derives from proportion at all since it is essentially additive rather than multiplicative.

But note that b is a geometric mean of a and c if $(a - b)/(b - c) = a/b$

whereas b is an arithmetic mean of a and c if $(a - b)/(b - c) = a/a$.

We would be better with the Greek word, "analogia".

Ratio. Its Latin ancestry connects it with rational, relational. The Greek word is "logos".

This is a later piece of jargon. In Mediaeval times writers used proportio for a:b and proportionalitas for the equality of a:b and c:d. The first uses of ratio restrict it to a ratio of integers.

My typewriter keeps reminding me that as well as fraction and proportion there is the word ration.

A plague on all these uses.

3. Verbal IQ tests frequently contain items of the form: "p is to q as w is to ... " Justification for these items stems from Spearman (who posited a "factor g" of general intelligence); he asserted that purest estimates of g could be obtained from questions of this type, that he called the "eduction of relates and correlates". The solver has to educe (induce?) a directed relation between p and q, than apply this relation to the term w and find its correlate.

Note how comfortably all this could be expressed in the language of mappings and functions.

4. Definition 20 of Book VII of Euclid says, "Four numbers are in proportion when the first and the third of these are obtained from the second and the fourth by multiplying them by the same whole number, or dividing them by the same whole number, or by doing one thing and the other."

i.e. $a:b = c:d$ if $a = \frac{n}{m} b$ while $c = \frac{n}{m} d$.

Definition 5 of Book V says, "Two magnitudes A and B are said to stand in the same ratio to two others C and D, where, for any pair of numbers for which we have

$$mA \gtreqless nB \quad \text{we also have} \quad mC \gtreqless nD.$$

Note that the first refers to numbers, the second to magnitudes. Why does not Euclid use the same definition for magnitudes as he uses for numbers? (It is only part of the answer to say that Books VII and X are based on Theatetus whereas Book V derives from Eudoxus.)

The Eudoxus definition for magnitudes could, it would seem, have led immediately to the notion of a Dedekind cut. There is probably no simple reason why it did not, but Lakatos points out that the theory of proportions for the Greeks served to translate arithmetic into geometry. He talks of "changes in the dominant theory". On this view it is not until Cauchy and Weierstrass are busy changing the dominant view back from geometry to arithmetic that the Dedekind cut emerges.

(What happens to a theory of proportion in a world of mini-calculators?)

5. A clinical interview. The investigator establishes that the child prefers candies of one colour (say red) to candies of anther colour (say green). She explains that she will put some candies into a bag, that the child may put his hand in and draw one out (and, presumably, consume it). She points to one set of candies, say one red and one green, and to another, say two red, and asks the child which he would prefer her to put into the bag. The investigator changes the number of pieces of candy in the two groups and repeats the question several times ....

The interest of this question is that it implies a comparison of ratios (or proportions) but is independent of any formal knowledge about ratios or fractions.

We would classify it as a probability question, no doubt. One of the earliest formal propositions in probability theory is due to Nicole Oresme (1325 - 1382) whose theorem says, "It is probable that two proposed unknown ratios are incommensurable because if many unknown ratios are proposed it is most probable that one would be incommensurable to another."

6. Clifford ("The commonsense of the exact sciences") treats proportion of quantities. He takes it as axiomatic that they obey the principle of continuity - i.e. that they are indefinitely divisible. He shows that the problem of finding a fourth proportional to three given quantities is equivalent to the problem of constructing a similar triangle to a given triangle. It is easy to see (the construction employs parallel lines) that being able to make this construction is equivalent to accepting the parallel postulate.

In turn, the parallel postulate expresses a linearity condition - the fact that the sum of the angles of a triangle is 180° is equivalent to the fact that a rotation through the sum of two angles is equal to the sum of two rotations through the separate angles, i.e. rotations are linear transformations.

The mathematics of proportion could be replaced by the mathematics of linearity.

> Note, for example, elementary trigonometry (and the fact that we talk of trigonometrical ratios and trigonometrical functions). We may treat the problem of finding a side of a triangle, say, by interpretating cos 36 as a ratio of lengths of sides of a triangle, or by interpreting it as the length of a side of a "unit triangle". In the latter case the particular length required is found by using a scale factor - i.e. by enlarging the "unit triangle" to the required size. Enlargement is, of course, a linear transformation.
>
> There is no doubt of the mathematical equivalence of "proportion" and linearity, but there may be questions about psychological equivalence.

7. Most of the pedagogical problems in the teaching of ratio and proportion lie, I suggest, in the entirely unobvious equivalence of $s_1:s_2 = t_1:t_2$ and $s_1:t_1 = s_2:t_2$.

David Wheeler, Mathematics Department,
Concordia University.

APPENDIX D (Working Group A)

List of Participants, Group A

Alexander, David	Faculty of Education, University of Toronto.
Crawford, Douglas	Faculty of Education, Queen's University.
Hoffman, Martin	Department of Mathematics, Queen's College, CUNY.
Kieren, Thomas	Faculty of Education, University of Alberta.
Muller, Eric	Department of Mathematics, Brock University.
Mura, Roberta	Faculte des Sciences de l'Education, Universite Laval.
Wheeler, David	Department of Mathematics, Concordia University.

MINICALCULATORS AND THE MATHEMATICS CURRICULUM

Leader

C. Gaulin,
Faculte des Sciences de
l'Education,
Universite Laval.

Reporter

D. Drost,
Department of Curriculum
and Instruction,
Memorial University.

Appendices

A: Realistic Problems through Grade 8 - Connelly and McNicol

B: List of Participants, Group B

Report of Working Group B

Prior to setting the "agenda" which would be followed, the participants in the group discussed and agreed to several assumptions under which the group would function. These included the following:

1. Calculators are widely available to students of all age groups and will be used in the classrooms of our schools and in other school related activities.

2. How calculators will be used in an educational setting depends on the level of a student. Four groups were identified as being early elementary, upper elementary and junior high, senior high, and college or university. It was agreed to restrict the discussion to the pre-university student. Throughout the deliberations of the group, the majority of the discussions centred on grade levels 7 through 12. However, it should be noted that the group felt that the calculator could be used beneficially at all grade levels, including the early elementary years.

3. A wide range of calculators are available including those with only the 4 basic arithmetic functions

together with other keys such as percent, square, root, and reciprocal; scientific models which contain keys for the trigonometric functions together with other functions; programmable models; and pseudocalculators such as the Little Professor and Dataman. The choice of a calculator would depend upon the level of the student for whom it was intended as well as the purpose of the situation in which it was to be used. For the purposes of the study group, it was assumed that all calculators would have a memory facility. This memory could be accessible by a M^+ or a STO key or both.

4. The starting point of our deliberations was an examination of what presently exists in the way of curricular materials and other calculator resources. It was recognized that a plethora of material exists, however much of it is of minimal value in the classroom. It was suggested that a major value of the calculator lies, not only in how it can be used with the curriculum that now exists, but also with new options which have not previously been included in the curriculum due to excessive calculations. Some examples of possible additions are included later in the report.

The "agenda" for the remainder of the session was agreed to as follows:

1. An investigation of existing materials
2. An examination of various "good" ways to use the calculator in schools and the impact of these ways on curricula and teaching methods.
3. A discussion of vehicles to sensitize teachers, parents and administrators to the use of calculators in education. The role of research and both inservice and preservice education of teachers was to be included in the discussion Statements of policy would also be considered.

Existing Materials

The majority of the first session of the study group was spent examining various resources available concerning calculator usage in education. Materials were considered using the following general categories: general resources, periodicals, books on games and tricks, books on curriculum, and other materials. Following are brief remarks on each category together with examples of resources in that category. Mention of a particular resource should not be considered as a recommendation for its use but simply as an example of which a member of the study group was aware.

1. General resources:

   This category included resources that would be important for leaders in education. In this category would be teacher educators, provincial departments of education, school supervisors, and others actively involved in the selection and implementation of curricular change as well as in pre and in-service education. An important source of information for these people is:

   THE CALCULATOR INFORMATION CENTER,
   1200 Chambers Road,
   Columbus, Ohio 43212.

   This center is associated with ERIC and is under the direction of Marilyn Suydam. Information bulletins are published as information becomes available and includes information on bibliographies, suggested activities, research being conducted, criteria for selection, as well as several other areas. Most references suggested below are documented in one or more of these bulletins.

   Other general references would include:

   Caravella, J.R. Minicalculators in the Classroom, National Education Association, Washington, D.C. 1977.

   Electronic Hand Calculators: The Implications for

Pre-College Education Final Report prepared for the National Science Foundation, 1976. (ERIC document number ED 127 205)

Report of the Conference on Needed Research and Development on Hand-Held Calculators in School Mathematics. 1976 (ERIC document number ED 139 665)

The second reference listed above includes extensive appendices which contain several useful suggestions of possible directions for calculator usage.

Under this section, it should be mentioned that several sessions on the use of the calculator will be included on the program of Fourth International Congress on Mathematical Education to be held in Berkeley during August 1980. Also, the Second International Mathematics Study will include an examination of the availability and use of the calculator in schools.

2. Periodicals:

The journals mentioned above are readily available to classroom teachers and are an important vehicle in the sensitization of teachers to the use of the calculator. They also include articles more appropriate for the personnel listed above in the discussion of general resources.

The Arithmetic Teacher (see special issue November 1976)
The Mathematics Teacher (see special issue May 1978)
School Science and Mathematics
Mathematics Teaching (a British journal)
Mathematics in Schools (a British journal)

Various provincial mathematics council newsletters. For example, the June 1978 issue of the B.C. publication VECTOR included several articles on the use of the calculator.

3. Books on Games and Tricks:

These books are available in most bookstores and are therefore available to the public at large. They are not written for direct classroom use, however many of them contain ideas which may be adapted to the classroom. Teachers must appraise these materials critically and select games and activities from them that are mathematically meaningful and assist in meeting classroom objectives.

Schlossberg, E., Brockman, J. The Pocket Calculator Game Book, New York, William Morrow 1975.

Hyatt, H.R., Feldman, B. The Handheld Calculator - Use and Applications, New York, John Wiley.

Mullish, H. How to use a Pocket Calculator, New York, Avco Publishing Co. 1977.

4. Books on Curriculum:

These materials are usually written with the school mathematics curriculum in mind and contain many suggestions for effective use of the calculator in the classroom. Several provincial and state mathematics councils have published monographs on calculators.

Rade, L., Kaufman, B.A. Adventures with your Handheld Calculator, Cemrel Inc., St. Louis, 1977.

Bell, Alan et al A Calculator Experiment in a Primary School, Shell Centre for Mathematical Education, Nottingham, 1978.

Nebel, K. Using the Calculator as a Teaching Aid in the Classroom, Western Springs Public Schools, Western Spring, Illinois.

Neufeld, K.A. Calculators in the Classroom. Monograph No. 5, Edmonton, Alberta Teachers Association, 1978.

5. Other Materials:

This section includes commercial materials prepared for use in the classroom. These materials include kits and booklets of worksheets which can easily be duplicated for use in the classroom. Teachers will find many useful ideas from the material but, as with than we really do of what our students are thinking. I believe that for all teachers, clinical interviews and careful analysis of video-tapes from the classroom can reveal major conceptual differences between their students' ideas and what they thought their students' ideas were. In that sense, the teacher in Lakatos' Proofs and Refutations represents a false ideal. The reason is simple: his teaching is a rational reconstruction of what is, to a significant extent, an irrational process. That is the "missed" communications enhance the struggles of the students to sort their ideas out organically and to find their own expressive metaphors, because this process has been encouraged in the classroom. Unfortunately, this kind of teaching and learning rarely happens because expression of deviant ideas is so rarely accepted, and it cannot be significantly increased except by using it as the means of educating teachers, and, recursively, the means of educating teachers of teachers, including the education of educational researchers, etc., etc., etc. The bright hope of the future lies only in the enormous generative power of branching recursive processes. Such a process connects the way we do research along many paths with what teachers and children do in classrooms.

previously mentioned sources, must be selective in choosing from ideas available.

Sharp, J. Norman The Calculator Workbox, Don Mills, Ontario, Addison Wesley 1977.

Judd, Wallace P. Problem Solving Kit, Willowdale, Ontario, Science Research Associates 1977.

Educators at all levels should consider the materials listed above, as well as similar materials, as being a starting point for the use of the calculator in the classroom. As suggested earlier the calculator can be an aid in many areas of current curricula. It will no doubt also give us the opportunity to include topics in the curriculum which, up until now, have not been feasible due to excessive calculations. What these topics are, only the future can tell.

WAYS TO USE THE CALCULATOR

The second day of our deliberations was spent discussing the use of the calculator in solving various mathematical problems. We were challenged by our group leader to develop problems in the following four areas.

1. Problems which could be used to develop understanding of concepts and generalizations.
2. "Realistic" problems.
3. Problems in which the calculator can be used to explore patterns and properties.
4. Problems where the machine is used as an aid in developing algorithmic thinking.

Examples of problems discussed include the following:

1. To add and multiply 2/3 and 4/5 on your calculator, multiply 2003 by 4005. The answer is 8 022 015. The desired sum is 22/15 and the produce is 8/15. Investigate this process further. Can you explain it? Are there any limitations?

2. Using your calculator determine the repeating portion of the decimal equivalent of 1/17. Use paper only to record your answer.

3. Find an exact answer for 845735281 x 2278345587 on your calculator.

4. Display 137 in base 7 on your calculator.

5. Find the value of

$$1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cdots}}}}$$

6. Find the highest common factor of 51822 and 4794 using the Euclidean algorithm.

7. Postage costs 17 cents for surface mail and 35 cents for airmail. If you have an unlimited supply of 15 cents and 35 cents stamps what values of mail can you post?

8. Measure the height and armspan of all your classmates and hence determine the ratio of height to armspan. Graph the results.

9. Determine the amount of water which is released into the ground on the Saskatchewan Prairie if there is a certain amount of snow covering a certain area. The data is to be collected by the students.

10. The multiplication key on your calculator is broken. How would you multiply 259 by 243 by using other keys?

Our discussions of these problems and others centred on two main topics. First, our interest was more on the process of mathematics than on the product. Secondly, when was a problem "realistic"?

We were concerned with how the classroom teacher would perceive the problems which we examined. On the surface the problems might appear to be somewhat removed from the so-called core curriculum. Yet as we investigated the problems it became clear to us that the problems would be excellent vehicles to aid in focusing on this very core. For example, problem 3, which involves multiplying two large numbers on the calculator, requires a thorough understanding of both place value and the multiplication algorithm. The product of the multiplication itself is insignificant in the problem. What is more important in our opinion was the opportunity to focus on the process of mathematics. The question still remains however, as to how classroom teachers would perceive such problems. Unfortunately, we suspect that they would see them as being interesting asides, to be done only after the regular material is covered, if at all.

One advantage often cited for the use of the calculator is that it allows more realistic problems to be solved. Our group was of the opinion that many problems cited as examples of "realistic" problems are not realistic at all. In many instances whether or not the problem is realistic depends on how it arose. To illustrate this consider problem 9 above. Should the problem be presented by the teacher as one of several similar problems, many students would not see it as being realistic. However, if the problem arose from the student's experiences either in the classroom or on a field trip, as in fact was the case, then the problem would certainly be a realistic one. Although it is certainly not a necessary condition that to be a realistic problem, the problem must be suggested by the student, it could be strongly argued that if the problem arises out of experiences that the student has had, it will more often than not be realistic to that student.

The two areas discussed above are closely related. In solving realistic problems, the focus will most often be on the process of mathematics. Our group chose to discuss the use of the calculator through problem solving, yet we recognize the value of the calculator in other areas of mathematics teaching. It can be used as an efficient checking device, a table of trigonometric functions, or a machine to do routine calculations that occur in the mathematics classroom. An example of the latter would be with work on area and volume. In teaching a unit on volume, the excessive calculations involved in finding the volume of a cone could be easily completed with a calculator. Much time would thus be freed to allow students to experiment with the concepts of volume by actual measurement processes.

SENSITIZING EDUCATIONAL PERSONNEL TO THE USE OF THE CALCULATOR

The earlier section of this report on existing materials was concerned primarily with sensitizing various educational personnel to the use of the calculator. It should be noted that the choice of material must depend on the audience to whom it is directed. One further vehicle for sensitization, particularly for those in leadership positions, is the reporting of research. The Use of Calculators in Education: A State-of-the-Art Review by Marilyn Suydam of the Calculator Information Center, mentioned earlier, contains an up-to-date summary of research findings on the topic. The most widespread finding is that the use of calculators does not harm students' mathematical achievement.

Our attention was also drawn to research being conducted by Shumway at Ohio State and by the Wheatleys at Purdue. Reports of their work is forthcoming in various journals.

An important method of sensitizing teachers is through in-service education. During the final day of our study group we examined in detail an inservice package prepared by Claude Gaulin and Roberta Mura of Laval University for teachers in Quebec. This material, written in French, is a self-contained kit which is administered at the local level by a resource person (the animateur). An animateur's guide is included in the kit.

The material is present in four phases. In the first phase, the teachers become familiar with their own calculator, through a series of exercises which they work on during their own time. They receive this material approximately two weeks prior to meeting in a group of 15 teachers.

During the second phase, which consists of a 3 hour group session conducted under the direction of the animateur, the teachers review work done in the first phase, as well as work through a series of five worksheets. The worksheets contain examples of activities that can be carried out with students in the schools. Included are discussions on certain pedagogical issues relevant to each of the activities.

The third phase is again carried out by the teachers during their own time. It consists of a project done either individually or in a team situation. The project includes topics such as a survey of teacher attitudes, analyzing some available curricular material or calculator books, or trying out a small project in the classroom.

During the final phase, which, like the second phase, is a 3 hour group meeting conducted under the guidance of the animateur, the reports of the projects are discussed, a second series of worksheets is completed and an overall summary is given. The worksheets contain activities more advanced than those previously used but which still contain ideas that may be adapted for use in the classroom.

It was the general feeling in the group that the kit was an excellent example of a good inservice project and provided a superb overview of current thinking on the use of the calculator in our schools. The kit contained examples of activities dealing with "realistic" situations, large numbers, estimation, development of concepts, logical thinking, exploration of patterns as well as other areas. It gave the teacher ample opportunity to become familiar with a calculator and materials with which it could be used. The objective of sensitizing teachers to the use of the calculator was well achieved.

APPENDIX A (Working Group B)

Realistic Problems Through Grade 8

Before presenting a few examples of the type of problem for which the availability of a calculator is a significant advantage, a brief discussion of what constitutes a "realistic" or "relevant" problem is in order. With the easy availability of calculators, there have been numerous spokesmen quoted as saying, "We can now pursue more realistic problems." Those involved in problem-solving research emphasize the need for "relevant" problems. We feel there is a great danger of going overboard on both points. For example, take the following problem:

> The PC government has promised legislation making interest payments on mortgages tax deductible, with 25% of the first year's interest being tax deductible. If a person has a $36,000 mortgage, amortized over 20 years at 11%, and is taxable at a rate of 46% on his income "top dollars", what would be his tax saving during the first year of the PC program?

We can attest to both the realism and relevance of the above question, but realistic and relevant to whom? We would be greatly surprised to find a "general" math class highly enthused with carrying out the solution to the above problem, but somehow many people tend to assume that if a problem deals with taxes, interest, mortgages and the like, it is loaded with realism and relevance (in which case the above problem is a bonanza!) Having seen elementary and junior high students enthusiastically pursuing problems seeming to have virtually no realism/relevance, we would argue that the primary concern in presenting a problem to students is in presenting it in such a way as to motivate its solution. Once the situation is presented, students themselves will usually be more than happy to provide an agenda for the problems that can be created from it. Therefore, the sample problems provided below are intended to be examples of situation from which virtually unlimited problems can be drawn, with the "realism/relevance" dictate left to the creator of the situation.

During the working group, a member presented the problem, "How high will 7 billion hamburgers reach if each hamburger is 4 cm thick?" as an example of a ridiculous, unrealistic/irrelevant problem. We disagree. We would suggest that it was the way in which the problem was presented, not the problem itself, that's at fault. In a recent edition of the Arithmetic Teacher, a teacher described a situation starting with the observation of a McDonald's sign on which the statement "over 7 billion served" had been written. With very little effort on the part of the teacher, the students started generating all kinds of questions about this figure (e.g., if hamburger buns come 144 to a box (a fact observed while the group was on a field trip to MacDonald's), how many cases of buns would it take to make 7 billion hamburgers?) Questions about the amount of ketchup, mustard, etc. needed, and, yes, even how high a stack of 7 billion hamburgers would reach (in cm, m, and km) were all eagerly pursued. I tried the same situation in a grade 6 class (without the field trip) and the students were unbelievably enthusiastic. I'd also suggest that for many of them, a much clearer picture emerged of just how great a number 7 billion was. Objectives related to understanding large numbers, problem-solving (when to use which operation, for example), and others could be identified in this situation, and the calculator was invaluable for providing quick answers to the flood of questions generated.

Another situation was described where a problem, also involving MacDonald's and calculators, resulted in a highly successful integrated project in a class of ten-year-olds. A question was asked: 'how many hamburgers do you think are eaten in our town daily?' After a brief guessing period it was decided that a more systematic approach would be needed. Suggestions were quick to follow: a survey could be made of sales in school cafeterias, MacDonald's ... Interest heightened when further questions were raised by students: 'how many hamburgers are eaten in town, in the country, state, country ... during a day, week, month, year? how many cattle are needed to produce enough meat? how many people are employed? how much money is spent on buying hamburgers? how much must MacDonald's make in a year? And a final question was asked by an excited boy who wondered how much profit his 'beef-ranch uncle' must make in a year and how wealthy must he be? It was obvious that a major study was under way. Committees were formed to obtain information from the library, town hall, restaurants, meat plant, agronomist, and rancher. Investigators, reporters and checkers were named. The original question which was posed by the teacher and which involved such subjects as mathematics, economics, statistics, science, social studies and such skills as estimation, approximation, computation and reading, had motivated the class to learn enthusiastically from a positive experience. And calculators would play an important role.

Another "situational" example: Would it be "better" to paint your home or have aluminum siding installed?

What's needed (most of this will come from the students): definition of "better" (e.g. cheaper), term (5 yr., 20 yr.), decision making (painting it yourself or having it painted), surface area of house, cost of paint (or estimate from a painter), durability of paint used, cost of aluminum siding, durability, etc. This example, when used in a junior high classroom, raised some excellent questions, and went beyond the obvious objectives to things like recognizing problems with insufficient data, research skills, etc.

A similar situation can be generated using insulation (which type is "best"?)

A different type of situation can be explored at the upper elementary/junior high level using the writing of fractional numbers as decimals. Even B.C. (before calculators), I had developed a two-week unit with this topic, and the calculator is a natural for exploring the intricacies of repeating decimals. After discussion of why rational numbers written as decimals have to either terminate or repeat, finding the decimal representation of a fraction like 1/17 (=.0588235294117647) using just a basic 8-digit (or less) display calculator can be explored. Although some intermediate steps are involved in the above problem, the discovery that once 1/17 is obtained in this manner, 2/17, 3/17 , ... , 16/17 can be obtained immediately from one operation on the calculator generates much enthusiasm. Hypotheses regarding things like the length of the repetend can be explored with 17ths (16 places), 23rds (22 places), 29ths (28 places), 47ths (46 places), and things like the role of counterexamples (37ths, 3 places) observed.

"Cycles" of repetends, e.g. $1/7 = .\overline{142857}$

$2/7 = .\overline{285714}$

$3/7 = .\overline{428571}$

$4/7 = .\overline{571428}$

$5/7 = .\overline{714285}$

$6/7 = .\overline{857142}$ can be explored (17ths also has a cycle; 13ths has <u>2</u> cycles of 6 digits each, etc.)

and observations made regarding these examples. Calculator "flaws" ($1 \div 7 \times 7 \neq 1$) can be examined.

There are many other examples that could be given, but hopefully the preceding ones will have served to get people thinking in terms of problem "settings" (as opposed to "one-shot deals") in which problems can be explored and in which the calculator is a most useful, if not vital, aid. Realistic? Relevant? You decide.

Ralph Connelly, Department of Mathematics,
Brock University.

Shirley McNicol, Faculty of Education,
McGill University.

APPENDIX B (Working Group B)

List of Participants - Group B

Connelly, Ralph	Department of Mathematics, Brock University.
Drost, Dale	Department of Curriculum and Instruction, Memorial University.
Gaulin Claude	Faculte des Sciences de l'Education, Universite Laval.
Kapoor, Don	Faculty of Education, University of Regina.
Kieren, Carolyn	Department of Mathematics, Concordia University.
McNicol, Shirley	Faculty of Education, McGill University.
Mura, Roberta	Faculte des Sciences de l'Education, Universite Laval.
Scoins, Ron	Department of Mathematics, University of Waterloo.
Thompson, Tony	Department of Mathematics, Dalhousie University.

IS THERE A MATHEMATICAL METHOD?

Group leaders and reporters:

William C. Higginson
Faculty of Education
Queen's University

Peter D. Taylor
Department of Mathematics
Queen's University

Appendices

The theme of the study group was partly inspired by the remarkable work, Proofs and Refutations, of Imre Lakatos. In this book Lakatos takes a number of examples, the main one being Euler's formula for polyhedra, and traces the historical development of the ideas. He shows that at each stage the result is formulated in what seems to be a final form with a rigorous proof, but then at a later stage, new examples or insights bring a realization that something has been overlooked. Perhaps the result must be more carefully formulated, or qualified, or the proof altered. The point Lakatos seeks to make is that the essential mathematics is not the formulation at any particular stage, but the way in which the examples and insights drive us from one formulation to the next. Thus the errors and the oversights contain at least as much that is mathematically interesting and important as the results they ultimately create. If we throw them aside once the new result is found, not only do we rob ourselves of something of value (the way we lose vitamins when we refine flour), but we lose the essential nature of the product (so we no longer have any flour at all but only wonder bread?)

So, for example, Euclid's marvellously ordered books do not tell us nearly as much about Euclidean geometry as they would if they had contained some of the false starts of the great geometer. Of course similar (but perhaps paler) false starts can be simulated by a good teacher or experienced by a resourceful reader who is willing to strike out on his own. This takes time, and less of the book is read, but the student may succeed in drawing closer to the heart of the subject. Paul Halmos makes the same point when he observes that a good way to read a mathematics book is with a paper and pencil at the side. A better way is to have the book at the side and paper in the middle. Best of all is to throw away the book.

We were fortunate to have Joseph Agassi as a member of our group, and he focused our attention on a number of important aspects of the problem. At one point he provided us with a metaphor which was to become most useful. He introduced the distinction between the workshop and the shop window. It is a variation on the process-content dichotomy, but seemed to serve our purposes much better. In the shop window of mathematics, are displayed the theorems, lemmas and proofs, and the standard tools, techniques and algorithms. The purpose of the workshop is to design and construct the objects to be put in the window. Most mathematics books describe almost entirely objects in the window, whereas Lakatos' point is that mathematics, its nature and methods, can only be discovered in the workshop. For Lakatos the contents of the garbage can (errors and thwarted forays) and scrap heap (fragments which have enough substance to be of possible use later on) both of which can only be seen in the workshop, merit our examination as much as do the objects in the shop window.

Here is an example that was presented to illustrate the metaphor. Suppose for an input α , the output is the largest root x of the equation $x^2 - 6x + \alpha = 0$. In particular, for $\alpha = 8$ the value of x is easily seen to be 4 . The problem is to do a sensitivity analysis for values of α near 8: as α moves away from 8 , how does x behave? This type of problem arises in practical applications and can be presented in a fairly concrete manner, accessible to high school students. (That is, they can be made to understand what we are trying to do.) Let us present a shop window and a workshop approach to the problem.

<u>A Shop Window Approach</u> Differentiate the equation with respect to α :

$$2x \frac{dx}{d\alpha} - 6 \frac{dx}{d\alpha} + 1 = 0$$

$$\frac{dx}{d\alpha} = \frac{1}{6-2x}$$

$$\left.\frac{dx}{d\alpha}\right|_{\substack{\alpha=8\\x=4}} = -1/2 \quad .$$

We conclude that as α moves away from 8 , x will move away from 4 in the opposite direction at approximately 1/2 the rate.

Most first year calculus students could come up with this solution, but they relate to it as if it were an incantation, and even after they have "done" implicit differentiation, they are not comfortable until they have asked whether it's all right to do it here. Some may use the quadratic formula first to get x as an explicit function of α and then differentiate, but this is a special technique which is not available for higher degree equations. The main thing about the shop-window solution is that it is slick, sophisticated (not accessible to most high school students) and cryptic (a couple of important ideas are hidden).

<u>A Workshop Approach</u> A calculator can be used to find values of x for α near 8, either by trial and error with a programmable machine, or with the quadratic formula. A table can be constructed by changing α in multiples of, say, 10^{-4}. A striking pattern is noticed immediately, and that is that the function relating x to α appears to be affine, at least for values of α near 8. It appears that for any α of the form $8 + h$, x will be approximately $4-h/2$, at least for small h. This appears to answer the problem, but we

α	x
8	4
8.0001	3.99995
8.0002	3.99990
8.0003	3.99985
8.0004	3.99980
7.9999	4.00005
7.9998	4.0001
7.9997	4.00015

may go on to ask whether there is some way we might have predicted this relationship without doing the calculations. We get the idea of letting an $x = 4 - k$ correspond to $\alpha = 8 + h$ and try to show $k \cong h/2$. Plugging into the equation,

$$(4-k)^2 - 6(4-k) + (8+h) = 0 .$$

$$k^2 - 2k + h = 0 .$$

For values of k near 10^{-4} , k^2 will be near 10^{-8} and will be negligible. We solve $k = h/2$. Incidentally, we perceive now that our tabulated values of x are not exact, but are out in roughly the 8th decimal place.

The workshop approach is accessible to almost any high school student. If he gets this far, he is well on his way to discovering calculus, not perhaps in today's polished form, but in a form more like that used by Newton and quite serviceable for the problems, such as above, he may encounter.

What has all this to do with mathematical method? Not much, perhaps. If the question of mathematical method concerns the nature of the activity in the workshop, then we had very little to say about this, at least in general terms. We did agree that the mathematician does not build the results in the shop-window in the same logical, ordered way they appear to be constructed. The process of building these is long, uncertain, somewhat disorganized, and contains many pitfalls. All that can be said from this is that very little can be learned about mathematical method simply by examining the objects in the shop-window.

We did not venture much further into mathematical method, and this is perhaps just as well. Perhaps not very much of a general nature could have been accomplished at this time. What we did do is provide a few interesting examples for one another. These examples, some of which appear in the appendices, were discussed in terms of mathematical method.

But we did skirt around the edges of the subject. For example, suppose we agree that a goal of the mathematician is the discovery of order. Is it the knowledge of the order, or the act of producing it that he really seeks?

The former suggests he works to have lovely objects in his shop window rather like a miser who must have his gold to count, while the latter suggests that it is the act of creating these objects that he craves, and once they are put in the shop window they are of interest, other than historical, only insofar as they provide tools for new creative activity.

There is, of course, something to be said for both sides. One thing that is perhaps worth saying is that the shop-window is of definite interest other than as a repository of potential tools. It provides the source of much of our activity. For example, in the Lakatos' view, yesterdays results are frequently taken out of the window, brushed off, and extended, improved, clarified, or corrected. It is perhaps a pity that when the new model is ready to go back in the window, the old one is not often put in beside it with a tag which sets it into the stream of ideas.

APPENDIX A (Working Group C)

An Introductory Problem in Math Modelling

This problem is given to the group of students in their opening class. No lead up remarks are made by myself. They are given 15-20 minutes to work in groups of 3-4. At the end of this time the groups report and the results are compared and discussed.

The mathematical background of these students ranges from high school graduates (Manitoba, hence, no calculus) to 3rd year Mathematics/Computer Science double majors.

The Problem

Given: A rabbit-fox population system about which the following facts are known.

1. Each pair of rabbits has 4 young per month.
2. Each pair of foxes has 2 young per year.
3. Each fox needs 20 rabbits per month as food.
4. Each month 1% of the foxes die of rabies.

Required: Describe a process of finding the number of foxes and of rabbits at the end of a one month period, given the numbers at the beginning.

Aims:

1, Realization that, in real world problems, assumptions have to be made since there is usually not enough data.

2. Realization that, since assumptions are made, the solution is not unique. To find the "best" solution, non-mathematical means are necessary.

3. To affirm the existence of a type of problem for which there is no answer in the numerical sense. The "formula" or algorithm is an answer; i.e., mathematics as a descriptive language, not as a set of rules of calculation.

Observations:

1. The students make assumptions without realizing it: e.g. rabbits are eaten before (after) giving birth, foxes die before (after) eating 20 rabbits, etc. These assumptions are pointed out from their formulae; usually the question has never entered their minds.

2. A crack appears in the belief held by some students that mathematics is perfect and always gives precise, unique, well-defined answers to problems. Usually they want to know which of the formulae suggested in right (fan-fare of trumpets). Some (usually the Mathematics/Computer Science majors) are upset when I say "I have no idea - probably none of them". This leads to a discussion of the use of scientific method to choose between different formulations.

Extension:

1. The problem is used later to lead to ideas of iterated transformations to extrapolate, equilibria, cycles, etc.

2. After the observation of zero death rate for foxes when the formulae are supplied to (rabbits = R, foxes = $F \leq 50$) natural questions lead to simulation and probabilistic models with deterministic models as limiting cases for large populations.

Summary:

The continuing use, modification and revision of this model/system to answer varying questions in various circumstances throughout the course casts lights (usefully, I think) on the evaluation of a mathematical model through time in response to different "environmental pressures" (different questions asked - different data given). The course I teach is based on this idea that a mathematical model is a result of a particular system, questions about that system, knowledge of that system and the mathematical vocabulary of the problem solver.

Don Eastman, Brandon University.

Comments on Mathematical Method

The question "Is there a mathematical method?" should be answered "No" at the level of detailed analysis, on the grounds that there are many mathematical methods. It seems hard to give an all encompassing meta-method which seems to fit all that we call "mathematics" or "applications of mathematics" which is not so general as to be either meaningless or tautologous.

Yet, the understanding of, and training in, mathematics is commonly supposed to confer some benefits on the recipient beyond the technical content of particular topics. Phrases such as "the ability to think clearly" and "understanding functional relationships" are often used in this connection. Surely this implies the belief that there is something which can be called "mathematical method", especially if we interpret "method" rather broadly to include attitudes and approaches.

It was suggested that, at the level of generalities, a large part of the answer to our question was that mathematicians are students of exact structures and that their method consists of trying to fit these structures, or others which they may invent for the purpose, onto the problems (perhaps initially of a non-exact nature) which they try to solve. The emphasis here on exactness is crucial and has many implications. In the area of applications of mathematics, the method referred to amounts to what we call "mathematical modelling" - an activity which is as old as mathematics, but which has only recently had the explicit recognition it deserves. Perhaps this recent recognition is due, not to intrinsic significance (which has been there all along) but to relatively recent increases in power made possible by advances in computer technology.

Economics got along for a long time as a subject without the qualities of exactness. The raw data were inexact because they were only available through sampling. Concepts such as "inflation", "depression", "the forcing up of prices", "pressure on the dollar" were doubtless real, but certainly not precise. What is usually referred to as "the mathematical study of economics" began when simple exact models of (admittedly primitive) economic systems were created. The work of the Club of Rome is particularly well-known in this regard. Economics is still, of course, inexact, but the furtherance of its understanding by means of exact approximating models is a characteristically mathematical approach, and one in which the future of the subject seems to lie.

Exactness is not necessarily connected with measurement and the real number system. Another approach to exactness is to be found in the (once again characteristically mathematical) method of axiomatization. Systems of relations need not be numerical in order to be axiomatizable. Even if the relations are not exactly definable, as with economics, the use of exact axiomatic systems in studying them can be very fruitful. For example in dealing with a problem of classification - say one in which the most natural categories seem to overlap - it would be natural for a mathematician to try to compare the given situation, or some part of it, with a suitable Boolean Algebra.

Discussion of this topic began with a (totally unscientific!) personal observation having to do with the behaviour of various groups of academics in dealing with the business of their faculty councils. Is there a hangover from one's strictly mathematical or humanistic activities which shows at the level of discussion of university affairs? It was suggested that there were, at least, times when the answer seemed to be "Yes". Mathematicians seemed to be impatient with discussions which remained for very long in a "jungle" of imprecise notions and seemed to suffer from philosophical timidity, whereas arts professors seemed to be much more at home in an (initially, at least) environment of "soft" concepts and more willing to take part in philosophical discussions even if the likelihood of progress were remote. These observations may have little value as evidence, but the issues which they raise are worth further study.

I should perhaps state explicitly that my interpretation of the question differed from that of the other participants.

Their interpretation was internal relative to mathematics. Given a mathematical problem, how do mathematicians go about solving it?

My interpretation was at the level of the relation between the method of mathematicians in approaching problems generally and that of others.

Problems such as counting the number of regions produced by joining all pairs from n points on a circle illustrate beautifully many principles of problem-solving, but I claim that the methods ["... the incessant improvement of guesses by speculation and criticism, by the logic of proofs and refutations." (Lakatos)] are general and logical (in the broad sense) rather than specifically mathematical. What distinguishes "Mathematical Method" is the use of exact structures of relations. What makes us think of this "circle problem" as connected with mathematical method is not so much the method as the problem itself. The problem is entirely mathematical to start with, so, of course, everything we do with it will have a mathematical flavour. If there is a specifically mathematical aspect of the solution, it is (after the dust of trial-and-error has settled) the final product, which is a proof by exact deductive logic.

Ralph Staal,
Faculty of Mathematics,
University of Waterloo.

APPENDIX C (Working Group C)

A Model for Voting Bodies

In exploring "mathematical method" in our Working Group, we discussed how a mathematician thinks about and ultimately solves problems that are presented to him as mathematical problems. Don Eastman moved a little way away from that in the last hour in telling us about his mathematical modeling problem with the fox and rabbit populations.

I had intended to bring up for discussion the mathematizing of basically non-mathematical situations. How does the mathematician decide what mathematical questions it will be profitable to ask? How does he design an appropriate mathematical model? How does he make decisions about its appropriateness? Can he remain aware of what aspects of the real situation have been lost in its translation to mathematics?

The example I had in mind was the one used by the Comprehensive School Mathematics Program (St. Ann, Missouri) in their "Book A". The situation describes voting bodies, which we would describe as a set of n voters. How many votes are assigned to each voter? "One person, one vote" is one situation but there are others. We might have a city government situation where the mayor is awarded (say) 3 votes, the council persons are awarded votes according to the size of their constituencies, the chairman of the School Board is awarded one vote (etc.).

How can a resolution pass in a given voting body? Some situations might require a yes vote from a simple majority of members present; others might require, for example, a 2/3 majority of all members. Or there are more complicated rules for passage of a resolution: in the United Nations Security Council, a resolution passes if it receives a yes vote from at least 9 of the 15 members provided that none of the 5 permanent members vote no.

Given any voting body, the vote power of each of its members, and the rule for passing a resolution, one question we might ask is what are all possible subsets of the voting body that constitute a winning coalition (where a winning coalition is a block of voters who, if they vote as a block, cannot lose regardless of the actions of the other voters). We would certainly then be interested in "minimal winning coalitions", i.e., a coalition that wins if it votes together and loses if any of its members vote against it (and no other voter votes with it). Further questions might concern individual members of the voting body: we might define as "powerless" an individual who is not a member of any minimal winning coalition; an individual who a member of every minimal winning coalition might well be described as "dictator" - but if more than one such member exists in the voting body, "dictator" will not have the conventional meaning.

Thus we have 3 essential components for our mathematical model. First, we have a set V whose elements represent members of the voting body. Second, we have a function, p , that assigns to each element i of V a number p(i) , the number of votes assigned to i (or some number taken to be an accurate description of the power of i's vote). Third, we have r , the rule that determines passage of a resolution. How are we to model the rule?

The rule must specify all the winning coalitions; however, a complete listing of these may not always be enough to determine the rule. If, for example, the rule for passing is that a resolution must simply receive more yes votes than no votes, a resolution may pass without the votes of every member of any winning coalition. This could occur, for instance, if a group of 15 voters under a power distribution of "one person, one vote" voted 7 yes, 6 no, 2 abstentions. The bloc of 7 is not a winning coalition by our definition since they would be defeated if the 8 others voted no.

In general, the rule r is a relation between the sum of all p(i) (for all i in V or else for all i in the subset of V participating in an event) and the sum of all p(y) where $y \in Y$, $Y \subseteq V$, where the elements of Y are those elements of V that vote "yes" on a given resolution. Certain restrictions might be placed on the relation as in the case of the U.N. Security Council. In the case described just above (the 7-6-2 vote) we could state the rule as $w > 1/2n$ where w is the number of voters who vote yes and n is the number who participate.

Thus our model of voting bodies will contain for its elements ordered triples (V,p,r) where V is a set; p is a function that maps elements of V to a number, $i \rightarrow p(i)$; and r is a relation between the sum of all $p(i)$, $i \in V$ and the sum of all $p(y)$, $y \in Y$ as defined above, with (perhaps) certain restrictions.

Once V , p , and r were clearly defined, we could give precise mathematical definitions for coalitions, minimal winning coalitions, minimal winning coalitions, powerlessness, etc. A winning coalition would be a subset A of V such that when $Y = A$, r is satisfied.

Such a model would be a useful mathematical tool for answering questions and evaluating situations. We would use it to answer questions about the composition of minimal winning coalitions, questions about whether any members of a voting body are powerless, questions about the relative power of various subsets of the voting body.

We would also want to be able to evaluate our model. Not only do we want to know whether it is the best possible mathematization and how well it helps us answer certain questions; we also want to consider what aspects of the real situation we are likely to lose sight of by viewing only the model. We might, for example, lose sight of the Sociological implications of certain coalitions, the Charismatic powers of certain members, and perhaps more.

Rosalie Dance, Queen's University

SOME LESSONS IN THE SUN

by Fernand Lemay

I. Identifying the set S.–

Here is a set (*fig 1*).

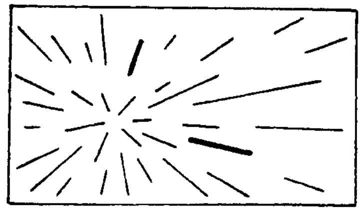


Fig 1

A pupil says that it looks like a sun – the lines are rays.

So that is what we call it: a SUN.

II. The core of the sun.–

What can be said about this set?

... There are lines in all directions – some long and some short – some black – two green ...

Certainly the lines (SEGMENTS) differ in colour, in length and in direction. But have they not anything in common?

They all pass through the same point.

(Since the segments are fixed it is not strictly true to say that they pass through a point. Inaccuracy of this sort is common and does not matter as long as it is understood).

Would it make sense to say that the SUPPORTS of the segments pass through a special point?

Could you add new segments to the sun? How many?

An infinity, asserts a pupil. Thus the picture cannot be completed. But nothing stops us from imagining the final COMPLETE SUN and it is to this that we will always refer. Fortunately a few segments or even just the centre are enough to represent it.

Now that we know what a sun is and can imagine it should we not write out an official description defining it so that we do not have to keep on remembering it?

As the construction of definitions had become for various reasons a very popular activity, someone soon offers to start framing one.

SUN: several supports passing through a centre.

Marie does not find this quite right and modifies it:

SUN: several segments whose supports pass through a centre.

But had we not said that we were going to think of the complete sun? ...

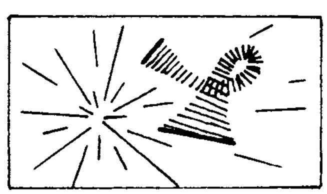
Gradually the pupils elaborate a text that satisfies them. It becomes official by being duplicated, distributed to all and added to the mathematical dictionary which other investigations had led us to compile. This dictionary is the arbiter in discussion. If at any time an entry in it is no longer sufficiently discriminating then it has to be improved.

III. Paths through the sun.—

Show a path joining the two green segments — using for instance an elastic band stretched to start with to the length of one of the segments.

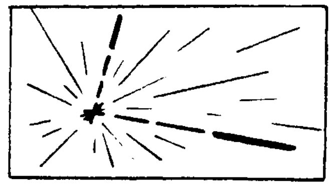
Such a path could be represented by an appropriate number of segments suggesting various positions reached during the transformation from the initial position to the final one (*fig 2*)

Fig 2



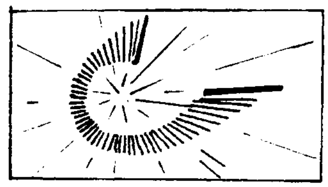
That is fine but now I would like a path WHOSE COMPONENTS ARE ALWAYS CONTAINED IN THE SUN (*fig 3*).

Fig 3



This first slightly cautious suggestion is followed by more and more adventurous ones (*fig 4*).

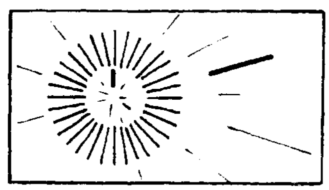
Fig 4



IV. Barricades.–

Here is a red barricade (*fig 5*).

Fig 5

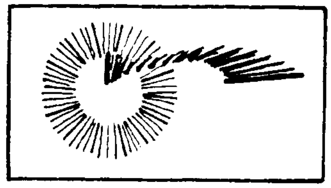


Is this obstacle going to stop one green segment finding the other by moving along a path that can be traced in the sun itself?

Anyone who has immediately seen a way THROUGH the red segments needs to be reminded that the figure was only a sketch – the CROWN should be complete. (How delicate is communication – a casual remark can here force the framing of an official definition of a crown.)

Before trying an actual solution the situation has to be explored in imagination. At first there will be disagreement about the existence of a path joining the two green segments. After discussion it happens that opinions converge to an agreement (*fig 6*).

Fig 6



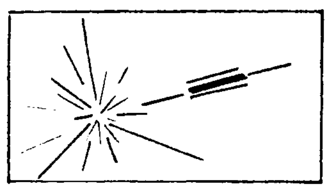
The discovery of a path joining the two green segments shows that there is a path made out of segments not lying in the crown. (Path ∩ Crown = ∅)

V. The beam.—

Having shown that the crown is permeable by segments of the sun we still have the problem of constructing an impenetrable frontier which would separate the two green segments — a frontier which would intersect all paths joining the two segments.

Someone believes at first that four POLICEMEN placed round a green segment would stop it escaping (*fig 7*).

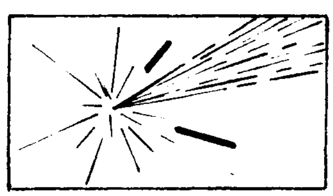
Fig 7



Eventually another suggests colouring red ALL the segments other than the two green ones. This solution, for it is one, seems unnecessarily radical.

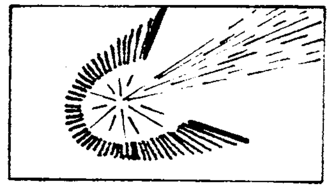
Then a sort of comet is suggested; this is made out of all the segments contained in a certain sector (*fig 8*).

Fig 8



But it is soon recognised that despite its strength this obstacle does not cloud the whole horizon (*fig 9*).

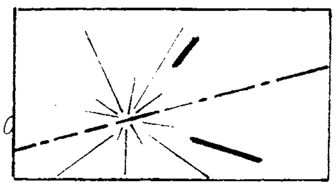
Fig 9



As various ways of modifying the latter barricade arise there is also an interest in finding the most economical barricade.

Threading the needle, as it were, a girl invents the BEAM, namely the set of segments supported by a particular line (*fig 10*).

Fig 10

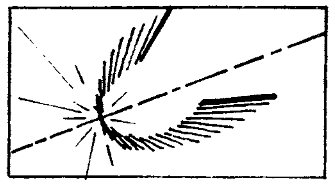


VI. Crossing the beam.—

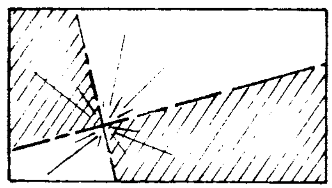
Among the suggestions then there is this very thin barricade, the beam (*fig 10*). Does this set make an efficient frontier? In other words, are all the paths joining the green segments intercepted by this set?

Once again opinion is divided; the class falls into two nearly equal groups. Those who believe this barricade is efficient have the task of seeking amongst themselves an argument that might convince their opponents. The others have to invent a path crossing the beam which they would be able to display to the whole class.

After some time each group comes back enthusiastically convinced they are right. *It is impossible to cross it*, says the spokesman for the first group, *since in passing from one side to the other the green segment has to take the direction of the beam and at that moment it will lie in the beam.* Unmoved by this, the spokesman for the other group presents a subtle path invented by two members of the group (*fig 11*).

Fig 11 

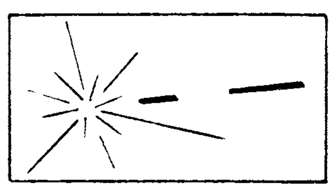
He shows however that TWO appropriately placed beams would provide an impenetrable obstacle (*fig 12*)

Fig 12 

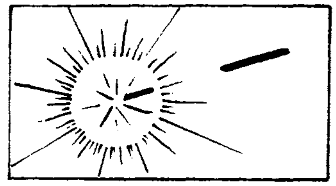
(But the members of the first group are now no longer interested in understanding where they might have gone wrong ...).

VII. Seeing a strategy.—

The last barricade is only useful as long as it can be inserted between the green segments. But there are some cases where this is not possible (*fig 13*).

Fig 13 

A new frontier has to be invented for this case. It does not take long before someone invents the following set — spontaneously called AN ECLIPSE (*fig 14*).

Fig 14 

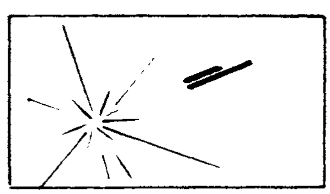
But is it really an impenetrable obstacle? A girl doubts it and asserts that she sees a route. She reduces the length of the moving segment until it becomes a point! Yet if such DEGENERATE SEGMENTS are permitted then they also occur in the eclipse itself .... Thus the class gets involved in defining the term 'segment'. It is decided that the extremities of a segment must be distinct. (This is the eventual definition but it was difficult to reach because the class felt it necessary to make the meaning of the word 'line' more precise).

Not only is the eclipse permitted as a frontier but it is also recognised to be a THIN frontier in the sense that if a single segment were removed at least one path would be made available. Moreover the inventor of the eclipse will have provided an oral demonstration of this. (Certainty creates the need for proof – J.L.Nicolet)

It may rightly be supposed that the terms eclipse, frontier and so on will eventually appear in the dictionary for inevitably they arise in the course of involved disagreements. Meanwhile there is nothing to stop us undertaking this task in any case.

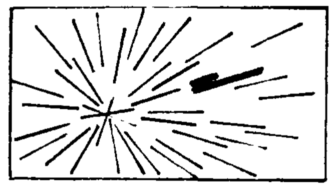
VIII. Sabotage of the eclipse.–

Pursuing the strategy of sabotage let us submit the case of two partially superimposed segments as shown (*fig 15*).

Fig 15 

It is now no longer possible to separate these two segments with beams or with an eclipse and new forms of frontiers must now be looked for. Concentrating on the fact that any path joining the two green segments must contain some segments of every intermediate length will produce the notion of a frontier that prohibits one of these lengths (*fig 16*).

Fig 16



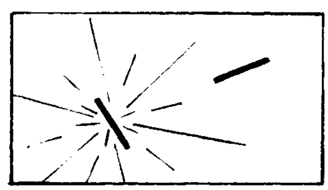
This new frontier, THE STAR, is a subset of the sun made out of segments of some previously chosen length.

This is indeed only one solution among many - one might have thought for instance of an inverted eclipse whose segments lie inside the underlying circle.

IX. Sabotage of the star.-

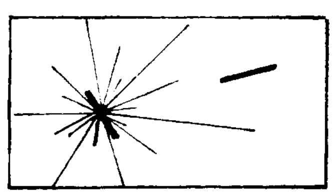
Notice another special case which will exclude the use of the star - two green segments of the same length (*fig 17*).

Fig 17



Though it is possible to call on a pair of beams or an eclipse we can interpose the elegant SPARKLER, namely that part of the sun made out of segments having one extremity at the centre (*fig 18*).

Fig 18



The sparkler – as indeed the star, the eclipse or the pair of beams – splits the sun into two isolated parts.

X. Descending to the empty set.–

Sabotage is no longer so important since there are now available a variety of frontiers that can split the sun whatever the given pair of distinct green segments.

The study of the various paths joining two segments of the sun can be repeated for each of the new sets which have appeared as frontiers.

Here, by way of illustration, is a 'descent' starting from the sun (*fig 19*).

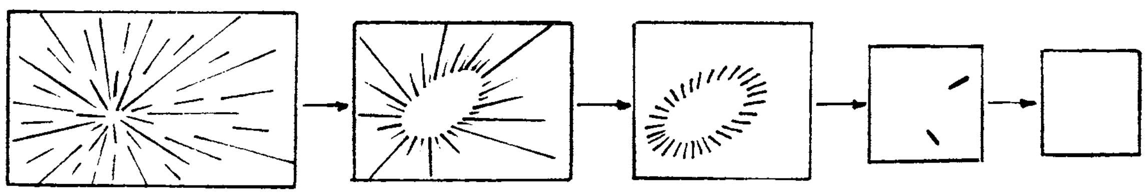


Fig 19

But this is only one among many as the following scheme suggest (*fig 20*).

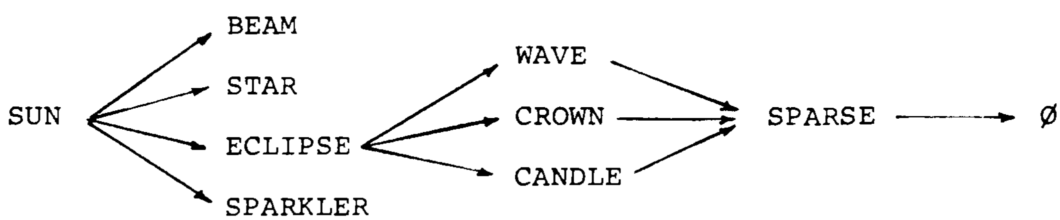


Fig 20

XI. Poincaré's problem.–

Just as the 'atomic matter' of conventional geometry merges into complexes that are called geometrical figures so does the basic matter of the universe of segments permit the construction of such geometrical objects as suns, stars, sparklers, beams, eclipses and so on. The children explored about twenty such sets and gave them delicious names. One day

I wrote a very serious letter to the thirty-two children in the class, daring to propose that they establish a clasification of such sets and of those that remained to be invented.

The empty set would have to be the sole member of the first category.

The next category, to be designated $\mathscr{C}_0$, would contain, besides the empty set, the SPARSE SETS, that is to say those which are already isolated from each other without introduction of any frontier.

There would then be a category $\mathscr{C}_1$, containing sets which could be separated from each other by frontiers taken from the previous category $\mathscr{C}_0$. (This category certainly contains the empty set and all sparse sets but are there other sets in it?)

There would follow a category $\mathscr{C}_2$, made up of sets which could be separated from each other by frontiers taken from $\mathscr{C}_1$. And so on – to form farger and larger categories.

$$\ldots \mathscr{C}_n \supset \mathscr{C}_{n-1} \ldots\ldots \mathscr{C}_3 \supset \mathscr{C}_2 \supset \mathscr{C}_1 \supset \mathscr{C}_0 \supset \{\emptyset\}$$

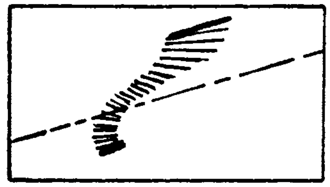
This classification was conceived by Henri Poincaré (*1854-1912*), one of the greatest mathematicians of our time. The sets which are in $\mathscr{C}_n$ without being in the preceeding categories are said to be OF DIMENSION n. (There is no need to define a dimension for the empty set).

XII. Ascending to the universe.–

Through frontiers we have descended from the sun as far as the empty set. The latter serves as a frontier for the sparse sets and in turn each set could be considered to be a frontier to larger sets, for example to the set of all plane segments, the UNIVERSE.

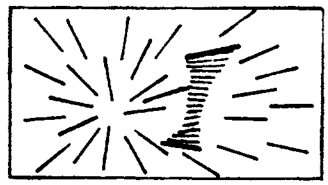
Given the freedom to move about in the whole universe, the beam seems such a weak obstacle that it can be surmonted almost absent-mindedly (*fig 21*).

Fig 21



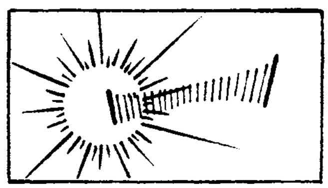
We can pass just as easily under the stars (*fig 22*),

Fig 22



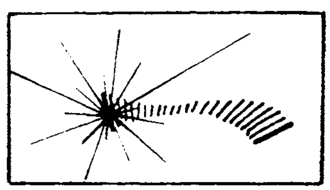
or slip through the eclipse (*fig 23*),

Fig 23



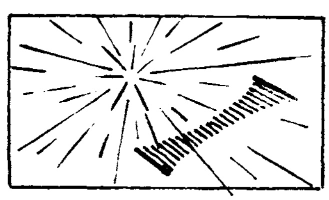
or even escape the sparkler (*fig 24*).

Fig 24



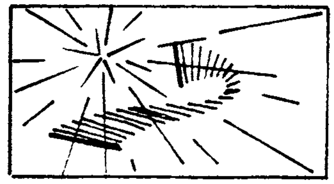
But all these paths could be intercepted by an adequately situated sun (*fig 25*).

Fig 25



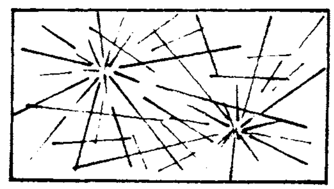
Nevertheless the sun is not impenetrable (*fig 26*).

Fig 26



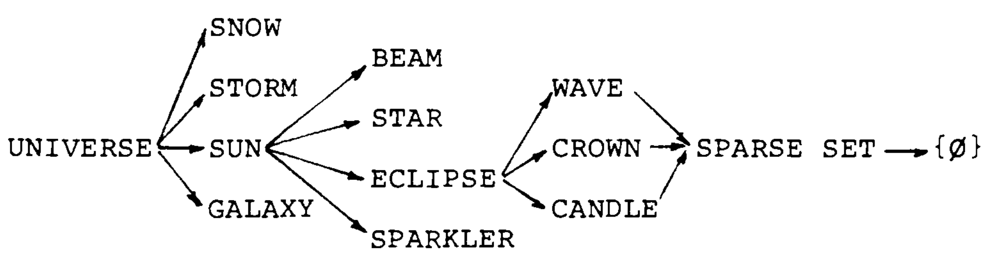
But as some pupils point out the rays of two suns double their effet and do finally just trap the segments (*fig 27*).

Fig 27



XIII. The dimension of the universe.—

These were only 'some lessons in the sun'. Many others have been evoked for the sun could have been sabotaged and after working at various tasks and sharing information the class might have made some lessons in the SNOW, in A STORM, or in A GALAXY ...



Since the various descents to Ø tally the universe (of line segments in the plane) is then seen to be of dimension four.

Université Laval, Québec.

APPENDIX E (Working Group C)

Is there a mathematical method? Brief notes for a discussion

The answer to the question is not in books. Indeed, there isn't a yes-or-no answer to the question.

The question is useful to the extent that it leads to a fruitful discussion; literature is useful to the extent that it heightens awareness or opens the mind to some un-thought possibilities.

The group members are as capable as anyone of tackling the question; it requires no particular expertise, only some experience of mathematics and a willingness to reflect about it. Authorities should not be cited. Participants should not exchange received opinions. One of the chief functions of the group leaders is to keep calling speakers back to the evidence for their views.

Why is the question interesting/significant?

Philosophically (of course)

"The traditional central concern of philosophy of science generally, and philosophy of mathematics especially, has been the problem of justifying our beliefs, certifying them to be objectively true. In mathematics, this is to be done by finding some central core which is "clear", and then deriving all else from that by irrefutable logic." (Reuben Hersh, Math Intelligencer Vol.1, No.3)

(Is this fair comment?)

Hersh uses this to introduce the quite different concerns of Lakatos, one of which certainly seems to have been to delineate "a method" of mathematics.

Another way to look at this is to say that traditional philosophy is concerned with the end products of mathematics, whereas Lakatos (and, say, Hadamard and Polya) are concerned with the methods by which mathematicians arrive at their results. Yet another way to distinguish is to separate the philosophy of formal mathematics from the philosophy of informal mathematics. And so on.

The distinction is worth keeping in mind, even in naive and non-professional discussions. Presumably, since both "philosophies" are about mathematics, they are not unrelated. But that is a question philosophers do not yet seem to be interested in.

Two possible shortcomings of the Lakatos position (though without in any way detracting from the genius of his contribution):

(i) it does not seem that fallibilism can account for all of the different sorts of mathematical development (see Agassi's criticisms in "The Lakatosian Revolution")

(ii) it can fall into a kind of neo-Platonism.

One may eventually prefer to answer the question of mathematical method with a "no". This "anarchic" position is the one that Feyerabend takes in "Against Method" (though he is talking about science, not mathematics).

Pedagogically

The group could spend a lot of time on this. There are two fairly obvious connections, one being much easier to trace than the other. Students' responses in problem-solving situations, especially where the problems are of a more-or-less "open" type, are sufficiently "like" the Lakatos story to suggest that aspects of the "mathematical method" are "natural", that they occur without the learner needing to be instructed in their use - that, in an informal mathematical situation, anyone acts in ways rather similar to those adopted by mathematicians. However obvious this may seem, it may be worth spelling out in detail since it is by no means embedded in the pedagogical folklore.

The harder question may be to seek for connections between the ways in which mathematics is developed and the ways in which learning develops. For example, can we find connections between mathematical methods and, say, Piaget's view that all knowledge is a special case of biological adaptation? Thom, for instance, says that "mathematical structures exist independently of the human mind which thinks them ... their type of existence is no doubt different from the concrete, material, existence of the external world, but is nevertheless subtly and profoundly tied to objective existence ... how else can we explain their decisive success in describing the universe?" This is, I believe, patently nonsense, and denied every day by our experience. Fortunately Piaget's approach - and other "genetic" variants - allow us to avoid this position while still accepting that there is a strong element of "necessity" in the mathematics that the learner develops: as if, in Spencer Brown's formulation, mathematics is part of the data imposed by the external world. Thom forgets the evolutionary nature of knowledge and is still stuck with Platonism - which, God knows, continues to be an albatross about the neck of pedagogy.

Mathematically

If there is a mathematical method, or if there are mathematical methods, it follows that it (or they) must guide mathematical research and be involved in the evaluation of research. Pursuit of this question will depend on the presence of mathematicians able to bring some specialized and concrete examples to the discussion. Certainly one might want to attempt to answer two tricky questions in this area:

(i) how are (should be) mathematical researchers trained? (Choquet wrote a provocative paper on this.)

(ii) in a time of limited resources, what criteria should guide the choice of mathematical research to be encouraged or promoted?

Historically

It seems on the face of it as if we will learn about method from history; but at the very least there is interaction between the history we choose to pursue and our understanding of the mathematical process. History is not facts but interpretation, and it is fairly clear that the kind of history (history of mathematics) we know is that which has been written by people who asked very simple and naive questions about mathematics. As Lakatos puts it, "actual" history is a parody of the rational reconstruction of mathematical knowledge. This is unlikely to be a matter that the group will want to do more about than just keep it in mind at present. But at some time not too distant we will find that most of the history we "know" is no use to us - that Kline and Boyer are no less fallible than E.T. Bell and D.E Smith.

If there is a mathematical method, it was there all the time and we shall find it present in what we already know about mathematics, though probably not expressed in the same kind of terms. It may therefore be worth looking at what might be called "characteristics" or "characteristic attributes" of mathematics and see if they yield anything when studied from this fresh point of view.

A few possibilities are:

Heuristics This is the most obvious, though probably the least important, connection with "method". Least important because heuristics is so ill-defined that it's impossible to define anything else in terms of it.

What would happen if ... ? This seems useful - it connects with thought experiments, with modelling, and with the kind of problem-generating that Brown and Walter talk about.

Substitutes, including schematic representation Mathematics characteristically uses substitutes because it needs a high degree of abstractness, a high level of generality, and because it is more concerned with the structure or skeleton of situations than with all their particularities. Its objects are never "things" but always representations of things or of relationships between things.

Concern with solving problems "in principle". The most obvious example is the whole range of arithmetic computation. Mathematics has "solved" the problem of multiplying "any" two numbers - yet in practice, for most choices of the two numbers, there is no one who will actually get the computation right. (The "in principle" attribute is behind potential infinities - e.g. Archimedes methods of summing certain infinite series. But, indeed, it seems almost like a necessary component of mathematical thinking of any kind.)

Self-evidence and construction Leibniz, discussing Descartes and "pure" concepts whose fundamental quality is existence: "It is not at all certain whether what is thinkable also has real existence: Of this type are such concepts as the number of all numbers, infinity, smallest, largest, most perfect, totality, and other notions of this type which are not by their nature self-evident and become fit to use only when clear and unambiguous criteria for their existence have been established. It all amounts to our making a truth mechanically, as it were, reliable, precise and so irrefutable: that this should at all be possible is an all but incomprehensible sign of grace."

"The axiomatic method" Is this a "method" in the sense we want? At least it cannot be overlooked that mathematics is the only field where axiomatization is possible. (This has been talked about so much that it is hard to get a new grip on it.)

The possibility of rigorous proof Again, this has been done to death and is embedded in the folklore. Moves to modify the special character of mathematics by stressing the social nature of proof - that it is what a certain group of people agree to - are not totally convincing, to say the least.

Elegance One may be ambivalent about this. On the one hand it is a sign of the high caste activity of the mathematician
on the other, it is an empirical fact of mathematical activity (connected with Ferenczi's condensation, and with Krutetskii's curtailment of reasoning, etc.). Intuitively, one feels that a careful study of the source of this urge for "elegance" would be instructive about mathematical activity in general. Do any studies of this phenomenon exist?

A description of mathematical method, in addition to its philosophical interest, might assist talk about the way mathematics comes into being, whether in the learner or the inventor.

David Wheeler,
Department of Mathematics
Concordia University

APPENDIX F (Working Group C)

List of Participants, Group C

Dance, Rosalie	Faculty of Education Queen's University
Eastman, Don	Department of Mathematics Brandon University
Guenther, Grant	Faculty of Education Queen's University
Higginson, William	Faculty of Education Queen's University
Hillel, Joel	Department of Mathematics Concordia University
Lemay, Fernand	Faculte des Sciences de l'Education Universite Laval
Mendoz, Lionel	Faculty of Education Memorial University
Robitaille, David	Faculty of Education University of British Columbia
Staal, Ralph	Faculty of Mathematics University of Waterloo
Taylor, Peter	Department of Mathematics Queen's University

TOPICS SUITABLE FOR MATHEMATICS COURSES FOR ELEMENTARY TEACHERS

Group leaders and reporters

Bernard R. Hodgson
Département de Mathématiques
Université Laval

Hugh A.J. Allen
Faculty of Education
Queen's University

Appendices

Report of Working Group D

Topics Suitable for Mathematics Courses for Elementary Teachers

At the 1978 CMESG meeting, one of the working groups considered the topic "Mathematics Courses for Teachers." Their report is concerned with the tone and the overall approach and intent of such courses, regardless of specific content. Working Group D at the 1979 meeting took this report as a starting point and saw its task to be the identification of specific topics suitable for inclusion in a mathematics (as opposed to 'methods') course for elementary school teachers. After some discussion, we agreed not to attempt to distinguish between courses for pre-service teachers and those for in-service teachers. We also agreed that although we would restrict our attention to mathematics topics, when teaching these topics to teachers (either prospective or practising) it would be advisable to consider the pedagogical implications of the topics vis-a-vis curricula in schools. (For example, some topics taught to teachers may be modified and adapted for use with children, whereas other topics are not intended to be adapted and presented to children but are intended to provide background for the teacher.) In considering suitable mathematics topics for elementary school teachers we had to keep in mind the fact that the mathematical background of many elementary school teachers is extremely limited. Many have had no senior high school mathematics.

After some consideration the group decided to concentrate on geometrical topics. Elementary teachers tend to regard geometry as a deductive system characterized by axioms, postulates and proved theorems, whereas the geometry they are asked to teach in elementary schools is largely an informal study of space and shape. Much of the geometry taught in elementary schools is three-dimensional geometry, and few teachers have had much experience with this topic. Geometrical topics can be presented in a manner consistent with the aims and suggestions contained in the report of Working Group A at the 1978 CMESG meeting.

The group began with a consideration of the general topic of polyhedra. From this beginning a number of related and suitable topics emerged and were discussed. What follows is an outline of suggested topics and approaches that were considered by the group. As indicated, greater detail concerning specific topics can be found in the various appendices to this report.

A formal definition of the term 'polyhedra' is not necessary at the outset. Indeed, a definition is only meaningful after one has been shown sufficient examples and non-examples to be able to appreciate what something is and what it is not. By examining many geometrical solids and attempting to determine why some are classified as polyhedra and some are not, an understanding of the concept of a polyhedron can be achieved that is sufficient for students to do productive work with polyhedra. The name 'polyhedron' merely serves to identify solids of this particular type. One may now examine various properties of polyhedra. Topics which might be included are

- the characteristics of edges, faces and vertices and the relationships between the number of these
- the regular and semiregular polyhedra
- the symmetries (rotational and reflective) of regular polyhedra
- the various 'sections' made by the intersection of a plane and a polyhedron (e.g., a cube)
- the representation of polyhedra in two dimensions (Schlegel diagrams).

The last topic led the group to consider a related topic, namely, that of making two-dimensional sketches of three-dimensional solids. One particularly useful reference for this topic is a set of work booklets associated with the Fife Mathematics Project. More information on these booklets is provided in Appendix A. C. Cassidy introduced the group to the commercial game STRUCTURO. A description of the game and some of Cassidy's imaginative variations may be found in Appendix B. Activities with the Soma Cube provide excellent opportunities for visualizing and constructing three-dimensional models as well as representing the models in two dimensions (see Appendix C). Another source of mathematical ideas associated with representing the three-dimensional world on a two-dimensional plane can be found in topographic maps. Some ideas may be found in Appendix D. One idea that emerges from an examination of profiles for various paths from A to B on a heavily contoured topographic map, is that the best route from A to B is not necessarily the direct path. This activity led the group to consider shortest paths on a sphere (geodesics) as a suitable topic for elementary teachers.

B. Hodgson described a different metric in two-dimensional space which gives rise to a non-Euclidean geometry known as Taxicab Geometry. The group felt that this would be a more suitable non-Euclidean geometry for elementary teachers than the usual non-Euclidean geometries obtained by altering the parallel postulate. A description of Taxicab Geometry can be found in Hodgson's Appendix E.

The routes of the taxicab reminded C. Verhille of a related problem in geometry and number. This attractive investigation is described in Appendix F. Verhille went on to describe the teaching possibilities inherent in the topic generally known as polynominoes. See Appendix G for more information and for references.

The last geometric topic considered by the group was one presented by J. Bergeron in which he described some work done by his students in relating art and modular arithmetic. For further details see Appendix H.

Concluding Remarks

The group was aware of the fact that one of the major problems of teachers is to motivate children to do mathematics and that in this respect geometry is an attractive subject as it is probably easier to arouse interest in geometric explorations than in many "traditional" curricular topics. In future years we would hope that a similar group might wish to consider suitable topics for elementary teachers from the areas of number and probability and statistics.

Finally, although it is difficult to express it in this summary, it is important to record the fact that this group enjoyed working together for nine hours. It was the unanimous opinion of the group that we had had valuable and significant sessions and we look forward to future meetings.

Appendix A (Working Group D)

Sketching Three-Dimensional Solids

A particularly attractive source for teaching 3-D sketching is the set of workcard booklets (and associated materials) devised for the Fife Mathematics Projects by Geoff Giles. This material is one of the Mathematical Aids designed by DIME (Development of Ideas in Mathematical Education) and is distributed by Oliver & Boyd, Croythorn House, 23 Ravelston Terrace, Edinburgh EH4 3TJ.

The five booklets on 3-D sketching make use of a set of solids as models to be represented by students on isometric dot paper. Book 1 is concerned with the interpretation of isometric drawings and provides practice in recognizing solids from the drawings. Subsequent books in the series deal with making 3-D sketches of solids on isometric paper, visualizing solids from different viewpoints and sketching the results, and visualizing and sketching the results of toppling, turning, and reflecting solids. Book 5 (Wedges) deals with sketching solids not made up of cubes and gives further practice involving toppling, turning, and reflecting solids.

Hugh A. J. Allen
Mathematics Education
Queen's University

Appendice B (Working Group D)

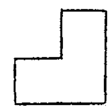
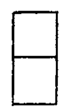
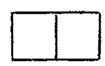
Le jeu STRUCTURO

Le jeu STRUCTURO a été conçu par André Clavel et il est distribué par l'éditeur français Fernand Nathan, 9, rue Méchain, Paris 14°.

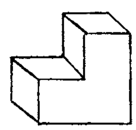
La composition de ce jeu est très simple: 53 cubes en bois, 62 cartes-problèmes et 62 cartes-solutions. Voici un exemple typique de problème suggéré:

Problème:

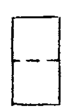
Construisez un solide ayant les caractéristiques suivantes:

- vue de face: 
- vue de gauche: 
- vue de dessus: 

L'enfant essaie alors de construire le solide en question puis il vérifie sa solution à l'aide de la carte-solution correspondante sur laquelle il retrouve le dessin suivant:



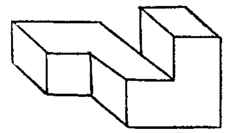
Afin de préciser un peu l'une des conventions du jeu signalons que, pour le solide illustré ci-dessus, la vue de droite aurait été représentée par:



Il n'est pas difficile d'imaginer une grande variété d'exercices à partir de ce matériel. En voici quelques exemples, uniquement à titre indicatif:

.../2

Sur la carte-solution numéro 36, on aperçoit le dessin:



On peut poser des questions comme les suivantes:

- Dessinez la vue de face, de gauche, de droite, etc...

- Imaginez l'objet illustré construit sur une table transparente de façon à ce que vous puissiez le voir par dessous. Si vous vous glissiez sous cette table, quelle vue auriez-vous de cet objet? Dessinez la vue d'abord puis vérifiez ensuite expérimentalement.

  (On peut poser la même question en demandant de se glisser sous la table de plusieurs façons différentes.)

- Si vous placiez un miroir derrière l'objet illustré, vous en verriez son image. Dessinez la vue de face, de gauche, etc... de cette image. Vérifiez ensuite expérimentalement.

- Il est également intéressant de demander à quelqu'un de construire un solide avec au plus 7 ou 8 cubes. Les autres personnes examinent ce solide un certain temps. On met alors une boîte de carton par-dessus le solide afin que personne ne puisse le voir. On peut alors demander de dessiner plusieurs vues différentes de ce solide. Quand ce processus est terminé, on vérifie sa solution en enlevant la boîte.

Il est certainement possible de trouver facilement un substitut au jeu STRUCTURO, le coût de ce jeu étant malgré tout relativement élevé. Il n'en demeure pas moins que le jeu est présenté de façon très attrayante et qu'il est susceptible de fournir des idées d'exercices à la fois accessibles à de jeunes enfants, parfois difficiles à faire même par des adultes mais toujours formateur quelque soit l'âge de l'utilisateur.

Charles Cassidy,
Département de Mathématiques,
Université Laval.

Appendice C (Working Group D)

Le jeu SOMA (Parker Bros.)

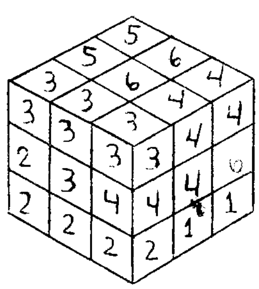
Le jeu SOMA est constitué de 7 pièces différentes qui sont numérotées de 1 à 7. Ces pièces peuvent être assemblées de bien des façons afin de pouvoir former un cube; la chose n'est pourtant pas si facile! Ce jeu est relativement bien connu et il a été mentionné plusieurs fois comme matériel éducatif, notamment par Martin Gardner dans le Scientific American.

Chacun peut facilement imaginer un grand nombre d'activités différentes à l'aide de ce jeu. Nous aimerions cependant signaler les deux publications suivantes:

- Marguerite Wilson, Soma Puzzle Solutions, Creative Publications, Inc.. Palo Alto, California;
- J. Meeus et P.J. Torbijn, "Polycubes", Collection "Les Distracts" 4, Editions CEDIC, Paris.

Dans chacune de ces deux publications, on retrouve la liste de toutes les façons possibles de construire le cube SOMA (et il y en a beaucoup...). La façon d'énumérer toutes ces possibilités diffère entre les deux publications. Mentionnons cependant qu'il est déjà intéressant en soi d'étudier de quelle façon les auteurs ont procédé afin de coder l'ensemble des solutions; il y a déjà là matière à imaginer une foule d'exercices. Le livre de Meeus et Torbijn propose, de plus, une foule d'autres activités parfois avec SOMA et parfois avec d'autres types de polycubes.

La brochure accompagnant le jeu SOMA donne également des suggestions de solides qui peuvent être construits avec ce matériel. Nous ne sommes pas convaincus, cependant, que tout ce qu'on peut gagner en construisant ces solides justifie le temps qu'on doit y consacrer, surtout si cela devait être fait en classe.



Il est cependant possible de donner des suggestions sur la façon de réaliser certaines constructions. L'illustration de gauche donne une certaine idée sur la façon de construire le cube mais ne dit pas tout! L'enfant pourrait utiliser les renseignements donnés sur cette figure pour commencer la construction du cube puis il pourrait ensuite la terminer en utilisant plus de déduction que de tâtonnement! On pourrait de même pré-solutionner partiellement certaines constructions illustrées

.../2

dans la brochure SOMA afin de diminuer un trop long travail de tâtonnement qui pourrait avoir pour effet de lasser l'enfant.

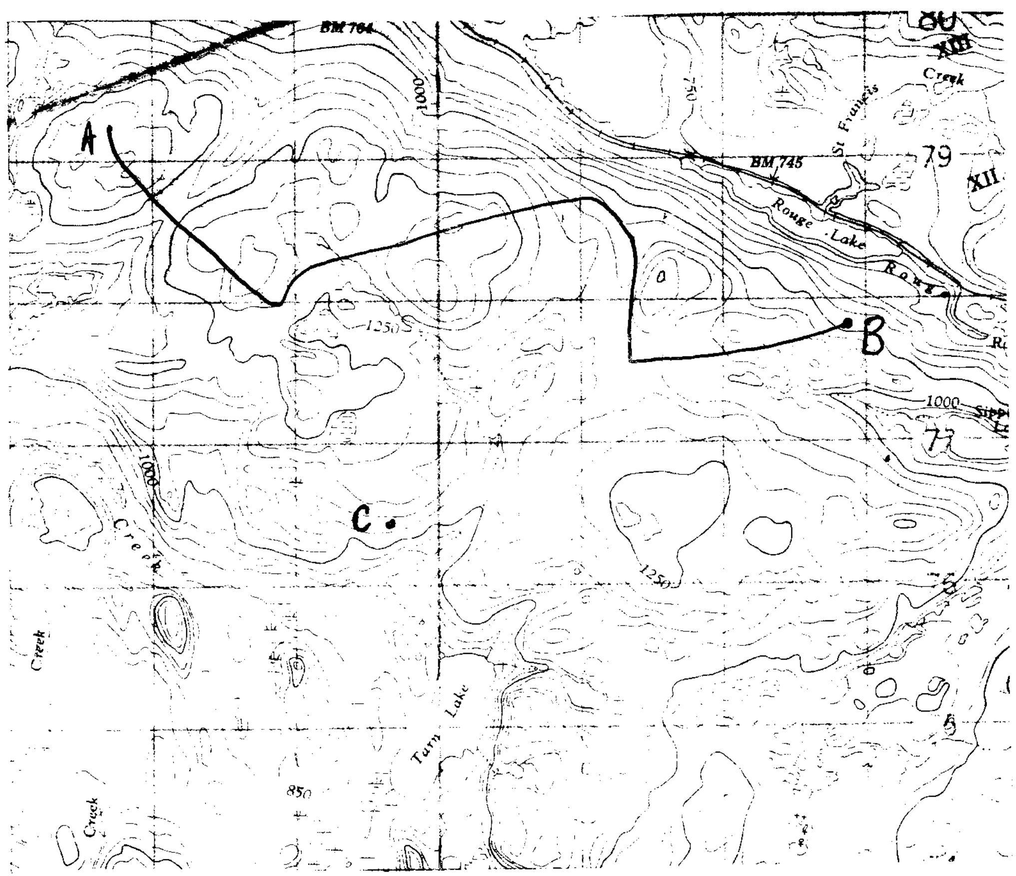
Charles Cassidy,
Département de Mathématiques,
Université Laval.

Appendix D (Working Group D)

Geometry in Topographic Maps

Because a topographic map is a planar representation of the three-dimensional world, it is a rich source of geometric ideas. Two examples are given below.

On the portion of the 1:50,000 topographic map shown below and on which the contour interval is 50 feet, a route is indicated from point A to point B. It is an interesting exercise to draw a graph showing the elevation profile of the terrain along the route as a function of the distance from A along the route. Another problem requires the students to imagine a steady source of water flowing out of the ground at the point C indicated on the map and to describe the path that the water would flow along.

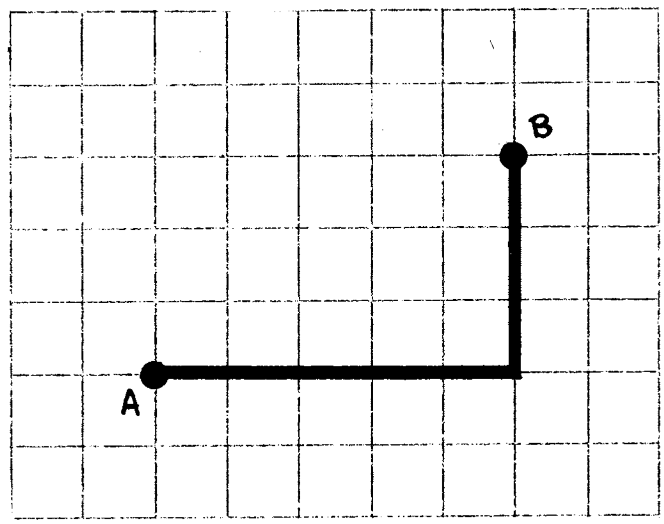


Hugh A. J. Allen
Mathematics Education
Queen's University

Appendice E (Working Group D)

LA GEOMETRIE DES CHAUFFEURS DE TAXI

Supposons que le quadrillage ci-dessous représente un réseau de rues et d'avenues et qu'un taxi veuille aller depuis l'intersection A jusqu'à l'intersection B. Un des chemins possibles est le suivant:



La distance parcourue (8 unités) est donc différente de celle obtenue en allant de A à B à "vol d'oiseau". Cette constatation sert de point de départ à l'exploration d'une géométrie non-euclidienne: la géométrie des chauffeurs de taxi ("taxicab geometry"). De façon plus générale, étant donné n'importe quels points $A = (x_1, y_1)$ et $B = (x_2, y_2)$ du plan, on définit leur taxi-distance

$$d_T(A,B) = |x_1 - x_2| + |y_1 - y_2|.$$

En plus de fournir un contexte naturel pour l'étude de problèmes de combinatoire (comme, par exemple, rechercher le nombre d'itinéraires permettant d'aller, sans détours superflus, depuis un point jusqu'à un autre), la géométrie du taxi permet à l'enfant d'approfondir la connaissance qu'il a de certaines notions de géométrie euclidienne en les opposant à ce qui se passe dans un univers non-euclidien, et ce dans un contexte plus simple que celui des géométries non-euclidiennes usuelles. De plus, il s'agit d'un contexte "nouveau" pour l'enfant qui lui donne l'occasion de se valoriser à ses propres yeux en effectuant quelques petits morceaux de recherche authentiques. (Le livre de Krause est écrit selon cette optique.) Enfin, les situations rencontrées sont d'une grande ri-

.../2

chesse et permettent aux enfants d'effectuer des explorations adaptées à leur niveau. Ainsi l'enfant du primaire se concentrera surtout sur l'étude de quelques "taxi-figures" simples (cercle, médiatrice,...), tandis que celui du secondaire pourra considérer les taxi-coniques, ou s'intéresser à l'aspect axiomatique de cette nouvelle géométrie, ou encore s'attaquer à des problèmes de répartition optimale d'un logement en fonction de divers lieux de travail, etc...

REFERENCES

- E.F. Krause, Taxicab Geometry, Addison-Wesley, 1975;
- D.R. Byrkit, Taxicab Geometry, A Non-Euclidean Geometry of Lattice Points, The Mathematics Teacher 64 (1971), 418-422;
- E.F. Krause, Taxicab Geometry, The Mathematics Teacher 66(1973), 695-706.

Bernard Ralph Hodgson,
Département de Mathématiques,
Université Laval.

Appendix F (Working Group D)

Spirolaterals

To draw a basic spirolateral use a sheet of regular graph paper, establish a set of rules and proceed. To draw the basic pattern of 1, 2, 3 using 90° right turns you first draw a segment of unit length (see Fig. 1(a)). Turn right through 90° and draw a segment two units long (see Fig. 1(b)). Again turn right through 90° and draw a segment three units long (see Fig. 1(c)). The pattern (1, 2, 3) has been established - repeat the same steps again continuing from the outer end of the three-unit segment. The result is Fig. 1(d).

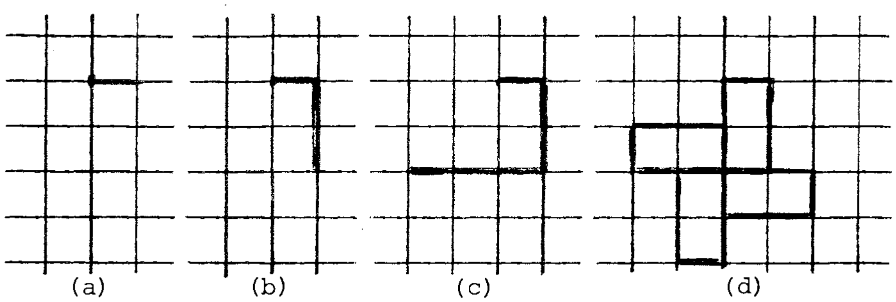


Fig. 1. Construction of a (1, 2, 3) Spirolateral

The above is but a taste in introducing an area for some neat exploration. The following references provide more thorough and detailed information.

Gardner, Martin. Mathematical games. Scientific American, 1973, 229(5), 116.

Odds, Frank C. Spirolaterals. Mathematics Teacher, February 1973, 66, 121-124.

Schwandt, Alice K. Spirolaterals: Advanced investigations from an elementary standpoint. Mathematics Teacher, March 1979, 72, 166-169.

Charles Verhille
Faculty of Education
University of New Brunswick

Appendix G (Working Group D)

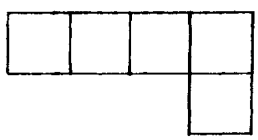
Visualizing a Box

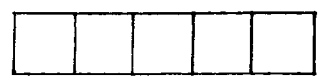
Close your eyes.
Visualize a box. Keep your eyes closed.
How many sides does your box have?
How big is the box you visualized?
Can you think of a bigger (or smaller) box?
Can you think of a longer (or shorter) box?
Imagine a box all of whose sides are squares.
Take the top off your box (are your eyes still closed?) so that you have a box without a top.
Visualize filling your box with sand. Pour the sand out again.
How many sides does your box have now?
Now imagine that a box manufacturer wants to ship his boxes flattened out. Can you imagine how such a box looks flattened out?

OPEN YOUR EYES!

Draw how a box with five square sides looks flattened out.
What did you draw?

Do the patterns shown below fold into a box without a top?





Can you think of other patterns made of five squares, regardless of whether they fold into boxes or not? Draw as many as you can find.

Which of the (twelve) patterns fold into boxes without tops? Check by cutting out all the patterns and folding the paper.
Can you predict which squares form the bottom? Mark it with an X.

Now draw all of the "box-makers" and label them.

Choose one of the patterns and write its number on the bottom of a cut milk carton. Try to cut the milk carton to obtain this pattern.

For the above and much more see:

Walter, M.I. Boxes, Squares and Other Things. Reston, VA: National Council of Teachers of Mathematics, 1970.

(Also see the bibliography in this reference.)

Charles Verhille
Faculty of Education
University of New Brunswick

MATHEMATIQUE ET ART

Juin 1979

L'arithmétique modulaire comme génératrice

de patterns décoratifs au primaire

par

Jacques C. Bergeron

Section primaire
Faculté des Sciences de l'Education

Université de Montréal

MATHÉMATIQUE ET ART

L'arithmétique modulaire comme génératrice de patterns décoratifs à l'intention des élèves du primaire.

DOCUMENT N° A103 - juin 1979.

Jacques C. Bergeron.

INTRODUCTION

A la session d'hiver 1979 deux étudiantes de troisième année, se destinant à l'enseignement des Arts plastiques se sont inscrites au cours *Résolution de problèmes* offert aux futurs-maîtres du secteur primaire. Nous avons essayé de trouver un projet de recherche qui leur permette de comprendre à la fois la démarche du mathématicien, les concepts mathématiques du programme du primaire, et en même temps de développer des attitudes et certaines habiletés pouvant servir dans leur champ de spécialisation.

Nous présentons les principales étapes de la démarche qu'elles ont poursuivie pendant environ quatre semaines.

Nous espérons que cet exemple illustrera d'une part combien, du côté mathématique peut être fructueuse pour l'étudiant l'exploitation de son propre champ d'intérêt, et d'autre part, comment cette méthodologie des projets peut s'insérer naturellement dans le cadre d'un cours régulier de la formation des maîtres.

Les dessins, et le projet *l'éclipse de soleil* sont entièrement l'oeuvre de ces deux étudiantes. Notre rôle en a été un de guide, de susciteur d'intérêt, et de rédacteur du compte-rendu qui suit.

Nos plus sincères félicitations à *Lucie Grégoire* et *Danièle Pomerleau*.

1 NATURE DU PROBLÈME.

Dans la livraison de mars 1972 de la revue *The Mathematics Teacher* sous le titre de *Residue Designs*, Phil Locke décrit une activité qu'il a proposée à ses élèves du niveau secondaire et qui consiste à traduire en dessins une table de multiplication modulo N.

Nous avons voulu reprendre cette recherche dans le but de produire d'une part des dessins intéressants, jolis à regarder, et de découvrir d'autre part un moyen de programmer en quelque sorte des recettes ou des formules qui génèrent de belles figures, de beaux *designs*.

Voici brièvement l'idée exploitée:

La figure 1 décrit la procédure utilisée pour générer à partir de tables de multiplication modulo N les *patterns* à l'intérieur de cercles.

Le produit modulo N peut être défini comme sa distance à la valeur inférieure la plus près d'un de ses multiples. Ainsi, dans la table de la figure 1 (a), le produit de 8 et 8 donne 1 puisqu'en considérant la suite des multiples de 21, soit (0,21, 42, 63, 84, 105, 126,...), 8 x 8 se situe à une unité au-dessus de 63 et l'on écrit dans ce cas 8 x 8 = 1 (mod 21). De la même manière, comme 8 x 14 se situe sept unités au-dessus de 105, on obtient que 8 x 14 = 7 (mod 21). Ce produit s'obtient rapidement comme étaïnt aussi le reste de la division par 21. En effet, 8 x 8 = 64 et 64:21 = 3 (reste 1), ce qui donne 8 x 8 = 1 (mod 21).

On pourrait encore considérer que 8 x 8 représente une ficelle de longueur 64 que l'on désire enrouler autour du cercle présenté à la figure 1 (c) en commençant à 0 et en procédant dans le sens contraire des aiguilles d'une montre. Où sera l'extrémité de la dite ficelle? Elle se situera en 1 puisque l'on pourra accomplir trois tours complets et il restera une longueur 1. Après la construction de la table viennent les étapes suivantes:

- Deux colonnes sont choisies, lesquelles déterminent N-1 couples, soit 14 dans le cas présent (voir la figure 1 (a) et (b)).
- Un cercle est dessiné puis divisé en N parties égales (voir figure 1 (c)).

Note: Si N est premier il n'y a pas de zéros dans la table et l'on peut diviser le cercle en N-1 parties égales si l'on veut.

- Des cordes correspondant aux couples identifiés sont tracés (figure 1 (b) et (d)). Par exemple, au couple (10,17) correspond la corde joignant les points 10 et 17.
- La règle de coloriage consistant à remplir les triangles formés par les cordes est appliquée (figure 1 (e)).
- La figure 1(f) illustre la possibilité d'obtenir des dessins si complexes qu'il devient difficile d'appliquer une règle simple de coloriage. Il s'agit ici d'un dessin obtenu en utilisant les colonnes 1 et 2 de la table de multiplication modulo 65, notée (x65,1,2).

2 **DÉMARCHE EXPLORATOIRE.** A partir de l'idée exposée ci-dessus nous avons identifié les variables jouant un rôle dans ce processus constructif et puis nous les avons fait varier tour à tour pour voir ce qu'elles contribuent au motif final. C'est ainsi qu'ont été explorées les variables colonne, opération arithmétique, module, longueur des arcs, nombre de superpositions, figure-matrice, combinaisons linéaires de colonnes, et règle de coloriage. Enfin, ces nouvelles connaissances acquises ont été mises à l'épreuve dans l'élaboration d'un projet, soit une éclipse de soleil.

2.1 EXEMPLES DE VARIATIONS.

2.1.1 LA VARIABLE OPÉRATION ARITHMÉTIQUE. Comme le montrent les figures 2 et 3, les opérations d'addition et de multiplication ne donnent pas les mêmes dessins.

2.1.2 LA VARIABLE COLONNE. On peut voir dans les figures 2 et 3 que, dépendant du choix des colonnes, des dessins différents sont obtenus. Les colonnes 1 et 4 , ainsi que les colonnes 1 et 3 donnent les dessins 2(a) et 2(b) respectivement pour l'addition. Les colonnes 1,4 et 1,5 donnent les dessins 3(a) et 3(b) pour la multiplication.

2.1.3 VARIATION DU MODULE. La variation du module produit des variations intéressantes dans les dessins. On peut voir par exemple un modulo 65 pour la multiplication à la figure 1 (f), des modulos 30 et 12 pour la multiplication et l'addition à la figure 4, un modulo 29 pour la multiplication à la figure 5, un modulo 24 pour l'addition à la figure 6, et des modulos 15 et 20 pour l'addition à la figure 7.

2.1.4 VARIATION DU NOMBRE DE SUPERPOSITIONS. La figure 4 montre le résultat de la superposition de deux dessins pour l'addition et pour la multiplication. Les superpositions sont indiquées par un S réunissant les expressions correspondantes. En 4(a) par exemple, on a superposé le dessin correspondant aux colonnes 0 et 5 dans l'addition modulo 12, (+12,0,5) à celui des colonnes 0 et 3 (+12,0,3) et que nous écrivons (+12,0,3)S(+12,0,5).
Note: Pour l'addition nous avons ajouté à la table, la colonne 0.

2.1.5 VARIATION DE LA LONGUEUR DES ARCS. La division du cercle en arcs inégaux selon sa propre fantaisie ou suivant une progression arithmétique ou géométrique produit des effets intéressants. La figure 5(a) illustre une division inégale et symétrique du cercle. On peut aussi combiner superposition et arcs inégaux dans une même figure (voir 5(b)).

2.1.6 VARIATION DE LA FORME DE BASE. D'autres figures que le cercle peuvent être considérées. Dans les figures 6(a) et 6(b) le triangle et le carré ont été utilisés en combinaison avec une superposition.

2.1.7 COMBINAISONS LINÉAIRES DES COLONNES. Soit (XN,i,j) un dessin correspondant aux colonnes i et j pour la multiplication modulo N. On peut modifier le dessin en ajoutant terme à terme une troisième colonne K à i ou à j, ou en additionnant ou en multipliant une colonne par une constante c, ce qui donne les possibilités suivantes: (XN,i+k,j) ou (Xn,i+c,j) ou (XN,c(i+k),j). La figure 7(a) illustre la rotation produite par l'addition de la constante 4 à la colonne 7 dans le cas de la multiplication modulo 24 et des colonnes 7 et 14. Nous symbolisons cette opération par (X24,7,14)T(i+4). On remarque que l'angle de rotation τ est deux fois plus grand que le rapport de la constante 4 au modulo 24, soit $\tau = 2 \times \frac{4}{24}$, soit $\frac{1}{3}$.

2.1.8 RÈGLES DE COLORIAGE. La façon de colorier les configurations obtenues peut varier à l'infini, allant de règles mathématiques strictes à de l'expression libre. Par exemple, en 1(e) les triangles sont coloriés; en 2, 3, 4, 5(a) des règles bien définies sont suivies; tandis qu'en 5(b), 6, 7 et 8, l'inspiration de l'artiste est débridée. On voit comment l'utilisation de contrastes (remplir ou laisser vide), de la symétrie, du rythme, et de l'équilibre est riche de possibilités.

2.2 UN PROJET: UNE ÉCLIPSE DE SOLEIL. Essai de rendition du phénomène de l'éclipse de soleil au moyen des dessins modulaires. On peut imaginer la lune qui avance lentement pour cacher le soleil et puis qui s'éloigne (figure 8). Ce projet a nécessité l'étude des effets produits par les diverses variations mentionnées ci-dessus. La figure 8 ne donne qu'un faible *reflet* du travail produit en couleurs.

2.3 OBSERVATIONS D'ORDRE PLUS THÉORIQUE.

2.3.1 DESSINS IDENTIQUES

a) Certaines combinaisons de colonnes donnent naissance à des dessins identiques. Par exemple, dans la table de multiplication modulo 15 (figure 1(a)) les colonnes 1 et 8, ainsi que les colonnes 7 et 14 conduisent au même résultat soit

(X15,1,8)=(X15,7,14). Nous exprimons cette propriété par l'expression

(XN,i,j) = (XN,N-j,N-i)

b) De plus, (XN,1,i) = (XN,1,j) si i x j = 1 (mod N) c'est à dire que si deux nombres comme 7 et 13 ont comme produit 1 dans la multiplication modulo 15 (voir figure 1(a)), alors les deux dessins (x15,1,7) et (X15,1,13) sont identiques.

c) Les dessins correspondant aux colonnes i+j=N s'avèrent moins intéressants puisqu'ils consistent en un ensemble de cordes parallèles. (XN,i,j) où i+j=N donne des cordes parallèles.

2.3.2 NOMBRE DE DESSINS DIFFÉRENTS POUR UN N CHOISI.

a) Afin d'aider les étudiantes dans leur processus de recherche de patterns intéressants nous avons écrit un programme d'ordinateur permettant de visualiser sur écran cathodique toutes les configurations possibles et d'en faire imprimer les modèles désirés. S'est alors posé le problème de savoir combien de modèles différents existent. Par exemple, la table de multiplication modulo 21 nous avait fourni 210 modèles dont plusieurs se répétaient. (1,2--1,3--1,4--...1,20--2,3--2,4--...2,20--....19,20) soit 20+19+...+2+1=210.

Une analyse du problème nous a montré que si N est impair, le nombre de dessins différents (D) est obtenu par la formule $D = (\frac{N-1}{2})^2$ ce qui, dans le cas de N=21 réduit de 210 à 100 le nombre de figures à examiner. On peut encore réduire ce nombre de 4 si l'on utilise la propriété 2.1(b) décrite plus haut qui permet d'éliminer les cas 1-11, 1-16, 1-17 et 1-19 qui sont identiques à 1-2, 1-4, 1-5 et 1-10 puisque 2x11 = 4x16 = 5x17 = 10x19 = 1 (mod 21) . Ceci résulte finalement en 96 cas différents pour N-21.

b) Dans le cas où N est pair, on trouve que $D = \frac{1}{4}N(n-2)$ moins les cas éliminés en vertu du cas 2.1(b).

Note: Si N est premier ce nombre à soustraire de D semble être $\frac{(N-3)}{2}$ (à vérifier).

2.3.3 DESSINS SEMBLABLES. Il y a des familles de dessins exhibant un lien de parenté entre elles.

a) Par exemple, nous avons mentionné la famille des dessins formés de cordes parallèles obtenus quel que soit N pourvu que i+j = N.

b) Il y a la famille des cardioides données par les formules (XN,1,2) ou encore (XN,i,2i) comme dans la figure 7.

c) Enfin, il y a les épicycloides données par (XN,1,i) où i est petit par rapport à N. Dans cette famille, le nombre de coupes ("cusps") égale i-1. La figure 1(f), de formule (X65,1,2) donne 1 coupe.

Nous n'avons qu'effleuré le sujet, beaucoup d'autres explorations sont permises.

3 PROLONGEMENTS ENVISAGÉS.

3.1 VARIANTES MATHÉMATIQUES.

- utiliser d'autres lois que l'addition et la soustraction: moyenne arithmétique, moyenne géométrique, moyenne harmonique, fonctions circulaires, lois de physique telles que celles du pendule, du plan incliné, de la gravitation, etc...

- subdiviser la frontière de la figure suivant une progression arithmétique, géométrique, Fibonaci,etc
- joindre les points par des lignes non droites: paraboles, ellipses, cercles, sinusoides, cycloides, etc
- utiliser d'autres figures de base que des cercles: carrés, triangles, parallélogrammes, cardioides, cycloides, etc,...
- faire exécuter le traçage et le coloriage des figures par ordinateur: programmes TEKTRONIX, PLOT10, VRSATEC, ou PLATON

3.2 APPLICATIONS

- courte-pointes, cerf-volants, murales,vitraux, assemblages, jeux éducatifs

4 CONCEPTS ET HABILETÉS IMPLIQUÉS.

4.1 CONCEPTS ET HABILETÉS MATHÉMATIQUES

- l'arithmétique modulaire
- l'addition et la multiplication
- topologie:intérieur, extérieur, frontière
- figures géométriques:droite,cercle,carré,triangles,cordes,..
- traçage de figures géométriques
- subdivision de frontières en N parties égales
- symétries, transformations
- programmation, combinatoire

4.2 CONCEPTS ET HABILETÉS ARTISTIQUES

- forme,couleur,teinte,ton,pleins,vides,rythme

5 CONCLUSIONS. Nous avons voulu rapporter en détails le cheminement de deux parmi trente élèves inscrites au cours *Résolution de problèmes* pour montrer qu'il est possible dans une classe assez nombreuse de laisser des étudiants poursuivre une recherche qui les intéresse sans pour autant sacrifier la qualité de l'apprentissage.

Cette formule,croyons-nous, possède de plus l'avantage de pouvoir faire naître chez le futur-maître, ou du moins de rendre plus positive son attitude vis-à-vis la mathématique.

De telles activités lui permettent, non seulement de redécouvrir des propriétés, mais à l'occasion d'explorer des terres vierges. On se rendra compte qu'il n'y a pas de limite à l'imagination et que tout lui est permis.

Pour l'artiste, la mathématique peut s'avérer, sinon un outil de création, du moins un guide pour l'exploration et l'expérimentation méthodique d'un domaine nouveau. De même pour le professeur de mathématique, l'esprit artistique, caractérisé par la gratuité d'une démarche, par la stimulation de l'imagination, de l'intuition, du sens de l'esthétique et de l'harmonie, peut le pousser à entamer certaines activités pour le simple plaisir de voir ce que ça donne, de trouver différentes façons de parvenir à un but, d'apprécier la beauté d'une démarche, d'une structure.

Finalement, à la suite de quelques années de travail dans ce sens, nous sommes parvenu à nous convaincre qu'il n'est pas du tout nécessaire que tous les futurs-maîtres (et par ricochet les enfants) aient étudié exactement le même programme, aient fait les mêmes exercices, aient suivi le même rythme. Ce qui compte c'est de vivre au moins une fois dans sa vie une expérience complète de recherche qui donne du plaisir et qui aiguise son insight sur la vraie nature des concepts mathématiques, des processus de pensée, des lois de l'apprentissage et de l'enseignement.

APPENDICE A

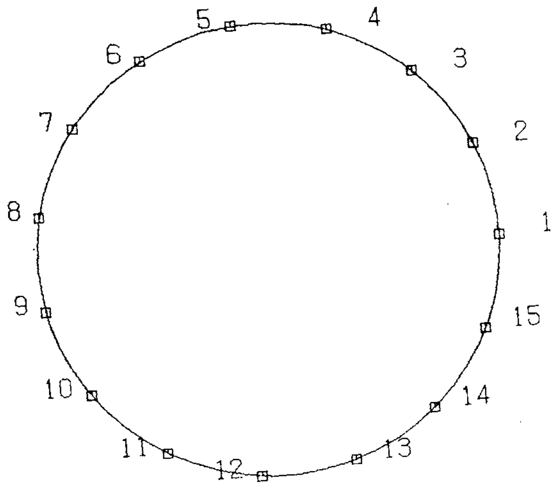
DESSINS MODULAIRES.

A.1 **FIGURE 1. ETAPES DE LA GÉNÉRATION DE PATTERNS.** (a)table de multiplication modulo 15; (b)couples correspondants; (c)subdivision du cercle en 15 arcs égaux; (d)cordes données par les couples; (e)coloriage des triangles formés par les cordes; (f) dessin complexe:colonnes 1 et 2, multiplication modulo 65.

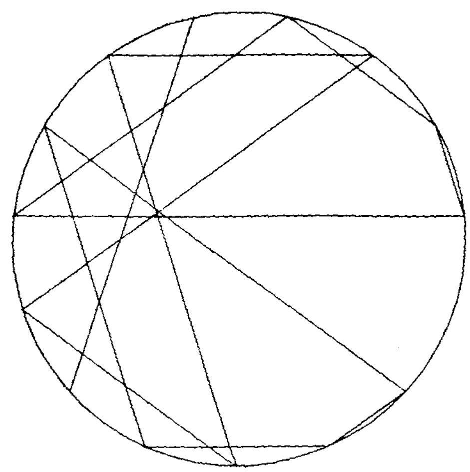
1	2	3	4	5	6	7	8	9	10	11	12	13	14
2	4	6	8	10	12	14	.1	3	5	7	9	11	13
3	6	9	12	0	3	6	9	12	0	3	6	9	12
4	8	12	1	5	9	13	2	6	10	14	3	7	11
5	10	0	5	10	0	5	10	0	5	10	0	5	10
6	12	3	9	0	6	12	3	9	0	6	12	3	9
7	14	6	13	5	12	4	11	3	10	2	9	1	8
8	1	9	2	10	3	11	4	12	5	13	6	14	7
9	3	12	6	0	9	3	12	6	0	9	3	12	6
10	5	0	10	5	0	10	5	0	10	5	0	10	5
11	7	3	14	10	6	2	13	9	5	1	12	8	4
12	9	6	3	0	12	9	6	3	0	12	9	6	3
13	11	9	7	5	3	1	14	12	10	8	6	4	2
14	13	12	11	10	9	8	7	6	5	4	3	2	1

(1,8)
(2,1)
(3,9)
(4,2)
"
"
"
"
"
"
"
"
"
(14,7)

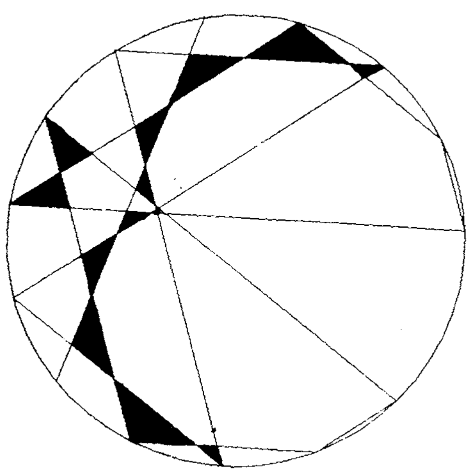
(a) **(b)**



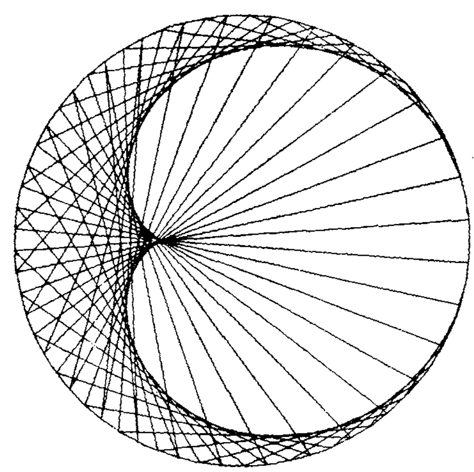
(c)



(d)

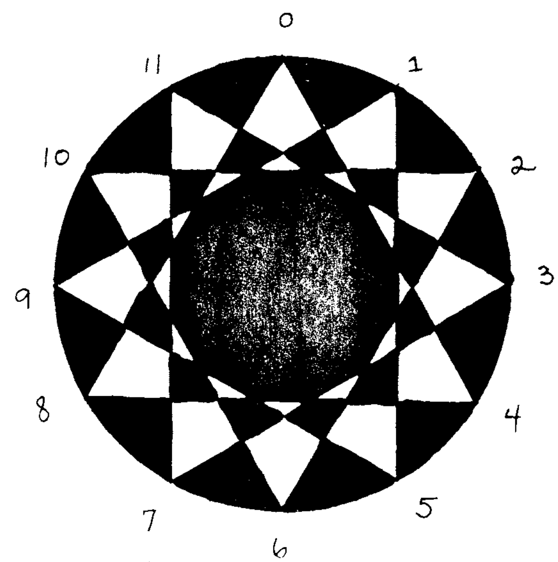


(e)

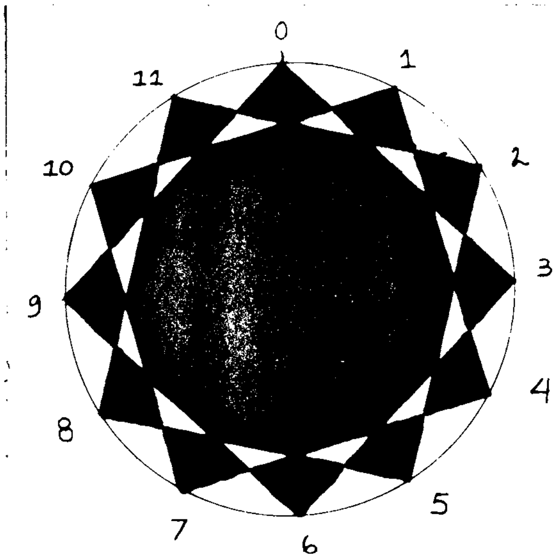


(f)

A.2 **FIGURE 2. VARIATION DES COLONNES.** Addition modulo 12. (a) colonnes 1-4; (b) colonnes 1-3.

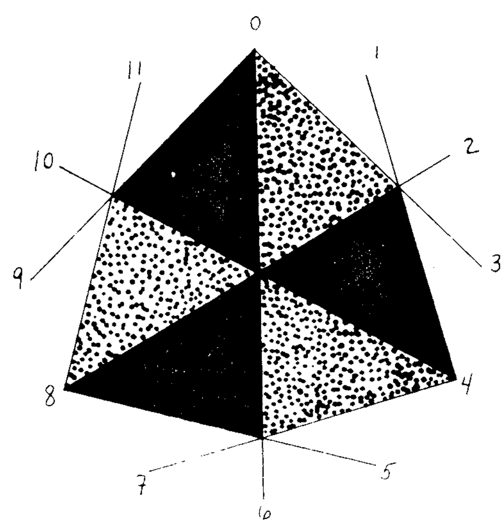


(a) (+12,1,4)

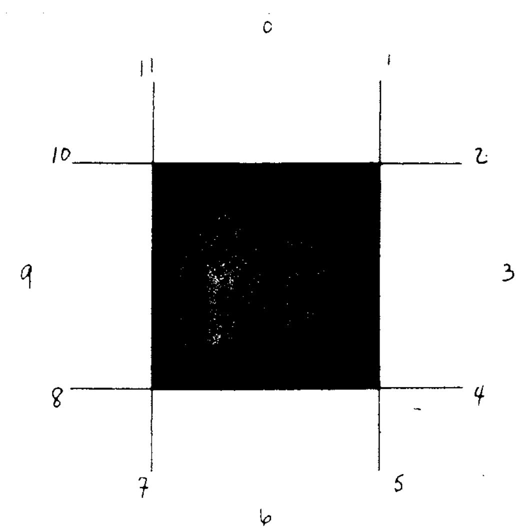


(b) (+12,1,3)

A.3 **FIGURE 3. VARIATION DES COLONNES.** Multiplication modulo 12. (a)col. 1-4; (b) col. 1-5.

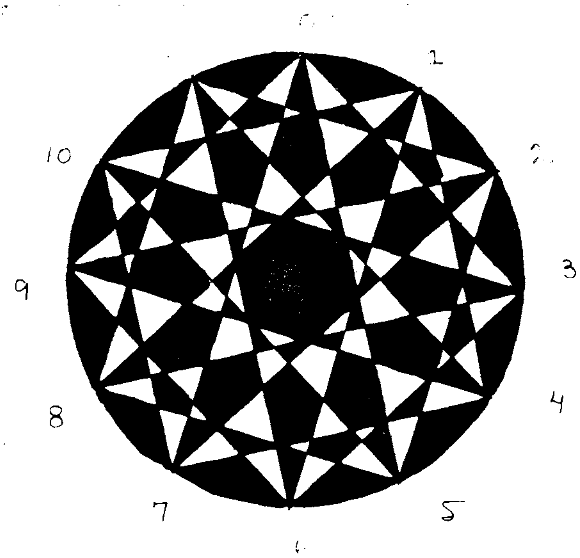


(a) (×12,1,4)

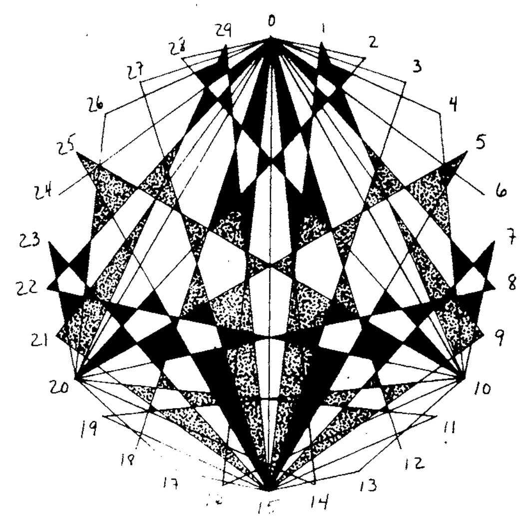


(b) (×12,1,5)

A.4 **FIGURE 4. SUPERPOSITION DE DESSINS.** (a) superposition des colonnes 0-3 et 0-5; (b)superposition des colonnes 7-10 et 1-15.

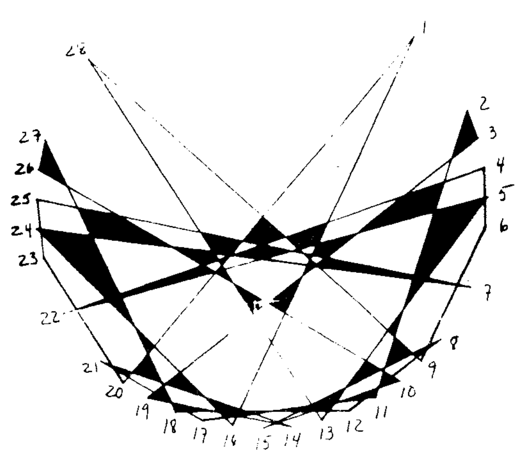


(a) (+12,0,3) S (+12,0,5)

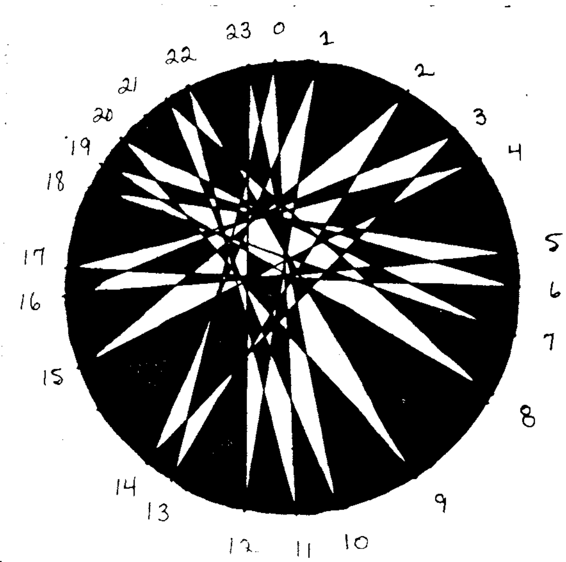


(b) (x30,7,10) S (x30,1,15)

A.5 **FIGURE 5. ARCS INÉGAUX.** (a)arcs décroissants de 0 à 15 et croissants de 15 à 29; (b) arcs inégaux combinés à une superposition.

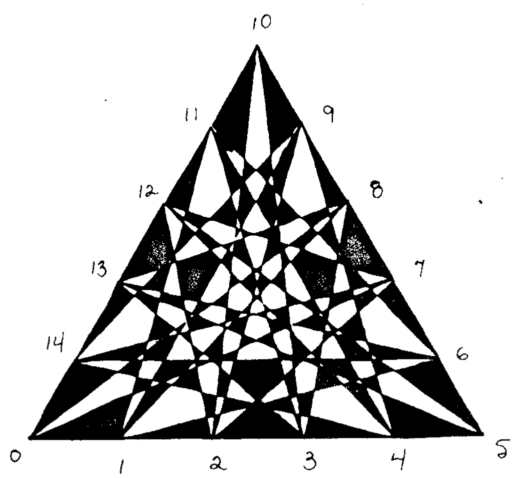


(a) (x29,7,25)

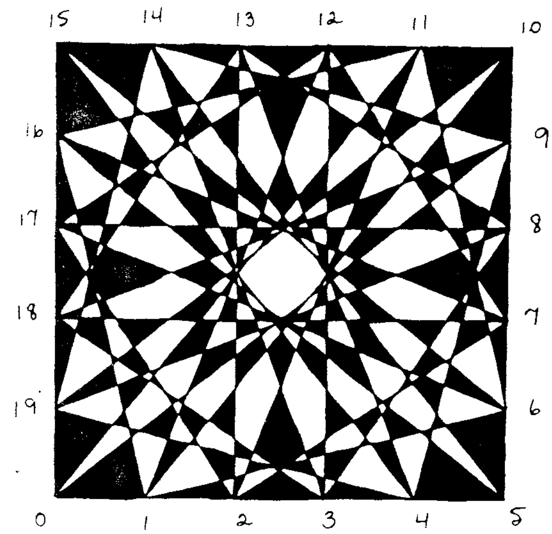


(b) (+24,0,11) S (+24,0,13)

A.6 **FIGURE 6. VARIATION DE LA FIGURE DE BASE ET SUPERPOSITION** A partir d'un triangle en (a) (addition modulo 15), et à partir d'un carré en (b) (addition modulo 20).

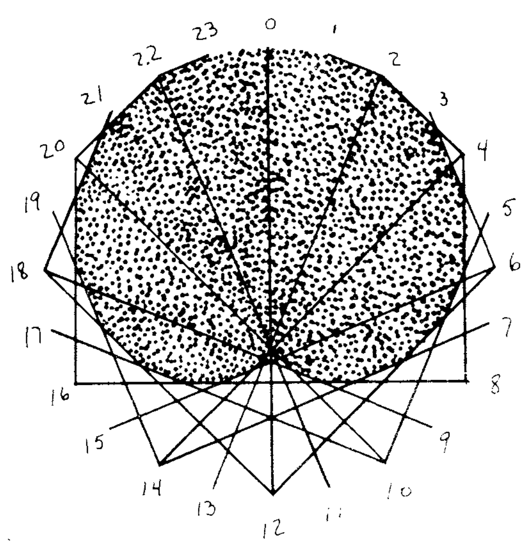


(a) (+15,0,5) S (+15,0,8)

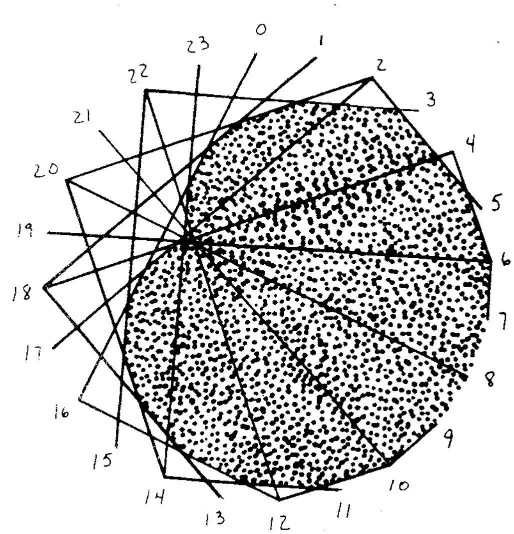


(b) (+20,0,5) S (+20,0,9)

A.7 **FIGURE 7. ADDITION D'UNE CONSTANTE À UNE COLONNE** En ajoutant 4 à chaque élément de la première colonne on produit une rotation du dessin de deux fois $\frac{4}{24}$, soit $\frac{1}{3}$ de tour autour du centre du cercle, et dans le sens de rotation des aiguilles d'une montre.



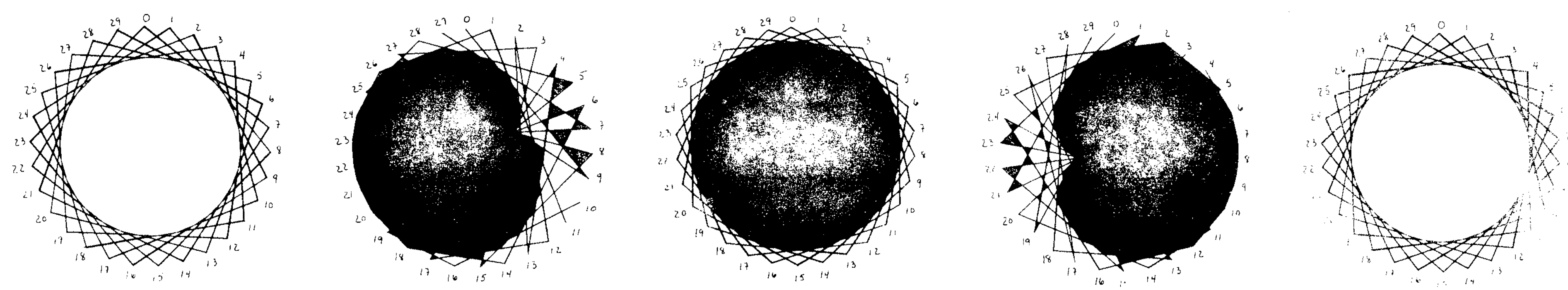
(a) (x24,7,14)



(b) (x24,7,14) (i+4)

12

A.8 **FIGURE 8. UNE ÉCLIPSE DE SOLEIL** (a)avant; (b)partielle; (c) totale; (d)partielle;(e)après.



(a) (+30,7,14)   (b) (x30,7,14)(i+11)   (c) (+30,7,14)   (d) (x30,7,14) (i+4)   (e) (+30,7,14)

Appendix I (Working Group D)

List of Participants, Group D

Allen Hugh	Faculty of Education Queen's University
Bergeron, Jacques	Faculté des Sciences de l'Education Université de Montréal
Boswall, Alberta	Department of Mathematics Concordia University
Cassidy, Charles	Département de Mathématiques Université Laval
Cristall, Eleanor	Department of Mathematics Brandon University
De Flandre, Charles	Département de Mathématiques Université du Québec à Montréal
Hodgson, Bernard	Département de Mathématiques Université Laval
Jeffery, Gordon	Nova Scotia Teachers College Truro, Nova Scotia
Kleiner, Israel	Department of Mathematics York University
Ripley, Ron	Faculty of Education Queen's University
Roy, Ghislain	Département de Mathématiques Université Laval
Trivett, John	Faculty of Education Simon Fraser University
Verhille, Charles	Faculty of Education University of New Brunswick

RESEARCH ON PROBLEM SOLVING

In the presentation, my aims were as follows:

a) to describe briefly the main thrusts in research on problem solving,

b) to provide a short bibliography of survey articles on research,

c) to initiate discussion on some questions which would serve as a basis for a future Working Group.

I have summarized the research under several general categories which clearly have a wide overlap.

I. Research related to the identification and clarification of variables which effect problem solving.

The complexity of the problem solving situation is manifested by the almost forbidding list of variables. The main components are the Task, the Solver and the Situation. For example, Task variables include Syntax, Content, Context, Structure and Heuristic Behavior variables, each of which can be further broken down. (Goldin and McClintock provide a complete survey of the variables and related research.)

II. Research related to instruction in heuristics and the observation of heuristic behavior.

Some of the research is related to methodological aspects such as techniques of observation, classification and codification of the different heuristics. The main thrust of the research is in the teaching of heuristics and assessing the effectiveness of the teaching. The research varies widely over the range of situations (in class, out of class, group, individual), length of instruction period (from several weeks to a year's course) and the kind of problems used. Most research indicates that heuristic instruction improves problem solving ability somewhat, though certainly not in a dramatic way. The teaching of heuristics is most effective if a small set of heuristics is taught in conjunction with a related class of problems, and least effective when general heuristics are taught and their use is tested over a large range of tasks.

Possible questions for discussion: Are we going about heuristic teaching the wrong way? Are there alternatives to Polya's model? Some observations are in order, namely:

a) Heuristics are deceptively simple (as sentences in the language) yet are really quite sophisticated instructions. Landa has already made the point that even something as simple as 'Look at the given' may be meaningless to an uninitiated problem-solver (the 'given' may not be explicit), though I doubt that Landa's approach of breaking a heuristic into a sequence of semi-algorithmized steps can be carried far.

b) There are too many potentially useful heuristics and choosing an appropriate one becomes a new task. Schoenfeld suggests a 'managerial strategy', a kind of second-order heuristic to negotiate among the list of heuristics. This makes the whole heuristic enterprise look formidable.

c) Little effort seems to have been made in integrating heuristics teaching slowly and throughout the curriculum, starting with the simplest heuristics as applied to easily solved problems.

d) Little attention has been paid to the willingness of a solver to make an assigned problem his own problem (the affective aspect).

III. General theories of problem solving.

As in cognitive psychology in general, I.P. (Information Processing) theories of problem solving have dominated. Whether espousing their point of view or not, most researchers have adopted I.P. language (Processing Unit, Memory Unit, Executive, Retrieval, Search, Storage, etc.).

Claims by I.P. theorists of having simulated human problem solving behavior seem to be greatly exaggerated. The class of problems that are considered (i.e. Tower of Hanoi, Cryptarithmetic, Missionary and Cannibals, etc.) are very narrow, goal-oriented and locally finite (i.e. at each point, there are only finitely many decisions to make, usually two), problems which are tailor-made to I.P. kind of analysis of problem solving.

Possible questions for discussion: What are viable alternatives to I.P. theory, keeping in mind that:

a) the problem solving act seems to be very unstable (small changes can produce large effects) and highly idiosyncratic,

b) the range of tasks one would like to consider is quite wide and should definitely include open-ended problems.

IV. Research Methodology

Most researchers have accepted the idea that one must observe the whole problem solving episode rather than simply look at the actual attempted solution. The 'thinking aloud' technique has become fairly standard, with or without the aid of video). The tapes are transcribed and the transcription is followed by a detailed analysis of the written protocols. The amount of data collected is extremely rich and, indeed, creates for the researcher a problem of focusing as a multitude of interpretations is possible.

Possible questions for discussion: It would seem profitable to have a discussion of experiences in problem solving research so as to try to tackle certain methodological problems related to interviewing techniques, for example:

a) How does one interpret the vast amount of data?

b) How does one handle the fact that the most important processes in problem solving seem to take place exactly when the solver is silent?

c) How can one verify whether the verbal accounts by the solver are simply post-rationalization?

A brief list of review articles on problem-solving research and theories.

B. Kleinmuntz (ed.) Problem Solving: Research, Method, and Theory (Wiley, 1966) (in particular the articles by B. Green; Current Trends in Problem Solving and by G. Forehand; Epilogue: Constructs and Strategies for Problem Solving Research).

J. Kilpatrick Problem Solving and Creative Behavior in Mathematics, Review of Educational Research, 1969.

E. Simon, A. Newell Human Problem Solving: The State of the Theory in 1970, American Psychologist 1971, 26.

R.E. Mayer Thinking and Problem Solving (Scott, Foresman, 1977)

J. Kilpatrick Variables and Methodologies in Research on Problem Solving in L. Hatfield (ed.) Mathematical Problem Solving, ERIC/SMEAC 1978.

G. Goldin
E. McClintock (eds.) The Classification of Problem Solving Research Variables, ERIC/SMEAC (in press).

Joel Hillel
Department of Mathematics
Concordia University

PROVINCIAL OBJECTIVES FOR MATHEMATICS EDUCATION

In the past two years, a number of provincial and nation-wide surveys of current trends and practices in mathematics education have been conducted. These include a Canada-wide survey sponsored by the Council of Ministers of Education, as well as smaller scale surveys conducted by the authors of this report.

The purpose of the Review Group was to examine and discuss the findings of these various surveys and to attempt to identify common findings. The McNicol survey indicated that three major areas of concern to mathematics educators, Ministry of Education personnel, and teachers were in-service education, assessment procedures, and lack of communication both between and within provinces. Results of the Robitaille survey showed that all three groups mentioned above felt that the major trends for the immediate future of mathematics K-12 in Canada concerned the impact of hand-held calculators and computers, consumer mathematics, declining enrollment, and decreased levels of financial support. Topics such as the impact of metrication on the mathematics curriculum and sex-role stereotyping were rarely mentioned. In all of the provinces from which responses were obtained, the "back-to-basics" movement is quite apparent, particularly among Ministry personnel. Respondents from all provinces indicated the existence of some form of province-wide assessment program, although the nature of such programs varied considerably. The Council of Ministers report, which was available only in draft form at the time of the CMESG meeting, indicates that every province has either adopted a new mathematics curriculum (K-6) since 1977-78 or is in the process of revising its curriculum. There would appear to be a good deal of similarity among the provinces insofar as the elementary school mathematics curriculum is concerned, and there exists "little evidence to support the popular belief that standards are falling." At the secondary school level there is a much greater degree of diversity in courses and curriculum among the provinces than is the case at the elementary level. There is a nation-wide trend to de-emphasize mathematical rigour and proof, and, on the other hand, a countervailing trend to give increased emphasis to consumer skills. In certain provinces, there is a continuing attempt to integrate the various branches of mathematics.

David F. Robitaille, University of British Columbia.

Shirley McNicol, McGill University

APPLICATIONS OF MATHEMATICS FOR HIGH SCHOOLS

Members present attempted to isolate important aspects of "Applications".+ The application should

(1) help us to understand the world we live in. It should be real to someone even though a particular student may have no interest in it;

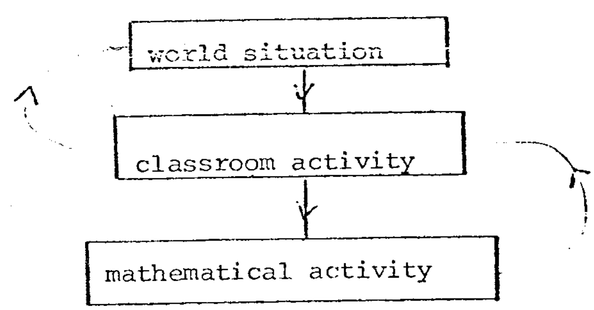
(2) not be confused with an example of a particular mathematical technique, i.e. a highly contrived situation placed in the form of a problem at the end of a chapter;

(3) not be taught as part of the curriculum but it should be employed to underline the process;

(4) whenever possible demonstrate that realistic situations are - ill-behaved - poorly formulated - messy - require numerous assumptions before they can be modelled and often have no solution or a non-unique solution.

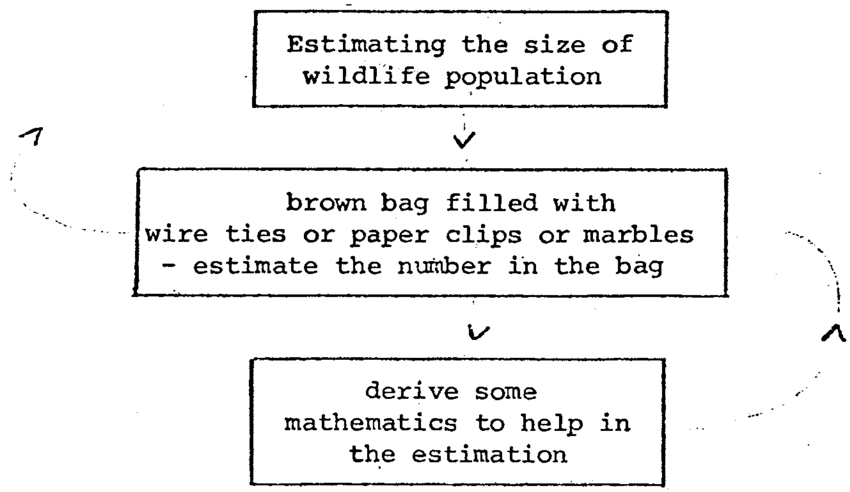
There was some concern expressed that the many so called "applications" appearing in textbooks were just word problems on a particular topic, they did not tie together various mathematical concepts, they did not involve the student, and the solutions did not reflect the numerical difficulties encountered in practice.

There was a general interest in a workshop at next year's meeting aimed at generating materials of the following type:



+ In general the group appeared to prefer the term mathematical modelling, but spent no time on the definition of these terms.

An example of this could be



Reference: "Statistics by Example - Exploring Data"
Mosteller et al. Addison Wesley 1973

(cf. article 12, page 99, by Samprit Chatterjee).

Eric R. Muller, Department of
Mathematics,
Brock University.

INSERVICE COURSES FOR ELEMENTARY TEACHERS

The session began with a review by D. Alexander of the results of a survey of teachers' views of inservice conducted by the Commission on the Education of Teachers of Mathematics of the National Council of Teachers of Mathematics. (Reference: An Inservice Handbook for Mathematics Education, N.C.T.M.).

Major complaints of elementary teachers (p.p. 29-30):

It does not fit my needs in the classroom.
The program was too theoretical.
I did not help select topics.
Materials used in the inservice courses were too expensive for practical classroom use.

Topic Selection

Elementary teachers who helped identify inservice topics were significantly more likely to have realized satisfaction --- and have a positive in-service experience --- than those who had not.

Timing

Only 36% of respondents agreed that "In-service education programs are short and to the point" while 93% felt they ought to be. (p. 39)

Follow-up

Only 18% felt "Systematic follow-up in the classroom is provided after an in-service program" but 83% felt that there ought to be. Moreover "Follow-up activities appear to build a significantly greater feeling of satisfaction with in-service education --- and to provide for a positive experience". (p. 41)

Usefulness

30% responded that "I can use in my teaching most of the mathematics I learn in in-service programs" while 92% felt that they ought to be able to do so.

28% responded that "I have been able to use in my teaching most of the methods demonstrated in in-service education" but 90% felt that they ought to be able to do so. (p. 41)

Goals of an Inservice Program

Among the goals identified by the Commission are (p.p. 6-8):

1. To provide teachers the opportunity, the time, the means, and the materials for improving their professional competencies.

2. To assist teachers in applying to themselves new insights into the learning process.

3. To help teachers expand their perceptions of mathematics.

4. To assist teachers in developing creative instructional approaches (a) that are meaningful and mathematically correct and (b) that inculcate in students an enthusiasm and a satisfaction in learning and using mathematics.

5. To implement significant innovative curricular and instructional practices.

Universities and Inservice Programs

The Commission notes some problems in the attempt of universities to respond to the inservice needs of teachers.

> "An academic institution proffering graduate credit must respect academic goals, but these goals are not exactly congruent with teachers' needs in the schools."

> "--- many institutions are discovering two problems. First, in-service education concerned with methodological and curricular problems and issues often requires attention at a level not consistent with the academic goals of graduate work. Many of the important problems and issues endemic to the schools require solutions that do not have the characteristic theoretical, research-oriented goals valued (appropriately) by institutions of higher education. Second, as an institution of higher education markets an in-service program based on need, it typically must operate on a very general level in order to attract students (i.e., teachers) from a large number of schools. But needs are specific to schools and vary from one setting to another. The in-service programs of universities and colleges are often taken to task by teachers because the generality of the program fails to meet the specific needs of a particular school."

> "The shift is from mathematics to methodology, from the appeal to general needs faced by teachers in most schools and on-campus classes to programs marketed for one school (or a limited number of schools) and conducted in the school(s)." (p.p. 14-15)

R. Mura then described the Laval plan (see Appendix) which appeared to respond to many of the concerns identified by the Commission survey. In the discussion it was emphasized that the program was a costly one and was only possible because of the support of the university administration. It was also pointed out that the majority of elementary teachers in Quebec do not currently have a university degree (although their diploma course would be almost equivalent in terms of time) and that there is a salary raise associated with every 30 credits accumulated.

John Trivett reported that during an inservice conference in British Columbia the following were identified as guidelines on which government policy would be based:

1. The problems of education were not going to be solved in preservice programs.
2. Inservice must deal with the re-education of society.
3. Episodic inservice must go.
4. Cooperative planning must be part of inservice.
5. There must be a long term commitment of mutual inservice interests by all parties concerned.

D.W. Alexander, Faculty of Education,
University of Toronto.

APPENDIX (Review Group K)

The PPMM programme: an interesting innovation in inservice teacher education

PPMM ("Plan de Perfectionnement des Maîtres en Mathématique") was created in 1978 at Laval University, in Québec City. It is a school-focused inservice teacher education programme with the following characteristics:

(a) It is an inservice programme in mathematics for elementary school teachers from about 30 school boards located in the region around Laval University.

(b) Courses are developed in cooperation by a team of university professors (both math educators and mathematicians) and a few elementary school teachers (or math. coordinators) whose full-time services have been lent by their school boards.

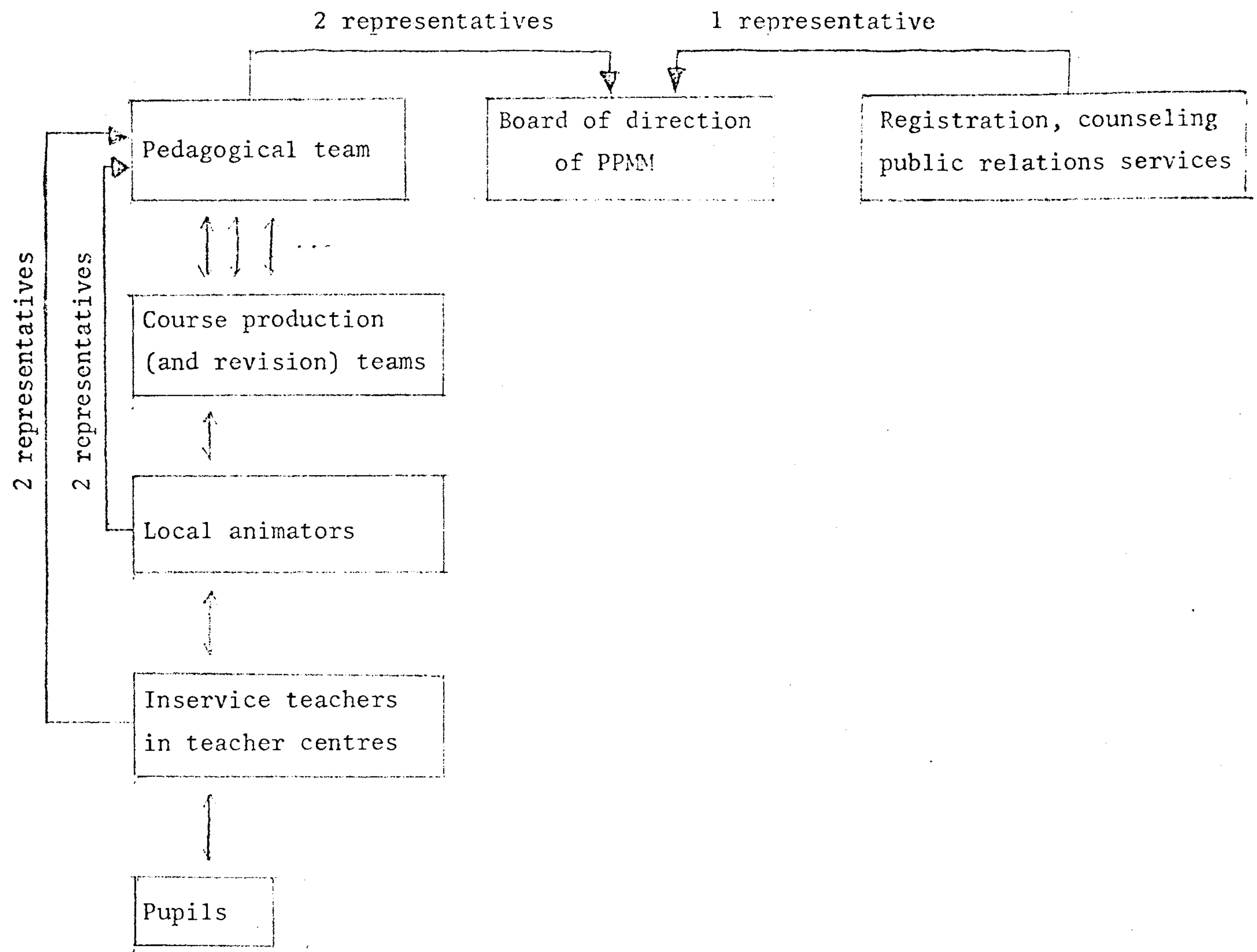
(c) Courses are school-based, i.e. offered off-campus at local teacher centres.

(d) Material for the courses is produced at the university, but the courses are under the responsibility of "local animators" (instead of university professors).

(e) Systematic efforts are made to consult school teachers and math. coordinators about subject-matter, activities and pedagogical approach most suitable for each course being produced or revised.

Remark: PPMM is the first programme of its kind in Quebec, although a few similar programmes (PPMF's) have been in existence for some time in French as mother tongue.

STRUCTURE


2 representatives
1 representative
Pedagogical team
Board of direction of PPMM
Registration, counseling public relations services
Course production (and revision) teams
Local animators
Inservice teachers in teacher centres
Pupils
2 representatives
2 representatives

1. How it all started.

A group of mathematics coordinators in the region of Quebec City asked Laval University to organize an inservice programme for elementary school teachers. The University made a survey to estimate the number of possible customers and their needs and finally decided to create six courses: two on Natural Numbers, two on Geometry and two on Rational Numbers and Measurement (*).

Two departments were entrusted with the task of carrying out this plan jointly: the Department of Mathematics (in the Faculty of Science and Engineering) and the so-called Department of "Didactics" (in the Faculty of Education).

2. How a course is produced.

For each course a production team is formed consisting of half a dozen people including maths educators (from one department), mathematicians (from the other department) and a few elementary school teachers (or maths coordinators).

The first task of this team is to assess the needs of the elementary school teachers in the region with respect to the subject-matter of the particular course. On the basis of the results of this investigation, the team prepares a course outline both with regard to content and pedagogical approach.

After this outline has been approved by the pedagogical team, the actual production work gets under way.

The product consists mostly of written material (some of it for the students and some for the local animators), but it might include material of a different kind, e.g. videotapes.

The first course was ready to be offered in the fall term of 1978. A new course is to be added at each subsequent term.

(*) More courses might be added in the future.

3. How a course is offered.

When a group of teachers (*) in a given school board wants to take a course, the school board has to submit to the University the name of one person who could assume the responsibility for the course, e.g. the mathematics coordinator or another person. If this person meets certain criteria s/he is hired by the University as a course animator and the school board frees him/her for 7-8 full days during the term.

The animators from the different school boards come to spend these days on campus, where they get prepared to play their role adequately. Such a role includes organizing various activities, stimulating work and discussions, acting as a leader and a resource-person, etc., according to the plan outlined in the written material for the course.

The registered inservice teachers take the course outside their working hours. They meet once a week for 3 hours during one term, for a total of 45 hours. The courses carry university credits and can be integrated into a programme leading to a university degree. They can also lead to reclassification of the teachers and therefore to career advantages.

4. How a course is evaluated and revised.

When a course is offered for the first time, student satisfaction is evaluated twice, in the middle and at the end of the term. Three students are also asked to make a detailed evaluation of all the material supplied, and the local animators' task includes giving feed-back on all aspects of the course.

After the course has been given once, a revision team is formed (it may or may not consist of the same people who produced the course). On the basis of the feed-back received from the students and from the local animators, this team produces a sketch of the changes deemed necessary, and, after having this sketch approved by the pedagogical team, it proceeds to make such changes. Revising a course may be almost as long and demanding as producing a new one.

R. Mura, Dep. de
didactique, F.S.E.,
Universite Laval.

(*) The group size must be between 20 and 25.

SELF-EDUCATION FOR MATHEMATICS

This report includes, by permission, extracts from a chapter, 'Educating Everyone's Mathematical Self,' by John V. Trivett from the book on Self-Education compiled by members of the Faculty of Education of Simon Fraser University. It will be published shortly. Further details may be obtained from Dr. M. Gibbons, S.F.U., B.C., Canada, V5A 1S6.

After referring to the 'sad state of affairs' in Canada in the understanding and learning of mathematics in schools[1] the chapter continues.....

Who teaches a child to talk?

Parallel to (the traditional experiences of mathematics teaching) however, every child has met his first language, learnt to speak it without direct teaching from anyone and progressed in its fluent use. If this is not apparent in his school work one needs only to listen to the speech of a five year old with that of a child four years younger. Why is it, therefore, not more readily understood that if children can and should talk about so many daily interests, they cannot or should not also talk mathematically? For mathematics, as a study of relationships and of relationships of relationships, is a language which in its codification communicates such awareness, the learning of all languages necessitating just such understandings and practices.

A child learning its 'mother tongue' has to understand relationships between sounds it makes and sounds it hears; has to sort one attribute from another; accept and reject; notice variations of order; be aware of associations, correspondences, likenesses and differences, transformations, sets and subsets, additions and removals. On the one hand these are notions of mathematics. On the other they are necessary attributes every child has to study in his acquisition of language, to talk about nothing in particular, about 'shoes and ships and sealing wax and cabbages and kings'-the common language - or about numbers, points, lines and what can be done with them - the mathematical language.

Initially young learners do not need teachers for this because they bring to the task of coming to grips with themselves and their universe, rich endowments of energy, perception, motivation and inner thought. They self-learn.

Self-learning is not, of course, learning alone as though on a desert island, marooned. There are always people around. Neither is self-learning an approach solely used by infants, though for them it is often easily seen to be so. Inevitably, everything learned by every individual has to be self-learned. No one else can learn for me. It is 'I' who does the thinking, learns to hear, see, feel, relate to and although other people undoubtedly have experiences almost like my own, the 'almost' makes the point. My experiences are unique. They are never really shared.

As I write here the ink is shaped on to the paper as a result of an intent of mine with sensitivity focussed at my finger tips in holding the pen (which I had to learn to manipulate) to form words. It is my eyes that have to watch and monitor what written shapes are made, my decision as to what should be left, what replaced by something better (in my opinion), my will which oversees the whole activity.

There are several omni-existent yet identifiable aspects to self-learning, all relevant to language and mathematics.

First, when no one other than the learner has any role to play except for purposes of general care, sustenance and comfort. Digesting, seeing, hearing, tasting, turning over my body, crawling, standing, walking, finding the itch in my back - all these I learned entirely by myself. Parents did not teach me. They did not demonstrate. If they had tried to do so, it would have availed me little. I had to experience for myself whatever was needed, the feelings in my throat, in my muscles, the sensitivities of body balance, the image of my back which I have never seen.

For language I had to notice that I could make sounds by propelling air up from my lungs, past my vocal chords, varying the effect with different arrangements of tongue, mouth, teeth, lips. I acquired more and more skill in making the sounds when, or immediately after, I wanted to.

I had to examine long drawn out sounds and short ones, sounds of high pitch and low pitch and the repetition of sounds. Sometimes I tried to make a sound acceptable to _me_ as equivalent to one heard which I knew _I_ did not make. Much practice was needed.

Mathematically I learned that my parents were 'larger than me,' though I did not yet have the language to say so, that some orders of objects, sounds or movements of my head could be reversed with impunity, while with other things a change of order mattered. In floor-play I recognized areas smaller than others, increases in temperature, differences in densities between my knee and the contacted table-leg!

I knew who were in my family, assigning membership by associating together certain books, sounds, smells and touch. I was also frequently aware of the different numbers of people in the room. Whether potentially such experiences developed into mathematics or not, I initiated them myself and no one gave me the awarenesses which were mine.

For the second phase of language progress I did, however, need the presence of others. It was they who provided the spoken environment. It was my task to take advantage of that. I listened, experimented, tried this and that, just as I had done for quite a time but now I noted the effects of my attempts on those others around me. I began to be aware of which sounds I could make corresponded to what mother and father said and seemed to say purposefully. I noticed the cadences, the rhythms and stresses in the long flows of sounds they uttered, characterising the music of their language. But still no one taught me. Indeed they may have had no idea when precisely I was busy concentrating on the sounds, their combinations and correspondences, or even that I was spending time and energy on a task which has been described as the most difficult intellectual task each of us ever succeeds in.

> The speed with which children accomplish the complex process of language acquisition is particularly impressive. Ten linguists working full time for ten years to analyse the structure of the English language could not program a computer with the ability for language acquired by an average child in the first ten or even five years of life. In spite of the scale of the task, and even in spite of adverse conditions - emotional stability, physical disability and so on - children learn to speak[2]

Why should it be different for French, Spanish, any of the 2,000 world languages or for mathematical language? Learning and using number names and counting certainly occurs in phase 2, even if perfection or words, their pronunciation or their conventional order is not yet attained. So does the learning of many of the words and phrases corresponding to the already-grasped meanings from phase 1: 'greater than,' 'smallest,' 'that's mine,' 'pair,' 'difference,' 'same number,' and so on.

The third phase can be said to begin when other people intentionally help - or try to - although they do so incidentally or informally. Learning according to the principles of the other two phases continues,but parents do talk with their children in the common language well knowing that this will increase language powers in both self-expression and communication. Another incentive is that it is enjoyable. In other skills father will hold his son's bicycle while the boy learns to balance and coordinate the complex essential feelings and activities for the machine to become an extension of the rider's body. But father's effort at teaching is quite limited.

It is possible for others to help incidentally with mathematical language though it seldom does, apart from a few words, because parents and teachers seem very worried the moment junior makes an error. They feel too soon the compulsion to correct the outward words of number talk.

> Children's errors are essential data for students of child language because it is the consistent departure from the adult model that indicates the nature of a child's current hypotheses about the grammar of language... It seems to be virtually impossible to speed up the language-learning process... Courtenay B. Cazden of Harvard University found that children benefit less from frequent adult correction of their errors than from true conversational interaction. Indeed correcting errors can interrupt that interaction[3].

If adults therefore realized the importance of arithmetic conversation about numbers, lines, lengths, sizes and knew how to concentrate on that, rather than upon perfection of detail, children would learn a great deal more.

The fourth phase of self-learning needs little mention here for it is through the traditional approach of formal teaching. With the language analogy it is reflected in reading, writing, spelling, composition, literature study and poetry - language arts. In mathematics most of us have experienced it for years without perhaps being fully aware that all the other three classifications had importance and reality at the same time.

One final comment is necessary regarding self-learning before turning more practically to the implications of the mathematics classroom. Every individual is unique. Unique in his inherited characteristics, in his particular experiences since conception, in the complex manifold of those uniquenesses. There exists consequently a much greater complexity with any group of learners, even with one student-teacher pair.

If it is true, as Carl Sagan quotes[4], that there are $2^{(10^{13})}$ possible states for one human brain, this number being far greater than the number of elementary particles in the universe, educators can no longer hold without very careful re-examination hypotheses about whether students of a certain grade should and can only learn materials alleged to be for that grade; that there is only one way of talking about anything; that because only some students understand it is sufficient reason why those who do not are culpable; that teachers can ever other than crudely 'know where a student is.'

> Moreover, Man . . . cannot have a permanent and single I. His I changes as quickly as his thoughts, feelings and moods, and he makes a profound mistake in considering himself one and the same person; in reality he is always a different person, not the one he was a moment ago.[5]

In summary, we can assert the following learning principles, for all learning of all that is ever learned, and for mathematics in particular:

1. Every human is unique and as a learner, continually operates uniquely.
2. As a consequence there is no alternative but self-learning, though other people can and do have vital roles to play.
3. Everyone learns partly on his own; partly from others, sometimes without intention, at other times, intentionally; and partly when they agree to share the formal and sustained efforts of others.
4. Every young learner at first has to take the responsibility for his own learning, particularly in language because no one else can help.

Given the acceptance of these principles, there follows one for every teacher:

5. Any person approaching another intentionally as a teacher, would be well advised to examine the care and sensitivity necessary, because of these principles and because of his inability to monitor, other than crudely, the complexities of anyone else's mind.

Becoming Conversant with Mathematics

Turning now to mathematics we can suggest what needs to be emphasized in keeping with the stated self-learning principles. Mathematics, being a language with words, phrases, sentences, punctuation, ambiguity of meaning, can be taught as a language. Talk, discussion, conversation are essential - between students, between teachers and students. Perfection in the various concomitants of the language, either spoken or written, should be seen as an aim of long growth, not insisted upon at every confrontation. For many meetings with mathematics are complex and no learner can easily attend simultaneously to all the details which 'ultimately' lead to an accepted perfection.

Creating one's own mathematical sentences, paragraphs and stories is also essential just as we expect in language classes. There we do not merely hand over phrase-book sentences to be rote-accepted. Instead, one aim is fluency in making up for oneself the sentences and paragraphs. In mathematics, too, this can be done at all stages and in all topics.

The contexts for the mathematical essays may be quite different, being situations where shape, size, number, relationships and form play the characters and their inter-relationships, deductions which can be made, and conclusions arrived,at replace the common language developments of plot, the shape of the story and the inter-relationships of the people and their environment.

Because learning partly takes place when no one else has any role to play, patience for students and teachers also become necessary strategies. No one must expect that answers to problems, except the simplest, should be given immediately. Each individual has to take time dealing with his own perception of each challenge. It may take hours or days and therefore should be allowed for.

Different kinds of models of mathematical concepts may be helpful to accommodate students' differing insights, preferences and changing moods. The models will sometimes be the traditional conventional signs and symbols but in recent years many others have proved their value: fingers, counters, coloured rods, multibase blocks, geometry boards, films and videotapes, cardboard models of geometrical solids, etc.

Some examples here, embodying mathematical applications of the general learning principles, may help to suggest that there is a growing knowledge of how all this can be exemplified, with details, in all topics of current curricula, from preschool to twelfth grade and beyond.

Embodied in the student implementation of each example is a combination of principles previously considered. For 'Computation' there are analogies to synonym construction, to the expressing of equivalent sentences and to conversation. Under 'Subtraction' the emphasis is on patterning based on the associated inter-connected meanings, evoked possibly from actions on concrete materials. The fraction work shows a way of dealing with tentative hypotheses and self-correction, with respect for every individual's contributions.

'Equations' parallels syntactic rules and transformations, as does 'Quadrilaterals' too, though with different images. The 'Multiplication Table' in its systematic display of nouns lends itself to far more complexity and richness, just as a person's words can indicate his complexity and value. Finally 'Other bases' strikes to the heart of the common structures of all languages.

Instead of the examples included in the forthcoming book, I alluded in Kingston to some which arose during our CMESG meeting. It is therefore appropriate to use them here with thanks to those who provided such valuable help. I add notes, with fragments of what a teacher can do in an approach of 'games, patterns and self-learning':

Example		Suggestions for Teachers
1. A student who said, 1.60 = 1.06	Jack Easley's lecture	Accept student hypothesis. Ask, 'Is 1.61 = 1.16?' If 'yes', ask student to develop 1.61-1.60 and 1.16-1.06. Is .01 = .10? If 'yes', see what happens when decimal form for 1/100 is needed!
2. Development of difference of squares, by a sequence of arithmetic examples, some considered 'hard' (99^2-98^2), others 'easier' ($10^2 - 8^2$).	Jack Easley	De-stress the need for standard names which lead to allegations of 'hard' and 'easy'. Introduce generalisation of the difference of squares as a result of manipulation of coloured squares and rods.
3. 'They do better the second day'	Jack Easley	Realise that each complex self takes different durations to assimilate, date, understand and be ready for a reply.
4. An example of students discussing elephants sitting on a table (!) interrupted by teacher, but subsequently continuing their study of stress and strain as if <u>not</u> interrupted.	Jack Easley	Study the art of knowing when to interrupt students and when not to.
5. Taxicab geometry. Soma cube problems.	Hugh Allen	Have students play games with drawings or models. Taxicabs move only along lines of a square grid. 'Soma cubes' can produce many different models, each attainable by <u>different</u> engineering.
6. $\lim_{x \to 8} \frac{1}{x-8} = \infty$ ∴ $\lim_{x \to 5} \frac{1}{x-5} =$ ᔕ	Hugh Allen	Understand why this occurs sincerely in Phase 2 learning and that it happens less dramatically but continually in classroom mathematics.

	Example		Suggestions for Teachers
7.	'A sheet of paper has, in fact, six faces, not two!'	John Trivett in work group	Ask, as someone did, 'Do you include the three holes?' Result: John corrected himself to '9 faces.'
8.	$\frac{2}{3} + \frac{4}{5} = \frac{6}{8}$	John Trivett	Accept the hypothesis of the student who gives this. Ask him to generate more names for 6/8. Each student produces, e.g., 4/3 + 2/5, 1/2 + 1/2 + 1/2 + 3/2 until each individual meets what to him is a contradiction.
9.	'Three, two, nine, six'	John Trivett	A 3 year old saying this, counting on her fingers, is probably aware of (one, one)-correspondence and that these words are (in English) number words. That's a great deal at age 3. What she does not know perfectly yet is the English language!
10.	'...the barrier of language'	Fernand Lemay	Recognise that the richness of language implies its ambiguousness. Consider carefully whether 'explanations' are the only or the best means of inducing mathematical awareness.
11.	'The mistakes of the students need to be taken as the meat of the course'	Joseph Agassi	Ponder and act upon the hypothesis that mistakes provide valuable evidence of serious though perhaps tentative hypotheses, of effort being made, and of indication of what mistakes are never made. (e.g. 6x9=45 is common, 6x9=3000, never!)
12.	'I don't know'	Joseph Agassi	Overcome the cultural prejudice that teachers must not admit any ignorance; that no one can ever know a mind other than his own. The best any teacher can do is to grope, try and endeavour to follow each student.

	Example		Suggestions for Teachers
13.	'catched the tail'	Joseph Agassi	Do nothing! Accept this hypothesis of the formation of the past tense of all English verbs. (It's charming, anyway!).
14.	7 13 17 19 1 13 23 31 37 41 43 1 17 31 43 53 61 67 71 73 67 79	Marty Hoffmann	'What is <u>the</u> pattern of these numbers?' is not the best question. 'What patterns <u>can</u> be seen?' is better!

Conclusion

Some basic principles of all learning have, very briefly been suggested. They must, if true, apply to the learning of mathematics for all people. They are of such a nature that, when accepted, significant changes will follow in the teaching of the subject with student outcomes of much higher quality than at present and more important, their school mathematics will have contributed to ennobling their lives.

Mathematics can be learned by all because everyone has the beginnings of what is needed, mathematics is a language and all people show their ability to learn their first language between the ages of 0 and 5.

Teachers need to recognize how these basic principles can be applied to their particular specialisations.

(a) The essence of mathematics being virtual, abstract, non-material, of the nature of thought itself, rather than a set of outward forms or rituals, the sensitivities required in teaching the subject are more subtle than in some other disciplines.

(b) Emphasis on process, on actions, is more important than static products. Algebra precedes arithmetic as it does in language learning. Recognition of this is essential to success in arithmetic.

(c) The mathematics studied by students of all ages has to be considered valid by mathematicians.

(d) School mathematics must nevertheless be approached in ways appropriate to the development and self-learning, motivations, interests and language of the students with all the complications of those realities, not necessarily in the sophisticated terminology, order or other conventions passed down historically.

Ironically self-learning is not a new set of awarenesses, generally or for mathematics. Galileo said,

> You cannot teach a man anything. You can only help him discover it within himself.

And in an arithmetic text of 1874 among 'suggestions for teachers' this advice is given,

> Seek to cultivate the habit of self-reliance. Avoid doing for him (the student) anything which, either with or without assistance, he should be able to do for himself. Encourage and stimulate his exertions, but do not supersede them.
> Never permit him to accept any statement as true which he does not understand. Let him learn not by authority but demonstration addressed to his own intelligence. Encourage him to ask questions and to interpose objections. Thus, he will acquire that most important of mental habits, that of thinking for himself.[6]

John V. Trivett, Faculty of Education,
Simon Fraser University.

THE SECOND INTERNATIONAL STUDY OF MATHEMATICS ACHIEVEMENT

In 1964 the International Association for the Evaluation of Educational Achievement (IEA) conducted a study of mathematics achievement in 12 countries. Canada did not participate in that study, but the reports of it have been relatively widely read in this country.

The Second IEA study of mathematics achievement is viewed both as a follow-up to and an expansion and improvement upon the design of the first study. Two populations of students are to take part in the study:

Population A: Students enrolled in the grade where the modal number of students have attained the age of 13.0 - 13.11 by the middle of the school year.

Population B: Students who are in the normally accepted terminal year of secondary education and who are studying mathematics as a substantial part of their academic program.

The study will have several major components: a curriculum analysis which is designed to portray each participating country's curriculum against an international backdrop; an assessment of student outcomes, both cognitive and attitudinal; a detailed inventory of teaching practices in mathematics by means of data gathered through very detailed, topic-specific questionnaires developed for use in the Study; an Opportunity-to-Learn measure which will indicate differences among participating countries as regards treatment of specific topics.

To date, several provinces have indicated varying degrees of interest in participating in the study. Several attempts to interest the Council of Ministers of Education in the project have not borne fruit, and it seems unlikely that there is sufficient time or money available to organize nation-wide participation. Individual provinces are free to participate on their own provided that they can find the funds necessary to pay for the costs of such participation.

It was agreed by those present that D. Robitaille would serve as a resource person for the Study and that queries and requests concerning the Study should be directed to him.

David F. Robitaille,
University of British Columbia

MEDIC[1] - THE UBC MATHEMATICS CLINIC

The Department of Mathematics Education of the Faculty of Education at the University of British Columbia has operated a mathematics clinic for about ten years. Since 1971 the clinic, which is known as Medic, has been directed by the author. The Faculty of Education provides funds for a research assistant for MEDIC as well as for operating expenses.

MEDIC serves four major purposes. First, clinical services are provided to children who are in need of specialized assistance in mathematics. Secondly, training in diagnosis and remediation of learning problems in mathematics is provided to graduate and undergraduate students either through work in the clinic or through in-service courses and workshops offered by the department. Thirdly, a number of research studies in the area of diagnosis and remediation of learning difficulties in mathematics have been conducted through MEDIC, and several of these have been published in various forms. Finally, MEDIC serves as a clearinghouse for standardized tests and teaching materials in mathematics. Each of these functions is described in detail below.

<u>Clinical Services</u>:

Each year approximately two dozen children, most of whom are in elementary school, receive diagnostic and/or remedial assistance in mathematics through MEDIC at no cost to them, their parents, or the school. Some of these children are referred directly to the clinic by their parents; others are referred by their teachers, counsellors, or learning assistance teachers. In every case, both the parents and the school are consulted. The majority of these

---

[1] MEDIC is the acronym for the Mathematics Education Diagnostic and Instructional Centre.

children receive diagnostic/remedial assistance under the auspices of the two courses which the department offers and which are described below. The remainder are referred to the Research Assistant who is assigned annually to MEDIC. The availability of such assistance is not widely advertised since neither the facilities nor the personnel are currently available to cope with the demand which such advertising would undoubtedly create.

Diagnostic interviews are conducted on an individual basis with every child who is accepted as a referral, using one or more of the Check Lists which have been developed in MEDIC over the years. Three such check lists are currently available.

MEDIC Check List of Prerequisite Skills

MEDIC Check List for Primary Grades

MEDIC Check List for Intermediate Grades

These Check Lists are working documents and they are revised periodically in the light of our continuing experience with them.

Wherever possible, parental permission is obtained to make a videotape of the diagnostic interview. The videotape can then be used by the interviewer to re-examine the child's work or explanations, or to verify the appropriateness of the questions asked and the materials used. In addition, these tapes are sometimes used to demonstrate diagnostic techniques to teachers in in-service workshops or courses.

Courses and Workshops:

The department of Mathematics Education offers two courses in the area of diagnosis and remediation of learning problems in mathematics. The first of these, Education 471, is an introductory course at the undergraduate level; the second, Education 561, is available for graduate students who are specializing in the field of mathematics education or of learning assistance. An important component of each of these courses is supervised work with children who are experiencing difficulty in mathematics.

In addition to these campus-based courses, members of the department have offered a large number of in-service courses and workshops in locations throughout British Columbia. Education 473 has been offered as an in-service course for university credit at several locations in the recent past, and many short courses and one-day workshops or seminars have also been given.

Research:

Over the past several years a number of research studies have arisen from work with children in MEDIC. These studies have led to the production of a number of research reports which are issued periodically by MEDIC, to the publication of articles in journals, and to presentations at scholarly conferences. To date the studies have been concerned with areas such as the development of the MEDIC Check Lists, categories of students' errors in whole number computation, sex differences in mathematics achievement, and students' confidence in their incorrect computational algorithms.

Clearinghouse:

Over the years a comprehensive collection of standardized tests in mathematics has been organized and maintained in MEDIC; the collection currently contains approximately 150 tests. A computer file has been created and contains pertinent information about each test including appropriate grade levels for use, references to evaluations in Mental Measurements Yearbook, and a list of key-words describing the mathematical content of the test. Prospective test users can build search requests describing the attributes of the tests they require and obtain a printout of the information concerning any such tests which are available through MEDIC.

MEDIC also serves as a storehouse for materials which are used for remedial work with the children referred to the clinic. This material ranges from manipulative materials used in re-teaching important mathematical concepts, to paper-and-pencil workbooks and drill kits. Also on hand are a number of games, hand-held calculators, and calculator-like devices for drill and practice. Both the test file and the materials collection are open to the public, and a number of groups of teachers have spent time in the clinic examining the materials to be found there.

David F. Robitaille,
University of British Columbia