CANADIAN MATHEMATICS EDUCATION STUDY GROUP

GROUPE CANADIEN D'ETUDE EN DIDACTIQUE DES MATHEMATIQUES

PROCEEDINGS OF THE 1981 ANNUAL MEETING

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BY

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PROCEEDINGS

APPENDICES

INTRODUCTION

These Proceedings are a partial record of the fifth annual meeting of the Canadian Mathematics Education Study Group. They are intended primarily as a reminder and a resource for those who participated in the meeting, but they may nevertheless contain material which will speak to people who were not there.

A lot is missing, of course, since the Study Group's work does not consist entirely, or even mainly, in listening to prepared presentations. Presentations are much easier to record and pass along than the lively, unpredictable discussions that take place, particularly in the Working Groups. Yet it is the latter that indicate that people are really working and generating new insights and new ideas. The writers of some sections of these Proceedings have tried to convey a sense of this mental activity, but inevitably with only very modest success.

The Study Group manages to hold together, better than some other collections of people who meet to talk about the teaching of mathematics, the essential strands of mathematics education. Whatever else it concerns itself with, mathematics education must attend to:

(i) psychological matters, such as the cognitive and affective aspects of human learning, thinking and problem solving;

(ii) mathematical matters, such as the content of mathematical knowledge, the historical and cultural aspects of mathematics, and the nature of mathematics as a human activity;

(iii) epistemological matters, such as how particular mathematical concepts and skills are generated and apprehended.

No doubt the Study Group has so far only managed to keep these three axes in view rather than integrate then into a solid structure, but it is certainly moving in a direction towards this difficult, and perhaps distant, goal. Any advance will strengthen discourse and research in the field of mathematics education immeasurably. Whether there will be any impact on the practices of mathematics teaching is more speculative. It is doubtful if anyone anywhere really knows how to bring about intended changes in our classrooms.

David Wheeler

Chairman: CMESG

EDITOR'S FORWARD

The organization of these proceedings reflects the organization of the meeting itself. The agenda included two lectures, four working groups, three special groups, and two panel groups, as well as the possibility of production/ad hoc groups. The proceedings are organized around the contributions of these groups.

The two lectures were given by Kenneth Iverson and Jeremy Kilpatrick. Dr. Kilpatrick's lecture is presented in its entirity, however Dr. Iverson's lecture consisted mainly of a demonstration of the use of the computer language APL via a computer terminal and was therefore not available as a paper. A brief comment by Dr. Iverson is included in the proceedings and the reader is directed to the references included with that comment for further information on his views.

Each of the various group leaders was asked to provide a short summary of his/her sessions and these are included in the appropriate sections of the proceedings. Reports were available for all of the groups with the exception of the special group on the art of solving and posing problems. In addition to the short summaries, the working group and panel group leaders submitted various contributions made by individual members of their groups. These contributions are not included in the body of the proceedings, however are included in the appendices.

Finally, two production/ad hoc groups were formed and the presentations are included in Appendix G.

Dale R. Drost
Editor

SUMMARY OF THE CONFERENCE

Canadian Mathematics Education Study Group
Groupe canadien d'étude en didactique des mathématiques

The fifth annual Meeting of the CMESG/GCEDM took place at the University of Alberta from June 5th to 9th, 1981. Approximately 50 people attended, most of them mathematicians and mathematics educators with positions in Canadian universites, and from every province except PEI.

The guest speakers this year were Dr. Kenneth Iverson (I.P. Sharp Associates), who challenged the Meeting with his view that existing computer languages are not mathematically equivalent alternatives, and that both mathematicians and educators will live to regret the pervasive effects of BASIC and other limited languages, and Dr. Jeremy Kilpatrick (University of Georgia), who examined the theoretical and methodological reasons why research in mathematics education has so far made little impact on classrooms. Other speakers were Dr. Murray Klamkin (University of Alberta), who shared some of his insights into the art of posing and solving problems, and Professor Fernand Lemay (Université Laval), who successfully undertook the unusual and difficult task of showing in an hour the stages of development of awareness in his conquest of Rubik's Cube.

Two panels of speakers introduced discussions of "Mathematics and language" and "The relation between the history and the pedagogy of mathematics". Groups discussed the teaching of geometry in elementary schools, the character of mathematics and education in China, looking ahead to ICME-V, and possible cooperation with the Science Council's project in science education.

This Meeting, following the precedent of other years, gave central importance to the Working Groups. The topics studied this year were (1) Mathematics education research and the classroom, (2) Computer education for teachers, (3) Issues in the teaching of calculus, and (4) Revitalising mathematics in teacher education programmes. Each member of the conference chose one Group and worked in it for a total of 9 hours. The size of the Groups ensures that everyone can actively participate, and the time alloted is enough for each Group to go beyond the obvious first stages in a topic and begin to work at the harder questions. There is no doubt that this feature contributes a great deal to the generation of the atmosphere of serious and friendly cooperation which characterises these Meetings.

Anyone wishing further information about this or future Meetings may write to Joel Hillel, Department of Mathematics, Concordia University, 7141 Sherbrooke Street West, Montreal, Quebec H4B 1R6.

David Wheeler

LECTURE I

MATHEMATICS AND COMPUTERS

DR. KENNETH IVERSON

APL AS A MATHEMATICAL NOTATION

Kenneth E. Iverson
I.P. Sharp Associates
Toronto, Canada

The Speaker's remarks were based largely upon his Notation as a Tool for Thought [1], and are perhaps best summarized in the following excerpt from the introduction to that paper.

> The importance of nomenclature, notation, and language as tools for thought has long been recognized. In chemistry and in botany, for example, the establishment of systems of nomenclature by Lavoisier and Linnaeus did much to stimulate and to channel later investigation. Concerning language, George Boole in his Laws of Thought [1, p.24] asserted "That language is an instrument of human reason, and not merely a medium for the expression of thought, is a truth generally admitted."
>
> Mathematical notation provides perhaps the best-known and best-developed example of language used consciously as a tool of thought. Recognition of the important role of notation in mathematics is clear from the quotations from mathematicians given in Cajori's A History of Mathematical Notations [2, pp.332,331]. They are well worth reading in full, but the following excerpts suggest the tone:
>
> > By relieving the brain of all unnecessary work, a good notation sets it free to concentrate on more advanced problems, and in effect increases the mental power of the race.
> >
> > A.N. Whitehead
>
> > The quantity of meaning compressed into small space by algebraic signs, is another circumstance that facilitates the reasonings we are accustomed to carry on by their aid.
> >
> > Charles Babbage
>
> Nevertheless, mathematical notation has serious deficiencies. In particular, it lacks universality, and must be interpreted differently according to the topic, according to the author, and even according to the immediate context. Programming languages,

because they were designed for the purpose of directing computers, offer important advantages as tools for thought. Not only are they universal (general-purpose), but they are also executable and unambiguous. Executability makes it possible to use computers to perform extensive experiments on ideas expressed in a programming language and the lack of ambiguity makes possible precise thought experiments. In other respects, however, most programming languages are decidedly inferior to mathematical notation and are little used as tools of thought in ways that would be considered significant by, say, an applied mathematician.

The thesis of the present paper is that the advantages of executability and universality found in programming languages can be effectively combined, in a single, coherent language, with the advantages offered by mathematical notation.

The cited paper is also available in A Source Book in APL [2], which includes other relevant material. Interested readers may wish to consult not only the cited paper, but "Algebra as a Language" (pages 35-45), the discussion of the mathematical roots of programming languages in the conclusion to "The Evolution of APL" (pages 70-74), the discussion of conventions governing the order of evaluation in mathematics (pages 29-34), and the discussion of the inductive method of teaching (pages 131-139).

References

1. Iverson, K.E., "Notation as a Tool of Thought", CACM 23, August 1980, Copyright 1980, Association for Computing Machinery, Inc.

    1. Boole, G. "An Investigation of the Laws of Thought". Dover Publications, NY 1951. Originally published by Walton and Maberly, London, in 1954, and by MacMillan and Co., Cambridge.

    2. Cajori, F. "A History of Mathematical Notations", Volume II Open Court Publishing Co., La Salle, Illinois, 1929.

2. Falkoff, A.D. and Iverson, K.E., "A Source Book in APL", APL PRESS, Palo Alto, (Suite 201, 220 California Ave., Palo Alto, CA 94306), 1981.

LECTURE II

THE REASONABLE INEFFECTIVENESS OF RESEARCH IN MATHEMATICS EDUCATION

DR. JEREMY KILPATRICK

The Reasonable Ineffectiveness of Research in Mathematics Education

Jeremy Kilpatrick

University of Georgia

You may recognize the title above as a play on the title of an article by Richard W. Hamming, "The Unreasonable Effectiveness of Mathematics," which appeared in the *American Mathematical Monthly* in February 1980. In his article, Hamming asks the question, "Why is mathematics so unreasonably effective?" He offers some partial explanations but ends by saying that these explanations are so insufficient "as to leave the question essentially unanswered" (p. 82). I am going to follow the same strategy but with a different question. My question is, "Why is research in mathematics education so ineffective?" I shall offer some tentative thoughts on this question, but you will soon discern that I, too, must leave my question "essentially unanswered." In addressing my question, I shall make two claims: (1) much of the ineffectiveness of research in mathematics education is more perceived than real, and (2) most of the perceived ineffectiveness is reasonable.

The Effectiveness of Research in Education

Before asking whether research in mathematics education is really ineffective, I should like to broaden the context and ask whether research in education is, or has been, effective.

In 1978, the National Academy of Education published a thick book--672 pages--edited by Suppes (1978), whose title suggests the positive claims that are made in its pages: *Impact of Research on Education: Some Case Studies*. The book contains nine studies that purport to show how educational research has contributed to school practice. Scriven (1980) has argued that the book does not address the "pay-offs" from educational research; it

> turns out to be . . . concerned with such questions as whether the results of basic research in psychology and the social sciences trickle down to the educational research journals (which, you will not be surprised to hear, they do). That does not show that educational research *benefitted* from that other research and, more importantly, it does not show that educational research *benefitted anything else*. (p. 10, italics in original)

Whether or not you agree with Scriven's analysis, the somewhat defensive tone Suppes adopts in the book's preface makes it clear that even in his view the effects of educational research have not been overwhelming:

> I like to think of the case studies presented here as representing only a sampling of the beginning of educational research in its early years of development. What the future holds should be brighter and better, because it will be able to build on the kind of work reported in this volume and, as that building takes place, the impact on practice should become more marked. (p. xvi)

Some of the difference of opinion between Scriven and Suppes concerns whether one should talk about effects (impact) or benefits. Scriven is pushing a much more demanding criterion than is Suppes. But even if one limits the question to effectiveness, one has only to glance at the pages of journals such as the Educational Researcher over the past couple of years or attend some of the sessions at the annual meetings of the American Educational Research Association to sense that there is a crisis of faith in educational research. Have we been doing the wrong things? Have we failed to make contact with school practice? Who, if anyone, is listening to what we have to say? Is it all an empty exercise? These are the kinds of questions one senses are below the surface of the articles and talks. Many of these same questions undoubtedly occur to researchers in mathematics education, but we do not seem to be addressing them in even the oblique fashion of our colleagues in educational psychology.

The issue of whether or not educational research is effective seems to get tangled up, in some people's minds, with the distinction between pure and applied research. What is the difference between pure research and applied research in education? Which is more likely to have a greater impact on educational practice? Two models seem to be implicit when people make the distinction between pure and applied research.

The first model is hierarchical. Basic research is at the top--so high in status that it is often seen as "up in the clouds." Below it is applied research, and below that somewhere is the mundane world of practical affairs. The simplest version of this model is what Greeno (1978) calls "the pipe-line model":

> According to this model, fundamental knowledge and theories are like crude oil, which gets pumped out of the ground in basic research. Basic knowledge is shipped to applied research settings where it is transformed into something more useful; this is like shipping crude oil to refineries, and transforming the product into useable forms. Finally, results of applied research are shipped to developers and disseminators who use the knowledge in making products for use in school and send the stuff around to school users. This is analogous to sending refined gasoline to filling stations, where customers can drive up and fill their tanks. (pp. 7-8)

Greeno argues that this model is at best a weak reflection of the relation between science and technology in any field, and that it is grossly misleading in education. He argues that basic and applied research are not hierarchical; they overlap substantially, each contributing to the other, and the significant research questions are both basic and applied. I shall return to this argument after further consideration of the hierarchical model.

People who do not adhere to the oversimplified pipeline model may still view basic research as higher in status because it is theory-oriented and because it aims at generalization, whereas applied research, with its lack of theory and specificity, is necessarily lower in status. Suppes (1967), in an article written for a booklet that was the forerunner of the Journal for Research in Mathematics Education, seemed to adopt this hierarchical view. He argued that basic research could have a direct impact on practice, that we needed more basic research in mathematics education, and that we needed both theoretical and empirical basic research. He seemed to be suggesting that basic research in our field should be largely concerned with how students learn mathematics, that applied problems in our field include more effective ways of organizing the curriculum, and that basic research will necessarily be more helpful than applied research in addressing such curriculum questions. I should not have to point out that this is a somewhat restricted view.

Tyler (1981) seemed to assume the same sort of hierarchy between basic and applied research recently when he spoke to some directors of projects sponsored by the National Science Foundation. Where he uses "science education," one can read "mathematics education":

> Research needed in science education is not only basic research which results in widely generalizable concepts and principles but also applied research, that is, inquiries focused on particular situations and particular kinds of students, teachers, and institutions, which furnishes information of importance in improving science education in the particular circumstances where efforts to improve science education are being made. (p. 6)

Again, one has the connection between basic/general and applied/specific and, it seems to me, an implied status differential.

The other dominant model for thinking about basic and applied research might be called the complementarity model: the two types of research are seen as complementary, each with its own domain and its own agenda, and equal (supposedly) in status. When one sees basic research characterized as "conclusion-oriented" and applied research

as "decision-oriented," one can be fairly sure these types of research are being viewed as separate but equally valid--for their own domain of relevance. Other ways to assert their complementarity are to characterize basic research as descriptive and applied research as prescriptive, or basic research as being concerned with learning and applied research with teaching.

Adherents of the complementarity view (Gelbach, 1979; Greeno, 1978) seem agreed that the distinction between the two types of research has become increasingly blurred in recent years. Gelbach argues that "it is now almost impossible to discriminate" (p. 9) between basic and applied research because of: (1) new capabilities in research methodology (such as multivariate statistical methods) and design that permit the investigation of practical instructional problems in natural classroom settings with the same "scientifically respectable levels of precision" (p. 9) that one has in investigating basic research problems; and (2) important arguments recently advanced by people such as Glaser, who claim that our theory-building should be prescriptive, not descriptive. Snow has argued that we should abandon attempts at general theory construction in favor of less ambitious, but more achievable, "local" theories; for example, "theories that apply to the teaching of arithmetic in grades 1-2-3 in Washington and Lincoln schools in Little City, but perhaps not to the two other elementary schools in that town" (Snow, quoted by Gelbach, 1979, p. 9). Gelbach criticizes this view, saying that "local theory development should be our last resort rather than our next move" (p. 9, italics in original). Certainly, Snow's position implies a convergence between basic and applied research.

I propose a third model for the difference between basic and applied research that can be used to address the question of the effectiveness of research. Although it may appear to conflict with the other two models, I think it is equally valid. Recall the story of the married couple who went to the rabbi to help resolve their quarrel. The wife went in to the rabbi and told her story, whereupon the rabbi nodded and said, "You are right." Then the husband told his side of the story to the rabbi, who nodded and said, "You are right." After the couple had gone on their way, the rabbi's wife, who had heard both exchanges, said to her husband, "They cannot both be right." And the rabbi nodded and said, "You are right." It is in this sense that I claim my model is right.

When people talk about research as being "basic" or "applied," "conclusion-oriented" or "decision-oriented," "descriptive" or "prescriptive," and so forth, they are referring not to anything that can be considered an intrinsic quality of the research study itself, but rather to either the researcher's purpose in conducting the study or the uses to which the study is put. In other words, the same study can be either basic or applied, depending on who is doing the labeling and for what purpose. Basic research is not defined by whether it is conducted in a laboratory rather than a school, nor is it defined by whether analysis of variance is used rather than chi-square. One cannot unambiguously label a piece of research as either basic or applied; one can only ask what connection the research study appears to have to theory and what connection to practice. Both of these, to a large extent, are in the eye of the beholder.

Consider a hypothetical situation: a researcher conducts a study and then writes a report. Her purpose may have been to understand and explain some phenomenon that has to do with the learning of

mathematics. She may hope to derive some generalization from the results. Her purpose is basic research--conclusion-oriented research. That is her purpose, but that does not mean the study was basic research in some intrinsic sense. We are speaking only of her purpose. When the report is written, of course, her view will be woven, more or less explicitly, into the account. A reader of her report brings his own frame of reference to it. Although it is important to distinguish between a research study and the report of the study, especially when writing about one's own work, there is a sense in which, for the reader, the study _is_ the report. The reader ordinarily has no firsthand knowledge of a study other than what is contained in the report. More precisely, for the reader, the study consists of the report plus the frame of reference in which he embeds it. A given reader may see our researcher's study as applied research, despite her avowed purpose. She may have failed to make a clear link with theory, even though she intended to do so. The reader may have a practical problem to solve and may be able to use the results of the study to help solve it.

With respect to a particular piece of research, which can only be known to a public through some report, the classification of basic versus applied depends upon the perspective of the reader of the report. This model might be called a _lens_ model--a study may be basic or applied depending upon the lens you use in reading a report of it. This lens ought to be understood as incorporating your purposes and intentions in extracting information from the report. If the report helps you formulate a theory, then the study is functioning as basic research, regardless of the author's intentions. If the study helps you solve a practical problem, then the study, for you, has been applied research.

This point of view has implications for the effectiveness of educational research in general and research in mathematics education in particular. From this point of view, effectiveness, too, is relative. It is relative to both the criterion used and the person using the criterion. Consider the argument presented by Getzels (1978) for the effectiveness of basic research in education. He characterizes basic research as follows:

> Basic research comprises studies in which the investigator formulates his own problem regarding a phenomenon or issue, and his aim is primarily to conceptualize and understand the chosen phenomenon or issue and only secondarily, if at all, to do anything about it. The work is theory oriented rather than action oriented. Although the distinction here is not foolproof, in this view basic research (like fine art) deals with "discovered problems" and applied research (like commercial art) deals with "presented problems." (p. 480)

This is a nice formulation, but it hinges on the "aim," the intention, of the investigator. It characterizes the research study as seen by the investigator, but not necessarily as seen by others. Getzels continues as follows:

> Despite the belief that basic or theory-oriented research has little effect on practice--a belief on which the sacrifice of basic research for other activities is founded--the fact is that basic

> research can have powerful effects on practice. (p. 480)

Getzels then gives several specific examples of such effects.

How is it that someone like Getzels can so confidently assert that basic research can have powerful effects on practice, when our own experience as practitioners has suggested to most of us that such effects are rare, if not unknown? Is it a difference between mathematics education and the rest of education? Or is it a difference attributable to our perspective, as opposed to Getzels'? Here, too, I shall argue that both are right.

The Ineffectiveness of Research in Mathematics Education

Let us turn to research in mathematics education and ask why it appears, to many, to be ineffective. One reason may be that, despite what appears to be a flood of research in our field, we actually have very little in the way of research to go on in drawing implications for practice. As many observers (e.g., Greeno, 1978) have noted, the amount of money spent on research in education by the federal government in the United States is a small fraction of that spent on other areas of research such as national defense, space research, or atomic energy. Moreover, the amount spent on research in education compared to the amount spent on education in general is an even smaller fraction; one estimate is that it is less than 0.4% (Getzels, 1978, p. 478). Mathematics education, although it has done comparatively well, has shared in this dearth of funding. Again, it is a matter of perspective. Viewed one way, a lot of money has been spent by the US federal government over the last decade or so to support research in mathematics education. Viewed another way, however, the funds have not been nearly enough to do the job properly.

We face another paradox, too. It sometimes seems as though, amid the frantic activity of research in mathematics education, we must have more than enough data to answer important questions that face us. Begle (1979) surely had this impression as he undertook the massive research synthesis that resulted in his book, *Critical Variables in Mathematics Education*. Yet one can convincingly argue that we do not have enough data--certainly not enough of the right sort of data. As Bauersfeld (1979) noted: "We have a shortage in the midst of abundance" (p. 210). Sanders (1981), speaking of educational research as a whole, recently argued that we lack

> a body of systematically observed, factual knowledge about the way education operates in its natural settings. . . . Despite rampant empiricism, there is no extensive "figuration of facts" . . . observed regularities of the empirical world which must be accounted for by scientific explanations. Although we have large quantities of census-type data, achievement and other test data, there is very little trustworthy data representing the facts of the educating process. (p. 9)

Sanders calls for more case studies and naturalistic investigations "to redress this weakness" (p. 9). His argument applies as well to research in mathematics education as to educational research in general.

Another partial explanation for the apparent ineffectiveness of research in mathematics education has to do with our lack of what

Bauersfeld (1979) termed our "self-concept." Researchers in mathematics education do not constitute a true community. In North America, we have the Canadian Mathematics Education Study Group, the Special Interest Group for Research in Mathematics Education, the North American branch of the International Group for the Psychology of Mathematics Education, the Research Council for Diagnostic and Prescriptive Mathematics, the informal networks spawned by the Georgia Center for the Study of Learning and Teaching Mathematics, the research sessions at the meetings of the National Council of Teachers of Mathematics, the *Journal for Research in Mathematics Education*, and so on. Despite all of these activities--and perhaps because of all of them--we lack a strong common identity; we are not truly a community.

You have undoubtedly heard the refrain that most of the research studies in our field are conducted as part of the requirement for a doctorate and that most of these are done by people who will never do another piece of research. It is an old refrain, but unfortunately it seems to be as true today as it ever was. The annual surveys of research that have been published in the *Journal for Research in Mathematics Education* during the last decade show that, although the growth in the number of dissertations cited may have lagged a little behind the growth in the number of journal articles, there are still something like two dissertations for every article. Further, it appears that the overwhelming majority of the dissertations do not come from departments of mathematics education, nor are they conducted under the supervision of people who are recognized researchers in mathematics education. They may be good dissertations, and they may come from worthy programs in worthy institutions. But they do not arise from what one might call "the research community in mathematics education." They do not partake of issues that concern this community; they do not arise from common concerns, shared knowledge, mutual interaction. Is it any wonder that collectively they do not add up to very much?

Insufficient funds, insufficient support, insufficient knowledge, insufficient collegiality--are these not good reasons for the perception of research in mathematics education as ineffective? Is it not reasonable that research conducted under such conditions would fail to influence school practice? There are two additional reasons, however, that appear most compelling of all: (1) our lack of attention to theory, and (2) our failure to involve teachers as participants in our research.

Attention to Theory

I recently examined the 35 (out of 38) articles in the ten issues of the Journal for Research in Mathematics Education from July 1979 to May 1981 whose authors had affiliations with US institutions only. I looked at each article to see if an attempt had been made to link the question under investigation to some theoretical context. For 20 of the articles, I could find no such attempt. I may have been too harsh in some of my judgments, and I may have been somewhat hasty, but this and other observations convince me that a lack of attention to theory is characteristic of US research in mathematics education. This conclusion may not apply to research done in some other countries, but the problem is not unique to the United States.

Why is this lack of attention to theory such a serious problem? I contend that it is only through a theoretical context that empirical research procedures and findings can be applied. Each empirical

research study in mathematics education deals with a unique, limited, multi-dimensional situation, and any attempt to link the situation considered in the study with one's own "practical" situation requires an act of extrapolation. Extrapolation requires, however, that one embed the two situations in a common theoretical framework so that one can judge their similarity in various respects. As the old adage has it, "There is nothing so practical as a good theory." Kerlinger (1977) has argued that "the basic purpose of scientific research is theory" (p. 5) and that "there is little direct connection between research and educational practice" (p. 5). The effect of research on educational practice is indirect; it is mediated through theory. As Kerlinger points out, two factors that in the long run hinder the effectiveness of educational research are the twin demands for payoff and relevance. Such demands short-circuit the theory-building process.

Let us consider some examples of how theory has, or has not, affected practice in mathematics education. A frequently cited example (Cronbach & Suppes, 1969; Resnick & Ford, 1981) is E. L. Thorndike's influence on the teaching of arithmetic during the early years of this century. There is no doubt that Thorndike, through his research, his teaching, and, most especially, his analysis of the psychology of arithmetic, substantially influenced the teaching of school arithmetic in the United States. He was one of the few educational theorists to be actively concerned with the nuts and bolts of curriculum building. His theoretical ideas had an impact in the classroom largely because he himself (and his students) analyzed textbooks in the light of his theory and made concrete suggestions for changes. His theory was his hammer; he looked around and saw the arithmetic curriculum as something to pound. One should perhaps note that he did not have much competition at the time and that he was extremely energetic in his efforts to apply his theoretical ideas. He was not, strictly speaking, a mathematics educator, and his research, strictly speaking, was not research in mathematics education, but we put it there quite happily. He is a notable exception to the charge that researchers do not influence practice in our field.

A second example is Piaget, also--needless to say--not a mathematics educator. Groen (1978) has assessed the impact of Piaget's theoretical ideas on educational practice, and he devotes one section of his assessment to mathematics. Groen begins by noting that "the hard core of Piagetian theory is replete with mathematical analogies" (p. 299), and consequently, "it is not surprising that there are many parallels between mathematics education and Piaget's own ideas" (p. 299). Groen contends that, usually, rather than Piaget influencing the teaching of mathematics, it was the other way round--mathematics influenced Piaget's thinking. Groen then raises the issue of discovery learning and argues--with considerable justification--that on this issue the influential theorist has been not Piaget, but Polya. Further, he argues that with respect to "the notions of mathematical competence underlying the 'new math' curricula" (p. 300), the applied research done under the Piagetian influence dealt with highly specific problems and was difficult to generalize from. Groen concludes with an analysis of Copeland's book for elementary school teachers on the teaching of mathematics. He claims Copeland gives a one-sided view of Piagetian theory that emphasizes its static aspects and that tends to confuse mathematical structure with Piaget's more dynamic view of structure.

One might reasonably conclude from Groen's assessment that Piaget's ideas have not had much influence upon mathematics teachers. A more valid conclusion is that Piaget's ideas, as the teachers understand them, have had a profound impact, but this impact is often difficult to discern clearly. In countless classrooms today, mathematics teachers are dealing with children and teaching their subject matter in the light of what they believe to be Piaget's ideas. It is part of the professional baggage they picked up in college that is still with them, and it is heavily reinforced by the professional culture in which they live. Although Groen apparently could not find much of an overt Piagetian influence on mathematics education, the influence has been substantial, but largely covert and indirect.

Let us consider a final example of the influence of theory on practice in mathematics education. Several years ago, Stake and Easley directed a series of case studies of science and mathematics teaching for the National Science Foundation (see Fey, 1979). They found a number of secondary school mathematics teachers who offered, as justification for teaching their subject, the argument that the study of mathematics improves one's ability to think logically:

> "I can teach them to think logically about real problems in their lives today."

> "Mathematics can teach the student how to think logically and that process can carry over to anything. To be able to start with a set of facts and reason through to a conclusion is a powerful skill to have." (quoted in Fey, 1979, p. 498)

These teachers had clearly rejected Thorndike's findings concerning the lack of transfer of the disciplines--if indeed they had ever heard of these findings--and had adopted a view that has echoes of faculty psychology. Presumably this view was not dominant in their preservice education program, which doubtless gave them much sounder, more scientific justifications for the teaching of mathematics. These justifications either had not survived or had never been accepted. The educational psychology textbooks are fairly clear on this issue: one cannot train logical reasoning ability through specific school subjects like mathematics. This is part of the received wisdom of the school-of-education culture, and these teachers must have been taught it. We have here a case in which current theories have not had much impact on teachers' thinking, and presumably their practice.

These three examples are intended to illustrate some of the various and perhaps perverse ways in which theory influences practice in mathematics education. As Kerlinger and others have noted, the influence is primarily indirect. Unless someone forceful and dominant such as Thorndike acts on the system, one must look hard to detect how the influence is occurring. A common procedure is for the theorist to set forth his views and then for a transmitter, such as Copeland, to provide a simplified, and perhaps somewhat garbled, version for a larger public of teachers. The transmission network, however, is complex. A Piaget introduces a new idea, which resonates for someone else, who incorporates it into a talk, paper, or book, and other mathematics educators begin to use it in their speaking or writing. Gradually, the idea comes into the culture of mathematics education and is picked up by teachers in practice. Sometimes the idea is banned from colleges of education--like faculty psychology--but lurks in the culture like a virus to strike down the receptive practitioner.

Sometimes the force of theory is felt merely by providing a name for a construct that people have been grappling with but have not articulated. Attribution theory and expectation theory seem thus far to have contributed to mathematics education in this fashion; researchers in mathematics education are intrigued by the constructs, but they have not been much concerned with following out the ramifications of the theories. Naming, however, is a powerful force, as Adam must have discovered. Hadamard (1947), in discussing Newton's contributions to the calculus, said it aptly:

> The creation of a word or a notation for a class of ideas may be, and often is, a scientific fact of very great importance, because it means connecting these ideas together in our subsequent thought. (p. 38)

Pimm (1981, p. 48) quotes Higginson's anagram, "re-nameing is re-meaning," from which it follows that "nam(e)ing is meaning." We need the constructs and networks of theory to help us think about things--about the phenomena we confront as mathematics educators. We ought to be giving more serious attention to the theoretical underpinnings of our work, and we need to make more explicit and coherent the assumptions we are making, the point of view we are adopting, and the frame of reference that surrounds the picture we are trying to paint. As long as we ignore the theoretical contexts of our research work in mathematics education, it will remain lifeless and ineffective.

Teachers as Participants in Research

Consider now the teacher's role in research. First, we should quickly note that "research" should be given the broadest possible connotation; we should not limit it to controlled experimentation or even to empirical research, as is often done. Research in mathematics education should include historical studies, philosophical studies, and analyses of curriculum topics, as well as surveys, case studies, clinical studies, and the like. What makes a study research is not the methodology but the attempt to be systematic and to put the study in a larger context of theory, if possible. (This is true even for what some would term "applied research.") "Disciplined inquiry" is perhaps a better term in some respects than "research" since it emphasizes the process and not just the form.

From this perspective, one can see that much of what mathematics teachers do every day comes close to being research; it is just not quite so deliberate, systematic, or reflective. As Alan Bishop (1977) has pointed out, teachers can borrow three things from researchers: their procedures, their data, and their constructs. What do researchers do when they do research? If they are conducting an empirical study, they might observe, formulate hypotheses, observe some more, and try to test their hypotheses. If possible, they try to vary the situation systematically to see what the effects of variation might be. They formulate constructs and models of how these constructs might be related, and then they gather data to test the constructs and models. They develop instruments to help them gather data. These are activities that teachers can do. They can borrow these procedures and use them to study their own teaching. They can also borrow researchers' data. As Bishop points out, you do not have to gather data yourself for them to be of value to you. The value of data is in the process of understanding and interpreting them. Teachers can interpret the data from a research study in the light of

their own situation and experience.

Teachers can also borrow a researcher's constructs and the accompanying models and theories. Bishop refers to the work of George A. Kelly, the developer of the psychology of personal constructs, which is a theory of personality functioning. In Kelly's theory, we are all researchers, creating constructs as interpretations of our world and testing the predictive validity of these constructs. When we teach, we are concerned with the students we are teaching and with the ideas we are trying to teach them. Our behavior is shaped by the constructs we have about the students and the ideas. The students, in turn, have their own constructs about us and about the ideas as they understand them. Kelly sees behavior as an experiment. To understand a child's behavior, says Kelly, try to figure out what question she is asking of the world. What hypothesis is she attempting to test? To change the child's behavior, try to figure out ways of getting her to form new constructs. To change one's own behavior as a teacher, try to create alternative constructs for interpreting the world. If you cannot create such constructs, try borrowing some. The great value of the work of theorists such as Piaget, Dienes, and Gagne is in the interpretive lenses they give us for looking at familiar phenomena in new ways.

Too many mathematics educators have the wrong idea about research. They give most of their attention to the results. They think it is primarily important for teachers to know the results of the research on a given topic. They give a high priority to summarizing and disseminating research results so that teachers can understand them. In a nontrivial sense, however, the results are the least important aspect of a research study. Note that Bishop did not include results among the things to be borrowed from researchers.

The most important aspect of a research study is the constructs and theories used to interpret the data. A landmark research study is one that confronts us with data analyzed and organized so as to shake our preconceptions and force us to consider new conceptions. A researcher makes a contribution to our field by providing us with alternative constructs to work with that illuminate our world in a new way, and not simply by piling up a mass of data and results.

This view suggests why teachers should be active researchers, why they should develop a research attitude. Teachers should not stop at being borrowers; they should become collaborators. Research is not something to be left to people who understand randomized block designs and analysis of covariance. Research in our field is disciplined inquiry directed at mathematics teaching and learning. It is stepping out of the stream of daily classroom experience and stopping to reflect on it. It is becoming conscious of the constructs we are using and then trying other constructs on for size.

Research in mathematics education has increasingly been moving out into the classroom. This has been, in general, a healthy move. It would be better, however, if teachers were working more closely with researchers in formulating their problems and interpreting their findings and not simply in helping them gather data. The teachers would benefit, with respect to both their professional attitudes and their effectiveness, and so would the researchers.

Sanders (1981) has suggested that in no other profession is the community of researchers more sharply differentiated from the community of practitioners than in education. Researchers tend to identify with, and publish for, communities that do not include

practitioners, and vice versa. The self-correcting mechanisms of science, however, require that the knowledge it claims is reliable be presented to a community of peers for review and correction. If incorrect or incomplete theories become institutionalized, asks Sanders, where will the impulse for correcting them come from? At present, the theory builders in our field, such as they are, do not see the consequences of their ideas in practice; and the teachers, who have been trained to depend on experts for answers, have little impetus to correct these ideas and improve their own understanding.

Certainly the interests of the teacher and the researcher are not necessarily congruent. Neither one should expect too much from the other (Phillips, 1980), but this by no means invalidates the argument that each can profit from a closer association with the other.

A Contrast between Mathematics and Research in Mathematics Education

Hamming (1980) offers four partial explanations for the effectiveness of mathematics that may help to explain further the ineffectiveness of research in mathematics education. Hamming argues, first, that we see what we look for; "we approach . . . situations with an intellectual apparatus so that we can only find what we do in many cases" (pp. 88-89). The phenomena we see arise from the tools we use, and mathematics has been highly creative in inventing tools. Research in mathematics education, on the other hand, has not. Hamming relates a parable he attributes to Eddington: Some men went fishing in the sea with a net and, upon examining what they caught, found that there was a minimum size to the fish in the sea. In research in mathematics education, our nets have been rather coarse; our instruments, rather blunt.

Second, we select the kind of mathematics to use, and "it is simply not true that the same mathematics works every place" (p. 89). When the mathematics we have does not work, we invent something new. Hamming gives the illustration of how, when scalars did not work for representing forces, vectors were invented, followed by tensors. In research in mathematics education, we have not shown the same ingenuity in adapting our tools to our problems.

Third, science in fact answers comparatively few questions. "When you consider how much science has not answered then you see that our successes are not so impressive as they might otherwise appear" (p. 89). If one considers the questions associated with truth, beauty, or justice that mathematics cannot answer, says Hamming, one sees that almost none of our experiences fall under the domain of mathematics. Applying this same argument to the realm of research in mathematics education, one concludes that perhaps mathematics educators have not recognized the limits in the classroom to the kinds of questions that research might be able to answer. Perhaps one reason for the perceived ineffectiveness of research in mathematics education is that too much has been expected of it.

Fourth, Hamming contends that the evolution of man has provided the model for mathematics by selecting for "the ability to create and follow long chains of close reasoning" (p. 89). We have been, to some extent, selected according to the models of reality in our minds. For example, we think very well about problems pertaining to things that are about our size, says Hamming, but we tend to have trouble if the problems concern very large or very small things. Just as there are some light waves we cannot see and some sounds we cannot hear, perhaps there are some thoughts we cannot think. Although evolution has not

had much chance to operate over the few generations of scientists in the history of science, perhaps there has been some selection for the ability to follow chains of reasoning. The history of research in mathematics education is much shorter, and evolution has not had time to select for a research attitude. Many people who do research in mathematics education have, in fact, been "selected"--or have selected themselves--because of their mathematical abilities. These abilities may not be, and probably are not, the same abilities that are needed for effective research in mathematics education.

Davis and Hersh (1981) argue that the Platonist, formalist, and constructivist views of mathematics are no more than different ways of looking at the same thing. They use the analogy of how one can sit at the console of an interactive graphics system and learn about a hypercube by looking at pictures of the hypercube, rotating it so as to see how one view transforms into another. The viewer gradually builds up a comprehensive view of the thing itself out of the various partial views displayed. Similarly, one can build up a picture of mathematics itself by integrating the various pictures of it that are offered by the various philosophies of mathematics. Research in mathematics education may also be something like the hypercube, except that we are just beginning to note various views of it. Partial views are offered in several recent sources such as Begle (1979) and Shumway (1980), but a comprehensive image remains elusive.

The parallel between mathematics and research in mathematics education ought not to be pushed too far, however. Applying educational research to mathematics teaching practice is not an engineering problem like applying mathematics to a practical situation. For too long researchers have been misled by this engineering metaphor. The improvement of mathematics teaching is not a technological problem; it is a human problem. Kristol (1973), writing about the inability of the reforms of the 1960s to have much impact on the educational process, put it this way:

> There are some who will say that this state of affairs merely shows how obstinately conservative our "educational establishment" is. I think this misses the point. When there is so much will to change, so much dedication to effecting change, and so little effectual change, the more reasonable conclusion is that we are dealing with a network of human relationships that does satisfy, if only in a minimal way, certain basic societal needs, even if we don't quite know why or how it does. . . . That this should surprise us indicates how deeply our thinking about all subjects has been suffused with the technological mystique. We are inclined to believe that our power over nature and humanity is, or ought to be, limitless. We tend to assume that the will to transform our human condition is a sufficient condition for such a transformation to occur. Everywhere, we hear the refrain: "We can go to the moon, can't we? Well, why can't we do something equally marvelous about the ghettos or education or whatever?"
>
> The answer is, of course, that going to the moon

> is easy whereas improving our system of education is hard. The one is nothing but a technological problem, the other is everything but a technological problem. Doing something about education means doing something about people--teachers, students, parents, politicians--and people are just not that manipulable. They are what they are and do not become new people to suit any new ideas we might have. (p. 62)

If researchers in mathematics education are to become effective in improving the practice of mathematics teaching, they should: (1) develop a stronger sense of community, which would include practicing teachers as collaborators in research; (2) create their own theoretical constructs for viewing their work; and (3) recognize the limits of their domain as well as its complexity.

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WORKING GROUP A

RESEARCH AND THE CLASSROOM

TOM E. KIEREN
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RESEARCH AND THE CLASSROOM

A Report of the CMESG/GCEDM Research Working People

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The title of the group suggests a number of possible connotations:

- what is true both in research and practice?
- what is the bridge(s) from research to practice or vice versa?
- what are implications of research for practice or vice versa?
- *what are ways in which research interests can be projected to the classroom?

It is this latter question, inspired by a presentation at the 1979 CMESG/GCEDM meeting by Jack Easley, which provided a focus for the group.

Because of the nature of the group, some 19 university persons interested and active in mathematics education research, such a question is both appropriate and well within the scope of such persons' responsibility and perview.

The indicated question above leads to a variety of sub-questions examples of which are:

a) what are "spin-offs" from research useful in classrooms?
b) how can teachers be involved and interested in research?
c) how might research work effect various aspects of teacher education?
d) what are the researcher's responsibilities in relating his/her work to practice?

All of these questions are obviously related to the general topic

and like it have "direct" and "indirect" components. That is, in each case one can ask "is the impact or projection directly on to an aspect of practice or indirect, requiring further interpretation and elaboration?

In its deliberations, the working group focussed mainly on the indirect aspects of questions a,b, and c above. There were a considerable number of comments on question d; this will be addressed later. The working style of the group was to address these questions by looking at five examples of recent Canadian mathematics education research, each presented by one of the group members who had worked on it.

Kieren gave the example of the Rational Number Thinking Test as a research tool which teachers might use. This test is based on previous work by Gerald Noelting as well as Kieren, Nelson and Southwell, (which had been discussed at previous CMESG/GCEDM meetings in 1978, 1979 and 1980). The test contains four parts based on different mathematical interpretations of rational numbers. Within parts the questions or tasks represent a heirarchy of thinking. Research evidence based on use with over 1500 children and young adults indicates that the test might allow a teacher to observe the thinking tools that a student uses with respect to rational numbers, to see her/his differential reaction to various mathematical situations, and to see the interaction of language use and problem setting. In addition, according to Brindley's work in Calgary, the test can be used to classify students as concrete, transitional or formal with respect to rational numbers.

The group's discussion of this "spin-off" of research for practice revolved around three points. The first was how this test could be used in classrooms. The second was should it be used. That is, is such a test sufficiently related to a teacher's or a student's work to prove useful. Third, what messages from research are contained in such a "spin-off" test. With regard to this latter question, Gaulin made several stimulating observations. The first was that the test is in itself a reflection of both theoretical and empirical research and in that sense carried research ideas to the teacher. Second, the test above, with documentation of uses and results could inform practice in an indirect way. Third, the content and structure of the test contained a "hidden message" about how the curriculum for rational numbers might be organized. Thus an instrument or curriculum piece which comes from research and is used in practice might represent a projection at three levels:

- the theory behind the material is carried by the material itself,
- the results of research use could be projected to further use in the classroom,
- the material might be prototypic for other curriculum ideas.

Harrison's presentation of the Calgary Junior Mathematics Project focussed the group's attention on how teachers can stimulate and participate in research. This project had its theoretical underpinnings in the work of Piaget, Bruner and Skemp and the curriculum research of Bell. Harrison had a research group of teachers who both studied prepared process oriented materials and developed process oriented techniques which they would use in the research in their classrooms.

Following his carefully prepared summary of the development and implementation of the Project, discussion centred on the ways in which this research can be projected in the classroom. The group concurred with Harrison's response that the Project allowed for a) teacher involvement in the orientation and workshop sessions (where over 200 teachers shared materials), b) spin-off benefits to school colleagues and pupils, c) the dissemination of results at conferences in Berkeley, St-Louis and Grenoble.

The third example of research considered by the group was that of David Wheeler on problem solving, specifically "an investigation of the mental operations of high school students in solving mathematical problems". In his report on the uncompleted Project, Wheeler stressed:

a) difficulties involved in identifying "mental operations" - preferring instead to use the term "strategies", though with less satisfying results.

b) difficulties in identifying and analysing appropriate protocols & preferring to use "episodes" rather than each spoken contribution.

c) benefits of the "clinical interview" techniques especially when the interviewer takes a positive role in the situation.

In the brief discussion which followed, the group attempted to answer the question of how this research can contribute to mathematics teaching in the classroom. In addition to the obvious value to the teachers involved, as evidenced by their surprise at students' responses, it was generally agreed that this type of research is of greater interest to the constituency of researchers. (A more detailed account of Wheeler's Project is included in Appendix A).

The fourth research example took the form of video-taped interviews of children relating number ideas to a series of geometric tasks. This research done by Lunkenbein and presented by him was considered by the group for its implications for teacher education as well as implications for elementary school curriculum. The video tape itself shows a way in which teachers might use research. In a classroom setting a teacher rarely has the chance to observe individual students in a detailed or elaborate way. Such a video tape represents a controlled way for a teacher to get an opportunity to observe children doing mathematics. This control stems from the fact that the mathematics, the task, and the protocol have been predetermined by the researcher. The teacher should be able to find out the researcher's assumptions, but then observe the child for their own purposes; that is the teacher can predict behaviors of children and then test those ideas through observations. The group also saw such video tapes and related materials as a way of injecting a research objective or a sampling of functioning in a research made into the teacher education curriculum. Use of tapes would allow a person to get an image of the theory of the researcher, and to observe the materials and techniques which could be modified for informal use in the classroom.

Gaulin presented a longitudinal study of the use of calculators in upper elementary school mathematics. The group's discussion touched on a wide variety of questions related to research and the classroom. Gaulin continued to remind the group of the many audiences for research projections or implications.

Thus, for example research might have a "political" implication for provincial education authorities. In the case of the research example, research results comparing calculator and non-calculator groups on a number of dimensions of achievement. Thus, while the researcher might be interested in the more phenomenological aspects of calculator use, he or she reasonably may be obligated to collect information for particular constituents or funding bodies.

In the area of "spin-off", Gaulin reported that teachers were very interested in the collection of problems for calculator solution. While these were used in research to provide children with potentially rich settings for calculator use, the teachers saw them as useful in the curriculum in a more general way.

Conclusions

As a group, a main conclusion from our work was that we did not get very far in an effort to relate research and practice. Yet a number of useful observations did arise out of our sessions:

1. Spin-off materials tests, curriculum pieces, techniques - bear (or should) information about research and theory for their users.
2. Most research materials contain messages about potential change in the curriculum or in the actions of teachers or children. These could be made more explicit than is currently done.
3. Teachers involved as research colleagues help define research in "practical" terms. Such teacher/researchers can communicate the particular research in which they are involved to others as well as helping develop a research "face" in other teachers.

4. The fact that much current research involves detailed observations of children and young adults in mathematical settings, can be used to great advantage in teacher education.
5. There are many constituencies for research projections. The researcher must be alert to the possible interests these various practitioners might have.
6. Mathematics education researchers can interpret the mathematical relevance and implications of their work to others.
7. The example studies presented were alert to the interests of the child and seemed to engage children in activities in which they partook willingly and from which they profited in some direct way.

These observations should be tempered by the observations of the group (some of which are in the attached material) on the researcher's responsibility in making the projects of his/her work. Some were:

a) What are the researchers' responsibilities to the users of his/her work? This entails both the "selling" of her/his ideas and alerting users to limitations or misapplications.
b) Researchers' need to make their reasons for doing the research and their assumptions in its doing clear to others.
c) Research should (a value judgment) aim to improve practice. Things one knows should be clearly and directly presented to teachers.
d) What are the ways and ethics of presenting research results so as to appropriately impact practice?

These questions indicate that deep issues remain to be explored in this area of research and the classroom. Yet the group, despite the disappointment in not at least partly resolving such issues, felt that the time was well spent in relating the 5 example studies to the issues at hand. In addition, most liked the opportunity to discuss in detail the variety of interesting research which is ongoing in Canada today.

WORKING GROUP B

COMPUTER EDUCATION FOR TEACHERS

DALE BURNETT
MARVIN WESTROM

COMPUTER EDUCATION FOR (MATHEMATICS) TEACHERS:

A Statement for the Working Group on Computers in Mathematics Education

J. Dale Burnett

The form of this report reflects a desire to encourage exploration rather than to achieve closure. Many of the points made in our discussion were as valuable for the questions they raised as for any solutions that were offered. Hence many of the "phrases" are only partially completed, simply that they may be placed on the table (or in the reader's consciousness).

The following questions and points specific to the confluence of computer technology and mathematics education were noted at various points in the CMESG proceedings (on the plane to Edmonton, at the actual working group session, or in informal get-togethers).

1. Computer technology permits the display of data structures, both arithmetically (i.e. matrices) and graphically (traditional plus newer techniques such as Tukey's Exploratory Data Analysis).
2. Glass boxes. An idea championed by Howard Peelle (University of Massachusetts) - essentially the opposite of a black box - manifested by a program or an algorithm that easily reveals the nature of the procedure.

3. Student and teacher manipulation of simulation and game models Not only by specifying parameters and observing the results but also by modifying the model itself.
4. Debriefing sessions. Particularly valuable after students have had an opportunity to run a simulation. Is likely to be of value at the conclusion of any (programming or mathematics) assignment that permits alternative approaches or answers.
5. Interactive graphics. Simple examples could include the student specifying a, b and c and having an immediate display of the corresponding quadratic graph. A number of such displays could be superimposed on one another to facilitate comparisons. A more sophisticated (technologically) example could involve the simple placing or deleting of points on a scatter-plot (using a touch sensitive surface, or even games paddles) and having the resulting correlation coefficient displayed after each alteration.
6. Video-disc technology will soon be available on computer systems.
7. To what extent does familiarity with a computer language facilitate the learning of a natural language (or a second language)?
8. What are some of the important features (from the point of view of computing science, mathematics, linguistics, psychology, education,...) of various languages such as BASIC, PASCAL, APL, LOGO, LISP...?

9. What are some of the important technological features of emerging computer systems (colour, sound, graphics tablets, touch sensitive surfaces, joysticks, music boards, high resolution graphics,...)?
10. What are some of the underlying educational philosophies of different technological approaches?
11. What are the important elements in the mathematics curriculum that could be enhanced by utilization of computer technology?
12. We appear to be improving our authoring languages (NATAL, PILOT...) for tutorial programming. We also need to improve our usage languages (APL, LOGO,...).
13. What is an appropriate role for computers in (mathematics) testing? (e.g. sophisticated drill and practice based on an analysis of previous performance, item banking, branching tests,...).
14. What features of a computer-managed-instruction (CMI) system are suitable for a school system? (See paper by M. Westrom for an explanation of CMI.)
15. What types of resource centres and clearing-houses for computer-based materials are needed?
16. One should not forget about the adjunct use of other non-computer-based materials (course manuals, worksheets, photographs, handouts,...) while using a computer. Even within a specific task, some components may be better handled in other ways.

17. There should be structure in the (mathematics) curriculum at the molar level but choice (student and teacher) at the micro level.
18. Teacher receptiveness to atypical uses of a particular program may open new avenues. For example, even a simple game such as brick-out (a computerized form of solitaire ping-pong) can lead to the exploration of various strategies and an analysis of why some strategies should work better than others.
19. Development of interesting environments for student exploration:
    - let student input any figure (using a graphics tablet, games paddle or a light pen). Have computer keep track of the relationship between area and perimeter.
    - see point #5 for other examples using interactive graphics.
    - Logo (turtle geometry)
    - APL
    - billiard ball math from Jacob's book, Mathematics: A Human Endeavour.
20. Dr. Gerald Nolting, at a previous CMESG meeting in Kingston presented a paper on the use of different proportions of orange juice and water for studying students' understanding of fractions. A natural extension of this idea to computerized screen displays, where the student alters the parameters and the shade of the resultant mixture is adjusted accordingly on the screen. Dr. Kieren indicated they are already working on this at the University of Alberta.

21. Various games are now being placed on the computer (chess, checkers, backgammon, go, tic-tac,toe, Rubiks cube,...). However the real trick is to have this set up to facilitate the writing of "strategies" for playing these games.
22. In addition to numeric manipulation, computer usage can increase the need for symbolic manipulation (e.g. algebra) and graphic manipulation (e.g. use of colour within group theory).
23. Odometer displays for different bases.
24. We need software to display program execution one step at a time (e.g. displays highlighting, perhaps by flashing or reverse video, both the line of code being executed and the change to any data structure).
25. The introduction of this technology should require mathematics educators to identify the key concepts within the curriculum.
26. Use of computer generated films to show complex mathematics phenomena (e.g. turning a sphere inside-out).
27. To date, most mathematics has been "static", in the sense that it represents only snapshots, not motion. We are now on the verge of having a genuine "dynamic" mathematics environment involving motion on a screen.
28. It should be fun. We want people to enjoy mathematics.
29. Just because computers are in the classroom, is this a prima-facae reason that they should be extensively used?

30. Some uses of computers may actually be retrogressive (e.g. some languages {Basic?} may be so discordant with actual thought as to interfere with the key conceptual processes in question - for example, matrix multiplication).
31. A distinction should be made between computing science and computers augmenting mathematics.
32. What components of the mathematics curriculum become trivialized? What areas become accessible (with the advent of computer technology)?
33. We need to develop our abilities to explore.
34. There is a perceived need for guidance on how to evaluate software.
35. Attention needs to be directed to the level of the classroom teacher. How does one use a micro in today's math class?
36. We need some exemplary demonstration programs.
37. We must be careful not to limit ourselves to today's technology.

WORKING GROUP C

ISSUES IN THE TEACHING OF CALCULUS

RALPH STALL

CMESG GROUP C 1981

REPORT

Attendance at the Group C sessions rapidly stabilized to become the following group of contributors.

Robert R. Christian

H.N. Gupta

Bikkar Lalli

Ian McDonald

Ralph Staal (Chairman)

Hugh Thurston

Yvan Roux

All participated in the discussions. The papers referred to in this report were written by the leaders of the related discussions and are included in Appendix C.

1. Agenda Possibilities

The chairman opened the proceedings with the presentation of Paper #1 (Agenda Possibilities), the contents of which had previously been available to registrants. It was not expected that all 15 suggested topics would be discussed, and they were not, but nearly all were at least touched on, sometimes as side comments to a major item.

In this report, we will treat the topics mainly in the order in which they appear in Paper #1: chronologically, there was quite a bit of back-tracking as given issues were returned to when they interacted with each other.

II. Is Calculus the same as Introductory Analysis?

It was agreed that the answer to this (admittedly somewhat rhetorical) question should be NO. Much of the material may be the same, but the focus

emphasis and motivation are different. The break-up of the (at one time) closely knit package of mathematics and science has been associated with greater compartmentalization. Mathematics has often been left standing more by itself, with a resulting shift toward Introductory Analysis which is premature, inadequately motivated, and, in spite of its intentions, not really very strong theoretically.

Calculus, in contrast with Introductory Analysis, is by its very nature heavily involved with applications - they are part of it. The value of the applications here is not "practical" (in the usual sense); rather it lies in aiding the mathematical understanding and in generating (not just applying) the mathematical ideas.

One should speak of "interactions" rather than "applications", as this would suggest more strongly a reciprocating, or symmetric, relationship between the mathematics and the areas of "application".

See also R.R. Christian's Paper #7, item 8.

R.R. Christian (see Paper #7, pp. 1,2) answered the question with a firm "No!" and suggested replacing rigor by honesty.

H.N. Gupta (see Paper #6, p.6), in the matter of rigor counselled waiting for an appropriate time, referring to the quotation:

"Give them chastity and continence, only not yet "

III. What should be done about Differentials?

This question precipitated extensive discussion. First, it had to be sharpened by distinguishing between the differential notation and its various interpretations, and by treating it in different contexts.

The chairman presented three brief papers (#2, #3, and #4). #2, titled "Logarentials" was a satirical (with apologies) account of how some of the perplexities associated with most "standard" accounts of the differential notation and concept can be transferred, by a reasonable analogy, to the subject of logarithms - the point being that the unnecessary confusion inherent in the original would (it was claimed) be more easily seen in this way.

#4, titled "By Parts", was intended to indicate by examples (the full story presumably being familiar to the Group members) that the message implied by many textbooks that integration by parts involves differentials in an essential way is entirely false, both theoretically and in terms of convenient manipulation.

#3, titled "Setting up Integrals", compared two popular ways of arriving at the integral for an area under a curve. The case was put that the method which referred to Δx's and to forming a limit of a sum was, although incomplete, a clear indication of the nature of the process involved, whereas the "differential" approach was confusing.

H. Thurston responded to the question with a substantial presentation which in expanded form, appears in Paper #5,"The Leibniz Notation". R. Staal responded in part, by claiming that the difficulties which Professor Thurston resolved by using the definition

$$dx(\alpha,\tau) = x'(\alpha)(\tau)$$

could all be resolved equally well, in his opinion, without any such definition of dx in isolation, provided one interprets only the whole of "$\int_a^b \ldots dx$," rather than the parts "$\int_a^b$" and "dx" separately,

In Paper #6, pp.3-5, H.N. Gupta viewed the use of differentials from another point of view - as tools of approximation and discovery - and urged

that they not be banished from introductory Calculus. This was not actually in conflict with Professor Staal's point of view, as the latter's comments were in a different context. The use of finite (not "infinitesimal") approximations to finite increments, followed by a limiting process was not what Professor Staal was proscribing. His concern was essentially with those uses of differentials which appeared to avoid limiting processes.

In Paper #7, item 3, R.R. Christian distinguished between the differential (manipulative) and the infinitesimal (little bit of) aspects of differentials. He emphasized the importance of honesty, heuristics and usefulness (e.g. as mnemonics).

On the whole, there was agreement that many, or most, current standard treatments of differentials at the elementary level were notably unsatisfactory but there was more to the issue than this.

IV. How should $\ell n\ x$ be introduced?

R.R. Christian, (see Paper #8 for a detailed account) pointed out the advantages of approaching logarithms by means of the function

$$\ell(a) = \lim_{h \to 0} \frac{a^h - 1}{h}.$$

This approach was not generally familiar to the Group members, but Professor Gupta commented (Paper #6, pp. 1,2,3) that it resembled the treatment given by De Morgan in 1842, and elaborated on the history of the topic, and on the desirability of making this history more widely known among our students.

Professor Christian, on subsequently referring to De Morgan's work (see Paper #7, pp.1) found some overlap with his own approach, but also some considerable differences. (See paper #9, for the relevant parts of De Morgan's paper).

V. How should one sketch Polar Graphs?

R.A. Staal raised the issue of sketching polar graphs as one which normally fits into an introductory Calculus course and which, although perhaps mathematically trivial, leads to the important question "Why do we drill students in graphing, and then ignore good opportunities to use these graphs?" He outlined a method which invoved

(a) sketching the (usually familiar) Cartesian graph of $r = f(\theta)$, or perhaps a familiar related graph on which the values of $f(\theta)$ could be easily read.

(b) plotting the Polar graph of $r = |f(\theta)|$, but dotting the portions of the curve on which $f(\theta)$ was negative.

(c) reflecting the dotted portions of the graph in (b) in the origin.

(This method has been outlined, with illustrations, in the Ontario Secondary Schools Mathematics Bulletin, V. 17 No. 3, Sept. 1981, pp. 5,6).

H. Thurston (see Paper #10) has since added a note on the use of this method in suggesting where a polar graph might involve a change in concavity.

VI. What role should historical matters play?

The main responses to this question were the following:

H.N. Gupta, in Paper #6, already referred to, exemplified the historical approach to logarithms.

Y. Roux, in Paper #11 (written jointly with M. Lavoie) presented the chairman with a paper which, although written in another context, overlapped significantly the concerns of the Group. This paper emphasized the importance of having two points of view.

(a) the mathematical: dealing with internal subtleties and logical aspects

(b) the historical: emphasizing the genesis of ideas and the influence of history on teaching.

VII Why is Calculus an Important Subject?

A member of the group asked for a response to students who asked why they spent so much time on Calculus. Why, if at all, does it have a special value?

The response was that, in addition to more obvious reasons, Calculus brings together in an interesting way nearly all the mathematics to which the student has already been exposed, and its benefits are accordingly very broad. It is also a particularly powerful tool, covering a wide range of applications.

The students should be asking "Why do we spend so little time on Calculus?

It was pointed out that a very large portion of the time given to lecturing in Calculus is in fact devoted to matters algebraic, geometric, etc.

Mathematics teachers were urged to keep educational and mathematical matters uppermost, and to resist pressure to give courses on "Calculus for Chemists", "Calculus for Accountants" "Calculus for Economists" and others which are designed merely for training students in specialized applications. If given at all, such courses should be handled by Chemists, Accountants, Economists, etc.

WORKING GROUP D

REVITALISING MATHEMATICS IN TEACHER EDUCATION COURSES

HUGH ALLEN

Revitalalising Mathematics in Teacher Education Courses

Report of Working Group 'D'
H.A.J.Allen
Faculty of Education
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Working group 'D' met to discuss ways of overcoming the negative attitudes towards mathematics of many teacher education students. Several participants had brought along examples of topics and/or approaches that they had found to be effective in reducing students' anxiety. Through seeing some of these examples, and through the subsequent discussion, the group was able to clarify the framework within which constructive examples might apply. This report attempts to summarize the discussion that took place within the working group.

One aim of teacher education courses is to produce teachers who have (i) reasonable confidence in themselves and in subject matter, (ii) an interest in the subject matter, and (iii) the confidence to try new ideas, both mathematical and pedagogical. While some of our inservice and preservice candidates do possess these characteristics, most do not, and those that do not can be classified into two main types according to their attitude toward mathematics; namely apprehension or complacency.

Students whose attitude toward mathematics is one of apprehension (or fear, or sometimes outright rejection) have had very little mathematics in their academic background (often as little as grade 10 mathematics) and have little incentive to do any mathematics. Typically, these students are preservice or inservice elementary school teachers, and they are usually women.

The other type of student (the complacent ones) usually have a great deal of mathematics in their academic bacground-frequently including several university level courses in mathematics. Their "know-it all" attitude toward elementary mathematics makes them reluctant to try any <u>new</u> mathematics. They are products of the system, and the system rarely asked them to think. Typically, these students are secondary school teachers (practicing or intending).

To accomplish the three aims discussed earlier in this report it may be necessary to treat the apprehensive student differently from the complacent student. We want both students to do mathematics and to see mathematics as process. With the apprehensive student it is important to reduce his/her anxiety and to have the student experience lots of success. On the other hand, with the complacent student it may be necessary to <u>increase</u> anxiety initally and to have the student experience some lack of success before he/she is willing to do any new mathematics. With both types of students we are attempting to change attitudes toward mathematics.

It is the opinion of the working group that changes in attitudes are most likely to be achieved by engaging in activities that (a) provide insight into a mathematical topic, content, or procedure that was not previously understood, (b) promote a different view of mathematics - one in which mathematics is seen as "process" rather than a set of results, and (c) provide an opportunity for the teacher to behave in such a way as to make the student (in this case the preservice or inservice teacher) more comfortable with the material. These, then are desirable characteristics in an activity and hence are to be considered in choosing the kind of activity to be used with teachers.

In addition to considering the kind of activity, it is most important to consider the content of the activity and in particular, what can be done with this in teacher education. In mathematics courses, one teaches mathematics. That is the sole content. By contrast, in teacher education one teaches both mathematics and the vehicles for teaching mathematics; i.e.,pedagogy. In addition, the teaching involves an examination of the teaching by instructor and students. This examination includes a look at the teacher's attitude toward the subject matter, toward the learner, and toward himself. In teacher education one also examines the learning that has taken place with particular reference to <u>how</u> the learning took place and to the feelings of the learners. <u>So,</u> in attempting to change teacher's attitudes toward mathematics, we seek activities that maximize the potential to change attitudes (as described in the previous paragraph), and that maximize the kinds of things we can and should do in teacher education (as described in this paragraph).

The participants in this working group tend to favor the "investigation" activity and particularly those investigations that are reasonably open, since these activities seem to provide the maximum opportunity to (i) view mathematics in a different light, (ii) see the possibility of the teachers behaving in a less authoritarian manner and more as a learner and a resource person, and (iii) examine one's attitude toward the teaching that has taken place.

Several investigations and other activities were discussed by the working group. Some of these activities appear 'non-mathematical' (i.e., devoid of number) and are designed to demonstrate some of the processes of mathematics in a non-threatening form. Examples are "word chains" (described in Appendix A of the Report of Working Group 'A', CMESG conference Proceedings, Kingston, 1978), and the example "Animals" in Appendix D of this report. Other examples considered by this group are also included in Appendix D of this report.

SPECIAL GROUP P

THE PLACE OF GEOMETRY IN
THE ELEMENTARY SCHOOL

DIETER LUNKENBEIN

Canadian Mathematics Education Study Group (CMESG)
Groupe Canadien d'Etude en Didactique de Mathematique (GCEDM)
1981 Meeting

University of Alberta, Edmonton
June 5-9, 1981

Special group P: The place of geometry in the elementary school
(D. Lunkenbein)

"Geometry at the present time does not seem to be an important part of mathematics teaching. Is plays a minor role in official curricula and an even smaller role in most classrooms. This situation seems to be in sharp contrast to the intrinsic worth attributed to geometry teaching, on the one hand, and to the potential of informal geometry as a valuable vehicle for arousing the interest and strengthening the confidence of both teachers and children, an the other hand".[1] Many reasons may be given for such a situation. Certainly, the clear distinction between arithmetical and geometrical activities made in mathematics teaching from the very beginning on is one of the most important reasons for such a misconception of geometry in school curricula. Arithmetical and geometrical activities are but two different ways of approaching mathematical phenomena or problems and, in most cases, both types of activities are inseparable. In the course of acquiring elementary mathematical knowledge, arithmetical representations of geometry and geometrical representations of arithmetic go hand in hand and, from this point of view, the separate programming of geometrical and arithmetical activities appears the be rather artificial and prejudicial for the learning of mathematics.

1. Geometry in the Elementary and Junior High School Curriculum, Report of Working group C, Proceedings of the 1980 Meeting of the CMESG at Laval University, CMESG, Montréal, March 1981, page 74.

For a balanced mathematical education at the primary level, it seems to be of great importance, that such historical, epistemological and conceptual links between geometry and arithmetic be made explicit particularly to the teacher, so that they may be taken into account in classroom activities.

It was the aim of this group to investigate the possibility to communicate such links directly to the teacher in order to motivate him/her to integrate geometrical activities in day-to-day classroom work. Based an the outcomes of last year's meeting (Report of Working Group C, Proceedings of the 1980 - meeting), the original production or the new presentation of teaching material was to be considered in view of such genetic links between geometry and arithmetic with the intention to produce an appropriate document which would be accessible for and attractive to elementary or junior high school teachers.

Initially, the discussion centered around an exploratory activity concerning polygonal shapes composed of congruent equilateral triangles[1]. The example was intended to show a possibility of the gradual unfolding of a conceptual context by both geometrical and arithmetical means through continuous investigations over several years of schooling. Among the topics discussed were the nature and originality of possible themes, types of activities that would be most suitable for the purpose, general goals of such activities and possible formats of presentation. It was generally felt that there is no shortage of examplary subject matters or themes for such an entreprise, but that the most difficult task lies in the way of communicating the important ideas to the teacher. Such communication would be the most effective if presented in a ready-to-teach form and well commented as to the investigative

1. D. Lunkenbein, Deltagons: Shapes and Numbers, Département de mathématiques et d'informatique, Université de Sherbrooke, Sherbrooke, Québec, mai 1981.

character of the activity, the nature of the links between geometry and arithmetic, the possible insertions of the activity into current curricula and the general goals to be attained. The format of presentation may vary a great deal; the teacher's guides of the South Nottinghamshire Project[1] were mentioned as possible examples.

These discussions did seem to stimulate the enthusiasm and the creativity of the participants to a point that we agreed to give it a try and to work on it during the coming year. For this purpose, first-draft papers would be sent to the group coordinator by the end of July 1981 who would then in turn circulate these papers among participating authors in order to stimulate the discussion. Hopefully, we will be able to finalize these papers at the next CMESG - meeting and to publish them for wider circulation.

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1. Bell, A., Wigley, A., Rook, D., Journey into Maths, Teacher's Guides 1 and 2, (The South Nottinghamshire Project). Bishopbriggs, Glasgow: Blackie and Son, 1978, 1979.

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PANEL GROUP R

AN EXAMPLE OF MATHEMATISATION: THE RUBIK CUBE

FERNAND LeMAY

CANADIAN MATHEMATICS EDUCATION STUDY GROUP

UNIVERSITY OF ALBERTA, EDMONTON

(June, 5-9, 1981)

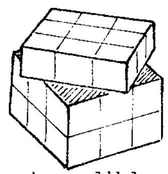
AN EXAMPLE OF MATHEMATISATION: THE RUBIK CUBE

F.Lemay

(Outline of Edmonton's Talk)

THE CUBE.—

Rubik's cube appears to be a clever setting of *27* smaller cubes or *"elements"* (or is it *26*, since the interior one is never to be seen?) whose visible faces borrow their colors to the *6* faces of the main cube (in its natural homogeneous form) and whose *9* "slices" can be rotated globaly and independantly



thus giving rise to an incredible collection of *multicolor forms*.

THE PROBLEM.—

The problem? Of course to find its way back out this labyrinth from any multicolor form or, conversely, to reach any pattern that might be proposed.

Or, if some multicolor forms are out of reach, to invent *accessibility criteria*. To determine the *orbit* of any form, that is the set of all its accessible forms. To describe ways of connecting any two forms.

To find didactical applications. To gather eventually algebraic subproducts.

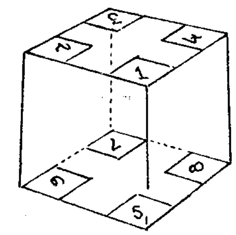
To proceed to an epistomological prospection.

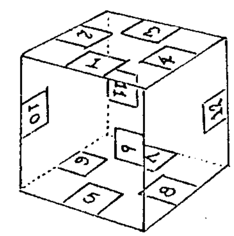
Etc.

SOME NOTATIONS.—

The structure of the cube does not allow the small central squares of any of the *6* faces of the main cube to leave their positions; therefore their constellation of colors acts as a *reference* and *coordinates system* for all elements of the cube: those lying at the vertex (*vertex elements*) shall be determined by *3* colors, those lying at the middle of edges (*middle elements*) by two, and of course the central elements of various faces by one only.

For a given element, the position is determined by that of any of its particular faces. Twenty-four positions can thus be assigned to a vertex element and twenty-four others to a middle element. Let's index these two sets of positions arbitrarily by giving ourselves a *"monitor"*





All mobile elements of the cube then also acquire an index: that associated with their natural position. The set of vertex elements can then be written (†)

$$S_0 = \{1,2,3,4,5,6,7,8\}$$

and the set of middle elements as

$$A_0 = \{1,2,3,4,5,6,7,8,9,10,11,12\}$$

Other vertex positions derive from these by rotations of one third of a turn ("screwing in" the cube) thus leading to the set of *"polarized positions"*

$$S = \{1,1',1'', \dots 8,8',8''\}$$

Similarly the set of all possible positions of middle elements shall be denoted

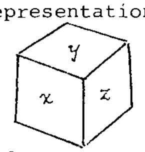
$$A = \{1,1',2,2', \dots 12,12'\}$$

(†) *S and A being the initial letters of the corresponding french words "sommet" and "arête".*

INVENTORY OF ACTIONS.–

In order to generate all multicolor forms, only the *9* rotations of the *9* slices are available; but these reduce essentially to the *6* rotations of the external faces since the rotation of a central slice is equivalent to rotations (in opposite directions) of the two parallel slices. The rotations of these external faces (quarter of a turn, screwing in the cube) shall be named after the characteristic colors of the external faces.

In every representations that follow



x, y, z shall always denote rotations of the left, of the upper and of the right faces respectively, while x', y', z' shall refer to the opposite faces. Finally the inverse of a rotation x, shall be denoted simply $\bar{x}$.

TRANSITIVITY IN S AND A.–

Our six rotations allow any vertex element to move freely about the cube, but once an element has reached its target, the three colors liable to push it further have to be *"confiscated"*. Taking this strong constraint under consideration, it will not be possible, in general, to assure more than the *"3-transitivity"* (that is the transitivity of triples of vertex elements in the set of all polarized positions) which, as experience proves, is far from our objectives.

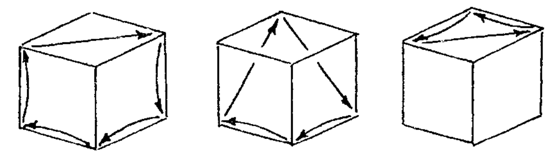
Perhaps should we have started by carrying the middle elements first? This time we would reach *3*-transitivity in A_0 but not in the polarized set A.

COMPLEX ACTIONS.–

We might conjecture that all accessible multicolor forms must remain "within *3*-transitivity", but once again experiments shall very soon destroy that conjecture.

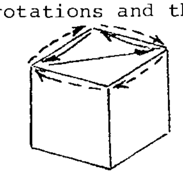
Solutions seem out of reach!

While nothing else than our six rotations is available, we must become aware of the fact that a "chain of rotations" can be seen as a new *complex action* having eventually new properties. Thus the composites of two "adjacent" rotations, or of the three rotations "around a vertex", or of the four rotations "around the walls"

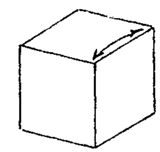


xz $\qquad$ xyz $\qquad$ $\boxed{xz} = xz'x'z$

give rise to new cycles, and the actions resulting from "conflicts" between rotations and these



y, $\boxed{xz}$

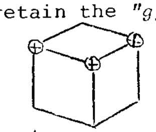


$y \cdot \boxed{xz}$

finally provide us with the means to realize complete transitivity in S_0 (that is *8*-transitivity on the set of vertex elements, neglecting polarization).

GYRATORS. DIPOLES.–

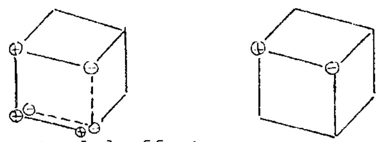
A more detailed description of $\alpha = xyz$, for instance, reveals that, while *4* elements circulate through *4* stations, *3* others "spin" on the spot; consequently by iterating properly that action, we can retain the *"gyrator"*



$\alpha^4 : S \longrightarrow S$

(⊕ indicates a spin of one third of a turn).

Now combining this gyrator with its inverse, one gets a *"dipole"*



which has a less extended effect.

It is now possible to control the polarization of all but one vertex and to stand with the firm conjecture that *it is impossible to rotate a single vertex without affecting others.*

CATASTROPHE.–

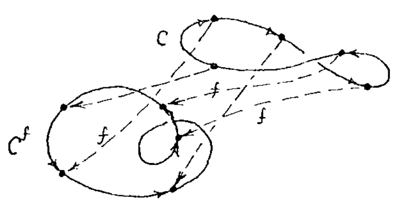
We could expect the structure of the cube to be so strong that the complete relocation of all vertex elements would necessarily carry the intermediate or middle elements back to their natural positions; but experience denies such a conjecture and we are confronted with *the problem of acting on these intermediate elements while leaving the vertex absolutely inert.*

THE DISCOVERY OF MALLEABILITY.–

Having gradually modified the underlying substance of our investigations from cubic elements to rotations, to complex actions, ... we still must go one step further toward a new vision and elect a new underlying *"matter"* on which to act.

The *cyclic paths* followed by the various elements of the cube are also "objects", complex objects, which can be submitted to our actions and which will thus reveal their *"malleability"*.

If a cycle C for instance undergoes a transformation f



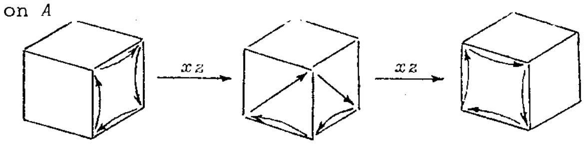
then there will result a new cycle

$$C^f = \bar{f}Cf \quad \text{(read from left).}$$

In particular the rotation z, transformed twice by the action xz, becomes the rotation x, so that a "conflict" between them

$$\bar{x}\cdot(z^{xz})^{xz}$$

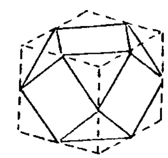
shall neutralise their mutual effects on S while it will continue to act on A



giving rise to a *circular permutation of three middle elements.*

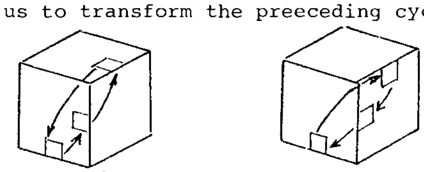
A "UNIVERSAL" MAP.–

Taking malleability into account this last cycle enables us now to move freely on the *"universal map"* connecting all middle elements

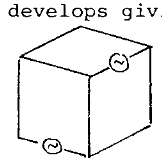


and to execute any *even* permutations of A_0 that we might wish to realise.

As for the polarization of middle elements, malleability also enables us to transform the preeceding cycle in a similar one



from which a "conflict" develops giving rise to an *"alternator"*.



We therefore have gained control over *all permutations* *in* S_0 over *all even permutations in* A_0 and over the *polarization of all elements but one vertex and one middle elements.*

From this we can deduce the existence of at least

$$\tfrac{1}{2}\times 12!\times 8!\times 2^{11}\times 3^{7} = 43.\ldots\times 10^{18}$$

accessible multicolor forms.

CONFIRMATION OF THE CONJECTURES.–

Any multicolor form exhibits a particular transformation of the original cube and they therefore determine a certain subgroup

$$\mathscr{C} \subset \mathfrak{S}(A\times S)$$

of the group of permutations of $A\times S$ which I shall call the *"group of polarized permutations of the cube"* (†).

Among these "targets", only those are *accessible* which belong to the subgroup generated by our six rotations

$$\mathfrak{A} = \langle x,y,z,x',y',z'\rangle$$

The action of this group can be "transferred" to any target; which amounts to say that it is *"transportable"* or, technically, that $\mathfrak{A}$ *is a normal subgroup of* $\mathscr{C}$. The quotient group $\mathscr{C}/\mathfrak{A}$ then represents the *class of orbits* of the various multicolor forms. In other words, accessibility or reducibility from one form f to another g, is expressed algebraically by

$$f \equiv g \quad (\mathrm{mod}\ \mathfrak{A})$$

Our proposition is the following:

Theorem.– $\mathscr{C}/\mathfrak{A} \approx Z_2\times Z_2\times Z_3$

In order to prove this, let us construct an homomorphism

$$\Gamma : \mathscr{C} \longrightarrow Z_2\times Z_2\times Z_3$$

(which we shall call the *characteristic of multicolor forms*) defined in the following way (with respect to an arbitrary monitor).

(†) *Consisting essentially in a pair of permutations of* A_0 *and of* S_0 *coupled respectively with two maps (the "alternance" and the "gyration")*

$$\begin{cases} \alpha : A_0 \longrightarrow \{0,1\} \\ \gamma : S_0 \longrightarrow \{0,1,2\} \end{cases}$$

expressing the eventual reorientation of the smaller cubes.

Definition.– For any multicolor form $f\in\mathscr{C}$, π *shall be the parity of the multicolor form, that is the parity of the permutation induced by* f *on* $A_0\cup(S_0+12)$ (†) *Secondly* $\ominus$ *shall be the sum mod 2 of the values of the function* α. *Finally* $\oplus$ *shall be the sum mod 3 of the values of the function* γ. *The characteristic is then defined by* $\Gamma=(\pi,\ominus,\oplus)$.

One shows first that Γ is independant of the particular monitor that we have chosen and that it is an homomorphism

$$\Gamma(fg) = \Gamma(f) + \Gamma(g).$$

The rest of the proof amounts to showing that the kernel of Γ is $\mathfrak{A}$.

SOME COROLLARIES.–

The computation of Γ is very simple and permits us to formulate a certain number of corollaries:

1) There exists an *"auto-dual quantum"*, ultimate germ of all actions (a double transposition acting on a pair of vertex elements on one hand, and on a pair of middle elements on the other hand).

2) The simplest of *"pure cycles"* are permutations on three elements of one kind.

3) The shortest gyrator is a dipole.

4) The shortest alternator acts on a pair of middle elements.

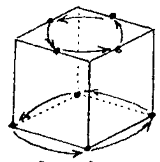
5) There exists a *"universal cycle"* involving all elements (that is a double cyclic permutation of the *8* vertex and of the *12* middle elements respectively).

6) For any sequences of distinct vertex elements and of distinct middle elements, there exists a double cyclic permutation of these sequences at the sole condition that the total number of elements in these sequences be even.

(†) *The use of* S_0+12 *amounts to reindexing temporarily the set* S_0 *in order to eliminate overlappings between* A_0 *and* S_0.

FISSION.-

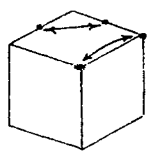
As an illustration, let us observe the striking "explosion" of the rotation y under the action of $\overset{\star}{z}\overset{\star}{y}{}^{2}\overset{\star}{z}{}'$ (where $\overset{\star}{z}$ refers to the rotation of the *central slice* which parallels z, ...)



leading to the *fission* of y *into two disjunct rotations in A and S.*

THE AUTO-DUAL QUANTUM.-

The occurrence of this first fission into *"dual actions"* is the starting point for the search of other actions acting *"symmetrically" in both A and S,* and leading to the discovery of a *"quantum" of action*



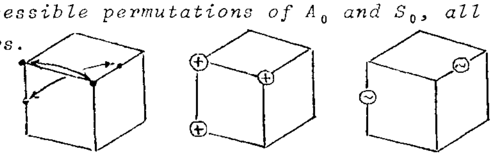
whose orbit generates all accessible multicolor forms.

ULTIMATE GERM OF ACTION.-

Finally by observing the fission of the rotation y more carefully, one is lead to the discovery of an other quantum acting in $K_0 = A_0 \cup S_0$:

$$\nu = (y \cdot y^{zx'})^{z'} . \bar{y}$$

The orbit of ν has the most interesting property of producing *all accessible permutations of A_0 and S_0, all gyrators and all alternators.*



ν ν^4 ν^6

We have thus obtained a basis for a *"dual treatment"* of the mathematics associated with the Rubik's cube.

Laval University, Quebec.
June 1981.

PANEL GROUP X

MATHEMATICS AND LANGUAGE

MARTIN HOFFMAN

Panel Group X: Mathematics and Language

Panel members: A. J. Dawson
W. C. Higginson
Moderator: M. R. Hoffman

The stated purpose of the Panel Groups was "to present themes which, if there is enough interest, may become the subject of working Groups at subsequent conferences". Panel Group X, attended by approximately 25 CMESG members, consisted of presentations by the panel members, followed by questions and comments by those in attendance. This report will briefly highlight the main themes of each presentation and the following discussion. Papers prepared by the panel members appear in the appendix of these proceedings, and should be consulted by those interested in pursuing these topics in greater depth.

Bill Higginson's presentation, entitled "X Arguments for the Consideration of Language by Researchers in Mathematics", after offering several interpretations of "X", offered an overview of several areas for further study: The relationship of mathematics and language with respect to context, communication, content and cognition. Each aspect was briefly discussed and supported with examples and relevant bibliography.

Sandy Dawson's presentation, entitled "Words triggered by Images, Images triggered by Words", focused on one aspect (imagery) of one of the areas (cognition) of possible study noted by Bill Higginson. The crucial role of images in the learning of mathematics was discussed. It was argued that

teachers must work with the images possessed by their students before the words of the teachers can be used to reliably transmit images to them.

At this point the entire group was engaged in an image making activity led by Dawson. They were asked to imagine a lemon and then to perform various manipulations on it. The discussion which followed indicated the wide range of images generated by the single set of instructions.

Several well-considered points were made during the discussion following the two presentations. Among those concerned with the question of extending the Panel Group to a future Working Group were the following:

1. For the purposes of CMESG Working Groups, visual imagery and language (in the general sense of Higginson's presentation) should be separated due to time considerations.
2. The general considerations presented by Higginson would need more focus to fit the restrictions of the Working Group format.
3. The potential for group participation (as in Dawson's presentation) was favorably noted.
4. The need to have working mathematicians attend these (projected) Working Groups was expressed.
5. It was hoped that one of the future invited speakers would focus on these topics.
6. Several participants suggested that, if such a Working Group is established, sufficient lead time be given by the Working Group leaders to allow prospective participants to read relevant articles and to gather examples from their own teaching experiences for presentation in the group.

PANEL GROUP Y

LES RELATIONS ENTRE L'HISTOIRE ET LA DIDACTIQUE DES MATHEMATIQUES

BERNARD HODGSON

Groupe de discussion Y.

LES RELATIONS ENTRE L'HISTOIRE ET LA DIDACTIQUE DES MATHEMATIQUES.

Bernard R. Hodgson

Certains aspects des relations existant entre l'histoire des mathématiques et la didactique des mathématiques ont été présentés par les trois invités. Leurs textes figurent en appendice. Louis Charbonneau (Université du Québec à Montréal) s'y questionne sur les leçons que l'étude du développement historique des mathématiques nous fournit à propos des stratégies d'apprentissage avec les enfants; un parallélisme y est fait, par exemple, entre certaines conclusions tirées d'une expérimentation avec les enfants à propos de la numération et une étude historique de ce même sujet. Hara Gauri Gupta (University of Regina) propose qu'à l'instar des étudiants en philosophie, histoire ou littérature, les étudiants en mathématiques devraient, eux aussi, avoir l'occasion de remonter aux sources et de "lire les classiques". David Wheeler (Concordia University) présente une série de commentaires sur certains aspects généraux du lien histoire-didactique. Il conclut en soulignant que non seulement une démarche didactique peut tirer profit d'une vision historique, mais également, de façon inverse, le travail de l'historien peut être enrichi par les interrogations présentées par le didacticien.

Les trois présentations ont été suivies d'un échange de vues avec les participants. On y a fait ressortir divers avantages d'une connaissance de l'histoire pour l'enseignant. Par exemple, l'histoire des mathématiques permet de voir celles-ci comme une science en évolution et non pas comme une science achevée. Une telle perception aide l'enseignant à voir les mathématiques de façon dynamique et le rend sensible aux besoins pédagogiques des étudiants quant à l'importance de créer les mathématiques pour eux-mêmes. Cet acte de création est souvent l'objet de nombreux échecs partiels; mais ces difficultés de parcours trouvent leur pendant dans la démarche même des grands mathématiciens d'autrefois et fait partie intégrante du processus de découverte des mathématiques. Les enseignants de tout niveau, même du primaire, peuvent "humaniser" l'enseignement des mathématiques en introduisant judicieusement certaines parenthèses historiques ou, à tout le moins, certains éléments faisant partie du bagage culturel et "folklorique" de notre civilisation: division de l'heure en 60 minutes, de l'année en 12 mois, etc.

Plus d'une douzaine de participants ont pris part au groupe de discussion.

LIST OF PARTICIPANTS