GROUPE CANADIEN D'ETUDE EN DIDACTIQUE DES MATHEMATIQUES

CANADIAN MATHEMATICS EDUCATION STUDY GROUP

PROCEEDINGS OF THE 1982 ANNUAL MEETING

QUEEN'S UNIVERSITY

KINGSTON, ONTARIO

JUNE 3-7, 1982

EDITED

BY

DALE R. DROST

CMESG/GCEDM

DECEMBER 1982

GROUPE CANADIEN D'ETUDE EN DIDACTIQUE DES MATHEMATIQUES
CANADIAN MATHEMATICS EDUCATION STUDY GROUP

PROCEEDINGS OF THE 1982 ANNUAL MEETING

QUEEN'S UNIVERSITY
KINGSTON, ONTARIO

JUNE 3-7, 1982

EDITED
BY
DALE R. DROST

CMESG/GCEDM DECEMBER 1982

GROUPE CANADIEN D'ETUDE EN DIDACTIQUE DES MATHEMATIQUES

CANADIAN MATHEMATICS EDUCATION STUDY GROUP

PROCEEDINGS OF THE 1982 ANNUAL MEETING

QUEEN'S UNIVERSITY

KINGSTON, ONTARIO

JUNE 3-7, 1982

EDITED

BY

DALE R. DROST

CMESG/GCEDM DECEMBER 1982

CMESG/GCEDM
1982 Meeting

PROCEEDINGS

INTRODUCTION

It is a pleasant task to write a few remarks to introduce the proceedings of the sixth annual meeting of the Canadian Mathematics Education Study Group (Group canadien d'étude en didactique des mathématiques), not least because no one could be sure at the time of the first meeting that a means would be found to sustain and consolidate a small common interest group in such geographically difficult circumstances. The Study Group owes an immense debt to the Science Council of Canada and subsequently to the Social Science and Humanities Research Council for their generous support during the past five years - a support that has always been given without strings and without interference in the management of the Study Group.

The Group has benefited, too, from the loyal support of a sizable core of members who have, over the years, established a "tone" for its meetings that makes an immediate impact on first-time participants - a tone that conveys that everyone's contribution is welcome, that listening is a positive action, and that disagreements can be aired without abrasiveness. A lack of the competitive, individualistic and self-regarding climate of so many scholarly and professional group meetings is one of the Study Group's more remarkable achievements.

This atmosphere does not prevent good work being done, as these and earlier proceedings demonstrate. But one might wish that some of this work could have more impact on mathematics education in Canada (although I don't forget the unnoticed effect brought about just by the participants taking back to their own situations what they themselves have learned). Can the Study Group "externalise" some of its activity and make it publicly available in an accessible form? This seems to be a question that the Group is more and more wanting to address.

I believe that in the future we must think about, and experiment with, a suitable form which such public interventions might take. The Study Group has worked out a form for its meetings which facilitates interaction and productive work. It should not be too difficult to give the same careful attention to the business of communicating productively with others involved in the task of teaching mathematics.

David Wheeler
Chairman/Président
CMESG/GCEDM

EDITOR'S FOREWARD

As with previous proceedings, the organization reflects the organization of the meeting itself. The agenda included two lectures, four working groups, two topic groups, and a panel group. Provision was also made for a computer workshop as well as other ad/hoc groups.

The two lectures were given by Philip Davis of Brown University and Gerard Vergnaud of the Centre National de la Recherche Scientifique in Paris. Each of these lectures is included in these proceedings.

Reports from each of the study groups, the topic groups and the panel group are also included. These reports include statements by the group coordinators of the group's activities or copies of addresses by the group participants. A copy of Dick Tahta's remarks was not available for Topic Group Q however an abstract of the session is included.

A report from D. Lunkenbein of the deliberations of the Geometry subgroup is also included. This subgroup has grown out of a study group from prior meetings of CMESG/GCEDM and more detail on the group can be found in previous proceedings.

Dale R. Drost
Editor

Lecture I

TOWARDS A PHILOSOPHY OF COMPUTATION

Speaker: Philip J. Davis, Brown University

Despite the omnipresence of computation in today's world, a philosophy of computation has hardly emerged. It is appropriate for us to promote such philosophical discussion, for, as mathematicians, and scientists, we need to know what methodological and conceptual roles computation plays with respect to our subject. As citizens we need to know what kind of a physical and mental world computation will bring about.

A number of classical dichotomies will first be described. These include the mind/matter split, the objective/subjective split, the finite/infinite split, the deterministic/stochastic split, the liberty/security split, the temporal/nontemporal split, the spatial/symbolic split, etc. Are these dichotomies merely pseudo-problems or are they fundamental tensions out of which the creativity of mankind flows?

The role the computer plays in these basic dichotomies will be discussed.

TOWARD A PHILOSOPHY OF COMPUTATION

by

Philip J. Davis
Division of Applied Mathematics
Brown University
Providence, R.I.

Ladies and Gentlemen:

I am delighted to be here this evening and I am greatly honored by your invitation to speak. When the arrangements committee asked me to talk about the role of the computer in mathematical education I drew back. Although for the past fifteen or twenty years, I have been propagandizing for more computer involvement in the college mathematics curriculum, my current concern is less about the specifics of courses, of hardware and software, than about the philosophical implication of the increasing mathematization and computerization of the world. At any rate, the committee was agreeable to a broader and fuzzier address, and out of it, hopefully, may come something of use in education.

The word "toward" in the title of the address emphasizes that at the present moment there is hardly a subject as "the philosophy of computation"; it also stresses my own inadequacies in the philosophical line. When I have done speaking, I shall not have presented a systematic and fully developed position, but only a few groping remarks on what I consider to be the important issues that such a philosophy should deal with.

1. This Cartesian Age.

I need hardly tell this audience that computers and computer science is where the action is now. In this area there is activity, ideas, motivation, energy, enthusiasm, money, jobs, prestige. The accomplishments of computers span the range of human concern. And, as if to make the whole enterprise even more characteristically human, there are now computer millionaires and computer criminals.

Will the field saturate? The present signs read: not in the forseeable future. But, of course, all this might have been said of the railroads 125 years ago.

There is considerable evidence that the brightest and the best students who in my generation would have elected a career in mathematics or physics are now selecting computer science. Mathematics is losing to computer science in a number of ways; gaining in others; and some computer imperialists assert that mathematics can not only be exploited by computers, but can also be created by them.

When did this computer revolution which sweeps us all along in its stream begin? Shall we say it began in 1944 when Howard Aiken and the Harvard group designed the Mark I? Shall we push the date back to 1889 when Herman Hollerith patented a machine for tabulating population statistics? Shall we move the date further back to 1833 to Charles Babbage's Analytical Engine or to 1805 with the automation of the Jaquard Loom?

Let me suggest that the computer revolution began on November 10, 1619 in a well-warmed room in a small German village. On that date, René Descartes a Frenchman, then twenty three years old, experienced a revelation. We may, with some justification, date the modern world from that moment.

The experience, Descartes tells us, was preceded by a state of intense concentration, excitement and agitation. His mind caught fire. He was possessed by a Genius and an idea was revealed to him in a dazzling and almost unendurable light. Later, in a state of exhaustion, he went to bed and dreamed three dreams that had been predicted by the Genius. He was so struck and so bewildered by all of this that he began to pray. He vowed that he would go on a pilgrimage. He vowed that he would put his life under the protection of the Blessed Virgin.

What was the idea that Descartes saw in a burning flash? It was no less than the unification and the illumination of the whole of science, the whole of knowledge even, by one and the same method. And this method would be applicable in an almost automatic fashion.

As with all good programmers, Descartes provided us with uneven documentation.

Eighteen years passed before he gave the world more details of this grandiose vision; such as he gave are contained in the famous "A Discourse on the Method of properly guiding the Reason in the Search for Truth in the Sciences; also: the Dioptrics, the Meteors and the Geometry, which are Essays in this Method". In this "Discourse" Descartes reveals that his method for "properly guiding the reason" is the method of mathematics. Mathematics, that is, the science of space and quantity, with the logical underlay provided by the Greeks, was the simplest and surest of all the conceptions of the mind. Why should it not form the proper basis for a universal method?

Descartes was a brilliant mathematician. He thought of himself first and foremost as a geometer and claimed that he was in the habit of turning all things into geometry. Ironically, though Descartes felt himself to be a geometer first, his method, by its very success, reduced geometry, in the visual and humanistic sense, to a minor role. The drive to quantification denigrates all that can not be thus treated. It splits the world into what is numerical and formalizable and what is not, and often compels the latter to masquerade as the former.

As we know, the reduction of geometry to algebra by use of Cartesian coordinates represents a turning point in the history of mathematics and in the history of ideas. As the French poet and essayist Paul Valéy observed "It won him the most brilliant victory ever achieved by a man whose genius was applied to reducing the need for genius".

In geometry itself, the culmination of the victory occured in 1931 when the mathematical logician Alfred Tarski proved - in the theoretical sense at least - that all problems of elementary geometry can be "automated out".

The vision of Descartes became the new spirit. Two generations later, the mathematician and philosopher Leibnitz, with even worse documentation than Descartes, talked about the "characteristica universalis". This was the dream of the universal method whereby all human problems, e.g., those of law, could be worked out rationally, systematically, by logical computation.

In our generation these visions of Descartes and Leibnitz are implemented on every hand.

If we can locate a flow of information that we want to process in some logical sense, then we are inclined to call it computation. The view of computer science, currently in a state of extreme euphoria, is that practically everything is computation. This provides a unitary view of the universe. The slogan of the neo-Cartesians is

"Computo ergo sum".

I compute, therefore I am. This is neat. But it is also dangerous.

It confronts us with loss of meaning. Loss of meaning in our personal lives. Loss of meaning in how we relate to our fellow humans. Loss of meaning in how we relate to and interpret the cosmic processes. In science there are no theoretical truths, only models from which conclusions may be inferred with some reliability. In mathematics there are no truths, just paradigms of non-contradictory systems. In our social outlook, we are afraid to specify preferred behavior. We live by cultural relativism in the name of tolerance.

Against this background which has been developing for several hundred years, against the "Cogito ergo sum", the assumption that ultimate individual identity and meaning lies in the intellectual process itself, the computer scientists, the artificial intelligencers, the mechanical brain people, now stand up and assert that what we previous thought was uniquely human, our thought, our intelligence, is easily simulated by a silicon chip and a bit of electric current. We feel debased and diminishedby this view. Though religions had assured us that we are dust and will return to dust, we imagined for ourselves a moment of intellectual glory in between when we thought,we reasoned, we computed. No wonder we shriek to the heavens when this uniqueness is threatened. We seek other avenues to meaning; through utility, through aesthetics, through ethics, through assertions of free will, through religious values all of which seem momentarily free from the processes of digitalization. Or, we allow ourselves to be co-opted by the process, and assert that the medium is the message, the transformation of formal symbols is itself the only and ultimate meaning.

2. The Philosophy of Computation and Its Pursuit.

What is the philosophy of computation? The subject hardly exists except as a subset of the philosophy of mathematics or as a label for interminable discussions as to whether the computer thinks. Perhaps we can help to create such a philosophy.

The word 'philosophy' has popular sense. One says: my philosophy of life is early to bed, early to rise makes one healthy, wealthy, and wise. Or, in computer science one says: my philosophy of programming is never to use a 'go to', or my philosophy of ordinary differential equations is always to use the Runge-Kutta method. This is practice, strategy; and this is not what I want to talk about.

What, then, should the philosophy of computation talk about? Well, classical philosophy in the hands of Aristotle, discusses the true, the good, the beautiful. So we may ask, as starters: what is true about computation; why should I believe a computation? What is good about computers; why should I allow the computer revolution to continue in its course? What is beautiful about computation; where and in what way does the computer create and enhance aesthetic

values? One knows also that classical philosophy discussed many other things; for example, the nature of knowledge and its relation to perception, the idea of perfection, the idea of the divine. These discussions, too, should be capable of admitting computer extensions.

Another philosophical question is "Does the computer think? Is the brain a computer? What, in fact, is thinking? This question has preempted the bulk of the philosophical writings, and the infant philosophy of computation in many minds is synonymous with it.

There are many ideas, then, which the philosophy of computation ought to discuss and one should insure, especially in its early days, that it does not focus on some few at the expense of wide coverage. We can learn from the philosophy of mathematics. In the last seventy years, although this subject might profitably have discussed a wide variety of things, the philosophy of mathematics has come to be almost synonymous with discussion of the foundations of mathematics The focus has primarily been on one question: why is mathematics true and how do we know it is true?

I believe there are two reasons for this preoccupation with an ultimately unprofitable question. The first reason is to be found in Bertrand Russell, in his work and personality. One of the very great figures in the modern philosophy, Russell lived an unusually long life and a colorful life.

Courageously eccentric, an extraordinarily brilliant writer when he chose to popularize, he was influential for more than two generations. Why did Russell stress the question about the truth of mathematics? We know the answer, because he tells us in his autobiographical writings. The intellectuals and scientists of his generation experienced a loss of faith in the revealed truths of religion. Religious truth having vanished for them, they sought to find a firm basis for truth in mathematics. Hence the question: is mathematical knowledge indubitable?

The second reason reflects a change in philosophers of science. A hundred years ago, the philosophy of science was pursued by the scientists themselves; men like Peirce, Maxwell, Mach, Whitehead, Russell, Poincaré, Hilbert; but around fifty or sixty years ago, philosophy of science became a subject in its own right, pursued by people who were not scientists, and who had limited personal experience with what it meant and felt to create and discover new things. In mathematics this second group latched onto the principal topic: foundations, and never let go. There is some evidence of a turn around. We have begun to see articles such as "Restoring the Mathematician to the Philosophy of Mathematics".

3. <u>Some Topics for the Philosophy of Computation</u>.

I have mentioned how the true, the good, and the beautiful might profitably be discussed within the context of

world computerization. Let me now suggest some topics with a more technical aspect.

<u>What is the relationship between mathematics and the computer</u>? As mathematicians and educators, we ask "How has each enhanced the other?" "How has each conflicted with the other?" "What is the relationship between computable structures and existential structures, between the finite and the infinite?" For example: how is it that we are able to use the computer productively in mathematical analysis considering that the statements of real variable theory make no sense on a real world computer?

There are currently three distinct formulations of calculus which are lobbying for exclusive rights in the classroom. There is the traditional way of limit theory with its deltas, and epsilons. There is the way of nonstandard analysis with its hyperreals, restoring infinitesimals at the cost of installing filters, ultrafilters, or ideals and quotient rings. There is the way of constructive, algorithmic, computable mathematics. Will the real calculus please stand up? All three are correct in context, but none of them provides an accurate description of what really goes on in the practise of numerical analysis.

What are the logical limitations of computation? How does the programming environment, sensibility and intuition, compare with that of mathematics? What is the relationship between language, symbols, mathematics and computation?

What is the relationship between computation and physics?

Should a theory of physics be computable? What reasons do we have for believing the Church Thesis and the physical Church thesis? These theses assert that all computation is reducible to that which can be carried out on a Turing machine and its physical realizations. What are the limitations on computation implied by quantum physics, by relativistic physics, by thermodynamics? In view of the fact that in a model one part of the universe is modelled by another part (physical or mental), why should it all reduce to a very special kind of mechanism known as the digital computer? Is the computer just a big pencil and a big wad of paper or is it more than that? Does quantity and speed of computation change quality? Can quantity of computation go "critical" and become "supercomputation"? What is the relationship between time and mathematics and computation? Didier Norden has pointed out that "Mathematics has been defined as the one scientific subject in which time is irrelevant". Norden says this is an inadequate view, and so do I, but explication is required.

Finally, is the whole universe a computer? This is not a recent view put forward by computer enthusiasts; on the contrary, it goes back as far as Pseudo-Aristotle (De Mundo, 300 B. C.). According to this writer, God is a mathematician.

He makes his will known through computations which drive the world in a mechanistic fashion. His will can be known if we study mathematics and make the proper models. Give or take God, this is still a popular view of the "unreasonable effectiveness" of mathematics in physics.

4. The Way of Dichotomy.

There is yet another way in which the philosophy of computation might be developed. Open the fundamental document of Cartesianiam, the "Discours de la Méthode" to its title page. There, beneath the title has been placed a colophon which depicts a man cultivating his garden and acting under the divine inspiration of Jehovah. We may take this little picture to be symbolic of the split of the universe into the world of mind and the world of matter, or, in a later but parallel terminology, into software and hardware.

Philosophy is confronted with numerous dialectical dichotomies or splits. There is the mind/matter split (the distinguished philosopher of science Sir Karl Popper has recently found it useful to split this into three). There is the subjective/objective split. There is the finite/infinite. There is the deterministic/probabilistic, the spatial/kinaesthetic/symbolic-linguistic, the time/space, the temporal/eternal, the freedom/constraint splits, and numerous others.

Do these dichotomies lead only to pseudo-problems or are they the fundamental tensions out of which the creativity of mankind flows?

At any rate, a description and interpretation of these splits as they operate within computation is a program of the first importance.

5. Why Philosophize?

Speaking jocularly, one might say that all decent subjects need a philosophy. Art has a philosophy as does literature. Science has a philosophy, mathematics has a philosophy. Computer science needs a philosophy. In it would lie the cachet of respectability.

As a mathematician, living in a world that is being increasingly mathematized and computerized, I want to know where my subject stands with respect to this trend. I want to know what meaning I can give to the symbols that are created by humans and increasingly processed by machine, or created by the machine and consumed by humans.

Finally, there is the area of social concern. I want to know whether there is salvation or damnation in the computer. Or is it just one more thing that we play with? Claims have been staked out on either side of the line. The brilliant mathematician I. J. Good sees in the computer, and in the "superthought" of the "supercomputer", the only way out of the mess that mankind finds itself in. Here is the computer as messiah.

On the other side, the humanistic writer on technology,

Lewis Mumford, worries that raw technology, computer driven, will destroy the human elements of civilization. Joseph Weizenbaum points to the instabilities created by the automation of decisions. A generation ago, the religious convert Simone Weil, sister of the Princeton mathematician André Weil, wrote "Money-mechanism-algebra. The three monsters of contemporary civilization". Money, mechanism, algebra. Put them together and they spell - not mother - but computers.

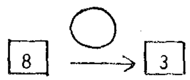
****

I began my talk with the vision of Descartes of a universal,automatic method. Cartesianism - rationalism - judges men by their activities. Systems of exclusive salvation,which the age of enlightenment overthrew,judged men by their beliefs. I live in the Cartesian age and am glad that I do. But actions, when formalized and automatized tend to become devoid of meaning. They lead to formalism and emptyness.

It is great to be a Cartesian; but one shouldn't push it too far.

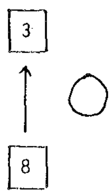
- or as a difference relationship

  . between states (included in each other)



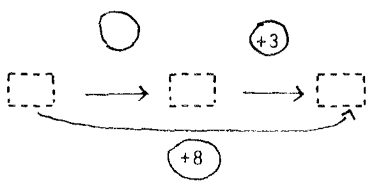
Example 4: Robert had 8 marbles before playing with Ruth. He has now 3 marbles. What has happened during the game?

  . between compared quantities (no inclusion relationship)



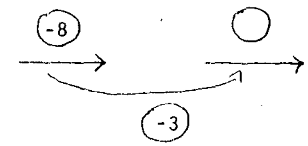
Example 5: Susan has 3 dollars in her pocket. Betty has 8. How much less does Susan have? Or how much more does Betty have?

  . between transformations



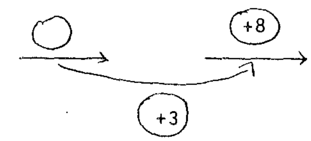
Example 6: Fred has played two games of marbles. In the second game he has won 3 marbles. He does not remember what has happened in the first game. But when he counts his marbles, he finds that he has won altogether 8 marbles. What happened in the first game?

Other examples could be:



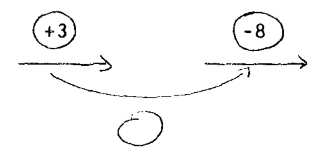
find the second transformation

or



find the first transformation

or



find the overall transformation

and so on.

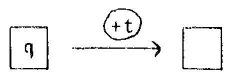
One can easily imagine the difficulties that children may meet in extending the meaning of subtraction from their primitive conception of a "decrease", to all these different cases. Each case requires from the child some relational calculus (calculus of relationships) enabling him to choose the right arithmetic operation 8 - 3.

Psychologists and maths educators have already devoted much work to the study of such problem-solving situations (see Carpenter, Moser, Romberg, 1981). The analysis of students' procedures and failures is most enlightening. It shows a slow development, over years and years, of children's conceptions of addition and subtraction. A psychogenetic approach is relevant and valuable not only for the elementary school level, but also for the secondary level. Think of the difficulties raised by "Chasles relationships" for 14-and 15-year-olds.

$\overline{AB}$ = abs (B) - abs (A)

$\overline{AC} = \overline{AB} + \overline{BC} \Longrightarrow \overline{AB} = \overline{AC} - \overline{BC}$

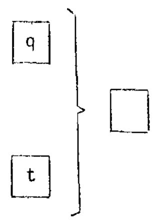
There may even be a complete decalage between the student's conception and the mathematical model. For example, young children's conceptions of addition and subtraction fit better to a unary-operation model than to a binary-operation model. As a matter of fact, addition can be viewed either as an external and unary operation of Z upon N (in the discrete case), which models very well the increase of a quantity,



$q \in \mathbb{N}$ $(+t) \in \mathbb{Z}$ discrete case

$q \in \mathbb{R}_+$ (or $\mathbb{D}_+$) $(+t) \in \mathbb{R}$ (or $\mathbb{D}$) continuous case

or as an internal binary composition of two elements of $\mathbb{N}$, which models very well the combination of two quantities.



$q \in \mathbb{N}$ $t \in \mathbb{N}$ discrete case

$q \in \mathbb{R}_+$ (or $\mathbb{D}_+$) $t \in \mathbb{R}_+$ (or $\mathbb{D}_+$) continuous case

Both models are not totally equivalent to each other for students. This discrepancy has heavy consequences for symbolic representations of addition and subtraction, as we will see later.

Teachers cannot just ignore the fact that students' conceptions are shaped by situations in ordinary life and by their initial understanding of new relationships. They must do with this fact and know more about it. It is an absolute necessity for them to know what primitive conceptions look like, what errors and misunderstandings may follow, how these conceptions may change into wider and more sophisticated ones, through which situations, which explanations, which steps.

It is essential for teachers to be aware that they cannot solve the problem of teaching by using mere definitions, how good they may be; students' conceptions can change only if they conflict with situations they fail to handle. So it is essential for teachers to envisage and master the set of situations likely to oblige and help students to accommodate their views and procedures to new relationships (inversion and composition of transformations for instance) and new types of data (large numbers or decimal numbers ...). This is the only way to make students analyse things more deeply and revise or widen their conceptions.

Solving problems is the source and criterion of operational knowledge. We must always keep this idea in mind and be able to offer students situations aiming at extending the meaning of a concept, and at testing students' competences and conceptions. This idea is crucial for researchers in France, at the present time, in our effort to provide a theory of didactic situations and operational knowledge.

Most obviously, this view leads to practical considerations in mathematics, and to practical goals of education. But I also want to stress that there is no opposition between practical and theoretical aspects of knowledge. They are both faces of the same coin, and one cannot think of many practical competences in mathematics that would refer to no theoretical view whatsoever; competences are always related to conceptions, how weak these conceptions may be, or even wrong. I do not know of any algorithm or procedure, that would develop and live by itself, free of any idea on the relationships involved. Reciprocally, theoretical concepts or theorems are void of meaning if they cannot be applied to any practical situation.

Still a problem is not necessarily practical. It may also be theoretical; for instance, the extension of multiplication and division to negative numbers is mainly a theoretical problem: multiplying a negative by a negative does not refer to any practical problem (for 12-14 year-olds anyway), unless you consider the use of algebraic calculus as a practical problem. Theoretical questions may be related to different level competences. For instance, the conception of addition as a unary operation modeling the increase of a quantity, versus a binary operation modeling the combination of two quantities, concerns the practical problem of representing many real life situations; the extension of multiplication to directed numbers concerns the coherence of algebraic calculus. Both theoretical problems are related to competences, but not at the same level.

2. A developmental approach

Conceptions and competences develop over a long period of time. This is true not only for general characteristics of thinking such as studied by Piaget and other psychogeneticians, but also for specific contents of knowledge. For example, the concepts of fraction and ratio have their roots in activities that are meaningful for 8-year-olds, for simple values as $\frac{1}{2}$ or $\frac{1}{4}$; and still the concept of rational number is a big and long-lasting source of difficulty for 15- or 16-year-olds and many adults.

As regards additive structures, although the first principles of addition and subtraction are understood by 3 or 4-year-olds, 75% of 15-year-olds still fail problems like the following ones:

Example 7: John has received 45 dollars from his grandmother. Then he goes to the store and buys different things. When he counts his money, he finds 37 dollars less than he had before receiving money from his grandmother. How much did he spend?

Example 8: Mr Dupont drives along the Loire valley 35 km westwards; then he drives eastwards. When he stops, he is 47 km eastwards from his departure point. How long was his second drive?

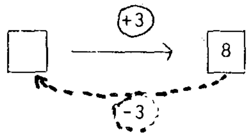
The same 15-year-old students are taught Chasles' relationship $\overline{AC} = \overline{AB} + \overline{BC}$, which is directly related to example 8. But nothing is said, in the curriculum, about composition of functions which would help them in understanding better these examples.

Psychologists have described some general stages of intellectual development, but they have not paid enough attention to the detailed paths taken by students to develop specific competences like solving addition and subtraction problems of all kinds. A good example though, is Noelting's work on ratio, but the problems envisaged are too limited, and Noelting's interpretation of results tends to rely more on logical characteristics than on mathematical ones.

A priority for research in maths education and psychology is to make experiments on a large variety of problems, so as to make a better picture of the steps by which students handle different classes of mathematical problems and use different procedures, not equally powerful, to solve them.

First of all, it is essential to recognize the variety of problem-structures and to analyze the relationships involved and the operations of thinking and procedures necessary to solve each class of problem. For instance it is not the same operation of thinking to invert a direct transformation, or to find a complement (see examples 2 and 3 above).

It is not either equivalent to solve a find-the-initial-state problem like example 3 by inverting the direct transformation and applying it to the final state (dotted arrow),



or by making a hypothesis on the initial state (4 for instance), applying the direct transformation, seing the difference from the expected final state and then correcting the hypothesis.

4 $\xrightarrow{+3}$ 7 (does not fit)

5 $\xrightarrow{+3}$ 8 (OK)

Obviously, these procedures are not equally powerful and the second one has only a local value, depending heavily on the numerical characteristics of the variables involved. Small whole numbers have properties that help children to solve many problems by non-canonical procedures. A way to get students to move to more powerful procedures is to use larger numbers for which the canonical procedure is the only practical possibility. In some other problems, it might be necessary to use decimal or rational numbers. Brousseau (1981) has insisted a lot upon the fact that changing numerical values is an important didactic means of making students' conceptions move from primitive to more sophisticated ones.

But of course, this is not independent of the intellectual development of students; and the description of the comparative complexity of problems and procedures relies very strongly upon a developmental approach of mathematics learning. However, this description may be very partial and not even be

understandable if the range of problems considered is too small and the period of development too short. For the same class of problems one may find a decalage of 3 or 4 years among students in the use of the same procedure, correct or wrong; and very often the behavior in one problem cannot be understood unless it is related to the behavior in other problems. There are strong correlations, strong hierarchies and also lots of metaphoric substitutions in the handling of problems. This consideration has led me to the firm conviction that it is necessary to study the formation of rather large pieces of knowledge: conceptual fields.

But before I explain below what a "conceptual field" consists of, I need to stress my third point on invariants and theorems in action. I also need to give a short conclusion to this short part on development.

The slowness of concept development is heavily underestimated by teachers, parents, and curriculae. For instance it is very often accepted that once students have studied a chapter of mathematics they should know it, or at least a high percentage of them should know it; and therefore it should not be necessary to come back to it during the following school-years. Empirical studies show that it would be wiser to study the same field year after year, going deeper into the field each time, meeting new aspects, and coming again on aspects studied before. This should be widely done through problems to be solved. Different problems usually require the mastery of different properties of the same concept. It is essential to recognize this fact in a developmental approach.

3. Theorems in action and invariants

It is usually accepted, in education, that action and activity of students should be favored, in order to push them to construct operational knowledge, i.e. lively and efficient knowledge. But it is not so usually recognized that action in situations and problem-solving is concept formation. In what sense is it concept formation? Here we come to a very important theoretical point.

Mathematicians know what invariants are and I do not have to explain in detail that, under certain sets of transformations (or variations), some quantities or some relationships remain invariant: actually invariants are means of characterizing sets of transformations. But mathematicians and maths educators have not yet fully acknowledged the fact, which is more specific of cognitive developmental psychology, that very simple invariants, about which adults do not even think that they might vary, are not invariants at all for young children and are even most obviously varying, as first shown by Piaget:

- the number of eggs when you change the arrangement;
- the quantity of orange juice when you transfer it from a wide glass into a narrow one;
- the weight and volume of a piece of plasticine when you change its shape.

Isn't it obvious that there is more water in the narrow glass, since water gets to a higher level?

Invariants are a recurrent topic in Piaget's work, down to the problem of young babies' development, with the "permanent object scheme". And yet Piaget has not fully recognized the importance of many invariants, most important in mathematics and physics; relational invariants. By "relational invariants" (this is an idiosyncratic expression) I mean relationships that remain the same over a certain set of transformations or a certain set of variations (a range of values for instance).

Let me give an example in parenthood relationships: it is not easy for a young boy to grasp the idea that the relationship "son of" is all the same true for himself and his father, himself and his mother, his friend Matthew and Matthew's parents, and even his own father and his grand parents: how can his father be both father and son?
Similar problems are raised for spatial relationships (behind, to the west of ...), numerical relationships (bigger than, bigger by n than, multiple of ...) and others.

Beside these binary relationships, children at an early age meet higher level relationships, which we usually call theorems. They do not meet them in a real mathematical shape of course; but they nevertheless do have to handle these theorems in action and problem-solving, at least for certain values of the variables. This is my reason for calling them "theorems-in-action".

The essential purpose, for a cognitive analysis of tasks and behaviors, is to identify such theorems in action, even if it is not easy to do so and come to an agreement upon their behavioral criteria.

Let me start with the example of the third axiom of the theory of measure.

$$m\ (x * y) = m\ (x) + m\ (y)$$
$$\forall x, y \text{ for } * \text{ adequately chosen.}$$

In the case of discrete quantities and cardinals this axiom becomes:

$$\text{card}\ (X \cup Y) = \text{card}\ (X) + \text{card}\ (Y)$$
$$\forall X, Y \text{ provided } X \cap Y = \emptyset$$

Such an axiom is necessarily used by a young girl laying the table, when she counts the persons in the lounge and the persons in the garden, and add both numbers to find out how many persons there are altogether.

It is a more efficient method than getting all the persons in the garden and counting them all.

A large amount of research work has been done on the initial learning of addition that shows the emergence of such a theorem in action.

The "counting-on" procedure, for instance, is a crucial step in the discovery of such a theorem (or axiom). It conveys another important theorem:

$\forall$ m, n
At least for small values

$$m \underbrace{\xrightarrow{+1} \xrightarrow{+1} \xrightarrow{+1} \cdots \xrightarrow{+1}}_{n \text{ times}} \text{ is equivalent to } \xrightarrow{m+n}$$

as an intermediary step between the counting-all procedure, which does not suppose the axiom, and the usual procedure in its final state, when m + n is known as a fact or as a result of the addition algorithm (for more details, see Carpenter, Moser, Romberg, 1981, and especially Fuson's paper).

There are several other axioms and theorems involved in the construction of the natural number concept as the measure of discrete quantities. Some of them can be appropriated by 3, 4 or 5-year-olds. Some by 6 or 7-year-olds. Some are still difficult for many 9-year-old children.

- $2 \xrightarrow{+1} 3$ whatever objects are counted
  $3 \xrightarrow{+1} 4$
  ...
- $2 \xrightarrow{+1} \xrightarrow{+1}$ equivalent to $2 \xrightarrow{+2}$
  ...
  $n \xrightarrow{+1} \xrightarrow{+1}$ equivalent to $n \xrightarrow{+2}$
- $2 \underbrace{\xrightarrow{+1} \cdots \xrightarrow{+1}}_{n \text{ times}}$ equivalent to $2 \xrightarrow{+n}$
  ...

$$m \underbrace{\xrightarrow{+1} \cdots \xrightarrow{+1}}_{n \text{ times}} \quad \text{equivalent to } m \xrightarrow{+n}$$

for $n + m \leqslant 10$

- conservation of discrete small quantities
- card $(A \cup B)$ = card (A) + card (B) provided $A \cap B \neq \emptyset$ and $A \cup B$ not too big.
- $X \subset Y \Longrightarrow$ card $(X) \leqslant$ card (Y) (inclusion problem)

Many other theorems can be identified that are necessary to the solution of addition and subtraction problems as illustated above in examples 1 to 8. It would be boring to enumerate all of them; but I have said enough to conclude that the landscape is more complicated than one expects it to be at a first glance.

The study of multiplicative structures shows also the emergence of solutions to simple and multiple proportion problems that can be interpreted as the emergence of theorems in action: these solutions have not usually been taught to students and are not usually explicitly expressed by them. This is the case for the isomorphic properties of the linear function.

$$f(x + x') = f(x) + f(x')$$
$$f(\lambda x) = \lambda f(x)$$
$$f(\lambda x + \lambda' x') = \lambda f(x) + \lambda' f(x')$$

It would probably be nonsense to teach these theorems and procedures formally to students. It is better to face them with problems in which they may find it natural to use them (for simple values of the variables for instance) and then help them to extend the procedure to other values of the variables.

Anyway these isomorphic properties are more easily used than the constant coefficient property.

$$f(x) = ax$$
$$x = \frac{1}{a} f(x)$$

even in the case when the numerical value of a is simple (3 or 4).

The psychological (and epistemological) reason for this, is that variables x and f(x) are not pure numbers for students, but magnitudes; and it is not easily accepted by students to look for the ratio of magnitudes of different kinds $\frac{f(x)}{x}$ or $\frac{x}{f(x)}$ (distance and time, costs and goods, weight and volume ...).

The above isomorphic properties of the linear function do not raise the same sort of difficulty, because the extracted relationships relate magnitudes of the same kind (for further details, see Vergnaud, in press).

The isomorphic properties of the linear function are more easily used than the constant coefficient, even when the only procedure taught is the constant coefficient one. So we meet the paradox that a theorem in action that has never been taught as a theorem may be more naturally used than a theorem that has been taught but has not really become a real theorem in action.

This raises two questions:

- How can we get theorems become theorems in action?
- How can we get theorems in action become theorems?

Let us never forget that theorems in action are relational invariants. Like other invariants studied by psychologists, they are associated with the feeling of obviousness: they are at a certain stage of development taken as obvious properties of situations.

How can we make students catch as obvious the relevant properties of situations for simple values of the variables, and then generalize?

4. Conceptual fields

This is another idiosyncratic expression. Is it necessary? and what does it mean?

An interactive conception of concept formation considers a concept as a triplet (S, I, $\mathcal{S}$).

S : set of situations that make the concept meaningful
I : set of invariants that constitute the concept
$\mathcal{S}$: set of symbolic representations used to represent the concept, its properties and the situations it refers to.

I will deal with the symbolic representations in the next part of this paper.

Several considerations can be made now:

First: a given situation does not involve all properties of a concept. If you want to address all properties of a concept, you must necessarily refer to several (and even many) kinds of situations.

Second: a given situation does not usually involve just one concept; its analysis requires several concepts. For instance, additive structures require the concepts of measure, transformation, comparison, difference and inversion, the concepts of unary and binary operations, the concepts of natural and directed number, the concepts of function , of abscissa and others.

Third : the formation of a concept, especially when you look at it through problem-solving behavior, covers a long period of time, with many interactions and many decalages. One may not be able to understand what a 15-year-old does, if one does not know the primitive conceptions shaped in his mind when he was 8 or 9, or even 4 or 5, and the different steps by which these conceptions have been transformed into a mixture of definitions and interpretations. It is a fact that students try to make new situations and new concepts meaningful to themselves by applying and adapting their former conceptions.

As a consequence of these three reasons, I consider that psychologists and maths educators must not study too small-sized objects, because they would not understand the complex process by which children and adolescents master, or don't master mathematics.

A "conceptual field" is "a set of situations, the mastering of which requires a variety of concepts, procedures and symbolic representations tightly connected with one another.

This definition does not intend to be rigorous: it refers to a set of problems (not strictly defined) more than to a set of concepts; the description of its contents requires both the analysis of situations and problems (i.e. the relationships involved) and the analysis of students' procedures when dealing with these situations. Symbolic representations such as diagrams, algebra, graphs, tables ... may be crucial for the extraction of relevant relationships, but may also be misinterpreted by students and misleading. I will come to this point in the next part.

The best example of "conceptual field" I can give is the field of "additive structures" in which I have taken many examples. But there are many other examples.

. multiplicative structures: understood as a set of problems requiring multiplications and divisions. Although not independent of additive structures, they make a specific field including simple proportion problems (even simple multiplication and division problems are proportion problems) and multiple proportion problems which are (unfortunatly) not often analyzed as such: area and volume for instance are rarely studied with the help of the n-linear function, which is actually essential for their understanding. Ratios of magnitudes of the same kind, and ratios of magnitudes of different kinds lead to the concept of rational number and also to dimensional analysis (product and quotient of dimensions). Vector-space

theory is also involved in the analysis of linear combinations and linear mappings.

Different sorts of symbolic representations may be useful to represent these problems: they are not equally meaningful to students. It depends on the problems and on the students' level of analysis: tables, graphs, equations have different properties.

. Spatial measures: such as length, area, volume make a specific field across additive and multiplicative structures, involving both geometrical representation of space and arithmetization of space.

. Dynamics: such as coordination of distance, time, speed, acceleration and force make a conceptual field of its own, very important for physics, but also important for mathematics in the development of such concepts as measure, function, dimension ... For instance, it has been shown by Laurence Viennot that the primitive conception of force, as proportional to speed, is still alive among University students and even, in tricky situations, among highly educated physicists.

. Classes, classifications and boolean operations: constitute another important conceptual field, related to other fields, but having its own specificity. It also develops from infants' first categorizations to the mastery of boolean operations and inclusion relationship.

5. Representation and the problem of adequacy between signifier and signified.

Let me recall, in Table I, the main relationships involved in additive structures and the three criteria that give account of the differences between the first three cases.

- Table I around here -

(Ed note: Table I is on page 50)

I have used distinct symbols to represent combination of measures, transformation of a measure, comparison of measures, and to represent natural numbers (measures) and directed numbers (transformations, comparisons and other static relationships such as debts and abscissas).

This choice facilitates communication. If I had used algebraic expressions, some information would have been lost. All these relationships and problems can be represented by equations in $\mathbb{Z}$ (or $\mathbb{R}$) but at the cost (and profit) of identifying different mathematical objects to one another.

- identifying natural numbers to positive numbers (and decimal and real measures to $\mathbb{R}_+$).

- identifying the sum of measures, the application of a positive transformation, the combination of transformations, the combination of static relationships, and the inversion of a negative transformation to the same binary law of combination in $\mathbb{R}$ and its signifier "+".

- identifying the application of a negative transformation, the difference between measures, between states, or between transformations, and the inversion of a positive transformation to the same minus operation in R and its signifier "-".

- identifying the equality sign to different meanings: is the same element as, outputs, is equivalent to.

This is all very good and necessary, but under which conditions, which explanations, and when?

The three criteria used in Table I to differenciate cases I, II and III show that case II is characterized by a dynamic aspect, the presence of a unary positive or negative operation and the presence of a part-whole relationship between the initial and final states. If case II is the primitive model of addition and subtraction for children, as most empirical results show, then one can expect some misunderstandings and some difficulties in the handling of other cases.

For instance, the equality sign may have for students a different meaning from the mathematician's one, which is mainly shaped by case I (internal law of composition in N).

By the way, Table I shows that, although distinct, three criteria do not make eight cases but only three: this comes from the fact that they are not independent. Still all three of them are useful tools to understand the different difficulties met by students.

But let me give more examples:

<u>Example 9</u>: Peter has 5 marbles. He plays a game with friends and wins 7 marbles. How many marbles does he have now?

<u>Example 10</u>: Robert has just lost 7 marbles. He counts his marbles and finds 5. How many marbles did he have before playing?

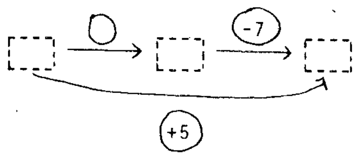
<u>Example 11</u>: Thierry has just played two games of marbles. At the second game he has lost 7 marbles. When he counts his marbles in the end, he finds that he has won 5 marbles altogether. What has happened at the first game?

First, it is important to know that, although all three problems can be solved by the addition 5 + 7, example 10 is solved about 1 or 2 years later than example 9, and example 11 is failed by 75% of 11-12 year-olds.

This can be explained by the fact that example 9 is a "find-the-final-state-problem" (case II)

whereas example 10 is a "find-the-initial-state-problem" (case II).

and example 11 is a "find-the-first-transformation-problem" (case IV)

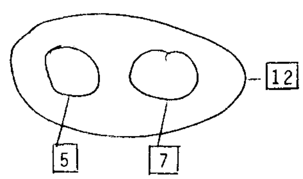


I will illustrate with three different symbolic systems (equations, arrow diagrams, Euler-Venn diagrams) some ambiguities of the solutions that can be offered by students:

Example 9 (see above)

$$5 + 7 = 12$$

$$5 \xrightarrow{+7} 12$$



All three representations can be accepted and represent both the problem and the procedure to solve it.

Example 10 (see above)

problem

$\square - 7 = 5$

$\square \xrightarrow{-7} 5$

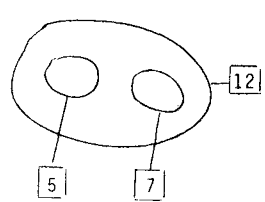
? ? ?

No Euler-Venn representation of negative transformations

procedure

$5 + 7 = 12$

$5 \xrightarrow{+7} 12$

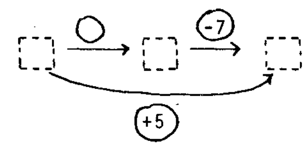


The symbolic representations of the problem and the procedure are different. Most students, at the elementary school level, represent the procedure and not the problem. The Euler-Venn symbolism is still worse because it does not allow the representation of negative transformations.

Example 11 is still more difficult to handle in Euler-Venn symbolism, and the two correct representations of the problem are either the equation in Z

$$x + (-7) = (+5)$$

or the arrow diagram:



The solution that students usually write

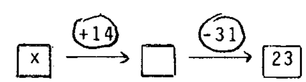
$$5 + 7 = 12$$

(when they find it) has nothing to do with the representation of the problem.

Here is a last example:

<u>Example 12</u>: Janet has played marbles in the morning and in the afternoon. In the morning she has won 14 marbles. In the afternoon she has lost 31 marbles. When she counts her marbles in the end, she finds 23. How many marbles did she have before playing?

The arrow-diagram of the problem is the following



Among the solutions, right or wrong, that I have found, I will cite four of them:

A. 23 + 31 = 54 - 14 = 40

B. 23 + 31 = 54
   54 - 14 = 40

C. 31 - 14 = 17 + 23 = 40

D. 14 - 31 = 17
   17 + 23 = 40

Solution A exhibits a treatment of the problem which is quite fair: starting from the final state, adding what has been lost and subtracting what has been gained. But the writing violates both symetry and transitivity of the equality sign.

Solution B does not violate any property of the equality sign. It may be considered as better. But it is essentially the same procedure as A. Like A it represents the procedure and not the problem. The equality sign is probably taken as an output symbol, not as an equality relationship.

Solution C leads to another comment. Although it also violates symetry and transitivity of the equality sign, this procedure consists of steps which are different from those used in procedures A and B: it combines transformations first, and then applies the result of the combination to the final state.

Solution D, which is essentially the same as C, shows a new mistake 14 - 31 = 17. But if you think of it, it is not a mistake in the context of the particular procedure: the problem is to find the difference between two transformations, one positive (+ 14) and one negative (- 31). Is the student right to be happy with his solution? My answer is yes.

So one can see the many sources of decalages between signifiers and signified in additive structures and problems.

Some symbolic systems are quite unable to represent problems that imply certain relationships. Some symbolic systems are not likely to help students to distinguish between the repre-

sentation of problems and the representation of solutions. And finally some symbolic systems may convey meanings that stand far away from the mathematical standards. One big problem for research on teaching is "how can we fill the gap, or help students to fill it?

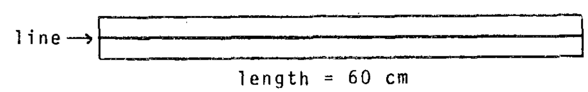
I will now turn to a different example, which is clearly related to additive structures, although it is more important in geometry and graphs: the real number line.

The real number line considers numbers as dots on a line, positive on the right hand side and negative on the left hand side of an origin called 0. This is clearly a symbolic system. What operations of thinking does it require from students to read it and use it?

We have made experiments on this in Paris. The report has not yet been published and only a short description has appeared (Vergnaud and Errecalde, 1980).

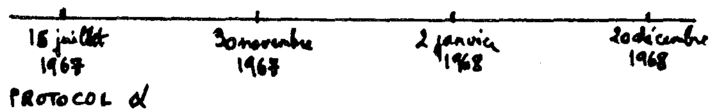
I will only mention the main aspects of our results.

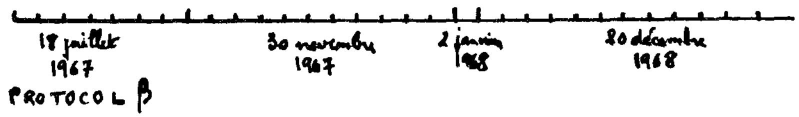
Having to place numerical data (weights, distances) or quasi-numerical data (ages, dates of birth) on a long strip of paper, with a line in the middle



students from 10 to 13 meet many kinds of difficulties. We collected more than 600 protocols, and not less than a dozen criteria led to 50 or 60 different categories.

Let me start with a few questions about two protocols concerning dates of birth.



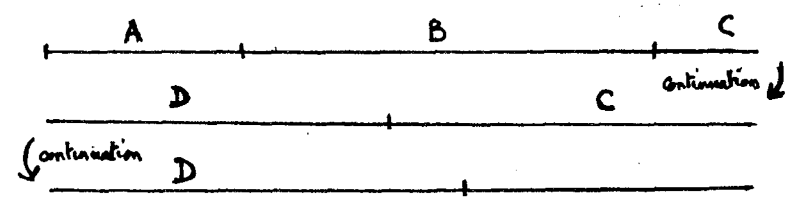


The author of protocol α is satisfied with ordering regularly dots representing the dates, taking no account of the different durations between dates.

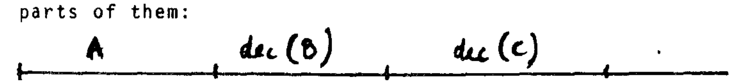
The author of protocol β represents dates as segments, the length of each segment corresponding to the rank of the month in the year. Segments are placed end to end (no inclusion relationship between durations). The day and the year are ignored. There is a confusion between event and duration and between the ordinal and cardinal aspects of the data.

Which protocol is nearer the final concept of scale or number line? It would be hard to say because none of them is close to the final concept, which requires the synthesis between the concept of order and those of distance and interval. But I consider protocol β as a very first attempt to take durations into account. There would not be much to say about all this, if protocols α and β were just anecdotic exceptions. But nearly one third of 10-year-olds' protocols are in categories similar to α and β.

A very important obstacle for students of that level consists in the inclusion principle. The end-to-end protocols illustrate the principle: "distinct signifiers for distinct signified", or else "no inclusion between excluded quantities". When they represent weights a, b, c, d ... of newly-born babies, many students produce such protocols as the following:

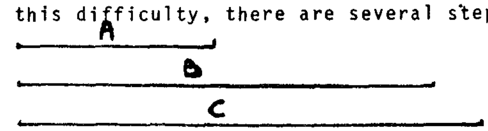


or they may cut the data and place end-to-end only the decimal parts of them:

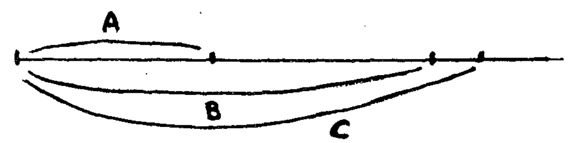


To overtake this difficulty, there are several steps

either



or

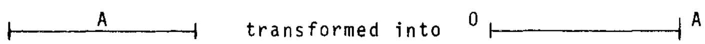


or

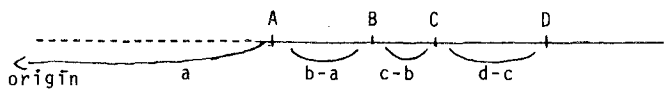
The last protocol is ambiguous because B is just above the segment that represents the difference b-a (and not b). The same applies to C.

At this stage, there are still some steps to go.

One of these steps consists of the identification of segments with the right hand extremity, the left hand extremity being identified as the origin.

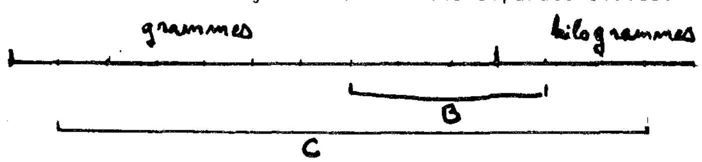


Another step consists of reasoning on dots and intervals whatever the origin may be, even if it is not present on the strip of paper.

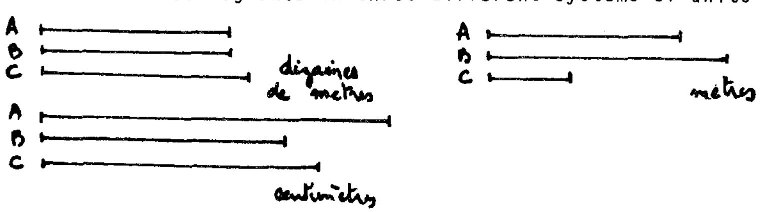


In two of the tasks used, the choice of the data and the scale did not make it possible to place both the origin and the data on the same line. So students were obliged to change origin and/or reason on differences (b-a, c-b, d-c ...). Most of them were unable to do so, even after five or six lessons on scales.

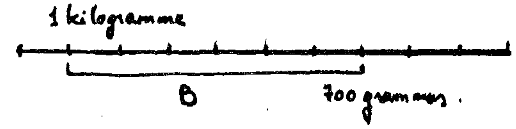
Other protocols show that the coordination between two systems of units (meters and centimeters, kilogrammes and grammes, years and months) also raises big difficulties. Some protocols represent the same data by two dots on two separate scales.



or even three segments in three different systems of units



or even two dots on the same scale (reading it with one unit above, and with the other unit beneath).



In summary, reading and using a symbolic system like scales, directly related to the very important mathematical concept of "number line", requires several operations of thinking that children and young adolescents do not find natural at all.

Whereas it is fairly natural for children to represent ordered magnitudes by ordered dots on a line, or magnitudes by separate and distinct segments, it is most difficult for them to coordinate both points of view and to accept the inclusion principle: OA does not represent only a, but also parts of all other data

$$a < b < c \implies OA \subset OB \subset OC$$

even when O does not appear on the line.

6. Problems of methodology

Methodology is usually important in the emergence of a new scientific field. This is also the case for research in maths education. Psychology provides us with some sufficient tools, for example clinical and critical interview techniques, designed experiments and questionnaires.

It is fair and very important to use these methods as often as possible. They can provide reliable and analysable information. But they probably miss an essential point: the description and analysis of the process taking place in the class-room, where interaction with new situations and new objects on the one hand, interaction with the teacher and the other students on the other hand, make the picture more complex and more evolutive.

This paper is already very long, and I don't want to repeat well-known advantages and defects of interviews, questionnaires and designed experiments. All of them are useful techniques. Each of them can be used, according to the question raised. Interviews are better for the understanding of students' conceptions, designed experiments for reliable comparaisons and for the control of the comparative importance of different factors, questionnaires for large-scale assessments.

I would rather insist upon the most specific method of didactics: experimenting in the class-room. This is not an easy job. It requires a long preparation, a good-sized and well-trained group of teachers and observers and costly registration devices.

A time-consuming part of the work consists of reading the recorded tapes.

What is the most important, still, is the necessity to take steps to get reliable information. Many observers in the class-room only see anecdotes, the importance of which can be hardly estimated. The challenge is to promote didactic experimenting in the class-room as a reliable method providing repeatable facts.

I would like to stress three important needs:

1. the need to make as explicit as possible the cognitive objectives of the sequence of lessons undertaken;

2. the need to dismiss carefully beforehand the choice of the situations, the reason for sequencing them so and so, the modalities in which information is given and the question asked, the numerical values of the variables, the symbolism used and the explanations that should be provided;

3. the need to make explicit hypothesis about the behavior of students and the events that might happen.

One cannot observe well what one is not prepared to observe. This presupposes that the contents, and the situations through which these contents are conveyed, is clearly analyzed beforehand so that one may be prepared to "see" the meaning of events and behaviors observed. It would be unrealistic to expect to reach a high degree of reliability and repeatability immediately. Psychologists were faced with the same problem when they started experimenting and interviewing: they were also faced with the extreme variety of subjects' behavior. But when interviewing subjects, you notice that regularities appear, and the more subjects you see, the stablest the different patterns of behavior appear to be.

Researchers are not in a position to give many examples of repeated experiments in didactics. But it is a fact that, when sequencing and observing the same series of lessons in different class-rooms, with the same teacher or with different teachers, at different levels or at the same level, one can observe that some events happen and happen again and the same coherently and hierarchically organized behaviors appear and appear again.

Errors, procedures, spontaneous explanations and formulations, ways of designating and representing things are most enlightening.

The specific and rewarding aspect of the method is the observation of the process of discovery by students, of the contradiction between their initial conception and the situation to handle, of the conflict between different students' points of view (working in groups of three or four), of the evolution of conceptions and procedures.

I can't see how we would be able to develop a scientific approach of maths education without experimenting in the class-room, under conditions that make this experimentation as scientific as possible.

The cost is heavy, and many improvements have still to be found. But when you have to pay the price, the only thing to do is to pay the price. Experimenting in the class-room is an unescapable issue.

Conclusion

My conclusions have already been given all along this paper. I will not repeat them again. I can only insist once more upon the analysis of the mathematical contents involved in situations. This analysis is essential. I also want to stress the importance of a behavioral and developmental approach of learning, and the importance of understanding students' actual conceptions, behind their behaviors and verbal explanations. I finally think that one must try to avoid all kinds of schematism, because the landscape is complicated, and because meaning and understanding cannot be handled without taking into account metaphors, misunderstandings and the strange relationship between words and meanings, between signifiers and signified.

I have sometimes been told by North-American friends that they were surprised by the "struggle for theory" in France. We may "struggle" too much. But some examples I have given are convincing examples and show that symbolic representations are really a crossing point in maths education research.

REFERENCES

BROUSSEAU, G. (1981) Problèmes de didactique des décimaux. Recherches en Didactique des Mathématiques, 2, 37-125.

CARPENTER, J.P., MOSER, J.M. and ROMBERG, T.A. (Eds) (1981) Addition and Subtraction: a cognitive perspective. Hillsdale, N.J., Lawrence Erlbaum.

FUSON, K. (1981) An analysis of the counting-on solution procedure in addition. In: Addition and Subtraction: a Cognitive Perspective, Carpenter, T.P., Moser, J.M., Romberg, T.A. (Eds), Hillsdale, N.J.: Lawrence Erlbaum.

VERGNAUD, G. (1981) A classification of cognitive tasks and operations of thought involved in addition and subtraction problems. In: Addition and Subtraction: a Cognitive Perspective, Carpenter, J.P., Moser, J.M., Romberg, T.A.(Eds), Hillsdale, N.J.: Lawrence Erlbaum.

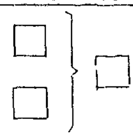
VERGNAUD, G., ERRECALDE, P. et al. (1980) Some steps in the understanding and the use of scales by 10-13 year-old students. Proceedings of the Fourth International Conference for the Psychology of Mathematics Education. Berkeley, Ca: University of California, 285-291.

VERGNAUD, G. (in press) Multiplicative structures. In: Acquisition of Mathematics Concepts and Processes, Lesh, R., Landau, M. (Eds), Academic Press.

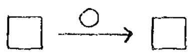
VIENNOT, L. (1979) Le raisonnement spontané en dynamique élémentaire. Paris: Hermann.

TABLE I

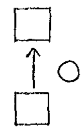
Main additive relationships



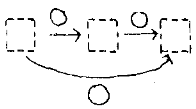
I Combination of measures



II Transformation of a measure

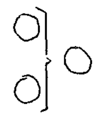


III Comparison of measures



IV Combination of transformations

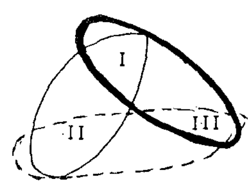
V Transformation of a relationship



VI Combination of relationships

Criteria differentiating cases I, II and III

- presence/absence of a dynamic aspect
  II against I and III
- absence/presence of a directed relationship
  I against II and III
- absence/presence of a part-whole relationship (inclusion)
  III against I and II



Working Group A

THE INFLUENCE OF COMPUTER SCIENCE ON UNDERGRADUATE MATHEMATICS EDUCATION

Leaders: Bernard Hodgson, Université Laval
Tony Thompson, Dalhousie University

In the last twenty years, there has been considerable attention given to the general role played by mathematics in cumputer science and to the kind of mathematics required in an undergraduate computer science curriculum. But little seems to have been done about the influence in the other direction: what effect on undergraduate mathematics education does or should the recent development of computing and computer science have? For example, should the traditional calculus-linear algebra sequence which was developed for the needs of Physics and Chemistry still be considered as the starting point for a mathematics curriculum? Could discrete mathematics be considered as either an alternative or as a preferable basis? More generally, to what extent is the "finite" character of the computer a motivational source for changes in the traditional "infinite" approach to the education of mathematics majors? With the discussion of these and other related issues, this Working Group aims at identifying some general trends in the subject while focusing on the actual situation in Canadian universities.

WORKING GROUP A

The Influence of Computer Science on Undergraduate Mathematics Education

Bernard R. Hodgson
Tony Thompson

List of participants:

Lee ADLER	(Concordia)
Ed BARBEAU	(Toronto)
Josef BRODY	(Concordia)
Bernard R. HODGSON	(Laval)
Peter MANUEL	(Western Ontario)
John POLAND	(Carleton)
Fran ROSAMOND	(Cornell)
Peter TAYLOR	(Queen's)
Tony THOMPSON	(Dalhousie)
Charles VERHILLE	(New Brunswick)

The Undergraduate mathematics curriculum generally, but especially the Calculus sequence in the first two years has remained remarkably constant over the last 15-20 years. This is in spite of substantial changes in the high school curriculum (most Provinces include a large introduction to Calculus in the final year), the very widespread use of calculators, and now the rapid deployment of microcomputers.

It appears that this last will bring significant changes. This is because (a) there is now a large body of scientific users of mathematics (the Computer Scientists) with different problems and different needs from the traditional "continuous" approach needed in Engineering, Physics and Chemistry. (b) Computer Science is generating genuinely new problems for mathematics which are unlikely to be solved by Calculus methods; (c) Computer Science challenges and competes with traditional mathematics for students; (see Appendix 2); there is a danger that mathematics departments could follow philosophy departments into "prestigious isolation and irrelevance" (W.F. Lucas in Mathematics Tomorrow p. 68). Clearly the interests of students seeking a good education are not best served by the mathematics department ignoring Computer Science nor vice versa. There is urgent need for dialogue to find a good new approach to the two subjects together. (see Appendix 1 by Lee Adler). The Working Group began by recognizing this situation. The question then is not whether some changes are necessary but what those changes should be. The discussion of this question began with an examination of the article <u>Computer Science, Mathematics and the Undergraduate Curriculum in both</u> by A. Ralston Amer. Math. Monthly, 88 (1981) 472-485. The group found Ralston's suggestions unsatisfying. A brief summary of the views of the group are found in Appendix 3 by John Poland.

Before being able to offer alternatives to the Ralston proposals the group found it necessary to discuss the broad aims of mathematics education. Though the conclusions were rather too general to be very useful - the aims are to develop <u>critical thinking</u>, to develop a logical approach to problem solving, to understand 'why' rather than 'how' -

they did reinforce the view that the traditional first-year Calculus course needs change. Since too often it is training in a long sequence of skills rather than in understanding and critical thought.

The group thought that Ralston did not avoid this 'how to' approach. Indeed his long list of topics, which would necessarily be covered superficially, encouraged it. The group decided that to avoid this the first year course should be based on a general theme rather than lots of topics.

Several themes were suggested: Algorithms, Optimization, Approximation, Applied Problems. In the limited time available, individual group members worked on these themes and gave suggestions as to how they might be developed in practice.

These are contained in Appendices 3, 4, and 5. These form the main outcome of our deliberations.

It was realized that the nature of a course, especially the emphasis on 'how to', is very often determined by the examination given at the end. Student attitudes are certainly very dependent on this. Ed Barbeau developed some interesting alternatives to the usual examination and these are given in Appendix 6.

Throughout the discussions it was emphasized (mainly by Joe Brody and Peter Manuel) that University mathematics departments have not yet come to terms with the Calculator, let alone the Computer. Dr. Brody gave four reasons for making heavy use of calculators in his Calculus classes

(1) Replaces the tedious
(2) Allows one to do new things
(3) Clarifies the underlying theory
(4) Makes one think in new ways

and suggested that the Computer will do the same (except that its very speed and power and the fact that one is not aware of all the intermediate steps can be disadvantages). These views are elaborated in Appendix 7.

Throughout the discussion the participants made reference to articles and books they had found stimulating. These are gathered in the bibliographies in Appendices 7,8 and 9. There is a little overlap (surprisingly how little!) but I thought it not worthwhile to edit them into a single list. I have also appended Ralston's bibliography as Appendix 10.

Finally, I was asked to include an article by Jules Gribble and my views on a combined mathematics, statistics, Computing Science department. These are the last two Appendices.

Tony Thompson
28/6/82

Working Group A Report:

Appendix 1 June 15, 1982

This note is meant to serve as an appendix to the CMESG report from the Working Group on the Influence of Computer Science on Undergraduate Mathematics.

Computers affect all of our lives and probably will have an increasingly larger impact on the lives of people who have studied mathematics than they have on the lives of the population at large. I believe that it is our responsibility to make sure that our students learn to compute; then, since our students have computation skill, to put into our courses some of the mathematical ideas which generate these techniques.

To solve or to mathematize problems which are formulated by others, it is necessary but not sufficient to have a broad mathematical repertoire. Mathematicians rarely work alone, usually form part of a team, and are required to work on problems articulated by people with limited mathematical training but experience in computing or data processing. Many companies think of computing as mathematics and vice versa, and they put no clear line between the two. Opportunities for interaction between mathematics and its applications in industry or government are often affected by computational techniques of increasing levels of mathematical sophistication. To compute at all students have to know how to use the computer correctly and efficiently; so they probably should know a few

programming languages, the principles of programming, some basic data structures and techniques for manipulation in these data structures. Therefore, so that mathematics students know some computer science, they should be counselled to take computing courses. Computer science thus should influence the choice of electives of students in undergraduate mathematics programmes.

Mathematics students who have taken computer courses have usually enjoyed those courses and have acquired skills and techniques. Including in our courses those mathematical ideas which unify many of the methods learned in computer courses would allow students to see the theory from which at least some of their expertise follows almost automatically, and would give them examples of the applicability of mathematics to the "real" world. It has been said that the particular choice of topics in a course is often less important than the opportunity for students to learn to do something difficult, have fun learning it, and thereby gain the confidence to tackle another formidable task. In particular, for those students who are in search of an education, have always enjoyed mathematics courses, may someday find a job due to their functional expertise in mathematics, but whose careers will most likely evolve along more general lines, a mathematics programme offers opportunities to: stretch their imagination and intuition; practise logic, precision and accuracy of thought; and learn the rewards of patience and perseverance. Unfortunately many students can justify choosing a programme less rigourous than mathematics based on the excuse that teaching is the only job open to a mathematician. We must remind our own colleagues and assure our students that this claim is not true. Linking computer techniques to mathematical theory would be one operational way of providing the counter-evidence. In addition this coupling could harness some of the synergistic energy which comes from teaching material which is on the periphery of things which are fun to do, and use it to attract more students into mathematics programmes.

In conclusion, I would like to recommend that the CMESG Study Group try to find what computer knowledge our students should have, identify the mathematical ideas which generate this knowledge, debate whether this mathematics should be included in our curriculum, and, if so how and at what point.

L.S. Adler
Concordia University

LES ÉTUDIANTS DES SCIENCES MATHÉMATIQUES AU PREMIER CYCLE UNIVERSITAIRE

Bernard R. Hodgson

Département de mathématiques
Université Laval

Comme le souligne Tucker [6, p. 40], la valorisation des études en mathématiques et en sciences provoquée par le lancement de Spoutnik en 1957 a eu comme effet une augmentation considérable des effectifs étudiants en mathématiques; ainsi, aux Etats-Unis, le nombre de diplômés en mathématiques au premier cycle universitaire était de 23 000 en 1970 contre 5 000 en 1956 (la proportion était semblable aux 2e et 3e cycles). La tendance semblait telle que le CBMS n'hésitait pas, en 1970, à prédire que le nombre de finissants serait, en 1975, de 50 000. C'était sans compter avec le ressac anti-science provoqué aux Etats-Unis par la guerre du Vietnam: il n'y eut en fait que 18 000 grades de premier cycle décernés en mathématiques en 1974-75!

L'étude récente du CBMS [2] fournit des renseignements intéressants sur la situation des diverses sciences mathématiques aux Etats-Unis. Ainsi (voir [2, p. 11]), en comparant les diplômés de 1974-75 et ceux de 1979-80, on constate une diminution de 42% en mathématiques et en statistique contre une augmentation de 145% en informatique, de sorte que près des 2/5 des finissants en sciences mathématiques en 1980 étaient des informaticiens (voir Tableau 1 ci-dessous). De plus, alors qu'en 1966, 4,5% des "freshmen" indiquaient vouloir se spécialiser en mathématiques, ce pourcentage n'était que de 0,6% en 1980; cette même année, 4,9% des "freshmen" donnaient comme spécialisation probable l'informatique.

Le tableau suivant donne, pour les Etats-Unis, le nombre de grades décernés au premier cycle universitaire en sciences mathématiques (voir [2, p. 30]).

TABLEAU 1

	1974-75	1979-80
Mathématiques	17 713	10 160
Statistique	570	467
Informatique	3 636	8 917
Actuariat	70	146
Mathématiques appliquées	886	801
Enseignement secondaire	4 778	1 752
Autres	164	580

Au Canada, on ne semble pas disposer de statistiques aussi détaillées qu'aux Etats-Unis sur les étudiants des sciences mathématiques. Qui plus est, certains des renseignements disponibles de différentes sources ne paraissent pas concorder (par exemple [1, Tableau A.2] et [3]). Néanmoins, il semble permis de croire que certaines des tendances rencontrées aux Etats-Unis seraient les mêmes ici.

Le Tableau 2 donne, pour le Canada, le nombre de grades décernés au premier cycle universitaire en sciences mathématiques; les données de la partie (A) proviennent de [1, Tableau A.2], celles de la partie (B) de [4] et celles de (C) de [5].

TABLEAU 2

(A)	1960	138	
	1965	263	
	1970	699	
	1971	911	
	1972	1 181	
	1973	1 138	
(B)	1974	2 415	
	1975	2 353	
(C)	1976	2 223	
	1977	2 422	
	1978	2 423	
	1979	1 663	mathématiques (pures, appl., stat., act., r.o.)
		1 009	informatique

D'après les renseignements que j'ai recueillis auprès de Statistique Canada, ce n'est qu'à partir de 1979 que la classification a été modifiée pour isoler les étudiants d'informatique, de sorte qu'il ne semble pas possible de distinguer davantage le champ de spécialisation pour les années antérieures à

1979. (On trouve dans le Tableau C.1 de [1] des projections, par sous-champ de spécialisation, pour les années 1974 à 1977; mais il apparaît difficile de contrôler ces projections.)

On peut s'attendre à ce que cette évolution de la clientèle étudiante des sciences mathématiques aura un impact certain sur les départements de mathématiques et leur corps professoral. L'étude du CBMS commente en ces termes [2, p. 34]:

> "Taking numbers of course enrollments as a measure, the mathematical science departments are currently prospering. Reasonable projections suggest that this prosperity will continue into the near future. However, the pattern of enrollments is far from optimal for the preferences of most faculty -- with the decline in advanced mathematics students and increase of less attractive, lower level courses. Those students, greatly reduced in number, who continue to elect a mathematics major are concentrating in applied areas, statistics, and computing which are not the specialties of most current faculty. The decline in numbers of potential secondary school mathematics teachers is also an ominous sign for the long-term improvement of school mathematics."

Références

1. K.P. Beltzner, A.J. Coleman et G.D. Edwards, Les sciences mathématiques au Canada, Conseil des sciences du Canada, 1976. (Etude no. 37)

2. J.T. Fey, D.J. Albers et W.H. Fleming, Undergraduate Mathematical Sciences in Universities, Four-Year Colleges and Two-Year Colleges, 1980-81, Conference Board of the Mathematical Sciences, 1981. (Report of the Survey Committee, volume VI)

3. Grades, diplômes, certificats décernés par les universités et collèges (1970, 1971), Statistique Canada, Catalogue 81-211 annuel.

4. Grades, diplômes et certificats décernés par les universités (1974, 1975), Statistique Canada, Catalogue 81-211 annuel.

5. Universités: inscriptions et grades décernés (1976 à 1979), Statistique Canada, Catalogue 81-204 annuel.

6. A. Tucker, Redefining the mathematics major, in: L.A. Steen, éd., Mathematics Tomorrow, Springer-Verlag, 1981, pp. 39-48.

Appendix to "Influence of Computers on Undergraduate Mathematics Education".
Canadian Mathematics Education Study Group, 1982 Annual Meeting, Kingston, Ontario.
By John Poland, Department of Mathematics & Statistics, Carleton University, Ottawa, Canada.

How can the first (and second) year university mathematics courses use the momentum of the hype of the "computer age" to swing more students into an appreciation for the relevance, power and importance of the mathematical perspective? This judo approach to curriculum revision would be more intelligent than simply ignoring or fighting against the convictions of our young students that computers are the new frontier. Here are some possible alternatives:

(1) The Ralston First Course: the following is an outline of the detailed syllabus appearing in A. Ralston "Computer Science, Mathematics and the Undergraduate Curricula in Both", Technical Report 161, Department of Computer Science, SUNY at Buffalo, 1980. The course is broken into twelve units:

Ralston's First Year Discrete Mathematics Course:

- (i) Basics (Σ-notation, sequences and series,...)
- (ii) Algorithms and their analysis (basics about what it is)
- (iii) Logic (Propositional calculus, boolean algebra, predicates, relevance to the verification of algorithms)
- (iv) Limits (convergence, limits of discrete functions, series)
- (v) Induction
- (vi) Discrete number systems (fixed and floating point number systems, number bases)
- (vii) Combinatorial analysis (permutations, combinations, binomial theorem, Stirling)
- (viii) Difference equations and generating functions
- (ix) Discrete probability (random number generators, basic queuing theory, analysis of algorithms)
- (x) Graphs and trees (tree enumeration, binary trees)
- (xi) Iteration and recursion
- (xii) Basic Automata theory (finite state machines, Turing machines)

Critique:

Scattered topics, treated superficially? The outline has the appearance of the first, "background" chapter of a textbook, outlining the definitions and notations to be used in the coming development. The power and importance of mathematical ideas would be difficult to elucidate - in fact, no topic would have the time to be explored in depth. For example, generating functions are notoriously difficult to grasp; without any exploration of their use and power, a student entering university would end up frustrated and unappreciative of the concept of a generating function.

However, the text:

Alan Tucker "Applied Combinatorics" John Wiley & Sons, 1980, does generating functions early, with many later applications, covers many of the Ralston topics, and has an excellent problems-oriented style. Worth exploring.

(2) A more narrow approach, covering about half of the topics of Ralston First Course but very basic to much of the mathematical aspect of algorithms, could be developed by using the enjoyable and accessible text:

G. Berman and K.D. Fryer "Introduction to Combinatorics" Academic Press 1972

(3) A more purely algorithmic approach could be devised around two basic topics

(a) Algorithms for sorting and searching
(b) The mathematical model of a computer

The idea of topic (a) would be to bring to the first-year level Chapters Two and Three (Design of efficient algorithms, and Sorting and order statistics) of: A.V. Aho, J.E. Hopcroft and J.D. Ullman "The Design and Analysis of Computer Algorithms" Addison Wesley 1974. For example: (a1) Sorting - various situations; what it is; what ways can you think of to sort?

(a2) Notion of an algorithm, some sorting algorithms; use of graph theory to discuss them.

(a3) Comparing sorting techniques: which is better, what does better mean? introduce $o(n)$ and $O(n)$.

(a4) The mathematics of comparing and calculating finite sums; approximations; asymptotic behaviour, Stirling's Formula.

The idea of topic (b) may be easier but more abstract. For example (b1) Turing machines and some basic things they can do.

(b2) More complicated calculations on a Turing machine
(b3) The universal Turing machine and the unsolvability of the halting problem
(b4) Other decision problems - an outline
(b5) The number of Turing machines: countable versus uncountable
(b6) There are only countably many computable reals.

I have taught all these latter topics to high school students with great success.
Reactions of some of my colleagues has been:

(c1) Topic (a) is too close to what is taught in computing science courses; there is too much overlap. But others have claimed that only the algorithms appear in the computing science course, not the detailed efficiency comparisons. This should be examined.

(c2) The concepts $o(n)$ and $O(n)$, and the comparison of algorithms, is too difficult and narrow a topic to present to students in their first year at university. Most hardly know anything about computers and algorithms.

(4) Another approach could be to modify present calculus or linear algebra courses to include more computing-related topics:

G. Strang "Linear Algebra and its Applications", Academic Press, 1976, is an exciting read in applied linear algebra with computational discussions (including the stability of the algorithms, and a note on LINPACK, the linear algebra computer package, in an appendix to the second edition).

Warner Steinberg and Robert J. Walker "Calculus: a computer oriented approach", an experimental textbook produced by the Center for Research in College Instruction of Science and Mathematics, Florida State University, 1968, begins with the flow-chart approach to computers, approximating solutions of equations, convergence, basic results (including the Squeeze Theorem) which lead to finding areas and volumes by approximating; then integration and into differentiation, with many numerical and algorithmic ideas.

Thomas Wonnacott "Calculus: an applied approach" John Wiley & Sons, 1977, begins with approximating and defining e and has much interplay between the differencing situations and the differentiation situation.

Finally, I would like to add that finding ways in which computers are helpful, in our mathematics courses (as in linear algebra or calculus), may not be a significant indication of the usefulness of a mathematical perspective to computer-bound students. Instead, it may be more useful to search for topics, like the analysis of sorting algorithms or more interactive computer-and-mathematics problems, in which mathematics offers a mathematical perspective on computing. "Ask not what computer science can do for you; ask what you can do for computer science"!

DECREASING THE EMPHASIS ON CALCULUS IN THE FIRST YEAR

E.J. Barbeau

Although calculus should continue to be part of the first year programme in mathematics, it should not be the focus. Rather, various themes might be chosen which involve mathematical ideas and techniques from many areas including calculus. Three possibilities are approximation, optimization and evolution of systems. For these, I will sketch out some topics which might be covered; some of these will require a considerable amount of background material.

A. APPROXIMATION

(a) Approximating values of functions: distinguishing between linear and nonlinear approximations,
using interpolation and extrapolation formulae (finite differences),
discussion of appropriateness of techniques used,
use of the Mean Value Theorem of differential calculus to give an approximate value, estimate possible error and determine whether an over- or under- estimate is found,
Taylor expansion with remainder.

(b) Approximating solutions of equations: use of Intermediate Value Theorem for continuous functions (method of bisection), analytical and graphical treatment of Newton's Method, example to illustrate possible behaviour of successive "approximations", conditions for covergence of successive approximations.

Hensel's Lemma - an adaptation of Newton's Method for solving polynominal congruences (Let p(x) be a polynominal over Z, q be a prime. Thinking of a number as being "close" to zero if it is divisible by very large numbers, we can approximate roots of p(x) by solutions of $p(x) \equiv 0$

(mod N) where N is large. By the Chinese Remainder Theorem, it is enough to look at N equal to prime powers. Hensel's Lemma gives us a technique for solving the congruence for successively large powers of a given prime. Thus, let u be a solution of $p(x) \equiv 0 \pmod{q^r}$ for which $p'(u) \not\equiv 0 \pmod{q}$. Choose w so that $w\,p'(u) \equiv 1 \pmod{q^r}$. Then $u - wp(u)$ is a solution of $p(x) \equiv 0 \pmod{q^{2r}}$. Proof: Write $p(x) = p(u) + (x-u)p'(u) + (x-u)^2 f(x)$ where f(x) is a polynomial over Z.)

(c) Using integrals to approximate sums: the sum of the first n kth powers, Stirling's formula (A.J. Coleman, Amer. Math. Monthly 58 (1951), 334-336 = Selected papers on calculus, p. 325)

(d) Approximate integration: estimates of log 2, π

(e) Use of special approximating functions: piecewise linear, polynomials (Weierstrass Approximation Theorem, Berstein polynomials Lagrange polynomials), trigonometric polynomials (statement of Dirichlet's Theorem; Gibb's phenomenon)
spline functions

(f) Least squares regression

(g) Approximation in mean

(h) Functions of more than one variable

B. OPTIMIZATION

(a) Statement of the extreme value theorem for continuous functions

(b) Use of differential calculus: tests for identifying whether exterme value of a single- or multiple- variable function is maximum or minimum - use of second derivatives,
checking signs of first derivative, examining difference between maximum/minimum of a function and the function to see if it is readily identifiable as always positive/negative,

basic reasoning of the following type - f is continuous; it is known, say using the extreme value theorem, that it has a minimum within a certain interval; it can be seen the minimum cannot occur at the end points; hence, f' vanishes at a minimum; it is easily seen f' vanishes at a single point; hence, this must be where the minimum occurs

(c) Optimization under constraints: linear programming (preceded by Gaussian elimination and some geometric perception of polyhedra and planes which intersect these)
Lagrange multipliers

(d) Approximate optimization methods: use of lattice points
steepest descent

C. EVOLUTION OF SYSTEMS

(a) Recursive relations: $x_{n+1} = f(x_n)$, stability, fixed points, contraction mapping, graphical analysis, $x_{n+1} = ax_n + bx_{n-1} + \dots$, difference equations,
characteristic polynomials

(b) Differential equations: exponential growth and decay
logistic growth
qualitative analysis of solutions using graphical techniques (isoclines)
solutions in closed form, separation of variables, linear first order homogeneous and non-homogeneous equations
approximate solutions, polygonal method

(c) Financial mathematics: bonds, annuities, sinking funds
life contingencies, Gompertz' Law

(d) Kepler's Laws derived from Newton's Laws (O. Toeplitz, The Calculus: a genetic approach)

It can be seen that a thorough treatment of these topics involves not only calculus, but a considerable amount of algebra. The aim is to develop a sound intuition and flexibility in thinking; proofs are given only insofar as they advance this aim. There is quite a bit of scope for the use of a calculator.

The Second Semester of Calculus

In the 1981 CUPM report on the Mathematical Sciences, the subpanel on Calculus argues that Ralston (AMM 88 No. 7 1981) has gone too far in suggesting calculus be replaced by discrete mathematics, and suggests that in the freshman year the first semester of calculus provide a basic introduction to differentiation and integration (as at present) but the second semester explore applications of calculus to discrete mathematics and modelling. I agree with this, and wish to comment on this second course.

It should present problems, hopefully from the "real world" (scientific or aesthetic), which tend to be quite naturally modelled as discrete problems. But, as so often happens, discrete methods may have insufficient power to analyze these problems, whereas if they can be modelled as continuous problems, they often yield to the methods of calculus.

Let me mention briefly three examples.

1. Generating Functions. In problems of enumeration or probability one often wishes to find a sequence (a_i) where a_i is the number of type i, or the probability of event i etc. It may be very hard to calculate the a_i directly or discover their asymptotic properties. An analysis of the real-valued function $G(x) = \Sigma a_i x^i$ often solves some of these problems. For example Knuth in the Art of Computer Programming Vol. 1, 1968, shows how G can be used to calculate the nth term of of the Fibonacci sequence 1, 2, 3, 5, 8, ...

2. Differential vs Difference Equations. In modelling the change of gene frequency under the process of natural selection, it is most natural to take a discrete time model, and write the gene frequency p' in one year in terms of the frequency p the year before. A non-linear difference equation $p'-p = sp(1-p)/(1+sp)$ is obtained which is difficult to analyze. A continuous time model leads to the corresponding differential equation $\frac{dp}{dt} = sp(1-p)$ which can be solved.

3. Sums vs Integrals. In a population let ℓ_n be the probability of living (from birth) to age n and let m_n be the number of offspring an individual can expect to have between age n and $n+1$. These

3. numbers are called life history parameters and population biologists are interested in analyzing them. For many purposes, their analysis is facilited by using a continuous variable x for age, with m_x now being the instantaneous expected birth rate at age x. For example let $E(n)$ be the expected future number of offspring of an individual of age n. At birth $E(0)$ is small due to the probability of death before maturity. For large n, $E(n)$ is small because of the effects of senility. In between $E(n)$ is large and we ask when it is a maximum. This requires analysis of the sum $E(n) = \sum_{i=n}^{\infty} \frac{\ell_i}{\ell_n} m_i$. The continuous model maximizes the integral $E(x) = {}_x\int^{\infty} \frac{\ell_y}{\ell_x} m_y \, d_y$ which calculus can easily handle.

Peter Taylor

ALTERNATIVE MODES OF EXAMINATIONS

E.J. Barbeau

Examination questions might be given to the students in advance. At the actual examination itself, no notes or aids would be allowed. The student should have assimilated the solutions sufficiently to write down in a restricted amount of time a coherent and accurate solution. One might object that the more parasitical students may simply try to memorize someone else's solution; however, it has been found that very often a student writing down something he really does not understand very often reveals this in his exposition.

Alternatively, the student might be given a passage from a book to study (perhaps a book in some applied area). At the examination, he would be presented with an excerpt and asked to fill in missing details, explain the procedure used or adapt a method to a similar situation.

A multiplicity of examinations might be available for a given course. During the course of the year, the student should develop an opinion of his ability, and on this basis can choose to write an examination of one of the following types:

Type 1: Very straightforward questions, successful completion of which gives a maximum gradt of 70%.

Type 2: Basically straightforward with some multiple-step problems, successful completion of which gives a maximum grade of 80%.

Type 3: Problems of a more penetrating type, successful completion of which gives a maximum grade of 100%.

The rationale behind this regime is to encourage the student to get some sense of his level of understanding of the material studied.

CONCORDIA
UNIVERSITY

June 16, 1982

Dr. Tony Thompson
Department of Mathematics,
Statistics and Computer Science
Dalhousie University
Halifax, Nova Scotia
B3H 4H8

Dear Tony:

I am sending the required appendix and list of references. The appendix is not self-explanatory, for it would require another page or two. It is already slightly longer than the limit of two pages required - my apology. If, in the references to this appendix, you would like to add some information, then maybe these points should be stressed.

a) In the appendix, we wanted to attempt to illustrate that there is much more to the presence of the calculator in the classroom than calculation. The calculator is a tool which may stimulate and help sustain thinking, as well as expand its spectrum.

b) The examples prevent a variety of problems picked from different areas of Mathematics, for example, great loss in number of significant digits may be fatal in,
   (i) Finite difference methods for solution of partial differential equations,
   (ii) Solution of quadratic equations, $ax^2 + bx + c = 0$, where $b \gg ac$, etc.

c) The teaching of Mathematics using the calculator has already shown an improvement in:
   (i) understanding the theory - for example, definitions, theorems,
   (ii) interest in the subject itself.

d) Since a calculator soon becomes a tool which everybody will carry as easily as a pencil, the question of re-evaluation of the undergraduate curriculum may become of crucial importance.

I believe that I have helped a little, but if at any time you think I can do something more within the aims of the study group, please do not hesitate to ask.

JB/rw
Encl.:

Sincerely yours,

P.S. If it is possible, send me a copy of the report as soon as you can. Thanks.

J. Brody
Associate Professor
Department of Mathematic

SIR GEORGE WILLIAMS CAMPUS
1455 DE MAISONNEUVE BLVD. WEST
MONTREAL, QUEBEC H3G 1M8

THE ROLE OF THE POCKET CALCULATOR IN TEACHING OF MATHEMATICS

Jozef Brody
Concordia University

We consider a few examples in order to demonstrate the various contributions a calculator can make to the process of solving problems.

EXAMPLE 1: Find the value of: $\sqrt{19820000} - \sqrt{19819999}$ to 10 significant digits (or on as many as the calculator in use permits).

EXAMPLE 2: Evaluate $(-8)^{1/3}$

EXAMPLE 3: Find the derivative y' if $y = \cos\sqrt{x^2 + 2}$.

EXAMPLE 4: A box with a square bottom and volume of $32m^3$ costs 20¢/m^2 for the top and the bottom, and 15¢/m^2 for the sides. What are the dimensions of the least expensive box?

EXAMPLE 5: Evaluate the definite integral $\int_0^4 (1-\sqrt{x})/(1+\sqrt{x})\,dx$, using the integral rutine on the calculator.

In the first example the direct calculation gives much less than the example requires (a good answer is usually $1.123\cdot 10^{-4}$). It is a nice illustration of the necessity for rationalization in order to get the required accuracy as:

$$\frac{1}{\sqrt{1982000} + \sqrt{1981999}} = 1.123099373\cdot 10^{-4}$$

In the second example one usually tries to calculate the required value using the following sequence of keys:

-8 [x^y] 3 [1/x] [=] .

Some calculators will display the error message. Why? Is $(-8)^{1/3}$ a well-defined expression? What about $(-8)^{2/6}$? Is the value the same as the one of $\sqrt[3]{-8}$?

In the third example one uses the calculator to evaluate the function and separate the value of the variable x from the actual computation. The value for x is therefore stored in the memory first : x→M followed by the computational sequence:

[RM] [x^2] [+] 2 [=] [$\sqrt{\ }$] [cos] .

This sequence is illustrated in the computational scheme in the Figure 1. The problem asks for the derivative dy/dx i.e. the change of the output variable respect to the change of the input one. The basic computational steps change the value on the display and therefore ask for different variables along (x,u,v,w and y). The basic transformations (functions) offer the following derivatives:

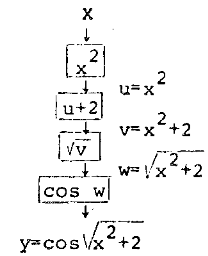


Figure 1

$\frac{du}{dx} = 2x$, $\frac{dv}{du} = 1$, $\frac{dw}{dv} = 1/2\sqrt{v}$, $\frac{dy}{dw} = -\sin w$; giving the required answer as: $\frac{dy}{dx} = \frac{dy}{dw} \times \frac{dw}{dv} \times \frac{dv}{du} \times \frac{du}{dx} = -\frac{x \sin w}{\sqrt{v}}$.

This gives a specific model for the chain rule in the differential calculus, and the proof using the limits becomes clear and obvious.

The fourth example belongs to the category of so called word problems. The first step may be a choice (estimate) of dimensions feasible with repect to the conditions (constraints) in the problem, let's say 4 by 4 by 2. To remember these one asks the student to put one of them into the memory, let's say 4 (→M). This way the dimensions can be calculated as: RM , RM , 32 ÷ x^2 , and the validity can be checked by the calculator. The total cost C is then:

C = ([RM] [x] [RM] 2 [x] 20 [+] [RM] [x^2] 32 [÷] [x^2] [x] 4 [x] 15 [=])

= 1120¢ = $11.20.

Algebraically then: $C(x) = 40x + \frac{1920}{x}$, which simplifies the computation to:

40 [x] [RM] [x^2] [+] 1920 [÷] [RM] [=]

The stress is on the concept of a function being a transformation expressed as a sequence (an algorithm) of basic transformations. Using the differentiation ($C'(x) = 0$), one gets $x = 2.8845...$. The above algorithm (after 2.8845... → M) gives the total cost C = $ 9.98 - a significant improvement to $ 11.20 .

Using the CASIO fx-180P (or fx-3600P) calculator, the required integral in its original form when executed displays the error message. However if transformed to the integral in terms of $u = \sqrt{x}$, i.e. $2\int_0^2 (1-u)/(1+u)\,u\,du$ can be evaluated using:

P1 : [→M] 1 [−] [RM] [=] [x] [RM] [÷] [(] 1 [+] [RM] [=] or

P2 : [→M] [−] [RM] [x^2] [=] [÷] [(] 1 [+] [RM] [=]

The execution of the integral using P1 takes about 35 seconds, while using the P2 only 27 seconds. However P1 used the nested multiplication and P2 does not. The question of complexity in actual computation versus theoretical ones comes to mind.

The above examples illustrate that using a calculator

1) shows the need for algebraic transformations of algorithms;

2) supports the concept development of notions and techniques ;

3) changes the way of thinking about mathematics;

4) makes one think about things one would probably hardly ever try.

Let me also include the references concerning the muMath, we discussed in Kingston :

1) The fellow,who reported the curricular changes in Toronto on the annual meeting of NCTM was:

James T. Fey from University of Maryland, College Park Maryland.

2) There was also a very interesting article in Mathematical Monthly, January 1982 on: The disk with the college education by Herbert S. Wilf.

In this article Wilf talks about a very miniature version of muMath. Much more sophisticated versions are available and used. They require more memory capacity and are a little more expensive.

REFERENCES

1. Elich J, Elich C.J. - Trigonometry Using Calculators - Addison Wesley, 1980.

2. Elich J. Elich C.J. - Precalculus with Calculator Applications - Addison Wesley, 1981.

3. Gattegno G. - Elegant and Satisfying Solutions - AEDS Monitor, Vol. 19, N. 789, Feb. 1981.

4. Glayman M. - Initiation aux Méthodes itératives et utilisation de Calculateurs avec des enfants - Bulletin A.M.Q., May 1979.

5. Green D.R., Lewis J. - Science with Pocket Calculator - The Wykeman Science Series, 1978

6. Judd W. - Games, Tricks and Puzzles for a Hand Calculator - Dymax, Marlo Park, California, U.S.A., 1974.

7. Kahan W.M. - Handheld Calculator Evaluates Integrals - Hewlett-Packard Journal, August 1980.

8. Mayer R.E., Bayman P. - Psychology of Calculator Languages a Framework for Describing Differences in User's Knowledge - Communications of the ACM, Vol. 24, N. 8, 1981.

9. Mura R. - Résolution d'équations par des procédés itératifs à l'aide de calculateurs de poche - Bulletin A.M.Q., May 1980.

10. NCTM - An Agenda for Action-Recommendations for school mathematics of the 1980's - NCTM, Reston, Va., 1980.

11. Ridge L.M. - Mathscope 1, 2, 3 and 4 - Prentice-Hall, Canada Inc., Scarborough, Ont., Canada, 1981.

12. Robert D.M. - The impact of Electronic Calculators on Educational Performance - Rev. Ed. Res. 50, 1, Spring 1980.

13. Scandura A.M., Lowere G.F., Veneski J., Scandura J.M. - Using Electronic Calculators with elementary children - Ed. Tech. 16, 8, August 1976.

14. Schultz J.E., Waits B.K. - A new look at some old problems in light of the calculators - The two-year college mathematical journal, January 1979.

15. Suydam M.N. - State of the art review on Calculators, their use in Education - Calculator Info Center, Rept. N.S., Columbus, Ohio, 1980.

16. Szetela W. - Using Calculators: What are the best Published Materials? - NCTM - Calculato Information Center - Info. Bulletin No. 12, January 1982.

17. Wertheimer M. - Productive thinking - Harper & Row, New York, 1959.

SOME ITEMS FOR A BIBLIOGRAPHY ON
"COMPUTER SCIENCE AND MATHEMATICS CURRICULUM"

Bernard R. Hodgson

Département de mathématiques
Université Laval

A general background on the subject can be found in the well-known report:

K.P. Beltzner, A.J. Coleman and G.D. Edwards, Mathematical Sciences in Canada, Science Council of Canada, 1976. (Background Study no. 37)

A recent study of the trends in the United States is presented in:

J.T. Fey, D.J. Albers and W.H. Fleming, Undergraduate Mathematical Sciences in Universities, Four-Year Colleges and Two-Year Colleges, 1980-81, Conference Board of the Mathematical Sciences, 1981. (Report of the Survey Committee, volume VI)

Some highlights of this report have been presented in the Notices of the AMS, February 1982. One can also compare with previous issues of this study (volume IV: 1970-71 and volume V: 1975-76).

The book Mathematics Tomorrow, edited by L.A. Steen (Springer-Verlag, 1981), contains some articles related to our subject. For example:

Alan Tucker, Redefining the mathematics major, pp. 39-48;
William F. Lucas, Growth and new intuitions: can we meet the challenge?, pp. 55-69.

The following report of the CUPM was published recently:

Recommendations for a General Mathematical Sciences Program, Committee on the Undergraduate Program in Mathematics, MAA, 1981.

It is based on the idea that the previous CUPM mathematics major curriculum (defined in 1965 and revised in 1972) should be substantially revised and broadened to define a mathematical sciences major. This report contains se-

veral recommendations that should have great influence in the next years.

As an illustration of the possibility of using finite techniques and the computer to replace much of what is done in continuous applied mathematics, one can consider:

D. Greenspan, Arithmetic Applied Mathematics, Pergamon Press, 1980. From the preface: "In this book we will develop a computer, rather than a continuum, approach to the deterministic theories of particle mechanics. (...) At those points where Newton, Leibniz, and Einstein found it necessary to apply the analytical power of the calculus, we shall, instead, apply the computational power of modern digital computers. (...) The price we pay for [the mathematical simplicity of our approach] is that we must do our arithmetic at high speeds."

I. Introduction

I think that there is a need for a continuation of the type of workshop that we had at Kingston and was glad to hear at the closing plenary session that such a possibility is being seriously considered. All undergraduate mathematics teaching at the University level should be assessed from time to time; this certainly includes our service teaching. Perhaps a more formalized group put on a continuing basis on this subject would get the attention of our colleagues at the University so as to give more recognition to the problems of updating the mathematics curriculum and making it more relevant to today's needs.

What follows is a listing of texts and articles that are related to the subjects that we discussed in the working group on "Computer Science and Undergraduate Mathematics Education" held in Kingston at the sixth annual meeting of the Canadian Mathematics Education Study Group from the third to the seventh of June, 1982. I have broken this bibliography down into two classes; texts and articles, and have have added notes to each section. Very few of these references are directly related to the Ralston article*, but indirectly, most are very pertinent to his theme as well as to the tenor of our general discussion. Included are a few references to mathematical and/or computer modelling. Some of these references will be of little interest to some; you will have to pick and choose.

II. Pertinent Texts

A: Science with Pocket Calculators; D.R. Green and J. Lewis; Wykeham Publications (London) Ltd., 1978.

*Ralston; Computer Science, Mathematics, and the Undergraduate Curricula in Both; Amer. Math. Monthly, 88 (1981), 472-485.

B: Pocket Calculator Supplement for Calculus; J.B. Rosser and C. de Boor; Addison Wesley, 1979.

C: Computational Analysis with the HP-25 Pocket Calculator; P. Henrici; Wiley Interscience, 1979.

D: Calculus: An Applied Approach; T.H. Wonnacott; Wiley, 1977.

E: Adventures with your Pocket Calculator; L. Rade and B.A. Kaufman; Penguin Books.

F: Mathematical Models: Mechanical Vibrations, Population Dynamics, and Traffic Flow; R. Haberman; Prentice-Hall, 1977.

G: Scientific Analysis for Programmable Calculators; H.R. Meck; Prentice-Hall, 1981.

H: Introduction to Computational Methods for Students of Calculus; S.S. McNeary; Prentice-Hall, 1973.

I: Riddles in Mathematics; E.P. Northrop; Penguin Books.

III. Comments on Texts

A: Soft cover edition about $9.

- Useful chapters on the basic mathematics functions and their applications; solving non-linear equations, simultaneous equations, numerical approach to calculus, population dynamics, case studies in physics, chemistry and biology, probability and statistics.
- Population growth from discrete approach only.
- Programs in both algebraic and reverse polish notation.

B: Soft cover edition about $10.

- Can be used in conjunction with any calculus text.
- I particularly like his chapters on Sources of Error and on Root Finding.

To find roots of a polynomial very accurately, where error due to cancellation of terms is a problem, he expands the polynomial in a Taylor Series about an estimate of the root. This important application of Taylor Series is not stressed enough.

- Programs in both algebraic and reverse polish notation.

C: Main interest to me is not in his actual programs (reverse polish) but in the way he uses many areas of mathematics to produce algorithms that are short enough to be programmed on the HP-25, a calculator with a very limited number of program steps. Also many of his programs are likely to be of more interest to the mathematician than those in most numerical analysis texts.

D: This calculus text stresses the "finite calculus along with the infinitesimal calculus". Many of his examples are from economics. He doesn't really integrate his book with the calculator but his overall approach is pertinent. He is a very good teacher and this shows in his text, as it is very readable and makes the student think at each step; he gets the student involved.

E: Paperback about $3.

- The title is very appropriate. Book is of interest to various mathematics students.
- I especially like his sections on π, Fibonacci and the Golden Section, and on Iteration.
- Book divided into two parts; an introduction and then a study in further detail.
- Good bibliography for interesting students in these subjects.

F: A very good text for a mathematical modelling course. By concentrating on three main themes, the book has enough material to be used at a variety

of undergraduate levels. In the section on population dynamics, he uses both the discrete and the continuous approach.

G: Paperback about $10.

Particularly interesting sections on approximating functions on a calculator and on evaluating improper integrals. In his section on improper integrals, he emphasizes the need for transformations and, although several transformations may work, some are more suitable computationally than others.

- Programs are in algebraic notation.

H: Older text, much of which is largely displaced by book B.

- Still quite interesting.
- Novel to me are his examples using continued fraction expansions.

I: Paperback.

- He discusses various paradoxes in mathematics. Some of these paradoxes can be integrated with the calculator;-rearranging conditionally convergent series, paradoxes in calculus.

IV. Articles of Interest

1: A Discrete Approach to the Calculus; S.P. Gordon; Int. J. Math. Educ. Sci. Technol., 10, 21-31 (1979).

2: A Laboratory, Computer and Calculus Based Course in Mathematics; H.M. Schey, J.L. Schwartz, W.U. Walton and J.K. Zacharias; Int. J. Math. Educ. Sci. Technol., 1 115-130 (1970).

3: An Approach to the Teaching of Ordinary Differential Equations; A.C. Bajpai, I.M. Calcus and G.B. Simpson; Int. J. Math. Educ. Sci. Technol., 1, 39-54 (1970).

4: Mathematical Models in Science; J. Hyslop and R. Lewis; Int. J. Math.

Educ. Sci. Technol., 1, 375-388 (1970).

5: Recurrence in Numerical Analysis; J.J. Davidson; Byte; 20-30, April (1961).

6: Computer Algebra; R. Pavelle, M. Rothstein and J. Fitch; Scientific American; 136-152, Dec. (1981).

7: Strange Attractors: Mathematical Patterns Delicately Poised Between Order and Chaos; D.H. Hofstadter; Scientific American; 22-43, Nov. (1981).

8: Personal Calculator Has Key to Solve Any Equation f(x) = 0; W.M. Kahan, Hewlett Packard Journal, Dec. (1979).

9: Handheld Calculator Evaluates Integrals; W.M. Kahan, Hewlett Packard Journal, Aug. (1980).

V. Comments on Articles

1: This article seems to me to be more pertinent to our discussion than the Ralston paper. Many of his concerns were our concerns. I liked his phrase that he is "using the results of the calculus of the finite differences and finite sums both as motivation and as a significant tool leading to applications."

2: Another computer oriented calculus course — "experimental calculus." Their course is more computer oriented and seems similar to the "Keller plan" of Physics.

3 and 4: Mathematical modelling courses using differential equations.

5: Stability related to recurrence relations.

6: Demonstrates the interplay between large computers and mathematics in the area of algebra.

7. A very interesting article showing the applications of fixed point algorithms, iteration of functions, to new ways of thinking about nonlinearity in various fields of science.

8 and 9: Kahan's articles are interesting to show anyone that there is much more than just using text-book algorithms if you want to produce "almost fool-proof" calculator buttons for finding the roots of an equation or for integrating any definite integral. His articles contain a lot of material for anyone interested in numerical methods to think about.

Any comments or further references would be gratefully appreciated.

Peter W. Manuel
Associate Professor
Department of Applied Mathematics
University of Western Ontario
London, Canada
N6A 5B9

References

1. ACM Curriculum Committee on Computer Science, An undergraduate program in computer science - preliminary recommendations, Comm. ACM, 8 (1965) 543-548.

2. _____, Curriculum 68 - Recommendations for academic programs in computer science, Comm. ACM, 11, (1968) 151-197.

3. ______, Curriculum 78 - Recommendations for the undergraduate program in computer science, Comm. ACM, 22 (1979) 147-166.

4. A.T. Berztiss, The why and how of discrete structures, SIGCSE Bulletin, 8 (1976) 22-25.

5. R.P. Boas, Jr., "If this be treason ...." this MONTHLY, 64 (1957) 247-249.

6. E.W. Dijkstra, Programming as a discipline of mathematical nature, this MONTHLY, 81 (1974) 608-612.

7. W. Givens, Implications of the digital computer for education in the mathematical sciences, Comm. ACM, 9 (1966) 664-666.

8. S.P. Gordon, A discrete approach to computer-oriented calculus, this MONTHLY, 86 (1979) 386-391.

9. P. C. Hammer, Computer science and mathematics, in Proceedings of the IFIP World Conference on Computer Education, vol. 1, IFIP, Amsterdam, 1970, pp. 53-55.

10. IEEE Computer Society Education Committee. A Curriculum in Computer Science and Engineering, IEEE Computer Society, 1977.

11. J. Kemeny, J. Snell, and G. Thompson, Introduction to Finite Mathematics, 1st ed., Prentice-Hall, 1956; 3rd ed., 1974.

12. D.E. Knuth, The Art of Computer Programming, vols. 1-3, Addison-Wesley, 1968, 1969, 1973.

13. ________, Computer science and its relation to mathematics, this MONTHLY, 81 (1974) 323-343.

14. L.S. Levy, Discrete Structures of Computer Science, John Wiley, 1980.

15. MAA Committee on the Undergraduate Program in Mathematics, A General Curriculum in Mathematics for Colleges, CUPM, 1965.

16. _____, Recommendations for an Undergraduate Program in Computational Mathematics. CUPM, 1971.

17. _____, Recommendations on Undergraduate Mathematics Programs in Computing, CUPM, 1972.

18. MAA, PRIME 80, Proceedings of a Conference on Prospects in Mathematics Education in the 1980s, 1978.

19. J.E. McKenna, Computers and experimentation in mathematics, this MONTHLY, 79 (1972) 294-295.

20. National Research Council, The Role of Applications in the Undergraduate Mathematics Curriculum, 1979.

21. H.O. Pollak, quoted in The Chronicle of Higher Education, 15 May 1978, p.3.

22. A. Ralston, Computer Science, Mathematics and the Undergraduate Curricula in Both, Technical Report 161, Department of Computer Science, SUNY at Buffalo, 1980.

23. A Ralston and M. Shaw, Curriculum 78 - Is computer science really that unmathematical? Comm. ACM, 23(1980) 67-70.

24. J.B. Rosser, Mathematics courses in 1984, this MONTHLY, 79 (1972) 635-648.

25. D.F. Stanat and D.F. McAlister, Discrete Mathematics in Computer Science, Prentice-Hall, 1977.

26. O.E. Taulbee and S.D. Conte, Production and employment of Ph.D.'s in computer science, Comm. ACM, 20 (1977) 370-372.

27. J.P. Tremblay and R. Manohar, Discrete Mathematical Structures with Applications to Computer Science, McGraw-Hill, 1975.

THE STRUCTURE OF UNDERGRADUATE APPLIED MATHEMATICS PROGRAMMES AT DALHOUSIE UNIVERSITY

JULES DEG. GRIBBLE

ABSTRACT: In this article recent changes in the structure of the "applied" mathematics courses in the Department of Mathematics, Statistics and Computing Science at Dalhousie, and the reasons for these changes are reported on.

INTRODUCTION AND BACKGROUND

There is quite a lot of literature discussing how undergraduate mathematics and applied mathematics curricula should be structured so that the student will receive an unbiased, up-to-date and employable education. See, for example [1-5]. At present there is considerable interest in ensuring that applied and applicable mathematics curricula remain pertinent and up-to-date. This has been spurred by the effect of computers in all branches of mathematics, the many new and exciting areas of mathematics that are gaining increasing importance, and the disturbing fact that enrollments in undergraduate programmes seem to be declining, despite good employment prospects. At the June, 1981 S.I.A.M. meeting, at Rensselaer Polytechnic Institute in Troy N.Y., there was an open panel discussion of some of the proposals that the current Committee on Undergraduate Programmes in Mathematics (C.U.P.M.) committee put forward on course content and programme structure. See [5] for more details.

At Dalhousie, we have also been taking a critical look at our applied mathematics curriculum. (We have however, had to make compromises with what our ideal programme would be due to the obvious constraints of manpower and, in the short term at least, our expectations of student numbers.)

The major motivation for reviewing our "applied" course structure came from noticing that over the last few years some ideosyncracies had crept into the structure and that some classes had no well defined objectives. In the immediate, the catalyst for making some changes was that the Department had recently introduced two Co-operative Education Programmes, one in Mathematics and the other in Mathematics and Computing Science combined. These programmes are both Honours programmes. The structure of these programmes requires that classes taken by students enrolled in their fourth year all be half credit, i.e. one semester, classes. This required, at the very least, some renumbering and subdivision of classes.

A loosely structured committee was organized by myself with participation encouraged by anyone who was interested. A write-up of the discussions was distributed to every member of the Department after each meeting, with invitation for comment. The committee met five times beginning in early April and by mid-summer the issue had been resolved. The changes that we have made will all be implemented in the 1982/83 academic year.

To help our courses in a more general context, I will outline the relevant Faculty of Arts and Science regulations which we have to abide by. The normal course load each year for a full-time undergraduate student enrolled in our faculty is five full credit classes. A full credit class typically (in our Department) meets for three lecture hours, with the possibility of some additional tutorial hours, each week of the academic year. These classes have an "R" following their number. A half credit class would typically meet for one semester, and will have an "A" or a "B" following its number. To obtain a concentrated Honours degree a student must satisfactorily complete between 9 and 11 full credit classes, beyond the first year level, in the major field. At least 2 of these classes must be of fourth year level. To obtain a combined Honours degree in two allied subject the student must complete between 11 and 13 classes above first year level in the two subject areas, with at least 4, and at most 7, classes being in either subject. At least one of these classes must be of fourth year level in one of the subject areas. For the purposes of major areas of study, Mathematics and Computing Science are considered to be distinct, despite the fact they come under the aegis of the one Department in our case. A more detailed listing of the Faculty regulations can be found in the Faculty of Arts and Science Calender.

PHILOSOPHY

After some initial discussions, we decided that there are four groups of students we are trying to cater for:

(1) Co-operative Education Programme Students.

(2) Students from other disciplines who are required or encouraged to take math courses, particularly students from engineering, physics and related areas.

(3) Students, of superior ability, enrolled in a math programme, i.e. those who are probable Honours graduates and potential graduate students.

and (4) The majority of students, enrolled in a math programme, who are competent but not outstanding.

Since we do not have a large enough department to cater for each of these groups entirely independently of each other, there followed several themes which we needed to keep in mind whilst designing our course structures, particularly in third and fourth year;

(a) As mentioned in the introduction, all classes should be half credits.

(b) The classes, whilst forming an integrated whole, should have as few interdependencies (particularly co-requisites) as possible, thus providing flexibility in class choices.

We then began the more difficult job of deciding what the student needed to get out of our programme and what we felt was essential for him or her to know at the end of a course of study. The following principles provided the foundation on which we built:

(i) There should be a core of classes in first, second, third and even fourth years which our "applied" students are required to take (and pass).

(ii) The objective of the programmes should be, as much as possible, an exposure to solutions of real and challenging problems. This means that attention is paid to how the mathematical problem is derived,or abstracted, from its original environment; to techniques which are used to solve the mathematical problem; and last but not least to how the results obtained can be interpreted in the context of the original problem.

(iii) The programme should be arranged so that the student has an exposure to various techniques of problem solving, i.e. we want to provide a broad based education rather than a specialization in any one aspect of problem solving.

(iv) We would design the programme with the students to categories (1) (2) and (4) in mind and then make sure that the students in category (3) receive, as minimum, an education at least as good as they presently obtain. In other words, we wanted to design the course structure for the more average student and then cater to the exceptional student instead of giving priority to the exceptional student and simply giving the average student a pruned version of an exceptional students programme. In the end we feel that this approach will raise the standard of our courses and the quality of all our graduating students. It is important to note here that this principle is essentially a statement of an underlying philosophy and a comment on the attitude with which we hope the classes will be taught.

(v) The students should be encouraged, and have sufficient time available, to take a reasonable sequence of courses in at least one other appropriate discipline.

THE CORE CLASSES

The simplest way to report our final conclusion is pictorially, in Figure 1, with some explanatory comments below. Detailed syllabuses will be available in the 1982/83 Faculty of Arts and Sciences Calendar (available early 1982).

In Year I the student completes a class in Differential and Integral Calculus (100A/101B) and an introductory Computing Science course (CS140A/CS141B).

In Year II the core classes are an Intermediate Calculus course (200R or 220R), an introduction to Matrix Theory (203A) and Linear Algebra (204B) an Introduction to Statistics (207A/208B) and an Introduction to Numerical Linear Algebra (227B). Students will also be encouraged to take the non-credit Co-op Seminar (this is compulsory for Co-op students) in which some aspects of industrial mathematics are discussed; we have some talks by people working in industry, some talks by Co-op students about their work experience, and, amongst other things, talks on the writing of curriculum vitae and on interview techniques. A strong recommendation will be made for students to take the computing science Year II core courses (CS245A/CS261B). If a student has an interest in Statistics, then having taken 208B would complete the Year II statistics core. If a student is an Honours student, then 220R or 200R, 203A and 204B, are replaced by the higher level classes 250R, (Introductory Analysis) and 213R (Linear Algebra) respectively.

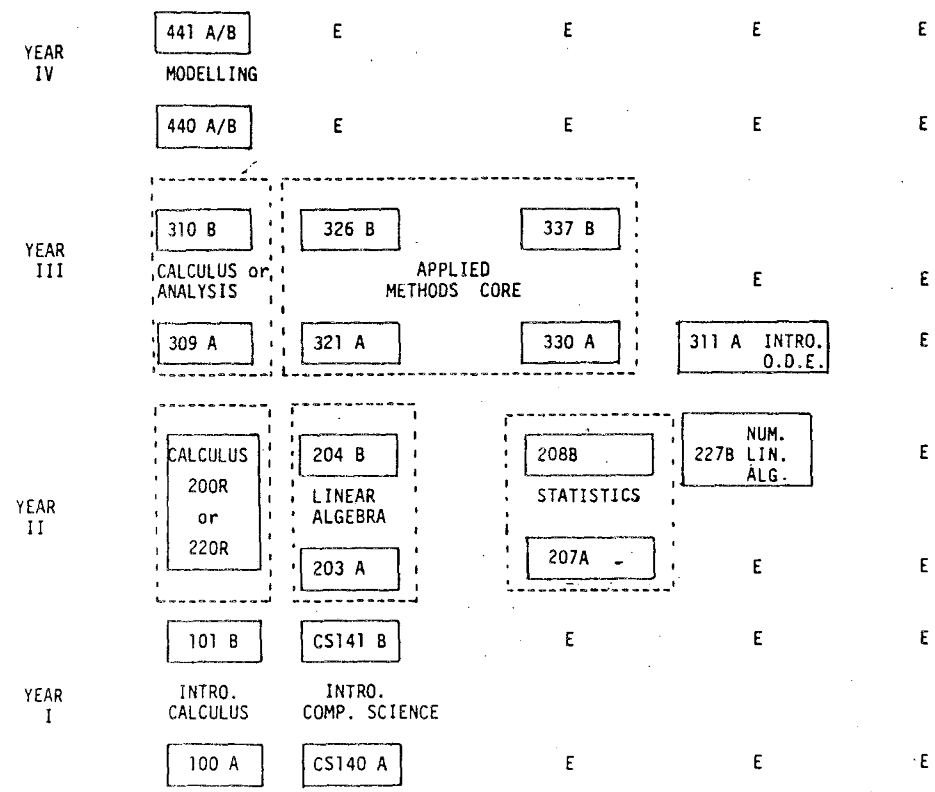
In Year III we require the student to take an Advanced Calculus sequence (309A/310B) and an Introduction to Differential Equations (311A). An Honours student would take 350R (Intermediate Analysis) in place of (309A/310B). The four half credit classes

321A Introduciton to Numerical Analysis
326B Foundations of Applied Mathematics
330A Linear and Integer Programming
337B Stochastic Processes

provide the necessary basis for the breadth requirement of our applied undergraduate programme, providing the tools for modelling via discrete methods, analytic methods and non-deterministic methods. Since we are encouraging the student to take a sequence of classes in another discipline, we would be prepared to allow the student to take one or two of the above four half classes in Year IV, although this is perhaps not the most desirable option.

In Year IV the only core class would be one of 440 or 441 (or perhaps both if they were both offered), which are Applied Mathematics Modelling courses. This class is the objective and culmination of the programme. Its aim is to allow the student to use as many of the tools as he or she has aquired from the Year III core classes and to gain experience in seeing a problem dealt with in depth and in its entirety. The problem would be

FIGURE 1 : CORE COURSES IN APPLIED MATH.



E: an elective half credit class

A class number followed by an A and/or B or an R means that the class is given in the first and/or second semester or throughout the year.

presented in its original context, abstracted so that a mathematical model can be developed; the necessary computations and analysis would be done in the form of projects and then the results interpreted in the original context. We expect that at least two different modelling techniques would be used and compared during the class.

It is our hope that on completing a core course, like the one outlined above, that the student will have been taught an appropriately critical yet practical way of thinking, and that with this attitude of mind will be able to apply his or her knowledge and expertise outside the halls of academia with success.

Finally, note that if a student wishes, by the end of Year IV, he or she can also become a competent statistician or computer scientist, whilst completing the core classes in applied mathematics. There are of course a variety of Year III and Year IV classes in the applied area which the student may take, depending on the chosen area of specialization.

The core classes are intended to represent the minimum number of classes that have to be taken and some flexibility in when the student takes these courses is naturally acceptable. There are clearly many other classes a student could (and may well) profitably take; we felt, however, that the student should not be restricted more than absolutely necessary.

C.U.P.M. RECOMMENDATIONS

At the June SIAM meeting in Troy N.Y., mentioned earlier, a brief summary of recommendations for programme philisophy and coursework were given as a handout. The handout is reproduced below. We feel that our new course structure essentially meets all the given recommendations. It is important to note that there is emphasis placed on the importance of the role of the instructor in making the courses relevant, interesting, and catering to the needs of the majority.

Handout of Summary of C.U.P.M. Recommendations (distributed at June '81 SIAM meeting)

PROGRAM PHILOSOPHY

1. The curriculum should have a primary goal of developing the attitudes of mind and analytical skills required for efficient use and understanding of mathematics. The development of rigorous mathematical reasoning and abstraction from the particular to the general are two themes that should unify the curriculum.

II. The mathematical sciences curriculum should be designed around the abilities and academic needs of the average mathematical sciences student (with supplementary work to attract and challenge talented students).

III. A mathematical sciences program should use interactive classroom teaching to involve students actively in the development of new material. Whenever possible, the teacher should guide students to discover new mathematics for themselves rather present students with concisely sculptured theories.

IV. Applications should be used to illustrate and motivate material in pure and applied courses. The development of most topics should follow the paradigm: applications → mathematical problem-solving → theory → applications. Theory should be seen as useful and necessary for all mathematical sciences.

V. First courses in a subject should be designed to appeal to as broad an audience as is academically reasonable. Most mathematics majors do not enter college planning to be math majors, but rather are attracted by beginning mathematics courses. Broad introductory courses are important for a mathematical sciences minor (see section 8).

COURSEWORK

VI. The first two years of the curriculum should be broadened to cover more than the traditional four semesters of calculus - linear algebra - differentia equations. Calculus courses should include more numerical methods and non-physical-sciences applications. Also, other mathematical sciences courses, such as computer science and applied probability/statistics, should be an integral part of the first two years of study.

VII. All students should take a set of two upper-division courses leading to the study of some subject(s) in depth. Rigorous, proof-like arguments are used throughout the mathematical sciences, and so all students should have some proof-oriented coursework. Real analysis or algebra are natural choices but need not be the only possibilities. (conflicting points of view on the role of "proof" courses in a mathematical sciences major is discussed in section 6.)

VIII. Every mathematical sciences student should have some coursework in the less theoretically structured, more combinatorially complex mathematics associated with computer and decision sciences.

IX. Students should have the opportunity to undertake "real-world" mathematical modeling projects, either as term projects in an operations research or modelling course, as independent study, or as an internship in industry.

X. Students should have a minor in a discipline using mathematics, such as physics, computer science, or economics. In addition, there should be a sensible breadth in the physical and social sciences. For example, a student interested in statistics might minor in pyschology but also take beginning courses in economics and biology (heavy users of statistics).

The report of the C.U.P.M. committee, containing detailed recommendations is expected to be available late in 1981.

For further information about other CUPM documents and related MAA mathematics education publications one should contact: Director of Publications, The Mathematical Association of America, 1529 Eighteenth Street, N.W., Washington, D.C., 20036.

ACKNOWLEDGEMENTS

I would like to thank all those members of the Department who contributed to our discussions. In particular I would like to single out John Clements, Ken Dunn, Chris Field, Carl Hartzman, Lee Keener, Dick Sutherland, and our Chairman, Tony Thompson.

REFERENCES

[1] "Applied Mathematics in the Undergraduate Curriculum" committee on the Undergraduate Program in Mathematics (C.U.P.M.) of the Mathematical Association of America, 1972 (copies available free of charge from : C.U.P.M., P.O. Box 102A, Berkeley, California 94701).

[2] "The Role of Applications in the Undergraduate Mathematics Curriculum", ad hoc Committee on Applied Mathematics Training, Assembly of Mathematical and Physical Sciences, N.R.C., 1979 (copies available from: Office of Mathematical Sciences, N.R.C., 2101 Constitution Avenue, Washington, D.C., 20418).

[3] "Preparing Statisticians for Careers in Industry", Report of the Committee on Training Statisticians for Industry, American Statistical Association, section on Statistical Education, presented at A.S.A. meeting Aug. 13-16, 1979, Washington D.C.

[4] "Curriculum '78: Recommendations for the Undergraduate Program in Computer Science", Report of the A.C.M. Curriculum Committee on Computer Science, Communications of the A.C.M. Vol. 22, No. 3, p. 147-165, March 1979.

[5] "C.U.P.M. Recommendations for a General Mathematical Sciences Program", C.U.P.M. Panel on a General Mathematical Sciences Program, to appear (late 1981). Information on the Reports availability will appear in the September 1981 issue of FOCUS, the MAA newsletter.

Working Group A Report:
Appendix 12

At the close of our discussions in Kingston it was suggested that it might be a useful contribution to the debate if I, as Chairman of a combined department of Mathematics, Statistics and Computing Science, were to try and evaluate the advantages and disadvantages of such a combination.

The main disadvantages are the practical, administrative ones with which I am only too familiar. These centre on the differences between mathematics and computing science. The latter is, to a large extent, an experimental science with quite different criterion of excellence from mathematics. This makes matters of promotion and merit somewhat tricky. The result of this is that Computer Scientists probably prefer to be in a separate department which, in turn, makes an already tough recruiting situation a bit worse.

The advantages are more wide spread but perhaps less obvious. First, on the same question of staffing, having the various mathematical sciences within one department makes the teaching schedule more flexible. Many mathematicians can teach some computer science classes. Thus, if staffing does not keep pace with changes in student demand from mathematics to computer science it is somewhat easier to switch manpower than it would be in separate departments.

Secondly, the fact that there are mathematicians teaching some sections of the large elementary computing science classes has I think a positive effect on both these classes and the other mathematics classes these same instructors teach. There is a cross-fertilization of ideas which is beneficial. 'Pure' computer scientists and 'pure' mathematicians might disagree with this!

Thirdly, having people together in the one department makes discussions of curriculum questions such as this group aimed at, rather easy to carry out. (Group A was lacking any computer scientists!). Moreover, since people know each other well, they understand each other's different points of view. There is a tendency for 'pure' mathematicians to adopt an uncompromising academic point of view on curriculum and for 'applied' mathematicians and computer scientists to emphasize a pragmatic point of view on 'market demands'. With both groups strongly represented one tends to get a more balanced view between the poles of this contradiction.

Fourthly, at the research level, Computer Science is generating new problems for mathematics. It is interesting to see Computer Scientists talking to algebraists and combinatorists, just as Statisticians talk to Analysists and Measure Theorists.

Though Computer Science, Statistics and Mathematics are distinct subjects with different outlooks and methods there are no hard and fast lines between them. Rather one merges into the other. Thus, though the practical difficulties of trying to adequately look after all sections often seem overwhelming, I think the rational view is that it is better to keep the mathematical sciences together.

Tony Thompson

Working Group B

THE APPLICATION OF RESEARCH IN MATHEMATICS EDUCATION TO TEACHER-TRAINING PROGRAMS

Leader: Dale Drost, Memorial University

There exists an extensive body of research in the field of mathematics education dealing with how students in primary, elementary, and secondary schools learn mathematics. There also exists considerable research on the effectiveness of various teaching techniques for various educational goals at each of these levels. A largely unanswered question is how can teacher educators most effectively apply the results of this research into teacher-training programs. Should the teaching methods employed by teacher educators parallel those they advocate for use in the schools? Should different approaches be used with primary/elementary school teachers than with secondary school teachers? Should different approaches be used with preservice teachers than inservice teachers? The working group will use these and related questions to address the issue of how results from research in mathematics education can best be applied to teacher-training programs.

Report of

Working Group B

The Application of Research in Mathematics Education to Teacher-Training Programs

Dale Drost

This working group, although small in number, was composed of people who energetically discussed the issues implied by the title of the group. The participants included Stanley Erlwanger, Helene Kayler, Israel Klenier, Andre Ladouceur, Medhat Rahim, Ron Ripley, and Dale Drost.

Cynics in education might argue that results from research in mathematics education are often so ambivalent that any attempt to apply these results to teacher-training programs could only lead to chaos. Such was not the feeling of the participants of Working Group B. During the nine hours of discussion the group debated a number of ways in which both methods of research and results from research might be used in teacher-training programs. Due to the broadness of the area and the limited amount of time, the discussions were often general and the essence of them was difficult to capture on paper. What follows is an attempt to capture the flavour of the discussions and not the specifics. Future working groups might find this general discussion a useful starting point from which to focus on more specific aspects of the area.

The group began its deliberations by considering some possible directions proposed by the group coordinator. These possibilities were presented as suggestions only and the group was free to delete or to add others as they wished. The suggested directions were as follows:

1. How do we as 'teachers of teachers' deliver results of research to our students so that they will incorporate these results into their own teaching strategies.

2. How do we use results from research in organization and structure of the mathematics and mathematics education components of our teacher education programs.

3. What does research tell us, if anything, about the overall role of mathematics education in teacher education programs.

4. What does research tell us about the proportion of our time that we should spend on inclass activities, on oncampus laboratory work, on inschool individual or small group work, and on student teaching.

5. - - - - - -

6. - - - - - -

7. Before taking any of the directions suggested above, do we need to conduct research on that direction. In other words, is any research available to assist us in making decisions.

It was also suggested that in our discussions that we should consider the issue of how much time is actually available for mathematics education in the various teacher-training programs. In most programs time is restricted to two or fewer one-semester courses and this tends to impose a limitation on many activities. Also, it was suggested that the group should consider restricting itself to pre-service or inservice education as well as to differentiating

among primary, elementary, junior high, and senior high mathematics education programs. A final suggestion was that discussion might be restricted to particular areas of research such as problem solving, learning, or evaluation. After considering these suggestions, it was decided that we would not impose these restrictions at the beginning, but would attempt to relate our comments to the different subgroups of students and different areas of research when applicable.

Much of the first two days of discussion was taken up by attempting to find a structure under which we could organize the topic as well as to come to agreement as to the more specific topics which we wished to discuss within that structure. Basically the outline presented in figure 1 can be used to describe how we saw the application of research to teacher-training programs.

The major areas of application of research were seen as being application to program content and application to program method. Program here is to be interpreted as being the mathematics education program, although it is useful to take a broader interpretation at times and include the entire teacher education program.

Within the area of application to program content, three general areas were identified. First, the content of research should be part of the mathematics education curriculum. This content was subdivided into two areas; content about student learning, and content about the teacher.

Figure 1

Outline of Application of Research in
Mathematics Education to Teacher Training Programs

A. <u>To Program Content</u>

A.1 From the content of research

A.1.1 <u>About Student Learning</u>

- psychogenesis :evolution of math concept
  :formulation
  :obstacles

A.1.2 <u>About the Teacher</u>

- attitudes
- questioning
- use of time

A.2 From the methods of research

- interview method
- tests
- error analysis

A.3 From the environment

- socioeconomic factors

B. <u>To Program Method</u>

- student work with children
- "discovery" method
- material manipulation
- content analysis

With respect to student learning the results of research, such as that described by Professor Vergnaud in his address to the conference which is included earlier in these proceedings, should be conveyed to teachers in training. Research such as this provides the student teacher with valuable information on the evaluation and formulation of mathematical concepts by children as well as obstacles to the learning of such concepts. Throughout our deliberations we discussed several specific concepts in light of the learning research. Among these were fractions, the number line, the equal sign, different bases, equations, and dissections. The topics discussed were illustrative of most topics in the curriculum. Since, within the time frame under which our teacher-training programs operate, it is not possible to study every mathematical topic in detail, it is necessary that students be able to generalize findings and methods from research across the mathematics curriculum. Content from research about the teacher also was discussed. Topics such as teacher attitudes, questioning techniques, and use of time were debated. Some of these will be dealt with briefly later in this report.

Second, with respect to program content, knowledge of the methods of research is an important part of the mathematics education curriculum. This is particularly true where the methods of research are closely related to methods of teaching. Much research has been conducted recently using interview methods and methods of error analysis both of which can be considered to be useful techniques for the classroom teacher. Methods of testing used by researchers also have potential benefits for classroom teachers.

A third area of research from which program content may be derived is research about the environment. Very little time was spent on this aspect in the working group, however, it was felt that results of research dealing with environmental aspects such as socio-economic status need to be part of the teacher training curriculum.

The other major area of application of research was to program methods. What methods of teaching should we as teacher educators use in our classes? An issue here was, do we practice what we preach. Many, perhaps most, teacher educators advocate extensive use of manipulative materials and the use of activity or discovery techniques of teaching. Research is often cited to support such suggestions yet the same teacher educator often fails to use the methods which are being advocated. One method advocated by the study group was the active participation of the teacher in the research process. This strategy was felt to be particularly applicable in the inservice aspect of teacher training. In both inservice and preservice programs, active involvement with children was considered to be a critical element in the teacher training process.

Considerable time was spent on discussing a means of incorporating both the results and methods of research into our curriculum. One possibility is described below which

requires that the university student have access to a child or a group of children. The student teacher prepares a test or selects a suitable test, activity, or items from other sources. The test could be a commercially prepared one, a test used in some research study, or a collection of Piagetian activities. The specific content could be selected by the student.

The student teacher then administers the instrument to a child or a group of children. If several children are involved, a pencil and paper instrument is a more efficient means of collecting the initial data. After administering the pencil and paper test, an individual child, or several children could be interviewed to gain insight into their thinking and methods of operation. The interview should be short and the topic of concern quite specific. The student should then analyze the result of the test and the interview and compare them with results from the related research.

After having collected data and given some thought to the findings, a discussion can be held with the professor, either individually, or with a small group of classmates or with the entire class. Such a discussion should reflect on the methods used and how similar methods are used by researchers, as well as on how the results compare with those from research on similar topics.

Depending on the situation, it might be desirable to interview again and/or retest the child after the discussion with the professor. If this is done, the student teacher would then analyze the new results and then, if possible, discuss them again with the professor. The procedure could be extended to include remediation of any problems that the child is having, with the student teacher providing appropriate activities.

Within the structure described earlier in this report the group also considered discussing several topics related to applications of research. These included: (1) Follow-up on research reported by Professeur Vergnaud. (2) How to make the best use of time. (3) Using research on questioning. (4) Different treatment of "Different" students. (5) Attitudes and perceptions in Primary/ Elementary and Secondary. (6) Error research. (7) Awareness of Procedures rather than "just" results. (8) Testing (9) Methodology in Teacher Education classes.

The topics were not considered in any particular order nor were all given equal attention. Several participants in the group had attended a session given by Marilyn Suydam at the NCTM meeting in Toronto in April, where she discussed many of the same issues. Reference was often made to her comments.

Some time was spent on discussing the results from research on attitudes. One member of the group, Ron Ripley, had conducted a study on the attitudes of prospective elementary school teachers towards mathematics. Of most interest to the study group were the procedures used in the study. Professor Ripley used his own teacher education

students as the subjects in the study and interviewed them on different occassions throughout the course with respect to their attitudes. He reported that this procedure resulted in most students developing positive attitudes towards mathematics. This finding was consistent with the study group's contention that the education students should be actively involved in the research process. Participation in a study, such as Ripley's, has the potential for helping teachers learn methods by which they can help their future students develop positive attitudes.

The group also spent some time on the topic of questioning. Again the discussion was general but focused on issues such as product versus process questions; the type and structure of questions; leading questions; following questions; the relationship of questions to content. Although questioning was considered to be a topic more directly related to the teacher, the same mathematical content topics were used in the discussion. Examples from mathematical topics such as equations and dissections were used to discuss the various issues involved in questioning.

Different treatment for different students was discussed briefly as was error research where it was suggested that education students should collect and analyze errors made by school children. This could be done by using a procedure similar to that described earlier in this report.

With respect to many of the issues discussed, concern was expressed as to what can be done within the classroom between teacher educator and student teachers. One possibility was to discuss certain events as they occur in our own classrooms. For example, when a particularly good, or perhaps bad, question is asked, the opportunity exists to discuss what research says about such questions. Similar opportunities exist with respect to many of the issues discussed thus far, such as use of time, use of manipulative materials, differentiating instruction, and management techniques.

As stated at the beginning of this report, the discussions within the group were often very general. To some extent the discussions were exploratory in nature and for this reason, it is recommended that the group be continued at a future meeting of CMESG/GCEDM.

In conclusion, the group agreed that much more research is needed in areas like questioning and consideration of individual differences in the classroom, both at the teacher-training level and at the public school level. Much can be gained by involving students in teacher-training classes in the research process. The student not only learns about the content of research but also about the methods of research which can be used in actual classroom teaching. Also, by participating in the research process, teachers and prospective teachers, might well develop more positive attitudes to other findings from research. And finally, teacher educators should, as much as possible, adhere to ideas from research in their own classrooms.

Practice what you preach and discuss with students the research related to your practice.

Working Group C

IMAGERY AND MATHEMATICS

Leaders: Sandy Dawson, Simon Fraser University
David Wheeler, Concordia University

Many mathematicians have reported that their most creative thinking was coloured, or even borne along, by imagery. Probably the mathematical thinking of anyone, at any level, employs imagery, though educators have generally given very little attention to it. Perhaps we feel we know so little about the part played by imagery in mathematical thinking because we don't usually look for it, and because the mathematics we produce (unlike the poetry or the sculpture) does not seem to externalise in an obvious way the imagery we may have experienced.

The group will attempt to study imagery, how it is generated and how it is transformed into mathematics. Although examples and exercises may be drawn from any sources, the main object of the group will be to work on the question directly using the resources of the participants. Attention will be given to the part played by imagery in learning and in teaching.

Intending participants are invited to prepare for the group sessions. Suggestions of questions to think about and activities to try with students can be obtained from the first-named co-leader (c/o Faculty of Education, Simon Fraser University).

Working Group C

Imagery and Mathematics

Sandy Dawson

David Wheeler

This working group involved eighteen to twenty people for the three half-day sessions scheduled. As a result of the work done by the group, one sub-group was formed whose focus is on imagery and polynomials, a second sub-group met and issued a report on imagery and the calculus, and a third sub-group met and reported to the Plenary session on imagery and geometry but did not submit a report.

One of the main difficulties facing the working group was to define the word imagery. Though no definitive conclusions were reached about the meaning of this term, viewing imagery as "giving meaning to the unknown", or "representing the unknown" came to be working definitions for the group. Moreover, imagery came to be seen as a "way to look forward". These preliminary ways of conceptualizing imagery evolved over the three days work.

When the group commenced its work, the decision was made to focus on how (a) each of the participants used imagery in their own teaching and research, and also (b) to focus on trying to capture the feeling surrounding the power imagery provides for attacking problems, and for educating one's self.

Consequently, the first session on day one dealt with a problem in non-Euclidean geometry, the leader inviting the participants to use imagery to try to get a feel for the problem and a possible solution to it. An invitation to share images with the group was issued, and these contributions were discussed at length. The second half of the first day was devoted to a discussion of strategy to be used by the group to define its goals, and how it wished to operate in attempting to achieve those goals. One suggestion made was to create activities for the group which dealt with, if possible, non-visual imagery.

As a result, day two commenced with an activity (the walnut problem) which, it was hoped, would generate non-visual imagery. What became clear from this activity was the influential role played by previous experience, and how that experience could in effect 'block' fresh images. However, it was suggested, and accepted by many of the participants, that the energy available for concentration and for the generation of images was greater when the activity was strictly mental. When participants shifted from strictly mental imagery to using drawings or algebraic writings, energy was released to the writing and hence did not seem as focused on the imagery. Nonetheless, once again the activity produced evidence that people have a tremendous variety of ways of using their selfs, and that the imagery process though probably universal is highly ideosyncratic. The morning concluded with the group involved in a kinesthetic-affective-intellectual combinatorial problem dealing with a $(3 \times 4) - 1$ array of individuals. This exercise gave rise to the idea of local and global imaging (a la Lakatos) as a means to problem solution.

The third day of work was divided into two sections once again with the overall strategy being to image about specific mathematical topics. Working as a large group for the first half of the morning, discussion focused on the topic of "limit", each participant being invited to share the images each had with respect to this topic, and to also share the images they tried to evoke in their students if they taught the limit concept. The variety of images share was truly amazing, some sixteen to eighteen images in all, some obviously touched by strong emotional commitments and/or reactions.

One question which arose from the discussion about these various images was, does learning occur when one overcomes conflicting images? In this regard, it was argued that the limit concept has both a static (limit IS) and a dynamic (limit TENDS TO) quality, and that students have to reconcile these two in order to 'learn' about limits. This session provided some truly rich images for the limit concept, as well as evoking in the participants images about images.

In the final session (latter part of day three) three topic areas were identified, subgroups were formed, and they set to work on creating images for their chosen topics. One group chose to focus on polynomials, a second on the calculus, and a third on geometry.

The report of the imagery and calculus group is attached as Appendix B.

The polynomial group undertook to write some short papers with the long range view of preparing a monograph on imagery and polynomials. At the time of writing this report, seven papers have been prepared, and they are now undergoing critical scrutiny and rewriting. The goal is to have the papers completed by November 1, 1982.

A written report was not received from the imagery and geometry group.

In early July, 1982, Mario Lavoie, a participant in Working Group C, submitted a paper entitled Utilisation Des Images Dans L'Apprentissage du Concept de Limite au Niveau Collegial (Lavoie, Lepage, and Roux), and it is included here as Appendix C. Appendix A is a selected bibliography on Imaging, Imagination, and Mathematics prepared by Sandy Dawson and Arthur Powell.

While the working of Group C seemed to go quite well, there was some frustration on the part of a few participants that more progress had not been made. It was suggested that work on the topic at next year's meeting should focus very specifically on a few mathematical topics (eg. polynomials, calculus, and geometry), and that prior to that meeting work should be done with students (public and university or college) so that research data would begin to be accumulated which could inform the participants in the group's work.

It was strongly felt by some participants that the work of the group should result in some publications, and to this end the polynomial sub-group undertook the preparation of the monograph.

APPENDIX 1

A Selected Bibliography

on

Imaging, Imagination, and Mathematics

prepared by

A. J. (Sandy) Dawson
Simon Fraser University
Vancouver, B. C.

and

Arthur B. Powell
Rutgers University
Newark, N. J.

for the

N. C. T. M. Annual Meeting
SIG/RME Pre-Session
Toronto, Canada
April 14, 1982

Revised for CMESG Annual Meeting
June 3-7, 1982,
Kingston, Ontario

General Books

Bateson, Gregory: Steps to an Ecology of Mind, Ballantine, New York, 1972

_______________: Mind and Nature, E. P. Dutton, New York, 1979

Bentov, Itzhak: Stalking the Wild Pendulum, Bantam Books, New York, 1979

Bohm, David: Wholeness and Implicate Order, Routledge & Kegan Paul, London, 1980.

Capra, Fritjof: The Tao of Physics, Bantam Books, 1975

_______________: The Turning Point, Simon and Schuster, New York, 1982

Dubos, René: The Wooing of Earth, Charles Scribner's Sons, New York, 1980

Ferguson, Marilyn: The Aquarian Conspiracy, J. P. Tarcher, Inc., Houghton Mifflin Company, Boston, 1980

Gattegno, Caleb: What We Owe Children: The Subordination of Teaching to Learning, Outerbridge & Dienstfrey, New York, 1970

_______________: The Universe of Babies: In The Beginning There Were No Words, Educational Solutions Inc., New York, 1973

_______________: Of Boys and Girls, Educational Solutions Inc., New York, 1975 (Restricted Printing)

_______________: Evolution and Memory, Educational Solutions Inc., New York, 1977 (Restricted Printing)

Jantsch, E. & Waddington, C.H. (eds.): Evolution and Consciousness, Addison-Wesley, New York, 1976

Jantsch, Erich: The Self Organizing Universe, Pergammon, New York, 1980

Koestler, Arthur: Janus: A Summing Up, Hutchinson & Co. (Publishers) Ltd., 1978

Papert, Seymour: Mindstorms: Children, Computers, and Powerful Ideas, Basic Books, New York, 1980

Pelletier, Kenneth R.: Mind as Healer, Mind as Slayer, Delta, New York, 1976

_______________: Toward a Science of Consciousness, Delta, New York, 1978

Pirsig, Robert M.: Zen and the Art of Motorcycle Maintenance, Bantam, New York, 1975

Popper, Sir Karl R.: Logic of Scientific Discovery, Basic Books Inc., New York, 1957

_______________: Conjectures and Refutations, Basic Books Inc., New York, 1962

Popper, K. R. & Eccles, J. C.: The Self and Its Brain, Springer International, New York, 1977

Prigogine, Ilya: From Being to Becoming, Freeman, San Franciso, 1980

Roszak, Theodore: Person/Planet, Anchor Press, Doubleday, New York, 1978

Samples, Robert: The Metaphoric Mind, Addison-Wesley, Don Mills, 1976

_______________: The Wholeschool Book, Addison-Wesley, Don Mills, 1977

_______________: Mind of the Mother, Addison-Wesley, Don Mills, 1981

Sheldrake, Rupert: A New Science of Life, J. P. Tarcher, Inc., Los Angeles, 1981

Teilhard de Chardin, Pierre: The Future of Man, Fontana Books, London, 1969

_______________: The Phenomenon of Man, Fontana Books, London, 1959

Watson, Lyle: Gifts of Unknown Things, Simon & Schuster, New York, 1976

Young, Arthur: The Reflexive Universe, Delacorte Press, San Francisco, 1976

Zukav, Gary: The Dancing Wu Li Masters, Wm. Morrow and Company, New York, 1979

Books on imaging

Block, N. J. (ed.): Imagery, Bradford, Montgomery, Vermont, 1980

Casey, E. S.: Imagining: A Phenomonological Study, Indiana University Press, 1976

Dennett, D. C.: Brainstorms, Bradford, Montgomery, Vermont, 1978

Downs, R. M. & Stea, D.: Maps in Minds, Harper & Row, New York, 1977

Gattegno, Caleb: For the Teaching of Mathematics, Educational Explorers, London, 1963

____________: Towards a Visual Culture, Outerbridge & Dienstfrey, New York, 1969

____________: The Mind Teaches the Brain, Educational Solutions Inc., New York, 1974 (Restricted Printing)

Gazzaniga, M. & LeDoux, J.: The Integreted Mind, Plenum, New York, 1978

Hofstadter, Douglas R.: Godel, Escher, Bach: An Eternal Golden Braid, Vintage Books, New York, 1979

Hooper, F.: Imaging and Visual Thinking, The Open University, Milton Keynes, England, 1976

Horowitz, M. D.: Image Formation and Cognition, 2nd edition, Appleton-Century-Crofts, New York, 1978

Klinger, E. (ed.): Imagery, Vol. 2: Concepts, Results, and Applications, Plenum, New York, 1981

Kosslyn, S. M.: Image and Mind, Harvard University Press, Cambridge, 1980

Leapfrogs: Families of Circles, Tarquin Publications, Norfolk, England.

McKim, R. M.: Experiences in Visual Thinking, Brooks-Cole Monterey, Ca., 1972

Randhawa, B. S. & Coffman, W. E. (eds.): Visual Learning, Thinking, and Communication, Academic Press, New York, 1978

Segal, S. J. (ed.): Imagery: Current Cognitive Approaches, Academic Press, New York, 1971

Sheidk, N. A. & Schaffer, J. A. (eds.): The Potential of Fantasy and Imagination, Brandon House, New York, 1979

Sheikh, A. A. (ed.): Imagery: Current Theory, Research and Applications, Wiley, New York, 1981

Shore, J. E. et al, (eds.): Imagery, Vol. 1: Its Many Dimensions and Applications, Plenum, New York, 1980

Singer, J. L. & and Pope, L. S.: The Power of Human Imagination, Plenum, New York, 1978

Sommer, R.: The Mind's Eye, Dell, New York, 1978

Valle, R. S. & von Eckartsberg, R.: Metaphors of Consciousness, Plenum, New York, 1981

Warnock, M.: Imagination, Faber & Faber, London, 1976

Articles on Imaging

Alesandrini, K. L. "Imagery-eliciting strategies and meaningful learning," Journal of Mental Imagery, (in press)

Anderson, J. R. "Arguments concerning representations for mental imagery," Psy. Review, 1978, 85, 249-277.

Clements, K. "Visual Imagery and School Mathematics," For the Learning of Mathematics, 1981, 2, #2, 2-10. (The conclusion of this article will appear in the March, 1982 issue. A complete bibliography will be included.)

Dawson, A. J. (Sandy) "The Work of Lakatos and Gattegno: Implications for Mathematics Teaching," Proceedings, ICME IV, Berkeley, 1980

____________ "Words triggered by images: images triggered by words," Mathematics Teaching, March, 1982

Ernest, C. H. "Imagery ability and cognition: A Critical Review," Journal of Mental Imagery, 1977, 1, 181-216

Feinke, F. A. "Levels of equivalence in imagery and perception," Psy. Review, 1980, 87, 113-132

Gattegno, Caleb "Perception and Action as Bases of Mathematical Thought," For the Teaching of Mathematics, Vol. 2, 1963

_______________ "Mathematics and Imagery," Mathematics Teaching, Winter, 1965, #33, 22-24

_______________ "Foundations of Geometry," For the Learning of Mathematics, 1980, 1, #1, 10-15

_______________ "Mathematics: Tangible and Visible," Educational Solutions Newsletter, New York, 1980, IX, #3, 2-16

Hilgard, E. R. "Imagery and Imagination in American Psychology," Journal of Mental Imagery, 1981, 5, 5-65

Kosslyn, S. M. et al "On the demystification of mental imagery," Behavioral and Brain Sciences, 1979, 2, 535-581

Kosslyn, S. M. "The Medium and the Message in Mental Imagery: A Theory," Psy. Review, 1981, 88, 46-66

Lean, G. A. & Clements, M. A. "Spatial ability, visual imagery, and mathematical performance," Educational Studies in Mathematics, 1981, 12, #3, 267-299.

Marks, D. F. "Mental Imagery and Consciousness: A Theoretical Review," in A. A. Sheikh (ed.), Imagery: Current Theory, Research, and Applications, Wiley, New York, 1981

Nicolet, J. L. "Intuition mathématique et dessins animés," in C. Gattegno et al, L'Ensignement des Mathématiques, Vol. 2, Delachanz et Nristlé,: Neuchâtel, 1958, 63-99

Norman, D. N. "Twelve Issues for Cognitive Science," Cognitive Science, 1980, 4, 1-32

Paivio, A. & Cohen, M. "Eidetic Imagery and Cognitive Abilities," Journal of Mental Imagery, 1979, 3, 53-64

Palmer, L. "On the Animated Geometry Films," Educational Solutions Newsletter, New York, 1980, IX, #3, 16-19

Powell, A. "A Report on Animated Geometry Training Sessions," New York, 1981. Unpublished manuscript.

Pressley, G. M. "Imagery and Children's Learning: Putting the Picture in Developmental Perspective," Review of Educational Research, 1977, 47, 585-622

Pylyshyn, Z. W. "What the Mind's Eye Tells the Mind's Brain:

A Critique of Mental Imagery," Psy. Bulletin, 1973, 80, 1, 1-24

Rasco, R. W. et al "Imagery Instructions and Drawings in Learning Prose," Journal of Educational Psychology, 1975, 67, #2, 188-192

Shepard, R. N. "The Mental Image," Amer. Psy., 1978, 33, 125-137

_______________ "Externalization of Mental Images and the Act of Creation," in Randhawa & Coffman (eds.), Visual Learning, Thinking, and Communication, Academic Press, New York, 1978

Shepard, R. N. & Podgorny, P. "Cognitive processes that resemble perceptual processes," In Estes, W. K. (ed.), Handbook of Learning and Cognitive Processes: V. Human Information Processing, Hillsdale, N. J., Lawrence Erlbaum, 1978, 189-237

Sommer, R. "Strategies for Imagery Research," Journal of Mental Imagery, 1980, 4, 115-121

Tahta, R. "About Geometry," For the Learning of Mathematics, 1980, 1, #1, 2-9

_______________ "Some Thoughts Arising from the New Nicolet Films," Mathematics Teaching, March, 1981, #94, 25-29

_______________ "Words, Images, and Objects," Mathematics Teaching, Sept., 1981, #96, 34-37

...ry movement, latency of reported movement, estimated direction of the movement, and magnitude of movement. Subjects who were judged to be high in hypnotic susceptibility had significantly lower latencies and reported significantly more direction changes. The remaining measures did not differ as a function of level of hypnotic susceptibility.

The difference in perceptual reports between subjects judged high and low in hypnotic susceptibility has been replicated with a high degree of reliability in other labs as well (e.g., Graham & Evans, 1977, Karlin, 1979, Crawford, 1981). Thus, the phenomenon appears to be robust. Yet, if verbal reports of a subject's perceptions of images were not permissable, the phenomenon would not have been reported at all and we would not know of its existence. It is because of such examples that I agree with Hilgard that the ultraconservatism of radical behaviorism with respect to data collecting can no longer be tolerated.

In order for the study of mental imagery to progress, the most valuable tool at our disposal for determining what subjects experience is the verbal report. How else can we determine what they perceive? For example, with respect to the study of eidetic imagery, potential eidetikers could not be identified without reliance upon verbal reports (Wallace, 1978). I am pleased to see that interest in this area has been rekindled, thanks to the excellent reviews of the literature in this area by Ahsen (1977a) and Haber (1979).

The employment of verbal reports also has enabled us to gain some insight in the area of identifying potential factors which comprise the characteristic of hypnotic susceptibility. It is not surprising that the ability to produce imagery is an important factor in determining whether one is judged high in hypnotic susceptibility (Wallace, 1980). Yet, without verbal introspective reports, how could this factor have been identified?

I would agree with Hilgard that the emergence of cognitive psychology, specifically a branch referred to as "consciousness psychology," (Ornstein, 1977) has permitted us to study images of all varieties with the introspective tool which for many years was not considered "scientific"; and that we can no longer ignore the importance of images in learning, memory, and perception. Since investigators have shown that images can be empirically studied (and reliably so), there should be no excuse for failing to go forward with the advancement of the study of mental imagery.

A COMMENT ON THE COMMENTS

Ernest R. Hilgard

It is a pleasure to find so much interest in and so many varied approaches to imagery as represented in the fifteen commentaries. The diversity made me curious about the degree of overlap in the references cited by the commentators. It is natural to expect a few self-references by the commentators because their published work was a reason for inviting them, and it is to be expected that they would cite a few references from the target paper in order to comment on them. When these are eliminated there are left some 50 books and articles cited by the fifteen authors. I recognize that there is social pressure against long lists of references in a commentary, so that the ones cited usually represent something that is fresh, pertinent, or telling in criticism. The diversity of viewpoints in these critical citations is shown by their little overlap. Three commentators cited studies bearing Kosslyn's name, two Hunter's and two Orne's, otherwise the field was wide open with all other citations unique to one set of comments. I do not wish to make too much of this, because authors who might seem narrow by this method of counting often referred also to their own books or reviews which contained references reflecting many viewpoints. Still, I believe that the choice of favored citations that overlap so little points to the varied preoccupations of investigators who give attention to imagery. If we care to use Kuhn's terminology, it appears that the theory of imagery is in a preparadigmatic stage.

My role has been served by catalyzing this rich discussion, and I do not propose to comment on the individual papers. For any reader coming new to this complex literature, I strongly recommend the earlier series in which a paper by Haber on eidetic imagery was followed by a number of comments (Haber, 1979). A forthcoming chapter by Marks (1981, in press) is another theoretical review with an excellent literature coverage, and Kosslyn's (1980) book-length survey deals with the major issues in detail.

REFERENCES

Agadjanian, K. *Le mécanisme des troubles perceptive-associatifs en rapport avec l'origine de l'hallucination et du délire.* Paris: Peyronnet, 1946. [JWB]

Ahsen, A. *Basic concepts in eidetic psychotherapy.* New York: Brandon House, 1968. [PA & EFC]

Ahsen, A. *Eidetic parents test and analysis: A practical guide to systematic and comprehensive analysis.* New York: Brandon House, 1972. [AA, ERH, DM, PAR & EFC]

Ahsen, A. Eidetics: An overview. *Journal of Mental Imagery,* 1977, 1, 5-38. (a) [AA, ERH, PMc, BW]

Ahsen, A. *Psycheye: Self-analytic consciousness.* New York: Brandon House, 1977. (b) [PA & EFC]

Ahsen, A. Eidetics: Redefinition of the ghost and its clinical application. *Behavioral and Brain Sciences,* 1979, 2, 594-596. [AA, RNH, ERH]

Ahsen, A. Imagery approach in the treatment of learning disability. *Journal of Mental Imagery,* 1981, 5, 157-196. [AA]

Allport, G. W. Eidetic imagery. *British Journal of Psychology,* 1924, 15, 99-120.

Anderson, J. R. Arguments concerning representations for mental imagery. *Psychological Review,* 1978, 85, 249-277. [JFK]

Angell, J. R. Imageless thought. *Psychological Review,* 1911, 18, 295-323. [ERH]

Aserinsky, E., & Kleitman, N. Regularly occurring periods of eye motility and concomitant phenomena during sleep. *Science,* 1953, 118, 273-274. [PMc]

Barber, T. X. Eidetic imagery and the ability to hallucinate at will. *Behavioral and Brain Sciences,* 1979, 2, 596-597. [ERH]

Barber, T. X. Additional data on the unusual characteristics of excellent hypnotic subjects. Paper presented at 88th Annual Convention, American Psychological Association, September 3, 1980. [ERH]

Bartlett, F. C. Psychology in relation to the popular story. *Folk Lore,* 1920, 30, 264-293. (a) [IMLH]

Bartlett, F. C. Some experiments on the reproduction of folk stories. *Folk Lore,* 1920, 31, 30-47. (b) [IMLH]

Bartlett, F. C. *Remembering: A study in experimental and social psychology.* Cambridge, England: Cambridge University Press, 1932. [AA]

Berger, G. H., & Gaumitz, S. C. Self-rated imagery and vividness of task pictures in relation to visual memory. *British Journal of Psychology,* 1977, 3, 283-288. [ERH]

Berger, G. H., & Gaumitz, S. C. B. Self-rated imagery and encoding strategies in visual memory. *British Journal of Psychology,* 1979, 70, 21-24. [DM]

Betts, G. H. *The distribution and functions of mental imagery.* New York: Teachers College, Columbia University, 1909. [ERH, DM]

Binet, A. *Psychologie des grands calculateurs et joueurs d'echecs.* Paris: Hachette, 1894. [DHH]

Block, N. J. (Ed.) *Imagery.* Montgomery, Vermont: Bradford, 1980. [RRH]

Boring, E. G. *A history of experimental psychology* (2nd Ed.). New York: Appleton-Century-Crofts, 1950. [ERH, PAR & EFC]

Bower, G. H. Analysis of a mnemonic device. *American Scientist,* 1970, 58, 496-510. [RNH, ERH]

Bower, G. H. Experiments in story understanding and recall. *Quarterly Journal of Experimental Psychology,* 1976, 28, 511-534. [IMLH]

Bowers, P. G., & Bowers, K. S. Hypnosis and creativity: A theoretical and empirical rapprochement. In E. Fromm & R. E. Shor (Eds.), *Hypnosis: Research developments and new perspectives* (Revised ed.). New York: Aldine, 1979. [ERH]

Brown, J. *Mind, Brain and Consciousness.* New York: Academic Press, 1977. [JWB]

Brown, J. *Image and object.* In preparation, 1982. [JWB]

Brown, R., & Kulik, J. Flashbulb memories. *Cognition,* 1977, 5, 73-79. [JFK]

Burt, C. *The factors of the mind.* London: University of London Press, 1940. [PMc]

Chase, W. G., & Simon, H. A. Perception in chess. *Cognitive Psychology,* 1973, 4, 55-81. [DHH]

Coltheart, M. Iconic memory and visible persistence. *Perception and Psychophysics,* 1980, 27, 183-228. [DHH]

Crawford, H. J. Hypnotic susceptibilty as related to Gestalt closure tasks. *Journal of Personality and Social Psychology,* 1981, 40, 376-383. [ERH, BW]

Creel, R. Radical epiphenomenalism: B. F. Skinner's account of private events. *Behaviorism,* 1980, 8, 31-54. [JSA]

Degh, H. *Folktales and society.* Bloomington: Indiana University, 1969. [IMLH]

Dennett, D. C. *Brainstorms.* Montgomery, Vermont: Bradford Books, 1978. [ERH]

Dennis, W. (Ed.) *Readings in the history of psychology.* New York: Appleton-Century-Crofts, 1948. [ERH]

Ehrlichman, H., & Weiner, M. S. EEG asymmetry during covert mental activity. *Psychophysiology,* 1980, 17, 228-235. [JSA]

Ellenberger, H. F. *The discovery of the unconscious: The history and evolution of dynamic psychiatry.* New York: Basic Books, 1970. [JFK]

Ericsson, K. A., Chase, W. G., & Faloon, S. Acquisition of a memory skill. *Science,* 1980, 208, 1181-1182. [RNH]

Esquirol, J. E. D. *Des maladies mentales.* Paris: Bailliere, 1838. [ERH]

Finke, R. A. Levels of equivalence in imagery and perception. *Psychological Review,* 1980, 87, 113-132. [DM]

Freud, S. Screen memories. In J. S. Strachey (Ed.), *The standard edition of the complete psychological works of Sigmund Freud.* Vol. 6. London: Hogarth Press, 1960. (Originally published, 1899) [JFK]

Galton, F. *Inquiries into human faculty and its development.* London: Dent, 1883. [ERH, DM, MTO & KMMc]

Gordon, R. An investigation into some of the factors that favor the formation of stereotyped images. *British Journal of Psychology,* 1949, 39, 156-167. [ERH, PMc]

Graham, C., & Evans, F. J. Hypnotizability and the development of waking attention. *Journal of Abnormal Psychology,* 1977, 86, 631-638. [BW]

Graham, K. R. Brightness contrast by hypnotic hallucination. *International Journal of Clinical and Experimental Hypnosis,* 1969, 17, 62-73. [ERH]

Gruber, H. E. *Darwin on man: A psychological study of scientific creativity.* New York: Dutton, 1974. [PMc]

Gur, R. C., & Gur, R. E. Handedness, sex and eyedness as moderating variables in the relation between hypnotic susceptibility and functional brain asymmetry. *Journal of Abnormal Psychology,* 1974, 83, 635-643. [ERH]

Gur, R. C., & Hilgard, E. R. Visual imagery and the discrimination of differences between altered pictures simultaneously and successfully presented. *British Journal of Psychology,* 1975, 66, 341-345. [ERH, DM]

Haber, R. N. Eidetic imagery — real or imagined? *Scientific American,* 1969, 220 (April), 36-44. [RNH]

Haber, R. N. How we remember what we see. *Scientific American,* 1970, 222 (May), 104-115. [RNH]

Haber, R. N. Visual perception. In M. R. Rosensweig & L. W. Porter (Eds.), *Annual review of psychology,* Vol. 29, 1978. Pp. 31-59. [RNH]

Haber, R. N. Twenty years of haunting eidetic imagery: Where's the ghost? *Behavioral and Brain Sciences,* 1979, 2, 593-630. [AA, BRB, RNH, ERH, DHH, JFK, PMc, DM, PAR & EFC, BW]

Haber, R. N. How we perceive depth from flat pictures. *American Scientist,* 1980, 68, 370-380. (a) [AA, RNH]

Haber, R. N. Perceiving space from pictures: A theoretical analysis. In M. A. Hagen (Ed.), *The perception of pictures. Vol. 1. Alberti's Window.* New York: Academic Press, 1980. Pp. 3-31. (b) [AA, RNH]

Haber, R. N., & Haber, R. B. Eidetic imagery: I. Frequency. *Perceptual and Motor Skills,* 1964, 19, 131-138. [RNH]

Haber, R. N., & Wilkinson, L. Perceptual components of computer displays. *Computer Graphics and Application,* 1981, 1, in press. [RNH]

Hatakeyama, T. The process of having identified an adult eidetic person and her eidetic experiences in daily life. *Tohoku Psychologica Folia,* 1974, 33, 102-118. [DM]

Hecaen, H., & Ropert, R. Hallucinations auditives au cours de syndromes neurologiques. *Annals of Medical Psychology,* 1959, 117, 257-306. [JWB]

Held, R. Action contingent development of vision in neonatal animals. In D. P. Kimble (Ed.), *Experience and capacity.* New York: New York Academy of Sciences, 1968. Pp. 31-111. [KHP]

Hering, E. *Zu lehre vom lichtsinne.* (Revised Ed.) Vienna: Gerold, 1878. [ERH]

Hilgard, E. R. *Divided consciousness: Multiple controls in human thought and action.* New York: Wiley-Interscience, 1977. [JFK]

Hilgard, E. R. Imagery and imagination in American psychology. *Journal of Mental Imagery,* 1981, 5, 5-62. [AA, RNH]

Hilgard, J. R. Personality and hypnotizability: Inferences from case studies. In E. R. Hilgard (Ed.), *Hypnotic susceptibility.* New York: Harcourt Brace & World, 1965. [BRB, ERH]

Hilgard, J. R. *Personality and hypnosis: A study of imaginative involvement.* Chicago: Uni versity of Chicago Press, 1970 [BRB, ERH]

Hilgard, J R *Personality and hypnosis A study of imaginative involvement* (2nd Ed) Chicago University of Chicago Press 1979 [BRB, ERH]

Hofstadter, D R *Godel, Escher, Bach* New York Basic Books 1979 [RRH]

Holding D H Brief visual memory for English and Arabic letters *Psychonomic Science*, 1972 28 241 242 [DHH]

Holding D H The evaluation of chess positions *Simulation and Games* 1979, 10 207 221 [DHH]

Holt R R Imagery The return of the ostracized *American Psychologist* 1964 19 254 264 [ERH]

Holton G *Thematic origins of scientific thought Kepler to Einstein* Cambridge Massachusetts Harvard University Press, 1973 [RRH]

Hoyenga, K B & Wallace, B Sex differences in the perception of autokinetic movement of an afterimage *Journal of General Psychology* 1979, 100, 98 101 [BW]

Hunter I M L The easel procedure and eidetic characteristics *Behavioral and Brain Sciences*, 1979, 2, 605 (a) [DHH]

Hunter, I M L Memory in everyday life In M M Gruneberg & P E Morris (Eds), *Applied problems in memory* New York Academic Press, 1979 (b) [IMLH]

Hurvich L M & Jameson D A quantitative theoretical account of color vision *Transactions of the New York Academy of Sciences*, 1955 18, 33 38 [ERH]

Jaensch E R *Eidetic imagery and typological methods of investigation* (2nd Ed) New York Harcourt Brace, 1934 (Originally published in German 1925) [ERH IWB PMc]

Jaynes J *The origins of consciousness in the breakdown of the bicameral mind* Boston Houghton Mifflin, 1976 [JFK]

Jones P D & Holding D H Extremely long term persistence of the McCollough effect *Journal of Experimental Psychology Human Perception and Performance*, 1975 1, 323 327 [DHH]

Karlin R A Hypnotizability and attention *Journal of Abnormal Psychology*, 1979 88 92 95 [BW]

Klüver H Neurobiology of normal and abnormal perception In P Hoch & J Zubin (Eds) *Psychopathology of perception* New York Grune and Stratton 1965 [JWB]

Kosslyn S M *Image and mind* Cambridge Massachusetts Harvard University Press, 1980 [ERH]

Kosslyn S M The medium and the message in mental imagery A theory *Psychological Review* 1981 88 46-66 [JFK, PMc]

Kosslyn S M Pinker S Smith G E & Schwartz S P On the demystification of mental imagery *Behavioral and Brain Sciences*, 1979, 2 535 581 [DM]

Kuhn T S The structure of scientific revolutions Chicago University of Chicago Press 1962 [PR & EC]

La Mettrie J L de *L'homme machine* Leiden, 1748 (Translated by G C Bussey & M W Calkins as *Man a machine* Chicago Open Court Publishing House, 1929) [RRH]

Leask J Haber R N & Haber R B Eidetic imagery II Longitudinal and experimental results *Psychonomic Science Monographics*, 1969 3 No 1(Whole No 35), 25-48 [RNH]

Lorayne H & Lucas J *The memory book* New York Stein and Day 1974 [RNH]

Lord A B *The singer of tales* Cambridge Massachusetts Harvard University, 1960 [IMLH]

Luria A R *The mind of a mnemonist* Translated from the Russian by L Solotaroff New York Basic Books, 1968 [RNH]

McIlwain D J Eidetic abilities A study of spontaneous and post-picture imagery Honors thesis, University of Otago, 1978 [DM]

McKellar P *Imagination and thinking A psychological analysis* New York Basic Books, 1957 [PMc]

McKellar P *Mindsplit The psychology of multiple personality and the dissociated self*

London J M Dent, 1979 [PMc]

Marks D Visual imagery differences in the recall of pictures *British Journal of Psycholog* 1973, 64, 17 24 [ERH, DM]

Marks, D Eidetic imagery Haber's ghost and Hatakeyama's ghoul *Behavioral and Bra Sciences* 1979, 2, 610-612 [AA]

Marks D F Mental imagery and consciousness A theoretical review In A A Sheikh (Ed *Imagery Current theory, research and applications* New York Wiley, 1981, in pre [ERH, DM]

Meichenbaum, D *Cognitive-behavior modification An integrative approach* New Yo Plenum Press, 1977 [PAR & EFC]

Miller, J G Discrimination without awareness *American Journal of Psychology* 1939, 5 562-578 [ERH]

Mischel, W Toward a cognitive social learning reconceptualization of personality *Psych logical Review*, 1973, 80, 252-283 [JFK]

Monteiro, K P , Macdonald H , & Hilgard, E R Imagery, absorption and hypnosis factorial study *Journal of Mental Imagery*, 1980 2, 63-82 [ERH]

Morel, F *Encephale*, July, 1936 [JWB]

Murch, B M , & Paulson J An extension of the Riggs projection colorimeter *Behavioral R search Methods and Instrumentation*, 1974, 6 403 406 [DHH]

Neisser, U *Cognition and reality Principles and implications of cognitive psychology* S Francisco W H Freeman, 1976 [JFK]

Norman, D N Twelve issues for cognitive science *Cognitive Science* 1980, 4 1 32 [JF

Orne M T The nature of hypnosis Artifact and essence *Journal of Abnormal and Soc Psychology* 1959 58, 277 299 [ERH, DM]

Orne M T Communication by the total experimental situation Why it is important, hov is evaluated and its significance for the ecological validity of findings In P Pliner, Krames, & L Alloway (Eds), *Communication and affect Language and thought* N York Academic Press, 1973 [JFK]

Ornstein, R E *The psychology of consciousness* New York Harcourt Brace 1977 [BV

Paivio, A *Imagery and verbal processes* New York Holt, Rinehart and Winston 19 [ERH]

Pear, T H *Remembering and forgetting* London Methuen 1922 [PMc]

Perky, C W An experimental study of imagination *American Journal of Psychology* 19 21, 422-452 [BRB ERH, MTO & KMMc]

Pinker, S Mental imagery and the third dimension *Journal of Experimental Psycholo, General*, 1980 109, 354-372 [JFK]

Pötzl O *Die optisch-agnostischen storungen* Leipzig Deuticke, 1928 [JWB]

Pribram, K H *Languages of the brain Experimental paradoxes and principles in neurop chology* Englewood Cliffs New Jersey Prentice-Hall, 1971 (2nd Ed Monterey Califo ia Brooks/Cole, 1977) [KHP]

Pylyshyn, Z What the mind's eye tells the mind's brain A critique of mental image *Psychological Bulletin*, 1973, 80, 1-24 [ERH, DHH, PMc, DM]

Pylyshyn, Z The imagery debate Analogue media versus tacit knowledge *Psychologi Review*, 1981, 88, 16-45 [PMc, DM]

Rahn, C The relation of sensation to other categories in contemporary psycholo *Psychological Monographs*, 1913, 16, No 67 [ERH]

Richardson, A Voluntary control of the memory image In P W Sheehan (Ed), *The fu tion and nature of imagery* New York Academic Press, 1972 [ERH]

Sakitt, B Psychophysical correlates of photoreceptor activity *Vision Research* 1976, 129 140 [DHH]

Sarbin, T R , & Juhasz J B The social context of hallucinations *Journal of Mental Image* 1978 2, 117-144 [JFK]

hacter, D L The hypnagogic state A critical review of the literature *Psychological Bulletin*, 1976, 83, 452-481 [PMc]

gal S J , & Nathan S The Perky effect Incorporation of an external stimulus into an imagery experience under placebo and control conditions *Perceptual and Motor Skills*, 1964, 33, 125-137 [BRB, ERH]

guin, E A clinical study of lateral hemianopsia *Journal of Nervous and Mental Diseases*, 1886, 13, 445 454 [JWB]

eehan P W A shortened form of Betts' Questionnaire Upon Mental Imagery *Journal of Clinical Psychology*, 1967, 23, 386-389 [ERH]

eehan P W , Obstoj, I., & McConkey, K Trance logic and cue structure as supplied by the hypnotist *Journal of Abnormal Psychology*, 1975, 84, 331-345 [ERH]

eehan, P W Hypnosis and the processes of imagination In E Fromm & R E Shorr (Eds), *Hypnosis Developments in research and new perspectives* New York Aldine, 1979 Pp 381 411 [MTO & KMMc]

epard, R N The mental image *American Psychologist*, 1978, 33, 125-137 [ERH, DHH, JFK]

or R E The frequency of naturally occurring "hypnotic-like" experiences in the normal college population *International Journal of Clinical and Experimental Hypnosis*, 1960, 8, 151-163 [ERH]

nger, J L *Daydreaming An introduction to the experimental study of inner experience* New York Random House, 1966 [PMc]

nger J L , & Pope, K S *The power of human imagination* New York Plenum, 1978 [PMc]

inner, B F *The behavior of organisms* New York Appleton-Century-Crofts, 1938 [ERH]

inner, B F *Science and human behavior* New York MacMillan, 1953 [BRB]

inner B F *About behaviorism* New York Alfred A Knopf, 1974 [BRB]

inner B F Why I am not a cognitive psychologist *Behaviorism*, 1977, 5, 1-10 [JFK]

mmer R *The mind's eye Imagery in everyday life* New York, Delacourte Press, 1978 [PMc]

erling G The information available in brief visual presentations *Psychological Monographs*, 1960 74, 11 [DHH]

erry R W Perception in the absence of the neocortical commissures *Perception and its disorders*, 1970, 48, 123-138 [KHP]

ampfl T Implosive therapy In P Olsen (Ed), *Emotional flooding* New York Human Sciences Press, 1976 [PAR & EFC]

anding L Learning 10 000 pictures *Quarterly Journal of Experimental Psychology*, 1973 25, 207 222 [RNH]

lly, J *The human mind, Vol 1* New York D Appleton, 1892 [ERH]

utcliffe J P , Perry, C W , & Sheehan, P W Relation of some aspects of imagery and fantasy to hypnotic susceptibility *Journal of Abnormal Psychology*, 1970, 76, 279-287 [MTO & KMMc]

ellegen, A , & Atkinson, G Openness to absorbing and self-altering experiences ("absorption"), a trait related to hypnosis *Journal of Abnormal Psychology*, 1974, 83, 268-277 [BRB, ERH]

itchener, E B *Lectures on the experimental psychology of the thought-processes* New York MacMillan, 1909 [ERH]

reisman A Focused attention in the perception and retrieval of multidimensional stimuli *Perception and Psychophysics* 1977, 22 (1), 1-11 [KHP]

ingerleider, L , Ganz, L , & Pribram, K H Size constancy in Rhesus monkeys Effects of pulvinar prestriate and inferotemporal lesions *Experimental Brain Research*, 1977, 27, 251-269 [KHP]

Walker, N S , Garrett, J B , & Wallace, B The restoration of eidetic imagery via hypnotic age regression A preliminary report *Journal of Abnormal Psychology*, 1976, 85, 335 337 [ERH, BW]

Wallace B Restoration of eidetic imagery via hypnotic age regression More evidence *Journal of Abnormal Psychology*, 1978, 87, 673 675 [BW]

Wallace, B Autokinetic movement of an imagined and an hypnotically hallucinated stimulus *International Journal of Clinical and Experimental Hypnosis*, 1980, 28, 386 393 [BW]

Wallace, B , Garrett, J B , & Anstadt, S P Hypnotic susceptibility, suggestion, and reports of autokinetic movement *American Journal of Psychology*, 1974, 87, 117-123 [BW]

Wallace, B & Hoyenga, K B Autokinetic movement of an induced afterimage *Journal of General Psychology*, 1978, 98, 173-178 [BW]

Watson, J B The place of the conditioned-reflex in psychology *Psychological Review*, 1916, 23, 89-116 [ERH, RRH, BW]

Whitehead, A N (1925) *Science in the modern world* (Revised Ed) New York Mentor, 1952 [RRH]

Wilson, S C Memory, hallucinatory ability, and imaginative histories of excellent hypnotic subjects Paper presented at 88th Annual Convention, American Psychological Association, September 3, 1980 [ERH]

Wilson, S C , & Barber, T X The creative imagination scale as a measure of hypnotic responsiveness Application to experimental and clinical hypnosis *American Journal of Clinical Hypnosis*, 1978, 20, 235-249 [ERH]

Wolpe, J *The practice of behavior therapy* New York Pergamon Press, 1969 [PAR & EFC]

Woodworth, R S Imageless thought *Journal of Philosophy*, 1906, 3, 701-708 [ERH]

Woodworth, R S *Experimental psychology* (1st Ed) New York Holt, 1938 [PMc]

From: Hilgard, E.R. Imagery and Imagination in American Psychology, Journal of Mental Imagery, 1981, 5-65

APPENDIX 2

Imagery and Mathematics

Working Group C

Imagery in Calculus

(Subgroup Report)

Points considered were:

1. We should concentrate on the uses of our images.
2. One begins with, usually, a dynamic view of limiting processes: this becomes static as it takes the fomr of an algebra of limits.
3. The algebra of limits is what we use for calculations or proofs.
4. To get perspective, one should use many images.
5. D_x produces different images when seen as
   a) a slope
   b) a rate of change
   c) a magnification factor under a mapping.
6. In what way does tangency have a meaning other than via a limit?
7. Two and three dimensional limit situations make some points quite clearly.
8. What images help us us to connect, say, differentiation, anti-differentiation, definite integration etc. (this needs more thought?)
9. Looking at the workings of an odometer and speedometer might help in understanding the Fundamental Theorem.
10. $D_x (u + v) = D_x u + D_x v$

    $D_x (u \cdot v) = D_x u \cdot D_x v$

    have explanatory images.
11. The hatchet planimeter might help in visualizing the measure of areas.

Ralph Staal

APPENDIX 3

UTILISATION DES IMAGES DANS L'APPRENTISSAGE DU CONCEPT DE LIMITE AU NIVEAU COLLEGIAL

Mario Lavoie, Ernestine Lepage, Yvan Roux

Université du Québec à Rimouski

juin 1982

(Projet de recherche subventionné par le ministère de l'Education du Québec. Subvention FCAC: EQ1841)

1. La problématique

Lors de la présentation du concept de la limite, au niveau collégial, un type d'images est généralement utilisé pour aider à la formation de ce concept. Ce type d'images peut être assez bien décrit par les vocables "cinétique", "dynamique". Cependant bien peu d'auteur, sinon aucun, ne se préoccupe, dans ce contexte, de l'utilisation de la potentiellité qu'a l'intellect, d'abstraire des concepts, même complexes, à partir de situations semi-concrètes. Un exemple éclairant, dans ce contexte, est l'utilisation de la notion de tangente à une courbe, qui a une sémantique claire bien que difficilement verbalisable, pour aider à la formation du concept mathématique de la dérivée.

Toutefois la situation du concept de la limite se complique encore plus, lorsque l'on doit l'utiliser. En effet, dans le contexte de l'apprentissage du calcul infinitésimal au niveau collégial, il est hors de question de ne développer uniquement que de façon hypothético-déductive, les propriétés de la limite de fonctions. On doit donc nécessairement prévilégier une approche intuitive à ce niveau. Cependant, les images cinétiques ou dynamiques qui étaient très utiles pour former, appréhender et consolider le concept de la limite, le deviennent beaucoup moins lorsqu'il s'agit de calculer effectivement des limites. Tout se passe comme si on avait tourné quelques pages du grand volu[illegible] dépositaire du corpus de la théorie et que l'on était dans un nouveau chapitre. Maintenant un tout nouveau type d'images est utilisé pour voguer dans ces eaux non reconnues.

Ce type d'images, que nous qualifions de "manipulations algébriques" et que les étudiants de niveau collégial ont souvent eu à utiliser, relève beaucoup plus du domaine de la représentation symbolique utilisée par l'intellect pour traiter ce type d'information, que de la représentation figurale. Cette discontiguité, cette brisure maintenant bien apparente dans le déroulement psychologique de la théorie, personne à notre connaissance, n'a réussi, ni même essayé de les réparer.

Notre hypothèse de travail consiste donc, à jeter un pont entre les deux parties psychologiquement séparées du même corpus de théorie. Ce pont sera établi, à notre avis, en utilisant le même type d'images "cinétiques" ou "dynamiques", que l'on a utilisé lors de la formation du concept de la limite, dans le but, cette fois-ci, de soutenir l'abstraction de certaines propriétés algébriques cardinales du concept de la limite, pour que l'apprenant puisse s'élaborer un outil fiable de localisation à l'intérieur de la théorie. En un mot, l'apprenant devrait avoir construit, au moyen de cette approche, son modèle de la théorie, ceci l'habilitant en retour, à simuler dans le cadre de ce modèle, d'abord localement puis plus globalement, des pans entiers de théorie.

2. Les motivations de notre approche

2.1 Le rôle de l'image

Plusieurs motifs nous incitent à privilégier l'image dans la présentation de la notion de limite. Une des premières raisons repose sur le rôle de l'image dans la préparation et le fonctionnement des opérations cognitives.

Cette conception s'appuie sur les études de Piaget qui considère "l'image comme un instrument de la connaissance". Piaget distingue deux aspects à l'image: un aspect figuratif, un aspect opératif.

L'aspect figuratif "caractérise les formes de cognitions qui du point de vue du sujet, apparaissent comme des copies du réel, quoique du point de vue objectif, elles ne fournissent qu'une correspondance approximative des objets ou des évènements" (cf. Piaget, 1969, p. 73).

L'aspect opératif caractérise "les formes de connaissance qui consistent à modifier l'objet ou l'évènement à connaître, de manière à atteindre les transformations et leurs résultats et non pas uniquement les configurations statiques correspondant aux états reliés par ces transformations" (Piaget, 1969, p. 73).

En considérant ces deux aspects des fonctions cognitives, il serait pédagogiquement raisonnable de fournir aux étudiants qui apprennent la notion de limite, des images dynamiques de cette notion de limite qui seraient susceptibles d'agir sur

l'aspect opératif de leur connaissance. Nous croyons que cette forme de présentation semi-concrète de la notion de limite permettrait non seulement la création d'images mentales nécessaires à la formation de ce concept mais agirait aussi sur l'aspect opératif de la connaissance. Ainsi l'étudiant pourrait arriver à transformer les images qu'on lui fournit de façon à dégager l'essence et les propriétés algébriques du concept de limite.

2.2 Les étapes de la formation d'un concept

L'approche proposée s'articule, de plus, à partir des étapes de la formation d'un concept, décrites par Woodruff. Selon cet auteur, la formation d'un concept s'élabore selon les cinq étapes suivantes:

- la perception d'objets ou d'évènements concrets;
- les images mentales;
- les concepts intuitifs;
- les concepts verbalisés;
- les concepts généralisés (Woodruff, in Siegel, 1967).

La première étape, que Paré appelle aussi l'imput référentiel, est la source de "percepts" qui deviennent des concepts par la suite. "Il faut que celui qui apprend puisse avoir des perceptions provenant du réel et non uniquement des présentations de symboles qui remplacent ces objets réels" (Paré, 1979, vol. II, p. 62). Ce n'est que progressivement que l'on parvient à se dégager de ces "perceptions référentielles" pour parvenir à la généralisation du concept. A ce moment l'apprenant devient capable d'appliquer et d'utiliser le concept ainsi formé.

2.3 Principes didactiques sous-tendant notre approche

D'autres raisons nous incitent à concourir à l'élaboration de la sémantique de la notion de limite à partir d'images. Ces raisons prennent appuies sur les principes didactiques suivants:

D'une part, tout concept s'élabore prioritairement par l'individu qui a à l'intérioriser. C'est en corollaire à ce principe qu'il est important de fournir aux apprenants le maximum d'occasions nécessaires pour qu'ils puissent individualiser, personnaliser, l'élaboration et l'opérationalisation du concept de limite.

D'autre part, un concept n'est pas statique; il s'enrichit au fur et à mesure de l'expérience de l'apprenant. Ainsi la notion de limite gagnerait beaucoup en sémantique à être abordée de façon continue sous différents aspects pour permettre aux apprenants de niveau collégial, d'en saisir le mieux possible le plus grand nombre de facettes.

3. Epilogue

Les considérations précédentes ont tracé la voie à un programme d'actions que les auteurs s'appliquent à concrétiser dans le cadre d'un projet de recherche subventionné. Cependant toute concrétisation de cette approche, pour qu'elle soit adéquate, doit s'inscrire dans le cadre de matériels didactiques ayant une ampleur telle, qu'il est hors de question de produire ici pour des raisons évidentes de manque d'espace. Peut être en résultera-t-il éventuellement, une monographie?

Working Group D

PROBLEM SOLVING IN THE SCHOOL CURRICULUM

Leader: Sol. E. Sigurdson, University of Alberta

The three sessions of the workshop will focus on three major topics. First, the basic research, historical and present, on problem solving is rich, with several divergent approaches being represented. The main people here are Polya, Krutetskii, Kantowski, LeBlanc, Lester, Lesh and Greeno. During this session we will attempt to criticise typical studies of these researchers. There will also be an opportunity for members to discuss their current research related to problem solving.

The second part will examine several curriculum proposals for introducing problem solving into elementary and secondary school classrooms. Such proposals range from Polya's suggestions and discovery teaching (mathematizing mode) to recent proposals such as the Calgary Project, the Ohio study TOPS (Immerzeel), "Agenda for Action", and efforts made by individual school systems. Again participants will be encouraged to contribute their own curriculum and inservice ideas.

The third part will examine our curricular obligation to problem solving in elementary and secondary school and explore future directions which seem promising in basic research, curriculum development, and teacher inservice courses, for fulfilling this obligation. The presenter will identify his own concerns with current trends and encourage the group to outline a "research program in problem solving" which could be endorsed by the whole conference. The latter, indeed, is more an ideal than a goal.

The question to be kept before the workshop group at all times is "What difference can this make to teachers and students in our mathematics classrooms?"

PROBLEM SOLVING IN THE CLASSROOM

Sol E. Sigurdson

The deliberations of the three day workshop in this area were focussed upon classroom implications of problem solving - curriculum and instruction - rather than upon a psychological understanding of the process of problem solving. Discussion centered upon a full elaboration of many possibilities, rather than upon specific recommendations to practice or specific proposals for research. The focus upon classrooms seems justified in that a great deal of attention is being given to teachers becoming more involved in problem solving in their teaching. Ultimately our concern should be with a psychological understanding of problem solving but the urgent need is to help teachers change classroom situations to enhance problem solving development.

Our first attack was to try to give meaning to the term problem solving, especially as it relates to the curriculum. Three points of view immediately emerged, all of them centre on how closely the problem solving process is related to mathematics. One view sees problem solving as a generalized process related to solving real problems outside the classroom; another sees problem solving as a generalized mathematics process capable of being developed independently of mathematical content; the third sees problem solving as a process closely related to mathematical content. This divergence of views concerns the goals classroom teachers should have for their charges, rather than the larger philosophical issue.

The question is complicated further by students of differing aptitudes benefitting differentially from each approach.

A second concern was to identify current popular analyses of classroom problem solving. The major American approach being promoted by the National Council of Teachers of Mathematics emphasizes the development of strategies, stemming from Polya's work. The PRISM (1) (Priorities in School Mathematics) was also identified as a document supportive of the classroom thrust in problem solving. The definition used by PRISM (see Appendix) is problem solving as "methods of thinking and logical reasoning" supported by a teaching method of assigning problems "designed to challenge students to think." A teaching method also high on the PRISM list was assigning "projects that involve real-life problem situations ... to in-individual or teams of students". If anything, the PRISM view (which primarily reflects views of Canadian teachers) or problem solving appears to be extremely global. A third bit of evidence indicating the popularity of problem solving is a bibliography (see Appendix) on problem solving from 3 or 4 leading educational (mathematics) journals and Dissertation Abstracts which numbered approximately 140 articles. The research reported in the articles does not have any identifiable single direction. Our discussion of popular analyses of problem solving touched on three other perspectives: the historical perspective of mental discipline (training in logical reasoning) and Eulidean geometry; Krutetski's analysis of the problem solving behavior of bright children; and the psychol-

(1) PRISM Canada, University of Alberta Printing Services, Edmonton, 1981

ical thrust of the creativity school emphasizing such behaviours as fluency, flexibility and originality. And, finally of special interest to our concern was the thesis by Marcucci on "A Meta-Analysis of Research on Methods of Teaching Mathematical Problem Solving." (in Bibliography, see Appendix).

The workshop participants agreed to focus on the following five themes which continue to show up in the teaching of problem solving:

1. A strategies approach as supported by NCTM and promoted by "An Agenda for Action."
2. Discovery teaching where the curriculum content is generated as a solution to a posed problem.
3. Laboratory teaching of mathematics closely related to discovery teaching but emphasing manipulative models and "applied" problems.
4. Diagnostic approaches to problem solving improvement including the direct-instruction research on the teaching of mathematics.
5. Problem solving and technology, using computers and calculators as instructional aids and working tools.

This report will not try to follow through the discussion of the workshop nor even develop an analysis of each of the five themes. Our intent here is to deal with several issues that each of these five approaches raise. Probably the only concensus of the workshop was that each of these themes (approaches) has merit as an approach to problem solving in the classroom. It became obvious as the discussion proceeded that certain workshop participants were predisposed to certain approaches. Time (3 - half days) did not allow a full debate on any of these themes nor a resolution of the topic in general.

Strategies

The Strategy Check List (see Appendix) identifies the strategies often suggested in this approach. Alberta Education's document, "Let Problem Solving be the Focus for the 1980's", typifying this approach has identified strategies and the grade level at which different strategies should be taught. The approach is similar to a "training in heuristics" approach which has been attempted several times at the senior high school level (see Hunt in Bibliography). Issues arising in relation to focusing on strategies were hotly debated. The first one concerns the over-emphasis on "knacky" problems. The use of problems that require a special "knack" tends to shift the emphasis away from mathematics and certainly away from the particular mathematics in the curriculum. Of course, some educators don't see this as a problem. In fact, instances were identified where "problem solving through strategies" was being taught as a course quire separate from mathematics. A second issue centered around how explicitly the strategies should be taught and reinforced. Several educators emphasize games through which the strategies may be developed. It would appear that if we are inter-

ested in making problem solving are integral parts of the mathematics curriculum the strategies approach will not do it for us. In the end however, the strategies approach could be quite useful, at least, for introducing problem solving into the curriculum.

Discovery Teaching

This idea has had almost as long (and certainly far less productive) a history as problem solving itself. The fundamental notion behind discovery teaching is that everything be taught in the context of a solution to a problem. Discovery teaching sees a problem solving process used throughout instruction. However, actual implementation of discovery teaching is problematic considering classrooms with 30 unmotivated students, a curriculum stressing skills not understanding (nor problem solving) and the wide divergence of mathematical ability in our classrooms. Again within the discovery mode, the problem solving benefit may come from the process activity in which the student engages or, in fact, it may be the result of the deeper content understanding resulting from discovery teaching. All of this is speculation, in any case, because of the lack of research effort in this area. As in all problem solving related areas, the lack of clear definitions in discovery teaching severely hamper effective research and even communication.

One area of interest related to the discovery teaching is the kinds of questions that students ask about mathematics (arithmetic). Does, for example, a discovery teaching treatment increase the number and quality of these questions? Discovery teaching is also related to inductive, intuitive and informal approaches to teaching. All of these approaches should contribute to the flexibility and fluency of students' mathematical ideas which in turn should contribute to their problem solving capabilities.

Another identified area of research interest is the social psychologists' venture into cooperative versus competitive or individualistic learning environments. David W. Johnson's article in the Bibliography (see Appendix) is typical of this research. In the case of research specifically on discovery teaching several issues arise: To what extent should it be student centered as opposed to teacher centered, how should the "ideal type" be modified to account for the practicalities of the everyday classroom, and, of course, the fundamental question of how discovery teaching contributes to the development of problem solving capabilities.

Laboratory Methods

This teaching approach is meant to include all attempts to relate mathematical concepts to real objects or to real (familiar) concepts (applications). Research does indicate that there is problem solving benefits from such activity. Richard Lesh in a recent article in the Arithmetic Teacher (December, 1981) reports problem solving improvement from student investigations of "realistic everyday situations in which substantive ideas are used." Lesh's project emphasizes problems which require 10-45 minute solutions. The questions arising from the laboratory method are similar to those of discovery teaching. The benefits of a laboratory method would seem

to arise from a deeper and broader understanding of mathematical concepts as opposed to a benefit from the actual process engaged in. The problems posed in the laboratory approach emphasize "substantive" concepts (curriculum-content related). A problem solving study in this area using "the questions students ask about mathematics" as criteria would seem to have considerable merit. Finally discovery teaching as an integral feature of the laboratory method is inevitably a possibility and is also inevitably an element confounding a full understanding of the method.

"Time on task" has recently been identified as an important feature of any instrictional treatment. Viewed in the light of this concept, discovery teaching and to a lesser intent a laboratory method shows up as rather inefficient teaching approaches. In fact, any approach which encourages student initiative (such as solving any verbal or non-verbal problems) shows up poorly on the time-on-task criterion.

Diagnostics and Problem Solving

The diagnostic approach to teaching problem solving focuses directly on areas of difficulty such as reading, computational skills and problem analysis. The Hollander articles in the Bibliography (see Appendix) give a good review of the research in this view. The diagnostic approach is closely related to some of the "direct instruction" researches into problem solving. Like most of the direct instruction treatments, it is the extreme advantage of relatively easy classroom implementation. Teachers can easily apply the guide-lines developed by the diagnostic approach in their classrooms. The lack of attractiveness of this approach lies in its reliance on the textbook word problems as the criteria. Most educators argue that problem solving in mathematics goes much beyond word problems. On the other hand, they find it difficult to deny the fact that success in solving word problems correlates closely with success in the larger arena of problem solving.

Technology and Problem Solving

Although very little time was spent on this area in the workshop, the impact of calculators and microcomputers was readily acknowledged. Theoretically, calculators decrease computational distractions and further allow more honest (and realistic) applications. Microcomputers through softward packages such as LOGO have great potential in problem solving improvement, while direct programming (in BASIC language, for example) of mathematical solutions to problems appears, also, to have considerable merit.

Reflections

The last hour of the workshop focussed on suggestions to the classroom teacher interested in a problem solving goal in mathematics. Concensus of the group did not materialize. Participants recommended a focus on computational algorithms as a means to eventual problem solving improvement; the use of heuristics and strategies in and direct attack on problem solving processes; employing applied problems involving curriculum-content mathematics (for example, a farmer growing various crops on various land areas for

various economic benefits); conceptual understanding as a key to problem solving improvement; asking teachers themselves to reflect on their own problem solving capabilities as well as those of their students and proceed accordingly; and any combination of the above.

As a last word, the writer wishes to apologize to the workshop participants for this inadequate report of three very demanding and exciting days. Several issues have been intentionally avoided for lack of the writer's comprehension of these and certainly some have been unintentionally missed. Problem solving in mathematics (and outside of mathematics) is still a vital area of concern. The workshop and this report of it have hopefully contributed to its promotion as a classroom reality.

* BIBLIOGRAPHY *

(Problem Solving)

JRME = Journal for Research in Mathematics Education

DAI = Dissertation Abstracts International

Alderman, Donald L. Artificial Intelligence and Educational Psychology: A Probabilistic Computing Model for Well-Defined Problems. (The University of Connecticut, 1977) DAI 38A: 4670; February 1978.

Alvarez, Manuel S. A Problem Sequence Development of Linear Partial Differential Equations for Advanced Undergraduates. (Columbia University, 1978.) DAI 39B: 4924; April 1979.

Andersen, Lee H. Development Effects in Learning Hierachy Structure for Problem Involving Proportional Reasoning. (University of California, Berkeley, 1979.) DAI 40A: 3879;January 1980.

Anthony, Barbara and Hudgins, Bryce B."Problem-Solving Process of Fifth Grade Arithmetic Pupils." Journal of Educational Research 72: 63-67; November/December 1978.

Babad, Elisha Y. and Bashi, Joseph."On Narrowing the Performance Gap in Mathematical Thinking Between Advantaged and Disadvantaged Children." JRME 9: 323-333; November 1978.

Ballew, Hunter and Cunningham, James W."Diagnosing Strengths and Weakness of Sixth-Grade Students in Solving Word Problems." JRME 13: May 1982.

Bana, Jack and Nelson, Doyal."Distractors in Nonverbal Mathematical Problems." JRME 9: 55-62; January 1978.

Barron, Eloise Dianne Thompson. The Effects of Stressing Heuristics in Mathematics Problem Solving with In-Service Elementary School Teachers. (Georgia State University-College of Education, 1978) DAI 39A: 6602-6603; May 1979.

Bell, Frederick H."Posing and Solving Verbal Problems" Mathematics Teacher 73: 653-656; December 1980.

Bellile, Elke G. Background, Attitude and Problem Solving Characteristics of College Students Whose Lowest ACT Scores Are in Mathematics. (The University of Iowa, 1980.) DAI 41A: 1450-1451; October 1980.

Bernard, John E. A Theory of Mathematical Problem Solving Derived from General Theories of Directed Thinking and Problem Solving. (The University of Texas at Austin, 1978.) DAI 40A: 1322; September 1979.

Blake, John A. Analyzing the Solution of Problems Involving Three-Dimensional Geometric Figures: An Exploratory Study. (The University of Tennessee, 1978.) DAI 39B: 5404; May 1979.

Bratton, George N. The Effects of Heuristic Instruction on Problem Solving Ability in College Algebra. (Mississippi State University, 1977.) DAI 38A: 4001; January 1978.

Bredie, Joseph Wilhelmus Bernardus. The Mediated Acquisition and Application of Heuristic Problem-Solving Strategies in a Defined Word-Problem Task. (The Catholic University of America, 1980.) DAI 41A: 160; July 1980.

Brey, Rita K. Effects of Problem Solving Activities and Calculators on Problem Solving and Computation in Grade Four. (The University of Michigan, 1980.) DAI 41A: 1914; November 1980.

Caldwell, Janet H. The Effects of Abstract and Hypothetical Factors on Word Problem Difficulty in School Mathematics. (University of Pennsylvania, 1977.) DAI 38A: 4637; February 1978.

Caldwell, Janet H. and Goldin, Gerald A. Variables Affecting Word Problem Difficulty in Elementary School Mathematics. JRME 10: 323-336; November 1979.

Canner, Jane. Imagery and Mathematics Word-Problem Solving in Elementary School Students. (Fordham University, 1979.) DAI 40A: 1352-1353; September 1979.

Carpenter, Thomas P.; Corbitt, Mary, Kay; Keppner, Henry S., Jr. Lindquist, Mary M.; and Reys, Robert E. Solving Verbal Problems: Results and Implications from National Assessment. Arithmetic Teacher 28: 8-12; September 1980.

Carpenter, Thomas P.; Corbitt, Mary Key; Kepner, Henry S., Jr.; Lindquist, Mary M.; and Reys, Robert E. NAEP Note: Problem Solving. Mathematics Teacher 73: 427-433; September 1980.

Carpenter, Thomas P.; Corbitt, Mary Key; Kepner, Henry S., Jr.; Lindquist, Mary M.; and Reys, Robert E. NEAP Note: Problem Solving. Mathematics Teacher 73: 427-433; September 1980.

Carpenter, Thomas P.; Hiebert, James; and Moser, James M. Problem Structure and First-Grade Children's Initial Solution Process for Simple Addition and Subtraction Problems. JRME 12; January 1981.

Casteneda, Alberta M.; Gibb, E. Glenadine; and McDermit, Sharon A. Young Children and Mathematics Problem Solving. School Science and Mathematics 82: 22-28; January 1982.

Christensen, Evelyn Brown. Reality versus Fantasy in Algebra Word Problems: An Investigation of the Context Preferences of Students. (University of Kentucky, 1980.) DAI 41A: 574; August 1980.

Christopher, Carolyn J. A Comparision of Stress as Measured by Heartbeat Rate of Sixth-Grade Students During Teaching-Learning Activities in Solving Verbal Problems in a Classroom Setting. (North Texas State University, 1979.) DAI 40A: 3843-3844; January 1980.

Clarkson, Sandra P. A Study of the Relationships Among Translation Skills and Problem-Solving Abilities. (University of Georgia, 1978.) DAI 39A: 4101; January 1979.

Clements, M. A. (Ken). Analyzing Children's Errors on Written Mathematical Tasks. Educational Studies in Mathematics 11: 1-21; February 1980.

Cohen, Martin P. Scientific Interest and Verbal Problem Solving: Are They Related? School Science and Mathematics 79: 404-408; May/June 1979.

Cohen, Martin P.; and Carry, L. Roy. Interest and Its Relationship to Verbal Problem-Solving. International Journal of Mathematics Education in Science and Technology 9: 207-212; May 1978.

Conan, Mara H. Mastery of a Formal Operations Problem-Solving Strategy: A Development Study. (New York University, 1978.) DAI 39B: 3017; December 1978.

Cook, Blair A., Jr. The Predicton of Algebra Word Problem Difficulty Using Structure Variables. (The Pennsylvania State University, 1979.) DAI 40A: 4937; March 1980.

Days, Harold C.; Whaetley, Grayson H.; and Kulm, Gerald. Problem Structure, Cognitive Level, and Problem-Solving Performance. JRME 10: 135-146; March 1979.

Days, Harrold C. The Effect of Problem-Solving Structure on the Processes Used by Concrete - and - Formal-Operational Students to Solve Verbal Mathematics Problems. (Purdue University, 1977.) DAI 38A: 6038; April 1978.

Economopoulos, Majorie P. A Study Interrelating Decision Theory, Situational Problem Solving, and Divergent Thinking Among Gifted Secondary Mathematics Students. (Georgia State University-College of Education, 1978.) DAI 39A; 4102-4103; January 1979.

Egolf, Kenneth L. The Effects of Two Modes of Instruction on Students'Abilities to Solve Quantitative Word Problems in Science. (University of Maryland, 1978.) DAI 40A: 778-779; August 1979.

Ehr, Carolyn K. Cognitive Style and Information Selection During the Solution of Mathematics Word Problems. (University of Georgia, 1979.) DAI 40A: 5768; May 1980.

Fisher, Stanley. The Role of Illustrative Errors in Learning to Solve Problems. (City University of New York, 1977.) DAI 38A: 4042; January 1978.

Flener, Frederick O. Reflections on a Problem Solving Study. International Journal of Mathematical Education in Science and Technology 9: 9-13; February 1978.

Forsyth, Robert M. and Spratt, Kevin F. Measuring Problem Solving Ability in Mathematics with Multiple Items: The Effect of Item Format on Selected Item and Test Characteristics. Journal of Educational Measurement 17: 31-43; Spring 1980.

Fowler, Evelyn C. A Study Interrelating Situational Problem Solving, Mathematical Model Building, and Divergent Thinking Among Gifted Secondary Mathematics Students. (Georgia State University-College of Education, 1978.) DAI 39A: 2111-2112; October 1978.

Gallagher, John P. The Effectiveness of Mixed-Intuitive Instructional Components in an Artifically Intelligent CAI System for Teaching Attribute Blocks Problem Solving Skills. (University of California, Santa Barbara, 1978.) DAI 39A: 6501; May 1979.

Giffune, Magdalene P. The Effect of Inservice Training in Reading upon Students" Ability to Solve Verbal Problems in Mathematics. (Boston University School of Education, 1 1979.) DAI 40A: 2572; November 1979.

Goldberg, Sheldon. The Effects of the Use of Strategy Games on the Problem Solving Ability of Selected Seventh Grade Students. (Temple University, 1980.) DAI 41A: 1990-1991; November 1980.

Graham, Verdell G. The Effect of Incorporating Sequential Steps and Pupils Constructed Problems on Performance and Attitude in Solving One-Step Verbal Problems Involving Whole Numbers. (The Catholic University of America, 1978.) DAI 39A: 2028; October 1978.

Gross, Ena. An Exploratory Study on the Use of Calculators and Problem Solving Heuristics with In-Service Elementary School Teachers. (Georgia State University-College of Education, 1980.) DAI 41A: 1451-1452; October 1980.

Hebbeler, Kathleen M. The Development of Addition Problem Solving in Young Children. (Cornell University, 1987.) DAI 38B: 6117-6118; June 1978.

Herrington, James P. Experimental Analysis of Cognitive Style and Information Processing Variables in Mathematical Problem Solving. (Southern Illinois Univer ity at Edwardsville, 1980.) DAI 41A: 973-974; September 1980.

Heyn, Jan E. Age and Sex Differences in Errors in Problem-Solving. (University of Georgia, 1980.) DAI 41B: 2364; December 1980.

Hicks, James L. An Evaluation of a Graphical Construction Technique, Designed for Secondary School Students with a L Limited Background in Mathematics, for Solving Problems in PSSC Physics. (Northwestern University, 1978.) DAI 39A: 6046; April 1979.

Hiebert, James, Corpenter, Thomas P. and Moser, James M. Cognitive Development and Children's Solution to Verbal Arithmatic Problems. JRME 13: March 1982.

Hohly, Richard W. Development of Basic Problem Solving Skills in Calculus Based Introductory Physics. (Kansas State University, 1979.) DAI 41A: 188; July 1980.

Hollander, Sheila K. A Literature Review; Thought Processes Employed in the Solution of Verbal Arithmatic Problems. School Science and Mathematics 78: 327-334; April 1978.

Hollander, Sheila K. A Review of Research Related to the Solution of Verbal Arithmatic Problems. School Science and Mathematics 78: 59-70; January 1978.

Houtz, John C. and Speedia, Stuart M. Processes Underlying Divergent Thinking and Problem Solving. Journal of Educational Psychology 70: 848-854; October 1978. (grade 5)

Hunt, Martha L. An Investigation of the Interactions of Student Ability Profiles and Instruction in Heuristic Strategies with Problem Solving Performance and Problem Sorting Schemes. (University of New Hampshire, 1978.) DAI 39B: 3363-3364; January 1979.

Ibarra, Cheryl G. An Investigation of Children'S Ability to Solve Simple Addition and Substraction Story Problems Prior to Formal Arithmatic Instruction. (University of Pittsburgh, 1979.) DAI 40A: 2524; November 1979.

Ireland, Susan C. The Effects of Certain Factors upon Third Grade Children's Ability to Solve Written Verbal Arithmatic Problems. (Loyola University of Chicago, 1980.) DAI 40A: 5769 -5770; May 1980.

Johnson, David C. and Harding, Robert D. University Level Computing and Mathematical Problem-Solving Ability. JRME 10: 37-55; January 1979.

Johnson, David W.; Skon, Linda; and Johnson, Roger. Effects of Cooperative, Competative, and Individualistic Conditions on Children's Problem-Solving Performance. American Educational Research Journal 17: 83-93; Spring 1980.

Johnson, Nancy P. Mathematics Problem Solving and Classroom Environment. (The University of Wisconsin-Madison, 1979.) DAI 40A: 3172; December 1979.

Karnie, Michael James. The Effects of Using Hand-Held Calculators on Mathematical Problem-Solving Ability Among Sixth Grade Students. (Oklahoma State University, 1977.) DAI 38A: 5311 5311; March 1978.

Knifong, J. Dan. Computational Requirements of Standardized Word Problem Tests. JRME 11: 3-9; January 1980.

Knippenberg, Gary P. An Exploatory Study of the Effects of Instruction in Heuristics upon the Mathematical Problem-Solving Ability and Attitudes of Preservice Elementary Teachers. (Michigan State University, 1978.) DAI 40A: 724-725; August 1979.

Kraus, William H. The Use of Problem-Solving Heuristics in the Playing of Games Involving. JRME 13; March 1982.

Kulm, Gerald and Days, Harald. Information Transfer in Solving Problems. JRME 10: 94-102; March 1979.

Lappan, Glenda and Winter, Marry J. Some Problems with Fractions for the Middle School. Mathematics Teacher 74: 102-104; February 1981.

Lee, Jong Seung. The Effects of Process Behaviors on Problem-Solving Performance in Various Tests. (The University of Chicago, 1978.) DAI 39A: 2149; October 1978.

Lee Kil S. An Exploratory Study of Fourth Graders' Heuristic Problem Solving Behavior. (University of Georgia, 1977.) DAI 38A: 4004; January 1978.

Lee, Kil S. Fourth Graders' Heuritic Problem Solving Behavior JRME 13; March 1982.

Lee, Wanda M. A Comparison of Verbal Problem Solving in Arithmatic of Learning Disabled and Non-Learning Disabled Seventh Grade Males. (University of Kansas, 1979.) DAI 40A: 4528; February 1980.

Lester, Fran K., Jr. Issue in Teaching Mathematical Problem Solving in the Elementary Grades. School Science and Mathematic 82: 93-98; February 1982.

Levine, Deborah R. Making Change: A Problem-Solving Activity. Mathematics Teacher 75: 114-117; February 1982.

Luchins, Abraham S. and Luchins, Edith H. Geometric Problem Solving Related to Difference in Sex and Mathematical Interests. Journal of Genetic Psychology 134: 255-269; June 1979.

Malin, Jane T. Strategies in Mathematical Problem Solving. Journal of Educational Research 73: 101-108; November/December 1979.

Mandell, Alan. Problem-Solving Strategies of Sixth-Grade Students Who Are Superior Problem Solvers Science Education 64: 203-211; April 1980.

McDaniel, Cynthia R. Memory and Strategy in Thinking. (University Cincinnati, 1979.) DAI 40B:1931; October 1979.

McDaniel, Mark A. Bottom-Up and Top-Down Acquisition of Expertise on River Crossing Problems: A Study of Knowledge Acquisition. (University of Colorado at Boulder, 1980.) DAI 41B: 1548; October 1980.

Marcucci, Robert G. A Meta-Analysis of Research on Methods of Teaching Mathematical Problem Solving. (The University of Iowa, 1980.) DAI 41A: 2485; December 1980.

Masse, Marie A. The Relationship of Class Inclusion, Computational Skill, and Mode of Presentation to Four Categories of Subtraction Problems. (Boston University School of Education, 1979.) DAI 39A: 7204-7205; June 1979.

McGinty, Robert L. and Meyerson, Lawrence N. Problem Solving: Look Beyond the Right Answer. Mathematics Teacher 73: 501-503; October 1980.

Metwali, Samia M. Expressed Interest and Extent of Practice as They Affect Outcomes of Instruction in Problem Solving. (The Ohio State University, 1979.) DAI 40A: 1934-1935; October 1979.

Mevarech, Zemira R. The Role of Teaching-Learning Strategies and Feedback-Corrective Procedures in Developing Higher Cognitive Achievement. (The University of Chicago, 1980.) DAI 41A: 1031; September 1980.

Meyer, Ruth A. Intellectual Abilities that Discriminate Good and Poor Problem Solvers. JRME 13; March 1982.

Meyer, Ruth A. Mathemathics Problem-Solving Performance and Intellectual Abilities of Fourth-Grade Children. Journal for Research in Mathematics Tducation. 9: 334-348; November 1978.

Michael, Nagi D. Cognitive Levels of Duration and Time of Classes on the Mathematics in Egypt. (University of Pittsburgh, 1978.) DAI 40A: 652; August 1979.

Moore, Claude S. Differential Effects of Arithmatic and Reading Abilities, Field-Dependence, and Mathematics Anxiety on Problem Solving Achievement of Developmental Mathematics Students in a Community College. (Virginia Polytechnic Institue and State University, 1980.) DAI 41A: 974-975; September 1980.

Moroz, Swan W. Mathematical Problem Solving Ability in Learning Disabled Children as a Function of Memory Capacity and Memory Organization. (Rutgers University The State University of New Jersy(New Brunswick), 1978.) DAI 39A: 2 2871; November 1978.

Moses, Barbara E. The Nature of Spatial Ability and Its Relationship to Mathematical Problem Solving. (Indiana University, 1977.) DAI 38A: 4640; February 1978.

Moses, Barbara E. Visualization: A Different Approach to Problem Solving. School Science and Mathematics 82: 141-147; February 1982.

Muraski, Virginia S. A Study of the Effects of Explicit Reading Instruction on Reading Performance in Mathematics and on Problem Solving Abilities of Sixth Graders. (Michigan State University, 1978.) DAI 39A: 4104; January 1979.

Oehmke, Theresa M. The Development and Validation of a Testing Instrument to Measure Problem Solving Skills of Children in Grades Five Through Eight. (The University of Iowa, 1979.) DAI 40A: 6179; June 1980.

Ortiz, Frances, L. A Selected Study on Mathematical Word Problem Problem Solving Processes. (Standford University, 1977.) DAI 38A: 5321; March 1978.

Peacock, Alistair A. Analysis of Processes Sequence Traces Observed in Mathematical Problem Solving. (University of Oregon, 1979.) DAI 40A: 4876; March 1980.

Pereira-Mendoza, L. The Effect of Teaching Heuristics on the Ability of Grade Ten Students to Solve Novel Mathematics Problems. Journal of Educational Research 73: 139-144; January/February 1980.

Petitto, Andrea. The Role of Formal and Non-Formal Thinking in Doing Algebra. Journal of Children's Mathematical Behavior 2: 69-82; Spring 1979.

Putt, Ian J. An Exploratory Investigation of Two Methods of Instruction in Mathematical Problem Solving at the Fifth Grade Level. (Indiana University, 1978.) DAI 39A: 5382-5383; March 1979.

Quickenton, Arthur J. Cognitive Style and Mathematics Problem Solving; An Exploration of the Relationship of Field-Dependence-Independence to Secondary School Students' Mathematic Problem Solving. (The University of Texas at Austin, 1979.) DAI 40A: 3905-3906; January 1980.

Quintero, Ana H. The Role of Semantic Understanding in Solving Multiplication Word Problems. (Massachusetts Institue of Technology, 1980.) DAI 41A: 2522; December 1980.

Rowe, William R. An Investigation of the Effectiveness of the Proposal of Richard A. Earle's for Teaching Word Problem in Mathematics in Selected Topics and Grade Level. (Auburn University, 1979.) DAI 40A: 5351; April 1980.

Rutledge, David E. The Effects of Two Decoding Programs upon the Mathematics Problem Solving Achievement of Fifth Grade Students. (The University of Nebraska-Lincoln, 1978.) DAI 39A: 4105; January 1979.

Sagiv, Abrham. Hierarchical Structure of Learner Abilities in Verbal Computation Problem-Solving Related to Strength of Material. International Journal of Mathematical Education in Science and Technology 9: 451-456; November 1978.

Savage, Sandra S. Selected Topics in Applied Mathematics for Middle Grade Mathematics Classes - An Exploratory Study. (Columbia University Teachers College, 1977.) DAI 38A: 6586-6587; May 1978.

Schenck, William E. Arithmatical Problem Solving and Recognition Memory for Sentences among EMR Children. (The University of Connecticut, 1978.) DAI 38A: 6653-6654; May 1978.

Schoenfeld, Alan H. Explicit Heuristic Training as a Variable in Problem-Solving Performance. Journal for Research in Mathematics Education 10: 173-187; May 1979.

Scott, Patrick B. A Curriculum Study in Applied Mathematics for Prospective Secondary School Teachers. (Columbia University, 1980.) DAI 41A: 145-146; July 1980.

Semmes, Patricia A. A Preliminary Investigation of Two Presentation Modes on Dynamic Problems in Algebra in Relation to Individual Differences. (The University of Texas at Austin, 1977.) DAI 38A: 4005-4006; January 1978.

Sheddan, Sharon S. A Study of Problem Initiation and a Schema for Analysis of Discrete Mathematical Problems. (Georgia University -College of Education, 1979.) DAI 40A: 4941; March 1980.

Shult, Douglas L. The Effect of the Hand-Held Calculator on Arithmetic Problem-Solving Abilities of Sixth Grade Students. (University of Oregon, 1979.) DAI 40A: 6179-6180; June 1980.

Siegel, Alexander W., Goldsmith, Lynn T., and Madson, Camilla R. Skill in Estimation Problem of External and Numerosity. JRME 13; March 1982.

Silver, Edward A. Student Perceptions of Relatedness Among Mathematical Verbal Problems. Journal for Research in Mathematics Education 10: 195-210; May 1979.

Silver, Edward A. Recall of Mathematical Problem Infoemation: Solving Related Problems. JRME 12; January 1981.

Silver. Edward A. Student Perceptions of Relatedness Among Mathematical World Problems. (Columbia University Teacher College, 1977.) DAI 39A: 734-735; August 1978.

Smith, Richard L. Figure Typer and Problem-Solving Achievement in Computational Activities. (The Ohio State University, 1978.) DAI 39A: 2706; November 1978.

Snover, Stephen L. and Spikell, Mark A. Problem Solving and Programming: The License Plate Curiosity. Mathematics Teacher 74: 616-617; November 1981.

Southall, John C. A Study for the Validation of an Instructional Sequence Designed to Teach Verbal Problem Solving in Elementary Mathematics. (The University of Texas at Austin, 1979.) DAI 40A: 1935-1936; October 1979.

Spratt, Eleanora R. Effects of the Location of Subordinate Clause on Comprehension of Arithmatic Problems by Sixth Graders. (Hofstra University, 1979.) DAI 40A: 179-180; July 1979.

Stephen, H. Jeannette. A Comparison of Two Methods of Teaching Ninth Grade Algebra Students to Solve Verbal Problems Containing Two or More Unknow Quantities. (The University of Iowa, 1979.) DAI 40A: 3851-3852; January 1980.

Szetela, Walter. Analogy and Problem Solving: A Tool for Helping Children to Develop a Better Concept of Capacity. Arithmetic Teacher 27: 18-22; March 1980.

Thibodeau, Gerald Patrick. The Effect of Question Order, Distractor Position, and Language Set on the Quantitative Verbal Problem Solving Performance of Educable Mentally Handicapped Children. (The University of Connecticut, 1978.) DAI 39A: 6706; May 1979.

Thomas, Ossie MacBanks. Direct Instruction on Three Reading Variables Related to Verbal Arithmatic Problem Solving of Educable Mentally Retarded Pupils. (Georgia State University-College of Educatuin, 1978,) DAI 39A: 229; July 1978.

VanDeusen, Robert M. The Effects of Humorous and Personalized Story Problems on the Achievement and Attitudes of Sixth Grade Students. (The University of Iowa, 1980.) DAI 41A: 2487-2488; December 1980.

Vest, Floyd, and Congleton, Card. Three Steps in Teaching Problem Solving. Mathematics Teaching 82: 18-20; March 1978.

Wallace, Edward C. Thought Processes in Low-Achievement High School Mathematics Students. (The University of Texas at Austin, 1980.) DAI 41A: 1454-1455; October 1980.

Ward, Dennis E. The Effect of the Electronic Calculator on Problem-Solving Achievement and Attitudes Toward Mathematics of General Mathematics Students. (University of Southern California, 1978.) DAI 39A: 4038; January 1979.

Waters, William M. The Use and Efficiency of a Scanning Strategy in Conjunction Concept Attainment. (University of Pennsylvania, 1979.) DAI 40A: 1331; September 1979.

Watson, Albert G. An Exploration of Mathematical Problem Solving Processes: Case Studies of the Heuristic Strategies Used by Concrete Operational and Formal Operational Students Involved in a Teaching Experiment. (Columbia University, 1980.) DAI 41A: 1993; November 1980.

Webb, Noreen M. An Analysis of Group Interaction and Mathematical Errors in Heterogeneous Ability Groups. British Journal of Educational Psychology 50: 22-28; November 1980.

Webb, Norman L. Processes, Conceptual Knowledge, and Mathematical Problem-Solving Ability. Journal for Research in Mathematics Education 10: 83-93; March 1979.

West, Tommie A. A New Look at an Old Assumption: TheRole of the Horizontal Number Sentence in Problem Solving by Young Children. School Science and Mathematics 80: 493-501; October 1980.

West, Stephen F. Problem Solving Processes in High School Geometry. (The University of Texas at Austin, 1979.) DAI 40A: 1331; September 1979.

Wheatley, Charlotte L. Calculator Use and Problem Solving Performance. Journal for Research in Mathematics Education 11: 323-334; November 1980.

Wheeler, Larry J. Syntactic Complexity and Its Effect on Writen Mathematical Problems. (The University of Oklahoma, 1978.) DAI 39A: 4188; January 1979.

Whitaker, Donald R. Mathematical Problem Solving Performance on Related to Student and Teacher Attitudes. School Science and Mathematics 82: 217-224; March 1982.

Wiley, Larry D. The Construction and Formative Evaluation of the Introductory Portion of a Modern Honors Geometry Curriculum for Secondary Schools. (Columbia University, 1978.) DAI 39A: 6008-6009; April 1979.

Wills, Robert L. A Comparative Study of Mathematics Achievement by Eihgt Graders, 1975 to 1979. (The University of Iowa, 1980.) DAI 41A: 2402; December 1980.

$SIGNOFF

Wongyai, Piyavadee. An Investigation of Two Methods of Instruction in Solving Standard Textbook Word Problems at the Second-Grade Level(PRATOM 2) in Thailand. (Indiana University, 1980.) DAI 41A: 2488; December 1980.

Zalewski, Claire J. An Investigation of Selected Factors Contributing to Success in Solving Mathematical Word Problems. (Boston University School of Education, 1978.) DAI 39A: 2804; November 1978.

Zimmerman, Charles W. Contextual Details and Mathematical Problem Difficulty. (The Ohio State University, 1979.) DAI 40A: 143-144; July 1979.

PRISM CANADA, 1981

- Exerpted from the Report

PROBLEM SOLVING

Techniques

Respondents felt these five techniques definitely should be included in both elementary and secondary school problem solving:

- construct a table and search for patterns
- write and solve a simpler problem; then extend the solution to the original problem
- teach primarily global problem solving ideas (e.g., read, plan, work, check).
- draw a picture, diagram or graph to represent the problem situation
- translate the problem into number sentences or equations

There was only moderate support for including:

- explore the problem by using flow charts
- start with an approximate answer and work backwards
- guess and test possible solutions

Goals

The major goal selected for problem solving was:

- to develop methods of thinking and logical reasoning

Other goals receiving strong support were:

- to develop the skills to approach new topics in mathematics independently
- to develop creative thought processes —
- to enhance the ability to apply mathematics in science
- to acquire skills necessary for living in today's world

Resources

The favoured resources were:

- a resource guide to real-life problems
- in-service training on problem solving methods for all teachers who teach mathematics
- materials in every class for modeling problems and problem solutions (e.g., graph paper, measuring devices, construction sticks, etc.)
- supplementary materials which contain many more problems like those in textbooks
- card files of problems
- materials for problem solving contests and competitions
- textbook modules for teaching appropriate problem solving strategies

Clearly rejected as a problem solving resource was:

- textbooks with all verbal problems in a single chapter

Methods

When asked to indicate degree of support for a variety of methods to use when teaching problem solving, 100% of the respondents agreed that:

- problem assignments are designed to challenge students to think

Also given strong support were:

- projects that involve real-life problem situations should be assigned to individuals or teams of students
- problems are used to introduce mathematical topics
- problems are given in which the use of physical materials will aid in the solution
- students work in small groups to solve problems

Who/Time

In terms of who should receive instruction in problem solving and when, respondents expressed strong agreement that:

- short problem solving units should be included after each mathematical topic is taught
- students should be taught to *find* problems within situations

Strong disagreement was evidenced for these statements:

- problem solving is important only for college-bound students
- problem solving should not be taught in the elementary grades
- problem solving is a function of intelligence and cannot really be taught except to gifted students
- different problem solving courses should be offered for girls

STRATEGY CHECK LIST

LOOK FOR A PATTERN ____________

CONSTRUCT A TABLE ____________

ACCOUNT FOR ALL POSSIBLITIES ____________

ACT IT OUT ____________

MAKE A MODEL ____________

GUESS AND CHECK ____________

WORK BACKWARDS ____________

MAKE A DRAWING ____________

SELECT APPROPRIATE NOTATION ____________

RESTATE THE PROBLEM ____________

IDENTIFY GIVEN, WANTED, NEEDED INFORMATION ____________

WRITE AN OPEN SENTENCE ____________

IDENTIFY A SUBGOAL ____________

SOLVE A SIMPLER PROBLEM ____________

CHANGE YOUR POINT OF VIEW ____________

CHECK FOR HIDDEN ASSUMPTIONS ____________

Topic Group P

WOMEN AND MATHEMATICS

Leader: Roberta Mura, Université Laval

I will present a survey of the literature and try to summarize what research says about the past and present situation of girls and women in mathematics: what variables have been investigated and what factors have been put forth as possible explanations for the sex-related differences observed. I will also mention some actions that have been undertaken in order to change the situation.

Participants who have done research or organized activities concerning this issue, or who plan to do so, will be invited to present their work or their projects.

All participants will be invited to discuss:

(i) implications for teacher training and

(ii) possible directions for future research.

CANADIAN MATHEMATICS EDUCATION STUDY GROUP

Kingston, June 1982

Summary of a topic group presentation on GENDER AND MATHEMATICS by ROBERTA MURA

INTRODUCTION

At the 4th International Congress on Mathematics Education held in Berkeley in 1980, there were several presentations and workshops on the topic of women and mathematics. At the end of the conference, many participants felt that it would be very useful to have an international survey of the field. The task was undertaken by Dr. Erika Schildkamp-Kundiger in cooperation with the Second International Mathematics Study (IEA), and the survey has recently been published by ERIC/SMEAC (Ohio State University) under the title "An International Review of Gender and Mathematics".

I was responsible for the Canadian report appearing in the review. One of my conclusions is that so far, in Canada, not much attention has been paid to this topic. It still happens that when I mention my interest in the problem of women and mathematics, people react by asking: "What problem?".

One extreme aspect of "the problem" ought to be obvious, if one thinks of the very low percentage of women among university mathematics professors. However this remarkably unequal distribution has not been particularly noticed or considered worth investigating, just as many other instances of the sexual division of labour which is still mainly unchallenged in our society. In this connection, it should be noted that in the United States - the country that has done the most research on gender and mathematics - interest in the topic was spurred by political groups who were fighting the occupational segregation existing in their country and who had identified mathematics as the critical filter preventing access to many jobs and careers.

Once one stops taking for granted the lower participation of women in mathematics, one faces a very complex phenomenon which is still little known, let alone understood. I would like to suggest that "gender and mathematics" has the potential of becoming a whole new area of research. From an international point of view, it has already become one.

The purpose of this paper is to give an overview of this field of research. I have tried to classify the various issues and results into three main categories: describing the situation of women with respect to mathematics, explaining it, and finding ways to change it, where change is called for.

DESCRIBING THE SITUATION

What sex-related differences are there in mathematics?

(a) Participation

What is the percentage of women among mathematics students and teachers at the secondary school level? and among mathematics students and professors at the university level?

Not enough data is available in Canada to completely answer these questions. However surveys carried out in some provinces indicate that participation of women in mathematics begins to decline as soon as mathematics becomes an optional subject towards the end of secondary school. The percentage of women among secondary school mathematics teachers varies between 13% and 35% in the various provinces (excluding Ontario).

Some data about the university level has been published by Statistics Canada. Mathematics is treated as forming one category together with physical sciences. The percentage of women among full-time university teachers in this category in 1979-80 was 4,5%. Their proportion decreases as one moves from assistant professor to associate professor to full professor. Within each rank, the median salary for women is lower than that of men. In 1980-81 the percentages of women among those earning a bachelor's degree, a master's degree, or a doctorate in mathematics and physical sciences were respectively 28%, 17% and 8% in Canada. The same pattern, if not the same figures, seems to recur in several other countries.

Of course, even if exact figures were available for all levels, they still would not constitute a satisfactory description of the situation. To give just one example, about women mathematics professors in universities, one could ask whether they are hired, promoted and tenured at the same rates as do men, whether their distribution among different subjects is the same as the men's, whether they teach the same kind and the same number of courses in the same ways, whether they do the same kind and amount of research, whether they have the same kind and amount of administrative duties, etc. On another level, one could ask: what has been their experience? How do they live their situation of members of such a tiny minority? What internal or external factors made them decide to become mathematicians and helped them succeed? One could ask what happened to the women who had the potential of becoming mathematicians, but didn't...

(b) Achievement

Many studies have attempted to assess differences in mathematics achievement between girls and boys at primary and secondary school level. Results have been inconsistent: differences observed have been usually small, sometimes in favour of girls, sometimes in favour of boys.

One of the studies involving the largest number of students (about 70 000) is the Second Mathematics Assessment of the National Assessment of Educational Progress, carried out in the United States in 1977-78. Three different age groups (9, 13 and 17) were tested at four different cognitive levels (knowledge, skills, understanding and applications. The largest difference in favour of girls (1,4 percentage points) was found within the youngest group and at the lowest cognitive level; the largest difference in favour of boys (5,04 percentage points) was found within the oldest group and at the highest cognitive level.

Similar differences were found in the British Columbia and Alberta provincial mathematics assessments.

(c) Attitudes

Differences in attitudes towards mathematics between girls and boys have been found fairly consistently. They occur not so much in generic attitudes (liking or being interested in mathematics), as in more specific aspects. Namely it appears that:
(1) boys are more confident than girls in their own ability to succeed in mathema-

tics; (2) boys perceive mathematics as being more useful than girls do; (3) both boys and girls perceive mathematics as a male domain, but boys do so more than girls; (4) girls see mathematics as more anxiety-provoking than do boys.

(d) Other variables

A few researchers have begun to pay attention to some less obvious variables. An Australian study has found some evidence of differences in students' preferences of learning methods, girls preferring more cooperative methods, boys more competitive ones. The same pattern recurs in female and male teachers' preferences of teaching methods. If this is true, the higher proportion of men among mathematics teachers could lead to the hypothesis that girls avoid mathematics courses not so much because of content, as because of teaching methods.

In the United States, some intriguing results have been obtained about differences in the strategies used by girls and boys to (correctly) solve mathematical problems, and in the choices of wrong answers made by boys and girls when answering objective tests.

EXPLAINING THE SITUATION

Quite a number of factors have been considered in trying to explain why and how the differences observed arise and develop. (The distinction made here between variables describing the situation and variables explaining it is somewhat artificial. It might be more realistic - and more difficult - to view them as a system of interrelated variables interacting with each other.)

(a) Sex role perception

The factor that is supported by the largest amount of empirical evidence is the perceived appropriateness of mathematics to one's own sex. As mentioned before, girls, as well as boys, although to a lesser extent, see mathematics as a male domain, and this affects negatively their decision to pursue the study of this subject.

(b) Ability and attitude factors

In spite of the sensational coverage of the topic in the popular press, sex-related differences in mathematical ability have yet to be proved. The difficulty resides both in the definition of "mathematical ability" and in the design of an experiment that would genuinely measure ability as opposed to achievement.

Some studies indicate that a difference in spatial ability in favour of boys develops during adolescence. Other studies however have not confirmed these results. Moreover the relationship between spatial ability and achievement in mathematics is still unclear.

Among attitude factors, those that have been quoted most often as being related to participation in mathematics are the self-esteem of one's mathematical ability, and the perceived usefulness of mathematics.

Other attitude factors that have been proposed as explanations of the lower participation of women in mathematics are: a higher fear of success among girls, parti-

cularly among talented ones, and a different perception of the causes of one's successes and failures. (Girls seem to have a tendency to attribute their success in mathematics to effort, luck, or easiness of the task, rather than to their own ability, while attributing their failures to lack of ability or lack of effort, rather than to bad luck or difficulty of the task; the pattern is reversed for boys. Thus girls feel less in control of their mathematics learning and are less likely to persist in its study.)

(c) Environmental factors

These include educational, social and cultural factors. Examples of educational factors identified through empirical research are:

- differences in the kind and amount of interactions between teachers and their male or female students (this may sound surprising, but it should be kept in mind that most such differential behaviour is unconcious);
- a higher rate of enrollment in science courses among boys (science courses give an opportunity of applying mathematics and this has a positive effect on mathematics achievement);
- sexism in textbooks (studies carried out in Quebec and in Manitoba, among others, prove that school textbooks give children an image of a society even more stereotyped than the real one: male characters outnumber female ones and are portrayed in five times as many different occupations, hardly any female character ever appears in a mathematics related occupation).

Social factors seem harder to quantify and have not yet come under the scrutiny of empirical research. However some researchers have pointed out that there seems to exist a conflict between the norms that school and society set out for girls (success in academic and professional life conflicting with success in private life and personal relationships).

A more obvious social factor is overt and subtle discrimination, including sexual harassment which is only now beginning to be talked about openly and whose extent is still difficult to evaluate.

Cultural factors, by their nature, probably cannot become the object of empirical studies. However they may well be the most influential factors of all. Our cultural tradition (in the form of religion, philosophy, literature, history, etc.) presents girls and women with an image of themselves that is not conducive to engaging in intellectual pursuits, especially scientific ones, nor striving for success in academic life. The most respected and admired writings of our civilization contain an astounding collection of passages which explicitely insult and ridicule women, in particular with respect to their intellectual capacity. Far from being a remnant of the past, this attitude toward the female human being keeps turning up day after day through all our media.

CHANGING THE SITUATION

There exists a double link between the task of explaining the situation and that of changing it. In fact, whether and how one thinks that the present state of affairs can be changed depends to a great extent on the kind of explanation one offers for it. Some factors can be acted upon more easily than others. For instance, the perceived usefulness of mathematics may be easily increased by informing all students about the mathematics needed in various occupations and study programs. On the other hand, the negative effect of our cultural heritage is obviously much harder to counteract.

Thus the design of an intervention program has to be based on some theory about the origins of the differences that it tries to eliminate. Conversely, the degree of success of such a program can give some indication about the validity of the theory behind it.

Hardly anything has been done in Canada in this domain. This contrasts with the United States where there exist at least two national associations whose goal is to increase the participation of women in mathematics: the "Association for Women in Mathematics" (AWM), founded in 1971 and aimed mainly at the university level, and the more recent "Women and Mathematics Education" (WME).

Again in U.S.A., during the last few years, many different programs have been created to increase the chances that secondary school students - girls as well as boys - will continue to study mathematics. Such programs disseminate printed materials and films, and help to organize conferences and workshops to make participants aware of stereotypes about mathematics, to inform them about the importance of mathematics in different jobs and professions, to have them meet with successful women who use mathematics in their work, and to let them participate in some enjoyable mathematical activity. Some of these programs have already produced very encouraging results.

In order to improve the situation in Canada, a lot has to be done both at the individual and collective levels. On one hand, teacher trainers should develop pre-service and in-service teachers' awareness of these issues. On the other hand, elementary and secondary teachers should do their best to give their female and male students equal chances of succeeding in mathematics and of continuing to study it; this implies monitoring their own behaviour as well as finding ways to counterbalance the negative influences to which girls are subjected outside the classroom.

Issues related to women and mathematics should also be publicized in varios ways by professional organizations. (At least three national conferences specifically about women and mathematics have already taken place in Australia.) Concurrently, research efforts should be intensified and appropriate actions (intervention programs, etc.) undertaken in various parts of the country.

REFERENCES

Associations

Association for Women in Mathematics
Women's Research Center
Wellesley College
828 Washington Street
Wellesley, Massachusetts 02181
U.S.A.

Women and Mathematics Education
c/o Education Department
George Mason University
4400 University Drive
Fairfax, Virginia 22030
U.S.A.

ERIC/SMEAC publications

Erika Schildkamp-Kundiger (editor), An International Review on Gender and Mathematics, 1982.

Marilyn Suydam and Vicky Kirschner, Selected References on Mathematical Anxiety, Attitude, and Sex Differences in Achievement and Participation, 1980.

Judith Jacobs (editor), Perspectives on Women and Mathematics, 1978.

Elisabeth Fennema (editor), Mathematics Learning: What Research Says About Sex Differences, 1975.

All the above are available from:

The ERIC Science, Mathematics and Environmental Education Clearinghouse
The Ohio State University
1200 Chambers Road, Room 310
Columbus, Ohio 43212
U.S.A.

Other publications

Elisabeth Fennema, Teachers and Sex Bias in Mathematics, The Mathematics Teacher, March 1980, pp. 169-173.

L. Fox, L. Brody, D. Tobin (editors), Women and the Mathematical Mystique, The Johns Hopkins University Press, 1980.

Ministry of Education of British Columbia, Gender and Mathematics/Science Education in Elementary and Secondary Schools, prepared by G. Erickson, L. Erickson, S. Haggerty, 1980. Available from: Learning Assessment Branch, B.C. Ministry of Education, 7451 Elmbridge Way, Richmond, B.C. V6X 1B8.

Science Council of Canada, Who Turns the Wheel?, prepared by Janet Ferguson, 1982. Available from: Publications Service, The Science Council of Canada, 100 Metcalfe St., Ottawa, Ont. K1P 5M1.

Phyllis Zweig Chinn, Women in Science and Mathematics Bibliography, Humboldt State University Foundation, Arcata, California, 1980.

Topic Group Q

EDUCATIONAL USES OF THE HISTORY OF MATHEMATICS
Leader: Dick Tahta, Exeter, England

Has the history of mathematics a relevance for mathematics education? Obviously there is an indirect relevance but people who are interested in the history of mathematics sometimes press more direct claims. It is not always clear what they are proposing other than some history units in mathematics degree courses. There doesn't seem to be much to say about history in schools other than that mathematics teachers could be more aware historically. In trying to say more I would like to share some thoughts in three areas by -

1) describing some experimental work with nine and ten year olds;
2) offering some defense of the anecdotal E.T. Bell-style of history now scorned by present more rigorous writers, especially in view of what any kind of history can only be about for adolescents;
3) proposing a central theme for investigation, by those who are interested in the history of mathematics, for the light it may shed on epistemology and pedagogy.

Panel Group

DEVELOPING MATHEMATICAL TALENT

Speakers: Edward Barbeau, University of Toronto
Claude Boucher, Université de Sherbrooke
Stanley Erlwanger, Concordia University

This group will not primarily be concerned with psychological questions such as whether mathematical talent is a specific human ability, and whether it is mainly innate or environmental, but rather with the practical questions that arise when it is thought desirable to foster superior mathematical talents.

The speakers will address three different, but related, ways of encouraging mathematically-talented students: (i) regional and national mathematics competitions, (ii) mathematics summer schools, and (iii) special attention to "gifted students" in ordinary schools. What do these activities hope to attain? Do they achieve what they set out to do? Are there any side-effects or undesirable consequences?

After the individual presentations, the group will be invited to discuss the issue, particularly as it affects Canadian students. The issue is controversial at the best of times; now it is especially sensitive as it raises difficult socio-political questions about the allocation of limited educational resources.

IDENTIFYING AND DEVELOPING MATHEMATICAL TALENT

E.J. Barbeau

An opinion of the ability of a student should be regarded as provisional and subject to further sharpening. Some students are extroverts and function well in front of an audience; others are quite reticent. In addition, a student who is strong in one mathematical area may be weak in others. Thus, some students have a good geometrical intuition, while others are more at home with symbolic reasoning. While contests are a reasonably reliable way of identifying exceptional ability, they can lead to underestimation of students who do not work well under pressure or approach problems deeply and thoroughly. For these students, essay contests, correspondence clubs and problems to be worked at leisure give a better indication. Seminars are a good way to find the really talented, since one can see a number of individuals interacting and make a comparison. In a private interview, a talented student is often one who is able to canvass many ideas very rapidly, to get an overview of the problem, to judge the efficacy of an idea and abandon nonpromising ones and to attempt methods from different areas of mathematics.

Teachers can be unreliable identifiers of talented students if they have had little experience at mathematical thinking themselves or if they put undue emphasis on high marks in classroom work. Parents often have a pretty good idea of the intellectual potential of their children, but not of their motivation. When I first hear from a parent about a student, I try to encourage the student himself to follow up the initial contact and define for himself how closely he wants to get involved with mathematics.

Many talented students do not appreciate the extent of their abilities. Some who overestimate their ability do not consolidate their mathematical knowledge and often fail at higher level mathematics. Others denigrate themselves too much and drop out from situations in which they could achieve a great deal of success. Consequently, those who work with a talented child should help that person gain an understanding of both his limitations and strengths. One way is to bring him together with other students with whom he can compare.

Some bright students are enchanted with computation and oddbeat ideas, but lack the discipline and the techniques to explore them deeply. A nonchallenging school regime may lead to the use of shortrun strategies and a disinclination to persevere with a challenging problem.

I believe it is often a mistake to accelerate bright students. This can create social problems and make it difficult for the student to fit into a university programme. There are enough interesting and important areas of mathematics not well-covered in university which a good student might penetrate into in order to broaden himself; the New Mathematical Library published by the MAA gives some possibilities.

When discussing talent, one should not think only of exceptionally gifted students. Rather, mathematics is like music. While only a few are destined for greatness, a large percentage of the population have some aptitude for it. A school curriculum which gives a watered-down version of a specialist course does not serve these students. There is a surprising amount of mathematics which is elementary and non-technical, yet interesting and challenging to the layman. Recognizing that the layman in his daily life needs only few mathematical techniques, we should try to develop curricula which give a sense of the nature of mathematical reasoning and its application in some important areas. Books, magazines, newspapers and the broadcasting media all have a role to play in bringing mathematics to a larger public. While many good books have appeared recently, the record of the media on matters mathematical is generally quite poor. Should there be a course in mathematical journalism somewhere?

DEVELOPING MATHEMATICAL TALENTS

THE ORGANIZATION OF THE ANNUAL CONTESTS and the MATHEMATICAL SUMMER CAMPS of THE ASSOCIATION MATHÉMATIQUE DU QUÉBEC

Claude Boucher
Département de mathématiques
et d'informatique
Université de Sherbrooke

Apart from the fact that I will address to you in a language which is not my mother tongue, my most important problem in accepting to speak to you this afternoon was to establish the content of my text. For me, it was a real huge problem. I cannot keep myself from being astonished to find me amongst an assembly of mathematical education specialists, people with whom I have always kept very friendly relationships, but people who work in a domain where I have never, I humbly confess, acquired any theoretical competence, if there is some to be acquired. Of course, I speak of the theoretical aspects of mathematical teaching, because, otherwise, a twenty-five year career at the university level has taught me quite a few tricks in the art of conveying mathematical notions into young developing minds, and convince me that nothing can replace those preliminary conditions of a good teaching which are a deep knowledge of the subject and the facility to communicate with others, through a simple, picturesque and humorous speech.

So here am I, invited to talk to you about the development of mathematical talents, while I have devoted an important part of my life

asking myself, without very often finding any answer, by which means I could develop the modest resources of my own.

But, to express the issue frankly, I am not sure at all there is any answer to the problem which has been put in front of our panel. Does there exist any knowledge, any method, any finite number of well defined steps (as mathematicians said of certain processes which helped them to sail away from the troubled seas of paradoxes) which would lead as a final product to a developed mathematical talent? Are there recognized means to develop one's mathematical talents and the talents of others, to advance in that country where there are no royal roads, according to the remark which Euclid, the great geometer, made once to Alexander, the great conqueror? Which is an indication that it is not since this morning that was brought up that question which interests you and troubles me.

By the way, I liked very much that remark of Euclid, when I heard it for the first time, but I am afraid it will have to be taken with a grain of salt, since, according to my dictionary, Alexander died in 323 before our era, while Euclid lived in the third century. But, as the Italian proverb goes saying: se è non vero, è ben trovato. If it is not true, it is well coined.

So, if there are in my mind some reluctance to admit the existence of such developing processes, or rather if there are in my mind some presumption of the non existence of such processes, how can I have been pushed by a so strange recklessness that I have accepted to be a member of the present panel? For the simple reason that I have been prompted by the circumstances to take responsability since a few years in the organization of the annual contests of the Association mathématique du Québec, and afterwards, of its summer camps which gather for a few weeks the students having written the best copies at the contests.

My intention is not to present you here a theory on the subject of our panel, but rather to narrate to you simply the relation of an experience, of what could be called in the present jargon of French speaking psychologists un vécu personnel.

First, let me give you a word from my sponsor. L'Association mathématique du Québec has just celebrated its first quarter of a century. It was founded by young university and college mathematics professors eager to promote the educational and cultural roles of their discipline. The AMQ has, since its foundation, played an important part in the modernization of the mathematical curricula in Quebec, and has created the PERMAMA program to promote the reorientation and the permanent education of mathematics teachers already entered in their careers. Its involvement has touched any level of mathematical education, from kindergarten to university, the diversification being accomplished through regional and special interest groups. Any member of the Association can and is encouraged to belong to those groups.

The AMQ has also given to itself the mission of promoting research in the field of pure and applied mathematics. For examples, it has in 1973 initiated the first Colloque des mathématiciens du Québec, which has since met at a semi-annual frequency, and has given birth to two periodical publications: les Annales des sciences mathématiques du Québec, and La Gazette mathématique du Québec.

Finally, the Association has since nearly its beginning turned its interest toward the promotion of mathematics as a leisure activity. This interest has been expressed by its participation in the Fédération québécoise du loisir scientifique, and by the creation of a committee of mathematical games which organizes periodically promotion campaigns to make public aware of the interests of mathematical and computer science games. Nothing expresses that activity better than les valises mathématiques, whose name has probably been borrowed to the most travelling diplomatic bags. They are big suitcases full of mathematical puzzles, riddle books, games, and so on, which can be easily carried to any congress, meeting, or shopping centre, and be presented as efficiently as the street peddlers do their business in an Oriental market.

But the promotion of mathematics as a leisure activity has also given birth, since nearly the beginning, to the organization of annual contests to which local representants of the Association in every school and college were invited to prepare and present their pupils. In the sixties, it was felt that it would be sensible, as a reward, to gather the winners of the contest for a few weeks (at that time, it was four), and invite practicing mathematicians to work with them intensively on a subject of their choice. The only conditions to be filled were dedication and easiness to communicate one's knowledge and experience.

I remember to have participated to one of those camps in which I initiated for a week twenty teenagers to the secrets of mathematical automata theory on which I was myself working for my Ph.D. thesis. There reigned in those camps an intense intellectual life and an extremely elating atmosphere. That handful of teenagers showed an insatiable thirst for knowledge, and those camps contributed to orient towards their present career many young mathematicians who now lecture and do research in Quebec or foreign universities.

Unfortunately, around 1967, the Lesage government having fallen, the Quiet Revolution having become even more quieter, the government grants which we used to receive having dried up, we were forced to put an end to those experiences. The AMQ continued to prepare contests, but could not reward its winners with anything more exalting than modest sums of money by which the participating fees of the contestants were meagerly redistributed after the organizing expanses had been subtracted. In fact, to adjust to the major reforms which were introduced in the educational system of the province during the sixties, reforms which distinguished between a secondary level and a cegep level, we even had to organize two annual contests, one for each of those levels. But, if the organizational burdens and details were multiplied by two, our funds remained terribly modest, and each contest depended upon very scarce resources to reward its winners.

So, for years, we were forced to let the camp embers to creep under the ashes. But, in 1978, following circumstances on which you will allow me, I am sure, not to elaborate, I became for a short period chairman of my department.

In fact, those circumstances could be summed up as a crisis of leadership which had held sway for weeks and even months. As a result, the chairman's desk was crumbling under pending files when I sat in the chairman's chair. Among those pending files I found was one concerning the organization of the annual contest for college level which had been,

a few years before, handled to somebody in our department by somebody in the AMQ. Don't ask me to give you details about that handling operation. It appears to be cast under the shadows of prehistory. Anyhow, the file was there. We were in the middle of February, and the contest used to be held at the end of that month, because of the interface with the Canadian Mathematical Olympiads, in which the best of our contestants at the first year of college level were automatically registered. And nothing, nothing for that year, had already been done. However, the contest was held, but with a certain delay, and the Olympiad train was caught, even if it was, as the expression goes, by the skin of the teeth.

Slowly, the affairs of the department settled, as far as human affairs can settle, and I was able to come back to private life. So far, so good. But my interest in the AMQ contests had not completely died down, and having heard that other contest organizing groups in the province did receive from the government appreciable grants for their winners, came to my mind the idea of reviving those mathematical camps which had passed away with the last whispers of the Union Nationale party.

At the fall of 1978, I prepared a memorandum submitted by the Association to the Ministry of Education, in which we were soliciting a grant to hold a camp during the summer of 1979. But as the State is, it's well known, the slowest of the slow monsters, we had to wait until December 1979 to hear that the grant was granted, and until 1980 to know which amount was granted.

Of course, during that time, we had to accept to live dangerously, to organize in parallel and conditionnally the whole thing, to announce conditionnally a camp for the winners, to recruit conditionnally eventual instructors, and so on. I had often the impression to ride on a cross-breed from a James Clavell suspense and a three dimensional puzzle with a deadlined explosion. To put some additional cherries on the top of the cream pie, we were also afflicted during the winter with a modern variation of the plague: irregular sporadic strikes in some of the public CEGEPs, the junior college level institutions, which of course complicated even more the scheduling issues. Anyhow, the camp was held on the campus of the Sherbrooke University with a certain delay, but it was held. It had a duration of three weeks, involving four professors, three assistants and 24 participants, and was, despite the obstacle race we had to run, quite a success, laying down the bottom step of a new tradition. The first week was devoted to the topology of paper surfaces, the second, to the study of Boolean algebra, and the third, to the applications of discrete mathematics to human sciences.

Armed with the experience we had gained in the previous year, - - Tolstoi once remarked that experience is the stack of arms by which one has been wounded - it was a lot easier to prepare the 1981 camp. We knew to which doors we had to knock at, we were ready to answer many questions, we had already gained some good name amongst the public servants involved in the file, and amongst the students who had heard from the participants favourable news on the atmosphere of the previous camp. Everything rolled on better bearings, and we received for two weeks the same number of participants, involving two assistants and two professors. We decided to have an unifying theme for the camp, which happened to be the Rubik's cube. It was at that time a real craze amongst the young people. In fact, the mathematical study of the cube rotations gives very interesting

developments in the domains of graph and group theories.

Finally, the 1982 mathematical camp, last summer, involved three professors, the same number of assistants, and again 24 participants. It was divided in two subjects: the first week was devoted to Young tables and their applications to computer science, and the second one to the structural topology of flexible polyhedra, a subject having links with mathematics and architecture.

Next year, if the government, despite its tight budget policies, gives us what was called the sinews of war, and could also be called the sinews of education, we hope to go on our fourth camp of the new era. I am optimistic: optimism is the sinews of realization.

Do such activities contribute to the development of mathematical talents? I do fervently believe so, although I would be in a very bad position if anybody asked me to give a constructive proof of it, or wanted a measurable mean to establish my assertion. Let me only say that those activities cannot create a Newton, a Gauss, or a Galois, but if there is somewhere a Newton, a Gauss or a Galois, it can help him to come to light, encourage him to become aware of his genius and prepare it to blossom to the full extent of his capabilities. Any money, even in those restrictive budget years, any effort, even in those times when everybody is busy and hurried, is worth such an endeavour. But more modestly (let us consider more probable events), to help every year young students to compete, get involved and become conscious of their tastes and talents, is already worth the money and the efforts we have put in those contests and those camps. That is what I do believe in my mathematician's heart, even if I could not prove it to my mathematician's mind.

Could I finish my intervention at this round table in narrating a little anecdote which illustrates how some people sometimes become competent in some field? A few years ago, a young Italian interested by skindiving noticed in the Naples Museum of Fine Arts a greek vase representing a man swimming underwater with in his mouth some pipe which appeared to him as a snorkel. His curiosity being aroused, he inquired to the museum direction about the practice and techniques of skindiving amongst the ancient Greeks. But nobody in the museum could answer his questions, and he was sent to diverse specialists of the history, the language, the arts, the literature, the science, the technology, the sports and the sailing of Ancient Greece. But nobody could satisfy his curiosity. And one day, exasperated as one can be when he has an Italian temper, he exclaimed: "But is there anywhere a specialist of that question?" And somebody replied him: "As far as we do know, there are none, but if you continue to get interested in the subject, some day you will surely become a specialist of the field."

It is in this manner that I humbly hope to gain someday some competence in the art, if not in the science, of developing mathematical talents.

DEVELOPING MATHEMATICAL TALENT
A POINT OF VIEW

Stanley Erlwanger

The education of gifted and/or talented students has recently become a popular issue. In Canada, the CEA survey: "The Gifted and Talented Students in Canada" (1980) shows there were over 13,000 students entrolled in programmes for the talented during 1978-79. This trend reflects the concern and interest of parents, teachers and the public in developing talent. This concern however, is not shared by everyone. The CEA survey also suggests there is little visible support in this area from universities or the federal and provincial authorities. Thus, despite the current popularity of programmes for talented students, much of the activity is at the local school or school board level. As such there is a wide diversity of programmes across Canada.

Mathematics has traditionally been associated with bright students and selection. It is thus not surprising that the development of mathematical talent has generated much interest,activity and controversy. Some of these have appeared in the Arithmetic Teacher (1981) and NCTM Yearbook: The Mathematical Education of Exceptional Children and Youth (1981). My involvement in this area suggests that not many programmes are based on the claim that they take into account the characteristics of talented students. Instead, they are often attempts to "do something" in response to public expectations. Moreover, teachers and administrators usually do not have the time, resources or expertise so that programmes tend to be based on general views about mathematics and talented students.

My observation is that the nature of mathematical talent is not well known so that it is difficult to identify mathematically talented students or to design suitable programmes for them. It seems that before decisions can be made about how to develop mathematical talent there should be an adequate knowledge of the general and unique characteristics of mathematically talented students; general and specific methods of identifying mathematical talent; and the role of mathematical content in the development of general and unique features of this talent. Although research by Keating (1976), Krutetski (1975), Getsels and Dillon (1973), Stanley (1974 and Avery (1979) describe some characteristics of talented students and the design of special programmes, their information cannot easily be incorporated into locally designed efforts. I also share Oskorne's view that:

> "most (of this research) fail to account for the unique characteristics of mathematics and mathematical thinking. Mathematical thinking is sufficiently different from other types of intellectual activities to require consideration of special types of research questions."

The point of view I would like to advocate is that mathematics educators should take advantage of the current interest in this area to undertake research. To continue to ignore this area or to become directly involved in local programmes and controversies is unproductive. Undertaking research however will demonstrate an objective interest in talented students in much the same way that mathematics educators have approached the problem of teaching mathematics to less able students. In addition, research in this area is likely to broaden our knowledge of the nature of mathematical thinking and the characteristics of mathematically able students. This should in turn influence the teaching and learning of mathematics.

I shall illustrate this point of view by discussing further two areas where I think research is needed. These are the identification of mathematically talented students and the development of programmes.

Identification of Mathematical Talent

It is widely accepted that talent should be identified early using multiple methods which include intelligence, achievement and personality tests, case studies, teacher evaluations and nomination by peers. My observation is that this is not an easy task for teachers and administrators. In practise identification usually consists of an ad hoc mixture of teacher evaluations and scores on tests. Although this approach appears adequate, the results may be misleading and lead to controversies and conflicts with parents.

I think these are symptoms of two underlaying difficulties. One is the problem of discriminating between intellectual ability and mathematical talent. High intellectual ability indicated by tests may be a necessary but not a sufficient condition for demonstrating mathematical talent. Secondly, the most reliable evaluations by teachers tend to be based on criteria that reflect high performance on tests. In short, the instruments and methods that are currently available appear to be inadequate for identifying mathematical talent. What is needed are instruments and methods that are good indicators not only of high intellectual ability but also of specific characteristics of mathematical talent. Such instruments should also be easy to use in the classroom.

Programmes for Talented Students

It is often stated that the regular school mathematics programme is too structured and narrow for talented students. Hence special programmes are needed which would take into account the nature of mathematics as well as the characteristics of talented students. A wide diversity of programmes has emerged such as special schools, special classes, and various grouping patterns such as acceleration, enrichment, streaming, setting and individualized instruction. There are also clubs, summer schools and camps, and competitions. The mathematics content in such programmes often supplements the regular programme by including new topics, extensions of regular topics, and units on some topics like problem solving, graphing, number patterns, calculators and computers. Material for this purpose is either developed locally by writing teams or selected from different sources.

These programmes reflect the commitment of teachers and resource personnel. The typical view of each group is that the design and content of its programme stresses important features such as originality and creativity, the organization of data, logical reasoning, the ability to formulate a problem and to solve it, and so on. However, there is little evidence that some programmes are more successful than others. In fact, it is safe to say that the design of programmes and the selection of content is based largely on the assumption that a collection of mathematics topics outside the regular programme contains features that are necessary for developing mathematical talent.

My observation is that aside from such claims and assumptions, these programmes are seldom based on what it means to develop mathematical talent or what it means to claim that a programme takes into account the characteristics of talented students. Our current knowledge of the characteristics of talented students makes it difficult to consider questions such as: Which characheristics are important in the development of mathematical talent? Are there characteristics which enhance or inhibit mathematical talent? What is the role of different types of mathematics content and learning experiences in developing mathematical talent? What is an appropriate balance between structured and open-ended experiences in developing talent? What is the role

of personal and group encounters? Which methods are useful for assessing the development of mathematical talent in individual students?

The list of questions above could be extended. But it is enough to illustrate the problem of designing programmes for talented students. On the one hand, such questions cannot be considered easily in a school setting by teachers and administrators because of time constraints and limited resources. On the other hand, we do not have adequate answers for some of these questions at present.

Surely, unless there is more evidence to guide decisions about the design and content of programmes for talented students there is little or no guarantee that such efforts are useful, that effective use is being made of scarce resources and personnel, or that providing more funds, resources and personnel will result in better programmes. Indeed, unless some research evidence is available most of these programmes will disappear in the near future. But, like other popular trends in education, programmes for talented students will reappear again at a later date. It is a moot point whether we will be better equipped to deal with them then than we are now. This is the challenge for mathematics educators.

Mathematics educators have in the past accepted the challenge of conducting research on problems concerning the mathematical education of less able students. I have tried to suggest here that the current concern with gifted and talented students provides an opportunity for doing research in another challenging area.

References

Avery, Linda D., "The Replication of the SMPY Project at Selected Sites in Illinois". Paper presented at the American Educational Research Association Annual Meeting, San Francisco, 1979. Mimeographed.

Barthwick, B et al. The Gifted and Talented Students in Canada. Results of a CEA Survey. The Canadian Education Associateion, Toronto: 1980.

Getzels, J.W. and J.T. Dillon. "The Nature of Giftedness and the Education of the Gifted." R.M.W. Travers (Ed.). Second Handbook of Research in Teaching. Chicago: Rand McNally and Company, 1973.

Keating, D.P. Intellectual Talent: Research and Development. Baltimore: John Hopkins University Press, 1976.

Krutetski, V.A. The Psychology of Mathematical Abilities in Schoolchildren. Chicago: The University of Chicago Press, 1976.

Osborne, A. "Mathematics for the Talented". The Arithmetic Teacher, Volume 28, 6, February 1981.

Ridge, H.L. and J.S. Renzulli. "Teaching Mathematics to the Talented and Gifted". V.J. Glennon (Ed.). The Mathematical Education of Exceptional Children and Youth. National Council of Teachers of Mathematics, Reston: 1981.

Stanley, J.C. et al. (Eds.). Mathematical Talent: Discovery, Description and Development. Baltimore: John Hopkins University Press, 1974.

CANADIAN MATHEMATICS EDUCATION STUDY GROUP

GROUPE CANADIEN D'ETUDE EN DIDACTIQUE DE MATHEMATIQUE

GEOMETRY SUB-GROUP/SOUS-GROUPE DE GEOMETRIE

Report of the 1982-meeting in Kingston, June 3 to 7

par D. Lunkenbein, group coordinator

The teaching of geometry, or better the lack of it, in elementary and junior high school, seems to be a persistent concern of quite a number of collegues. Such concern lead, since the 1980 original working group [1] met in Québec, to an ongoing discussion, specifically with regard to the problem of getting geometry into the classroom. When some of us met again in Edmonton in 1981, we decided to take some concrete action by writing or selecting some geometry activities which might stimulate teachers to include them into their classroom activities [2]. Such contributions were collected during the year 1981/82 and distributed amongst contributing authors for reflection and criticism. As it turned out, these suggestions not only included activities for the classroom, but also general reflections on the why, what and how of the teaching of geometry in a variety of approaches. It therefore seemed appropriate to edit these contributions into a CMESG/GCEDM-monograph on "Teaching Elementary Geometry: Why, What and How".

For reasons of the relative ineffectiveness of such a monograph with respect to its impact on school teachers and for possible organizational and financial problems, the participants of this year's meeting felt that it would be probably better to proceed in the following way before thinking of producing a more coherent and, to some extend complete, monograph.

1. Publication of commented classroom activities in provincial teacher journals: It has been felt, that short (up to 5 pages) classroom activities well commented on the bases of teaching experience will stand the best chance of stimulating teachers to include geometry into their work, at

1 Report on working group C, "Geometry in the elementary and junior high school curriculum" in the proceedings of the 1980 meeting of CMESG/GCEDM.

2 Report on the special group P, "The place of geometry in the elementary school", in the proceedings of the 1981 meeting of CMESG/GCEDM.

least occasionally. The most convenient and inexpensive way of reaching the teacher is by way of provincial teacher journals. It therefore was suggested, that such classroom activities be edited or created and sent to the members of the group (via the group coordinator) for commentary and, if not yet experimented, for possible experimentation in the classroom. After such consultation (and possible discussion on the occasion of future CMESG/GCEDM meetings), comments and experiment-results will then be taken into account by the author. The group coordinator will ultimately send the thus finalized activity to editors of provincial teacher journals for publication under the author's name with the remark that it has been discussed and commented by members of the CMESG/GCEDM geometry group. The following members of the group have already agreed to start this process by producing or editing one such activity in the next four weeks: Tom Bates, Martin Hoffman and Arthur Powell. It is hoped, that other group-members will join in this process, so that over the year we will already publish 3 or 4 such activities and collect some more for discussion at the next meeting of the study group.

2. <u>Publication of position papers in appropriate Math. Education journals</u>: In a process analogous to 1. , members of the group are invited to work on general considerations concerning the teaching of geometry either by producing or commenting position papers. In particular Tom Bates and Dieter Lunkenbein invite commentaries to their papers entitled "Activities and structure in elementary school geometry" and "Géométrie dans l'enseignement au primaire" respectively. These comments may be send directly to the authors or to the group-coordinator. By such process we attempt to encourage the production and publication of position papers in appropriate Math. Education journals by several members of the group.

3. <u>Geometry monograph of CMESG/GCEDM</u>: It is expected, that, after some time of producing documents of the types mentioned above, we will be in a better position to produce a monograph on the teaching of elementary geometry as a collection of such documents.

Here is a list of collegues who have manifested a particular interest in this group over the past two years. It goes without saying that every collegue, who wants to get actively involved in the work of the group may do so by contacting the group coordinator.