Tom Kieren

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CMESG/GCEDM

1985 PROCEEDINGS

JUNE 7-11, 1985

UNIVERSITE LAVAL

EDITED BY C VERHILLE

GROUPE CANADIEN D'ETUDE EN DIDACTIQUE DES MATHEMATIQUES

CANADIAN MATHEMATICS EDUCATION STUDY GROUP

PROCEEDINGS OF THE 1985 ANNUAL MEETING

UNIVERSITE LAVAL
QUEBEC, QUEBEC
JUNE 7-11,1985

EDITED BY CHARLES VERHILLE

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Editors foreward

The proceedings for the 1985 CMESG/GCEDM meeting have been delayed for a long time. It was necessary to wait until a major contribution was received, otherwise the proceedings would have been most inadequate.

The proceedings, following the format of previous years, include the major lectures presented by Heinrich Bauersfeld and Henry Pollak followed by working group and topic group contributions in reduced format. This meeting represented our first effort to plan a joint speaker with the CMS - a group with whom we have many interests in common.

This represents our second meeting at Laval. The University in particular as well as Quebec City in general provide pleasant surroundings for such a gathering. We are especially appreciative to Claude Gaulin and Bernard Hodgson for making the local arrangements.

Charles Verhille
Editor

Canadian Mathematics Education Study Group
Groupe canadien d'étude en didactique des mathématiques

1985 Meeting

The ninth annual meeting of the Study Group was held at Laval University, June 7 to 11, 1985. Fifty mathematics educators and mathematicans met in plenary sessions and working groups. This year the conference was deliberately arranged to follow immediately on the CMS Summer Meeting and the first of the two guest lectures, by Henry Pollak (Bell Communications Research, was planned in collaboration with the CMS Education Committee. Dr. Pollak spoke "On the relations between the application of mathematics and the teaching of mathematics". He identified four different meanings commonly attached to the words "applied mathematics", and considered the implications of each for curriculum and for pedagogy. Also co-sponsored by CMS Education Committee was a session, led by Peter Taylor (Queen's), on "Exploratory problem solving in the mathematics classroom".

The second guest speaker was Heinrich Bauersfeld (IDM, Bielefeld) who made "Contributions to a fundamental theory of mathematics learning and teaching". Setting out to answer the question: How do we manage to retrieve what we require and adapt it to a new situation?, Professor Bauersfeld wove an intriguing account of constructivist theories.

Other lectures were given by Fernand Lemay (Laval), who presented a masterly sweep through the historical developments of analytic and synthetic geometry, and by Jacques Désautels (Laval), who applied the epistemological theories of Gaston Bachelard to the learning of science. Three accounts of specific researches on teaching, gender and mathematics were given by Roberta Mura (Laval), Gila Hanna (OISE) and Erika Kuendiger (Windsor).

The working groups at this conference focused on a positive view of students' errors, a group led by Stanley Erlwanger (Concordia) and Dieter Lunkenbein (Sherbrooke); on more advanced activities with LOGO, a group led by Joel Hillel (Concordia). A third group investigated the possibilities of symbolic manipulation software, led by Bernard Hodgson (Laval) and Eric Muller (Brock); the fourth tackled feelings and mathematics, led by Fran Rosamond (San Diego) and John Poland (Carleton).

This bald summary may indicate the scope of the conference but may not make clear the special characteristics of its style. Most conferences of comparable length offer participants many more lectures and paper presentations. The result, as everyone knows, is that participants at conventional conferences are selective in their attendance at sessions; no one can sit through continuous periods of being talked at. Participants at Study Group meetings, where ample time is allowed for cooperative work and discussion, tend to follow the whole programme. This generates more of a sense of common interest, a bridging of differences rather than an accentuation of them.

David Wheeler
Concordia University
Montreal

IN MEMORIAM DIETER LUNKENBEIN

The mathematics education community has been deeply shocked to hear bout the sudden death of our colleague Dieter Lunkenbein, on September 11, 1985, at 48 years of age.

Born and educated in Germany, he had come to Canada in 1968 to work as a research assistant for Dr. Zoltan P. Dienes at the Centre de Recherche en Psycho-mathématique in Sherbrooke. He subsequently got a Ph.D. in mathematics education at Laval University and he bacame a regular faculty member of Université de Sherbrooke, where he has displayed strong leadership in teacher education as well as in research and development in mathematics education.

In 1982 he was awarded the "Abel Gauthier Prize" by the Association Mathématique du Québec in recognition for his significant and exceptional contribution to mathematics education in Québec. At the Canadian level, he has been very active in the annual meetings of the Canadian Mathematics Education Study Group, particularly in working groups about teacher education and about the field of mathematics education, and as a leader of many groups on geometry education -- an area for which he was a recognized expert.

Dieter is the author of more than 70 scientific lectures or papers, including articles in Educational Studies in Mathematics, For the Learning of Mathematics, Bulletin de l'A.M.Q., etc. At the international level, he has been involved in many conferences and for about ten years he has been very active as a coopted member of the Commission Internationale pour l'Etude et l'Amélioration de l'Enseignement des Mathématiques (CIEAEM), of which he was the President from 1982 to 1984.

For the mathematics education community, the death of Dieter Lunkenbein constitutes a great loss. Everyone will long remember his work and dedication to our field as well as his impressive human qualities.

LECTURE 1

CONTRIBUTIONS TO A FUNDAMENTAL THEORY OF MATHEMATICS LEARNING AND TEACHING

BY HEINRICH BAUERSFELD
UNIVERSITAT BIELEFELD

Contributions to a fundamental theory of mathematics learning and teaching

HEINRICH BAUERSFELD
IDM (Institute for Mathematics Education), Universitat Bielefeld, FRG

> "Perhaps the greatest of all pedagogical fallacies is the notion that a person learns only the particular thing he is studying at the time. Collateral learning in the way of formation of enduring attitudes, of likes and dislikes, may be and often is much more important than the spelling lesson or lesson in geography that is learned. For these attitudes are what fundamentally count in the future." JOHN DEWEY (1938)

1. A theory gap in school practice

A few years ago the report of an outstanding piece of research appeared: it is D.HOPF´s investigation on the teaching of mathematics in grade 7 of the Gymnasium[1)] (D.HOPF 1980). The study analyses data from 14 000 students, their teachers and parents, at 417 Gymnasien in the area of West Germany including West Berlin, and it is a representative sample. Detailed questionnaires were used in order to find out about the "social, cognitive, and motivational conditions under which learning outcomes and credits" are produced in mathematics lessons. In our view the most interesting results are:

* There is an overwhelming dominance of direct instruction, in particular the well-known game of teacher´s questioning and student´s response as well as teacher´s monologues (lecturing) and similar types of instruction; and
* it is not possible to identify "any more general structure" in the extremely rich data base "which would indicate the existence of overall concepts for the orientation of method

and teaching". Clearly, this came out quite contrary to the researcher´s expectation, that "at least some of the concepts which were under discussion in mathematics education for methods and teaching would appear more often than in single specific phases of the lessons only." (D.HOPF 1980, p. 192)
The lack of explicit theory in everyday school practice could prove to be a surface phenomenon: Perhaps teachers do not talk about theoretical backgrounds, but they may follow recipes for action rather consistently, which are based upon certain theoretical concepts. One might expect, therefore, that careful analyses could lead to reconstructions of a hidden though theory-based grammar of teacher´s decisions.

Reviewing various well-known concepts of mathematics education, the researcher thought about such analyses,but "found no reason for establishing a search for interpretations which could be traced back to more general concepts." (D.HOPF 1980, p.191). That is to say, the researchers found continuities and regularities in the processes of the mathematics classroom - e.g. the preference for direct instruction - but they could not find any relation with the concepts that appear in the theoretical debates of the mathematics education community.

Now we can ask more generally: If not through theoretical reflection, how then do the often documented and criticized patterns of teaching and learning in mathematics classrooms come into being (see the "recitation game", HOETKER and AHLBRAND 1969)? On the one hand the available theories obviously do not cover the practitioner´s needs; the theories do not have sufficient explanatory power. The hidden regularities of everyday classroom practice on the other hand function as if they arose from the subjective theories of the participants (teachers and students). So probably these hidden regularities are the outcomes of covert processes of optimization, that is, they may represent a bearable balance between the given actual, societal, institutional, and micro-sociological forces in the classroom (where bearable means: bearable for the participants). Provided this is an adequate description, then the hidden gensis of the regularities would explain the product´s tenacity and resistance against every reform.

The following remarks are grouped into three chapters. The main part, chapter 3, presents theoretical considerations from the many micro-analyses of teaching-learning situations in mathematics conducted by a research group at the IDM Bielefeld (BAUERSFELD, KRUMMHEUER, VOIGT). The thesis of the domain-specific orientation of a peron´s action leads to new views on (and descriptions of) abstraction/generalization, representation/embodiment of concepts, and learning.

The preceding chapter 2 can just as well be read after chapter 3, since the remarks on deficiencies and paradigms in theories of mathematics education may then be more understandable. It is meant as an introduction as presented here. The concluding chapter 4 relates the theoretical discussion to certain recent issues in problem solving. The application gives support to the thesis of the preceding chapter.

2. The paradigms of theories of mathematics education

The usual set of didactical questions: What is the nature of the subject? How is it learned? and How should we teach it? reproduces in itself disciplinary boundaries. Theories of mathematics education tend therefore to stress the relation either to the acting persons or to the subject matter of mathematics. Thus we receive psychological or mathematical-philosophical answers, such as student-centered "theories of learning" and teacher-centered "theories of instruction" or as subject-matter-centered theories of knowledge, of curriculum, of task analysis, of AI-simulations[2)] etc. Until very recently, linguistics, sociology, etc., were not disciplines to which the math ed community referred.

Both theoretical mainstreams use the stages metaphor when characterizing developmental aspects. Psychological approaches arrive at stages based on classes - or more precisely at progressive class-inclusions - of abilities (e.g. KRUTETSKII 1976), or of operations (e.g. PIAGET 1971), at levels of learning (e.g. VAN HIELE 1959) etc. Since mathematical abilities as well as the success of learning mathematics are described or measured through the quality of solving certain mathematical tasks, it becomes inevitable that the hierarchies of psychological constructs map subject-matter structures. They duplicate mathematical hierarchies, but do not create genuine psychological descriptions of the related actions. The subject-matter-centered theories on the other hand use mathematical structures directly for the modeling of stages. We can state therefore, that in both theoretical mainstreams the description of the field is dominated by mathematical means.[3]

But math educators will have to extend their fundamental theoretical questions, if at least a reasonable subset of classroom processes follows hidden regulations. The more since the regulations develop interactively rather than directly through the participant´s intentions, and with effects often inconsistent if not conflicting with the official aims. Then we will have to take into account not only that teachers and students enter and leave the classroom with certain individual dispositions, intentions, and expectations - which we do in order to draw inferences from the difference between the two cross-sections, but we will also have to ask <u>what</u> they make of it in a concrete situation, how they actually employ available states of knowledge, and <u>when</u> they activate and <u>how</u> they use schemata (and not only which ones, as is usually done). Cross-section analyses of input and output states with inferences about the process in between are no longer sufficient for an adequate understanding. If, as becomes evident, knowledge develops together with and as part of the knowledge, then this calls in question the process-product metaphor.

Furthermore the ongoing vivid interation in the classroom indeed leads to very personal (subjective) interpretations and constructions of meaning. But socially shared meanings and norms of content-processing are produced as well. And these are not just taken over like ready-made rules, rather they are constituted through the interaction, they become reality via the mutual processes of construction and negotiation. That is to say, we have to discriminate individual structures of potentially available knowledge from the interactional structuring of the actual actions. And on a social level we have to discriminate (so-called) objective subject-matter structures from the related meanings, norms, and claims for validity, as they are constituted in the course of the interaction in the classroom. This of course makes cause-effect analyses haphazard, because attributing cause to a single person´s action may become difficult.

There is a remarkable convergence in recent developments in mathematics and in cognitive science as well as psychology that supports the scepticism advocated here. In the view of cognitive psychologists the When and the How, as mentioned above, are mainly organized on metacognitive levels. The classical problem solving strategy from AI-developments - building up a hierarchy of operations or organizing control on a superordinate level - recurs here and has been the subject of intensive discussion in cognitive psychology recently (see ANN L. BROWN, J.C. CAMPIONE, and M.T.H.CHI in WEINERT 1984) under the name "metacognition". Investigations begin to focus on "dynamic learning situations" and on "interactive processes" (J.V. WERTSCH 1978, 1984; and A.L. BROWN in WEINERT 1984, p.101/102. One of the "many largely unsolved problems" in developing advanced intelligent computer systems for educational purposes that TIM O´SHEA has named is that "not enough thought has been given to represent inexpert reasoning". He has also pointed at the crucial role of "using natural language" (1984, p.266). Interestingly the attack comes from non-human information processing research, human understanding, learning, and reasoning in general and of mathematics in particular.[4)]

Even mathematics itself has been challenged from within the community, as by LAKATOS´ concept of "informal, quasi-empirical mathematics", an image of the discipline which he holds out against the counterpart of "authoritative, infallible, irrefutable mathematics"[5)] (1976, p.5). FREUDENTHAL has long since argued against the same enemy: "True mathematics is a meaningful activity in an open domain." (1983, p.39).

> "Why, come to think of it, do we have so few good ideas and theories about the mind? I propose the following answer to this question:
> 1. It may be the most difficult question Science ever asked.
> 2. It is made even harder because our first theories have led us in the wrong direction."
>
> MARVIN MINSKY (1982, p.35)

3. "Domains of Subjective Experience" and "Society of Mind"

In our research process the adoption of sociological methods and concepts has turned into a process of adapting the means to the end. Since we are interested in learning and teaching mathematics rather than in general social structures, as identified by sociologists across subject-matter, our analyses are focussed on the relations between the subject-matter aspects, as thematized by the participants, and the predominantly social nature of classroom processes and their conditions. This, we think, describes an important weakness in the dominating psychological and subject-matter-oriented theories.

Our micro-analyses of video-taped teaching-learning situations at different schools and with different ages have led to three related theoretical elements. GOTZ KRUMMHEUER has adapted GOFFMAN´S "frame analysis" in order to describe the participants´(teacher and students) definitions of situations - "frames" - and their stratified changes - "keyings" - in the flow of interactions. Complementary to these actual activities, my concept of "domains of subjective experience (DSE)" aims at the description of the sources and the organization of memory and of the related long-term effects called learning. These respresentations function as relatively stable dispositions and as the potential from which the individual´s actual orientation and action is coined and formed. JORG VOIGT has investigated the hidden regulations of classroom procedure as they are constituted among the participants. He describes "patterns of social interaction" and their relation to "moves under duress" and to (DSE-rooted) individual "routines". (See KRUMMHEUER 1983 and 1984, BAUERSFELD 1980, 1983 and 1985, and VOIGT 1984 and 1985.)

In the following I shall restrict myself to discussing the main aspects of the DSE-metaphor. It should be noticed that we offer alternative interpretations but do not claim to describe "the" reality. The theoretical elements offer a well-founded perspective on classroom processes among other theoretical perspectives, with which it competes. The theses and their substained connections are the products of "abductions" (C.S. PEIRCE 1965, J.VOIGT 1984a). Thus a specific understanding of the genesis of theories as well as of theory itself is functional in our approach.

1. Thesis — All subjective experience is domain-specific. Therefore all experiences of a person (subject) are organized in Domains of Subjective Experiences (DSE).[6)]

Whenever I have experiences, that is: I learn, actively and/or passively, this occurs in a concrete situation, something which I realize as context. Thus learning is situation specific, is learning-in-context. Learning is not limited to cognitive dimensions. Since I cannot switch off one or the other of my senses deliberately, all of my senses are involved, particularly the genetically older organs like the mid-brain (emotions) and the cerebellum (motor functioning). The stronger the accompanying emotions, the more distinct and richer are certain details and circumstances in the recollection. We therefore speak of the totality of experiences and learning.

Learning is also multidimensional: I learn how to do things, and along with that, though mostly indirectly, I learn about the when and the why. At all times I learn about myself and about others.

The specificity to situation, the totality, and the multidimensionality, give good reasons for the conjecture that all experiences of a person are stored in memory in disparate domains according to the related situations. Each DSE encloses all of the aspects and ascribed meanings which appeared to be relevant for the person who was acting within the situation. Encountering the same situation repeatedly contributes to the consolidation of the related DSE, but as well to its isolation from other DSE´s. When entering a specific known situation a person immediately ´knows´ very much, due to the activated DSE.

> An example: More than 25 years ago during teaching practice with student teachers in the country, I visited a little nongraded school of some twenty pupils ranging in age from 7 to 14. The teacher opened the first lesson with a series of spectacular actions. He called on the attention of the few 11 and 12 year olds and made the others work silently. Then he ostentatiously dropped a plate which burst into pieces. A defective teapot followed, and finally he broke a few wood sticks into pieces. His hand waved over the scene accompanied by the key question: "What is this?!" And a nice little girl answered: "It is the introduction to fractions!"

> Apparently she had experienced this happening repeatedly in her earlier school years and she knew it would end up with naming and calculating with fractions. From that she gave a clear definition of the situation.

Under a phylogenetic perspective the immediate availability of an adequate DSE guarantees survival. The complex nature of the DSE´s enables the activation of a specific one just through a smell, a touch, a word, a picture, an action etc., and in such a way provides for the instantaneous identification of a dangerous situation for quick and appropriate (re-)actions, and for a certain coping with possible consequences. Obviously many of the students reactions in mathematics lessons are examples of such direct and prompt concatenation, ensuring survival in the classroom and saving unpleasant effort and reasoning.

The ideals of mathematizing, on the other hand, are clearly related to critical distance, to analytic decomposition and reflected construction, and to operations with symbols and models. These arts do not develop along the elicitation-reaction line. In order to overcome the troublesome phylogenetic conditions (which we cannot change nor deny), instructional situations should therefore give more attention to indirect learning on higher levels rather than to behavioral responses/evoking through invitations on the bottom level of direct action and reaction only.

<u>2. Thesis</u> The domains of subjective experiences (DSE) are stored in memory in a non-hierarchically ordered accumulation, following M.MINSKY´s idea of a "society of mind" (1982). In a given situation the DSE´s function in competition for activation, independently from each other, and this the more intensively they have been built up initially.

The model represents a powerful description for a functioning organisation of the isolated DSE´s. According to the flow of personal impressions and activities the "society of mind" is under continuous change and development. Permanent and lifelong new DSE´s are formed[7)], older DSE´s are changing. The gradual fading away of DSE´s, not activated for a long time, diminishes the growing burden, the more, the lower, the emotional status and the frequency of activation of the DSE are.

Every activation produces change: Often activated DSE´s pass through many transformations: the meaning, the relations, and the importance of their elements may shift, the characteristics of the situations become less specific (they allow more variance, i.e. they generalize), and a hard core of routines, of easy meanings, and of preferred verbal or pictorial presentations is shaped. In an actual situation these well-developed DSE´s obtrude themselves through their smooth perspectives and therefore have the best chance to win the competition for reactivation. Thus success has stablizing and tracking effects, though not necessarily for optimal solutions, as an observer may note rather than relative personal optima. But since every situation is new in a certain sense, there is an opportunity for younger and less elaborated DSE´s ("soft state") as well as for easy and robust older DSE´s. There is no preference in principle in the activation game as the phenomenon of regression demonstrates: The relapse into certain pattern of understanding and action under stress, which are older and less adequate or less differentiated, but are functioning more quickly and more reliably, receives a simple explanation from within the "society of mind" model.

The model, by the way, leaves no room for an independent or superordinate authority in the "society", a "demon" or something similar, who selects and decidedly activates DSE´s. Clearly we can exercise a limited influence on our internal retrieval processes, but we are not in command of our memory as the many failures of mnemonics show. An idea suggests itself - or not.

Through microethnographical analyses a surprisingly high degree of separation between single DSE´s has been demonstrated (LAWLER 1979, Bauersfeld 1982). Outcomes from quantitative-experimental research work gives support also. Recently E.FISCHBEIN et. al. (1985) have investigated the solving of verbal problems in multiplication and division with 623 Italian pupils in grades 5, 7, and 9. They focussed their attention on the role of "implicit, unconscious, and primitive intuitive models." Such models, so goes their hypothesis, might mediate "the identification of the operation needed to solve a problem" and thus "impose their own constraints on the search process." (FISCHBEIN et.al. 1985, p.4). The authors arrive at the unexpected profoundness of the expected effects, which they describe as "a fundamental dilemma" for the teacher:

> "The initial didactical models seem to become so deeply rooted in the learner´s mind that they continue to exert an unconscious control over mental behavior even after the learner has acquired formal mathematical notions that are solid and correct."(p.16).

The authors identify two sources for the genesis of such personal (subjective) models. One is the direct relation to the concept and the operation as it was initially taught in school. As the other, they found a natural tendancy to produce subjective regularities and use them intuitively through continuous activities "beyond any formal rules one has learned" (p.15) and though they might be "formally meaningless and algorithmically incorrect" (p.14). This represents an example of the genesis of a DSE, pointing at the specifity of situation as well as at the totality and multidimensionality of subjective experience as stated above.

The rigid disparity of two DSE´s which from a teacher´s perspective should be extensively interrelated (as e.g. experiences with a special case and the general rule) characterizes not only the phase of initial development in the subject. Against the expectations of a natural growing together of separately-gained pieces of knowledge through repeated practice, the persistent subordination of knowledge to specific DSE´s remains effective and dominates the subject´s actions. The supposition that cognitive networks develop quasi-automatically through an adaptation to the logics of subject matters appears as an illusion. The "society of mind" model with its independently competing DSE´s allows a simple explanation for the persistence of disparate DSE´s for the "same" situation. This can happen even in cases where a DSE´s concepts and procedures are stored but not used though they are superior or more general in an observer´s view, because they do not cover the "same" problem under the subject´s perspectives. Even so-called general concepts stored in memory are inevitably related to the subject´s perception of the situation in which the concepts were built. And therefore ascertaining the "sameness" of two cases affords a comparing of elements from at least two different DSE´s (see thesis 4). Each activation from memory on the other side reinforces the activated DSE, but not an abstract relation to other DSE´s.

3. Thesis The activities of the subject and the related subjective constructions of meaning and sense, as these develop through social interaction, are the descisive fundamentals for the formation of DSE.

In mathematics education, in particular, the subjectively relevant activities are bound to the offered mediatization of the matter taught, to what is really done. Teacher and students act in relation to some matter meant, usually a mathematical structure as embodied or modelled by concrete action with physical means and signs. But neither the model, nor the teaching aids, nor the action, nor the signs are the matter meant by the teacher. What he/she tries to teach cannot be mapped, is not just visible, or readable, or otherwise easily decodable. There is access only via the subject´s active internal construction mingled with these activities. This is the beginning of a delicate process of negotiation about acceptance and rejection. That is why the production of meaning is intimately and interactively related to the subjective interpretation of both the subject´s own actions as well as the teacher´s and the peer´s perceived actions in specific situations. Via these processes the (social) norms of mathematical action are also constituted in the classroom, covertly, regarding acceptability, validity, completeness, relevance, and so on.

The doctoral thesis of G.FELLER 1984) gives an idea of how important the activities with embodiments and physical means (teaching aids) are for the formation of mathematical experiences. She tested mathematical achievement at the end of grade 2 in Berlin in order to find out the extent to which the aims of the mathematics curriculum had been attained. As a by-product the author was "startled by the strong impact of the manner of representation". Her final assessment:

> "The outcomes indicate that the acquisition of each different type of representation requires the learner´s explicit endeavour and connected rehearsal, an effort which is not less than is usually required for the learning of mathematical matter itself (like addition or subtraction)." (G.FELLER 1984, p.67).

In our terminology this would mean that, for many children, experiences with a new representation of subject matter, though perhaps well-known from other situations, lead to a new DSE, stored separately in memory and with weak if any relations to the older experiences.

A new DSE can also develop through the explicit connecting of elements from different older and available DSE´s. The "Aha" insight, flashing up suddenly while acting within the horizon of an activated DSE and producing the idea of essentially "doing the same" as in another context (DSE), is the announcement of a birth, for the person as well as an observer. But the "Aha" alone does not produce by magic a fully developed network of relations here and now. It takes time and continual activities to elaborate the new DSE. An "Aha" insight, not elaborated after the first appearance, can fade away in the continuous flow and only light up again much later accompanied by the feeling that something like that was known already.

DSE´s disappear only (and slowly too) if they do not receive reactivation. Growing interrelations and even integration are not necessarily weakening effects. "The mind never subtracts" (M.MINSKY, 1981). As is the case with regression very old DSE´s can prevail in the competition for activation under stress against younger DSE´s where so-called "higher", "super-ordinate", "more sophisticated" knowledge is stored.

In the mathematics classroom students are often asked to identify common characteristics between two events or cases, which in the view of the teacher appear to be two models for the very same mathematical structure. This is the task of producing a generalizing abstraction from different embodiments upon request. In our view the student then has to compare elements which are rooted in two different DSE´s; in other words: which are incorporated in two different contexts. What can form the basis of the required comparing activities?
Usually the perspectives of the separate DSE´s themselves do not cover such operations, due to the specificity of actions, language and meaning. So where do the aims come from? Which kind of similarity or commonness do I have to search for? The adequate basis has to be a third DSE, the elements of which are the means for comparison and the possible aims. Comparing common characteristics by abstracting and neglecting other ones is a complex and highly constructive activity. Without an orientation, at least a diffuse image of the potential results and of the relevant characteristics, as well as an idea of the adequate means, there is no reasonable chance for the student´s success.

An example may demonstrate the difficulties. What do the following three situations have in common?

a) You plunge your hand into a paper bag three times and take out two eggs each time.
b) You see three blocks of houses with two houses in each block.
c) Three boys and two girls dance. How many different pairs are possible? (old-fashioned style: one girl one boy per pair!)

The question can also be put this way: For which more general issue can these three situations serve as models? Is it enough to answer - like fourth graders perhaps would do - "It´s always six!" or "All are three times two" or "It is multiplication!" or ...? What is the meaning of the concept "multiplication of natural number"? How may it be explained?

The critical step is the crossing of the borderlines of the three related DSE´s. The interesting commonness is not with the same twos, threes, and sixes in each situation. What are the conditions for seeing the well-known elements differently, to dissociate them from narrow concreteness, to attach another meaning, another relation, a more general relation, to them? Obviously, we can get hold of what we call a common structure only by means of a model, of a certain description; no matter how concrete or illustrative this model might be, provided that it can work for us as the more general model, which we can identify in (map onto) each of the three situations given.

For the above example a possible fourth model can be

d) Three parallel lines are cut by two other parallel lines. The first three lines can then represent the 1., 2., and 3. selection or house-per-block or boy. The second two lines represent the 1. and 2. egg per selection or block or girl. And the intersections (modes) stand for the six eggs or houses or pairs in total.

Clearly the learner either has to reconstruct from related help and hints or he/she has to construct such a model on his/her own. It should be clear, too, that this construction is not by nature an integrated part of any one of the three situations. It is not part of the experiences within the three related DSE´s it is a new perspective.

From another point of view the geometrical configuration d) is nothing else than just another (specific) model for the multiplication of natural numbers. Under this view there are many more adequate models or descriptions, e.g. e) A table with three columns and two rows, including the three initial ones (more in H.RADATZ/W.SCHIPPER 1983, p.73).

From a developed understanding of the concept, each of the models can serve as description of "the" general structure of multiplication of natural number, at least potentially, and realizable through one-to-one coordination. Thinking about the available modes and possibilities for the representation and any structures at all, we might find that we cannot overcome the force of the use of models in communication. In principle there is no transgression. This brings us nearer to the relativity of so-called general concepts (see T.B.SEILER 1973). At this point, on the other hand, the common statement about "the best learning is learning by example" sounds somewhat tautological.

<u>4. Thesis</u> In terms of memory there are no general or abstract - i.e. context-free - concepts, strategies, or procedures. The person can think (produce) relative generality in a given situation. But the products are not retrievable from memory in the same generality or abstractness, that is, they are not activatable independently of the related DSE´s.

With advancing years the development of the "society of mind" leads to an accumulation of DSE´s and also to a growing network of relations among their elements through even the relations are realized and retrievable only in specific domains. Their genesis is bound to the considerable constructive activities of the person as well as to the situations of practice and to the qualities of social interaction. The perspective of a certain DSE may become integrated into a new DSE, together with elements from other DSE´s. In the perspectives of the new DSE the integrated older experiences may appear as subordinate and hierarchically lower elements. But in spite of that the older DSE still can compete for activation with the new DSE. R.LAWLER therefore speaks of a "structure of a mixed form, basically competitive but hierarchical at need" (1981, p.20), more precisely perhaps: hierarchically through special activation. General knowledge is available through special activation only, this is the meaning of thesis 4.

The disparity of the DSE´s marks not only the phase of their initial formation but also the later phases when detailed or more general knowledge has been required, which of course is stored in different DSE´s because of the differences in situation, as the investigations of FISCHBEIN et al. (1985) show. Microethnographical studies at preschool and early school ages substantiate the extent to which the ability for identifying two events as being "the same case" depends upon previous learning experiences and upon the subjective perception and definition of the actual situation. In several long-term studies R.LAWLER has documented and analyzed the encounters of his children with computers, arithmetic and geometry (1979, 1981, 1985). His early concept of "microworlds" is the cognitive shadow of the domains of subjective experience (DSE) as defined here (and elsewhere, BAUERSFELD 1982, 1983).

> LAWLER´s daughter Miriam e.g. has solved tasks of the type 75 + 26 = ? according to the specificities of presentation in at least three different and for long incompatible microworlds.
>
> If the task appears as "75 cents plus 26 cents" Miriam calculates the solution via her activated "Money world", like: "That´s three quarters, four and a penny, one-oh-one!" The presentation of "seventy-five plus twenty-six equals..." she solves in her "Serial world" like: "seven plus two,

nine, ninety-six, ninety-seven, ninety-eight, ninety-nine, hundred, one-oh-one!"

And if it is written as a vertical sum, Miriam adds up the columns and carries the tens (R.W.LAWLER 1981, H.BAUERSFELD 1983).

The "identical" arithmetical task, as a teacher would name it, is thus solved according to the activated special DSE using related but completely different procedures. For the child, obviously, the different presentations are perceived as different and independent tasks. The rigid disparity remains in effect even when all three representation are given consecutively. It is much later that through spectacular "Aha" events certain relations are produced.

The studies support the supposition that, in particular, the use of language is specific to the situation and hence to the activated DSE. In LAWLER´s protocols Miriam uses the phonetically same words "six", "seventy", "plus", etc. across the different situations, whilst her concrete actions indicate different specific meanings in correspondence with the different activated microworlds. For an observer therefore it is impossible to interpret an utterance without adequate reconstruction of the related subjective definition of the situation (DSE). Likewise it is impossible for Miriam to take a distancing and critical perspective against her specific procedures and interpretations from within the activated DSE. Evidently this is impossible in general - without having developed the distancing and critical perspective as an integrated habitual activity within the DSE. That is why a teacher´s urging for comparing, for controlling, for looking closer etc. has no effect when these activities are not developed in relation to the activated narrow DSE.

There is by the way good reason for the development of disparate DSE´s because of the strikingly different sensual characteristics of the concrete activities.

Miriam´s "Money world" is built upon her intensive experiences with her pocket money, with buying and change. What mathematicians call the operations of addition and subtraction is here embedded in a world with its own specific sensuality: colour, and coinage etc. and with specific non-number names like penny, nickel, dime etc. (see H.RUMPF 1981).

In contrast to that her "Counting world" is ruled mainly by word sequences which obey certain rules of construction ("twenty, twenty-one,...") and which are produced through one-to-one procedures of speaking and touching the objects to be counted.

The Paper-sums world" is a medium of quite another type of sensuality: Writing symbols on paper using a pencil (with the typical fine-motoric muscle tensions), reading, and operating with the symbols (see H.BROGELMANN 1983).

So we can state that meaning is attached to a word through certain activities in a certain situation but a word has no definite meaning per se. This is true with speaking, hearing, reading, and writing. Likewise we interpret a word heard in a concrete situation within the range of the actually activated DSE. There is no other chance for understanding without additional effort, e.g. the activation of other DSE´s. In this sense even the so-called universal language of mathematics is not universally available (retrievable) for a person.
Theories become helpful models for realities when and insofar as they generate constructive orientation. So more interesting than the disparity of DSE´s and the unthinkable purity of context-free concepts, perhaps, are both the totality and the principle of multidimensionality of learning in social interaction:

<u>5. Thesis</u> Whenever we learn, all of the channels of human perception are involved; i.e. we learn with all senses, learning is total. And: simultaneously we learn on all dimensions and levels of human activities, at least potentially; i.e. learning is multidimensional.

A smell therefore can activate a certain DSE later on, as can a pattern of motion or a sophisticated metaphor. In a given situation we not only learn about the subject matter, directly and attentively, the what-to-do - e.g. the theme, facts and procedure (declarative and procedural knowledge) - we also learn, more covertly, about the how and the when to do it - e.g. orientations of action, strategies, the fit and the adequacy of situations - we also learn about the why to do it - e.g. sense, reasons, attached values - we learn about ourselves - e.g. anxiety and motivation, personal identity - and we learn about the others and how they see us - e.g. social norms, the person´s social identify. The listing is far from complete. We also develop routines and pattern of habitual activities in all dimensions.

JOHN DEWEY already formulated this idea in 1938[7]:

"Perhaps the greatest of all pedagogical fallacies is the notion that a person learns only the particular thing he is studying at the time. Collateral learning in the way of formation of enduring attitudes, of likes and dislikes, may be and often is much more important than the spelling lesson or lesson in geography that is learned. For these attitudes are what fundamentally count in the future."

The continuous flow of conscious production only marks the surface of a much deeper stream of experiences which form the orientation of a person´s future actions. As DEWEY stated, the most important things are learned collaterally, across many activities and preconsciously, in a FREUDian sense. So what is learned beyond the official theme, this major and more powerful portion of learning appears as a core problem of classroom teaching. AUSUBEL´s classical and often quoted words may now be read with a somewhat more differentiated understanding:

"If I had to reduce all of educational psychology to just one principle, I would say this: The most important single factor influencing learning is what the learner already knows. Assertain this and teach him accordingly" (1968)

Notes

1) In West Germany (FRG) after four years in primary school about 25-40% of the 10-11 years old students enter a Gymnasium, where they normally pass grades 5-13 and end up with the Abitur, at age 19. The Abitur exam is the general pre-requisite for university entrance. The majority of the students enter grade 5 of Hauptschule, Realschule or Gesammtschule, the other types in the secondary school system.

2) These include not only direct simulations of mathematical content on the computer screen, but also simulations of the learning process, of the learner´s previous knowledge and strategies, because all this information is processed in the form of mathematical or logical rules and with unambiguous ascriptions (meaning).

3) This, clearly, requires more detailed discussion, which cannot be done here. My interest is to point out the limitations which are carried by the unreflective use of my categories or descriptors. They seem to be "at hand" (like metonymies) for what we think we see. But we usually do not reflect upon their origin or their context, which leads to covert, narrow pursuit, and not to novel ideas. As operations in context, describing and interpreting are dependent on the qualities of these bases of the teaching-learning processes.

4) T.O´SHEA stated that often "the attempt to automate an activity forces a better understanding of the activity itself" (O´SHEA/SELF /1983, p.267). And he ends his diagnosis by saying: "...it is easier to let children try to learn BASIC than to develop learning environments which facilitate intellectual discoveries; it is easier to write programs which treat students uniformly than to write programs which try to take account of an individual student´s interests, errors and aptitudes" (ibid., p.268).

5) Analysing the role of example and counterexample in "proofs and refutations" LAKATOS said: "...we may have two statements that are consistent in (a given language)L_1, but we switch to (a new language) L_2 in which they are inconsistent. Or we may

have two statements that are inconsistent in L_1, but we switch to L_2 in which they are consistent. As knowledge grows, languages change.

´Every period of creation is at the same time a period in which the language changes.´ (FELIX) The growth of language cannot be modelled in any given language." (I.LAKATOS 1976, p.93; brackets added from context, H.B.). LAKATOS identifies the change of language as "concept-stretching" (p. 93 f.). But "concept-stretching will refute <u>any</u> statement, and will leave no true statement whatsoever." (p.99) Indeed he denies the existence of "inelastic, exact concepts" as bases for rationality (p.102). There is no eternal truth, there is only "guessing" (p.76 f.) and "the incessant improvement of guesses" (p.5). D.SPALT (1985) discusses in detail the failure of LAKATOS´ solution to this fundamental problem: "mitigation" of concept-stretching (LAKATOS 1976, p.102 f.).

6) The notion of "subjective" experiences rather than "personal" experiences (which might be nearer to colloquial English) follows etymological considerations. The Latin origin, the verb "subjicere", means in the transitive sense that the person (the subject) actively subjugates something, makes it the person´s own through action. This of course describes the functioning of "subjective experiences". The active parts are at least the continuous constructions of meaning and the selecting and focussing in our changing definitions of the actual situation.

7) For this quotation I am indepted to HARRIET K. CUFFARO´s article in the Columbia Teachers College Record, summer 1984, p. 567, which interestingly critizes the present use of computers in schools.

The vigilant reader will find that chapter 4 as promised at the bottom of page 2 is missing here. The chapter will have to be added later on.

References

ADELMAN, C. (ed.) Uttering, Muttering - Collecting: using the reporting talk for social and educational research. London: Grant McIntyre 1981.

ANDERSON, J.R.: The Architecture of Cognition. Cambridge, Mass.: Harvard Univerity Press 1983.

BAUERSFELD, H. (ed.): Fallstudien und Analysen zum Mathematikunterricht. Hannover: Schroedel 1978.

BAUERSFELD, H.: Kommunikationsmuster im Mathematikunterricht. In: H. BAUERSFELD: Fallstudien ..., Hannover, Schroedel 1978, p.158-170.

BAUERSFELD, H.: Subjektive Erfahrungsbereiche als Grundlage einer Interaktions-theorie des Mathematiklernens und -lehrens. In: H.BAUERSFELD, H.BUSSMANN et al. (eds.):Lernen und Lehren von Mathematik. Koln: Aulis Deubner 1983, p.1-56.

BROGELMANN, H.: Kinder auf dem Weg zur Schrift. Konstanz: Faude 1983.

CUFFARO, H.K.: Microcomputers in education: Why earlier better? In: Teachers College Record 85., 1984, p.559-568.

DEWEY, J.: Experience and Education. New York: Collier Books 1963, p.48 (the original appeared in: KAPPA DELTA PI 1938)

FELLER, G.: Lernfelder in des Grundschule. In: Zentralblatt fur Didaktik der Mathematik 16., 1984, Heft 2, p. 63-67.

FISCHBEIN, E./DERI, M./NELLO, M.S./MARINO, M.S.: The role of implicit models in solving verbal problems in multiplication and division. In: Journal for Research in Mathematics Education 16., 1985, no. 1, p. 3-17.

FREUDENTHAL, H.: Didactical Phenomenology of Mathematical Structures. Dordrecht (Netherlands): Reidel 1983

GILMORE, P./GLATTHORN, A.A. (ed.): Children in and out of School. Washington: Center for Applied Linguistics 1982

HEYMANN, H.-W.: Mathematikunterricht zwischen Tradition und neuen Impulsen. Koln: Aulis Deubner 1984

HOETGER, J,/AHLBRAND, W.P.: The persistence of the recitation.
In: American Educational Research Journal 6., 1969, no.2, p.145-167

HOPF,D.: Mathematikunterricht. Stuttgart: Klett-Cotta 1980

JAHNKE, H.N.: Historische Bemerkungen zur indirekten Anwendung der Wissenschaften. In: H.G.STEINER, H.WINTER (eds.): Mathematikdidaktik - Bildungsgeschichte - Wissenschaftsgeschichte. Koln: Aulis Deubner 1985

KOKEMOHR, R./MAROTZKI, W. (eds.): Interaktionsanalysen in padagogischer Absicht. Frankfurt/Main: Peter Lang 1985

KRUMMHEUER, G.: Das Arbeitsinterim im Mathematikunterricht. In: H.BAUERSFELD, H. BUSSMANN et al. (eds.): Lerner und Lenren von Mathematik. Koln: Aulis Deubner 1983, p.57-106

KRUMMHEUER, G.: Algebraische Termumformungen in der Sekundarstufe I - Abschlubbericht eines Forschungsprojektes. Materialien und Studien Band 31, Bielefeld: IBM der Universitat 1983

KRUMMHEUER, G.: Zur unterrichtsmethodischen Dimension von Rahmungsprozessen. In: Journal fur Mathematikdidaktik 5., 1984, p.285-306.

KRUTETSKI, V.A.: The Psychology of Mathematical Abilities, Chicago: The University of Chicago Press 1976

LAKATOS, I.: Proofs and Refutations. London: Cambridge Univerity Press 1976, deutsch: Beweise und Widerlegungen. Braunschweig: Vieweg 1979

LANCY, D.F.: Cross-Cultural Studies in Cognition and Mathematics New York: Academic Press 1983

LAWLER, R.W.: One child´s learning. Unpublished doctoral dissertation, Cambridge, Mass: M.I.T. 1979

LAWLER, R.W.: Computer Experience and Cognitive Development. Chichester (England): Ellis Horwood 1985

LORENZ, J.-H.: Lernschwierigkeiten: Forschung und Praxis. Koln: Aulis Deubner 1984

LORENZ, J.-H./SEEGER, F. (eds.): Arbeiten zur Psychologie und Didaktik aus der UDSSR. Materialien und Studien Band 32, Bielefeld: IDM der Universitat 1983

MEHAN, H.: Institutional decision-making. In: B.ROGOFF, J.LAVE (eds.): Everyday Cognition. Cambridge, Mass.: Harvard Univerity Press 1984, p.41-66

MEHAN, H.: Learning Lessons, Cambridge, Mass.: Harvard University Press 1979

MEHAN, H./WOOD, H.: The Reality of Ethnomethodology. New York: Wiley 1975

MINSKY, M.: K-Lines: A theory of memory. In: D.NORMAN (ed.): Perspectives on Cognitive Science. Norwood, N.J.: Ablex 1981, p.87-103

MINSKY, M.: Learning Meaning. Cambridge, Mass.: M.I.T., Artificial Intelligence Laboratory 11/1982

O´SHEA, T./SELF, J.: Learning and Teaching with Computers. Brighton, Sussex: Harvester Pres 1983

OTTE, M.: Zum Problem des Lehrplans und der Lehrplanentwicklung in der Sekundarstufe I. In: Informationen uber Bildungsmedien in der Bundesrepublik Deutschland IX, Frankfurt/Main: Institut fur Mildungsmedien 1982, p.87-103

PIAGET, J.: Biology and Knowledge. Edinburgh: Edinburgh Univerity Press 1971

RADATZ, H./SCHIPPER, W.: Handbuch fur den Mathematikunterricht an Grundschulen. Hannover: Schroedel 1983

ROSLER, W.: Alltagsstrukturen - kognitive Strukturen - Lehrstoffstrukturen. In: Zeitschrift fur Padagogik 29., 1983, Heft 6, p.947-960

RUMPF, H.: Die ubergangene Sinnlichkeit. Munchen: Juventa 1981

SCHOTZ, A./LUCKMANN, T.: Strukturen der Lebenswelt, Band I. Frankfurt/Main: Suhrkamp 1979, stw 284

SEILER, T.B.: Die Bereichsspezifitat formaler Denkstrukturen-Konsequenzen fur den padagogischen ProzeB. In: K.FREY, M.LAND (eds.): Kognitionspsychologie und naturwissenschaftlicher Unterricht. Bern: Huber 1973, p.249-285

SPALT, D.: Der induktive Diskurs - Lakatos zum Induktionsproblem in der Mathematik. Darmstadt: Technische Hochschule, FB Mathematik, preprint no. 903, Mai 1985

STREECK, j.: Sandwich. Good for you. Zur pragmatischen und konversationellen Analyse von Bewertungen im institutionellen Diskurs der Schule. In: J.DITTMANN (ed.): Arbeiten zur Konversationasanalyse. Tubingen: Niemeyer 1979, p. 235-257

TERHART, E.: Unterrichtsmethode als Problem. Weinheim: Beltz 1983

Van HIELE, P.M.: Development and Learning Process. Groningen: J.B. Wolters 1959, Acta Paedagogica Ultrajectina XVII.

VOIGT, J.: Routinen und Interaktionsmuster im Mathematikunterricht - Theoretische Grundlagen und Mikroethnographische Falluntersuchungen. Weinheim: Beltz 1984

VOIGT, J.: Der kurztaktige fragend-entwickelnde Mathematikunterricht. In: Mathematica Didactica 7., 1984, Heft 3/4, p.161-186

VOIGT, J.: Pattern and routines in classroom interaction. In: Recherche en Didactique des Mathematiques, vol.VI, 1985, p.69-118, no.1

WATZLAWIK, P. (ed.): Die erfundene Wirklichkeit. Munchen: Piper 1981

WALSH, W. (ed.): Mathematische Aufgaben fur die Klassen 6-10. Berlin (DDR): Volk und Wissen 1983

WEINERT, F.E./KLUWE, R.H. (eds.): Metakognition, Motivation und Lernen. Stuttgart: Kohlhammer 1984

WERTSCH, J.V.: Adult-children interaction and the roots of metacognition. In: Quarterly Newsletter of the Institute for Comparative Human Development 1., 1978, p.15-18 (The following issues of the LCHC-newsletter of UCSD have published many more contributions sharing the same fundamental paradigm, through these do not specialize on mathematics).

WERTSCH, J.V./MINICK, N./ARNS, F.J.: The creation of context in joint problem solving. In: ROGOFF/LAVE 1984, p.151-171

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LECTURE 2

ON THE RELATION BETWEEN THE APPLICATION OF MATHEMATICS AND THE TEACHING OF MATHEMATICS

BY HENRY O. POLLAK
BELL COMMUNICATIONS
RESEARCH INC.

ON THE RELATIONSHIP BETWEEN APPLICATIONS OF MATHEMATICS AND THE TEACHING OF MATHEMATICS *

INTRODUCTION

Most mathematics educators believe in the importance of applications, but it is nevertheless very difficult to get applications into the curriculum. Why? One possible reason appears to be that there is no agreement on what is meant by applied mathematics. In the following we shall explore four different definitions, and their consequences both for the mathematics subject matter and for pedagogy.

1 THE DEFINITION OF APPLIED MATHEMATICS AND ITS VISUALIZATION

In discussions of applied mathematics, a large amount of unnecessary difficulty is sometimes created by differences in perception of the appropriate definition. These differences have come about quite naturally in recent years, since the variety of mathematics which has significant practical applications, the number of fields to which mathematics is applied, and the modes of applications have all undergone very rapid change. It is useful to think in terms of *four* different definitions.

* Dr. Pollak's lecture followed closely parts of the text of his paper "The interaction between mathematics and other school subjects", Volume 4, UNESCO. The appropriate parts of the text are reprinted here by permission of the author and UNESCO.

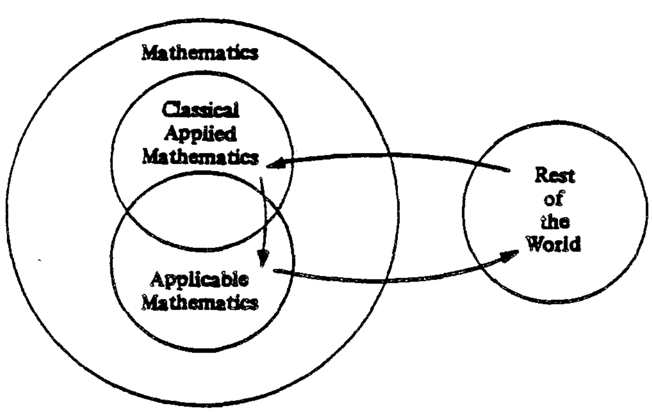
(1) *Applied mathematics means classical applied mathematics;* that is, the classical branches of analysis, including calculus, ordinary and partial differential equations, integral equations, the theory of functions as well as a number of related areas. It is sometimes convenient to annex those aspects of secondary mathematics which are essential prerequisites to calculus, in particular algebra, trigonometry and various versions of geometry. The fact that these branches of mathematics are the ones most applicable to classical physics is usually understood as part of this definition, but no actual connection with physical problems is implied.

(2) *Applied mathematics means all mathematics that has significant practical application.* This greatly enlarges the collection of mathematical disciplines included under (1). All the topics that have been considered world-wide for inclusion in the elementary and secondary school have significant practical applications – including sets and logic, functions, inequalities, linear algebra, modern algebra, probability, statistics and computing. Almost all the mathematics taught at the tertiary level (the undergraduate level at many universities) as well as much graduate mathematics are also included. In the views of many people, the most important areas of mathematics that are included in (2) but not in (1) are statistics, probability, linear algebra and computer science. There are many who feel that these topics are as important as classical analysis. Fields of potential applicability include more than physics, but, once again, only the mathematics itself is being considered.

(3) *Applied mathematics means beginning with a situation in some other field or in real life,* making a mathematical interpretation or model, doing mathematical work within that model, and applying the results to the original situation. Note that the other field is by no means restricted to lie in the physical sciences. In particular, applications in the biological sciences, the social sciences, and the management sciences have become extremely active. Many other areas of applications will also be considered.

(4) *Applied mathematics means what people who apply mathematics in their livelihood actually do.* This is like (3) but usually involves going around the loop between the rest of the world and the mathematics many times. An excellent example of the process involved in this definition of applied mathematics may be found in a report of the workings of the Oxford Seminar in the United Kingdom (Oxford, 1972).

A convenient aid in visualizing these four definitions is seen below:



In this picture the left-hand side shows mathematics as a whole, which contains two intersecting subsets we have called classical applied mathematics and applicable mathematics. Classical applied mathematics represents definition (1) and applicable mathematics, definition (2). Why doesn't (2) contain *all* of (1)? The overlap between these is great, but it is not true that all of classical applied mathematics is currently applicable mathematics. There is much work in the theory of ordinary and partial differential equations, for example, which is of great theoretical interest but has no applications which are visible at the moment. Such work is included in definition (1) as classical applied mathematics, since this contains all work in differential equations; on the other hand, if it is not currently applicable, it does not belong in definition (2).

The rest of the world includes all other disciplines of human endeavour as well as everyday life. An effort beginning in the rest of the world, going into mathematics and coming back again to the outside discipline belongs in definition (3). Definition (4) involves, as will be seen, going around the loop many times.

Other categorizations of applied mathematics have also been considered and can be examined in terms of the diagram. Typically, they involve a more detailed study of the process within mathematics itself than we shall undertake here. For example, applications of mathematics may consist of routine uses of mathematics, of mathematical reasoning as opposed to direct calculation, and of the building of models of various sizes going from small models through full mathematization of real situations to truly large-scale theories. Another very interesting way of slicing the pie may be found in Felix Browder (1976) "The relevance of mathematics". His categories consist of: (a) practical mathematics, that is mathematical practice in the common life of mankind in civilized societies; (b) technical mathematics, that is the use of mathematical techniques and concepts to formulate and solve problems in other intellectual disciplines; (c) mathematical research, that is the investigation of concepts, methods and problems of various mathematical disciplines including applied ones; and (d) mathematics as a universal pattern of knowledge, which means the science of significant form. His essay is highly recommended.

2 A DETAILED STUDY OF THE VARIOUS DEFINITIONS

2.1 The mathematics side of the diagram

The mathematical content of classical applied mathematics (definition (1)) and of applicable mathematics (definition (2)) have already been discussed. One recent trend has been the publication of books and articles showing the applicability of many of the mathematical disciplines which are not included in definition (1). To name just a few examples, Hans Freudenthal (1973) as well as M. Glaymann and Tamas Varga (1973) have written recent books on the applicability of probability; Tanur, Mosteller, Kruskal, Link, Pieters and Rising (1972) have edited a volume showing the great diversity of applications of statistics; R. H. Atkin (1974) in his book has included applications of topology, and Fred Roberts (1976) has devoted much space to applications of graphs and Markov chains. Journal articles are even more numerous; a few samples of particular interest follow – without the slightest pretence of coverage. Thus F. W. Sinden (1966) and Uwe Beck (1974) have shown some applications of topology; M. Dumont (1973) has discussed some uses of Boolean functions and J. H. Durran (1973) some applications of Markov chains. Recent applications of combinatorics and graph theory are examined, for example, by John Niman (1975), J. N. Kapur (1970) and W. F. Lunnon (1969).

A significant feature of applications of mathematics is that mathematical concepts and structures have important usefulness, not just mathematical technique. An interesting discussion of this point is given by H. G. Flegg (1974). Furthermore, since the relationship between mathematics and its applications is forever changing, there is a dynamic effect on mathematics itself. It has happened many times that areas of mathematics which were originally considered quite pure, and were developed with no thought of applications whatever, have turned out to be significantly useful. On the other hand, areas of mathematics which were invented only for application, with no thought of their possible contribution to core mathematics, have turned out to have an impact on pure mathematical disciplines. As an example of the former, the theory of entire functions has given notable insights in electrical communications; ideas of information theory, on the other hand, have been useful in such diverse fields as measure-preserving transformations and the theory of finite groups.

2.2 The rest of the world

Perhaps the outstanding feature of applications of mathematics in recent years is that the areas to which mathematics is applied have been increasing in number so rapidly. It is fair to say that no area of human endeavour is currently immune from quantitative reasoning or mathematical modelling. Besides the traditional physical sciences and engineering, the biological sciences, the social sciences, the management sciences, the humanities and everyday life are all arenas for interaction with mathematics. This is not meant to imply that mathematics is taking over all these other fields, but there *are* many interesting applications.

Perhaps the most extensive literature in recent years on applied mathematics from the point of view of the other disciplines has come in the biological sciences. An excellent overall survey appears in the book by J. Maynard Smith (1968). Books dealing with specific areas within the biological sciences include Victor Twersky (1967) on growth, decay and competition and R. M. May (1973) on the stability of ecosystems. Among the articles too numerous to summarize we note S. Karlin (1972a,b, the former jointly with M. Feldman) on genetics, S. P. Hastings (1975) on neurobiology, Arthur Engel (1971, 1975) and Beck (1975) on population models, W. D. Hamilton (1971) on the geometry of group behaviour, and several articles in "Computers in Higher Education" (1974) on the use of computers in biology. Not that new books and articles on mathematics in science have been lacking: We note particularly a little known volume by George Polya *Mathematical Methods in Science* (1963) as well as another portion of Victor Twersky (1967). Recent articles on mathematics in science include J. B. Griffiths (1976) on model building and mechanics, the conference report on "Modern Mathematics and the Teaching of Science" (1975), and the previously mentioned computer survey "Computers in Higher Education" (1974).

Another field which has recently flourished is the interaction of mathematics with the social sciences. Information on computers and statistics in the social sciences generally may be found in (Computers . . . , 1974) and (Teaching of statistics . . . , 1973); a fascinating and somewhat different viewpoint is represented in the article by H. R. Alker, Jr., "Computer simulations: Inelegant mathematics and worse social science?" (1974). The *Source book on Applications of Undergraduate Mathematics to the Social Sciences* (1977) contains descriptions of detailed mathematizations in many fields of the social sciences. To go on with specific fields, economics is extremely active for interactions with mathematics, although good expositions of the problems of model building in economics are not common. One nice example is "On the theory of interest" by David Gale (1973). Mathematical work in geography has also been quite popular in recent years, particularly in the United Kingdom. Again there are significant contributions in (Computers . . . , 1974) and (Source Book . . . , 1977), and an elementary treatment of weather forecasting in Durran (1973); see also King (1970). Mathematical psychology is represented by two recent survey articles by Anatol Rapoport (1976); *Source Book . . .* (1977) also contains extensive references to recent work. Besides their appearance in overall summaries, anthropology is represented by example in the book by L. Pospisil (1963) and the traditional mathematical theory of warfare by Arthur Engel in (1971). A magnificent example of mathematics applied to political science may be found in M. L. Balinski and H. P. Young (1975) "The quota method of apportionment". Mayer (1971) and Coxon (1970), for example, represent mathematical sociology.

The very large field of mathematical models in the management sciences including the entire area of operations research hardly needs description here. Sample articles of particular interest in recent years include those by F. J. Fay (1972), J. C. Herz (1973) and the delightful piece on mathematics applied to college presidency by J. G. Kemeny (1973). Mathematical models in medicine has been an increasingly active field; there is an excellent survey by J. S. Rustagi "Mathematical models in medicine" (1971). Mathematical linguistics has similarly become a major accepted field. Interesting particular articles appear, for example, as parts of Engel (1971) and *Source Book . . .* (1977), with Sankoff (1973) as another good source.

The penetration of mathematics into the humanities, including statistical and computer models, is a fairly recent event. Perhaps furthest advanced are mathematical analyses of art. We note, for example, A. V. Subnikov and V. A. Koptsik (1974) and a very valuable British summary of mathematical ideas and concepts in art by Beryl Fletcher (1976a). Mathematics applied to architecture is discussed by R. Fischler (1976) as well as in the summary work "Computers in Higher Education" (1974). Some examples of mathematical ideas in hobbies and handicrafts are given in Beryl Fletcher (1976b). Mathematical strategies for certain games such as NIM and the towers of Hanoi have long been familiar to, and enjoyed by, mathematicians. In recent years, there has been a great upswing in the discovery of optimal strategies for much more intricate games, and this has even provided one of the early applications of ideas from nonstandard analysis. We particularly note the work of E. R. Berlekamp and J. H. Conway, partly reported in Conway (1976). A nice example of optimal strategy for poker is given by W. H. Cutler (1975). Cryptanalysis has often been treated – see e.g. Sinkov (1968); for mathematics in sports see Klein (1972).

Besides the above-mentioned books and articles more or less devoted to specific areas of applications, there has been a trend in recent years towards the publication of excellent collections of articles and symposium reports which cover a broader spectrum. One of the earliest but still of great interest is the Utrecht colloquium "How to Teach Mathematics so as to be Useful" (Freudenthal, 1968). This was followed by the Echternach symposium "New Aspects of Mathematical Applications to School Level" (Echternach, 1973) and the Lyon seminar "Goals and Means Regarding Applied Mathematics in School Teaching" (Goals and Means . . . , 1974). Other noteworthy volumes of this kind include *Notes of Lectures on Mathematics in the Behavorial Sciences* edited by H. A. Selby (1973), *Topics in Behavorial Mathematics* by T. L. Saaty (1973), *A Source Book for Teachers and Students on Some Uses of Mathematics*, Max Bell (1967), *A Conference on the Applications of Undergraduate Mathematics . . .* (Knopp and Meyer, 1973) and *La Mathématique et ses Applications* by E. Galion (1972).

The preceding list well illustrates the current diversity of applications of mathematics in contrast with the historical monolith of applications to physics. It should not be assumed, however, that the arguments between those who stress the great variety of applications in recent years and those who feel that their total impact cannot compare to the 2000-year accumulation of success in mathematical physics have died down. In fact, this difference of instinctive value judgement underlies many of the arguments about mathematics education to which we will return later.

2.3 The model building process

When mathematics is actually applied to a situation in some other field, there are typically a number of distinguishable steps in the process. These consist of a recognition that a situation needs understanding, an attempt to formulate the situation in precise mathematical terms, mathematical work on the derived model, (frequently) numerical work to gain further insight into the results, and an evaluation of what has been learned in terms of the original external situation. This picture of the model building process has been widely accepted and there are many papers which elucidate the details from various points of view. Overall descriptions appear, for example, in the papers by M. S. Klamkin (1971), H. O. Pollak (1970) and P. L. Bhatnagar (1974). The same pattern, but applied specifically to operations research, appears in the paper by Gordon Raisbeck (1975) "Mathematicians in the practice of operations research"; its application to engineering may be found in A. C. Bajpai, L. R. Mustoe and D. Walker (1975), and again in the paper by H. G. Flegg (1974). M. E. Rayner (1973) in her paper "Mathematical applications in science" in the Echternach report describes in detail some of the difficulties in problem formulation. A quotation she gives from Eddington is particularly worth repeating, "The initial formulation of the problem is the most difficult part, as it is necessary to use one's brains all the time; afterwards, you can use mathematics instead". A proposal for better model building in mechanics is also given by J. B. Griffiths (1976). See also Wilder (1973).

The model building process has a number of interesting properties as well as pitfalls which we shall examine. A good model is one which is to some extent successful in explaining, or even predicting, external reality. If it fails to have this explanatory power then, no matter how satisfactory the mathematics itself, the model is not good applied mathematics and must be changed. This process can be quite painful for the mathematician but real progress in interdisciplinary efforts is often made through successive changes in the model. This is one of the reasons why definition (4) of applied mathematics involves going around the loop many times. Another phenomenon which sometimes happens is that a mathematical model predicts too much rather than too little. It may happen that phenomena observed in the other field are indeed explained satisfactorily, but that further logical implications of the model are not acceptable. For example, in the mathematics of communication a model of a signal which is of finite duration in time is very realistic. Similarly, a model of a communication signal using finite bandwidth comes up in many situations and gives quite satisfactory engineering results. Unfortunately, the two are contradictory and cannot be used at the same time in the same problem; models which do so unwittingly will lead to nonsense. On the other hand, attempts to understand this difficult situation fully have led to very interesting advances, see e.g. D. Slepian (1976).

Another feature of the model building process is that the purposes for which a mathematical model is created are also quite varied. In the physical sciences and engineering, the purpose is frequently very precise understanding which will in turn lead to action. In the social sciences, on the other hand, the purpose is often one of insight; you want to know whether a certain set of hypotheses could account for a particular observed phenomenon. It is often assumed, although not necessarily true, that these associations are in fact one-to-one correspondences. Physical models of why rivers meander, or why a rapidly slurped pice of spaghetti comes up and hits your nose, are not necessarily used for scientific decisions. On the other hand, mathematical models of shortest connecting networks and optimal pricing are often used for management action.

The overall picture of applications of mathematics would not be complete without a discussion of truly interdisciplinary activity. Much of the most exciting current work is in fact on the borderline between several fields, one of which being in the mathematical sciences. The above references will lead the reader to many examples of current interdisciplinary work.

3 EFFECTS OF APPLIED MATHEMATICS ON MATHEMATICS EDUCATION

3.1 Problems and problem solving in the schools

A framework for understanding the meaning of applied mathematics has now been established, and a number of ramifications of the various definitions have been examined. A look at effects of applied mathematics on education follows. It must be emphasized that many of the topics in this chapter represent ideas and experiments in various countries which cannot claim to be adopted on any large scale. Discussions at the Karlsruhe Congress did not bring forth any data which would substantiate broad use of applied mathematics in the schools.

Traditionally most of what was considered applied mathematics in the schools has been found under headings such as "word problems", "problem solving", etc. (This does not mean the "word problem" in the sense of modern algebra.) The meaning of such problem solving has been examined in a number of projects and articles in recent years. For example, the work of IOWO in the Netherlands is of particular importance. IOWO has also paid special attention to the differences in abstraction and precision between mathematical language and everyday language. The detailed meaning of problem solving is examined in papers by H. G. Flegg (1974), Beryl Fletcher (1976c) and H. O. Pollak (1969). Genuine applications of mathematics to other fields and to everyday life should ideally be in the character of definitions (3) and (4). It is often argued that a full presentation in the spirit of even definition (3) represents too large a project and takes too much time. In that case, the actual situation and numbers used in the word problem should at least be genuine extractions from an honest problem formulation. For example, estimates of crop yields and of times to complete a task should not be made to five significant figures, for this will never happen in real life. Too many plumbers in one room get in each other's way, and jobs are not always divisible. A current joint project of the National Council of Teachers of Mathematics and the Mathematical Association of America in the United States is producing a *Source Book* (1978) of hundreds of simple problems which are intended to be genuine in the above sense.

The opposite phenomenon is that the facts alleged in the statement of a problem are sometimes totally unreal. Problems which use wrong linguistics or impossible engineering or incorrect meteorology just to have some words from another discipline should be avoided. In this case, intent can nevertheless be important. Sometimes problems are clothed in a mantle of external vocabulary only for amusement, and the pretended application is not meant to be taken seriously. We shall call such problems whimsical problems. A strong argument in favor of such problems is made for example by Arthur Engel (1969) "Some examples are artificial, like fables. But just like fables, they have a moral, i.e., they facilitate insights into things that appear in the real world". For example, it can be quite effective to begin with an unsatisfactory oversimplification of a real situation, and to approach a genuine application in the sense of definition (4) through a series of increasingly realistic problems. Thus whimsical and unreal problems are not necessarily devoid of pedagogic value. However, if they are perceived as stupid, they may well be counter-productive. Similar discussions of real and unreal problems may be found in two particularly interesting papers by Margaret Brown (1972, 1973) and Mary Williams (1971). In particular, Mary Williams points out that the same difficulty of unreal models happens at a very advanced level as well as at the school level. See also section 1.1.5 of Chapter IV.

The increased awareness in many countries of the importance of teaching the applicability of mathematics has led to a number of very interesting attempts to collect real problems at various levels, and from various disciplines, and to make them available for teaching purposes. One collection at the school level (Source Book ... Secondary School, 1978) has already been mentioned. Other general collections have been made by Max Bell (1972), Ben Noble (1967),

D. A. Quadling (1975), and C. W. Sloyer (1974). Collections devoted to particular disciplines, mainly at the university level, include the series on statistics by example (Mosteller et al., 1973), the social sciences problem book (Source Book . . . Social Sciences, 1977) and the collection of mathematical models in biology (Thrall et al., 1967), although the realism of problems in the latter collection varies. Another text in the same-spirit, although it is organized as an actual course in engineering concepts, is *The Man Made World* (1971). It can be expected that very interesting collections of real problems in the above spirit will also be appearing in China. One such example of which we are aware contains, among other things, a number of excellent geometric problems from industry and agriculture (Applications . . . , 1975).

3.2 Mathematical subject matter in the schools

The diversity of applicable mathematics (definition (2)) which has emerged in recent years has greatly complicated the task of designing curricula for elementary and secondary schools. The traditional goals of preparing students for either shopkeeping or calculus (associated with definition (1)) cease to be uniquely valid when so many more areas in the mathematical sciences are of undeniable importance to so many of the world's people. As the number of reasonable choices increases, so does the difficulty of designing a curriculum. It has been argued by many that, for example, probability, statistics and computer science are as important for applications as the calculus. School materials for applications to many different disciplines have become available in recent years. Collections of materials involving applications to many different fields may be found, for example, in *Crossing Subject Boundaries* (Schools Council, 1970) and the materials from the Minnesota School Mathematics Center (Rosenbloom, 1963). The Chelsea Centre for Science Education project, "Science Uses Mathematics" (Chelsea) contains interesting applications to science which can be used in an interdisciplinary way, although this is not always done. *Applied Mathematics in the High School* by Max Schiffer (1963) also gives excellent examples of the relationship of mathematics and scientific applications from the point of view of the schools. A collection of examples which turn the tables and use physics to do mathematics has been made by Uspenskii (1961).

A major work examining curricular goals and pedagogy in the framework of an application to economics may be found in Damerow, Elwitz, Keitel, and Zimmer (1974). Biological applications may be found in Gibbons and Blofield (1971), and applications to geography in the materials by IOWO, in *New Ways in Geography* by J. P. Cole and N. J. Beynon (1968) and also in B. Fletcher (1976c). Applications to geography are also featured in the *Travaux d'Orléans (Les Mathématiques dans l'Enseignement* . . . , 1975), which in fact contains many other fascinating applications to a variety of fields throughout the curriculum, including economics, technology and medicine. This work also features references to recent work on applications in France and interesting philosophy on the usefulness of mathematics. An interesting application to political science may be found in Steiner (1966); environmental applications occur in the work of IOWO and in the book by Fred Roberts (1976). As we look at applications organized from the mathematical point of view, a superb collection of applications of linear algebra may be found in T. J. Fletcher (1972), and of statistics and probability in the work of Arthur Engel, e.g. (1970, 1973) and in *The Teaching of Probability and Statistics* edited by Råde (1970). *Mathematics Applicable* by the Schools Council (1975) also motivates much secondary mathematics through examples; the volume entitled *Logarithmic/Exponential* is a particularly interesting sample.

This great diversity of possible applications of mathematics, and of elementary branches of mathematics with significant applicability, has made the curriculum design problem very difficult. For example, topic A deserves to precede topic B in the curriculum if topic A is socially more important at this particular time, or if topic A is a prerequisite to topic B at this particular time.

As technology and social goals change, so should the ordering of importance. As available tools for teaching change, so will the order of prerequisites. These orderings will differ also from country to country. These facts make it even more difficult than it has been in the past to export curricula from one part of the world to another. Since an imported curriculum incorporates problems, situations and values which make no sense in a new country, this was probably never desirable, but is is even more questionable now.

3.3 The possible effect of applications on pedagogy

An appreciation for the different forms of applications of mathematics should affect not only the curriculum materials of the schools but also the pedagogy. If you examine even relatively simple uses of mathematics, you find that it is necessary to understand when and how and why the mathematics works in order to apply it correctly. There are several reasons for this. One is that mathematics which has been understood will be remembered better. Another more fundamental reason is the danger that mathematics which has been memorized without understanding will be misapplied. It is necessary to know where a particular method or formula comes from, exactly what kind of problem it will handle, and when and how it works in order to be sure that it will apply to a new situation. Curriculum reform in many countries has emphasized the "why" of mathematics in recent years on the grounds that it is essential for proper teaching of mathematics. What we see is that "why" is just as important for interactions of mathematics with other disciplines as it is for mathematics itself. The natural desire of mathematics teachers to emphasize understanding as well as technique is reinforced, not contradicted, by applications.

The model building process as developed through definitions (3) and (4) of applied mathematics interacts with mathematical pedagogy in a still deeper sense. Model building requires an understanding of the situation outside mathematics and of the process of mathematization as well as of the mathematics itself. You cannot hope to mathematize a situation without understanding it. Here we have yet another way in which "applied" problems which do nothing more than mouth words from another discipline are likely to mislead the student. A great weakness of some courses with titles like "Methods of Applied Mathematics" is that no attempt is made to provide an opportunity for the student to understand the situation and the mathematization process. This point has been particularly emphasized by H. G. Flegg (1974) and is further substantiated, especially from the point of view of future employment, in R. R. McLone (1973). Some of the college-level collections of real problems mentioned previously, for example Noble (1967) and *Source Book ... Social Sciences* (1977), take particular pains towards the understanding of the situation in the real world.

Another pedagogic implication of the interaction between mathematics and other disciplines as it is described in definition (4) is that such interactions are clearly open-ended. Open-ended teaching of mathematics itself has long been recommended by mathematics educators in many countries, although adoption is rare. What does "open-ended" mean in this context? Besides the usual activities of solving problems and proving theorems, students should have the experience of finding their own problems to solve and their own theorems to prove. Such experience is an important factor in the mathematical development of the student. But exactly the same argument holds in the context of applications. It is very valuable for the student to have open-ended modelling experience, which besides its great pedagogic value is an accurate foretaste of mathematical applications in the real world. Experiments in open-ended discovery teaching of mathematical applications, many in the form of truly interdisciplinary materials, are under way in surprisingly many countries. An outstanding example is certainly China, where a major practical problem will be used for reference and inspiration throughout a course in calculus or linear algebra. There are many other examples of open-ended and truly interdisciplinary activities at the tertiary level, represented, for example, by the *Case Studies in Applied Mathematics* (1976), the books by T. J. Fletcher (1972), Maki and Thompson (1973) and Roberts (1976). At the elementary level, an outstanding example is provided by the USMES project in the United States (Lomon et al., 1975) in which students attack a series of action-oriented challenges by appropriate combinations of mathematics, science and social science. Truly open-ended applications are particularly difficult to introduce at the secondary level, and corresponding materials are very scarce.

3.4 Applications and teacher training

As mathematics teaching changes in the light of the increasing applicability of the subject, so should teacher training. Teachers should become familiar with the new fields of applicable mathematics, with the process of model building, and with the associated pedagogic emphases on understanding and open-endedness. There is a general tendency world-wide to reverse certain recent trends and to include more experiences involving applications in the training of prospective teachers. Perhaps the most exciting development in this direction is the pattern pioneered in the United Kingdom and now also spreading, for example, to Australia (Fensham and Davison, 1972), i.e. to make an internship in industry part of the training of a mathematics teacher. In this way, it is possible for the teacher to learn something of how the mathematical sciences are really applied. Practising teachers also sometimes help with the preparation of new interdisciplinary, open-ended materials (see e.g. Case Studies . . . , 1976). Especially in those countries in which there is currently an ample supply of teachers, those prepared in the broader mathematical sciences and familiar with applications of mathematics enjoy a stronger position in looking for employment. In other countries, applied mathematics in the sense of definition (1) has always been a strong component in teacher training, but experience with applications in the sense of definitions (3) and (4) has been missing. Once again, major industrial or agricultural experience has become part of teacher training in China.

3.5 Vocational education

A further educational effect of applications of mathematics is in vocational education. As the importance of the mathematical sciences increases for many disciplines, so does the need the workers and technicians in these disciplines to learn the most appropriate mathematical techniques. Noteworthy vocational materials in a variety of technical fields have been developed in a number of countries. For example, of the order of a dozen volumes of applications of mathematics in different technologies (clothing, carpentry, metal work, etc.) have been produced in Hungary. A different development in the same spirit is the increasing popularity of special curricula for technicians in computer science and data analysis. These have become particularly prevalent, for example, in the United States.

3.6 The implications of truly interdisciplinary teaching

Teaching which is truly multidisciplinary is very difficult to achieve at any level, but perhaps nearest to reality in the elementary school, where – in many countries – a single teacher normally handles most if not all subjects. The evidence for this may be found in the many multidisciplinary materials for the elementary school which have been mentioned. Such activities, when actually carried out in the schools, are especially satisfactory for students because they strengthen the relationship between school and real life. Students are not always satisfied with the promise of future gratification inherent in such statements as "you will find out why this is useful later on", and are pleased with the applicability of mathematics to problems in which they are interested. This is particularly stressed, for example, by IOWO and USMES (Education Development Center, 1974, 1975). However, if the time during the school day is apportioned according to disciplines, it is necessary that the time for multidisciplinary activities be contributed by the various disciplines involved. This implies, at a minimum, that multidisciplinary projects must state what responsibility they will take for specific topics in the several disciplines. Appropriate teacher training at the elementary level is very necessary. On the secondary level, the implications for the structure of the educational system are much more severe. If a single unit involves mathematics, science, social science and language arts all in a significant way, who is going to teach the material, who will contribute the time, how should the school be organized? These problems have not been solved, although team teaching is one possibility; see also Rao (1975). They are discussed particularly in section 3.7 of Chapter III and in the *Report of the Memphis Conference* (Education Development Center, 1974). At the university level, multidisciplinary educational activities may take the form, for example, of genuine model building courses discussed previously, or of team teaching by faculty from mathematics and from a field of application of sections in basic courses such as calculus, linear algebra, and statistics. An example of a master's programme with multidisciplinary experience is Hunter College (1974).

REFERENCES

Alker, H.R. Jr., Computer simulations: Inelegant mathematics and worse social science, *Int. J. Math. Educ. Sci. Technol.*, Vol. 5 (1974), 139-155.

Applications of Mathematics in Industry and Agriculture, edited by Education Department of City of Peking, People's Republic of China, 1975.

Atkin, R.H., *Mathematical Structure in Human Affairs*, Heinemann, 1974.

Bajpai, A.C., Mustoe, L.R. and Walker, D., Mathematical education of engineers, Part I, A critical appraisal, *Int. J. Math. Educ. Sci. Technol.*, Vol. 6, No. 3 (August 1975), 361-380.

Bajpai, A.C., Mustoe, L.R. and Walker, D., Mathematical education for engineers, Part II, Towards possible solutions, *Int. J. Math. Educ. Sci. Technol.*, Vol. 7, No. 3 (August 1976), 349-364.

Balinski, M.L. and Young, H.P., The quota method of apportionment, *American Math. Monthly*, Vol. 82, No. 7 (August-September 1975), 701-729.

Beck, U., Anwendung der Eulerschen Polyederformel zur Bestimmung von Summenformeln und Ringstrukturen in Molekulen, manuscript, 1974.

Beck, U., Polulationsdynamik und Mathematikunterricht, Ein Beitrag zur Problemorientierung des Mathematikunterrichts, *Didaktik der Mathematik*, Vol. 3 (1975), No. 3, 194-212.

Bell, M.S., *Mathematical Uses and Models in our Everyday World*, Studies in Mathematics, Vol. XX, School Mathematics Study Group, 1972.

Bell, M.S. (ed), *Some Uses of Mathematics: A Source Book for Teachers and Students of School Mathematics*, Studies in Mathematics, Vol. XVI, School Mathematics Study Group, 1967.

Bhatnagar, P.L., The nature of applied mathematics, *The Math. Teacher, India* (1974), 12-22.

Browder, F.E., The relevance of mathematics, *American Math. Monthly*, Vol. 83, No. 4 (April 1976), 249-254.

Brown, Margaret, 'Real' problems for mathematics teachers, *Int. J. Math. Educ. Sci. Technol.*, Vol. 3, No. 3 (July-September 1972), 223-226.

Brown, Margaret, The use of real problems in teaching the art of applying mathematics, *Echternach Symposium* (June 1973), 65-73.

Case Studies in Applied Mathematics, Committee of the Undergraduate Program in Mathematics (CUPM) / Mathematical Association of America (MAA), CUPM/MAA, 1976.

Cavalli-Sforza, L., Teaching of biometry in secondary schools, *Biometrics*, Vol. 24, No. 3 (September 1968), 736-740.

Chelsea Centre for Science Education, Bridges Place, London, SW6 4HR.

Cohors-Fresenborg, E., Dynamische Labyrinthe, *Didaktik der Mathematik*, Vol. 1 (1976), No. 1, 1-21.

Cole, J.P. and Beynon, N.J., *New Ways in Geography*, Basil Blackwell, 1968.

Comprehensive School Mathematics Program, *Topics in Probability and Statistics,* Chapter 12 of Elements of Mathematics, Intuitive Background, Central Mid-western Regional Education Laboratory, Inc., 1971, 2-21 (1976).

Computers in Higher Education, *Int. J. Math. Educ. Sci. Technol.*, Vol. 5 (1974), Nos. 3 and 4 (Special issues on the Proceedings of a Conference on Computers in Higher Education, held under the auspices of NCC/CAMET, 1974).

Conway, J.H., *On Numbers and Games,* Academic Press, 1976.

Coxon, Anthony P.M., Mathematical applications in sociology: measurement and relations, *Int. J. Math. Educ. Sci. Technol.*, Vol. 1 (1970), No. 2, 159-174.

Cutler, W.H., An optimal strategy for pot-limit poker, *American Math. Monthly,* Vol. 82, No. 4 (April 1975), 368-376.

Damerow, P., Elwitz, U., Keitel, C. and Zimmer, J., *Elementarmathematik: Lernen für die Praxis?* Klett, Stuttgart, 1974.

Dumont, M., A propos des fonctions Booléennes, *Echternach Symposium* (June 1973), 227-243.

Durran, J.H., Markov Chains, *Echternach Symposium* (June 1973), 149-160.

Echternach Symposium, Les Applications Nouvelles des Mathématiques et l'Enseignement Secondaire (New Aspects of Mathematical Applications at School Levels), Proceedings of a Symposium held in Echternach June 1973, Imprimerie Victor, S.A., Gr. D. Luxembourg, 1975.

Education Development Center, Inc., *Report of the Memphis Conference on Formation of a National Consortium for Development of Real Problem Solving Curriculum for Secondary Schools,* Newton, Mass., 1974.

Education Development Center, Inc., *Comprehensive Problem Solving in Secondary Schools: A Conference Report,* Newton, Mass., 1975.

Elias, P., The noisy channel coding theorem for erasure channels, *American Math. Monthly,* Vol. 82, No. 8 (October 1974), 853-862.

Engel, A., The relevance of modern fields of applied mathematics for mathematical education, *Educational Studies in Math.*, Vol. 2, No. 2/3 (December 1969), 257-269.

Engel, A., *A Short Course in Probability,* Central Mid-western Regional Educational Laboratory, Inc., 1970.

Engel, A., Mathematische Modelle der Wirklichkeit, *Der Mathematikunterricht,* Vol. 17 (1971), No. 3.

Engel, A., *Wahrscheinlichkeitsrechnung und Statistik,* Klett Studienbücher Mathematik, 1973.

Engel, A., Computerorientierte Mathematik, *Der Mathematikunterricht,* Vol. 21 (1975), No. 2.

Fairley, W. and Mosteller, F., Trial of an adversary hearing: public policy in weather modification, *Int. J. Math. Educ. Sci. Technol.*, Vol. 3 (1972), No. 4, 375-383.

Fay, F.J., Extremwertprobleme im Wirtschaftsbereich und die Interpretation des Prinzips von Fermat als wirtschaftliches Prinzip, *Der Mathematikunterricht,* Vol. 18, No. 5 (December 1972), 76-94.

Fensham, P.J. and Davison, D.M., Student teachers discover mathematics in industry, *Int. J. Math. Educ. Sci. Technol.*, Vol. 3 (1972), No. 1, 63-69.

Fischler, R., A mathematics course for architecture students, *Int. J. Math. Educ. Sci. Technol.*, Vol. 7, No. 2 (May 1976), 221-232.

Flegg, H. Graham, The mathematical education of scientists and technologists – a personal view, *Int. J. Math. Educ. Sci. Technol.*, Vol. 5, No. 1 (January-March 1974), 65-74.

Fletcher, Beryl, *Illustrations of mathematical ideas and concepts in art,* 1976a.

Fletcher, Beryl, *Illustrations of mathematical ideas and concepts in handicraft, 1976b.*

Fletcher, Beryl, *A Report on Aspects of Interaction between Mathematics and Geography,* March 1976c.

Fletcher, T.J., *Linear Algebra through its Applications,* Van Nostrand Reinhold Company, 1972.

Floyd, F., *Mathematics for the Majority,* Institute of Education, University of Exeter, 1967-1972.

Freudenthal, H., Der Wahrscheinlichkeitsbegriff als Angewandte Mathematik, *Echternach Symposium* (June 1973), 15-27.

Freudenthal, H. (ed), Proceedings of the Colloquium How to Teach Mathematics so as to be Useful, Utrecht, August 21-25, 1967, *Educational Studies in Math.*, Vol. 1, No. 1/2 (Special issue, May 1968).

Gale, David, On the theory of interest, *American Math. Monthly*, Vol. 80, No. 8 (October 1973), 853-867.

Galion, E., *La Mathématique et ses Applications* (Troisième Seminaire International, Valloire 8-18 Juillet 1972), CEDIC, 1972.

Gibbons, R.F. and Blofield, B.A., *Life Size: A Mathematical Approach to Biology*, MacMillan, 1971.

Glayman, M. and Varga, T., *Les Probabilités à l'Ecole*, CEDIC, 1973.

Goals and Means regarding Applied Mathematics in School Teaching, Unesco Seminar, Lyon, France, February 4-8, 1974.

Griffiths, J.B., Towards the liberation of mechanics from bondage to Newton's laws, *Int. J. Math. Educ. Sci. Technol.*, Vol. 7, No. 1 (February 1976), 79-85.

Hamilton, W.D., Geometry for the selfish herd, *J. Theor. Biol.*, Vol. 31 (1971), 295-311.

Hastings, S.P., Some mathematical problems for neurobiology, *American Math. Monthly*, Vol. 82, No. 9 (November 1975), 881-894.

Herz, J.C., Exemples de mathématisation, *Echternach Symposium* (June 1973), 75-81.

Hunter College of the City of New York, *Master's Programs in Applied Mathematics*, 1974.

IOWO – Instituut Ontwikkeling Wiskunde Onderwijs, Tiberdreef 4, Utrecht/Overvecht, The Netherlands. (Information about IOWO, e.g. in *Educational Studies in Mathematics*, Vol. 7 (1976), No. 3.)

Kapur, J.N., Combinatorial analysis and school mathematics, *Educational Studies in Math.*, Vol. 3, No. 1 (September 1970), 111-127.

Karlin, S. and Feldman, M. (1972a), Mathematical genetics: a hybrid seed for educators to sow, *Int. J. Math. Educ. Sci. Technol.*, Vol. 3, No. 2 (April-June 1972), 169-189.

Karlin, S. (1972b), Some mathematics models of population genetics, *American Math. Monthly*, Vol. 79, No. 7 (August-September 1972), 699-739.

Kemeny, J.G., What every college president should know about mathematics, *American Math. Monthly*, Vol. 80, No. 8 (October 1973), 889-901.

King, C.A.M., Mathematics in geography, *Int. J. Math. Educ. Sci. Technol.*, Vol. 1, No. 2 (April-June 1970), 185-205.

Klamkin, M.S., On the ideal role of an industrial mathematician and its educational implications, *Educational Studies in Math.*, Vol. 3, No. 2 (April 1971), 244-269, and *American Math. Monthly*, Vol. 78 (1971), 53-76.

Klein, R.J., *Mathematics and Sports*, University of Chicago, August 1972, an English Translation of the German Work by Ernst Lampe, Teubner, 1929.

Knopp, P.J. and Meyer, G.H. (eds), *Proceedings of a Conference on the Application of Undergraduate Mathematics in the Engineering, Life, Managerial, and Social Sciences*, Georgia Institute of Technology, June 1973.

Lomon, E.L., Beck, Betty and Arbetter, Carolyn C., Real Problem solving in USMES: interdisciplinary education and much more, *School Science and Mathematics* (January 1975), 53-64.

Lunnon, W.F., A postage stamp problem, *Comput. J.*, Vol. 12 (1969), 377-380.

Maki, D.P. and Thompson, M., *Mathematical Models and Applications*, Prentice-Hall, Inc., 1973.

Maki, D.P. and Thompson, M., *Mathematical Models in the Undergraduate Curriculum* (Proceedings of a Conference held at Indiana University, October 30 – November 1, 1975), Indiana University, 1975.

Man Made World, The, Engineering Concepts Curriculum Project, Polytechnic Institute of Brooklyn, McGraw-Hill Book Company, 1971.

Mathematics and School Chemistry: Interim Report of a Working Party of the British Committee on Chemical Education, The Institute of Mathematics and its Applications, March 1974.

Les Mathématiques dans l'Enseignement Scientifique et Technologique, Travaux d'Orléans, *Bulletin de l'Association des Professeurs de Mathématiques*, September 1975.

May, R.M., *Stability and Complexity in Model Ecosystems*, Princeton University Press, 1973.

Mayer, T.F., Mathematical sociology: some educational and organisational problems of an emergent sub-discipline, *Int. J. Math. Educ. Sci. Technol.*, Vol. 2, No. 3 (July-September 1971), 217-232.

McLone, R.R., *The Training of Mathematicians*, Social Science Research Council, London, 1973.

Modern Mathematics and the Teaching of Science, A Conference Report, Mathematical Association in conjunction with the University of Leicester School of Education, *Int. J. Math. Educ. Sci. Technol.*, Vol. 6, No. 1 (February 1975), 89-119.

Mosteller, F., Kruskal, W., Link, R.F., Pieters, R.S. and Rising, G.R., (eds), *Statistics by Example*, Addison-Wesley Publishing Company, 1973.

Münzinger, W., *Projektorientierter Mathematikunterricht*, Project des Hessischen Kultusministers, Frankfurt, 1976.

Niman, J., Graph theory in the elementary school, *Educational Studies in Math.*, Vol. 6, No. 3 (November 1975), 351-373.

Noble, B., *Applications of Undergraduate Mathematics in Engineering*, The Mathematical Association of America, 1967.

Oxford Study Groups with Industry, 1968-1971, *Progress Report on Applications of Differential Equations*, Mathematical Institute, Oxford, 1972.

Pollak, H.O., How can we teach applications of mathematics? *Educational Studies in Maths.*, Vol. 2, No. 2/3 (December 1969), 393-404.

Pollak, H.O., Applications of mathematics, Chapter 8 in Begle, E.G. (ed), *69th Yearbook of the National Society for the Study of Education*, University of Chicago, 1970.

Polya, G., *Mathematical Methods in Science*, Studies in mathematics, Vol. XI, School Mathematics Study Group, 1963.

Pospisil, L., *Kapauku Papuan Economy*, 1963.

Quadling, D.A., *Contexts for applications of mathematics in education*, 1975.

Rade, L. (ed), *The Teaching of Probability and Statistics* (Proceedings of the first CSMP International Conference, Carbondale, Illinois, March 18-27, 1969), Wiley Interscience Div., 1970.

Raisbeck, G., *Mathematicians in the practice of operations research* (presented at the Short Course on Mathematics and Operations Research of the American Mathematical Society, 21 January, 1975).

Rao, C.R., Teaching of statistics at the secondary level, an interdisciplinary approach, *Int. J. Math. Educ. Sci. Technol.*, Vol. 6, No. 2 (May 1975), 151-162.

Rapoport, A., Directions in mathematical psychology, Parts I and II, *American Math. Monthly*, Vol. 83 (1976), No. 2, 85-106, and Vol. 83, No. 3, 153-172.

Rayner, M.E., Mathematical applications in science, *Echternach Symposium* (June 1973), 209-216.

Roberts, F.S., *Discrete Mathematical Models with Applications to Social, Biological, and Environmental Problems*, Prentice-Hall, Inc., 1976.

Rosenbloom, P.C., *Experimental Units of Minnesota School Mathematics Center*, Minnesota School Mathematics Center, University of Minnesota, 1963.

Royal Society – Institute of Biology, *Biological Education Committee Report of the Working Party on Mathematics for Biologists*, London, 1974.

Rustagi, J.S., Mathematical models in medicine, *Int. J. Math. Educ. Sci. Technol.*, Vol. 2, No. 2 (April-June 1971), 193-203.

Saaty, T.L., *Thinking with Models: Mathematical Models in the Physical, Biological and Social Sciences*, 1972.

Saaty, T.L., *Topics in Behavioral Mathematics;* Lectures given at the 1973 MAA Summer Seminar, Williams College, Williamstown, Massachusetts, Mathematical Association of America, 1973.

Sankoff, D., Mathematical developments in lexicostatistic theory; in Sebeok, T.A. (ed), *Current Trends in Linguistics*, Vol. 11, 93-113, The Hague, Mouton, 1973.

Schiffer, M.M., *Applied Mathematics in the High School*, Studies in Mathematics, Vol. X, School Mathematics Study Group, 1963.

Schools Council, *Crossing Subject Boundaries*, Hart-Davis, 1970.

Schools Council, *Mathematics for the Majority Continuation Project*, Schofield and Sims, 1974.

Schools Council Sixth Form Mathematics Project, Reading University, *Mathematics Applicable*, Heinemann Educational Books, 1975.

Selby, H.A. (ed), *Notes of Lectures on Mathematics in the Behavioral Sciences,* The Mathematical Association of America, 1973.

Sinden, F.W., Topology of thin film circuits, *Bell System Techn. J.,* Vol. 45, No. 9 (November 1966), 1639.

Sinkov, A., *Elementary Cryptanalysis, a Mathematical Approach,* Random House, 1968.

Slepian, D., On bandwidth, *Proc. IEEE,* Vol. 64, No. 3 (March 1976), 292-300.

Sloyer, C.W., *Fantastiks of Mathematiks,* 1974.

Smith, J. Maynard, *Mathematical Ideas in Biology,* Cambridge University Press, 1968.

Source Book on Applications of Mathematics for Secondary School, MAA and NCTM, forthcoming, 1978.

Source Book on Applications of Undergraduate Mathematics to the Social Sciences, MAA and MSSB, 1977.

Steiner, H.G., Mathematisierung und Axiomatisierung einer politischen Struktur, *Der Mathematikunterricht,* Vol. 12 (1966), No. 3, 66-68.

Steiner, H.G., *What is applied mathematics?* (Paper presented at the Unesco Seminar on Goals and Means regarding Applied Mathematics in School Teaching, Lyon, France, February 4-8, 1974.)

Subnikov, A.V. and Koptsik, V.A., *Symmetry in Science and Art,* Plenum Publishing Corp., 1974.

Tanur, Judith M., Mosteller, F., Kruskal, W.H., Link, R.F., Pieters, R.S. and Rising, G.R. (eds), *Statistics: A Guide to the Unknown,* Holden-Day, Inc., 1972.

Teaching of Statistics in the Social Sciences, *Int. J. Math. Educ. Sci. Technol.,* Vol. 4, No. 1 (January-March 1973).

Thrall, R.M., Mortimer, J.A., Rebman, K.R. and Baum, R.F. (eds), *Some Mathematical Models in Biology,* University of Michigan, December 1967.

Turner, A.D., Nuffield "Science uses Mathematics" Continuation Project, *Education in Science,* Vol. 66, January 1966, 17-18.

Twersky, V., *Calculus and Science,* Studies in Mathematics, Vol. XV, School Mathematics Study Group, 1967.

Uspenskii, V.A., *Some Applications of Mechanics to Mathematics,* Blaisdell Publishing Company, 1961.

Walton, W.U., *The interface between physics and mathematics at school level,* Trend paper No. 11a, International Conference on Physics Education, University of Edinburgh, July 29—August 6, 1975.

(This trend paper served as part of the background for the chapter with the same title in New Trends in Physics Teaching, Unesco, 1976.)

Wilder, R.L., Mathematics and its relations to other disciplines, *Mathematics Teacher,* Vol. 66, No. 8 (December 1973), 679-685.

Williams, Mary B., Letter to the Editor, *Notices of the American Math. Society,* Vol. 18, No. 3 (April 1971), 502-503.

WORKING GROUP A

LESSONS FROM RESEARCH ABOUT STUDENT'S ERRORS

GROUP LEADERS:

STANLEY ERLWANGER

DIETER LUNKENBEIN

LESSONS FROM RESEARCH ABOUT STUDENTS' ERRORS
by S.H. Erlwanger

PARTICIPANTS OF WORKING GROUP A

H. Bauersfeld, H. Bouazzaoui, V. Byers, B. Emmouna, H. Gerber, M. Hoffman, R. Kayler, D. Kirshner, A. Kramti, E. Kuendiger, D. Lukenbein, O. Mohammed, A. Powell, J. Vervoort, S. Erlwanger.

"Students' errors in mathematics learning have often been approached from a pathological point of view. In such an approach, the study of errors or error patterns is conceived as the study of the symptoms of some disease for which a cure has to be found or discovered. In other relatively recent studies, the phenomenon of errors in mathematics learning has been approached from a more developmental, cognitive point of view. In this latter approach, students' errors are seen more as signs of progress in learning, which may indicate an incompleted process, a deviation from an expected development or even a misconception but which essentially is a phenomenon of a cognitive process called learning. In this perspective, students' errors in mathematics are important indicators for the description of the learning process and its gradual development.

It is the intention of this working group to study and to discuss the phenomena of error in mathematics learning from the latter perspective.

1. by looking at some recent publications or research in this field;

2. by indicating the impact of particular results on the conception and description of models of the learning process;

3. by identifying some areas of research, where the described approach could be of particular interest."

The Working Group consisted of a diverse set of individuals with interests ranging from the elementary school to the university level.

1. Publications and Research

Several publications were on display for the group. Some of these are listed in the Appendix. In addition, several copies of reports and examples of students work were made available to members. There was unfortunately, not a wide enough range of material.

2. Areas of research identified as of interest were in the articles on display, especially two by Ginsberg which are discussed below. The following points of interest emerged from the discussions.

A. Error Analysis for Cognitive Purposes

There was some discussion as well as general agreement on the idea of using errors and error analysis to study cognitive processes. The article, "Cognitive Diagnosis of Children's Arithmetic" by Ginsburg was discussed as a good example of one application of error analysis. It was felt that Ginsburg's classification of cognitive purpose and integrative purpose was a useful way of considering error analysis.

Some of the members here felt that such analysis could be useful for theory building while others were more interested in using such analysis for remediation. These differences reflected individual members interest and experiences in the area of diagnosis.

One of the participants, H. Gerber, has aptly observed that "the session got off to a ponderously slow beginning. Perhaps it was due to the heterogenious nature of the groups, or the variety of biases, expectations, and concerns, that the first three hours were, to me at least, a waste of time. The meeting came alive at the start of the first session with your (Erlwanger) examples, confirming the theory that we should introduce a topic with a problem that interests the audience." The point being made is an important one in that it reflects the state of the art regarding error analysis in mathematics. The article by Ginsburg was a beginning attempt towards some sort of theory. However, it became clearer over the three days that we were all easily motivated by anecdotal examples and subsequent discussions remained at the descriptive level and led to different interpretations by individuals.

Over the remaining two days we attempted to follow a plan to discuss Procedural Errors, Conceptual Errors and finally errors in Problem Solving.

B. Procedural and Conceptual Errors

A distinction was made between these two types of errors by V. Byers. The former were errors in the steps of an algorithm and the latter were concerned with concepts. The discussion on procedural errors led to the following points:

(i) The article, "Cognitive Analysis of Children's Mathematics Difficulties" by Russell and Ginsberg was introduced by Byers as an example to illustrate procedural errors as specific defective procedures when executing a written algorithm. Members did not have enough time to digest the article. It was also felt that the paper summarized results rather than described individual children. However, the results did indicate that fourth graders with learning difficulties made a greater number of errors than their peers.

(ii) Two examples of children's work were shown by Erlwanger. One example showed a boy who made procedural errors with an algorithm but could do the example mentally in his head. Moreover, the same boy could handle fractions which were used in plans for his model constructions, but he made errors with fractions at school. The next example was of a boy who was able to use algorithms correctly as well as informal methods of working with percents for example. The examples were used to suggest that the so-called informal methods are used by both good and poor students, but it is only the latter who are often unable to use standard algorithm that are taught at school.

(iii) Two examples of conceptual errors were given. V. Byers used an example by R. Davis on the zero product principle which some high school students used incorrectly to solve $(x-7)(x-8) = 3$ as $x-7=3$ or $x-8=3$.

A set of examples by Erlwanger of children's interview responses to questions about the equal sign was distributed. The examples illustrated how children gave their own (different) interpretations of the equal sign in examples such as $2+3=5$, $3=3$ and $5+3=3+5$.

There was not enough time to consider these examples or any others further.

C. Errors in Problem Solving

There was no time to discuss this area at all.

D. Other aspects that were touched upon but not discussed at length were:

1. Articles by Byers and Erlwanger. One article on "Content and Form in Mathematics" raised the question whether errors should be attributed to either the content or the form, or both. A second article on "Memory and Mathematical Understanding" raised the question: What do good students in mathematics remember that poor ones do not? A related question is the role of memory in errors arising from spurious genralizations.
2. The article by Russell and Ginsburg supported some of the observations made by the group, namely: children with mathematics difficulties are not deficient in key informal mathematical concepts and skills, but they have trouble recalling addition number facts and they make procedural type errors.
3. The problem of how to minimise the occurrence of errors as well as how to use errors as they occur to assist students in learning mathematics was proposed by M. Hoffman.
4. Looking at errors from a broad context in which errors emerge as only one aspect of the totality of that student. (H. Bauersfeld)
5. The notion that errors are subject matter specific and reflect its content as well as its form. (Byers and Erlwanger)
6. The development approach in Geometry where errors are considered to be indicators of the level of development of children. (D. Lunkenbein) This raised the question that we speak of children's errors frequently in subjects taught at school while we seldom think of errors in subjects that are taught informally such as geometry.

7. Most discussions on errors remained at the descriptive level and did not lead to any theory. (D. Kirshner)

To summarize, the working group demonstrated once again that our knowledge regarding the learning of mathematics and the causes and nature of errors is still incomplete, fragmentary and far from a theory. Thus when the group met initially a great deal of time was spent in trying to evaluate each others views. It would probably have been advantages to have focussed on introducing each aspect by means of examples. But it turns out that finding examples to cover different levels e.g. elementary, secondary, college and university is quite difficult.

Individual comments by participants:

(a) E. Kuendiger

"I appreciated very much, that during discussions a variety of different views came up, as to how an error can be defined and what role it plays in the learning process. Depending on the chosen conceptual framework different aspects come to the fore.

I remember three different approaches, that partly are overlapping, partly exclusive.

(a) Stanley gave examples of a student, who could solve an addition sentence mentally in a non-school environment and could not do it when at school, neither mentally nor by using the standard algorithm. Looking back, I think Heinrich's domains of learning are very suitable to describe these difficulties: A cognitive structure is built up in one domain and by this is related to this domain and is not necessarily taken over as a successful strategy into another domain.

In this situation the tasks of the teacher would be to recognise suitable strategies a student posseses already, to enable the student to transfer this strategy into another domain and to demonstrate the relationship between strategies (standard algorithm - others).

There is another reason why I like the above mentioned example given by stanley: it demonstrates the relevance of the affective part of the learning process. This affective aspect is - as to my opinion - one of the most important characteristics of a domain, e.g. if a learning environment is supportive in a way that a student ventures to think, transfer of cognitive structures from another domain is more likely.

(b) Another aspect came in to the fore in Dieker's approach, that is the developmental one. Taking geometrical concepts as an example the development of a cognitive structure could be described. In this conceptual framework an error can be looked upon as indicator of the stage of development. By choosing appropriate tasks the teacher is able to support the development of the cognitive structure.

(c) The above, shortly described approaches complement one another, - the third one, introduced by David Kirshner is - as to my opinion - not compatible with the others. Frankly, I do not agree with David, I perceive his framework as too static: an error is defined as

deviation of a well defined norm system. Moreover the occurrence of errors has to be avoided. If this is not possible, the teacher - by intervening - has or will lead to the right way more or less immediately.

This approach makes it easy to identify errors and to classify them; but I think it is far from school learning or learning in general."

(b) D. Kirshner

"In this report, I focus on my own principal intervention in the Error Analysis Working Group, concerning the relationship of competence models to error analysis. The thesis consists of the following components:

1. Data available on students' errors are (usually and appropriately) analysed through comparison to, or as deviations from, competent behaviours.

2. Error patterns are less uniform and "stable" than competence patterns, both within and between subjects, because the class of deviations from a procedure is (in principle and practise) broader than the procedure itself. Also, errors may present an intermediate stage in the acquisition of competence. The latter is therefore an 'end point' of a developmental process.

3. The greater stability of competence data permits, in principle, more systematic and rigorous analysis of competence patterns than is possible, independently, of error or acquision patterns. The dominant paradigms in the psychology of mathematical skill (e.g. Information Processing) do not exploit this potential, but, model competence and error using the same tools and ascribing equivalent status to theories of error and theories of competence. The result is that "in most (IP) analyses there has been considerable obscurity in the boundary between what is meant to be true of all subjects, and what is meant to be true of a particular subject." (VanLehn, Brown, & Greeno, 1984, p.236)

4. More productive error analysis (i.e. more genralizable analyses) may have to attend the more rigorous modelling competence. In that case, error analysis may serve a new, subservient role directed to the evaluation and verification of competence models."

(c) H. Gerber

The conference was an excellent one, well-organized and with outstanding speakers. However, the session got off to a ponderously slow begining. Perhaps it was due to the heterogeneous nature of the groups, or to the variety of biases, expectations, and concerns, that the first three hours were, to me at least, a waste of time.

The meeting came alive at the start of the second session with your examples, confirming the theory that we should introduce a topic with a problem that interests the audience. From that moment, and the time when the francophones began to speak, our session was first-rate.

For me, the sessions opened a whole new aspect of teaching. I began to understand the problems, the terminology, and the present limitations on our understanding of errors. Moreover, I now have a bibliography on which to begin. The next time I see you, I intend to participate in such a meeting in a more intelligent fashion.

You wanted an example. Let me remind you of the one I gave. The average of the examination scores 22/30, 15/20, and 5/8, on tests is now calculated as (22+15+5)/(30+20+8)=42/58. My son thought that this was the same as the old percentage average. In that case the average of 60%, 70%, and 80% is (60+70+80)/(100+100+100)=210/300=70/100=70%. This confusion led him to believe that if his cumulative average after 3 months was 60%, and he got 70% on his next examination his new average would be 65%. He was bright enough to see his error as soon as I pointed it out to him."

(d) S. Erlwanger

"This is the second working group in five years that I have attended on the subject of errors. The first one in 1980 focussed on results based on tests and examples of remediation. This time we tried to focuss on the value of errors in cognitive analysis. I note that in each case the groups got off to a very slow start. This is probably a reflection of the different biases and interests of individuals.

Although the discussions did not go very far, I think they did reflect some development in this area that ought to be pursued by furture working groups - I hope in less than five years time.

I would suggest though that in future an attempt should be made to get members to contact each other before the conference. I think this is absolutely essential so that the group leaders can get some idea of the interests of the participants and perhaps arrange that participants bring one or more examples of errors for discussion."

APPENDIX

(a) Articles

4. Ginsburg, H.P., 1983, 'Cognitive Diagnosis of Children's Arithmetic', in J. C. Bergeron and N. Herscovics (eds): Proceedings of the Fifth Annual Meeting of the International Group for the Psychology of Mathematics Education, pp. 247-54.

6. Russell, R.L. and Ginsburg, H.P., 1984, 'Cognitive Analysis of Children's Mathematics Difficulties', Cognition and Instruction 1(2), pp. 217-244.

5. Matz, M., 1980, 'Towards a Computational Theory of Algebraic Competence, The Journal of Mathematical Behaviour, Vol.3, No.1, pp. 93-165.

1. Byers, V. and Erlwanger, S.H., 1984, Content and Form in Mathematics', Educational Studies in Mathematics 15, 259-275.

2. Byers, V. and Erlwanger, S.H., 1985, 'Memory in Mathematical Understanding', Educational Studies in Mathematics 16, 259-281.

3. Davis, R.B. et al., 1982, The Roles of "Understanding" in the Learning of Mathematics. Part II of the Final Report of the National Science Foundation, April 1982.

(b) Assortment of books.

WORKING GROUP B

LOGO ACTIVITIES FOR THE HIGH SCHOOL

GROUP LEADER:

JOEL HILLEL

Appendix

This appendix contains three examples of LOGO-related activities for the math classroom. The first activity relates to the topic of Pattern and can be used at varying levels of sophistication through the grades (elementary and secondary). The second activity relates to the topic of Least Common Multiple and can be used in late junior and intermediate level math classes. The third activity, relates to teaching about π and the circumference of a circle (intermediate grades).

Report of Working Group 'B': LOGO

The Group spent most of its time in examining and evaluating several LOGO inspired investigative situations which had strong links to the math curriculum. This was a follow up to last year's group (Working Group A: LOGO and the math-curriculum) in which the consensus emerged that the availability of such explicit 'microworlds' represents the best strategy for having LOGO accepted and used by most teachers. It is an approach taking the path of 'minimal resistance' since it calls on no special programming expertise by the teacher, nor does it require a major perturbation of the existing classroom setup or the existing curriculum. This is not an argument against other possible implementations of LOGO in the school, including a more inclusive Papertian vision of a fully implemented LOGO curriculum. Rather, it is based on the pragmatic realisation that the acceptability of LOGO to most teachers will be based, rightly or wrongly, on their perception of its relevance to what is currently taught.

Aside from an emphasis on specific math content, last year's group employed other criteria which were intended to reflect the advantages of LOGO-based environments. These included: modifiability, extensibility, the possibility of users writing their own procedures and following several lines of inquiry, etc. (see last year's report). At the risk of an oversimplification, we can say that two general types of situations were examined during the three days. The first type comprised those situations created specifically to enhance a topic

within the existing math curriculum. The second type comprised situations whose underlying math concepts are not traditionally taught, but yet seem accessible to students because of the graphical capabilities afforded by the computer.

Gary Flewelling of the Wellington County Board of Education produced many examples in which LOGO was used to generate "investigative situations" connected to topics in the math curriculum. These included investigations involving fractions, vectors, motion and acceleration, trig functions and statistics (see the appendices for some examples). These were viewed by the group, which discussed how they could be modified, or extended to allow the user more control.

Denis Therrien of Université Laval also demonstrated some software packages which dealt with number concepts such as divisors, prime, composite, odd/even numbers, etc.

A. Senteni of U.Q.A.M. demonstrated a non-turtle LOGO microworld - that of variations on Conway's Game of LIfe (designed by B. Silverman of L.C.S.I.). Here members of the group discussed briefly whether this kind of situation is only for 'buffs' or whether such an investigation could be used to launch into some important math. concepts such as 'state', 'action on states', 'stability', finite and infinite 'orbits', etc.

Finally, the possibility of using LOGO to investigate limiting processes was discussed. Here the group thought out several types of limiting behaviour which could be exhibited geometrically: limiting shapes (e.g. circle as a limit of n-gons), limiting points and lines of inspirals, numerical limits (e.g. ratio of perimeter to diameter of n-gon) and fractals using recursive procedures.

Members of the group:

R. Blake (U.N.B.)

G. Flewelling (Wellington County Board)

J. Girard (U.Q.A.C.)

J. Hillel (Concordia)

B. Kastner (S.F.U.)

H. Kayler (.U.Q.A.M.)

T. Kieren (U. of Alberta)

E. Lepage (U.Q.A.R.)

A. Senteni (U.Q.A.M.)

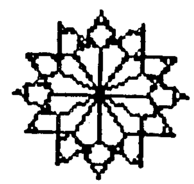
D. Therrien (Laval)

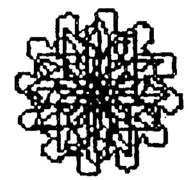
C. Verhille (U.N.B.)

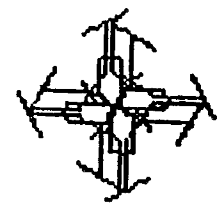
D. Wheeler (Concordia)

DESIGNS FROM LETTER PATTERNS

(Using LOGO procedures)







Developed by

Gary Flewelling
Mathematics Consultant
Wellington County Board of Education

DESIGNS FROM LETTER PATTERNS

LOGO	MATH
NONE	Letter Patterns Properties of 2D Designs

START UP INSTRUCTIONS

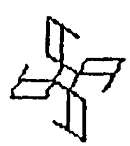
1. Load LOGO into your computer (see pin up card #1)
2. With Flewelling disk in drive, type
   READ "LETTERS [RETURN]
3. When the LETTERS file has been read in, type
   LETTERS [RETURN]

YOU WILL BE ASKED TO RESPOND TO ONE OR TWO INSTRUCTIONS.

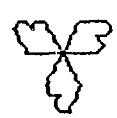
- After you have responded to the instructions on the screen, the alphabet keys you asked for will be activated.
- As each letter is typed in, it will appear in the upper left portion of the screen.
- In addition, a larger version of the letter will appear at midscreen. (see below)

- Each additional letter begins to be drawn where the previous letter stops being drawn. This gives rise to a large number of letter designs.
- Even very simple single-letter patterns will generate designs. (see below)

AAAA



BBB



- More complicated designs result from using two or more letters.

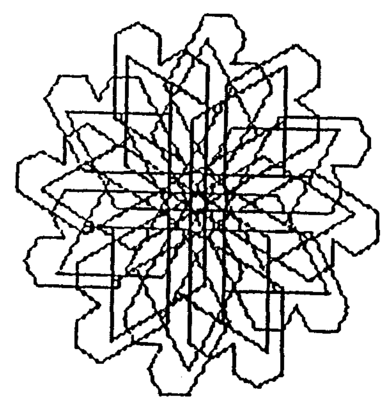
ABABABABABABAB

- If a key is hit in error and you wish to remove that letter from your design, just hit the [-] key. If you want to undo several letters, hit the [-] key several times.
- If for any reason you want to blow up or shrink a design, just hit the [/] key.

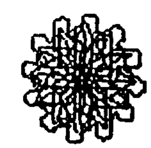
You will be asked, "What scale factor?" If you wanted to double its dimension, for example, respond by typing [2] [RETURN]. Had you wanted to shrink it to half size you would have responded by typing [.5] [RETURN]

To get back to original design size you must hit the [/] key and respond with a scale factor [1] [RETURN].

Below is the 'ABABAB' design blown up using a scale factor of 3.



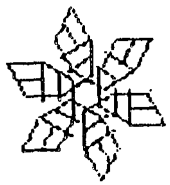
ABABABABABABABABABABABAB



- Hit the [*] key to erase the screen and start over again.

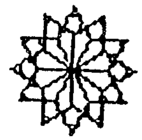
e.g. 1

MEMEMEMEMEMEME



e.g. 2

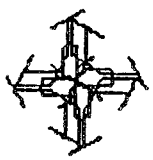
BOBBOBBOBBOBBOBBOBBOBBOBBOBBOBBOBBOBBOB



- Letter patterns need not be set out on just one line as above. A 2 dimensional array of letter patterns can also be created. This is achieved by pressing the [1] key at the end of each line of letters in the array.

  Keying ART1ART1ART1ART1 for example, would give the letter pattern and design shown below.

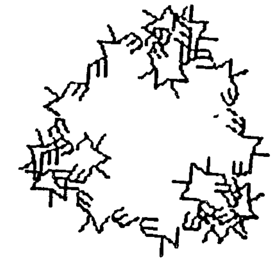
ART
ART
ART
ART



- Designs can be printed onto paper following the instructions on 'pin up card #9'.

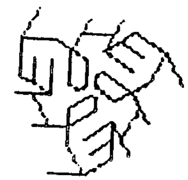
  A few sample print outs are shown on the next page.

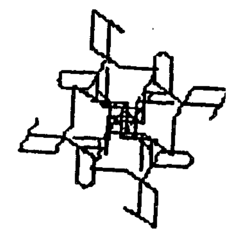
ZZWW
ZWWZ
WWZZ
WZZW
ZZWW
ZWWZ
WWZZ
WZZW
ZZWW
ZWWZ
WWZZ
WZZW
ZZWW



HELP
HELP
HELP

PIZZA
PIZZA
PIZZA
PIZZA



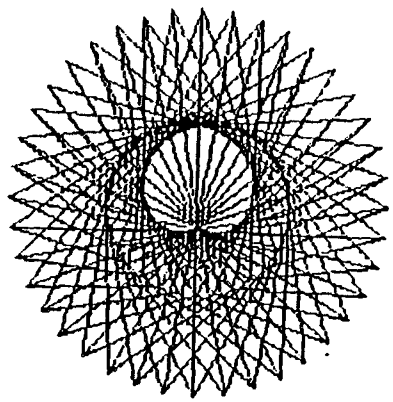


- See the supplement 'WHAT CAN I DO WITH THE PRETTY PICTURES?' for ideas on how to utilize these designs.

NOTE: Should something go wrong, for whatever reasons, and you want to start over again, hold the [CRTL] and [G] keys down, together, and type in LETTERS [RETURN].

LCM

(Using LOGO procedures)



Developed by

Gary Flewelling
Mathematics Consultant
Wellington County Board of Education

LEAST COMMON MULTIPLE

LOGO
NONE

MATH
Multiples
Least Common Multiple
Lowest Common Denominators
Common Factors
Coprime #'s
Composite #'s
Properties of 2D designs as gear ratios

START UP INSTRUCTIONS

1. Load LOGO (see pin up card #1)
2. With Flewelling disk in disk drive, type,
   READ "LCM [RETURN]
3. When LCM file is loaded, type,
   BEGIN [RETURN]

. You are first asked to type in the coordinates of the centre and radius for each of two circles.
I would suggest, in the beginning, typing,

0 0 [RETURN] and

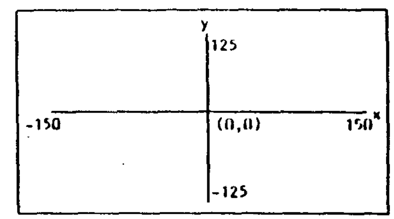
125 [RETURN]

for the first circle, and

0 0 [RETURN]

60 [RETURN]

for the second circle.

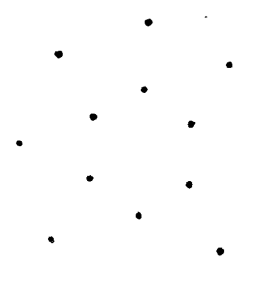


screen

(Keep the circles within the screen dimension shown above.)

. You are then asked to input two natural numbers. Initially, you should consider using one digit numbers. Had you typed, for example, **8** [RETURN] and then **6** [RETURN] you would see, on the screen, eight points on a large out circle and six points on an inner circle. (fig 1)

fig 1



. The procedures will cause the first point of circle one (C1) to be joined to the first point on circle two (C2), then the second point on C1 to be joined to the second point on C2, etc. Two coloured disks will appear on the points being joined. (fig 2 & fig 3)

fig 2

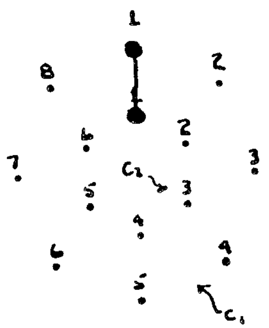
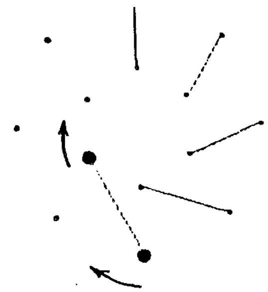


fig 3

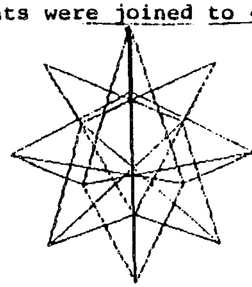


. You can control the action on the screen (type [S] [RETURN]) or let the procedure run continuously(type [R] [RETURN]).

Keying [S] and [RETURN] will activate the [*] key. Each time the [*] is pressed another pair of points will be joined. (fig 3 shows result of pressing [*] five times)

. In the above example, it will be noticed that the design won't be complete (fig 4) until 24 pairs of points have been joined. In this time, the disk on C1 will have made 3 trips around C1 and the disk on C2 will have made 4 trips around C2. (i.e. 3 sets of 8 points were joined to 4 sets of 6 points).

fig 4



Had the action been run continuously, you would see the two disks run around their circular tracks, with the disk on C1 completing 3 laps in the time that the disk on C2 completed four. (each touching 24 points)

. This hints loudly of the following

e.g. 1 $\frac{1}{6} + \frac{1}{8} = \frac{4\times1}{4\times6} + \frac{3\times1}{3\times8} = \frac{4}{24} + \frac{3}{24} =$ etc.

e.g. 2 a gear with 8 teeth turning a gear with 6 teeth

. Here 24 is the "least common multiple" of 6 and 8. LCM concepts can be introduced with this package.

. Students should be able to predict screen behaviors and outcomes given any two inputs.

e.g. 1 C1: 8 points and C2: 5 points (fig 6)
e.g. 2 C1: 8 points and C2: 4 points (fig 7)
e.g. 3 C1: 8 points and C2: 8 points (fig 8)

fig 6

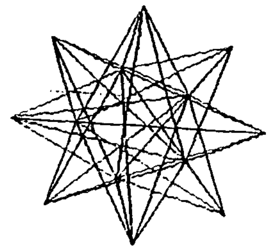


fig 7

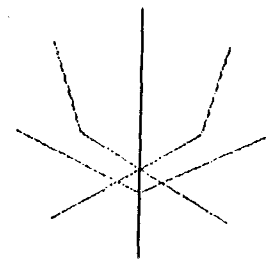
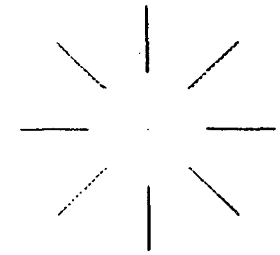


fig 8



. Natural numbers up to 100 can be entered (too large a number will result in an "out of memory" error).

. To print completed designs (fig 9-13) from to screen to paper follow these instructions.

1. Have Flewelling disk in disk drive and printer 'ON'.
2. Stop LCM procedures with [CTRL] and [G] keys held down together.
3. Press [P] key and the [RETURN] key.

(Figures 9-13 had both circles centred at (0,0), the centre of the screen.)

fig 9 C1: 4 C2: 50

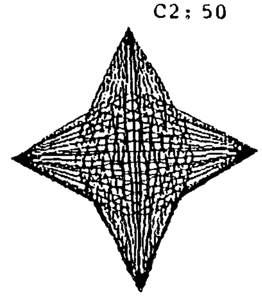


fig 10 C1: 67 C2: 2

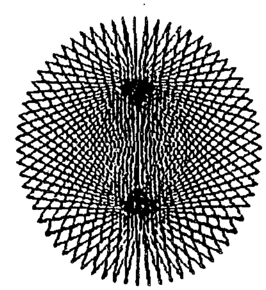


fig 11 C1: 6 C2: 32

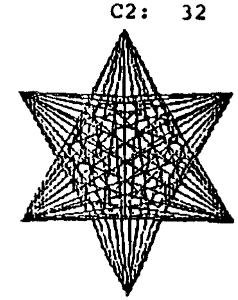


fig 12 C1: 88
C2: 11

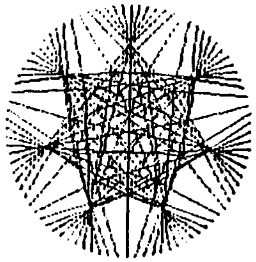


fig 13 C1:40
C2:30

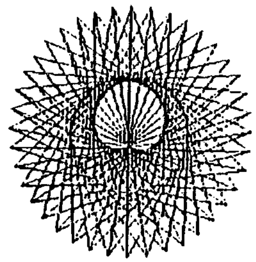


fig 14
C1: (0,50) r=60
C2: (0,-60) r=50
#'s 90 and 3

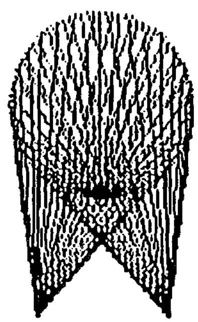


fig 15
C1: (0,50) r=75
C2: (0,-50) r=75
#'s 24 and 72

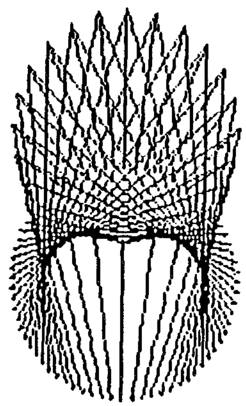
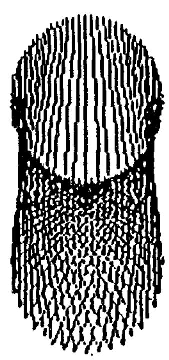


fig 16
C1: (0,50) r=50
C2: (0,-50) r=50
#'s 80 and 40



. Things other than LCM's and gear ratios can be investigated.

Q1. How do successive segments vary in length? (could directly measure each off screen and plot a graph, pair # v.s. length in mm)

Q2. Can you predict design characteristics given values of inputs? (e.g. C1:16 and C2:12)

Q3. Given design, can you determine input values?

Q4. Are there characteristic differences in designs where inputs:
a) are a multiple of the other (e.g. C1:24 and C2:8)
b) share a common factor (e.g. C1:24 and C2:8)
c) coprime (no common factors) (e.g. C1:7 and C2:5)

Q5. Are there characteristic differences between design C1:a, C2:b and C1:b, C2:a?

NOTE 1: prolonged use of the [*] key to step out a design will result in an "out of memory" error. At this point the design can be completed by typing DESIGN [RETURN].

NOTE 2: The procedure is not self-stopping. You must hold the [CRTL] and [G] keys down to stop the drawing action.

NOTE 3: To enter two new numbers without changing the size or position of the two circles, type,

LCM [RETURN]

NOTE 4: To start with two new circles, type,

INFO [RETURN]

NOTE 5: Should anything go wrong, for whatever reason, hold down the [CTRL] and [G] key then type, BEGIN [RETURN]

Make sure the Flewelling disk is in the disk drive.

CIRCLE ACTIVITIES

LOGO	
Activity 1, 2, 3	TO, REPEAT, FD, RT, LT
Activity 4	+ [↑]

MATH	
Activity 1	circles
Activity 2	circle designs dilatations
Activity 3	circular arcs designs
Activity 4	π circumference

ACTIVITY 4 (π AND THE CIRCUMFERENCE FORMULA)

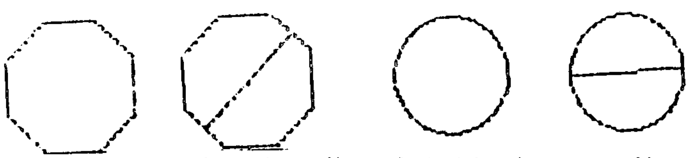
This activity does three things,

a) Gives meaning to π ,
b) gives the user a way of approximating π ,and
c) gives the user a method for working out a circle's circumference.

POLY2 #1 #2 is a procedure from the PI file that draws regular polygons just like the POLY1 procedure used in Activity 1. The only difference here is that once the polygon is drawn, the turtle moves to the centre of the last side drawn, turns inwards by 90°, and pulses the direction it is pointing in.

- If you now enter a command like FD 2 (or 1 or 5 etc.) and hit the [↑] [RETURN] keys repeatedly until you get to the opposite side of the polygon, you will have measured the polygon's width (if you count the number of steps taken x2 (or 1 or 5 etc.)
- Investigate how the width of a specific regular polygon compares to its perimeter.

Example



Each regular polygon (regardless of size) has its own peculiar constant (arrived at be dividing its perimeter by its width).

Polygon	Perimeter	Width	Constant P/W
Square			4
Pentagon			
Hexagon			
7-GON			
Octagon			
Circle			

(Use computer to do arithmetic calculation, just enter perimeter/width [RETURN])

This means that the perimeter of a regular polygon can be found simply by working out the answer to

width of polygon x polygon constant

This is a weird way of calculating a perimeter. Normally, you would just take the length of one side and multiply by the number of sides. And yet, it is a way of working out perimeter thats worth getting used to!

When the regular polygon becomes a circle, you have no choice but to use

width x circle constant !!

You are more familiar with the usual way of writing this formula.

Circumference of a circle = diameter x pi

WORKING GROUP C

IMPACT OF SYMBOLIC MANIPULATION SOFTWARE ON THE TEACHING OF CALCULUS

GROUP LEADERS:

BERNARD HODGSON

ERIC MULLER

Impact of Symbolic Manipulation Software on the teaching of Calculus

Influence des Logiciels a Calculs Symboliques sur L'Enseignement du Calcul Differentiel et Integral

Working Group C

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Annotated Bibliography	- Bernard Hodgson	7
Appendix 1	- Gilbert Morin - A useful introduction to muMATH.	15
Appendix 2	- Charles Latour - Part A - an excellent description on how muMATH was used to solve the curvature of light problem in general relativity - Part B - discusses the inportance "du calcul" or "general computational skills" in mathematics.	20
Appendix 3	- Noelange Boisclair - raises some general questions regarding the use of computers in calculus courses.	29
Appendix 4	- Edgar Williams - provides an extensive list of potential benefits which one can gain by using asymbolic manipulation software in teaching mathematics.	32
Appendix 5	- Dave Alexander - raises a number of questions and suggests a sequence for teaching differentiation with Symbolic Manipulation Software.	35

Leaders:	Bernard Hodgson	(Université Laval)
	Eric Muller	(Brock University)
Participants:	Dave Alexander	(Ontario Ministry of Education)
	Tasoula Berggren	(Simon Fraser University)
	Nöelange Boisclair	(Cégep Montmorency)
	Gila Hanna	(OISE)
	Charles Latour	(Cégep F.-X. Garneau)
	Fernand Lemay	(Université Laval)
	Hal Proppe	(Concordia University)
	Ghislain Roy	(Université Laval)
	Robert Sealy	(Mount Allison University)
	Bernard Vanbrugghe	(Université de Moncton)
	Edgar Williams	(Memorial University)

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Report

(In this report the terms Symbolic Manipulation Software (SMS) and Computer Algebra Software (CAS) are assumed equivalent. They refer to software which manipulates algebraic systems, uses rational arithmetic and can perform calculus operations.)

The group started by spending three hours obtaining first hand experience of the muMATH software in the Laval Mathematics Department's Microcomputer laboratory. The group followed a set of instructions developed by Gilbert Morin - a mathematics undergraduate at the Universite Laval (see Appendix 1).

A large number of shortcomings were found during this three hour session, the most serious of these being that wrong and incomplete answers were produced on the screen without comment. The general concerns of the group is that this particular software is not yet in a form sufficiently consistent and correct to be used in or with a first year class. The group is aware that such software as MAPLE and MACSYMA have been far more widely used and tested and that they do not contain some of the shortcomings of the muMATH. At present both MACSYMA and MAPLE require larger computer systems to operate. Nevertheless it is the opinion of workers in the field that both MAPLE and MACSYMA will be available on extended micros very soon. The group therefore was looking ahead to times when tested and powerful (computer algebra) symbolic manipulation systems will be readily available. Part A of Appendix 2, by Charles Latour, is a particularly good description of the experiences of an individual using muMATH for the first time to solve a specific problem.

At the end of the first session participants were asked to think about the impact of such systems on mathematics and to prepare a list of topics, concerns, etc., which could be studied and developed by the group.

The following list was drawn up at the beginning of the second session: (topics not in order of importance)

1. Develop problems (examples) particularly suited to solution using symbolic manipulation software.

2. Develop guidelines for the use of SMS systems as a check to one's work.

3. Determine whether an SMS system permits the introduction of more advanced ideas at an earlier stage, i.e. order within curriculum when SMS system is used .

4. Discuss the use of such systems for non-university bound students.

5. Identify either

   a) "routine" parts of the curriculum which can be undertaken by the SMS system and which have in themselves no value towards achieving the aims of the course

   or

   b) isolate the important parts of the curriculum which can be enhanced by, but not replaced by, the use of an SMS system.

6. Guidelines on how to use the SMS systems as a means for the exploratory development of mathematics

7. Isolate those skills which are necessary for using the system sensibly:-

   (a) Estimation
   (b) Sense of reasonableness
   (c) Knowledge of concepts
   (d) Are the procedures used in testing algorithms useful in testing solutions from an SMS package?

   (e) Use of graphical techniques as a check of reasonableness.

8. How much should one know about the algorithms and the computer language used in such packages? Do these algorithms and languages give any insight into the the mathematics?

9. What properties should an SMS system have in order for it to be useful in education (as opposed to a pure research tool) eg. capability to show intermediate steps etc.

The group then decided to isolate one topic within the differential and integral calculus and to discuss the use of SMS systems in the teaching of that concept. Without making any statement as to when or where within a calculus curriculum "limits" should be taught the group decided to look at the possible impact of SMS systems on the teaching and learning of limits.

(a) SMS systems and the teaching of Limits

SMS systems do not provide a rich environment for the teaching of the concept of limits. These systems can be used to simplify complicated algebraic expressions but generally numerical procedures provide a better medium to motivate intuitive ideas of limit concepts in calculus, which is of the type $\frac{0}{0}$. A useful numerical software package would have a split screen displaying graphical values on one side and algebraic representations on the other. The plotting of function values should be dynamic so that subsequent values appear one at a time. It should be simple to enlarge any interval of values so that intervals which initially are very small could be enlarged to fill the whole graphical portion of the screen. Such software would be used to present simple cases in class and would allow students to explore many different functions which are normally not accessible because either the student lacks the algebraic techniques, or the computations are extremely tedious.

Once the concept of limit is understood SMS systems should be used to motivate the laws of differentiation. Every effort should be made to present the derivative as a dynamic concept and not a numerical one. SMS software allows quick access to more meaningful applications and to the introduction of differential equations which provide life to the derivative.

(b) SMS systems and the teaching of Integration

When discussing integration techniques -- algebraic integration procedures -- two disparate points of view are expressed:

(a) Too much time is spent on integration techniques both in class and student assignments. These techniques tend to dominate the use of the student's time and mastery of these techniques does not translate into a better understanding of integration. Some argue that we can now dispense completely with integration techniques as they are largely algebraic manipulations which shed no new light or insight on the concept of integration.

(b) Integration techniques are a necessary part of any calculus course. A student faced with a particular integral is forced to consider alternative procedures for solving it. There is therefore a certain openess or trial and error situation. It is one of the few areas where students apply the algebraic skills they have acquired in school mathematics.

The group believes that the following points are sufficiently significant that they can form the basis of further thought and study in the use of SMS software in calculus courses.

When technology is available, course content, lecture presentation and student activities should shift to higher mental activities. Can calculus courses learn from the statistics experience? Initially statistics courses spent many hours on simplification of expressions involving sums of squares and cross product expressions. This was "good" for the students as they obtained experience using the Sigma notation and manipulation of indices to change the conceptual definition to the efficiently calculable form. This is rarely done now and more time is spent on the statistical concept and the where, when and how to apply it. Is the calculus curriculum so well established that it no longer has any flexibility for change? One way to review the Calculus curriculum is to firstly isolate those concepts which are essential to calculus and secondly to structure a curriculum with supporting activities restricted to those which reinforce the concepts and give a deeper understanding of calculus. It is likely that SMS software will play a major role in such supporting activities. Many students presently complete a calculus course and are unaware of integral tables. They have a very limited experience of integration techniques and many are unaware that the integral of the majority of functions do not have closed analytical forms. Hopefully SMS software will change this situation and will place students in a more experimental situation.

A reduced emphasis on algebraic manipulation in calculus courses should have a major impact on school mathematics courses as much of the school algebra is directed towards preparation for calculus courses.

It is clear that unviersity mathematics professors involved with first year calculus and linear algebra courses have a lot to learn regarding the use of SMS software in these courses. It is imperative that those who are experimenting with the use of such software in their courses communicate their findings. It is important that experimental use of such systems be well documented so that others can repeat these experiments in different settings. Either one of the leaders of this working group would welcome receiving such information and to circulate it to interested individuals.

ANNOTATED BIBLIOGRAPHY

ASSMUS 85 — E.F. Assmus, Jr., "Pi." Amer. Math. Monthly 92 (1985) 213-214.
An example of a "conceptual problem" (the existence of π) showing the importance of change of variables and integration by parts in studying integration.

BUCHBERGER 83 — B. Buchberger, G. E. Collins and R. Loos, Computer Algebra: Symbolic and Algebraic Computation. Springer-Verlag, 1983. (2nd ed.) First edition issued as a Supplementum to the journal Computing (1982). A basic book containing sixteen survey articles (with extensive references) about the theory and implementation of symbolic mathematical systems (the so-called "computer algebra").

COMP & MATH 84 — J.T. Fey, ed., Computing and Mathematics: The Impact on Secondary School Curricula. NCTM, 1984.
Report of a conference sponsored by NSF. Of particular interest are the chapters "Impact of computing on algebra" and "Impact of computing on calculus".

COSERS 80 — B.W. Arden, ed., What can be automated? MIT Press, 1980.
The Computer Science and Engineering Research Study. A huge report (nearly 1000 pages!) on all aspects of computer science. Pages 513-526 give a short introduction to algebraic manipulation.

COXFORD 85 — A. Coxford, "School algebra: what is still fundamental and what is not." In [NCTM.YB 85] pp. 53-64.
"The push to incorporate symbolic mathematical systems in algebra is questionable because we are not sure of the relationships between procedural knowledge and skill and the understanding of algebra. (...) I predict that more procedural knowledge will be needed to learn algebra than many would believe."

DAVENPORT 81 — J.H. Davenport, "Effective mathematics - the computer algebra viewpoint." In Constructive Mathematics, F. Richman, ed. Springer-Verlag, 1981, pp. 31-43. (Lect. Notes in Maths, no. 873.)
An introduction to the theory underlying symbolic mathematical systems.

FATEMAN 80 — R.J. Fateman, "Symbolic and algebraic computer programming systems." Proc. ICME-IV (1980) Birkhaüser, 1983. pp. 606-612.
A mini-course on symbolic and algebraic computer programming systems.

FEY & GOOD 85 — J.T. Fey and R.A. Good, "Rethinking the sequence and priorities of high school mathematics curricula." In [NCTM.YB 85] pp. 49-52.
"A small number of familiar and powerful mathematical ideas are at the heart of most common applications (...) A student assisted by [a symbolic mathematical system] need not endure a long skill-building apprenticeship in order to become an effective problem - solver - if the key organizing concepts are well understood."

HEID 83 — M.K. Heid, "Calculus with muMATH: implications for curriculum reform." Computing Teacher" 11 (1983) 46-49.
A condensed version of some issues discussed in [COMP & MATH 84].

HODGSON & AL. 85 — B.R. Hodgson, E. Muller, J. Poland and P. Taylor, "Introductory calculus in 1990." In [STRASBOURG 85] pp. 213-216.
"We propose ways in which the introductory Calculus curriculum might respond to the recent and rapidly changing computer ressources." Discussion stresses the use of a contextual approach, the qualitative analysis of functions in mathematical modelling and an interactive mode of classroom teaching.

HODGSON & POLAND 83 — B.R. Hodgson and J. Poland, "Revamping the mathematics curriculum: the influence of computers." CMS Notes 15(8) (1983)17-23.
Outcome of working groups at the CMESG meetings of 1982 and 1983. Raises the question of the relevance, in the context of the actual computer revolution, of mathematics courses taught in the traditional form. Proposes scenarios of reasonable solutions to the changes needed in undergraduate mathematics education.

HOSACK.1 85 — J.M. Hosack, "The effect of computer algebra systems on the curriculum." Preprint, Colby College (Waterville, ME), 1985. (5 pages)
A general discussion regarding the use of

symbolic mathematical systems in early courses (see also [LANE 85]) for a presentation of the Colby Curriculum Project).

HOSACK.2 85 J.M. Hosack, "A guide to computer algebra systems." Preprint, Colby College (Waterville, ME), 1985. (15 pages)
A comparison of the capacities of MACSYMA, Maple, muMATH, REDUCE and SMP.

HUBBARD & WEST 85 J.H. Hubbard and B.H. West, "Computer graphics revolutionizes the teaching of differential equations." In [STRASBOURG 85] pp. 29-36 (Supplement).
Illustrate the use of interactive high-resolution graphics for the (early!) teaching of differential equations.

ICMI 84 "The influence of computers and informatics on mathematics and its teaching." (An ICMI discussion document). <u>L'enseignement mathématique</u> 30 (1984) 159-172.
The discussion document prepared for the ICMI Symposium held in Strasbourg in March 1985 (see [STRASBOURG 85]). An expanded version of this paper, as well as a selection of papers submitted to Strasbourg or written by invitation following the meeting, will appear in the <u>Proceedings</u> of the Symposium, to be published by the Cambridge University Press.

KENELLY & AL.85 J. Kenelly, P. Henry and C.O. Jones, "The advanced placement program in calculus." In [NCTM .YB 85] pp. 166-176.
Some of the topics of the maths curriculum should <u>not</u> be treated with the computer. The authors make a parallel with machine translation of natural langages: "Here, the computer is very capable with mechanical substitutions but the rich subtleties are lost."

KUNKLE & BURCH 84 D. Kunkle and C.I. Burch, "Symbolic computer algebra: the classroom computer takes a quantum jump." <u>Mathematics Teacher</u> 77 (1984) 209-214.
Illustrates the use of muMATH for finding the sum of $j^k (j=1,...,N)$ for different values of k.

LANE 85 K.D. Lane, "Symbolic manipulators and the teaching of college mathematics: some possible consequences and opportunities." Preprint, Colby College (Waterville, ME), 1985. (13 pages)

Description of the Colby Curriculum Project integrating symbolic mathematical systems in the college curriculum. A condensed version appears in [MAA. PANEL 84].

MAA. PANEL 84 M.J. Siegel. ed., <u>Panel on Discrete Mathematics in the First Two Years</u> (Preliminary Report). MAA, Nov. 1984.
Included is a short "Report on the use of symbolic mathematics system in undergraduate instruction" by J. Hosack, K. Lane and D. Small.

MAPLE 84 B.W. Char, K.O. Geddes and G.H. Gonnet, "An introduction to Maple: sample interactive session." Research Report CS-84-04, Department of Computer Science. University of Waterloo. (15 pages)
An introduction to what can be done with a symbolic mathematical system, using the system Maple currently under development at the University of Waterloo.

MOSES 71 J. Moses, "Algebraic simplification: a guide for the perplexed." <u>Comm. ACM</u> 14 (1971) 527-537. "Symbolic integration: the stormy decade." <u>Comm. ACM</u> 14 (1971) 548-560.
Two papers from the Second Symposium on Symbolic and Algebraic Manipulation. These excellent expositions, although somewhat dated, give a lot of informations about the way computers can manipulate symbolic expressions and find antiderivatives. A good preparation for the reading of the more technical [BUCHBERGER 83].

NCTM.CONF 84 M. K. Corbitt and J.T. Fey, <u>The impact of computing technology on school mathematics</u>. (Report of an NCTM conference). NCTM, 1984, (5 pages)
A brief report from a conference held in March 1984. Includes recommandations related to the impact of computing technology on curriculum, instruction and teacher education.

NCTM.YB 84 J.Fey and M.K. Heid, "Imperatives and possibilities for new curricula in secondary school mathematics." In <u>Computers and Education</u> (1984 Yearbook), V.P. Hansen and M.J. Zweng, ed. NCTM, 1984. pp. 20-29.
Similar in spirit to [HEID 83] or [COMP & MATH 84]. Stresses "topics of diminished importance" and "topics of continued importance". (The whole 1984 Yearbook contains

27 papers divided in five parts: Issues; The computer as a teaching aid; Teaching mathematics through programming; Diagnostic uses of the computer; Bibliography.)

NCTM.YB 85 — C.R. Hirsch and M.J. Zweng, ed., The Secondary School Mathematics Curriculum (1985 Yearbook). NCTM, 1985.
Of special interest to symbolic computations are the papers [COXFORD 85], [FEY & GOOD 85], [KENELLY ET AL. 85] and [RALSTON 85].

NORMAN 83 — A.C. Norman, "Algebraic manipulation." In Encyclopedia of Computer Science and Engineering, A. Ralston et al., eds. Van Nostrand, 1983. pp. 41-50.
A quick overview of different symbolic manipulation systems.

POLAND 84 — J. Poland, "Computers and the impending revolution in mathematics education." Ont. Math. Gaz., 23(2) (1984) 26-29.
A fresh discussion of some of the issues raised by the presence of computers and symbolic manipulation systems.

RALSTON 85 — A. Ralston, "The really new college mathematics and its impact on the high school curriculum." In [NCTM.YB 85] pp. 29-42.
What changes should occur in the high school curriculum as a result of changes in the college curriculum (vg the role of discrete mathematics) and the direct impact of computers technology (vg symbolic mathematical systems) on the high school curriculum.

RAND 84 — R.H. Rand. Computer Algebra in Applied Mathematics: An Introduction to MACSYMA. Pitman, 1984.
An introduction to the use of symbolic manipulation systems (viz. MACSYMA) in higher maths. ("This book is aimed at a reader who has had at least three years of college level calculus and differential equations.") The syntax of MACSYMA is learned while working some examples. Contains a few exercises with detailed solutions. "It is expected that it will not be long before computer algebra is as common to an engineering student as the now obsolete slide rule once was."

SCI. AMER. 81 — R. Pavelle, M. Rothstein and J. Fitch, "Computer Algebra." Scient. Amer., Dec. 1981, pp. 136-152 (Version française: "L'algèbre informatique." Pour la science, fév. 1982, pp.90-98.)
A most influencial paper in making symbolic manipulation systems known to the general (scientific) public.

SIGSAM 85 — SIGSAM Bulletin 18(4) and 19(1) (1984-85).
A special issue of the Bulletin of the "Special Interest Group on Symbolic & Algebraic Manipulation" (SIGSAM) of the ACM. Contains papers from the session on "Symbolic mathematical systems and their effects on the curriculum" held at ICME-5, Adelaide, 1984. Sixty-two pages of interesting reading. Some of the papers report on experiments done in high school or university.

SQUIRE 84 — W. Squire, "muMATH system effective tool for algebra." SIAM News, Nov. 1984, p. 4.
"The situation may be described as a potential revolution waiting for a textbook."

STEEN 81 — L.A. Steen, "Computer calculus." Science News, 119 (1981) 250-251.
A short presentation of symbolic mathematical systems.

STEWART 84 — I. Stewart, Review of [BUCHBERGER 83] Math. Intell., 6(1) (1984) 72-74.
Some comments on the general question: Will the computer, with its symbolic manipulation capability, put all mathematicians out of business ?

STOUT 79 — D.R. Stoutemeyer, "Computer symbolic math and education: a radical proposal." SIGSAM Bull., 13(2) (1979) 8-24.
An interesting discussion of the use of symbolic manipulation systems in the teaching of mathematics. A revised and augmented version of this paper has appeared in [SIGSAM 85], pp. 40-53, under the title: "A radical proposal for computer algebra in education".

STOUT 83 — D.R. Stoutemyer, "Nonnumeric computer applications to algebra, trigonometry and calculus." Two-Year Coll. Math. J., 14 (1983) 233-239.
A general introduction to symbolic manipulation systems. Mentions somme applications in abstract algebra.

STOUT 85 — D.R. Stoutemeyer, "Using computer symbolic math for learning by discovery." In [STRASBOURG 85], pp. 155-160.

Some nice suggestions of projects using computer algebra for math discovery.

STRASBOURG 85 The Influence of Computers and Informatics on Mathematics and its Teaching. Supporting papers for the symposium organised by ICMI. Strasbourg, March 1985. (256 pages plus a Supplement of 52 pages).
The papers presented by the participants to the ICMI symposium. A new edition of these supporting papers is to be published by the IREM of Strasbourg. Copies can be ordered from F. Pluvinage, Département de mathématiques, 7 rue René-Descartes, 67084 Strasbourg Cédex, France. The price is FF100.

TALL 85 D. Tall, "Understanding the calculus." Math Teaching No. 110 (March 1985) 49-53.
How to use the graphical capabilities of the computer to illustrate basic concepts of the calculus. See also, by the same author, "Continuous mathematics and discrete computing are complementary, not alternatives", Coll. Math. J. 15 (1984) 389-391 and "Visualizing calculus concepts using a computer", in [STRASBOURG 85] pp. 203-211.

USISKIN 84 Z. Usiskin, "Mathematics is getting easier." Math. Teacher 77 (1984) 82-83.
"Some skills are clearly necessary, but (...) too much else should be learned about mathematics to waste time in practicing obsolete skills. Mathematics in getting easier [with muMATH]. We will not be able to keep this secret from our students forever."

WILF 82 H.S. Wilf, "The disk with the college education." Amer. Math. Monthly, 89 (1982) 4-8.
muMATH is coming! muMATH is coming! A paper intended as a "distant early-warning signal" for the mathematical community.

WILF 83 H.S. Wilf, "Symbolic manipulation and algorithms in the curriculum of the first two years." In The Future of College Mathematics, A. Ralston and G.S. Young, ed., Springer-Verlag, 1983, pp. 27-40.
Expands on the issues raised in [WILF 82]. "It can be very unsettling to realize that what we previously thought was a very human ability (...) can actually be better done by "machines"." (Also contains a description of a second semester sophomore course introducing algorithms.)

WINKEL 84 B. Winkelmann, "The impact of the computer on the teaching of analysis." Int. J. Math. Educ. Sci. Techn. 15 (1984) 675-689.
A basic discussion of the ways the computer capabilities (among others, the symbolic capabilities) will influence the teaching of calculus.

WINKEL 85 B. Winkelmann, "Some remarks on the teaching of elementary calculus in the computer age." In [STRASBOURG 85] pp. 1-7 (Supplement).
"So if it seems possible to master [differential equations] at a more elementary level than hitherto was possible, they should be regarded as the most appropriate final level and goal even for the teaching of elementary calculus at schools and colleges."

YUN & STOUT 80 D.Y. Yun and D.R. Stoutemeyer, "Symbolic mathematical computation." In Encyclopedia of Computer Science and Technology. J. Belzer et al., eds. M. Dekker, 1980. vol. 15, pp. 235-310.
A general discussion of symbolic manipulation systems. Includes a guide to some existing systems and a discussion of basic issues and alternatives for building up such a system. The last 30 pages are devoted to applications: algebra, nonscalar analysis, numerical analysis, celestial mechanics, general relativity, high-energy physics.

Appendix 1

AN INTRODUCTION TO muMATH

(A symbolic mathematics package for micro-computers)

presented to the CMESG meeting

by

G. Morin

Université Laval

Juin 1985

BASIC INSTRUCTIONS FOR THE USE OF muMATH SYMBOLIC PACKAGE

1- Insert the DOS 2.10 diskette in the left disk drive.

2- Put the power on the video screen and on the computer (right- side of the machine).

3- On the screen will appear: "ENTER NEW DATE:" ;just press the "return" key (↵) in response; same thing for the "ENTER NEW TIME:"prompt.

4- Remove DOS 2.10 diskette from disk drive, insert "muMATH 1" disket in that drive and place "muMATH 2" diskette in the right disk drive.

5- Type the word: MUSIMP on the keyboard, followed by a "return" key (↵).

6- Press the key: [Caps Lock] (for the use of capital letters, it's important), then press: [Num Lock] .

7- Following the question mark, type: LOAD (MUMATH); then press the "return" key.

You are now in muMATH.

N.B. In muMATH, always end a sentence by a semi-colon followed by a "return".

A BRIEF SURVEY OF WHAT muMATH CAN DO

Name of file	What it does
ARITH.MUS	rational arithmetic
ALGEBRA.ARI	elementary algebra
EQN.ALG	equation simplification
SOLVE.EQN	equation solver
ARRAY.ARI	array operations
MATRIX.ARR	matrix operations
LINEQN.MAT	simultaneous linear algebraic equations
ABSVAL.ALG	absolute-value simplification
LOG.ALG	logarithmic simplification
TRG.ALG	trigonometric simplification
ATRG.TRG	inverse trigonometric simplification
HYPER.ALG	hyperbolic trigonometric simplification
DIF.ALG	symbolic differentiation and Taylor series
INT.DIF	symbolic integration

```
INTMORE.INT.................extended symbolic integration
LIM.DIF.....................limits of functions
SIGMA.DIF.    ..................closed-form    summation    and
                            products
ODE.SOL.....................first-order ordinary differential
                            equations
ODENTH.ODE..................higher  order  ordinary differen-
                            tial equations
ODEMORE.ODE.................extended first-order ODE methods
VEC.ARR.....................vector algebra
VECDIF.VEC..................vector calculus
```

If you want to see a demonstration of one of the above items, type: RDS (<items's 1st name>,<item's 2nd name>,B);

For example if you want to know how to differentiate with muMATH, type: RDS (DIF,ALG,B); and wait for a few seconds. (140 seconds at most.)

After each demonstration the following will appear: Abort, Break, Continue, DOS?

Don't consider "Break" or "DOS", just press "C" if you want to continue with a different example or "A" if you want to abort the demonstration and do some of you own material (using the same punctuation and orthograph as in the demonstration of course).

N.B.: The "system file" named MUMATH has been built to include all the so-called "source files" above. When you have typed LOAD (MUMATH); as indicated above, you thus have just in the memory all the tools offered by muMATH. But if you want to see a demo, you need to type the RDS command above.

A BRIEF DEMONSTRATION OF muMATH

MuMATH does exact rational arithmetic. Try these examples on the keyboard.

? 1/2+1/3;

@: 5/6

? (-3)^(1/4);

@: 3^(1/4) #E^(#I #PI/4) meaning: $\sqrt[4]{3}\ e^{i\pi/4}$

You can assign an expression or a value to a "name", e.g.

? TOTO: Y+3*X;

@: Y+3X

Now to see that Y+3*X is really assigned to "TOTO":

? TOTO+Y;   N.B. "*" is the multiplication symbol which can often (but not in every case) be omitted and replace by a "space".

@: 2Y+3x

Remember, you can do symbolic mathematics so it is possible to handle variables who don't have values assigned to them.

Here's the trigonometric expansion function, "TRGEXPD":

? TRGEXPD (SIN(2*X),-3);

@:2 COS X SIN X

these parameters tell the machine how to do the expansion.

? TRGEXPD (2*COS(X)*SIN(X),30);

@: SIN(2 X)

If you want to know more about trigomometry on muMATH do: RDS (TRG,ALG,B);

SOME USEFUL muMATH COMMANDS

To do:	Type:
- $\int f(x)dx$ (indefinite integral)	INT (F(X),X);
- $\int_a^b f(x)dx$ (definite integral)	DEFINT (F(X),X,A,B,);

N.B. b can be positive infinity "PINF"
and a can be minus infinity "MINF"

- $\sum_{x=a}^{b} f(x)$	SIGMA (F(X),X,A,B);

where a and b can respectively be "MINF" and "PINF"

- $f'(x)$	DIF (F(X),X);
- $\frac{d^n f(x)}{dx^n}$	DIF (F(X),X,N);

- $\frac{d^{m+n}f(x,y)}{dx^n\ dy^m}$ — DIF (F(X),X,N,Y,M);

n^{th} degre Taylor expansion of f(x) about point A — TAYLOR (F(X),X,A,N);

- Solve poly. equ. P(x) = q(x) — SOLVE (P(X)==Q(X),X);

- Solve poly. equ. P(x) = 0 — SOLVE (P(X),X);

- Solve a system of n linear equations with respect to $x_1, x_2, \ldots, x_n$

LINEQN([equ1,equ2,...,equN],[x_1,...,x_n]);

To solve a differential equation.

For example if you want to solve:

$$(y'(x)+1)y'''(x) = (y''(x))^2$$

```
do: DEPENDS (Y(X));
    DIFVAR: 'X;
    SOLVE ((Y'+1)*Y'''==Y'''^2,Y);
```

N.B. in muMATH e=#E (e.g. LN#E=1)

π=#PI (e.g. SIN(#PI/2) = 1)

i=#I (e.g. #I^2 = 1)

and ARB(1) is the 1st arbitrary constant of an expression
ARB(2) is the 2nd arbitrary constant of an expression

.
.
.

to remove X from being "DIFVAR" and the dependency of Y upon x do:

```
DIFVAR: FALSE;
PUT ('Y,'X,FALSE);
```


Appendix 2

GCEDM 1985

Appendice au rapport du groupe de travail C sur les logiciels symboliques

Charles LATOUR
Département de mathématiques
CEGEP François-Xavier Garneau

PARTIE A

(Séance pratique tenue le 6 juin 1985 sous la supervision de Bernard R. Hodgson, Eric Muller et Gilbert Morin.)

Le document "An Introduction to muMATH" préparé par Gilbert Morin s'est révélé fort utile et tout à fait correct. Mentionnons cependant un petit oubli à la page 4: il faudrait y lire au nota bene
(e.g. # I^2 = -1) au lieu de (e.g. # I^2 = 1).

Au cours de cette session nous avons choisi d'explorer les "démonstrations" suivantes:

1- LOG. ALG. sur les simplifications logarithmiques. Nous n'y avons décelé rien d'inquiétant.

2- SIGMA. DIF. sur les sommations et produits. Cette fois nous avons pu faire une observation quelque peu surprenante. Pour la sommation $\sum_{j=1}^{n} j^2 3^j$, nous avons obtenu une expression de la forme

____ + ____ + ____ + ____ + A - A.

Les termes algébriques de la réponse se ressemblant tous, les A n'étant pas des expressions simples et si on y ajoute la difficulté de lire les réponses sous forme linéaire, il aura fallu un bon sens de l'observation pour se rendre compte de la possiblité de les soustraire. On a utilisé la fonction EXPAND pour faire disparaître les A et récupérer la réponse la plus réduite. On ne s'attendait pas à ne pas obtenir la meilleure réponse à l'intérieur d'une "démonstration". C'est d'ailleurs un problème constant dans l'emploi de ce logiciel de savoir si la réponse obtenue est la plus réduite possible. C'est sans doute un problème de grande taille pour l'étudiant qui apprend et qui ne possède donc par l'expérience requise pour évaluer la réponse.

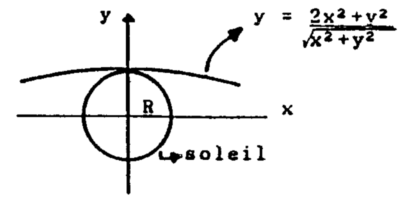
Laissant les démonstrations j'ai suggéré de résoudre le "vrai" problème suivant sur le mode autonome: de la formule

(1) $$y = \frac{2x^2+y^2}{\sqrt{x^2+y^2}}$$

où $y = y(x)$, trouver $y'(x,y)$ et $y''(x,y)$ puis calculer la courbure $K = \frac{y''}{[1+(y')^2]^{3/2}}$

Ce problème se pose en relativité générale. La formule (1) donne l'équation en approximation du premier ordre de la

trajectoire d'un rayon lumineux rasant les bords d'un massif (e.g. soleil). Le contexte physique indique que la courbe (1) devrait avoir l'allure suivante



Il s'agissait pour nous de confirmer l'allure de cette courbe par l'étude usuelle des dérivées première et seconde puis d'en calculer la courbure. Après quelques tâtonnements nous avons procédé de la façon suivante à l'aide de muMATH:

```
DEPENDS (Y(X));
DIFVAR: 'X;      [ligne peut-être superflue]
DIF (Y-(2*X^2+Y^2)/(X^2+Y^2)^(1/2),X);
          ↓
        [peut-être inutile]
C: @;
SOLVE (C==0,y');
C1: (2*X^3+3*X*Y^2)/((X^2+Y^2)^(3/2)-Y^3);
[cette expression est Y'=Y'(X,Y): poser simplement C1:Y'
ne fonctionne pas pour la suite]
DIF(DIF(Y-(2*X^2+Y^2)/(X^2+Y^2)^(1/2),X),X);
[je pense que DIF(C1,X); aurait été plus simple et plus
facile pour la suite]
C:@
```

SOLVE(C==0,Y");

[on obtient

$$Y'' = Y''(X,Y,Y')$$

$$= \frac{3Y^2X^2(Y')^2-6Y^3XY'+3Y^4}{2Y^2X^2(Y^2+X^2)^{1/2}-Y^3X^2+Y^4(Y^2+X^2)^{1/2}+X^4(Y^2+X^2)^{1/2}-Y^5}$$

pour obtenir Y" = Y"(X,Y), je réécris la formule précédente en substituant C1 à Y' et utilise EXPAND]

EXPAND ( );

[on obtient Y" = Y"(X,Y) en 18 lignes!]

C3: [je réécris l'expression pour Y"]

EXPAND (C3/((C1^2+1)^(3/2)));

[ce qui procure la courbure K = K(X,Y) en 3 1/4 pages!]

La façon de procéder évoquée ci-dessus est sans doute très grossière mais elle est juste. Elle a été testée sur

$$y = x^3 \;,\quad K = \frac{6x}{(1+9x^4)^{3/2}}$$

et $$x^{2/3} + y^{2/3} = a^{2/3} \;,\quad K = \frac{1}{3(axy)^{1/3}}$$

L'inconvénient majeur réside dans le fait de réécrire C1 et C3 au long. Je suis raisonnablement sûr (et satisfait) d'avoir obtenu les bonnes expressions pour Y', Y" et K.

J'ai aussi testé le calcul de dérivées plus complexes telle celle de la fonction $y = x^{1/x}$. On n'obtient pas la réponse la plus simple, comme c'est le cas pour $y = x^4$ par

exemple. Il faut utiliser EXPAND.

Au niveau de l'intégration muMATH ne peut prendre $\int \sqrt{x}\,\sqrt{1-x}\,dx$ directement mais il effectue très bien $\int \sqrt{x-x^2}\,dx$ qui est évidemment une forme équivalente. Mais c'est moi qui ai transformé $\sqrt{x}\ \sqrt{1-x}$ en $\sqrt{x-x^2}$! Le logiciel permet-il d'obtenir $\sqrt{x-x^2}$ de $\sqrt{x}\ \sqrt{1-x}$? Si non, l'étudiant ne peut s'appuyer sur le logiciel pour résoudre l'intégrale et doit alors s'entraîner de façon traditionnelle à manipuler les expressions algébriques. Cette situation amène la question plus générale suivante: "Etant donné qu'il est fréquent qu'on doive transformer légèrement les intégrales proposées pour utiliser les tables d'intégration, dans quelle mesure le système muMATH permet-il de le faire?" "Quels moyens offre ce logiciel pour écrire de façon différente une expression algébrique ?"

(A ce propos, on a soulevé au cours de l'atelier du lendemain la pertinence d'utiliser un logiciel de calcul symbolique dans l'étude de l'intégration par fractions rationnelles pour éviter la longue "digression" des calculs algébriques. Mon opinion à ce sujet est que je doute fort que muMATH puisse convertir, par exemple, $\frac{2x+3}{x^3+x^2-2x}$ en $-\frac{3}{2x} + \frac{5}{3(x-1)} - \frac{1}{6(x+2)}$. Je n'ai pas eu le loisir de tester muMATH à ce sujet.)

PARTIE B

(Sur l'utilisation de muMATH en classe ou en laboratoire)

En atelier il fut surtout discuté de l'emploi d'un logiciel symbolique dans l'étude de la notion de limite.

Mon opinion à ce sujet est que si l'on s'en tient à la limite "pure" du type $\lim_{x \to 3} (x^2+4) = 13$ par exemple, alors le logiciel symbolique est à toute fin pratique inutile. Seul le numérique est en jeu. Mais pour la définition de la dérivée par une limite, le logiciel peut se révéler utile. Par exemple

$$\lim_{\Delta x \to 0} \frac{f(x_0+\Delta x)-f(x_0)}{\Delta x} = f'(x_0)$$

pour $f(x) = x^4$ à $x_0 = 2$. L'étudiant peut demander le développement de $(2+\Delta x)^4-2^4$ puis le quotient par Δx (puisque $\Delta x \neq 0$). Ce serait à explorer en laboratoire.

On peut également penser à utiliser ce logiciel pour lever les indéterminations par la règle de l'Hospital.

Plus généralement, l'emploi de tels logiciels pose la question fondamentale suivante: "Les étudiants perdent ils quelque chose (si oui, quoi?) à ne plus s'entraîner à calculer de façon traditionnelle?"

Je crois que oui parce que l'étude des mathématiques et leur utilisation comportera toujours du "calcul" sous une forme ou une autre.

(a) L'élève de 3[e] année qui réussit une division s'exerce à un calcul.

(b) L'élève de secondaire III qui exécute $x^3-a^3/x-a$ s'exerce à un calcul d'un cran plus abstrait.

(c) Plus tard celui ou celle qui montre que $(g o f)^{-1}=f^{-1} o g^{-1}$ pour f et g satisfaisant des conditions appropriées s'exerce encore à un calcul plus abstrait.

(d) Lorsqu'on montre que dans un groupe, tout élément inverse à droite est aussi élément inverse à gauche, on calcule encore à un niveau plus abstrait.

(e) Lorsqu'on démontre les trois théorèmes d'isomorphisme en théorie des groupes, on calcule toujours (à mon avis du moins) à un niveau encore plus abstrait.

(f) Je soutiens que même en topologie, on "calcule" encore d'une façon particulière à un niveau plus abstrait.

Je suis d'avis que les mathématiques restent essentiellement une étude des formes de calcul (dans un sens large et avouons-le vague!)- la géométrie élémentaire pouvant faire exception quand elle n'est pas modelée par l'algèbre linéaire ou la théorie des groupes.

En conséquence l'étudiant chez qui l'entraînement au calcul sous forme traditionnelle aura fait défaut au profit d'un effort accru au niveau de la conceptualisation, de la mise en équation, de la "mathématisation" des situations présentées pourra, selon moi, souffrir de carences à un niveau supérieur, où l'on ne peut plus reléguer les calculs à la machine. En résumé, si dans un cheminement mathématique qui va de l'élémentaire à l'université, l'étudiant tronque les premières étapes de son entraînement au calcul - étant devenu surtout "spectateur" au plan "calculatoire" durant ces premières étapes, qui peuvent s'étendre jusqu'au collégial - il devient vraisemblable qu'il manifestera une faiblesse dans l'éxécution de toutes manipulations symboliques (lesquelles, rappelons-le, sont inévitables dans l'étude des sciences avancées.)

J'ajouterai que les démonstrations mathématiques (de préférence les longues) offrent une occasion singulière de former de longues chaînes de pensées ou d'idées en devant s'assurer d'un lien solide entre chacune. Je soutiens que cette habileté à "former des chaînes" est fondamentale à bien des égards dans l'exercice de la science. La mise en équation et la conceptualisation, bien que très importantes, n'offrent pas un tel intérêt de ce point de vue. Quant à la possibilité d'acquérir ailleurs cette habileté (en jouant aux échecs par exemple), je réponds qu'il est préférable que le cycliste acquière la forme sur son vélo plutôt qu'en nageant ou courant. (D'ailleurs, aux échecs, le lien entre chaque mouvement n'est pas toujours aussi étroit et solide que dans une démonstration mathématique).

En conclusion, je poursuivrai sûrement ma réflexion sur l'emploi de ce type de logiciel dans l'enseignement des mathématiques. Je le ferai autant par goût que par nécessité.

Non seulement il serait souhaitable que de tels logi-

ciels soient jumelés à des logiciels graphiques mais aussi à des logiciels de traitement de textes mathématiques, lesquels permettraient peut-être l'écriture "normale" des expressions algébriques dont le déchiffrement devient très pénible lorsqu'elles dépassent en longueur plus de deux lignes.

Appendix 3

Réaction au groupe de travail CMESG/GCEDM 1985
"INFLUENCE DES LOGICIELS A CALCULS SYMBOLIQUES SUR L'ENSEIGNEMENT DU CALCUL DIFFERENTIEL ET INTEGRAL"

Noëlange Boisclair

Cégep Montmorency

PREMISSES AUX COMMENTAIRES

Constats sur l'enseignement actuel des premiers cours de calcul: universités, colleges et cégeps

THEORIE: approche axiomatique et formelle
(cf: livres de référence des étudiants)

APPLICATIONS: prépondérance pour le calcul numérique à tendance acrobatique

ANALYSE: souci académique très vif mais, carence dans l'enseignement supérieur
(les mathématiques du secondaire et le développement de la pensée formelle s'harmonisent sur de courtes durées; exception peut-être pour la 12^{e} année du High School)

SYNTHESE: apprentissage confié à l'étudiant responsable de sa formation
(l'université lègue cette préoccupation au college ou au cégep; le college ou le cégep lègue au High School ou à la polyvalente;
de là, l'instance responsable devient muette!)

CURIOSITE ET EMERVEILLEMENT: efficacité et rendement s'approprient les honneurs du cours
(tout étudiant écoute une certaine musique; tout étudiant découvre les séries harmoniques; qui fait u . lien?)

COMMENTAIRES SUR L'ATELIER

Aucun participant n'étant spécialiste de l'incidence des logiciels à calculs numériques dans l'enseignement du calcul différentiel et intégral, il en ressort que les nomoreux thèmes soumis par les intervenants représentent une ébaucne intéressante qui nécessite cependant une classification suivant leur caractère pédagogique et leur priorité d'insertion dans une séquence d'apprentissage.

Toutefois, discuter du concept de LIMITE et/ou d'INTEGRATION tel que nous l'avons fait, m'apparaît une approche d'un dynamisme décroissant à court terme car, elle perd de vue la structure globale du calcul.
A mon sens, la LIMITE est comme un architecte, elle crée, elle engendre entre autres, la DERIVATION et l'INTEGRATION et elle se veut leur composante inhérente.

De ceci, ma réaction aux discussions est que l'on a eu tendance à protéger sa vision personnelle du calcul tout en manifestant un vif intérêt àvouloir répondre aux nouvelles exigences scientifiques. Comment?
A mon avis, on a exploré des moyens de moderniser les cours en gardant à vue le rythme traditionnel des concepts tels qu'enseignés aujourd'hui.

Nonobstant cette remarque, cette concertation a eu un aspect positif dans le sens qu'elle a répondu à une nécessité de poser une base de discussion qui se veut l'amorce d'une réflexion plus articulée au prochain congrès. Je partage l'idée qu'un tel débat mérite une démarche prudente et réfléchie.

SUGGESTION

Il me semblerait intéressant d'orienter le débat autour des besoins distincts qui rendraient pertinente l'utilisation des logiciels, qu'il s'agisse de MUMATH, de MACSYMA, de MAPLE ou d'autres.

Disons, en guise d'exemples, qu'un logiciel pourrait être considéré sous divers aspects, soient:

un outil pour alléger l'enseignement des notions reconnues acquises dans les cours préalables
v.g.: manipulation algébrique
domaine et image de fonctions élémentaires

un outil pour développer une représentation speciale des êtres mathématiques
v.g.: graphiques statiques
graphiques dynamiques; mouvement des ε et δ
familles de courbes

un instrument pour soutenir et/ou prolonger l'enseignement
v.g.: esquisces d'analyse
proposition de synthèse
interprétation des valeurs numériques

Et après, qu'est-ce qu'on met la-dedans?... un peu de génie et.. beaucoup de créativité à l'épreuve.

NOELANGE BOISCLAIR
collège Montmorency

Appendix 4

The Canadian Mathematics Education Study Group

Laval 85 Meeting

A Personal Report from Working Group C

The impact of symbolic manipulation software on the teaching of calculus.

Edgar R. Williams

Memorial University of Newfoundland

I suspect that for some of us in Working Group C, our first two sessions could be more appropriately labelled as Learning Group C. We did try to come to some conclusions during the last session and overall, I can honestly say that, for me, the learning, the discussions and the product of Working Group C made it one of the most fruitful and interesting sessions that I have attended.

Without going into a lot of detail, I would like to summarize some of the conclusions that I drew from this session. In what follows, the abbreviation CAS will refer to Symbolic Manipulation Software Programs or simply Computer Algebra Software (CAS).

1. CAS has the potential to provide Professors with the opportunity to spend less time in the classroom illustrating routine but time consuming computations and more time on deeper and more exciting Mathematical concepts.

2. CAS also has the potential to provide students with the opportunity to spend less time on lengthy and time consuming paper and pencil computations, as is normally required on assignments, and to spend more time doing real mathematics.

3. Simply put, CAS can be used as a tool to alleviate computational drudgery and allow more complex examples to be introduced and studied.

4. CAS can be used by both students and professors to check answers to assigned or completed homework.

5. CAS can be used to automate part of a task, for example, the computation of Taylor Series when the task is to examine questions of convergence, etc.

6. More examples can be done and done successfully when the computer takes over the chore of routine computation.

7. For exceptional students, CAS may permit the introduction of Calculus and other areas of study much earlier than is possible at the moment.

8. With the possibility of incorporating graphics capabilities into a CAS system, it may be possible to illustrate many concepts geometrically right before the students eyes in a very dynamic and interactive way.

9. CAS has the potential to improve student attitudes toward mathematics especially for those of average ability or below.

10. CAS has the potential to permit us to re-establish the importance of creative thought and problem solving in the mathematics curriculum.

11. The present generation of CAS Systems were developed for the use of Scientists and Engineers. However, with potential developments in Artificial Intelligence, the future potential for improvement in CAS designed for educational purposes, seems enormous.

12. CAS has the potential to provide opportunities for more individual attention to those students who need it.

13. Successful mathematics students today appear to learn by being "programmed by example", i.e. after observing enough examples, a methodological technique is inferred. Unfortunately, many (unsuccessful) Mathematics students never infer such rules, do not infer them correctly, or in some cases, never even realize such rules exist. CAS has the potential to convince weaker students that such rules exist and that even a dumb machine can be programmed to carry them out.

14. CAS can be used to provide enrichment and motivation in the mathematics classroom.

15. CAS has the potential to permit students to do exploratory mathematics on a scale never before possible.

16. CAS can be extended to include automatic drill, testing, and record keeping, an obvious advantage to those of us who have better ways to spend our time.

17. What are we going to do when many, or most, or perhaps all of our students will be able to come to class with a relatively inexpensive hand held computer with CAS capabilities? We must answer that question now. Otherwise, our students will answer it for us.

Appendix 5

Some thoughts on the "Impact of symbolic manipulation software on the teaching of calculus"

D.W. Alexander

What routines are unnecessary for understanding?

What routines are necessary for understanding?

How can the graphic capabilities and symbolic manipulation potential of computers be best used to enhance learning (of calculus)?

How might the availability of symbolic manipulation software (and graphics) effect priorities, order?

Can these be used to promote understandings, open-endedness a la Pollack?

How does this relate the Whitehead's cycle: romance, precision, and generalization?

Suggested sequence:

1. Graphical introduction to derivative:
   chords to tangent; "window" on screen;
   associate slope of tangent at a point;
   exploration - generalization for specific function,
   "derivative" (i.e. slope of tangent at any point).

2. Symbolic manipulation code for derivative
   Maximum/minimum problems
   - approximation (graphically)
   - precision (using derivative code)

Should problems be limited to polynominals or would students "understand" derivatives of other functions?

Should equation solving capacity of symbolic manipulation be used?

Is there need to explore second derivatives or does graphical capacity remove that need?

Could second derivative tests be introduced as a means of confirming computer graphs? (reasonableness of answers)

What other aspects of "curve sketching" techniques are still appropriate assuming availability of graphics packages?

Should inverse differentiation (differential equation) problems be introduced?

3. Generalization: Explorations of derivatives as given by symbolic manipulation to give $y' = nx^{n-1}$; derivative of sinx = cosx; derivative of cosx = - sinx; derivative of sinax, etc. Is this the time to introduce limit ideas as a basis for proof?

4. Other "Rules of Derivatives"
   - Derivative of a Sum
   - Product Rule for Derivatives
   - Derivative of Quotient
   - Chain Rule

Given the symbolic manipulator, how much of this is needed?

Could it be motivated by "need" to know how to get the results without the "black box"? By a desire to "understand" how the derivatives are obtained?

Would this be "optional" and only done with some students?

A fundemental issue: Do we desire to teach calculus as a "rigorous" development with the need for "proofs" or is our goal to use calculus in solving problems?

- If the latter, then 4. and perhaps 3. are unnecessary. (Is it only my conditioning that makes me suspicious of this conclusion?)

WORKING GROUP D

THE ROLE OF FEELINGS IN LEARNING MATHEMATICS

GROUP LEADERS:

JOHN POLAND

FRAN ROSAMOND

MATHEMATICS AND FEELINGS

List of Participants: Dorothy Buerk, Renee Caron, Claude Gaulin, Lars Jansson, Bill Higginson, John Poland, Pat Rogers, Fran Rosamond, Ralph Staal, Peter Taylor.

We began this working group by explaining that although there is a lot of literature touching on the role of feelings in learning mathematics, there is almost nothing directly on it. This is an important area to understand and we must rely strongly on personal examples shared in the group.

Next, each participant introduced him or herself to the group, explaining his or her connection to this workshop. This process of going around the group to share was a key component of the dynamics of our working group. The following excerpts from some of the introductions indicate the wonderful collection of minds and experiences in this group.

- Mathematics is connected with feeling the power of looking at new and significant ideas. There is the thrill of invention, of being able to name, of making up new words.
- There is the feeling of exploration and of uncovering new and exciting things. There is the eureka experience, the feeling of curiosity, of challenge, of aesthetic, and the philosophical side of uncovering real basic truths.
- I would like to see how the enthusiasium of the teacher can influence students in the classroom.
- I have team taught a math class with a poet. Half the class was spent in analysing a piece of poetry. The other half was spent analyzing a math problem. I would like to explore the feelings that are common to poetry, music, math.
- I see that the beginner's view of math is far different from the mathematician's vision and I would like to explore how to open up the latter vision to the beginners.
- I am interested in how the environment influences us. Also I would like to try to be specific about which feelings we pay attention to.
- There are not many people at my school with whom I can discuss these ideas. I feel isolated and would like this workshop to be the beginning of a support group.
- There is no such thing as non-emotional motives. People seem less inhibited to express feelings about music. Large groups of mathematicians love music. Is there a complimentarity here?

As a possible framework for our topic we introduced and discussed the Perry development scheme. Several handouts on the scheme that are attached to this paper were kindly supplied by Dorothy Buerk. As these indicate, it seems that at the college level we tend to teach to reinforce level 1 perceptions and expect students to evolve to level 4. Early levels collude with a view of the world that what is correct is restricted to ones immediate family, peers, school and is reflected in statements such as, "My teacher last year didn't do it that way."

In this regard, Lars Jansson drew our attention to articles discussing the problems of beginning teachers (see bibliography). We discussed but left unresolved whether emotions and feelings are more important at lower Perry levels than at higher levels. Does a change in pedagogy equate to a change in the Perry level of students perceptions? The experience of many in the group was that a feeling of community and caring in the classroom play an important role in Perry development. The role of community continued to be a theme on successive days.

Participants were asked to attend the Topic Group on Problem-Solving by Peter Taylor that afternoon. They were to keep careful track of their feelings during this experience. When we met on the second day, the sharing of these feelings was a great stimulus. Pat Rogers described how talking about feelings and the need to be aware of them, validated the experience of having them. This validation helped uncomplicate situations when negative or confusing feelings arose.

One feeling for example, was Pat's anger with her peers who were model students for the teacher during the Problem-Solving session. There also was an anger with the teacher for insisting on receiving his own answer from the students. The feeling of anger formed a block that kept Pat from being an involved participant in the Problem-Solving session. Others in the workshop described feelings such as anger, confusion, isolation, or competition. The owners of these negative feelings described being turned off or disengaged during the Problem-Solving session.

The sources of the negative feelings could, in some cases, be traced to specific incidents. Male-female differences were discussed in this context. It was noted that in grade five, across all subjects, research has shown that teachers pay more attention to the male students. Discussion shifted to the bad press mathematics has in general. An argument was made that boys who were not particularilly athletic could be accepted by peers if they excelled in math. Many questions such as the following were raised. Does mathematics always assume one's worst? Is this related to an authoritative perception of math? Is math unusually strong in the feeling of self-worth?

Words such as "cool", "controlled" have been used to describe mathematics. These often convey a remoteness or unconnectedness on the part of the learner. We discussed, "How have you used mathematics to suppress feelings?" In response to this question the image of "cool" was pleasant and positive. John Poland described his ability to focus attention on mathematics and thus distract himself during a painful illness. Fran Rosamond mentioned that she enjoyed mathematics as an adolescent because thoughts about mathematics could crowd out thoughts about sex. Others commented that mathematics is a way to remove oneself from interaction with peers.

Day two concluded with many of us eagerly describing characteristics of the best teachers we have known. Fran Rosamond felt it imperative that we also recognize and share our own successes. To this end we began the third day by spending two minutes, in pairs, talking about "Why I am a good mathematics teacher." Returning to the group, we spoke in turn about what we had heard our partner say that struck us as important to good teaching. Some of the conversation follows.

- With our students:
  We must make maximum effort to involve all and avoid preoccupation with just the bright students.

  When students come in to office hours I go over the next days lession with them. Then in class the next day they join the discussion because they have had a preview. This also helps me find their misconceptions in advance.

  To deal with disappearing students, I have the class split to small groups and then report back.

  I try to involve the students using modified Moore method. There are weekly assignments leading to big results. I play it by ear to give just the right amount of challenge and hints so the results become theirs.

  I want the students to learn to think wilder in the future. We brainstorm in class. When a person suggests an idea, that person is the idea. Rejecting the opinion is rejecting the person. In brainstorming, no ideas are evaluated until all have been listed and a sense of community has developed in the group.

- In the classroom:
  Provide closure. At the end of every hour point to the positive accomplishment, if it is only the asking of a good question. Look forward and backward in the class.

  I use two overhead projectors. One is used with prepared overheads and is a way for me to convey my enthusiasm with the math and also look at the students. The other is used to write spontaneously on.

  When discussing preassigned problems, keep posing other interesting problems that come up. Let these become optinal homework problems.

  I work with colleagues in team teaching. An English teacher jointly teaches my math class. We each take half of a three hour class. The English prof discusses what makes a poem work. Then I discuss what makes a math problem work. There is criticism of the writing experience as well as of the math problem-solving.

  I relate what we are studying in math to other areas in math. Take a problem and approach it from several areas in mathematics. Students sometimes resist discussion of biographical, historical or cultural aspects of the subject. They limit what they want math profs to talk about. Its as if they feel we have changed the ground rules on them.

  I give marks for attendance. I assume that progression or growth depends on attendance. If an exam seems hard, I look at the marks of those who attend regularily.

  I give feedback periodically in the what looks like a quiz but it is not for a grade. The students write out what they feel is important. I give feedback.

- As a teacher:
  There has to be harmony between being totally egocentric or totally out-going. The teacher must be in charge of what is going on in the classroom. Also, the teacher must listen to what the students say and how they say it In this way the teacher can hear misconceptions. The teacher can build on students' past experience.

  The teacher must hang on to a genuine egocentricity. Students don't want a teacher who disappears into the background. Students want to hear what you are saying because you've got something. Be yourself; that is what you have the most of. Concentrating on others requires a detachment from yourself.

When I go into a professors office to ask advice, usually he or she has a special personal metaphor with which to explain the concept. We should share our metaphors in the classroom. Talk about mathematics as if you are talking to a person while walking by a lake or while on a stroll through the woods.

It is important to be upfront about what we are doing. Say that we expect our students to move through the curriculum by first being able to do problems of one or two steps. Eventually they should be able to read on their own and enjoy what mathematics has to offer. This can be written in a handout and said in class.

It is important to build on students past experiences.

I accept with good grace my own mistakes. The ideal course is not one where teacher never makes mistakes.

Class must function as a support system. This must be clear to the student so there is no fear to opening-up. I begin first class with lots of self-disclosure and time in small groups. This lays the groundwork for discussion of feelings.

A sense of community was a dominant theme among our suggestions. Community provides safety and belonging. This empowers students. This allows them to be in contact with themselves, to know themselves. We see the classroom as a 'happening' (as in the '60's). We envisioned the superior high, the communal high as of passing the mathematics.

Time passed far too quickly and we have much to discuss. We are especially interested in which emotions belong strictly to mathematics and cannot be avoided because of the nature of the subject. Where or when do we see the "Ah ha" experience in ourselves, our students? We also want to explore strategies for effective teaching that build community. Our main goal is to empower our students.

Appendix to the Report of Working Group D

June, 1985

U. Laval

(R.A. Staal)

One of the by-products of working in this group was a heightened appreciation of the importance (and existence) of emotive aspects of learning in mathematics which have their source outside of the mathematics itself.

Within mathematical activity, there are numerous examples of what we might call "emotive" factors - while not strictly part of the mathematics, they are inseparably connected with it, and reflect the essential nature of the total mathematical experience. A few examples are: "Eureka!"; various forms of aesthetic satisfaction (pleasure at reducing the apparently complex and instructured to a simple, structure; appreciation of a beautiful and ingenious proof...); the feeling of security in dealing with a "clean", well-defined structure with clear criteria of success; the excitement and suspense of exploration; the sense and stimulation of mystery; the "down side", of frustration ("why doesn't this work", "why couldn't I have seen that?" I just wasn't meant to be a mathematician") etc. These examples are all pretty familiar, and come to mind rather easily.

At a less purely mathematical level, there are emotive aspects arising from interactions of mathematics with other subjects(Newton at the seashore). These are hard to list in a systematic way, and didn't surface in our discussions.

To come to the main point of this note: there are emotive aspects of the classroom experience which have nothing especially to do with mathematics per se - they apply to the classroom, rather than the subject - but their influence on the learning of whatever the subject might be isn't always adequately kept in mind. They have to do essentially with personal-interpersonal matters, and include such things as: participation in the development of material, participation as a functioning member of a group, getting approval versus being put down, being considered important as a person.

It is to be emphasized that here we are concerned with the role of these aspects in the learning of mathematics, and have no intention of following the path in which concern for "the whole child" is expressed via a de-emphasis on the learning of a subject.

The role of the teacher is brought to the fore in this. Self-study, using library materials, and computer-assisted-instruction (both of which from time-to-time are touted as in the forefront of educational progress) leave this aspect of learning virtually untouched, unless, of course, used as a supplementary tool at the hands of a teacher. A corollary of our thesis, then, is that the teacher is uniquely important.

The following description of four levels of teaching mathematics fits the above comments into a broader scheme.

Level I

The subject matter is presented, in logical form (Definition, Theorem, Proof), examples are worked out, problems are assigned and solutions corrected, and examinations are conducted.

Level II

As in Level I, but enriched by the addition of background material (biographical and historical material included), mathematical motivation and interconnections with other topics and subjects.

Level III

As in Level II, but in addition the students are brought into the picture as participants in the mathematical activity. (The details are fairly obvious - Socratic and similar approaches, the use of open and exploratory assignments, etc.)

Level IV

As in Level III, but, in addition, the students are considered more fully as persons, and the emotive aspects of the classroom environment are taken into account as part of the process of learning mathematics.

The Dualistic View

Prepared For

Project MATCH Conference (Davidson College)

6/19/85

By Dorothy Buerk
Ithaca College

Let us look more closely at the beliefs of those holding a Perry 2 view of mathematical knowledge. Students holding this view will have a number of the following beliefs:

- Right answers are known by an authority for all mathematical questions. There are no unsolved problems, and no multiple answers. Right answers are passed on, not created by the authority.

- There is one right method to attain the right answer and while students may be asked to find it for themselves, they know they are being asked to use THE method to find THE answer.

- Mathematics is learned by memorization and hard work and by doing every problem that is assigned, while following literally each instruction that the teacher (or the textbook) gives. We know how much practice they need.

- One is either good at mathematics or bad at mathematics. If you are good at it you will catch on very quickly. Otherwise you will not. (This is in contrast to the notion that one can come to understand over time.)

- One does not act on a problem and one does not bring one's experience to a problem. One brings the methods that have been taught for similar problems. Even the authorities learn this way.

- The student's role is to collect facts, not act on them, but to store them as they are received. One does not use one's intuition.

- There are no gradations of truth - no gray areas.

- The authority (teacher, textbook, etc.) is responsible when a student lacks knowledge.

- A broad education isn't necessary, since it "won't do me any good on my job."

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Strategies to Enhance Learning

By Dorothy Buerk
Ithaca College

- Provide time to experience and clarify a problem (situation) before focusing on solution. Let each person think about the question before anyone speaks. Respond to questions about interpreting the problem. This would include providing background for applications outside of the student's field. Focus on resolution only after each person "sees" the problem (question) clearly.

- Include the historical perspective to help students become aware of the person-made quality of mathematics. Concepts as "simple" as zero and negative numbers were controversial and adopted with great difficulty and yet students are expected to accept them without question.

- Acknowledge and encourage alternative methods and approaches, approximation, guessing, estimation, partial solutions, and the use of intuition.

- Answer questions with questions that both clarify the students' questions and that help the students realize their own potential as problem solvers and problem posers.

- Encourage students to share ideas, partial solutions, and different interpretations of problems with each other. Establish a workmode that encourages collaboration and the pooling of ideas to reach solutions and/or new questions. Sharing authority in the classroom is invaluable to the improvement of student learning.

- Encourage the asking of new questions and create an atmosphere where both teacher and student are free to wonder out loud. Students need to see their teachers asking, thinking, puzzling, and conjecturing out loud in class.

- Make concerted attempts to avoid absolute language.

- Set as a goal the development of each student's internal sense of power, of confidence, and of control over the material. Help students to realize that mathematics can be learned by thinking, not only by memorizing.

- Offer opportunities for students to reflect on paper about their ideas and feelings about mathematics. Often after acknowledging negative feelings and reactions a student can move on as if relieved of a burden. Writing out one's thoughts often brings a deeper clarity and/or a new insight and with these come a new sense of confidence.

- Don't rush closure. It is important to continue to think about a topic, a problem, an idea, a question, and even a possible answer, and often to leave resolution until the next or an even later class.

Our Expectations of our Students

Prepared For

Project MATCH Conference (Davidson College)
6/19/85

By Dorothy Buerk
Ithaca College

This year in talking with college mathematics faculty about their expectations of their students, I have concluded that mathematics students are expected to be:

- analytical and able to think quantitatively and deductively.
- able to do proofs, a skill which requires bringing together disparate bits of knowledge and seeing the situation from several perspectives. Working in both "top down" and "bottom up" modes is often necessary to complete a proof.
- able to see the relevance of applications to theory and theory to applications and, in addition, to understand the connections between theories.
- able to use problem solving heuristics to approach non-traditional problems and to have the patience to try out several approaches - to stay with a difficult problem.
- able to realize that one's intuitions are important and need to be respected; that these intuitions can be misleading and need to be tested against a theory or with evidence.
- able to learn on their own and from each other; have the internal sense of processing necessary to do that.
- able to make reasoned guesses, conjectures, and to estimate results in the process of inquiry.
- able to ask good questions - especially new ones (problem posing).
- respectful of the power of mathematics, but still willing to experiment, to try out ideas that may not work.
- able to write good definitions and to use them - to pull out relevant information and to be complete.

Proposal for 1986 Working Group on Feelings and Mathematics

The 1985 Working Group on Feelings and Mathematics initiated concept analysis to identify the meaning, role, function and workings of affect in mathematics instruction. In 1986 we propose to further develop the analysis of how affective issues are related to mathematics learning and teaching and to outline a theoretical framework to guide research in this area.

While most research in mathematics education and problem solving has focused on developing information-processing models of purely cognitive systems, there has been considerable recent recognition that affective dimensions are integrated with and stimulate the cognitive. Emotions and belief systems are two of the twelve major issues that Norman (1981) asserts needs to be addressed in future research in cognitive science. Noddings (1985) urges careful attention to the language and affect of instruction and says we badly need comprehensive and meticulous studies of affect in mathematics classes. In describing the implications from recent research on mathematics teaching for future research and policy, Good (1984) emphasizes the need to examine systematically how teacher belief systems and student belief systems in small-group and whole-class settings influence learning.

The immediate descriptors of affect are the physiological signs such as flushed cheeks, muscle tension or rapid heartbeat. McLeod (1984) has related to mathematics problem solving Mandler's theory that emotion results when an individual's planned behavior is interrupted. Mandler's theory is useful but may need to be expanded to include emotions such as surprise, relief and Ah! Ha! Eureka! described by participants of the 1985 Working Group. Emotion also is evoked by unconscious association of present activity with past events. Recall of early childhood memories is one way to raise level of awareness.

A cognitive interpretation of affective behavior will include the influence of belief and value systems. We will use the forms of intellectual and ethical development formulated by Perry (1970) as a first model of how student beliefs, feelings and learning are related. Work of Rosamond (1984), Buerk (1985) and Copes (1982) will demonstrate the relevance of Perry's scheme to specific mathematics courses.

Cognitive interpretation also will include elements of managerial decision making during problem-solving as described by Schoenfeld (1985). Concerns such as that voiced by Brown (1984) about student ability to generate mathematical questions and by Rosamond (1981) about making meaning and feeling the significance of mathematical ideas will be integrated into the framework. Also to be discussed is the effect of emotion on memory and retention.

A major impetus to modification of learning behavior is the learner's level of awareness. With a low level of awareness a learner may react automatcally to certain emotions while a higher level of awareness allows the learner to choose appropriate behavior. The Perry development scheme, problem-solving courses such as Schoenfeld's and texts such as that by Mason, Burton and Stacey (1982) will be examined to ascertain the pitfalls and benefits of efforts to influence level of awareness.

We recognize that we need one-on-one research in laboratory conditions to help us describe and characterize those aspects of affect that impact on student learning behaior. Such "ideal" conditions can skew findings however, and for classroom mathematics learning to be improved, methodology must be developed to examine affect under conditions of large-group, classroom instruction. The observations of Taylor( ) and Poland (1985) will be used to guide development of such methodology.

A matrix of our proposed work then, would include the following:

INDIVIDUAL	X	SETTING	X	PHYSIOLOGICAL STATES	X	COGNITIVE INTERPRETATION
student		specific or general math content		sensations		belief and value systems
teacher		social content 1-1, small or large group		level of awareness		decision making strategies

Research methodology will be developed to test and elaborate the framework and thus lead the way to formulation of classroom policy. The scope of work outlined in these paragraphs is far too large to be accomplished during the duration of one working group. This indicates our direction however, and we expect to make major progress.

BIBLIOGRAPHY

Brown, Stephen I. "The Logic of Problem Generation: from Morality and Solving to De-Posing and Rebellion," For the Learning of Mathematics, No. 4, February 1984, pp. 9-20.

Buerk, Dorothy. "The Voices of Women Making Meaning in Mathematics," To appear in Journal of Education, Fall 1985.

Copes, Larry. "The Perry Development Scheme: a Metaphor for Learning and Teaching Mathematics," For the Learning of Mathematics, Vol. 3, July 1982, pp. 38-44.

Good, Thomas L. "Recent Studies of Teaching: Implications for Research and Policy in Mathematics Education," Invited address to the Special Interest Group in Mathematics Education of the American Educational Research Association, New Orleans, 1984.

Mason, John & Burton,Leone & Stacey, Kaye. Thinking Mathematically. Addison-Wesley Publishers Limited, 1982.

McLeod, Douglas B. "Affective Issues in Research on Teaching Mathematical Problem Solving," in Teaching and Learning Mathematical Problem Solving: Multiple Research Perspectives, Edward A. Silver, Editor, Lawrence Erlbaum Associates, 1985.

Noddings, Nel. "How Formal Should School Mathematics Be?", Invited address to Special Interest Group in Mathematics Education of the American Educational Research Association, Chicago, 1985.

Norman, D. A. "Twelve Issues for Cognitive Science," in Perspectives on Cognitive Science, D. A. Norman, Editor. Norwood, N.J. Ablex, 1981.

Perry, William G., Jr. Forms of Intellectual and Ethical Development in the College Years: A Scheme. New York: Holt, Rinehart and Winston, 1970.

Poland, John. "A Modern Fairy Tale?," Department of Mathematics and Statistics, Carleton University, Ottawa, Canada, K1s, 5b6, 1985.

Rosamond, Frances. "Cognitive Change in Adult Women Enrolled in Basic Mathematics Review," Presentation for Special Interest Groups of Mathematics Education and Research on Women at the American Education Research Association, New Orleans, 1984.

Rosamond, Frances. Listening to Students in the Cornell Mathematics Suppport Center, Doctoral Dissertation, Cornell University, 1981.

Schoenfeld, Alan H. "Metacognitive and Epistemological Issues in Mathematical Understanding," in Teaching and Learning Mathematical Problem Solving: Multiple Research Perspectives, Edward A. Silver, Editor, Lawrence Erlbaum Associates, 1985.

Taylor, Peter. A Red Book for Queen's, Department of Mathematics and Statistics, Queen's University, Kingston, Ontario K7L 3N6, 1976.

Taylor, Peter. Exploratory Problem Solving in the Classroom.

Peter says he is happy to have anyone write for copies of these books. The Problem Solving book contains 6 problems he used with grade 12 and 13 (good students) plus references and discussion.

TOPIC GROUP A

EXPLORING PROBLEM SOLVING IN THE MATHEMATICS CLASSROOM

BY: PETER TAYLOR

Is Factorization Unique?

This was an exercise in cooperative theorem discovery, formulation and proof. What may not be clear from the following review of the session is that often considerable time was spent playing with the formulation and proof of the current result to make it satisfactory to me and to the group.

The topic was the uniqueness of factorization of natural numbers. I started by reviewing the notion of prime number and ensuring everyone was familiar with the process by which a prime factorization is obtained. I then talked about the amateur Canadian mathematician J.P. O'Reilly (1867-1949) whose hobby for many years was playing with large primes. In 1915 he discovered by chance that if he multiplied the primes

$$p_0 = 2648552897$$

$$q_0 = 9133228103 \quad .$$

Together, the resulting number n_0 was divisible by 19. He realized right away that if he factored the quotient $n_0/19$ he would get a second factorization of n. This he did, obtaining

$$p_0 q_0 = n_0 = 19.73.223.727.1481.2161.33613 \quad .$$

The number n_0 has been written as a product of primes in two different ways with no primes in common. This was a revelation to O'Reilly because it was at that time generally supposed that prime factorizations were unique (up to order); indeed this was known to be true for reasonably small numbers. O'Reilly's discovery received some attention from mathematicians, and for many years, n_0 was the only number of this type known. Indeed the following definition is now standard.

Definition. An O'Reilly number is a number with at least two disjoint (no primes in common) prime factorizations.

I asked the class for another example of a number with two not necessarily disjoint factorizations. After a moment they agreed that every multiple of n_0 had this property. They formulated:

Theorem 1. If n is an O'Reilly number, then for any k, kn has 2 different prime factorizations.

Happily, someone asked about the converse.

Theorem 2. If n has two different prime factorizations, then n is, or is a multiple of, an O'Reilly number.

It took a few moments to find the simple proof of this, based on cancelling common primes of the two factorizations.

At this point, one or two students declared some confusion. Is it

not the case that all numbers have only one factorization? I explained that while this was indeed the case for the numbers one met in everyday life, it can evidently(!) fail for large numbers. Indeed our task this session was to discover just how widespread this failure might be. One young man, Ian by name, was not satisfied. He insisted that 19 had to divide either p_0 or q_0. That cannot be, I replied, they are both prime. Someone had a calculator which took 10 digits and verified that 19 indeed was not a divisor of p_0 or q_0. The youth became confused and angry. (I knew him to be one of the brighter and more active members of the group.) O'Reilly must have made a mistake; 19 cannot divide into n_0. I patiently explained that although I had not checked this myself, such an error would surely have been noticed by now. He persisted; he was sure that factorization was unique. How do you know, I asked. He could not say. His fellows were embarrassed for him and asked him to sit down. He did but he was upset.

Someone asked whether all O'Reilly numbers were as big as n_0. Are there any smaller ones? I answered that although others have been found, they are all bigger than n_0. Indeed an American mathematician W.P. Smith used a computer in 1952 to verify that all numbers less than n_0 have unique factorization; n_0 is the smallest O'Reilly number.

Of course, I continued, it is not pleasant to have numbers for which unique factorization fails, and it is important to try to understand what it is about these numbers which gives them this property. The above two theorems tell us that to understand such numbers, it is enough to understand O'Reilly numbers. The task I am proposing is to find some theorems about O'Reilly numbers, which elucidate their properties.

To start them off, I suggested

Theorem 3. An O'Reilly number cannot be even.

We spent some time finding and being careful about the proof, for I knew that this was to serve as a model for other proofs to come. It seemed natural to start by contradiction. Suppose n is an even O'Reilly number. Since n is even it has a factorization which contains 2, but since it has two disjoint factorizations, it must have one that doesn't contain 2. Thus $n = p_1 \ldots p_k$ where the p_i are odd. But the product of odd numbers is odd. So n is odd. Contradiction. Actually, if you look carefully at this proof, you will notice that it does not really have to (should not?) proceed by contradiction, but can be done more elegantly directly. I will write subsequent proofs in this direct mode, though the ones produced in class were always by contradiction.

I asked for another theorem of this nature. The one I got was

Theorem 4. An O'Reilly number cannot end in 5.

The proof proceeds as above. An O'Reilly number n must have a factorization that does not contain 5. The primes in this factorization cannot end in 5 (or they wouldn't be prime). So n is the product of numbers which don't end in 5, and so can't end in 5 either.

The proof hinges on the fact that the product of two numbers which don't end in 5 can't end in 5, and I asked how they could be sure of this. They replied that one just had to check the possibilities. The key point is that multiplcation has the property that if you know the last digit of two numbers, then you know the last digit of their product. So you draw up a "last digit" multiplication table for numbers not ending in 5.

It was pointed out that, because a number ending in 5 is odd, the table need only be constructed for the odd last digits 1, 3, 7 and 9.

Table of last digit of product of two odd numbers not ending in 5

last digit of first number \ last digit of second number	1	3	7	9
1	1	3	7	9
3	3	9	1	7
7	7	1	9	3
9	9	7	3	1

I asked for more theorems. Someone put forward that an O'Reilly number could not be prime and I called this Theorem 5. I asked for a Theorem belonging to the same family as the previous two. It was remarked that they state that O'Reilly numbers are not divisible by 2 or 5. What about other small primes?

Theorem 6. An O'Reilly number is not a multiple of 3.

I gave the class some time to think about this. Can they do for three what they did for 2 and 5? It was realized that 2 and 5 worked out because they are the factors of 10 which is the base of our number scheme, and the ingredients of our proof were little facts about endings of numbers in this base. The required results were not available for 3.

Why not work in based 3? Let's try. The proof should begin as before: an O'Reilly number n must have a factorization which does not contain 3. If we write the primes of this factorization in base 3, then none of them will end in zero. So (I guess) their product cannot end in zero. So n is not divisible by 3, and that seems to do it.

We used the fact that in base 3, the product of numbers not ending in zero cannot end in zero. Is this true? Everyone said it was. Are you sure? I asked. After a moment, it was decided that you simply had to check the last digit table.

Base 3
Last digit table for product of two numbers not ending in zero

last digit of first number \ last digit of second number	1	2
1	1	2
2	2	1

Since no zeros appear, the product of two numbers not ending in zero (base 3) cannot end in zero.

We appeared to have an interesting "machine". What's next?, I asked. It was suggested we should try 7 next. But the young man, Ian, I had had some trouble with earlier, who had been sitting scowling for the last while, said quietly, let's try 19.

Theorem 7. An O'Reilly number is not a multiple of 19.

Of course, I hastily explained to the class, we know this theorem to be false. But in trying the above approach on it, we may, in failing to find the proof, learn something about why 19 is different from 2, 3, and 5. So off we went.

An O'Reilly number must have at least one factorization which does not contain 19. Think of these prime factors in base 19. None of them end in zero, so the possible endings are 1, 2, 3, ..., 17, 18. (Here we treat the numbers 10, 11, ..., 18 as single "digits".) Can the product of two such numbers end in zero? I asked the class.

Someone said no, of course not, but someone else argued yes. Indeed think of the base 19 representation of p_0 and q_0 above. They must both end in "digits" between 1 and 18. But their product n_0 must end in zero. Make up the table, someone said. I sketched out an 18×18 block. It's a big table, I said.

Ian had borrowed the 10 place calculator and was calculating the final "digits" base 19 of p_0 and q_0. He did this by dividing them by 19 and taking the remainder. He got 17 and 18 respectively. We multiplied them together and filled in that square of the table with a 2. The class was silent for a moment. I wonder what that means, I said. It means O'Reilly was wrong, said Ian immediately, and there was another silence.

I think what it means, I said after a moment, is that I've miscopied one or both of these numbers p_0 or q_0. I'm sorry, they must be wrong in my notes. Ian shook his head in dismay. Having tasted blood, he was not about to be put off. Fill in the table, he said; you won't find a zero. It's a big table, I replied again.

Okay, I said, after a moment, suppose we fill in the table. Suppose we get no zero. What have we got? We know O'Reilly's example is wrong, came the reply. No O'Reilly number is divisible by 19. Right, I said, where do we go from there? Do we do the same thing for other primes? How far can we get just by filling in larger and larger tables? Can we find any way of arguing directly that the table couldn't have a zero, without actually filling it in? Such an approach would be very powerful because it might extend to a large family of primes. Suppose there's a zero in the table. Can you see anything wrong with that?

This was a large piece of direction I had given them, and the groups worked away for awhile. Happily it was Ian who found the argument. It did much to restore his equilibrium.

If there is a zero in the base 19 table, say in the (h,k) position, with $1 \le h,k \le 18$, then $hk = 19s$ for some s. Since h and k are less than 19, this gives us two different factorizations for the same number, hence gives us a new O'Reilly number (possibly after cancelling common factors). This new number is certainly smaller than n_0, and contradicts the fact that n_0 was the smallest.

The class was respectfully silent. Notice what's happened, I said. Without filling in the base 19 table, we argued that it couldn't have a zero. But we used a piece of information we hadn't used before: the minimality of n_0. How generally can you make this trick work?

Everyone felt game to try to tackle:

Theorem 8. There are no O'Reilly numbers.

It took a bit of trial and error to get the proof right. It turns out that to generalize the n_0 argument there are really two important components: that n_0 be the smallest O'Reilly number and that 19 be the smallest prime factor of n_0.

Proof of Theorem 8. Supposing the theorem false, let n be the smallest O'Reilly number and let p be the smallest prime factor of n. Now n must have a prime factorization that doesn't contain p, say $n = p_1 \ldots p_m$ with all $p_i > p$. Replace each p_i by its final "digit" r_i in base p, and let $t = r_1 \ldots r_m$. Since n is divisible by p, the last "digit" of t (which is the same as the last "digit" of n) is zero (base p) and $t = pk$. This gives us two factorizations of t, which, after removal of common primes, gives us a new O'Reilly number less than n_0 (since each $r_i < p_i$). Contradiction.

At the end, some of the class were a bit bewildered by what had been discovered. I pointed out that unique factorization was indeed a property of the integers, and that that was in fact what Theorem 8 stated. What we had produced, in our explorations, was quite a reasonable proof of the unique factorization result. Had anyone, I asked, seen a proof of the unique factorization theorem before? One or two thought they had, but they weren't sure.

Notes. I have given this exercise to four different groups: high school students, high school math teachers, university math seniors, and university math educators. In all groups there was some initial confusion over the appearance of an example which appeared to contradict a firmly held belief. But if the example was properly dressed up with the right historical footnotes, I found my audience on the whole quite willing to "suspend their disbelief" and enter actively into a search for theorems.

The numbers p_0 and q_0 are chosen with care. I don't have any reason to believe they are prime, but they have no factors ≤ 61. If you

multiply them out with a 10 digit hand calculator you get 2.4190×10^{19} which is also what you get if you multiply out the "small" factorization of n_0. Also if you do a "last 3 digit" analysis of the two factorizations you get 391 for the product of both sides. That is, the factorizations are the same mod 1000.

The unique factorization result is usually (casually) mentioned in high school, and is proved in a first or second algebra course in university. [Nevertheless I had no trouble selling my example to the university students.] The usual proof uses the Euclidean Algorithm. There is a standard proof similar in spirit to our "discovered" proof of Theorem 8, which Nathan Jacobsen [Basic Algebra I, Freeman 1974, p. 22] attributes to Zermelo. [I am grateful to John Poland for this reference.] It goes as follows: let n be the smallest number with two disjoint factorizations

$$p_1 \cdots p_m = n = q_1 \cdots q_k$$

and suppose $p_1 > q_1$. Then

$$(p_1 - q_1)(p_2 \cdots p_m) = q_1(q_2 \cdots q_k - p_2 \cdots p_m) .$$

By completing the factorization of both sides we get two prime factorizations of a number smaller than n, one of which contains a q_1, the other of which does not (since q_1 cannot divide $p_1 - q_1$ or it would divide p_1).

Peter D. Taylor
Dept. of Mathematics & Statistics
Queen's University
Kingston, Ontario
K7L 3N6

TOPIC GROUP C

EPISTEMOLOGICAL FALLICIES WILL LEAD YOU NOWHERE

BY: JACQUES DESAUTELS

EPISTEMOLOGICAL FALLACIES WILL LEAD YOU NOWHERE!

Titre arrogant pour une conférence si,d'une part,on traduit littéralement "fallacies" par "faussetés" et si,d'autre part,on imagine que le conférencier invective ses auditeurs. Il perd cependant son impertinence si"fallacies"prend le sens d'illusion puisqu'il se transforme en lapalissade. Qui, en effet, oserait affirmer que l'on peut aller "quelque part" dans le domaine de l'apprentissage des mathématiques en se berçant d'illusions et d'illusions épistémologiques au surplus. Mais cette lapalissade n'en est pas vraiment une,car elle n'en a pas le caractère premier, soit l'évidence liée à l'univocité du sens qu'elle connote. Pourtant,pour ceux d'entre nous qui ont réfléchi à certains problèmes de l'apprentissage des sciences, elle a acquis un sens évident, mis graduellement au jour par des travaux qui forment un véritable programme de recherche que l'on reconnaît dans la littérature sous les étiquettes: représentations pré-scientifiques, conceptions ou représentations spontanées, etc. Le titre perd alors définitivement toute insolence ou prétention puisqu'il réfère à l'apprentissage des sciences et ne s'adresse donc pas, tout au moins directement, aux didacticiens des mathématiques. Mais nos travaux peuvent-ils vous être d'une certaine utilité?

C'est la question qui a orienté ma réflexion et je me propose de traiter succinctement avec vous des sujets suivants:

1) Quelques exemples de représentations spontanées.

2) M. Bachelard, ses obstacles et son profil épistémologique.

3) La droite, le point, le hasard.

1) QUELQUES EXEMPLES DE REPRÉSENTATIONS SPONTANÉES:
La chaleur, le mouvement, etc.

Lorsqu'on demande à des enfants d'une dizaine d'années d'expliquer pourquoi l'extrémité A d'une tige de métal devient chaude alors que la source de chaleur est située à l'extrémité B de celle-ci, on ne les prend pas au dépourvu — ils fournissent spontanément des explications. Celles-ci, bien que fort variées, présentent ceci en commun: il y a quelque chose qui se déplace du point B au point A, ce qui au demeurant est tout à fait logique. Mais qu'elle est la nature de ce quelque chose qui se déplace ainsi? Evidemment, c'est de la chaleur et jusque là on ne peut rien reprocher à l'explication. Si on poursuit le questionnement jusqu'à leur demander ce que c'est la chaleur, on découvre que pour eux, il s'agit d'une substance plus ou moins volatile,qu'ils comparent à l'air, à la fumée ou à un fluide quelconque. Ces explications ne correspondent pas à celles qui forment le champ de connaissance de la science moderne, bien que, dans certains cas,elles présentent des similarités étonnantes avec des théories antérieurement reonnnues [1] par les scientifiques, notamment la théorie du calorique. Cependant, ces explications enfantines, comme nous le verrons ci-après, font obstacle à l'apprentissage des sciences et, à ce titre, devraient être prises en considération dans l'élaboration de stratégies pédagogiques.

L'explication du mouvement fournie par des élèves d'une dizaine d'années constitue un autre exemple de représentation spontanée. Ceux-ci, à l'instar d'Aristote, ne peuvent concevoir qu'un objet puisse se mouvoir sans l'intervention d'une force qui non seulement initie le mouvement mais le maintient. D'autre part, si la vitesse d'un objet est constante, c'est que nécessairement la force agissante est constante, et plus celle-ci est grande, plus la vitesse est proportionnellement grande. Dans cette optique, un objet qui se déplace à grande vitesse doit nécessairement être mu par une grande force.

Rosalind Driver [2] utilise l'expression "children's science" pour désigner l'ensemble des explications que les enfants construisent spontanément pour rendre compte des phénomènes avec lesquels ils interagissent, avant toute éducation scientifique formelle, mais également pour souligner que ces explications forment une structure conceptuelle dont on doit tenir compte en pédagogie des sciences, ne serait-ce parce qu'elle permet aux enfants de donner un sens à leurs observations quotidiennes. Or, jusqu'à tout récemment, on a négligé de le faire, pensant qu'il suffisait de montrer la bonne solution pour que les élèves changent leurs explications. Les résultats de la recherche sont clairs [2], les élèves n'abandonnent pas leurs explications premières et les réutilisent très volontiers lorsque le contexte du problème qui leur est posé diffère de celui des problèmes de fin de chapitre dans un livre ; ce qui d'ailleurs ne les empêche pas de réussir aux examens. Mais que devient la connaissance scolaire quelque temps après les

études? N'ayant pas été vraiment assimilée, elle est reléguée aux oubliettes et lentement mais sûrement se transforme en vague souvenir — Ah! oui je me souviens... le principe d'Archimède — l'eau qui monte dans la baignoire... je n'avais pas vraiment compris.

Le spectre des raisons qui peuvent être invoquées pour expliquer l'échec de nos enseignements respectifs [3] est varié: formation des maîtres, matériel didactique, stratégies pédagogiques, nature des disciplines, développement intellectuel des élèves, sont autant de facteurs à examiner afin d'éclairer le phénomène en toutes ses dimensions. Or, parmi ceux-ci, je m'attarderai à la nature épistémologique intrinsèque du processus de la transformation de la connaissance, qu'on le considère du point de vue historique ou du point de vue de l'apprentissage individuel, ce qui me permettra de spécifier en quoi les représentations spontanées des élèves constituent des obstacles à leur apprentissage des sciences.

2) <u>Monsieur Bachelard, ses obstacles, son profil épistémologique</u>

Il est étonnant de constater que la publication du petit livre de Thomas Kuhn [4], *La Structure des Révolutions Scientifiques*, ait provoqué un tel remous chez les intellectuels de toutes les disciplines, alors que l'oeuvre magistrale d'épistémologie historique de Gaston Bachelard continue à être largement ignorée. Dès ses premières publications [5], ce dernier, en interrogeant les acquis récents de la relativité et de la théorie quantique, posait les jalons d'une épistémologie qui, à mon avis, est plus riche d'enseignement que l'oeuvre de Kuhn, non seulement au regard de la compréhension de la nature du savoir scientifique et de sa transformation, mais également du point de vue pédagogique, car s'il est devenu épistémologue, Gaston Bachelard a d'abord été professeur de sciences. On ne doit donc pas s'étonner de trouver tout au long de l'oeuvre de Bachelard une préoccupation pour l'enseignement scientifique; n'écrivait-il pas dès les années quarante:

> *"Les professeurs de sciences imaginent que l'esprit commence comme une leçon, qu'on peut toujours refaire une culture nonchalante en redoublant une classe, qu'on peut faire comprendre une démonstration en la répétant point pour point."* (6)

Il ne saurait être question d'épuiser en quelques pages une oeuvre aussi riche; je me contenterai donc d'évoquer quelques-uns des concepts construits par cet auteur, qui permettent, à mon avis, de saisir en quoi les représentations spontanées constituent des obstacles à l'apprentissage.

Pour Bachelard[7] seule une philosophie dispersée des sciences peut rendre compte de la transformation historique du savoir scientifique, et c'est à partir de la notion de masse qu'il illustre cette idée. Il affirme que l'on peut distinguer cinq stades dans la transformation de cette notion correspondant à autant de courants philosophiques, c'est-à-dire: le réalisme naïf, le réalisme-empiriste, le rationalisme, le rationalisme dialectique et le rationalisme complet. Au premier stade, la masse est conçue intuitivement comme une "appréciation grossière et comme gourmande de la réalité" [8]; au deuxième stade, la masse est définie empiriquement par l'opération de la balance et alors: "Peser c'est penser. Penser c'est peser" [9]. Ce n'est qu'au troisième stade que la notion prendra son envol, si l'on peut parler ainsi, et sera rationnellement conçue comme "... un corps de notions et non plus seulement comme un élément primitif d'une expérience immédiate et directe." [10], et définie comme le rapport de deux autres notions, la force et l'accélération. Cette belle assurance rationaliste s'estompera au moment de la complexification de la notion de masse qui devient relative à la vitesse de l'objet en plus d'être transformable en énergie. Enfin, pour satisfaire et à la logique théorique et aux exigences empiriques, il a été nécessaire d'accepter l'idée d'une masse négative.

La description de ces stades ne nous informe cependant pas quant au mécanisme responsable de cette transformation, et c'est pourquoi Bachelard a mis au point le concept de rupture épistémologique. Par exemple, le passage de la masse absolue à la masse relative suppose l'abandon de certaines prémisses épistémologiques dont celles d'espace et de temps absolu, et d'en accepter d'autres dont celle de vitesse limite. Il y a donc une rupture qui rend ces notions incommensurables, ce qui ne signifie pas pour autant que celles-ci ne soient pas utiles dans certains domaines spécifiques. D'une façon similaire, la théorie cinétique qui permet de définir en science la notion de chaleur exige que l'on cesse de considérer la chaleur comme une substance pour adopter le point de vue énergérique, beaucoup plus abstrait, puisque la chaleur est alors conçue comme l'énergie cinétique moyenne des atomes ou molécules, telle que donnée par l'équation $E = \frac{1}{2} mv^2$. Or, il s'agit d'une véritable rupture dans la mesure où il est nécessaire de nier les impressions sensorielles à partir desquelles, tout jeune enfant, on construit une certaine représentation de la chaleur, sans oublier, d'autre part, l'élimination de la notion de froid, qui n'a aucun sens dans le contexte des théories scientifiques. De même, l'enfant doit nier les expériences sensorielles premières, qui le conduisent logiquement à croire au repos absolu et à nier qu'un objet puisse se déplacer

sans l'action d'une force, pour accéder à la compréhension du principe d'inertie. C'est dans ce sens qu'il faut saisir le mot de Bachelard lorsqu'il dit qu'

> *"en fait, on connaît contre une connaissance antérieure, en détruisant des connaissances mal faites, en surmontant ce qui, dans l'esprit même, fait obstacle à la spiritualisation."* (11),

d'où la notion d'obstacle épistémologique qui mériterait à elle seule un long commentaire. Je rappelle seulement que ces transformations de la connaissance intrinsèque à l'apprentissage des élèves sont l'équivalent d'une mutation culturelle. Le rôle du pédagogue doit alors s'articuler aux exigences de telles transformations et on comprendra qu'il est alors nettement insuffisant de présenter la version officielle des sciences, même si la présentation est logique.

3) Le point, la droite

Les notions de l'épistémologie bachelardienne nous ont aidés à comprendre en quoi les représentations spontanées des élèves constituent des obstacles à leur apprentissage des sciences. En effet, elles nous révèlent que le procès de la transformation de ces connaissances exige la remise en question de postulats largement implicites qui forment la structure de base de la vision du monde à partir de laquelle les élèves règlent, avec un certain bonheur, leurs interactions avec l'univers matériel. Il est dès lors illusoire de penser que ces changements profonds s'opéreront au cours de quelques leçons bien faites. L'apprentissage des mathématiques pose-t-il des problèmes similaires?

Je ne me risquerais pas à affirmer que l'on retrouve exactement les mêmes problèmes au niveau de cet apprentissage, compte tenu de la précarité de ma culture mathématique et ce, tant au plan des notions elles-mêmes que de leur statut épistémologique. Cependant, il me semble qu'un certain nombre de notions de la géométrie euclédienne (la seule que je connaisse) présente des difficultés similaires à celles que j'ai évoquées ci-avant, au plan de leur apprentissage par des élèves.

Pour ces derniers, comme pour la plupart des gens, il n'y a pas à priori de distinction entre la ligne et la droite. Celles-ci correspondent au trait physique observable qui, manifestement, a une longueur et une épaisseur. Le point, quant à lui, s'il est minuscule, n'est quand même pas infiniment petit — il est là devant leurs yeux et bien visible. Et il est tout à fait compréhensible que, pou eux, un point qui se déplace dans l'espace engendre une ligne, au sens où il laisse une trace. Mais on sait que les définitions mathématiques du point et de la droite ne correspondent pas à ces représentations sensuelles. Il est fort difficile pour les élèves de s'en détacher et de concevoir un point sans dimension, correspondant à l'intersection de deux segments de droites qui n'ont qu'une longueur et pas d'épaisseur. Or, la compréhension des concepts de la géométrie euclidienne nécessite un détachement par rapport à ces représentations concrètes afin d'accéder à l'univers abstrait des constructions géométriques. Il s'agit là d'un premier saut qualitatif sans lequel on imagine mal comment les individus accéderont à ces univers "étranges" des géométries à **n** dimensions où la référence au sensible constitue un véritable obstacle épistémologique.

N'y aurait-il pas ainsi une kyrielle d'obstacles épistémologiques à identifier en rapport avec de nombreux concepts mathématiques au sujet desquels les élèves ont construit spontanément des représentations? Je pense, par exemple, aux concepts suivants: l'infini, le hasard, la relation et, pourquoi pas, le nombre? Mais identifier des obstacles à l'apprentissage est une chose, créer les stratégies pédagogiques pour les surmonter en est une autre.

Je ne suis pas certain que mes propos aient été parfaitement clairs, ni qu'ils soient tout à fait pertinents par rapport aux problèmes rencontrés dans l'apprentissage des mathématiques. Intuitivement, je pense que les concepts de l'épistémologie bachelardienne ont un certain à-propos eu égard à vos préoccupations de didacticiens des mathématiques. La discussion qui suivra permettra, je l'espère, d'approfondir ces questions.

Jacques Désautels
Professeur.

NOTES DE RÉFÉRENCES

1. Je ne suppose pas ici que l'ontogenèse est une simple récapitulation de la phylogenèse, bien que, dans certains de leurs aspects, il y ait des rapprochements étonnants.

2. DRIVER, Rosalind. The Pupil as Scientist?, 1983, Stony Stratford, The Open University Press.

3. Voir notamment:
BARRUK, Stella. L'Age du Capitaine, 1985, Paris, Editions du Seuil, Collection Science Ouverte.
DESAUTELS, Jacques. Ecole + Science = Echec, 1980, Québec, Québec Science Editeur.

4. KUHN, Thomas. La Structure des Révolutions Scientifiques, 1980, Paris Flammarion, Collection Champs.

5. BACHELARD, Gaston,. Le Nouvel Esprit Scientifique, 1983, Paris, Quadrige, Presses Universitaires de France, 15e édition.

6. BACHELARD, Gaston. La Formation de l'Esprit Scientifique, 1975, Paris, Librairie Philosophique J. Urin.

7. BACHELARD, Gaston. La Philosophie du Non, 1973, Paris, Presses Universitaires de France, 6e édition.

8. Idem, p. 22.

9. Idem, p. 26.

10. Idem, p. 27.

11. BACHELARD, Gaston, op. cit., 1975, p. 14.

TOPIC GROUP D

RECENT CANADIAN RESEARCH CONCERNING TEACHING, GENDER AND MATHEMATICS

BY: GILA HANNA
ERIKA KUENDIGER
ROBERTA MURA

SEX DIFFERENCES IN THE MATHEMATICS ACHIEVEMENT OF EIGHTH GRADERS IN ONTARIO

Gila Hanna
The Ontario Institute for Studies in Education

In the past two decades researchers have shown considerable interest in the relationship between the sex and the mathematics achievement of children in the upper grades of the elementary schools. Some have examined sex differences by comparing total test scores (Backman, 1972; Benbow & Stanley, 1980, 1983; Maccoby & Jacklin, 1974), while others have focused on the proportion of students who answered a particular item correctly (Armstrong, 1980; Fennema, 1978; Raphael, Wahlstrom & McLean, 1984). In a recent study by S.P. Marshall (1983) the analysis is based on a comparison of the kinds of errors made by male and female students

Some of the studies done to date purport to have established that by age 13 there is a significant difference in mathematical ability between the sexes, and that it is especially pronounced among high-scoring exceptionally gifted students, with boys outnumbering girls 13 to 1 (Benbow & Stanley, 1983), while others have argued the opposite: that very little difference exists, if any, and that when a difference is detected it favours boys only slightly (Fennema & Carpenter 1981) According to the *International Review on Gender and Mathematics* (Schildkamp-Kündiger, 1982), reporting on research carried out in nine countries, gender-related differences in achievement have been found to vary considerably both within and among countries.

The purpose of this study is to assess the scope of sex-related differences in the mathematics achievement of Ontario Grade 8 students, making use of the pool of data collected by the IEA Second International Mathematics Study (SIMS).

Test Instruments and Data

For the SIMS study, a random sample of 130 schools was selected from a total of 2511 Ontario schools after each school had been assigned to one of twenty-six strata based on the following four categories: (a) school size, (b) type of school (private, French, English Catholic, and English public), (c) rural or urban, and (d) geographical region of the province (In Ontario, virtually all thirteen-year-olds are enrolled in either a private or a public school.)

The present analysis does not use data for private schools (which are attended by 2 percent of all Grade 8 students). Since previous analyses (McLean, Raphael & Wahlstrom, 1983) had shown that students in private schools had much higher rates of success, and since there were three times as many boys as girls in the private-school stratum, it was decided to delete these data from the analysis. The sample retained for this study consisted of all the Grade 8 students not attending private schools for whom data were available for both the pretest and the posttest: 3523 in total, 1773 boys and 1750 girls

The IEA had developed 180 items for Grade 8, administered in five forms: a Core form of 40 items and four Rotated forms of 35 items each; for technical reasons six items were not part of the Ontario data. The 174 Ontario items covered five broad topics: Arithmetic (58 items), Algebra (31 items), Geometry (42 items), Probability and Statistics (17 items), and Measurement (26 items).

All students were administered both a pretest and a posttest. Each student responded to the Core form and to one of the Rotated forms A, B, C, or D on each occasion. Each student was given the same Core form at both the pretest and the posttest, but not necessarily the same Rotated form; the Rotated forms were administered randomly on both occasions, each form to one quarter of the class. As a consequence of this method, there are variations in number of respondents among the four Rotated forms and also between the two occasions for the same Rotated form. In addition, the Core form yields a greater precision of results, since it has about four times as many respondents as a Rotated form. Table 1 summarizes the pattern of responses to each of these test forms.

Table 1

Number of Respondents by Sex and Test Form

	Pretest		Posttest	
	Boys	Girls	Boys	Girls
Form A	455	417	459	417
Form B	427	470	465	444
Form C	447	426	433	437
Form D	444	437	416	452
Core Form (total)	1773	1750	1773	1750

Note. Total number of respondents 3523.

The items were five-alternative multiple choice (one correct response and four distractors). Every response to each item was coded into one of three categories: correct, wrong, or item omitted. For each item, three percent values (correct, wrong and omitted) were calculated separately for boys and for girls, with the student as the unit of analysis. (The percent correct of an item, for example, is the percentage of students who answered that item correctly.) Three mean percent values were then obtained for each topic by averaging the percent values for the individual items in that topic; these are shown in Table 2.

Table 2

Mean Percent Values (and Standard Deviations) per Category of Response by Sex and by Topic

		Pretest			Posttest		
		Correct	Wrong	Omit	Correct	Wrong	Omit
Arithmetic (58 items)	Boys:	49.5	48.0	2.5	54.9	43.7	1.4
		(18.1)	(17.0)	(2.7)	(16.6)	(16.0)	(1.5)
	Girls:	48.2	48.2	3.5	54.2	43.3	2.5
		(20.0)	(18.4)	(3.9)	(17.8)	(16.8)	(2.2)
Algebra (31 items)	Boys:	34.6	59.1	6.2	44.1	52.5	3.3
		(15.9)	(16.4)	(3.6)	(15.7)	(15.0)	(2.1)
	Girls:	33.5	57.8	8.5	44.6	50.7	4.6
		(17.2)	(17.7)	(4.7)	(17.3)	(16.3)	(2.2)
Geometry (42 items)	Boys:	36.4	56.9	6.7	45.0	51.2	3.9
		(17.6)	(15.1)	(5.1)	(17.8)	(15.8)	(3.4)
	Girls:	33.4	57.6	8.9	42.6	51.8	5.6
		(17.2)	(14.5)	(6.5)	(18.3)	(15.9)	(4.5)
Probability & Statistics (17 items)	Boys:	53.5	43.7	2.8	57.6	40.6	1.8
		(19.6)	(18.7)	(1.9)	(18.8)	(16.5)	(1.8)
	Girls:	52.9	42.9	4.2	56.7	40.4	2.8
		(22.2)	(20.1)	(3.0)	(18.8)	(17.3)	(2.2)
Measurement (26 items)	Boys:	45.9	51.1	3.1	53.4	45.0	1.7
		(21.8)	(20.6)	(2.6)	(19.7)	(19.0)	(1.3)
	Girls:	42.6	53.1	4.3	50.2	46.8	2.9
		(22.8)	(21.2)	(3.4)	(21.6)	(20.4)	(2.1)

Note. Due to rounding error the figures for Correct, Wrong and Omit may not add to 100.

Results

For each topic the difference between boys and girls in the mean percent of correct, wrong and omitted responses was analysed using the paired t-test with the item as the unit of analysis. In addition, a Wilcoxon matched-pairs test was performed to obtain the z-statistic and its two tailed probability as well as information on the number of items with positive or negative differences between boys and girls.

Differences in Correct Responses

As shown in Table 3 no statistically significant differences were found between boys and girls on either occasion for three of the topics (Arithmetic, Algebra, and Probability and Statistics). In Geometry and in Measurement, however, more boys gave correct responses on both occasions; in both these content areas the difference of about 3 percent is statistically significant at the .01 level.

In the pretest as a whole, boys were more successful on 100 items and girls on 60; boys and girls tied on 14. But in Geometry and in Measurement, boys did better than girls on more than twice as many items. This pattern of results was very much the same for the posttest.

Table 3

Differences Between Boys and Girls in Mean Percent Values by Topic

		Pretest			Posttest		
	df	Correct	Wrong	Omit	Correct	Wrong	Omit
Arithmetic	57	1.3	-0.2	-1.0*	0.7	0.4	-1.1*
Algebra	30	1.1	1.3	-2.3*	-0.5	1.8	-1.3*
Geometry	41	3.0*	-0.7	-2.2*	2.4*	-0.6	-1.7*
Probability & Statistics	16	0.6	0.8	-1.4*	0.9	0.2	-1.0*
Measurement	25	3.3*	-2.0	-1.2*	3.2*	-1.8	-1.2*

Note. A positive difference represents a higher mean percent for boys; a negative difference, a higher mean for girls.

*$p<.01$

Differences in Wrong Responses

There were no statistically significant differences between boys and girls at the .01 level on any of the topics; the two sexes gave wrong responses with similar frequency.

In the pretest as a whole more boys gave wrong responses on 84 items while more girls did so on 77; on 13 items the percent of wrong response was the same for both sexes. In Arithmetic, Geometry and Statistics boys and girls gave wrong responses on approximately the same number of items. In Algebra, however, the rate of wrong responses was higher for boys on 20 items, while for girls it was higher on 11; this pattern was reversed in Measurement, with the girls giving more wrong responses on 14 items and the boys on 9. The posttest results were very similar to those of the pretest in terms of the distribution of wrong responses.

Differences in Omitted Responses

As shown in Table 3, the differences between the sexes were negative, indicating that the percent of omitted responses for girls was greater than that for boys on all the subtests in both the pretest and the posttest. Furthermore, the t-test paired comparisons showed that all the differences between boys and girls were statistically significant at the .01 level.

In both the pretest and the posttest more girls than boys omitted responses. The rate of omission was higher for the boys only on 17 items (10% of the test), while it was higher for the girls on 116 items (70% of the test). The Wilcoxon analyses yielded z-statistics significant at the .01 level for all the topics, indicating that this trend was consistent from topic to topic.

A detailed examination of the omitted responses revealed that the percentage of students omitting items on the pretest ranged from 0 to 28 for girls and from 0 to 23 for boys; the medians were 4.5 and 3.0, respectively. Although there was a decrease in these values for both boys and girls in the posttest (that is, fewer students omitted items), the gap between the sexes was maintained. On the posttest the range was 0 to 21 with a median of 3.0 for girls, while it was 0 to 17 with a median of 2.0 for boys.

Differences in Gains

The gains are based on the difference between the mean percent of correct responses on each topic on the posttest and on the pretest, for each group taken separately, and could thus reasonably be taken to represent the growth in mathematics achievement for the group. The results shown in Table 4 would indicate that on average boys and girls improve at the same rate during Grade 8: there were no statistically significant differences (at the .01 level) between the two groups in terms of their gains in mean percent of correct responses by topic.

Girls showed greater gains on 93 items and boys on 63; girls and boys tied on 18. In Statistics and in Measurement girls had greater gains on approximately the same number of items as boys; in Arithmetic, Algebra and Geometry, taken together, girls had greater gains than boys on twice as many items.

Table 4

Gains in Mean Percent Values by Sex and Topic

	Boys	Girls
Arithmetic	5.4	6.0
Algebra	9.5	11.1
Geometry	8.6	9.1
Probability & Statistics	4.1	3.8
Measurement	7.5	7.6

Note. Differences between boys and girls not significant at the .01 level.

Discussion

The results of this study may be summarized as follows:

1. The mean percent of correct responses in two of the five topics (Geometry and Measurement) was slightly higher for boys than for girls. These differences, though not large, were statistically significant at the .01 level.

2. There were differences between boys and girls in omitted responses. All the t-tests were significant at the .01 level. Girls had much higher omission rates on all topics. On average the omission ratio of boys to girls was 2:3.

3. Examination of the gains indicated that instruction in Grade 8 had about the same influence on girls as on boys.

These three findings assume educational significance when one bears in mind that the boys and girls came from the same randomly selected schools in approximately equal proportions and thus can be considered matched on socio-economic level, on amount of formal training in mathematics, and on quality of teaching (ignoring possible differential treatment of the two sexes on the part of teachers). The results can thus be generalized to students attending public schools in Ontario, and any sex differences found must be attributed to factors other than socio-economic level, formal training, or quality of teaching.

It is conceivable that the boys had had a certain amount of informal training through out-of-class activities not normally pursued by girls (following instructions for building models, reading charts and graphs, and the like). Different informal training in mathematics could explain the differences in achievement in Geometry and in Measurement in particular.

According to McLean, Raphael and Wahlstrom (1983), Ontario teachers reported that only about half of the Geometry items had been taught at all, either before or during Grade 8. This would lend some support to the idea that out-of-class activities contributed to the disparities in achievement between the sexes. On the other hand, the other topic which showed differences between the sexes, measurement, was among four topics in which most teachers reported covering about 80% of the curriculum. Thus on the basis of the information available it is not possible to determine with any confidence whether out-of-class activities had an effect on the differences between the sexes in Geometry or Measurement.

Other causes sometimes cited for sex differences in mathematical achievement, such as the presentation of mathematics as a male domain (Becker, 1982) or the presumed social conditioning and different expectations for boys and girls (Fennema, 1978), might explain why more girls omitted responses than did boys. On the basis of the Grade 8 SIMS data no attempt could be made to determine the importance of these factors, or indeed of informal training.

References

Armstrong, J. M. (1980). *Achievement and participation of women in mathematics: An overview* (Report 10-MA-00). Denver: Education Commission of the States.

Backman, M. E. (1972). Patterns of mental abilities: Ethnic, socioeconomic, and sex differences. *American Educational Research Journal, 9*, 1-12.

Becker, J. R. (1982). Gender and mathematics in the United States. In E. Schildkamp-Kündiger (Ed.), *An international review on gender and mathematics* (pp. 131-141). Columbus, OH: ERIC Clearinghouse for Science, Mathematics and Environmental Education.

Benbow, C. P., & Stanley, J. C. (1980). Sex differences in mathematical ability: Fact or artifact? *Science, 210*, 1262-1264.

Benbow, C. P., & Stanley, J. C. (1983). Sex differences in mathematical reasoning ability: More facts. *Science, 222*, 1029-1031.

Fennema, E. L. (1978). Sex related differences in mathematics achievement: Where and why? In J. E. Jacobs (Ed.) *Perspectives on women and mathematics.* Columbus, OH: ERIC Clearinghouse for Science, Mathematics and Environmental Education.

Fennema, E. L., & Carpenter, T. P. (1981). Sex-related differences in mathematics. In M. K. Corbitt (Ed.), *Results from the Second Mathematics Assessment of the National Assessment of Educational Progress.* Reston, VA: National Council of Teachers of Mathematics.

Maccoby, E., & Jacklin, C. (1974). *The psychology of sex differences* Stanford, CA: Stanford University Press.

Marshall, S. P. (1983). Sex differences in mathematical errors: An analysis of distracter choices. *Journal for Research in Mathematics Education, 14*, 325-336.

McLean, L., Raphael, D., & Wahlstrom, M. (1983). The Second International Study of Mathematics: An overview of the Ontario grade 8 study. *Orbit 67, 14*(3).

Raphael, D., Wahlstrom, M., & McLean, L. (1984). Results from the Second International Mathematics Study: Are boys better at math than girls? *Orbit 70, 15*(2).

Schildkamp-Kündiger, E. (Ed.). (1982). *An international review on gender and mathematics.* Columbus, OH: ERIC Clearinghouse for Science, Mathematics and Environmental Education.

Appendix

Items with differences between the sexes of 10 percentage points or more - -

Item A Girls > Boys

Item B to O Boys > Girls

Note Item D - difference between sexes largest 20%

A. $\frac{3}{5} + \frac{2}{7}$ is equal to

A $\frac{21}{10}$

B $\frac{5}{12}$

C $\frac{10}{21}$

D $\frac{6}{35}$

E $\frac{31}{35}$

B. Which of the following is equal to a quarter of a million?

A 25 250

B 40 000

C $\frac{1}{4\ 000\ 000}$

D 250 000

E 2 500 000

C. There are 35 students in a class. $\frac{1}{5}$ of them come to school by bus, another $\frac{2}{5}$ come by bicycle. How many come to school by other means?

A 7

B 14

C 21

D 28

E 35

D. The speed of sound is 340 m/s. How long will it take before the sound of a car horn reaches your ears if the car is 714 m away?

A 0.21 s

B 2.1 s

C 21 s

D 210 s

E None of these

E. 20 is what percent of 80 ?

A 4%

B 20%

C 25%

D 40%

E None of these

F. In a school election with three candidates, Joe received 120 votes. Mary received 50 votes, and George received 30 votes. What percent of the total number of votes did Joe receive?

A $\frac{6}{10}$%

B 40%

C 60%

D 80%

E 120%

G. One bell rings every 8 minutes, a second bell rings every 12 minutes. They both ring at exactly 12 o'clock. After how many minutes will they next ring together?

A 8

B 12

C 20

D 24

E 96

Figure 8

H. The air temperature at the foot of a mountain is 31°C. On top of the mountain the temperature is −7°C. How much warmer is the air at the foot of the mountain?

A −38°

B −24°

C 7°

D 24°

E 38°

I. On level ground, a boy 5 units tall casts a shadow 3 units long. At the same time a nearby telephone pole 45 units high casts a shadow the length of which, in the same units, is

A 24

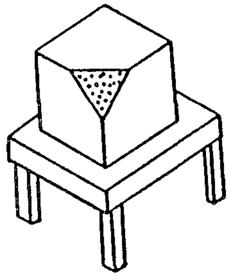
B 27

C 30

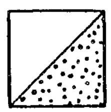
D 60

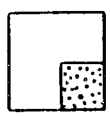
E 75

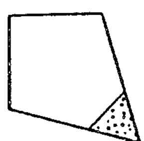
J.

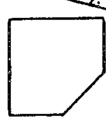


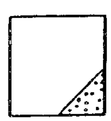
The figure above shows a wooden cube with one corner cut off and shaded. Which of the following drawings shows how this cube would look when viewed from directly above it?

A 

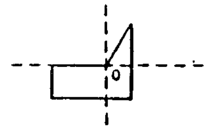
B 

C 

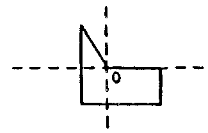
D 

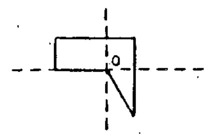
E 

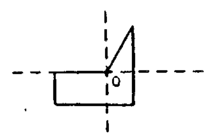
K.

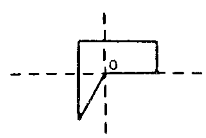


A half-turn (180°) about point O is applied to the figure above. Which of the figures below is the result?

A 

B 

C 

D 

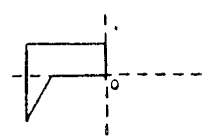
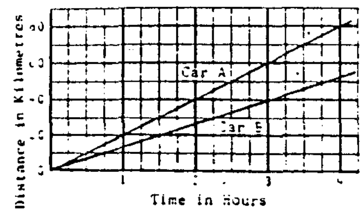
E 

Figure 9

L.



How much longer does it take for car B to go 50 km than it does for car A to go 50 kilometres?

A 1 h 15 min

B 1 h 30 min

C 2 h

D 2 h 30 min

E 2 h 35 min

N. What is the capacity of a cubic container 10 cm by 10 cm by 10 cm?

A 1 L

B 10 L

C 100 L

D 1000 L

E 1000 cm

M. How many pieces of pipe, each 20 m long, would be required to construct a pipeline one kilometre in length?

A 5

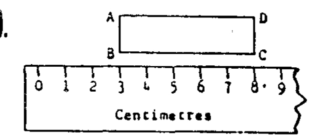
B 50

C 500

D 5000

E 50,000

O.



According to the scale shown, the length of side BC of rectangle ABCD (to the NEAREST CENTIMETRE) is

A 5 cm

B 6 cm

C 7 cm

D 8 cm

E 9 cm

Figure 10

Erika Kuendiger
University of Windsor

PERCEPTIONS OF PRE-SERVICE STUDENT TEACHERS ON MATHEMATICAL ACHIEVEMENT AND ON TEACHING MATHEMATICS. RESULTS OF A PILOT STUDY

The relevance of teachers' expectations in the learning process of students has been well recognized, since Rosenthal and Jacobson published their book 'Pygmalion in the Classroom' in 1968. Many attempts have been made to trace the channels by which teachers' expectations and students' achievement are linked together. In particular, this question has become of interest in research focusing on sex-related differences in students' mathematical achievement and course-taking behavior.

Let us assume for a moment a teacher has the following attitude: girls are not as able as boys when it comes to mathematics and, moreover, in their future profession they are not going to need it as much as boys do. According to this attitude the teacher does not expect the girls in his/her class to do very well in mathematics.
There are several possible ways in which this teacher's expectations might be communicated to the students. The teacher might openly display them when commenting on the poor work of a girl, and/or he/she might consciously or unconsciously use more indirect ways;e.g., praising a girl very much for correctly answering an easy question, asking mostly boys to solve really difficult problems ,and attributing good mathematical achievement by girls'to a lot of effort and by boys to ability.

Mathematics is not only a subject that more or less often gets sex-typed by teachers, parents and students themselves but it is also a subject that many people perceive as very difficult to learn.

Let us assume for a moment a primary teacher who succeeded in passing high school math fairly well. The teacher is probably female because most primary teachers are female. After high school she did not take any math courses at the university level.It was not until she entered the pre-service teacher training program that she had to deal with mathematics again. According to her personal experiences with mathematics she thinks that it is a difficult subject to learn and that she only got good grades because she worked very hard at it. Her self-esteem in relation to mathematics is low and she might even be convinced that somehow men are the better mathematicians because it appeared to her during high school that those students who seemed to have the least difficulty with mathematics and seemed to be the most self-confident in doing it were boys. Although they did not necessarily get the best grades, the boys did not seem to need a lot of effort to grasp the main concepts.

During her time in preservice teacher training our imaginary teacher pays particular attention to learn how to teach mathematics,for, as she sees it, this is the most difficult subject she is going to teach. She likes to collect as many teaching ideas as possible. The more ready made they are the better. She wants to be prepared for all possible situations and she is going to use her entire stock when teaching math. Being very hesitant about trying new things, in future years she will not give her students much scope to bring math problems they encounter outside school into the classroom. There is still a fear that she might not be able to solve unfamiliar ploblems.

Of course the learning history of our teacher could be completely different. Let us assume for a moment that she is a high school teacher. She is one of the few female high school teachers who have mathematics as a teachable subject. For her, doing math was always an enjoyable adventure. She is proud of her success in this subject and found it easy to teach right from the beginning of her career as a teacher.

Her self-confidence in her math ability being well established, she is not afraid of challenging questions from her students. On the contrary she appreciates them as they demonstrate that her students are interested in math. She often uses these questions as starting points for math investigations, of which she herself does not know the results in advance.

The above examples are hypothetical. In fact very little is known about the mathematical learning history of teachers teaching math at different grade levels, about the relationship of this history to their perceptions on teaching mathematics, and about their actual teaching. Moreover, the same is true for students entering a preservice program, that is for teachers to come. It seems to be reasonable to assume that the learning history strongly influences certain aspects of teaching and that it is worth investigating these variables.
In 1984/85 a pilot study was carried out at the University of Windsor. The study focused - among other things - on answering the following questions:

What is the personal learning history in mathematics of pre-service student teachers?

How confident are they in teaching mathematics?

What reasons do they give when the students they taught during practice teaching did not make much progress in mathematics?

Do the answers to the above stated questions depend on the sex of the student teacher and/or on the division he/she has chosen to teach?

Sample and Procedure

The teacher training program of the University of Windsor is a one year program and includes three divisions, these are the primary/junior (K - 6), the junior/intermediate (4 - 8) and the intermediate/senior (7 - 13) division. Students enrolled in the primary/junior program have to take the math education course offered for their division, while for students in the other divisions math education is an optional course .Students enrolled in either the junior/intermediate or the intermediate/senior division are grouped together for the analysis of the results referred to as jun/int/senior division.

Relevant information was gathered via a questionnaire.
The variable 'learning history in mathematics' was operationalized as follows: students were asked to evaluate their mathematical achievement during their schooldays and to attribute reasons to their achievement. The questionnaire used to measure the latter variable was developed by the author in the course of another research study (s. Schildkamp-Kuendiger 1980).

Moreover, the student teachers were asked to compare their achievement in mathematics and their confidence in teaching math with that in other subjects. They were also asked to report sex-related students'achievement differences they had found in schools. To evaluate the reasons the student teachers gave for the pupils they taught not makind satisfacory progress in mathematics, a questionnaire developed for the Second International Mathematics Study was used.

The questionnaire was answered by students during math educational classes after they had been out for their third of four practice teaching sessions. Students answered on a voluntary and anonymous basis.
Chi Square Tests were used to compare the responses of different groups of students; e.g. male and female student teachers enrolled in the primary/junior division.
In the graphs showing the results, arithmetic means are used to characterise the distributions.

Results

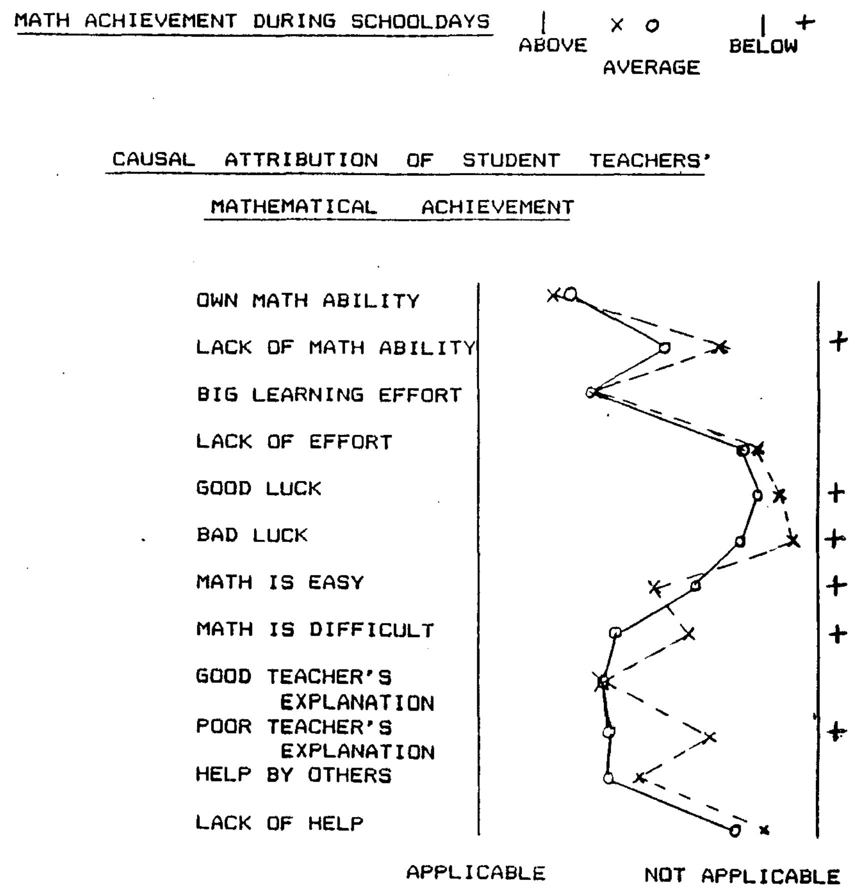
Overall 111 student teachers, enrolled in the primary/junior division, answered the questionnaire; 96 female and 15 male teachers.
The corresponding numbers for the jun/int/senior division are: overall 61 student teachers; 36 female and 25 male teachers.

Results will be discussed for primary/junior student teachers first. Information about their learning history in mathematics is presented in graph 1. As a group primary/junior teachers remember their math achievement during their schooldays as average and they account for it by a lot of reasons. The internal reasons stated are: math ability and learning effort; relevant external reasons are : math is difficult, good or poor teacher's explanation and help by others.
Significant sex-differences ($p < 0.05$) within the group are found for one variable only, that is lack of help of others. This variable is not considered as very relevant in general, but female student teachers judge it as even less relevant than male teachers.
Primary/junior student teachers remember their math achievement as being less good than their achievement in other school subjects (see grahp 2). This goes together with their perception of being comparatively less good in teaching this subject.
It is encouraging that they only sometimes encountered sex-related achievement differences in their pupils during practice teaching. Moreover, the questionnaire reveals that, if sex-related differences had been observed, they did not show in a particular subject like mathematics.
Male and female primary/junior teachers do not differ significantly as to the variables considered in graph 2.

Graph 3 displays the reasons teachers perceive as relevant when the pupils they taught during practice teaching did not make satisfactory progress in mathematics. Primary/junior student teachers mention two reasons the most: lack of ability of the pupil and lack of motivation. Lack of student ability is a revealing reason for the teacher; this is not his/her responsibility. Motivating to learn on the other hand is something that falls in the duty of a teacher.
Male and female teachers differ significantly in their evaluation of students' misbehavior and lack of motivation; female teachers judge these reasons as more important as their male colleagues.

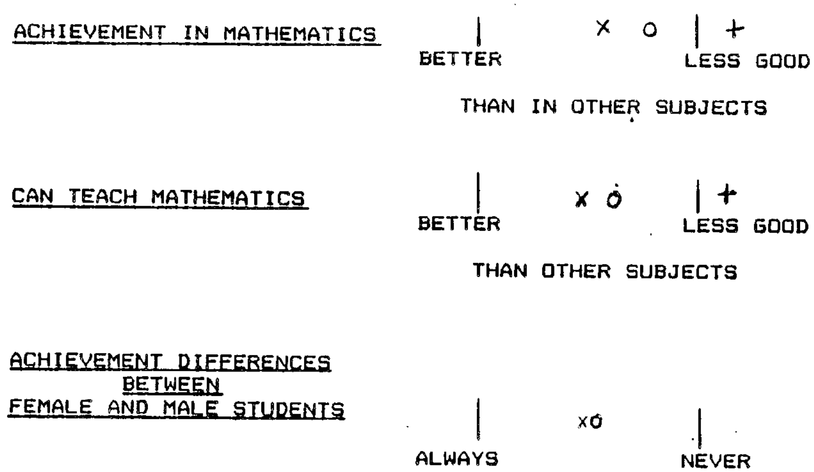
Overall, there are very few significant sex-related differences between male and female primary/junior student teachers. This is certainly partly due to the fact that there are very few male teachers in this sample. It seems as if teaching in the primary/junior grades will stay mainly a female affair. Whether or not male and female primary/junior student teachers can really be looked upon as having the same charcteristics as to the variables considered here has to be answered by subsequent research.

GRAPH 1



O PRIMARY/JUNIOR STUDENT TEACHERS, N = 111
X JUN/INT/SENIOR STUDENT TEACHERS, N = 61
+ INDICATES SIGNIFICANT DIFFERENCES BETWEEN THESE TWO GROUPS ($p < 0.05$, CHI SQUARE TEST).

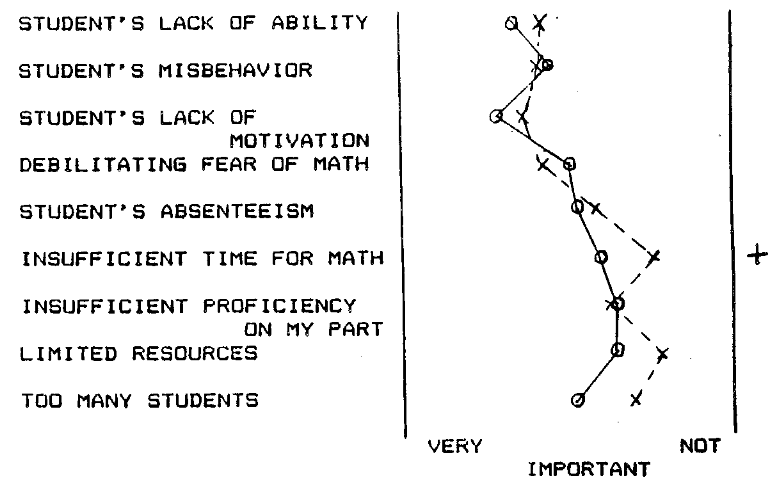
GRAPH 2



O PRIMARY/JUNIOR STUDENT TEACHERS, N = 111
× JUN/INT/SENIOR STUDENT TEACHERS, N = 61
+ INDICATES SIGNIFICANT DIFFERENCES BETWEEN THESE TWO GROUPS
($p < 0.05$, CHI SQUARE TEST).

GRAPH 3

REASONS THAT STUDENT TEACHERS GIVE FOR PUPILS NOT MAKING SATISFACTORY PROGRESS IN MATHEMATICS



O PRIMARY/JUNIOR STUDENT TEACHERS, N = 111
× JUN/INT/SENIOR STUDENT TEACHERS, N = 61
+ INDICATES SIGNIFICANT DIFFERENCES BETWEEN THESE TWO GROUPS
($p < 0.05$, CHI SQUARE TEST).

The learning history of jun/int/senior student teachers is quite different from that of the primary/junior group (s. graph 1). Jun/int/senior student teachers remember their achievement during schooldays as above average. It is significantly higher than that of the primary/junior group. Moreover, the jun/int/senior group states fewer reasons for this achievement. Lack of ability, good luck, bad luck, difficulty of math and poor teacher explanation are referred to significantly less as causes of achievement. At the time math is perceived as easier.

The jun/int/senior group remembers its math achievement as about as good as in other subjects and judges its ability in relation to the subject as average. For both variables the differences between the jun/int/senior teachers and the primary/junior teachers are significant.
The groups do not differ as to the extent achievement differences between boys and girls had been observed during practice teaching.

Graph 3 seems to indicate differences in the attribution pattern between jun/int/senior student teachers and primary/junior student teachers in the direction that the jun/int/senior teachers need fewer reasons to account for students not making satisfactory progress in mathematics. Only for the reason 'insufficient time for math' are the differences significant on the 5% level, but there is a trend ($p < 0.07$) for the reasons: students' lack of motivation, limited resources, and too many students.
Overall the two groups of student teachers seem to differ in several aspects considered in this reseach.

Significant differences between male and female primary/junior student teachers were rare; this is not the case for the jun/int/senior group. Although the whole group remembers its math achievement during schooldays as above average, this is even more true for the female teachers ($p < 0.01$). Moreover, female teachers evaluate their math ability and good teachers' explanations as more relevant a reason for their achievement than do male teachers ($p < 0.05$) ; whereas lack of effort is perceived as less a reason by female teachers ($p < 0.05$).

Finally there is another rather unexpected significant difference between male and female jun/int/senior student teachers: when it comes to explaining why their pupils did not make satisfactory progress female teachers more often perceive insufficient proficiency on their part to be the reason. This is the more astonishing as ,according to their learning history in mathematics, they should be even more self-confident than the male teachers.

Summary

The results of this pilot study reveal considerable differences between student teachers in the primary/junior division and those in the junior/intermediate/senior division. Student teachers of the latter division have a much more positive learning history in mathematics than the primary/junior group.
Of course, it can be argued that the group of jun/int/senior student teachers considered here would not have taken the math course, if they had not felt rather confident in this subject as they had a choice the primary/junior student teachers did not have. The situation becomes more delicate , if this learning history is looked upon as having important impact on the quality of teaching. Afterall all these student teachers will try hard to get a teaching postion after finishing the program. As a group the primary/junior student teachers will start teaching math with less confidence in their ability to teach this subject than to teach other subjects.
It can be expected that - in doing the job - they will get more confident in teaching mathematics. But the hypothesis can not easily be turned down that they might gain this confidence by following a rather rigid teaching method that minimizes the challenge of unexpected questions and problems.
Up to now the results of this pilot study indicate - at least as a trend - that the primary/junior student teachers need to call upon more reasons than the other group to explain why their pupils fail to learn mathematics. Further information about what student teachers think to be important to make math teaching more effective is available and will be analysed in the near future.

With regard to sex-differences the results indicate some astonishing differences for jun/int/senior students. It seems as if female student teachers only choose to teach mathematics if they feel very confident about their competence in the subject. This goes together with a readiness to explain failures in their pupils' learning more often by personal insufficient proficiency than do their male colleagues. The question remains open what hidden message these future female teachers are going to deliver to their female students.

Reseach about sex-related achievement differences at school have demonstrated that girls tend to have lower self-esteem in relation to their math ability, even when they have the same achievement as boys. It is worth investigating if there is a parallel at the teacher level in so far as the perception of personal teaching proficiency comes into play.

Literature

Rosenthal, R., Jacobson, L.: Pygmalion in the classroom. New York 1968.
Schildkamp-Kuendiger, E.: Learnig the concept of a function. In: W.F. Archenhold et al. (eds.): Cognitive Development in science and mathematics. Leeds 1980, p. 181-190.

Roberta Mura

Université Laval

Mécanismes d'actualisation de la sous-représentation des femmes en mathématique: présentation d'un projet en cours

Un sondage réalisé en 1981 sur l'état de la recherche concernant les différences reliées au sexe en mathématique au Canada, avait indiqué qu'à peu près dans tout le pays, la participation des filles aux cours de mathématique commence à décliner vers la fin du secondaire, mais qu'aucune recherche n'avait été effectuée pour tenter d'expliquer ce phénomène (Mura, 1982).

Cette constatation m'a incitée à concevoir une première étude exploratoire sur ce sujet. Comme il me semblait important d'étudier le phénomène dans sa globalité, j'ai cherché la collaboration de collègues avec des compétences en sociologie et en psychologie; Renée Cloutier et Meredith Kimball, ont accepté de se joindre à moi et notre projet a obtenu une subvention du Conseil de recherches en sciences humaines du Canada.

Au Québec, dans le secteur francophone, le phénomène de la sous-représentation des femmes en mathématique s'amorce au passage du secondaire au collégial (Cegep) -- c'est-à-dire de la 11ème à la 12ème année. D'après les statistiques fournies par le Ministère de l'Education du Québec, en 5ème secondaire (dernière année de l'école secondaire), même si les cours de mathématique ne sont pas obligatoires, depuis plusieurs années, les filles représentent 50,5% de la clientèle de ces cours. Au collégial par contre, à l'automne 1984, elles n'en constituaient plus que 42%. Toujours d'après le Ministère de l'Education, la réussite des filles, au secondaire comme au Cegep, est aussi bonne que celle des garçons, sinon meilleure.

Tout en étant conscientes que les racines des choix que les élèves font en entrant au Cegep peuvent remonter loin dans le passé, nous avons décidé d'aborder le problème en étudiant ce choix au moment de sa formulation, c'est-à-dire vers la fin de la cinquième année du secondaire.

La première phase de la cueillette de données a eu lieu de février à mai 1983 dans trois de ces classes de mathématique de cinquième secondaire. Pendant cette période les élèves faisaient, le cas échéant, leur demande d'admission au Cegep. Les mêmes élèves ont ensuite été contacté/e/s à nouveau un an plus tard.

Nous savions que le phénomène de la différenciation des choix scolaires selon le sexe était très complexe et nous avons choisi d'en brosser un tableau global, plutôt que d'en étudier plus en détail quelques aspects seulement. Dans cette perspective, nous avons opté pour l'emploi simultané d'une variété de méthodes de cueillette des données: questionnaires aux élèves, observations en classes, entrevues avec les élèves et avec leurs enseignant/e/s de mathématique.

Nous avons retenu un grand nombre de variables. Parmi les principales, on retrouve les suivantes:

- l'occupation et la scolarité des parents,
- l'écart entre l'image de soi et l'image d'une personne de science,
- la valeur intrinsèque et la valeur utilitaire attribuées à la mathématique,
- l'attitude envers le succès en mathématique et en français,
- la confiance en ses capacités en mathématique,
- les causes auxquelles les élèves attribuent leurs succès et échecs en mathématique et en français,
- les prévisions de réussite en mathématique,
- les aspirations scolaires et professionnelles,

- la présence de modèles de rôles scientifiques dans le milieu de l'élève,
- les cours suivis et les notes obtenues,
- les motivations du choix scolaire telles qu'exprimées par les élèves,
- l'attitude du milieu de l'élève envers son choix scolaire,
- les interactions entre les élèves et leur enseignant/e de mathématique,
- la perception que l'enseignant/e a du potentiel de ses élèves en mathématique, de leur intérêt pour cette matière et de leur niveau de confiance,
- les prévisions de l'enseignant/e à l'égard de la réussite de ses élèves,
- les causes auxquelles les enseignant/e/s attribuent les succès et les échecs de leurs élèves.

Dans le choix de ces variables, nous nous sommes en partie inspirées du modèle présenté par Eccles (1985) -- modèle qui était déjà disponible avant le début de notre projet.

Toutes les variables ont été analysées en fonction du sexe et du choix scolaire des élèves. Le choix scolaire a été défini à partir de la demande d'admission au Cegep faite par les élèves au printemps 1984; nous avons ainsi distingué les élèves qui ont choisi une orientation scientifique de ceux et celles qui ont choisi une autre orientation. Tel que prévu, le premier groupe comprenait proportionnellement moins de filles que de garçons. Cette définition du choix scolaire a le désavantage d'élargir le champs d'étude de la mathématique aux sciences, mais elle nous a semblé plus fiable qu'une définition basée sur les intentions de suivre des cours de mathématique exprimées par les élèves, car dans la demande d'admission l'élève spécifie le programme auquel il, ou elle, veut s'inscrire sans préciser les cours particuliers qui seront suivis.

Je présente ici seulement quelques résultats préliminaires à titre d'exemples, invitant les personnes intéressées à se procurer le rapport final à la fin de 1985.

Dans l'ensemble, nous avons trouvé plus de différences reliées au choix scolaire que de différences reliées au sexe. Ainsi, l'écart entre l'image que les élèves ont d'eux-mêmes, ou d'elles-mêmes, et l'image qu'ils, ou elles, se font d'une personne de science est plus petit chez les élèves qui s'orientent vers les sciences que chez les autres. De même, le premier groupe attribue une plus grande valeur intrinsèque et utilitaire à la mathématique et possède plus de confiance en ses capacités dans cette matière. Parmi ces quatre variables, la dernière est la seule qui a donné lieu à une différence entre filles et garçons, ces derniers manifestant un plus haut niveau de confiance.

Il existe toutefois quelques exceptions. Par exemple, à propos des causes auxquelles les élèves attribuent leurs succès et échecs en mathématique, nous avons trouvé des différences selon le sexe, mais non selon le choix scolaire: les filles attribuent très majoritairement leurs succès à leurs efforts, tandis que les garçons sont partagés entre leurs efforts et leur habileté. Pour ce qui est des explications de l'échec, la majorité des filles comme des garçons fait appel au manque d'effort, mais quelques filles invoquent aussi leur manque d'habileté ou la difficulté de la tâche. Les mêmes tendances se sont manifestées à propos des causes par lesquelles les enseignant/e/s expliquent les succès et échecs de leurs élèves. Nous n'avons pas trouvé de différence analogue entre filles et garçons dans leur perception des causes de succès et d'échec en français.

Une autre différence importante entre filles et garçons est apparue dans leurs propres projets d'emploi et dans ce qu'elles, ou ils, prévoient pour le conjoint, ou la conjointe, lorsque viendront les enfants: garçons et filles s'accordent majoritairement pour dire que ce seront ces dernières qui assumeront les responsabilités majeures au niveau des tâches familiales et limiteront leur emploi à l'extérieur au temps partiel ou même le suspendront complètement.

L'influence de ce facteur sur le choix scolaire est liée à l'image des sciences comme domaine particulièrement exigeant, où il est difficile de poursuivre des études ou une carrière à temps partiel, ou de les reprendre après une interruption.

Enfin, un dernier exemple de différence entre filles et garçons touche leur comportement en classe de mathématique: nous avons observé que les garçons participaient beaucoup plus vocalement que les filles, en répondant à 75% des questions de l'enseignant/e lorsque celles-ci n'étaient pas adressées à un/e élève en particulier (les garçons constituaient 44% de notre échantillon). Avant d'avancer des hypothèse sur le rôle de ce facteur dans les choix scolaires, il faudrait cependant effectuer des observations pour savoir si ce comportement ne se retrouve pas aussi dans des classes où l'on aborde des disciplines non scientifiques.

Références

Eccles, J. (1985). Model of students' mathematics enrollment decisions. Educational Studies in Mathematics, 16:3, 311-314.

Mura, R. (1982). Gender and mathematics in Canada. In E. Schildkamp-Kündiger (Ed.), An International Review of Gender and Mathematics, (pp. 32-43). ERIC Science, Mathematics and Environmental Education Clearinghouse, The Ohio State University, Columbus, Ohio.

Roberta Mura
Faculté des sciences de l'éducation
Université Laval
Québec, P.Q., G1K 7P4