CMESG/GCEDM

1986 PROCEEDINGS

JUNE 8-12,1986

MEMORIAL UNIVERSITY

EDITED BY C VERHILLE

UAlberta

GROUPE CANADIEN D'ETUDE EN DIDACTIQUE DES MATHEMATIQUES

CANADIAN MATHEMATICS EDUCATION STUDY GROUP

PROCEEDINGS OF THE 1986 MEETING

MEMORIAL UNIVERSITY

ST. JOHN'S, NEWFOUNDLAND

JUNE 8-12, 1986

EDITED BY CHARLES VERHILLE

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Editor's foreword

Never has the CMESG ventured so far east. Never before has the CMESG left the mainland. And never have we had a meeting more memorable than our visit to Memorial University in St. John's, Newfoundland. Special touches of Newfoundland hospitality have indelibly influenced those who were ritually initiated as honorary "SCREECHERS".

We are especially appreciative for the excellent representation on our behalf by Ed Williams. As our local organizer, Ed was instrumental in arranging the most enjoyable social agenda as well as the facilities for our professional agenda at Memorial University.

David Wheeler, one of the group instrumental in founding the CMESG, announced that he was stepping down as chairman of the CMESG. David has agreed to join the executive as past chairman in order that we may continue to benefit from his advice and interest in the group.

The major lectures were presented by Ross Finney and Alan Schoenfeld. Ross Finney kindly offered to present a lecture when the previously arranged speaker withdrew at the last minute. Alan Schoenfeld delivered a joint lecture to the CMS and the CMESG.

Charles Verhille

CMESG/GCEDM
1986 Meeting
PROCEEDINGS

Preface
Canadian Mathematics Education Study Group
Groupe canadien d'etude en didactique des mathematiques

The Study Group held its tenth annual meeting at Memorial University from June 8 to 12, 1986. Travelling distance and CAUT censure made this much the smallest of the Study Group's meetings. But the thirty-one participants managed to assemble a lively programme and to generate a comfortable working atmosphere.

Working Groups, as always the mainstay of the programme, this year covered **Affective aspects of problem solving** (led by Frances Rosamond, San Diego, and Peter Taylor, Queen's). **The problem of rigour in mathematics teaching** (led by Gila Hanna, OISE, and Lars Jansson, Manitoba), **Microcomputers in teacher education** (led by Charles Verhille, UNB) and **The role of the microcomputer in promoting statistical thinking** (led by Claude Gaulin, Laval and Lionel Mendoza, Memorial). In spite of the small numbers, each group managed to funcation and, miraculously, to flourish. It is worth repeating here, though it has been said in reports of earlier meetings, that the opportunity for a group to work for 9 hours on a single topic contributes powerfully to the productivity of the meetings and to the atmosphere of collaboration rather than competition that prevades them.

The principal guest speaker, Alan Schoenfeld (Berkley), threw himself into all aspects of the conference and delivered a dynamic address under the modest title of **Some thought on problem solving**. The lecture, jointly sponsored by the CMS Education Committee, gave extremely good value, being full of practical commonsense, critical analyses, cogent research results, and provocative speculation. Ross Finney (MIT), generously stepping in at the last minute to replace an advertised speaker, gave particpants several glimpses of the material collected by UMAP and COMAP, Harold Paddock (Memorial) refreshed the meeting with a witty and wide-ranging talk given from the prospective of a linguist and a poet on **Natural language and mathematics in human evolution.**

Other sessions included reports on the Second International Mathematics Study and the ICMI study on the impact of computers and informatics on the teaching of mathematics. Claude Janvier (UQAM) reviewed some of the research on representation, untaken by him and his colleagues. Several members gave brief surveys of the research activity in mathematics education in their provinces, and the final evening was rounded off with a dramatised reading of extracts from Lakatos' **Proofs and refutations.**

The local organizer, Ed Williams, by adding a banquet, a bus trip and (opportunistically) a run up Signal Hill, ensured that all the participants came away with pleasant memories of the host province, its capital, and its university.

David Wheeler
Chairman

IN MEMORIAM DIETER LUNKENBEIN

I would like us to use this opportunity to pause for a few minutes in order to pay tribute to one of our very dear colleagues: Dieter Lunkenbein, who was still among us at our conference last year and who died on last September 11th.

I had the chance to know Dieter and to start working with him shortly after his arrival to Canada in 1968, at the time he accepted a position as a research assistant to Professor Zoltan P. Dienes in Sherbrooke, Que. He was initially supposed to stay a few years in our country and then to return to Germany, his native land. But what happened is that he and his family decided to stay and live in Sherbrooke, where he had spent the last 17 years of his life. After taking his Ph.D. in mathematics education at Laval in the early 70s, he became the inspiring leader of a group of mathematics educators at the University of Sherbrooke, as well as a very active collaborator to the Quebec Ministry of Education and to the three major Quebec mathematics teachers associations.

In 1977, Dieter Lunkenbein was present at Kingston, Ontario, when the meeting that led to the creation of our Study Group took place. Since then he has been a regular participant to our meetings, making a remarkable contribution as a leader or a collaborator of many groups, particularly those on the development of geometrical thinking at the Elementary level, on research in mathematics education and on children's "errors" in mathematics.

In 1979, Dieter received the "Abel Gauthier Prize" in recognition for his exceptional contribution to mathematics education in Quebec. Besides his involvement in Canada, Dieter has also been quite active at the international level during the last ten years. In 1982, he was elected President of the "Commission Internationale pour l'Etude et l'Amelioration de l'Enseignement de la Mathematique" in Europe. But unfortunately, he had to resign from that position before the end of his mandate, after having gone through a heart operation.

Last year, Dieter had apparently recovered so well that in June he accepted a position as assistant dean of the Faculty of Education at the University of Sherbrooke, and that in July he participated in an international conference in Bielefeld, Germany. But two months later, alas, we heard the tragic news of his death at 48 years of age, at an age he still had so much to offer and to contribute.

To all those who have known Dieter Lunkenbein, his death means a great loss. On the one hand, we have lost a man with a rich personality and with remarkable human qualities: Dieter was friendly, generous, modest, and he had a great respect for others. On the other hand, we have lost a colleague with outstanding professional qualities: Dieter was a hard worker, with high standards of rigor and integrity, ever searching for truth and strongly dedicated to his work in mathematics education. Let us have good thoughts for him!

Claude Gaulin

LECTURE 1

CONFESSIONS OF AN ACCIDENTAL THEORIST

BY ALAN SCHOENFELD
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Confessions of an Accidental Theorist

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Confessions of an Accidental Theorist

David Wheeler had both theoretical and pragmatic reasons for inviting me to write this article. On the theoretical side, he noted that my ideas on "understanding and teaching the nature of mathematical thinking" have taken some curious twists and turns over the past decade. Originally inspired by Pólya's ideas and intrigued by the potential for implementing them using the tools of artificial intelligence and information-processing psychology, I set out to develop prescriptive models of heuristic problem solving -- models that included descriptions of how, and when, to use Pólya's strategies. (In moments of verbal excess I was heard to say that my research plan was to "understand how competent problem solvers solve problems, and then find a way to cram that knowledge down students' throats.") Catch me talking today, and you'll hear me throwing about terms like metacognition, belief systems, and "culture as the growth medium for cognition;" there's little or no mention of prescriptive models. What happened in between? How were various ideas conceived, developed, modified, adapted, abandoned, and sometimes reborn? It might be of interest, suggested David, to see where the ideas came from. With regard to pragmatic issues, David was blunt. Over the past decade I've said a lot of stupid things. To help keep others from re-inventing square theoretical or pedagogical wheels, or to keep people from trying to ride some of the square wheels I've developed, he suggested, it might help if I recanted in public. So here goes. . .

The story begins in 1974, when I tripped over Pólya's marvelous little volume *How to Solve It.* The book was a *tour de force,* a charming exposition of the problem solving introspections of one of the century's foremost mathematicians. (If you don't own a copy, you should.) In the spirit of Descartes, who had, three hundred years earlier, attempted a similar feat in the *Rules for the Direction of the Mind,* Pólya examined his own thoughts to find useful patterns of problem solving behavior. The result was a general description of problem solving processes: a four-phase model of problem solving (understanding the problem, devising a plan, carrying out the plan, looking back), the details of which included a range of problem solving *heuristics,* or rules of thumb for making progress on difficult problems. The book and Pólya's subsequent elaborations of the heuristic theme (in *Mathematics and Plausible Reasoning*, and *Mathematical Discovery*) are brilliant pieces of insight and mathematical exposition.

A young mathematician only a few years out of graduate school, I was completely bowled over by the book. Page after page, Pólya described the problem solving techniques that he used. Though I hadn't been taught them, I too used those techniques; I'd picked them up then pretty much by accident, by virtue of having solved thousands of problems during my mathematical career (That is, I'd been "trained" by the discipline, picking up bits and pieces of mathematical thinking as I developed). My experience was hardly unique, of course. In my excitement I joined thousands of mathematicians who, in reading Pólya's works, had the same thrill of recognition. In spirit I enlisted in the army of teachers who, inspired by Pólya's vision, decided to focus on teaching their students to think mathematically instead of focusing merely on the mastery of mathematical subject matter.

To be more accurate, I thought about enlisting in that army. Excited by my readings, I sought out some problem-solving experts, mathematics faculty who coached students for the Putnam exam or for various Olympiads. Their verdict was unanimous and unequivocal: Pólya was of no use for budding young problem-solvers. Students don't learn to solve problems by reading Pólya's books, they said. In their experience, students learned to solve problems by (starting with raw talent and) solving lots of problems. This was troubling, so I looked elsewhere for (either positive or negative) evidence. As noted above, I was hardly the first Pólya enthusiast: By the time I read *How to Solve It* the math-ed literature was chock full of studies designed to teach problem-solving *via* heuristics. Unfortunately, the results -- whether in first grade, algebra, calculus, or number theory, to name a few -- were all depressingly the same, and confirmed the statements of the Putnam and Olympiad trainers. Study after study produced "promising" results, where teacher and students alike were happy with the instruction (a typical phenomenon when teachers have a vested interest in a new program) but where there was at best marginal evidence (if any!) of improved problem solving performance. Despite all the enthusiasm for the approach, there was no clear evidence that the students had actually learned more as a result of their heuristic instruction, or that they had learned any general problem solving skills that transferred to novel situations.

Intrigued by the contradiction -- my gut reaction was still that Pólya was on to something significant -- I decided to trade in my mathematician's cap for a mathematics educator's and explore the issue. Well, not exactly a straight mathematics educator's; as I said above, math ed had not produced much that was encouraging on the problem solving front. I turned to a different field, in the hope of blending its insights with Pólya's and those of mathematics educators.

The first task I faced was to figure out why Pólya's strategies didn't work. If I succeeded in that, the next task was to make them work -- to characterize the strategies so that students could learn to use them. The approach I took was inspired by classic problem solving work in cognitive science and artificial intelligence, typified by Newell and Simon's (1972) *Human Problem Solving*. In the book Newell and Simon describe the genesis of a computer program called General Problem Solver (GPS), which was developed to solve problems in symbolic logic, chess, and "cryptarithmetic" (a puzzle domain similar to cryptograms, but with letters standing for numbers instead of letters). GPS played a decent game of chess, solved cryptarithmetic problems fairly well, and managed to prove almost all of the first 50 theorems in Russell and Whitehead's *Principia Mathematica* -- all in all, rather convincing evidence that its problem solving strategies were pretty solid.

Where did those strategies come from? In short, they came from detailed observations of people solving problems. Newell, Simon, and colleagues recorded many people's attempts to solve problems in chess, cryptarithmetic, and symbolic logic. They then explored those attempts in detail, looking for uniformities in the problem solvers' behavior. If they could find those regularities in people's behavior, describe those regularities precisely (i.e. as computer programs), and get the programs to work (i.e. to solve problems) then they had pretty good evidence that the strategies they had characterized were useful. As noted above, they succeeded. Similar techniques had been used in other areas: for example, a rather simple program called SAINT (for Symbolic Automatic INTegrator) solved indefinite integrals with better facility than most M.I.T. freshmen. In all such cases, AI produced a set of *prescriptive* procedures

-- problem solving methods described in such detail that a machine, following their instructions, could obtain pretty spectacular results.

It is ironic that no one had thought to do something similar for human problem solving. The point is that one could turn the man-machine metaphor back on itself. Why not make detailed observations of expert human problem solvers, with an eye towards abstracting regularities in their behavior -- regularities that could be codified as prescriptive guides to human problem solving? No slight to students was intended by this approach, nor was there any thought of students as problem solving machines. Rather, the idea was to pose the problem from a cognitive science perspective: "What level of detail is needed so that students can actually use the strategies one believes to be useful?" Methodologies for dealing with this question were suggested by the methodologies used in artificial intelligence. One could make detailed observations of individuals solving problems, seek regularities in their problem solving behavior, and try to characterize those regularities with enough precision, and in enough detail, so that students could take those characterizations as guidelines for problem solving. That's what I set out to do.

The detailed studies of problem solving behavior turned up some results pretty fast. In particular, they quickly revealed one reason that attempts to teach problem solving *via* heuristics had failed. The reason is that Pólya's heuristic strategies weren't really coherent strategies at all. Pólya's characterizations were broad and descriptive, rather than prescriptive. Professional mathematicians could indeed recognize them (because they knew them, albeit implicitly), but novice problem solvers could hardly use them as guides to productive problem solving behavior. In short, Pólya's characterizations were *labels* under which families of related strategies were subsumed. There isn't much room for exposition here, but one example will give the flavor of the analysis. The basic idea is that when you look closely at any single heuristic "strategy," it explodes into a dozen or more similar, but fundamentally different, problem-solving techniques. Consider a typical strategy, "examining special cases:"

> To better understand an unfamiliar problem, you may wish to exemplify the problem by considering various special cases. This may suggest the direction of, of perhaps the plausibility of, a solution.

Now consider the solutions to the following three problems.

Problem 1. Determine a formula in closed form for the series

$$\sum_{i=1}^{n} k/(k+1)!$$

Problem 2. Let P(x) and Q(x) be polynomials whose coefficients are the same but in "backwards order:"

$$P(x) = a_0 + a_1x + a_2x^2 + \ldots a_nx^n \text{, and}$$

$$Q(x) = a_n + a_{n-1}x + a_{n-2}x^2 + \ldots a_0x^n.$$

What is the relationship between the roots of P(x) and Q(x)? Prove your answer.

Problem 3. Let the real numbers a_0 and a_1 be given. Define the sequence $\{a_n\}$ by

$$a_n = 1/2\,(a_{n-2} + a_{n-1}) \quad \text{for each } n \geq 2.$$

Does the sequence $\{a_n\}$ converge? If so, to what value?

I'll leave the details of the solutions to you. However, the following observations are important. For problem 1, the special cases that help are examining what happens when where the integer parameter **n** takes on the values 1, 2, 3, . . . in sequence; this suggests a general pattern that can be confirmed by induction. Yet if you try to use special cases in the same way on the second problem, you may get into trouble: Looking at values n=1, 2, 3, . . . can lead to a wild goose chase. It turns out that the right special cases of P(x) and Q(x) you to look at for problem 2 are easily factorable polynomials. If, for example, you consider

$$P(x) = (2x + 1)\,(x + 4)\,(3x - 2),$$

you will discover that its "reverse," Q, is easily factorable. The roots of the P and Q are easy to compare, and the result (which is best proved another way) is obvious. And again, the special cases that simplify the third problem are different in nature. If you choose the values a_0=0 and a_1=1, you can see what happens for that particular sequence. The pattern in that case suggests what happens in general, and (especially if you draw the right picture!) leads to a solution of the original problem.

Each of these problems typifies a large class of problems, and exemplifies a different special cases strategy. We have:

Strategy 1. When dealing with problems in which an integer parameter n plays a prominent role, it may be of use to examine values of **n**=1, 2, 3, . . . in sequence, in search of a pattern.

Strategy 2. When dealing with problems that concern the roots of polynomials, it may be of use to look at easily factorable polynomials.

Strategy 3. When dealing with problems that concern sequences or series that are constructed recursively, it may be of use to try initial values of 0 and 1 -- if such choices don't destroy the generality of the processes under investigation.

Needless to say, these three strategies hardly exhaust "special cases." At this level of analysis -- the level of analysis necessary for implementing the strategies -- one could find a dozen more. This is the case for almost all of Pólya's strategies. In consequence the two dozen or so "powerful strategies" in *How to Solve It* are, in actuality, a collection of two or three hundred less "powerful," but actually usable strategies. The task of teaching problem solving *via* heuristics -- my original goal --thus expanded to (1) explicitly identifying the most frequently used techniques from this long list, (2) characterizing them in sufficient detail so that students could use them, and (3) providing the appropriate amount and degree of training.

[Warning: It is easy to underestimate both the amount of detail and training that are necessary. For example, to execute a moderately complex "strategy" like "exploit an easier related problem" with success, you have to (a) think to use the strategy (non-trivial!); (b) know which version of the strategy to use; (c) generate *appropriate* and *potentially useful* easier related problems; (d) make the right choice of related problem; (e) solve the problem; and (f) find a way to exploit its solution to help solve the original problem. Students need instruction in all of these.]

Well, this approach made progress, but it wasn't good enough. Fleshing out Pólya's strategies did make them implementable, but it revealed a new problem. An arsenal of a dozen or so powerful techniques may be manageable in problem solving. But with all the new detail, our arsenal comprised a couple of hundred problem solving techniques. This caused a new problem, which I'll introduce with an analogy.

A number of years ago, I deliberately put the problem

$$\int \frac{x\ dx}{x^2 - 9}$$

as the first problem on a test, to give my students a boost as they began the exam. After all, a quick look at the fraction suggests the substitution $u = x^2 - 9$, and this substitution knocks the problem off in just a few seconds. 178 students took the exam. About half used the right substitution and got off to a good start, as I intended. However, 44 of the students, noting the factorable denominator in the integrand, used partial fractions to express x/x^2-9 in the form $[A/x-3 + B/x+3]$ -- correct but quite time-consuming. They didn't do too well on the exam. And 17 students, noting the $(u^2 - a^2)$ form of the denominator, worked the problem using the substitution $x = 3\sin\theta$. This too yields the right answer -- but it was even more time-consuming, and the students wound up so far behind that they bombed the exam.

Doing well, then, is based on more than "knowing the subject matter;" it's based on knowing which techniques to use and when. If your strategy choice isn't good, you're in trouble. That's the case in techniques of integration, when there are only a dozen techniques and they're all algorithmic. As we've seen, heuristic techniques are anything but algorithmic, and they're much harder to master. In addition, there are hundreds of them -- so strategy selection becomes even more important a factor in success. My point was this. Knowing the strategies isn't enough. You've got to know when to use which strategies.

As you might expect by now, the AI metaphor provided the basic approach. I observed good problem solvers with an eye towards replicating their heuristic strategy selection. Generalizing what they did, I came up with a prescriptive scheme for picking heuristics, called a "managerial strategy." It told the student which strategies to use, and when (unless the student was sure he had a better idea). Now again, this approach is not quite as silly as it sounds. Indeed, the seeds of it are in Pólya ("First. You have to understand the problem."). The students weren't forced to follow the managerial strategy like little automata. But the strategy suggested that heuristic techniques for understanding the problem should be used first, planning heuristics next, exploration heuristics in a particular order (the metric was that the further the exploration took you from the original problem, the later you should consider it), and so on. In class we talked about which heuristic technique we might use at any time, and why. Was the approach reductive? Maybe so. But the bottom line is that this combination of making the heuristics explicit, and providing a managerial strategy for students, was gloriously successful.

The final examinations for my problem solving courses had three parts. Part 1 had problems similar to problems we had worked in the course. Part 2 had problems that could be solved by the methods we had studied, but the problems did not resemble ones we had worked. Part 3 consisted of problems that had stumped me. I had looked through contest problem books, and as soon as I found a problem that baffled me, I put it on the exam! The students did quite well even on part 3; some solved problems on which I had not made progress, in the same amount of time.

Thus ended Phase I of my work. At that point -- the late 1970's to 1980 -- I was pretty happy with the instruction, and was getting pretty good results. Then something happened that shook me up quite a bit. Thanks to a National Science Foundation grant I got a videotape machine, and actually looked at students' problem solving behavior. What I saw was frightening.

Even discounting possible hyperbole in the last sentence, one statement in the previous paragraph sounds pretty strange. I'd been teaching for more than a decade and doing research on problem solving for about half that time. How can I suggest that, with all of that experience, I had never really looked at students' problem solving behavior?

With the videotape equipment, I brought students into my office, gave them problems (before, after, and completely independently of my problem solving courses), and had them work on the problems at length. Then, at leisure, I looked at the videotapes and examined, in detail, what the students actually did while they worked on the problems. What I saw was nothing like what I expected, and nothing like what I saw as a teacher. That's because as teachers (and often as researchers) we look at a very narrow spectrum of student behavior. Generally speaking, we only see what students produce on tests; that's the product, but focusing on the product leaves the process by which it evolved largely invisible. (There's a substantial difference between watching a 20-minute videotape of a student working a problem and reading the page or two of "solution" that student produced in those 20 minutes. The difference can be mind boggling.) In class, or in office hours, we have conversations with the students, but the conversations are directed toward a goal -- explaining something the student comes prepared to understand, and knows is coming. The student is primed for what we have to say. And that's the point. When we give students a calculus test and there's a max-min problem in it, students know it's a max-min problem. They've just finished a unit on max-min problems, and they expect to see a max-min problem on the exam. In other words, the context tells the students what mathematics to use. We get to see them at their very best, because (a) they're prepared, and (b) the general context puts them in the right ballpark and tells them what procedures to use. By way of analogy, you don't discover whether kids can speak grammatically (or think on their feet) when you given them a spelling test, after they've been given the list of words they'll be tested on. (Even when I taught the problem solving class, I was showing students techniques that they knew were to be used in the context of the problem solving class. Hence they came to my final prepared to use those techniques.)

In my office, problems come out of the blue and the context doesn't tell students what methods are appropriate. The result is that I get to see a very different kind of behavior. One problem used in my research, for example, is the following:

Problem 4. Three points are chosen on the circumference of a circle of radius **R**, and the triangle with those points as vertices is drawn. What choice of points results in the triangle with largest possible area? Justify your answer as well as you can.

Though there are clever solutions to this problem (see below), the fact is that you can approach it as a standard multivariate max-min problem. Virtually none of my students (who had finished 3rd-semester calculus, and who knew more than enough mathematics to knock the problem off) approached it that way. One particular pair of students had just gotten A's in their 3rd-semester calculus class, and each had gotten full credit on a comparably difficult problem on their exam. Yet when they worked on this problem they jumped into another (and to me, clearly irrelevant) approach altogether, and persisted at it for the full amount of allotted time. When they ran out of time, I asked them where they were going with that approach and how it might help them. They couldn't tell me. That solution attempt is best described as a twenty-minute wild goose chase.

Most of my videotapes showed students working on problems that they "knew" enough mathematics to solve. Yet time and time again, students never got to use their knowledge. They read the problem, picked a direction (often in just a second or two), and persevered in that direction no matter what. Almost sixty percent of my tapes are of that nature. But perhaps the most embarrassing of the tapes is one in which I recorded a student who had taken my problem solving course the year before.

There is an elegant solution to Problem 4, which goes as follows. Suppose the three vertices are **A**, **B**, and **C**. Hold **A** and **B** fixed, and ask what choice of **C** gives the largest area. It's clearly when the height of the triangle is maximized -- when the triangle is isosceles. So the largest triangle must be isosceles. Now you can either maximize isosceles triangles (a one-variable calculus problem), or finish the argument by contradiction. Suppose the largest triangle, **ABC**, isn't equilateral. Then two sides are unequal; say **AC** ≠ **BC**. If that's the case, however, the isosceles triangle with base **AB** is larger than **ABC** -- a contradiction. So **ABC** must be equilateral.

The student sat down to work the problem. He remembered that we'd worked it in class the previous year, and that there was an elegant solution. As a result, he approached the problem by trying to so something clever. In an attempt to exploit symmetry he changed the problem he was working on (without acknowledging that this might have serious consequences). Then, pursuing the goal of a slick solution he missed leads that clearly pointed to a straightforward solution. He also gave up potentially fruitful approaches that were cumbersome because "there must be an easier way." In short, a cynic would argue that he was worse off after my course than before. (That's how I felt that afternoon.)

In any case, I drew two morals from this kind of experience. The first is that my course, broad as it was, suffered from the kind of insularity I discussed above. Despite the fact that I was teaching "general problem solving strategies," I was getting good results partly because I had narrowed the context: students knew they were supposed to be using the strategies in class, and on my tests. If I wanted to affect the students' behavior in a lasting way, outside of my classroom, I would have to do something different. [Note: I had plenty of testimonials from students that my course had "made me a much better problem solver," "helped me do much better in all of my other courses," and "changed my life." I'm not really sanguine about any of that.] Second and more important, I realized that there was a fundamental mistake in the approach I had taken to teaching problem solving -- the idea that I could, as I put it so indelicately in the first paragraph of this paper, cram problem solving knowledge down my students' throats.

That kind of approach makes a naive and very dangerous assumption about students and learning. It assumes, in essence, that each student comes to you as a *tabula rasa*, that you can write you problem solving "message" upon that blank slate, and that the message will "take." And it just ain't so. The students in my problem solving classes were the successes of our system. They were at Hamilton College, at Rochester, or at Berkeley because they were good students; they were in a problem solving class (which was known as a killer) because they liked mathematics and did pretty well at it. They come to the class with well engrained habits -- the very habits that have gotten them to the class in the first place, and accounted for their success. I ignore all of that (well, not really; but a brief caricature is all I've got room for) and show them "how to do it right." And when they leave the classroom and are on their own... well, let's be realistic. How could a semester's worth of training stack up against an academic lifetime's worth of experience, especially if the course ignores that experience? (Think of what it takes to retrain a self-taught musician or tennis player, rather than than teach one from scratch. Old habits die very very hard, if they die at all.)

Well, the point is clear. If you're going to try to affect students' mathematical problem solving behavior, you'd better understand that behavior. That effort was the main thrust of what (linear type that I am) I'll call phase 2. Instead of trying to do things to (and with) students, the idea was to understand what went on in their heads when they tried to do mathematics. Roughly speaking, the idea was this. Suppose I ask someone to solve some mathematics problems for me. For the sake of a permanent record, I videotape the problem solving session (and the person talks out loud as he or she works, giving me a verbal "trace" as well.). My goal is to understand what the person did, why he or she did it, and how those actions contributed to his or her success or failure at solving the problem. Along the way I'm at liberty to ask any questions I want, give any tests that seem relevant, and perform any (reasonable) experiments. What do I have to look at, to be reasonably confident that I've focused on the main determinant of behavior, and on what caused success or failure?

The details of my answer are xvi+409 pages long. The masochistic reader may find them, as well as the details of the brief anecdotes sketched above, in my (1985) *Mathematical Problem Solving.* In brief, the book suggested that if you're going to try to make sense of what people do when they do mathematics, you'd better look at:

A. "Cognitive resources," one's basic knowledge of mathematical facts and procedures stored in LTM (long term memory.) Most of modern psychology, which studies what's in a person's head and how that knowledge is accessed, is relevant here.

B. Problem solving strategies or heuristics. I've said enough about these.

C. Executive or "Control" behavior. [For the record, this behavior is often referred to as "metacognition."] I discussed this above as well. It's not just what you know (A+B above), it's how you use it. The issue in the book was how to make sense of such things. It's tricky, for the most important thing in a problem solving session may be something that doesn't take place -- asking yourself if it's really reasonable to do something, and thereby forestalling a wild goose chase.

D. Belief systems. I haven't mentioned these yet, but I will now.

Beliefs have to do with your mathematical *weltanschauung,* or world view. The idea is that your sense of what mathematics is all about will determine how you approach mathematical problems. At the joint CMS/CMESG meetings in June 1986, Ed Williams told me a story that illustrates this category. Williams was one of the organizers of a problem solving contest which contained the following problem:

"Which fits better, a square peg in a round hole or a round peg in a square hole?"

Since the peg-to-hole ratio is $2/\pi$ (about .64) in the former case and $\pi/4$ (about .79) in the latter, the answer is "the round peg." (Since the tangents line up in that case and not in the other, there's double reason to choose that answer.) It seems obvious that you have to answer the question by invoking a computation. How else, except with analytic support, can you defend your claim?

It may be obvious to us that an analytic answer is called for, but it's not at all obvious to students. More than 300 students -- the cream of the crop -- worked the problem. Most got the right answer, justifying it on the basis of a rough sketch. *Only four students* out of more than 300 justified their answer by comparing areas. (I can imagine a student saying "you just said to say which fit better. You didn't say to prove it.") Why? I'm sure the students could have done the calculations. They didn't think to, because they didn't realize that justifying one's answer is a necessary part of doing mathematics (from the mathematician's point of view).

For the sake of argument, I'm going to state the students' point of view (as described in the previous paragraph) in more provocative form, as a belief:

> Belief 1: If you're asked your opinion about a mathematical question, it suffices to give your opinion, although you might back it up with evidence if that evidence is readily available. Formal proofs or justifications aren't necessary, unless you're specifically asked for them -- and that's only because you have to play by the rules of the game.

We've seen the behavioral corollary of this belief, as Williams described it. Unfortunately, this belief has lots of company. Here are two of its friends, and their behavioral corollaries.

Belief 2: All mathematics problems can be solved in ten minutes or less, if you understand the material. Corollary: students give up after ten minutes of work on a "problem.")

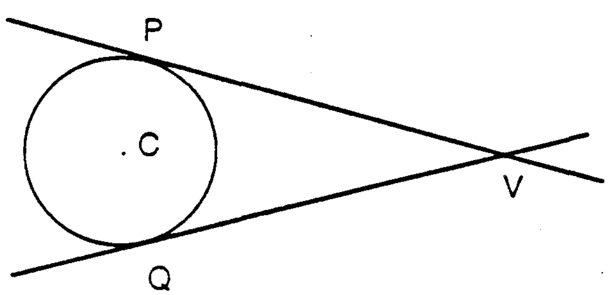
Belief 3: Only geniuses are capable of discovering, creating, and understanding mathematics. Corollary: students expect to take their mathematics passively, memorizing without hope or expectation of understanding.

An anecdote introduces one last belief. A while ago I gave a talk describing my research on problem solving to a group of very talented undergraduate science majors at Rochester. I asked the students to solve Problem 5, given in Fig. 1. The students, working as a group, generated a correct proof. I wrote the proof (Fig. 2) on the board. A few minutes later I gave the students Problem 6, given in Fig. 3.

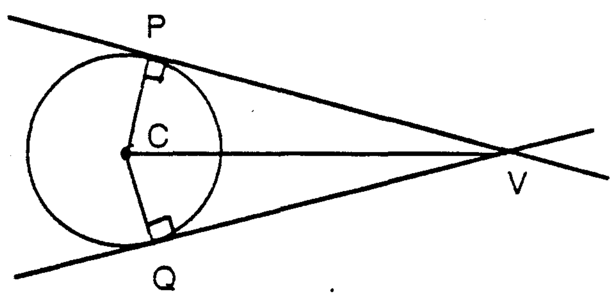
In the figure below, the circle with center C is tangent to the top and bottom lines at the points P and Q respectively.

a. Prove that PV = QV.

b. Prove that the line segment CV bisects angle PVQ.



-- Fig. 1 --



Proof:

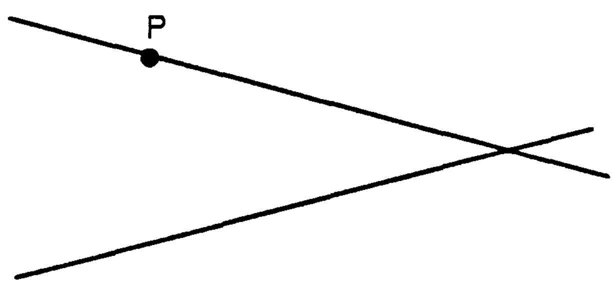
Draw in the line segments CP, CQ, and CV. Since CP and CQ are radii of circle C, they are equal; since P and Q are points of tangency, angles CPV and CQV are right angles. Finally since CV=CV, triangles CPV and CQV are congruent.

a. Corresponding parts of congruent triangles are congruent, so PV = QV.

b. Corresponding parts of congruent triangles are congruent, so angle PVC = angle QVC. Thus CV bisects angle PVQ.

-- Fig. 2 --

You are given two intersecting straight lines and a point P marked on one of them, as in the figure below. Show how to construct, using straightedge and compass, a circle that is tangent to both lines and that has the point P as its point of tangency to the top line.



-- Fig. 3 --

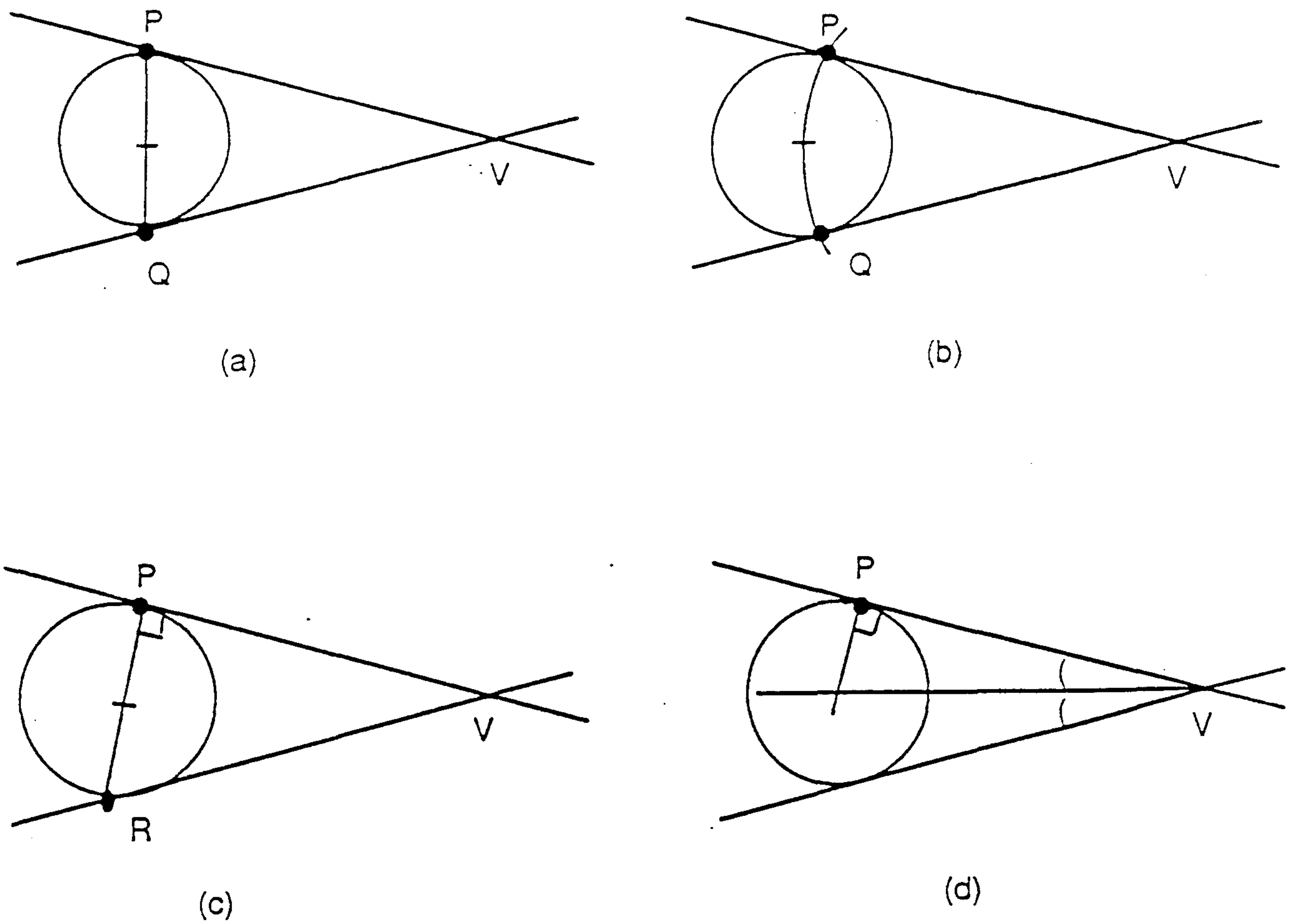
Students came to be board and made the following conjectures, in order:

a. Let Q be the point on the bottom line such that QV = PV. The center of the desired circle is the midpoint of line segment PQ. (Fig. 4a).

b. Let A be the segment of the arc with vertex V, passing through P, and bounded by the two lines. The center of the desired circle is the midpoint of the arc A. (Fig. 4b).

c. Let R be the point on the bottom line that intersects the line segment perpendicular to the top line at P. The center of the desired circle is the midpoint of line segment PR. (Fig. 4c).

d. Let L_1 be the line segment perpendicular to the top line at P, and L_2 the bisector of the angle at V. The center of the desired circle is the point of intersection of L_1 and L_2. (Fig. 4d).



-- Fig. 4 --

Students' conjectured solutions
(Short horizontal lines denote midpoints.)

The proof that the students had generated -- which both provides the answer and rules out conjectures a, b, and c -- was still on the board. Despite this, they argued for more than ten minutes about which construction was right. The argument was on purely empirical grounds (that is, on the grounds of which construction looked right), and it was not resolved. How could they have this argument, with the proof still on the board? I believe that this scene could only take place if the students simply didn't see the proof problem as being relevant to the construction problem. Or again in provocative form, Belief 4: Formal mathematics, and proof, have nothing to do with discovery or invention. Corollary: the results of formal mathematics are ignored when students work discovery problems.

Since we're in "brief survey mode," I don't want to spend too much time on beliefs *per se.* I think the point is clear. If you want to understand students' mathematical behavior, you have to know more than what they "know." These students "knew" plane geometry, and how to write proofs; yet they ignored that knowledge when working construction problems. Understanding what went on in their heads was (and is) tricky business. As I said, that was the main thrust of phase 2.

But enough of that; we're confronted with a real dilemma. The behavior I just described turns out to be almost universal. Undergraduates at Hamilton College, Rochester, and Berkeley all have much the same mathematical world view, and the (U.S.) National Assessments of Educational Progress indicate that the same holds for high school students around the country. How in the world did those students develop their bizarre sense of what mathematics is all about?

The answer, of course, lies in the students' histories. Beliefs about mathematics, like beliefs about anything else -- race, sex, and politics, to name a few -- are shaped by one's environment. Your develop your sense of what something is all about (be that something mathematics, race, sex, or politics) by virtue of your experiences with it, within the context of your social environment. You may pick up your culture's values, or rebel against them -- but you're shaped by them just the same.

Mathematics is a formal discipline, to which you're exposed mostly in schools. So if you want to see where kids' views about mathematics are shaped, the first place to go is into mathematics classrooms. I packed up my videotape equipment, and off I went. Some of the details of what I saw, and how I interpreted it, are given in the in-press articles cited in the references. A thumbnail sketch of some of the ideas follows.

Borrowing a term from anthropologists, what I observed in mathematics classes was the *practice* of schooling -- the day-to-day rituals and interactions that take place in mathematics classes, and (*de facto*) define what it is to do mathematics. One set of practices has to do with homework and testing. The name of the game in school mathematics is "mastery:" Students are supposed to get their facts and procedures down cold. That means that most homework problems are trivial variants of things the students have already learned. For example, one "required" construction in plane geometry (which students memorize) it to construct a line through a given point, parallel to a given line. A homework assignment given a few days later contained the following problem: Given a point on a side of a triangle, construct a line through that point parallel to the base of the triangle. This isn't a problem; it's an exercise. It was one of 27 "problems" given that night; the three previous assignments had contained 28, 45, and 18 problems respectively. The test on locus and constructions contained 25 problems, and the students were expected to finish (and check!) the test in 54 minutes -- an average of two minutes and ten seconds per problem. Is it any wonder that students come to believe that any problem can be solved in ten minutes or less?

I also note that the teacher was quite explicit about how the students should prepare for the test. His advice -- well intentioned -- to the students when they asked about the exam was as follows: "You'll have to know all your constructions cold so that you don't spend a lot of time thinking about them." In fact, he's right. Certain skills should be automatic, and you shouldn't have to think about them. But when this is the primary if not the only message that students get, they abstract it as a belief: mathematics is mostly, if not all, memorizing.

Other aspects of what I'll call the "culture of schooling" shape students' view of what mathematics is all about. Though there is now a small movement toward group problem solving in the schools, mathematics for the most part is a solitary endeavor, with individual students working alone at their desks. The message they get is that mathematics is a solitary activity.

They also get a variety of messages about the nature of the mathematics itself. Many word problems in school tell a story that requires a straightforward calculation (for example, "John had twenty-eight candy bars in seven boxes. If each box contained the same number of candy bars, how many candy bars are there in each box?"). The students learn to read the story, figure out which calculation is appropriate, do the calculation, and write the answer. An oft-quoted problem on the third National Assessment of Educational Progress (secondary school mathematics) points to the dangers of this approach. It asked how many buses were needed to carry 1128 soldiers to their training site. if each bus holds 36 soldiers. The most frequent response was "31 remainder 12" -- an answer that you get if you follow the *practice* for word problems just described, and ignore the fact that the story (ostensibly) refers to a "real world" situation.

Even when students deal with "applied" problems, the mathematics that they learn is generally clean, stripped of the complexities of the real world. Such problems are usually cleaned up in advance -- simplified and presented in such a way that the techniques the students have just studied in class will provide a "solution." The result is that the students don't learn the delicate art of mathematizing -- of taking complex situations, figuring out how to simplify them, and choosing the relevant mathematics to do the task. Is it any surprise that students aren't good at this, and that they don't "think mathematically" in real world situations for which mathematics would be useful?

I'm proposing here that thorny issues like the "transfer problem" (why students don't transfer skills they've learned in one context and use them in other, apparently related ones) and the failure of a whole slew of curriculum reform movements (e.g. the "applications" movement a few years back) have, at least in part, cultural explanations. Suppose we accept that there is such a thing as school culture, and it operates in ways like those described above. Curricular reform, then, means taking new curricula (or new ideas, or...) and shaping them so that they fit into the school culture. In the case of "applications," it means cleaning problems up so that they're trivial little exercises -- and when you do that, you lose both the power, and the potential transfer, of the applications. In that sense, the culture of schooling stands as an obstacle to school reform. Real curricular reform, must in part involve a reform of school culture. Otherwise it doesn't stand a chance.

Well, here I am arguing away in the midst of -- as though you haven't guessed -- phase 3. There are two main differences from phase 2. The first is that I've moved from taking snapshot views of students (characterizing what's in a student's head when the student sits down to work some problems) to taking a motion picture. The question I'm exploring now is: how did what's in the student's head evolve the way it did? The second is that the explanatory framework has grown larger. Though I still worry about "what's going on in the kid's head," I look outside for some explanations -- in particular, for cultural ones.

And yet *plus ça change, plus ça reste le même*. I got into this business because, in Halmos's phrase, I thought of problem solving as "the heart of mathematics" -- and I wanted students to have access to it. As often happens, I discovered that things were far more complex than I imagined. At the micro-level, explorations of students' thought processes have turned out to be much more detailed (and interesting!) than I might have expected. I expect to spend a substantial part of the next few years looking at videotapes of students learning about the properties of graphs. Just how do they make sense of mathematical ideas? Bits and pieces of "the fine structure of cognition" will help me to understand students' mathematical understandings. At the macro-level, I'm now much more aware of knowledge acquisition as a function of cultural context. That means that I get to play the role of amateur anthropologist -- and that in addition to collaborating with mathematicians, mathematics educators, AI researchers, and cognitive scientists, I now get to collaborate with anthropologists and social theorists. That's part of the fun, of course. And that's only phase 3. I can't tell you what phase 4 will be like, but there's a good chance there will be one. Like the ones that preceded it, it will be based in the wish to understand and teach mathematical thinking. It will involve learning new things, and new colleagues from other disciplines. And it's almost certain to be stimulated by my discovery that there's something not right about the way I've been looking at things.

Are there any morals to this story -- besides the obvious one, that I've been wrong so often in that past that you should be very skeptical about what I'm writing now? I think there's one. My work has taken some curious twists and turns, but there has been a strong thread of continuity in its development; in many ways, each (so-called) phase enveloped the previous ones. What caused the transitions? Luck, in part. I saw new things, and pursued them. But I saw them because they were there to be seen. Human problem solving behavior is extraordinarily rich, complex, and fascinating -- and we only understand very little of it. It's a vast territory waiting to be explored, and we've only explored the tiniest part of that territory. Each of my "phase shifts" was precipitated by observations of students (and, at times, their teachers) in the process of grappling with mathematics. I assume that's how phase 4 will come about, for I'm convinced that -- putting theories and methodologies, and tests, and just about everything else aside -- if you just keep your eyes open and take a close look at what people do when they try to solve problems, you're almost guaranteed to see something of interest.

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LECTURE 2

APPLICATIONS OF UNDERGRADUATE MATHEMATICS

BY ROSS FINNEY
MIT

Dr. Finney's lecture followed closely parts of the text of his paper "Applications of Undergraduate Mathematics" origionally printed in Mathematics Tomorrow, edited by Lynn Steen, Springer-Verlag, New York, 1981. The text is reprinted here by permission of the author.

Applications of Undergraduate Mathematics

Ross L. Finney

In recent years there has been a phenomenal growth in the professional use of mathematics, a growth so rapid that it has outstripped the capacity of many courses in our schools and colleges to train people for the mathematical tasks that are expected of them when they take employment. People who take jobs with the civilian government, the military, or industry, or who enter quantitative fields as graduate students or faculty, discover with increasing frequency these days that they lack acquaintance with important mathematical models and experience in modeling. Many of them also find to their distress that they have not been trained to be self-educating in the application of mathematics.

This discovery, perhaps I should say predicament, is not the exclusive domain of people who enter fields that depend for their progress upon advanced mathematics. In Louisville, Kentucky, the profession of interior decorating is highly competitive. To stay in business, a decorator must be able to make accurate cost estimates. To do so without delay requires facility with decimal arithmetic, fractions, and area formulas. People hired as stenographers by The First National Bank of Boston discover that the work is done not on typewriters but on computer-driven word processors. Many stores now use their cash registers for inventory control. The keys on business machines have multiple functions, and the functions must be performed in the right order. As these examples suggest, almost every professional field now uses mathematics of some kind.

Since 1976 the U.S. National Science Foundation has provided support for a unique multi-disciplinary response to the need for instruction in applied mathematics: the Undergraduate Mathematics Applications Project. UMAP, as the Project is called, produces lesson-length modules, case studies, and monographs from which readers can learn how to use the mathematical sciences to solve problems that arise in other fields. The applications presented by UMAP cover a broad range from chemistry, engineering and physics, to biomedical sciences, psychology, sociology, economics, policy analysis, harvesting, international relations, earth sciences, navigation, and business and vocational pursuits.

UMAP modules are self-contained, in the sense that anyone who has fulfilled the prerequisites listed inside the front covers can reasonably expect to read the modules and solve the problems without help. They cover about as much material as a teacher would put into an hour's lecture. There are exercises, model exams keyed to objectives, and answers. The modules are reviewed thoroughly by teachers as well as by professionals in the fields of application, revised, tested in classrooms throughout the world, reviewed by individual students to be sure they are as self-contained as they should be, and revised again before publication.

The modules are used for individual study, to supplement standard courses, and in combination to provide complete text coverage for courses devoted to applications of the mathematical sciences. These sciences, which I shall simply call mathematics, include probability and statistics, operations research, computer science and numerical methods as well as the elementary and advanced aspects of analysis, algebra and geometry.

UMAP case studies are not intended to be as self-contained as are the modules. The studies contain data and background information for a mathematical modeling problem as a field professional would collect it, but readers are asked to develop their own models for solving the problems. The data are real, the problems current. Teachers are given the solutions of the problems as they were originally worked out by the professional applied mathematicians who furnished the problems to the project. Each study has a teacher's guide developed through classroom use. The case studies are used in mathematical modeling courses, and may take several weeks to complete. One of their striking features is that, like the UMAP modules, they expect no previous experience with mathematical modeling on the part of either instructor or student. Nor do they require any previous knowledge of the applied field. Anyone with the right mathematical background can work through them successfully.

UMAP's expository monographs are works of eighty pages or more that make available to students in upper level courses, and to faculty in diverse fields, significant applications that are not in commercial texts. They also

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give users of standard texts access to additional and complementary professional methods. Like all UMAP materials, the monographs are written for students to read, and contain exercises with answers.

Although UMAP modules, case studies and monographs are similar to traditional texts in that they provide instruction for students with suitable examples and exercises, they differ dramatically in their objectives: a UMAP unit follows the logic of the practitioner, not the syllabus of a course; it presents mathematics as a natural constituent of a whole problem, not as a defined niche in a planned curriculum. Because of their allegiance to diverse masters, UMAP curriculum materials reflect both the excitement and disarray of current practice rather than the artificial order of traditional textbooks. They provide an entrée to the useful mathematics of the next decade. Here are some examples, taken from UMAP modules.

Measuring cardiac output

Brindel Horelick and Sinan Koont wrote *Measuring Cardiac Output* to teach an application of numerical integration in medicine.

Your cardiac output is the amount of blood your heart pumps in one minute. It is usually measured in liters per minute. A person awake but at rest, perhaps reading, might have a cardiac output of five or six liters a minute. A marathon runner might have a cardiac output of more than thirty liters a minute.

A change in cardiac output may be a symptom or a consequence of disease, and doctors occasionally want to measure it. One technique for doing so, one that works when the heart's output is fairly constant, calls for injecting a small amount of dye in a main vein near the heart. Five or ten milligrams will do. The dye is drawn into the heart and pumped through the lungs and into the aorta, where its concentration is measured as the blood flows past a Swan-Ganz catheter. Figure 1 shows a typical set of readings in milligrams per liter, taken every second for about twenty-five seconds.

You will notice in Figure 1 that the concentration stays at 0 for the first few seconds. It takes that long for the first of the dye to pass through the heart and lungs. The concentration then begins to rise. It reaches a peak at about 12 seconds, then declines steadily for another seven seconds. Instead of tapering to 0 at that point, however, the concentration rises slightly and holds steady. Some of the dye that went through first has begun to reappear.

The determination of the patient's cardiac output requires calculating the area under the curve that gives the concentration of the first-time-through dye. To find this curve, or at least to make a satisfactory version of it, one has to replace the real data points for the last few seconds by

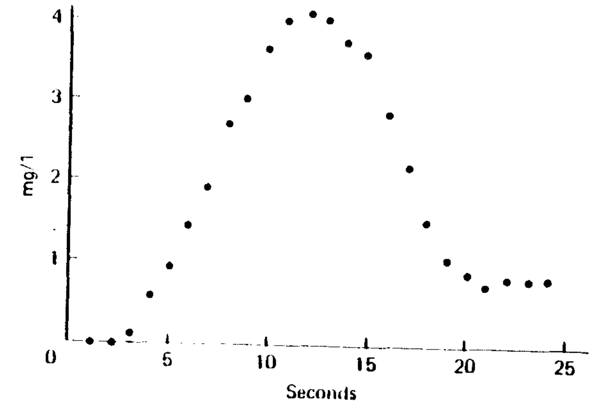

Figure 1. Typical readings of dye concentration in the aorta when 5 mg of dye are injected into a main vein near the heart at $t = 0$ seconds.

fictitious ones, as shown in Figure 2. The chosen points continue the downward trend of the points that precede them. The estimates involved in selecting the fictitious points seem reasonable, and any errors introduced by the replacement are likely to be small in comparison with other uncertainties in measurement.

The concentration curve can now be sketched, but there is no formula for it that can be integrated. This is often the case with data generated in the laboratory or collected in the field and there are standard ways to cope. On the data here there is no reason to use anything more sophisticated than Simpson's rule or the trapezoidal rule, which is precisely what Horelick and Koont proceed to do. The patient's cardiac output is then calculated by

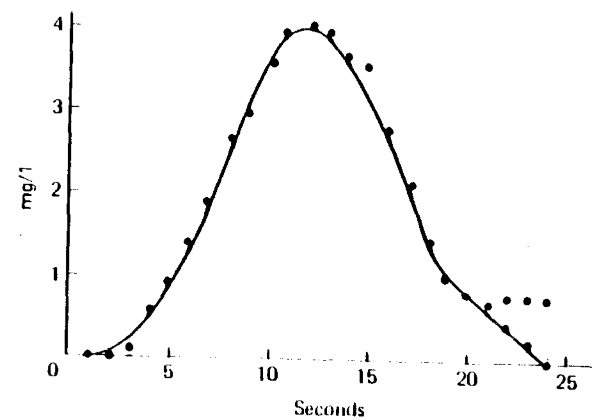

Figure 2. The curve shown here is fitted to the real and adjusted data points. Its height above the horizontal axis approximates the concentration of the injected dye passing the monitoring point in this patient's aorta for the first time.

dividing the estimate obtained for the integral (expressed in milligram minutes per liter) into the number of milligrams of dye originally injected. The result: 6.8 liters per minute.

Chemistry

Ralph Grimaldi's module, *Balancing Chemical Reactions with Matrix Methods and Computer Assistance*, shows how matrix methods may be used to balance chemical reactions. The unit gives a concrete setting for the concepts of linear independence and dependence in vector spaces of dimension four or more.

In the reaction

$$Pb(N_3)_2 + CR(MnO_4)_2 \Rightarrow CR_2O_3 + MnO_2 + Pb_3O_4 + NO,$$

which takes place in a basic solution, the atoms from lead azide and chromium permanganate combine into four other products: chromium oxide, manganese dioxide, trilead tetroxide, and nitric oxide. To find how much of each of the original reactants has to be present to produce how much of each of the products, we "balance" the reaction. That is, we find integers u, v, w, x, y, and z, with the property that u molecules of lead azide plus v molecules of chromium permanganate produce exactly w molecules of chromium oxide, x molecules of manganese dioxide, y molecules of trilead tetroxide, and z molecules of nitric oxide. Schematically,

$$u\ PB(N_3)_2 + v\ CR(MnO_4)_2 \Rightarrow w\ CR_2O_3 + x\ MnO_2 + y\ PB_3O_4 + z\ NO.$$

The numbers u, v, w, x, y, and z are integers chosen to make the number of atoms of each element the same on each side of the reaction. To balance the reaction, we balance the atoms.

To balance the atoms, we assign a basic unit vector to each element. It does not matter which vector we assign to which element, as long as we assign one apiece and keep track of the assignment. The assignment

$$\begin{aligned} Pb &= (1, 0, 0, 0, 0) \\ N &= (0, 1, 0, 0, 0) \\ Cr &= (0, 0, 1, 0, 0) \\ Mn &= (0, 0, 0, 1, 0) \\ O &= (0, 0, 0, 0, 1) \end{aligned}$$

will do as well as any. We use five-dimensional vectors because there are five elements.

We then replace the chemical reaction with the vector equation

$$u(1, 6, 0, 0, 0) + v(0, 0, 1, 2, 8) = w(0, 0, 2, 0, 3) + x(0, 0, 0, 1, 2) + y(3, 0, 0, 0, 4) + z(0, 1, 0, 0, 1).$$

You can see where the vector entries come from. For every u lead atoms in lead azide, $Pb(N_3)_2$, there are $6u$ nitrogen atoms; hence the $u(1, 6, 0, 0, 0)$ in the vector equation. For every v chromium atoms on the left side of the reaction, there are also $2v$ manganese atoms and $8v$ oxygen atoms. And so on for the other four integers, w, x, y, and z.

The idea now is to solve the vector equation for the integers u, v, w, x, y, and z: To do so we rewrite the equation as a system of five linear equations in six variables. Six variables are too many for a unique solution, but we can arbitrarily assign the value 1 to the variable u to match the number of unknowns to the number of equations. We may want to change the value assigned to u later, but $u = 1$ will do for now. The resulting system in matrix form is

$$\begin{bmatrix} 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ -1 & 2 & 0 & 0 & 0 \\ -2 & 0 & 1 & 0 & 0 \\ -8 & 3 & 2 & 4 & 1 \end{bmatrix} \begin{bmatrix} v \\ w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 6 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

This system of equations can be solved by a short computer program listed in Grimaldi's module. The solution given by the computer when $u = 1$ is

$$v = 2.93333,\ w = 1.46667,\ x = 5.86667,\ y = 0.33333,\ z = 6.$$

These values are not the integers we seek because they are not all integers. Once we notice that 0.03333 is about 1/30 and 0.06667 about 2/30, however, we know enough to scale everything by taking u equal to 30 instead of 1. The resulting solution is

$$u = 30,\ v = 88,\ w = 44,\ x = 176,\ y = 10,\ z = 180.$$

The module discusses what to do if at first you do not recognize the integer solution that underlies the computer's decimal solution. It also discusses an example in which reducing the number of variables to match the number of equations does not seem to work. The difficulty is traced to the fact that the reaction being balanced consists of two reactions that take place simultaneously, independently of each other. Each must be analyzed apart from the other.

Scheduling prison guards

James M. Maynard's *A Linear Programming Model for Scheduling Prison Guards* describes a linear program that Maynard developed for the Pennsylvania State Bureau of Corrections. As the newspaper clippings reproduced in Figures 3 and 4 show, the Bureau was concerned in the middle 1970's about the increasing cost of paying prison guards to work overtime. In the

At 8 State Prisons

Overtime Guard Pay Bills Keep Mounting

By The Associated Press

The Bureau of Corrections is still paying heavy overtime to keep guards on duty at the eight state prisons. Some guards are doubling their salaries through extra work.

A bureau spokesman said yesterday that in the year ended June 30 the agency paid out nearly $4 million in overtime, a boost of $750,000 over the previous year.

The bureau already had been strongly criticized for excessive overtime payments. Legislators and other officials think the state could save money by hiring more guards at regular salaries and reducing overtime payments at time and a half and double time.

Auditor Gen. Robert P. Casey, one of the critics, zeroed in yesterday on overtime at the state prison in Dallas, Luzerne County. More than $430,874 was paid during the fiscal year ended in June 1974.

Casey said 12 guards received between $10,144 and $6,707 in overtime. Eleven of the 12 guards had base salaries of $11,731. One guard has a $12,875 base salary.

"The new commissioner — William Robinson — is very aware of the problem and that, along with every other program, is being looked at very carefully ...," the correction bureau spokesman said. "He does want to cut down on the overtime."

He said former Commissioner Stewart Werner had a hiring freeze in effect because of the tight budget policy adopted by the Shapp administration.

But under Robinson, who assumed the post last month, the freeze has been lifted and 35 guard vacancies around the state are being filled, the spokesman said.

However, the overtime problem will linger.

Glen R. Jeffes, superintendent at Dallas, said vacancies alone don't govern how much overtime will be needed. Vacations and the fact that authorized staff levels are inadequate also are factors, he said.

"I have requested additional officer positions the last two years. I received no new positions' ..."

"Without additional officer positions I see very little impact on the reduction of overtime," he said.

Courtesy of the Associated Press.

Guards Due Windfall For Missed Breaks

From The Patriot Wire Services

About 1,700 state prison guards will be reimbursed for perhaps $1,000 each for missed coffee breaks, it was learned yesterday.

The windfall comes as a result of an arbitrator's decision earlier this month on grievances filed at eight penal institutions across the state. It may cost the commonwealth as much as $1.7 million.

Under the terms of their contract with the State Bureau of Corrections, the guards are allowed a 15 minute break every four hours.

But because of critical manpower shortages at the state's prisons, the men have not been able to take the breaks since before July, 1973.

Robert Saylor, executive director for the Bureau of Corrections, refused to comment on published reports of the reimbursement.

The guard answering the phone at the bureau's headquarters here said he had heard of the decision, but added, "We should be getting $2,000."

The arbitrator's decision was handed down on July 12, according to published reports, but the bureau made no announcement of it.

According to Jack Walsh, president of guards Local 2500 at Western State Penitentiary, the payment will be made to the guards sometime this month.

Courtesy of the Harrisburg Patriot.

Figure 3.

Senators Tour Prison Facility In Camp Hill

By MERRY BROOKS
Staff Writer

A fact-finding tour by members of the State Senate Prison Inquiry Committee yesterday at the State Correctional Institution at Camp Hill seemed more like a whirlwind campaign swing with senators shaking hands and eliciting opinions from prisoners and guards.

But in fact, the information sought by four state senators on the sixth tour of the eight state prisons yielded information that may assist the special committee in drafting prison-related legislation.

Sen. Freeman Hankins, D.-Philadelphia, committee chairman; Sen. Martin Murray, D.-Luzerne; Sen. Herbert Arlene, D.-Philadelphia, and Sen. James E. Ross, D.-Beaver-Washington, accompanied by a herd of reporters, breezed through the prison in Lower Allen Twp. in a three-hour VIP tour.

The senators engaged Ernest Patton, prison superintendent, in a give-and-take roundtable discussion before the tour began. They obtained the following information:

—The prison paid $353,000 in overtime to guards last year and expects to pay $461,000 in overtime this year. The prison needs an additional 67 guards to reduce the amount of overtime pay.

Courtesy of the Harrisburg Patriot.

Guards sought for Graterford

By The Associated Press

The head of the state Corrections Bureau says Gov. Shapp and the Legislature may be asked to provide from 80 to 100 guards at the Graterford State Prison.

The increase would raise to 400 the number of guards at the Montgomery County prison.

Corrections Commissioner Stewart Werner estimated the added guards would cost $500,000 annually.

The extra men could cut down on overtime payments to the current guards, now running about $20,000 a month.

★ ★ ★

Graterford, the largest of the state's eight correctional institutions, has about 1,600 inmates, about 200 below capacity.

Courtesy of the Associated Press.

Figure 4.

year ending June 30, 1975, for example, the Bureau paid nearly four million dollars in overtime pay, $750,000 more than it had paid for overtime work the year before. Some overtime work is to be expected, of course. It is expensive to keep a full-time staff large enough to cover peak loads, for a staff this large is likely to be underemployed much of the time. On the other hand, a staff so small that regularly scheduled guards have to work so many overtime hours that they sometimes double their salaries is also expensive, as the Bureau was finding out. Understaffing can be expensive in other ways, too, for fatigue and high inmate-to-guard ratios create dangerous tensions.

Legislators and other officials thought the State might save money by increasing the size of its regular prison staff. Maynard was hired to determine the size of the least expensive overall work force.

The goals of Maynard's investigation were to minimize the total cost of paying prison guards, while reducing the overtime work and establishing uniform work schedules in all prisons. He was able to meet the goals successfully with a linear program, the one described in his UMAP module.

Table 1 shows two work schedules for one of the Bureau's prisons, referred to here as Prison G. One schedule has parentheses, the other does not. The numbers with parentheses are the numbers of guards recommended by the linear program. The numbers without parentheses show how many guards were on duty at Prison G during the week ending September 30, 1973.

The schedules are weekly schedules divided into twenty-one periods, three shifts a day for seven days. Each box in the table shows the numbers of guards working at three different pay levels during the given shift: regular, time-and-a-half, and double time. The two numbers in the top line in each box are the numbers of guards working the shift as part of their regular weekly work schedule. The two numbers next in line are the numbers of guards working the shift at time-and-a-half. The last two numbers are the numbers of guards working at double time.

For example, Monday morning, September 24th was worked by 94 guards on regular schedules, 19 guards at time-and-a-half, 3 guards at double time. On Tuesday afternoon more than half of the 146 guards present were working overtime.

The numbers in parentheses proposed by the linear program are strikingly different from the 1973 figures. On Monday morning the model covers the work load with 117 regularly scheduled guards; where once there had been 22 overtime guards, now there are none. On Tuesday afternoon there are only 9 overtime guards where once there had been 76. The new work schedule is more equitable and less fatiguing than the old one. It is also more economical. If regular pay is calculated at $4 an hour, for instance, the new schedule for Prison G saves the State $5,216 a week.

Readers of Maynard's module are given an opportunity to follow the development of the linear program, to see the effects of various scheduling

Table 1. Data and Results from Prison G for the Week Ending September 30, 1973

Day	Shift					
	Morning		Afternoon		Night	
Monday	94	(117)	70	(131)	38	(74)
	19	(0)	61	(0)	40	(4)
	3	(0)	0	(0)	0	(0)
	116	(117)	131	(131)	78	(78)
Tuesday	94	(126)	70	(137)	36	(74)
	17	(0)	62	(9)	38	(0)
	15	(0)	14	(0)	0	(0)
	126	(126)	146	(146)	74	(74)
Wednesday	97	(116)	69	(137)	36	(74)
	19	(0)	68	(0)	27	(1)
	0	(0)	0	(0)	12	(0)
	116	(116)	137	(137)	75	(75)
Thursday	94	(128)	63	(98)	37	(74)
	41	(21)	24	(2)	34	(7)
	14	(0)	13	(0)	10	(0)
	149	(149)	100	(100)	81	(81)
Friday	74	(97)	45	(89)	37	(39)
	20	(0)	16	(0)	2	(0)
	2	(0)	0	(0)	0	(0)
	96	(97)	61	(89)	39	(39)
Saturday	57	(43)	37	(45)	26	(0)
	15	(33)	14	(6)	3	(29)
	4	(0)	0	(0)	0	(0)
	76	(76)	51	(51)	29	(29)
Sunday	53	(63)	36	(48)	25	(35)
	7	(0)	12	(0)	3	(0)
	3	(0)	0	(0)	2	(0)
	63	(63)	48	(48)	30	(30)

assumptions, and to develop a small-scale program of their own. As in the Grimaldi chemistry module, the program does not at first yield integer solutions, but by rounding the numbers of guards given by the computer to integer values and rerunning the program to determine the values of the remaining variables, one obtains a feasible solution that is close enough. It is not necessary to prove that the integer solution found this way is optimal. One can test its utility by evaluating the objective function, which gives the total amount of money paid to prison guards. If the value of the function for the integer solution is close to the value of the function for the original not-necessarily-integer solution, then the integer solution is good.

Continuous service in legislatures

Once a group of people has been elected to a legislature, the number of them who serve continuously from that time onward will normally decrease exponentially with each passing election.

The elections for the Senate of the United States are held in the fall of every even-numbered year. The senators, elected for six-year terms, take office the following January. Figure 5 shows the proportion of the 1801 Senate that remained in office after successive elections. They were all gone by 1811. The data are fitted nicely by the curve

$$y = e^{-0.029t},$$

where t is measured in months beginning in January 1801 with $t = 0$.

Thomas W. Cassteven's module, *Exponential Models of Legislative Turnover*, shows how exponential curves can be used to forecast election results, to speculate convincingly about what would have happened if a postponed election had been held on time, and to disclose suppressed data.

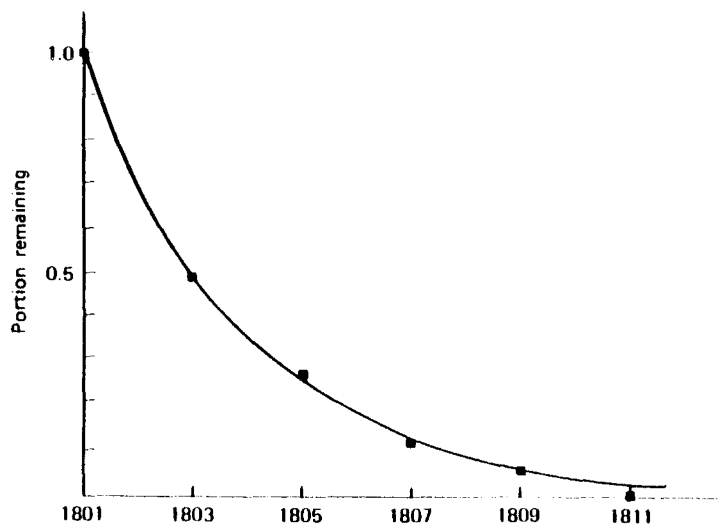

Figure 5. The proportion of the U.S. Senators taking office in 1801 who continued in office through subsequent terms. The pattern shown here, of discrete election data fitted by an exponential curve, is typical of legislative turnover. The data to be fitted may be either raw (as in Figure 6) or proportional (as in the figure above).

One of Cassteven's many interesting examples is the turnover in the membership of the Central Committee of the Communist Party of the Soviet Union. In 1957, First Secretary Nikita Khruschev, in some semi-secret infighting, succeeded in removing a number of his opponents from the Committee. Their identity was not made public, nor was their total number. Their number can be estimated, however, by a calculation based on election data from nearby years. There were elections in February 1956, October 1961, March 1966, and March 1971. From these one can calculate the exponential decay constant for the Central Committee's normal turnover. One can then calculate how many of the February 1956 cohort should have been present after the 1961 election. It turns out that there were about 12 too few of them there. At least a dozen full members were removed in Khruschev's purge.

It is intersting to note that the decay constants for the U.S. Senate and the Central Committee of the Communist Party of the Soviet Union have been nearly equal in recent decades. For the data shown in Figure 6, the best fitting values of the decay constants are about 0.0079 (Senate) and 0.0073 (CC/CPSU). If the twelve members purged by Khruschev in 1957 are added back in, the match is even closer.

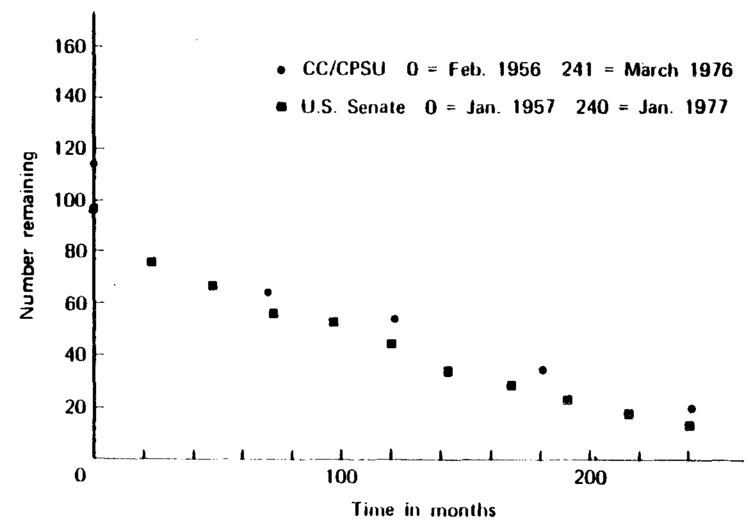

Figure 6. A comparison of continuous service in the U.S. Senate and the Central Committee of the Communist Party of the Soviet Union. The exponential decay constants of these two legislative bodies have been nearly equal in recent years. Membership in these two legislatures has been turning over at about the same rate.

Mercator's world map

Anyone who has ever wondered what the integral of the secant function is good for can find a satisfying answer in Philip Tuchinsky's UMAP module, *Mercator's World Map and the Calculus*. The unit explains how the integral of the secant determines the spacing of the lines of latitude on maps used for compass navigation.

The easiest compass course for a navigator to steer is one whose compass heading is constant. This might be a course of 45° (northeast), for example, or a course of 225° (southwest), or whatever heading is required to reach the navigator's destination without bumping things on the way. Such a course will lie along a spiral that winds around the globe toward one of the poles (Figure 7), unless the course runs due north or south or lies parallel to the equator.

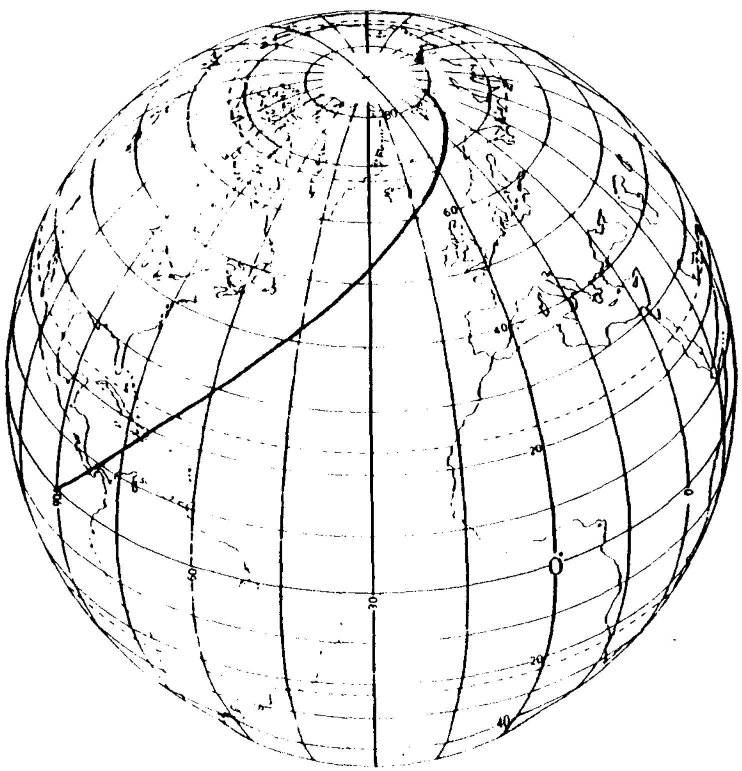

Figure 7. A flight with a constant bearing of 45° East of North from the Galapagos Islands in the Pacific to Franz Josef Land in the Arctic Ocean.

In 1569 Gerhard Kramer, a Flemish surveyor and geographer known to us by his Latinized last name, Mercator, made a world map on which all spirals of constant compass heading appeared as straight lines. This fantastic achievement met what must have been one of the most pressing navigational needs of all time. For from Mercator's map (Figure 8) a sailor could read the compass heading for a voyage between any two points from the direction of a straight line connecting them.

Figure 9 shows a modern Mercator map. If you look closely at it you will see that the vertical lines of longitude, which meet at the poles on the globe, have been spread apart to lie parallel on the map. The horizontal lines of latitude that are shown every 10° are parallel also, as they are on the globe, but they are not evenly spaced. The spacing between them increases toward the poles.

The secant function plays a role in determining the correct spacing of all these lines. The scaling factor by which horizontal distances from the globe are increased at a fixed latitude τ to spread the lines of longitude to fit on the map is precisely sec τ. There is no spread at the equator, where $\sec\tau = 1$. At latitude 30° north or south, the spreading is accomplished by multiplying all horizontal distances by the factor sec 30°, which is about 1.15. At 60° the factor is sec 60° = 2. The closer to the poles the longitudes are, the more they have to be spread.

The lines of latitude are spread apart toward the poles to match the spreading of the longitudes, but the formulation of the spreading is complicated by the fact that the scaling factor $\sec\tau$ increases with the latitude τ.

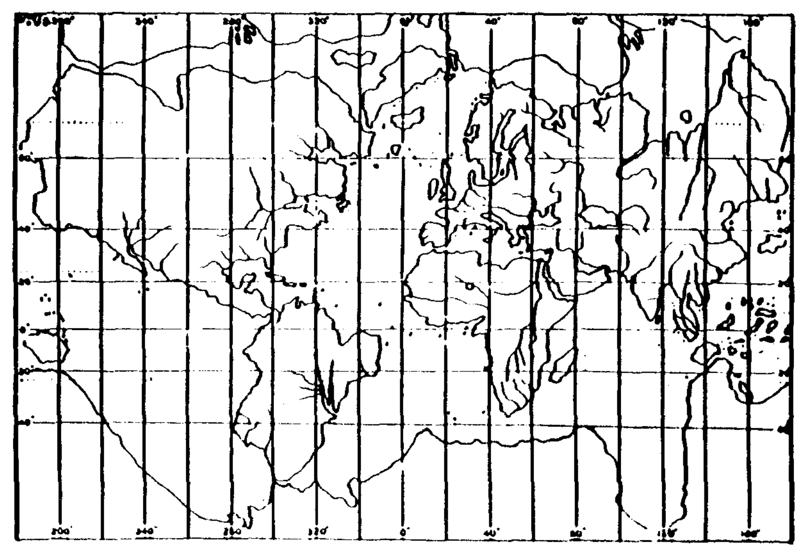

Figure 8. A sketch of Mercator's map of 1569.

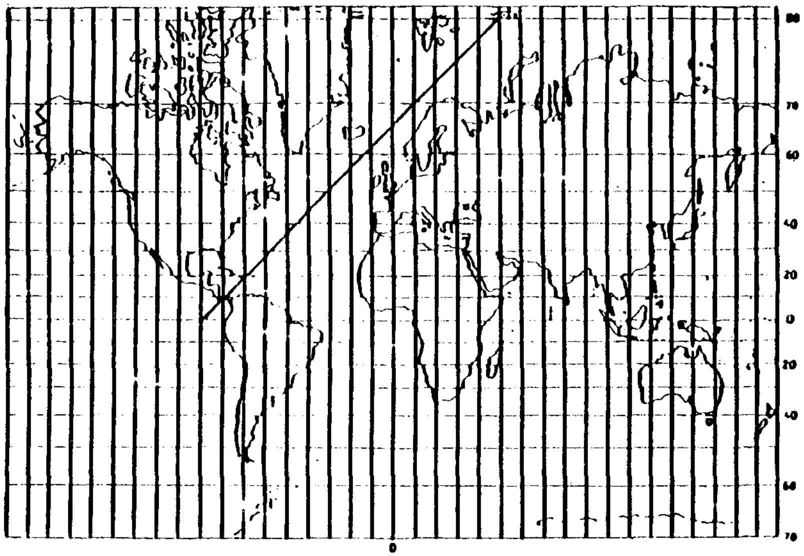

Figure 9. The flight of Figure 7 traced on a modern Mercator map. Courses of constant compass heading appear as straight line paths on a Mercator map. They are easily constructed, measured, and followed.

The factor to be used for stretching an interval of latitude is not a constant on the interval. This complication is overcome by integration. If R is the radius of the globe being modeled, then the distance D between the lines drawn on the map to show the equator and the latitude $a°$ is R times the integral of the secant from zero to a:

$$D = R\int_0^a \sec\tau \, d\tau.$$

The distance on the map between two lines of north latitude, say from $a°$ up to $b°$, is

$$D = R\int_0^b \sec\tau \, d\tau - R\int_0^a \sec\tau \, d\tau = r\int_a^b \sec\tau \, d\tau.$$

Suppose, for example, that the equatorial length of a Mercator map just matches the equator of a globe of radius 25cm. Then the spacing on the map between the equator and latitude 20° north is

$$25\int_0^{20} \sec\tau \, d\tau \approx 9 \text{ cm},$$

whereas the spacing between latitudes 60° north and 80° north, is

$$25\int_{60}^{80} \sec\tau \, d\tau \approx 28 \text{ cm}.$$

The vertical distance on the map between latitude 60° and latitude 80° is more than three times the vertical distance between latitude 0° and latitude 20°. The navigational properties of a Mercator map are achieved at the expense of a considerable distortion of distance.

Concluding thoughts

Mathematical reasoning penetrates scientific problems in numerous and significant ways. If the secret of technology, as C.P. Snow said, is that it is possible, then the secret of mathematical modelling is that it works. However, the process of developing and employing a mathematical model is both more subtle and more complex than is the traditional solution of mathematics textbook problems. Real models frequently have to be constructed in the presence of more data than can be taken into account; their conclusions are often drawn from calculations in which good approximations play a greater role than do exact solutions; very often there are conflicting standards by which solutions can be judged, so whatever answers emerge can only rarely be labelled as right or wrong. Students using UMAP modules, case studies, or monographs experience mathematics in its scientific context, and leave the classroom better equipped to face real demands of mathematical modelling in business, research, and government work.

ASPECTS OF CURRENT RESEARCH IN MATHEMATICS EDUCATION.

EDITED BY CAROLYN KIERAN
UNIVERSITE DU QUEBEC A MONTREAL

ASPECTS OF CURRENT RESEARCH IN MATHEMATICS EDUCATION*

Carolyn Kieran

Université du Québec à Montréal

The session devoted to "Aspects of Current Research in Mathematics Education" at the 1986 meeting of CMESG included reports of research being carried out in British Columbia, Alberta, Ontario, Quebec, and Newfoundland, with a special report being given by Jörg Voigt on his research in Bielefeld, West Germany. These reports were not meant to be a comprehensive survey of the mathematics education research being engaged in throughout the country, but were intended to give an idea of some of the main themes of current interest to researchers and to provide pointers to some of the work which is going on. More details can be had by corresponding directly with the researcher(s) involved. This article briefly summarizes those reports.

British Columbia

David Kirshner reported on the research projects of three colleagues, as well as his own work. There is no single theme which characterizes these projects. One study (D. Owens) involves intensive work with a small number of sixth grade pupils to see if meaningful understanding of decimal concepts can be achieved at that grade level. Another project (W. Szetela)

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* Thank you to all who contributed both to the session and to this article: David Kirshner (B.C.), Tom Kieren (Alta.), William Higginson (Ont.), Erika Kuendiger (Ont.), Claude Gaulin (Que.), Joel Hillel (Que.), Lionel Mendoza (Nfld.), and Jörg Voigt (Bielefeld, West Germany). Our apologies for misrepresenting anyone's research and for not being able to include mention of everyone's work.

deals with problem solving, more specifically, the improvement of teacher reliability ratings in the evaluation of students' protocols. Szetela is also carrying out a cross-cultural problem-solving study (Canada and Poland) involving 11- and 13-year-olds. The third study (D. Robitaille and G. Spitler) focuses on developing teaching materials and providing in-service training at the junior secondary level in the Dominican Republic. Kirshner's research in algebra is based on the assumption that symbol skill relies on procedures which are not related to mathematical theory, but rather to generative linguistics.

Alberta

Although Tom Kieren was not able to attend the meeting this year, he prepared a report for this session.* The thrust of the research being carried out in Alberta can be captured in the questions: How do persons build mathematical ideas? What curricular/instructional actions affect (positively and negatively) this knowledge building? A recently completed study in Calgary (L. Marchand, M. Bye, B. Harrison, T. Schroeder) looked at the match of school demands and knowledge building levels of pupils in elementary schools (1767 pupils). A "match" with student levels and demands was found for 64% of the cases, but there were significant divergences at the grade 5 level where the curriculum appeared to be rather formal. An Edmonton group of researchers (Y. Pothier and D. Sawada) is investigating partitioning and fractional numbers. Another team (T. Kieren, D. Sawada, B. Wales) has been looking at an image of mathematical

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* Kieren, T. "Mathematical Knowledge Building in the Classroom: A Report of Recent Mathematics Education Research and Development in Alberta". Copies of this report are available from T. Kieren, University of Alberta, Edmonton, Alberta, T6G 2G5

knowledge building and using it to interpret the fractional comparison abilities of young children (ages 6-8). Researchers (D. Sawada and A. Olson) are also involved with using the concept of auto-poeisis as developed by Maturana to explain how a person's mathematical knowledge system evolves.

As well, there has been considerable work on Logo and mathematics in Edmonton: Cathcart has looked at debugging strategies; Kieren and Olson have developed a theoretical model relating van Hiele geometry levels, levels of Logo use, and levels of language use from Frye; Ludwig and Kieren have tested this theory and used it to explain results in a Turtle Geometry development project involving transformational geometry with seventh graders; Dobson and Richardson have developed extensive curriculum materials on Logo and problem solving for preliminary elementary aged children.

Finally, there has been an interest in expert systems and mathematics. Balding has designed a system which allows teachers to analyze the ratio work of a consistent student work simulator and, thus, to identify aspects of student thinking patterns. Moreno is developing a problem solving helper which will use expert knowledge/strategies in a computer advisor to beginning calculus students.

Ontario

Some of the recent mathematics education research in Ontario has focused on interpreting the results of the Second International Mathematics Study (SIMS). For example, E. Kuendiger and G. Hanna have analyzed SIMS data according to sex differences. Another related area of research interest is Women and Mathematics (E. Kuendiger, G. Hanna, P. Rogers).

Kuendiger has developed a theoretical model accounting for sex differences in achievement and course-taking behavior. A current project (E. Kuendiger) examines relationships between preservice student teachers' perceptions of mathematics and their mathematics teaching. Another study (G. Hanna) focuses on instruction and achievement in eighth grade mathematics classrooms. Another project which is currently in progress (N. Hutchinson) involves the teaching of representation and methods of solution of algebra word problems.

A large number of Logo studies were incorporated into the "Creative Use of Microcomputers by Elementary School Children" Project (W. Higginson, D. Burnett, H. Carmichael, and others). Though the learning of mathematics was not the major focus of this project, the final report does provide several insights into children's geometry activity in various Turtle Geometry environments.

Quebec

Much of the research taking place in Quebec can be characterized as the study of the cognitive processes involved in learning mathematics.

Many of these cognitively-oriented studies investigate different aspects of mathematical learning within a Logo environement. One research team (J. Hillel, C. Kieran, S. Erlwanger, J.-L. Gurtner) is examining the use of visual and analytical schemas by sixth graders in the solving of selected Turtle Geometry tasks. Another group (H. Kayler, T. Lemerise, B. Côté) is investigating the evolution of logical-mathematical thinking among 10- to 12-year-olds in a Logo environment. A third study (R. Pallascio and R. Allaire) is focusing on the development of spatial-visualization skills by fourth graders using Logo-like computer activities involving polyhedra. In another study (E. Lepage), a modified version of Logo for the very young

serves as the setting for researching the learning of early number concepts. An Object-Logo computer programming environment is used by another researcher (G. Lemoyne) to examine the knowledge schemas used by 9- to 12-year-olds in their production of mathematical expressions.

Other studies use non-Logo computer settings for their investigations. One project (A. Taurisson) involves researcher-designed programs to be used as tools by elementary school children in order to develop their problem solving abilities. Another team (A. Boileau, M. Garançon, C. Kieran) is examining the use of computer tools and methods as a semantic support for learning high school algebra. A group of researchers (J.C. Morand and C. Janvier) is investigating the evolution of students' primitive conceptions of circles. Another study (C. Janvier and M. Garançon) is looking at the understanding of functions and feedback systems using microcomputers. Other researchers (M. Bélanger and J.-B. Lapalme) are creating exploratory computer learning environments in which children can develop problem solving strategies.

Other studies with a cognitive emphasis which are currently being carried out (or have only recently been completed) include the work of: N. Herscovics and J. Bergeron who are investigating the acquisition of the concepts of early number among kindergartners and of unit-fraction among older children; D. Wheeler and L. Lee on high school students' understanding of generalized algebraic statements; L. Chaloux on sixth and seventh graders' construction of meaning for algebraic expressions; B. Janvier on the use of dynamic representations in the learning of early arithmetic; N. Bednarz who is comparing constructivist and traditional approaches to the teaching of numeration; C. Girardon on conflictual conceptions of transformations; A. Boisset on the difficulties which college level students experience with calculus; B. Héraud on the concept of area among 8-year-olds; C. Gaulin and R. Mura on the effects of calculators on the achievement of fifth and sixth graders; C. Gaulin, E. Puchalska, and G. Noelting on students' understanding of the representation of 3-D geometrical shapes by means of orthogonal coding; N. Nantais on the evaluation of children's mathematical understanding by means of the mini-interview.

Another group of studies exists where the focus is on attitudes towards mathematics: J. Dionne has analyzed teachers' perceptions of mathematics and of mathematics learning; L. Legault is looking at the affective factors influencing mathematical difficulties; L. Gattuso and R. Lacasse are investigating mathematical anxiety at the college level. Several related studies have recently been carried out by R. Mura and her colleagues on Women and Mathematics.

Newfoundland

The mathematics education research which is presently underway in Newfoundland includes the work of L. Mendoza, E. Williams, and M. Kavanagh. L. Mendoza is involved in a study of error patterns associated with combining monomials. He is examining both the error patterns and the underlying rationale for these errors by means of written testing and in-depth interviews. M. Kavanagh is studying grade 12 students' perceptions of mathematics, comparing those of students from all male, all female, and co-educational schools. E. Williams' focus is the study of students writing mathematics competitions such as the Canadian Mathematics Olympiad, more specifically, the investigation of heuristic and executive strategies used by "good" mathematical problem solvers.

Special Report

The CMESG research information session in St. John's also included a special report by Jörg Voigt of Bielefeld on his own research. He provided us with a brief summary of his presentation which is reproduced here:

Patterns and Routines in Classroom Interaction: A Microethnographical Study in Mathematics Education *

Jörg Voigt
Universität Bielefeld, West Germany

Often the question-response teaching in mathematics classrooms is seen by the teacher as being a liberal discourse in which the students actively participate. In opposition to the teacher's view, microanalyses of the discourse processes point to concealed and stereotyped patterns of interaction and routines. Certain patterns and routines lead to misunderstanding of the teacher's intentions. On the one hand, the patterns and routines facilitate the "smooth" functioning of the classroom discourse; while, on the other hand, they produce undesirable effects on the students' learning.

For instance, the following pattern has been reconstructed across several videotaped situations. The teachers attempted to activate the students' everyday experiences as a starting-point for introducing a new mathematical content.

-- The teacher asks an open, ambiguous question hoping to elicit the students' non-academic ideas.
-- The students refer to their own subjective experiences from everyday life.
-- The teacher rejects the students' ("deviant") everyday idea using tactical routines. Although the students' idea could be a worldly wisdom, the teacher wants a different specific idea. He uses suggestive hints in order to make the students give the expected answer.
-- In effect, the students learn to isolate the mathematical concept in the classroom from their "truths" in everyday life.

While the teacher thinks that he used the students' experiences as a starting point, the opposite happened. The teacher and the students seem to be so skilled in how to deal with each other that the teacher does not become aware of the gap between his intentions and the routines taken for granted. Because of the latency of the routines, it would be helpful to develop the teacher's awareness of such microprocesses as they occur in these social interactions.

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* A fuller version of this study is reported in:
Voigt, J. "Patterns and routines in classroom interaction". Recherches en Didactique des Mathématiques, Vol. 6, No. 1, 1985, pp. 69-118.

WORKING GROUP A

THE ROLE OF FEELINGS IN LEARNING MATHEMATICS.

GROUP LEADERS:
FRANCES ROSAMOND
PETER TAYLOR

WORKING GROUP A

The Role of Feelings in Learning Mathematics

Frances Rosamond
Peter Taylor

List of Participants:

John BERRY	(Manitoba)
Annick BOISSET	(John Abbott)
Robert BRYAN	(Western Ontario)
Sister Rosalita FUREY	(Holy Heart, NF)
Carolyn KIERAN	(Quebec a Montreal)
Lesley LEE	(Continuing Ed, PQ)
Robert McGEE	(Cabrini)
John POLAND	(Carleton)
Marie POLAND	

This Study Group derives much of its excitement and cohesiveness from bringing together individuals who have long been concerned with topics involving affective aspects of mathematics education, but who have been developing their ideas almost in isolation. For me it meant reconnecting with two excellent foci: the positive part of affectivity, and the community in the classroom which sets the stage for these positive feelings through its cooperative organization. As we headed into our final hour, Peter Taylor summarized most beautifully our collected anecdotes in the following framework:

* Our belief in the sharing of goals; e.g. by the teacher, openly and honestly, with full frontal explicitness, reducing hidden agendas,
* Our belief in the sharing of our joy in doing math,
* Our caring for the people in the class and in the mathematics being done, and
* Our promotion of cooperative small group work.

John Poland

The Working Group focussed on two activities. We did much problem solving in pairs in an effort to identify and explore the emotions involved in problem solving. This activity is described and the findings are discussed in the appended paper.

Our second major activity was to share techniques for implementing the framework summarized above. The following paragraphs list some of the many creative techniques that have been devised and used by various members of the Working Group.

-The use of a monthly newsletter talking about the course, test results, who the teachers are as people, the positive aspects of doing mathematics, and where to get help.

-Taking small group or individual pictures at the beginning of the course and posting them (perhaps take them at an early informal gathering).

-Make a list of names and phone numbers of class members and get everyone a copy.

-In some way convey expilicitly to the students that they are a special group, perhaps breaking ground through some teaching or curriculum innovation you are sharing with them.

-Build in fun with math learning. Use Math games. Decorate the room. Use flowers, music and movement.

-Control the lighting. Some teachers turn off fluorescent lighting and use candles or lamps instead.

-Interview the physics, chemistry, etc. professors and put on big cardboard their answer to the question, "What I want students to know when they come into my course".

-"Algebra Arcade" (Wadsworth Electronic Publishers, 8 Davis Dr., Belmont CA 94002) was suggested for a first algebra course for groups of 3 or 4 students to work at a time or for one large demonstration screen.

-Allow students to suggest how they will be evaluated in the course. They must come to consensus. The discussion can extend over several days.

-Spend teacher energy on the positive. Emphasize the students who do achieve and their accomplishments.

-Talk about what understanding proofs does for them as people, that they can handle and generate arguments. Have positive expectations.

-Use ice-breaking techniques that help students learn the correct language and notation of mathematics. For example, put 4-5 students on a team to try to communicate to another team (without showing any writing) a given collection of math symbols.

-Seriously address the idea of math anxiety. The teacher can talk about his or her own feelings about mathematics. Alert students to use positive self-statements and other means to prevent emotions from overwhelming short term memory. Evaluating an emotion can take up so much student memory that little is left for mathematics decisions. Math thinking becomes confused with thinking about math.

We came up with many areas to explore further. We would like to know which ways of organizing classrooms and tests encourage students into good study, classroom and exam habits. How should we sequence questions, sets of problems that will provoke students to "review" as in Polya? How should small groups best be utilized? What is best size? How can writing be used in math classrooms?

We decided to ask colleagues to describe techniques they have used successfully. We plan to compile these anecdotes together with a bibliography of appropriate readings and disseminate the information in a fuure CMESG Newsletter.

AUTHORITY IN THE CLASSROOM

I should like to see the locus of authority in the classroom shift away from the teacher and the material (these should be regarded as resources - a less threatening category) and toward more inwardly generated forces such as beauty, excitement, challenge, communication.

Let me explain the difference. When consulting a resource, you are the boss, when consulting an authority, the authority is the boss. Alternatively, from a resource you take what you want; from an authority, you take what it wants. Early in the learning game, teachers have to be authoritative. But part of their purpose must be to gradually change themselves into resources (by changing the student) and substitute instead the criteria which guide active scholars through the question of whether they are working in the right things: is it beautiful? does it excite me, challenge me? does it lead to fruitful communication with my colleagues?

If we relate this to problem-solving, one thing we see is that the problems the student works on should much more often be generated within himself and the various sources of inward authority I listed above should increasingly be used to guide him on the questions of what time he should spend on the problem, and whether certain avenues should be pursued.

PGT
July 86

COMPETITION AND COOPERATION

When one studies a community, there are two types of forces one looks for: competitive or disruptive forces and cooperative or supportive forces. The mathematics classroom is a community in which often too much of the action is really of a competitive nature, either student against student, or student against a teacher's expectations, and the effect of this must often be to increase student anxiety.

We felt that such anxiety was not beneficial to the student. While it might enhance certain aspects of the student's performance, we felt it was not likely to increase his problem-solving abilities, and would certainly dampen the feelings of joy he might have when searching for the solution.

We made a number of suggestions for enhancing the cooperative atmosphere of the classroom, in short, the feeling that we're all on the same side. First it is important that the teacher be open and as explicit as possible: about the goals of the course, about his views on the subject matter, and about his own feelings about the class. It is important that the teacher care both about the subject and about the students, and be clearly enjoying the teaching experience. Second, the nature of and rationale behind, the methods of testing and evaluation, should be thoroughly aired. Thirdly the students should know and work with one another; often this can be facilitated with small group work. Other devices such as classroom games, attention to physical character of the room (lighting, decoration), and a monthly newsletter, were mentioned. It was suggested that experimental programmes often generate a very positive feeling of shared community. Perhaps we should more often be experimental; even if we have little flexibility in the content of the curriculum, we can experiment with style.

PGT
July 86

July 20,1986.

The "affect" workshop took place six weeks ago and even if some of the details have slipped away (thanks for the reminders Fran), I am still feeling the after-affects of having been with a group of student-centered math teachers who are interested in exploring affective elements in themselves and their students. Although I have thought a lot (and talked a lot) about this theme, I feel that the workshop broke my isolation.

The experience of doing individual problem solving in teacher-teacher pairs was new to me. I have done some introspective work, interviewed about a hundred students, and given the introspective problem solving exercise to many adults. It was interesting to see that as teachers and mathematicians we are not so very different from our students in affect during problem solving. Another memory of that experience is of several people indicating that their problem solving behavior was in some way indicative of their behavior in non-mathematical situations: "That's the story of my life." If this is so, it certainly would be worth exploring further.

Although not everything has been said about affect in individual problem solving, I feel I would like to move on to an exploration of group problem solving. In the workshop we all seemed to be interested in promoting cooperative models and group work in our classrooms. Yet problem solving in groups is much more complicated than individual work. Group dynamics and the politics of the classroom come into play. I, for one, feel a little insecure in initiating group work - which maybe why I rarely "find time" for it. I think that the affect workshop, because of its secure and supportive atmosphere, would be the least scarey place to start looking at group problem solving.

For this I'd even go to Kingston!

Lesley Lee

PATTERNS OF EMOTION WITHIN MATHEMATICS PROBLEM-SOLVING

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Paper prepared for the Panel on "Mathematics as a Humanistic Discipline" presented at the Joint Meetings of the Mathematical Association of America and the American Mathematical Society, San Antonio, Texas, January 1987.

I like the clever twists of logic that turn a two page proof into a one-half page proof. There are lots of clever little insights. There's something very satisfying about a nice tight argument that no one can doubt is correct...I've worked on a research problem for over six months with no results...now I'm starting to dream about it and that's too much..the mathematics is taking too much control over me. (Angrily.) (Rosamond, 1982)

Mathematics often is viewed as the ideal discip-line-pure rational thought dealing with ideal objects to produce irrefutable arguments, not coloured by any emotion. Training in mathematics is seen as producing students capable of such clear thinking in all disciplines. So why don't all mathematics teachers present mathematics in the ultimate, Bourbaki style? To mathematize is supposedly part of the human condition, so how can there be such a thing as math anxiety, when feelings should clearly not be a part of learning in mathematics?

Or does mathematic arouse emotion because it was conceived out of emotion in the first place?...What is the link between the affective and the cognitive? (CMESG Announcement, 1985)

PATTERNS OF EMOTION IN MATHEMATICS PROBLEM-SOLVING

In an effort to understand and explicate the feelings of satisfaction and anger expresssed by the mathematics graduate student in the first quotation, a Workshop on the Role of Feelings in Learning Mathematics was held during the Canadian Mathematics Education Study Group annual meetings of 1985 and 1986. We engaged in a problem-solving exercise that also was given to six mathematics education graduate students at a State college and on two occasions to six people who met in a private home.

We are all (with the exception of two people) involved in mathematics as professional mathematicians, as teachers, as graduate students or as people who use mathematics in our work. We believe that thinking, feeling and acting work together, that true understanding implies feeling the significance of an idea, and that our experiences are not far from that of our students. We decided to examine our own feelings in depth in hopes of finding outstanding commonalities that could be used to improve classroom teaching.

Studies on cognitive science (Davis, 1984. Papert, 1980), problem-solving (Silver, 1985), metacognition (Schoenfeld, 1983)

and belief systems (Perry, 1970) offer some insight into the role of emotions in problem-solving but only indirectly. We are not sure we have even a vocabulary with which to describe feelings at a specific moment as a function of many variables.

To begin with, we made a list of relevant positive and negative emotion descriptors (see appendix). This list was adjusted by the results of the exercise. The exercise is a simple one. We went in pairs to different parts of the room where one person agreed to be the problem-solver and the other the observer. The rules were 1) The solver do his or her best to provide a running commentary on feelings. 2) The observer keep quiet, pay attention, take notes.

After a fixed amount of time (15 minutes, in later sessions changed to 30 minutes) all gathered and each observer reported on what the solver had done, focussing on the feelings. The solver also reported.

The roles were then switched, observer became solver. Solver became observer. Another problem was presented and the observation and reporting process repeated.

We feel many positive emotions (challenge, hope, zest, satisfaction, etc.) when doing mathematics and wish to promote these in our students. Lazarus is a noted psychologist at University of California at Berkeley who has done extensive analysis of the theory of emotions. In his paper, "Emotions: a Cognitive - Phenomenological Analysis", he describes some of the contributions positive emotions make to coping. Before describing our exercise and the implications that we found for teaching, I will briefly outline some of Lazarus' position and make some connections to mathematics.

LAZARUS ON POSITIVE EMOTIONS

Lazarus points out that negative emotions have been studied almost exclusively. Some reasons for this are that emotions have been studied as evolutionary and that negative emotions such as fear or stress influence our capacity to survive. Another reason is that emotion is studied by therapists who may view emotion as pathological. In this case happiness may be seen as hysteria, concern as paranoia and hugs as evidence of nymphomania. A third reason is that it is more difficult to measure arousal for joy, delight, and feelings of peace than it is for rage, disgust or anxiety.

Because we are trying to promote good problem-solving, we feel it is appropriate to focus on the positive feelings associated with our goal: on hope rather than hopelessness, challenge rather than threat, zest rather than dispair although negative emotions do need to be recognized.

Positive emotions tend to be frowned upon or viewed as "childish." Not many people exhort optimism like Ray Bradbury does: "We are matter and force turning into imagination and will! I am the center of a miracle! Out of the things I am crazy about I've made a life!...Be proud of what you're in love with. Be proud of what you're passionate about! (Bradbury, 1986) It is even hard to hear people shout gladly onto the Lord; but we were just trying to hear people shout gladly about mathematics. People who exhibit positive emotions often are accused of playing, of not being serious.

Yet playing with ideas is inherent in mathematics problem-solving. What emotions should we expect to feel when engaging in problem-solving? Lazarus answers this by saying that the essence of play is that it is highly stimulating. It is accompanied by pleasurable emotions such as joy, a sense of thrill, curiosity, surprise, wonder, emotions exploratory in nature. We recognize that we do experience these positive toned emotions when doing mathematics.

As educators we wish to know the optimum conditions that encourage problem-solving. Lazarus says, "...exploratory activity ocurs more readily in a biologically sated, comfortble and secure animal than in one greatly aroused by a homeostatic crisis. The human infant will not venture far from a parent unless it is feeling secure, at which point it will play and explore, venturing farther and farther away but returning speedily if threatened or called by the mother." As shall be discussed in more detail in the next section, mathematics problem-solving requires playing in an almost "other-world" of intense concentration. Insecurities in terms of math ability or other issues (world peace) inhibits problem-solving by interferring with the level of concentration.

USES OF POSITIVE EMOTIONS

Lazarus sees at least three ways in which a person uses positive emotions: as "breathers" from stress, to sustain coping, and to act as restorers to facilitate recovery from harm or loss. Lazarus' discussion may be interpreted with mathematics in mind.

BREATHERS OR TIMES OF INCUBATION

"Breathers" are times when positive emotion occurs as during vacations, coffee breaks or school recess. They can also be thought of as times of incubation.

Lazarus quotes the noted mathematician Poincare to suggest that it may be the good feelings themselves that allow a solution to emerge from the subconscious to the conscious.

Poincare made the surprising comment that unconscious creative mathematical ideas "are those which, directly or

indirectly, affect most profoundly our emotional sensibility." By this he meant that, since creative thoughts are aesthetically pleasing, the strong, positive emotional reaction to such ideas provides an opening through which they are ushered into consciousness.

Lazarus reminds us of another relevant description of a "breather" made by the great German physicist Helmholtz:

> He (Helmholtz) said that after previous investigations of the problem "in all directions...happy ideas come unexpectedly without effort, like an inspiration. So far as I am concerned, they have never come to me when my mind was fatigued, or when I was at my working table...They came particularly readily during the slow ascent of wooded hills on a sunny day."

The acceptance of the role of a breather is reflected in the usual advice given by teachers to their students: "Concentrate long enough to get the problem firmly in your mind and to try several approaches. But then take a walk or do some pleasant activity and let your mind work on the problem for you."

SUSTAINERS OR MOTIVATORS

Positive emotions act to sustain problem-solving in the sense that good feelings build on good feelings. Mathematics and the word "challenge" often are linked together as in "The problem is a challenge." A challenge can be viewed as a threat and in our exercise, problem-solvers were momentarily worried about failure in front of an observer. However, in challenge, a person's thoughts can center on the potential for mastery or gain. This challenge is accompanied by excitement, hope, eagerness, and the "joy of battle." All these positive emotions were mentioned by problem-solvers. One solver summarized the feeling as "the joy of mental engagement and the bringing of all mental force to bear in a cohesive way." Solvers who perceived their problem as too easy felt disappointment even before they began to work on the problem. Those who felt the problem worth working felt an immediate joy even before proceeding. This joy was a signal to bring all mental force to bear on the problem, which in itself produced pleasure and therefore motivation to continue.

Lazarus describes "flow" ,an extremely pleasant, sustaining emotion, as in the case of the basketball player who is "hot" or the inspired performance of a musician.Lazarus claims flow arises when one is totally immersed in an activity and is utilizing one's resources at peak efficiency. Mathematical problem-solving requires total immersion and we found that a comfort with notation was important in maintaining this flow. Comfort with notation will be discussed later in this paper.

The positive emotion of hope also provides motivation to keep going. Occasionally during a problem-solving episode the solver lost control of the problem. Solvers said, "I've lost control of the problem." or "This is too complicated, too many angles to label." or "I feel this is getting a little out of hand. This one and that one cancel out and I haven't used fact that it's a prime." Hope, the belief that there is even a slim chance things will work out, helps one continue. Ambibuity nurishes hope. One cannot be hopeful when the outcome is certain. We would like to know how ambiguity can serve classroom mathematics. The emotions of challenge and hope are powerful motivations in problem-solving and deserve further research.

A more obvious way in which emotions sustain actions is in terms of longer range goals. The student who has a positive feeling solving one math problem is more likely to try another. The confidence that comes from understanding mathematics empowers the student to attempt new ventures also, as in the case of a geometry student who attributes his decision to help in crime prevention directly to his success in his geometry class.

RESTORERS

Lazarus offers a third function of positively toned emotions, that of restorer. Lazarus' descriptions of recovery from depression or restorations of self-esteem might be useful to the teacher dealing with math-anxious students. Lazarus quotes Klinger:

> At some time during clinical depression patients become unusually responsive to small successes. For instance, depressed patients working on small laboratory tasks try harder after successfully completing a task than after failing one, which is a pattern opposite to that of nondepressed individuals, who try harder after failure.

It would be worthwhile for the classroom teacher to know when small successes are more likely to evoke positive emotions. Offering a small task to a math anxious student may foster optimism and incentive while the same problem may seem trivial to a non-anxious student and provoke anger or disappointment. This is an area for more research.

Much of the information on emotion in problem-solving is obtained by having students fill out questionnaires. While the information is useful, a rating on a scale from one to five of confidence in doing math, liking for math, or usefulness of math is very general. Questionnaries also are remote from the actual process of problem-solving. Recollections of feelings might not be quite the same as the feelings at the time. Also, mathematical problem-solving requires intense attention to the problem. It is likely that without some help a solver will not even be aware of his or her emotions. The above reasons together with the belief that our own feelings when doing mathematics are the same as those of our students prompted us to do an exercise utilizing a close observer and introspection.

OBSERVATIONS FROM THE EXERCISE

Altogether the exercise of observing, reporting, solving, reporting was done by 19 pairs. Problems initially were of 'the puzzle variety (Gardiner,1967, 1979. Mott-Smith, 1954) but in later sessions more substantial problems were chosen from Honsberger. One person kept track of time for the whole group. A group of six people (three pairs) seems the best size. We posture...laughter...intent stillness" but that description is not used in this paper.

EMBARKING ON THE PROBLEM-SOLVING

Solvers accepted their problems with curiosity and positive anticipation. These were people who did formal mathematics frequently. Two people who had not done formal math recently reported terror.

The initial reading of the problem provoked a reaction to its type followed by a sense of its difficulty. "I anticipate I will enjoy this problem but may not make much progress." or "I loathe this type of problem . It is do-able but will require a big effort. I think I will have to go through many tedius decompositions."

The word "do-able" was used often and meant either that the problem was solvable or that progress could be made in understanding the question. For one of the people who reported terror, a person who rarely uses formal mathematics rarely and who was talked into coming to the workshop, considerable time was spent blocking the reading of the problem. Emotion can be regulated by avoidance or denial. This person acknowledged feeling bad but then felt bad about feeling bad so that "Even if I could do it I couldn't." Considerable time was spent recalling past history of problem solving failures all the while avoiding (somewhat consciously) making the decision to try to do the problem. Another solver also reported "I felt unhappy and then felt unhappy about feeling unhappy." Emotions tend to feed on and reinforce each other. The math oriented solvers were predisposed to extend effort on the problem. They had much more commitment to do math.

After reading the problem, all began to develop a notation, to draw a diagram or to write some hypothesis. This was the beginning of a cycle of attention on problem - attention on self or distraction by environment - attack on problem - attention to self or environment - problem - self - problem - self, etc.

When preparing to choose a method of attack, there was considerable emotion tied in with "not cheating." Each person placed the problem in a certain context and at a certain level of difficulty and felt it would be cheating, bad sport, to use a technique that was too powerful. One solver says, "Can I use fancy stuff?... Then I'll use Jordan Curve Theorem....laughs". Backtracks. "Maybe an easier way." Another solver resisted but finally made a grudging commitment to using calculus for a problem entitled, "An Obvious Maximization."

Using brute force was considered almost as bad as using a too powerful method. "I'm annoyed because I can't see any other way than brute force and that would not yield for me any understanding of the problem...there must be an easier way." Solvers wanted to find solutions that were generalizable. Using a too powerful method, brute force, or an "obvious method" brought forth comments of feeling embarrassed or annoyed.

A less conscious resistance to cheating was the seen in the imposition of ridiculous restrictions on oneself. For example, one solver had Honsberger's book in hand and was to "Use the 'Method of Reflection' to...". (Honsberger, p.70). The solver's reaction was, "I understand the problem but don't know this method...I wish I could read the chapter..." . Instead of simply reading the chapter, the solver tries to invent a plausible 'Method of Reflection'.

Another solver spent long moments seemingly aimless. "I'm feeling a little out of control of the problem...lots of parameters...seems to be a lot of ways to define this problem...I'd like to clarify the problem by asking whoever wrote it." Finally with a forced air, "I could break it up into cases myself and come to grips on my own terms and get partial solutions...got control back."

Self imposed restrictions would slow a solver down until there were reports of, "I'm squandering time. I really haven't done anything." Then there would be a squaring of the shoulders and a businesslike assertion to "...take a stand and try to prove it ..." even though this might mean grinding out a meaningless, albeit correct, solution.

INVOLVEMENT WITH THE PROBLEM

Once commitment was made to attempt the problem, there was a lorelai seductiveness about it, a delicious slipping off into another world. Solver became oblivious to self, observer, or environment. This total immersion was a wonderful release from daily life. Poland (CMESG, 1985) used mathematics to help him ignore the pain of an illness. Some people use the other-world quality of doing mathematics to avoid interaction with peers. Mathematics can help with depression as the famous mathematician Kovalevskaya said in a letter: "I am too depressed...in such moments, mathematics comes in handy, and one enjoys the existence of a world completely outside of oneself." (Knopp, 1985).

Mingled with the charm of seduction there was a dangerous quality, a frightening isolation if one stayed immersed too long. Rosamond (1982) gives examples in which the solver feels consumed by a too dominating mathematics. As one mathematics graduate student said with tears in his eyes, "What do you do if you are 80 - 90% mathematics? If you've let yourself become consumed by mathematics so that that is what you are. And then you want to let someone get to know you. What do you do when you can't explain that much of yourself to them?" The presence of the observer comforted the solver and lessened the dangerous quality in the isolation.

There was a letdown feeling of disapointment if the solution came so easily that little emotion needed to be invested in in the problem. Typical is the remark, "The problem must have been too easy, I got it. So what's the big deal? I feel let down."

or "It was fun but not intense because not a challenge. I feel let down because I didn't spend a lot of emotion." The complexity of the problem came like a revelation to one solver who then responded with a BIG smile. Overall, the amount of satisfaction with the problem correlated directly with the intensity of concentration. The perceived level of difficulty of the problem also influenced satisfaction and this will be discussed later.

However, one cannot maintain a constant level of intensity throughout the solving of a problem. The use of notation in a ritualistic manner provided a "breather" or moments of relaxation while allowing the solver to remain in the "other-world". When no progress was being made on a problem, the solver remained in the intense state by writing out some formal routine. Some solvers would rewrite the definition of the variable. One solver began, "There are two cases: a) the problem is solveable and b) the problem is not solveable." Almost everyone used x's and y's at one time and then decided to switch to a's and b's (or vice versa). Some would say, "I'm going to try induction." and then write out the induction hypotheses. The rote writing out of hypotheses or the rote switching of variables afforded a lull within the other-world state and continued the flow. The importance of these rituals was to help focus on the problem. To sit too long without progress or a ritual meant the solver would think about self again.

Other pauses also bump one out of concentration. When the solver paused overlong in appreciation of some success, then attention tended to turn to self or environment. The jolt of finding a counterexample to a hoped-for truth caused one to notice the ticking of the clock or the coldness of the room. Extended frustration of method caused recall of poor geometric visualization in the past and then embarrassment. Attention was diverted from the problem to the self. This usually was for a brief amount of time, less than a minute. Solvers would look around, sigh, stroke the pen, scratch, talk a little and then go back into the problem.

Most solvers were engrossed in the problem when time was called and these people were irritated at being interrupted. They almost all mumbled "I'll continue later." Solvers who were in an attention-outward part of the problem-solving cycle just prior to time being called generally sat back and waited out the time. They did not work on the problem further while waiting but mentioned that they would return to it later. There was reluctance to allow oneself to get lost in a train of thought and then yanked out of it.

IMPLICATIONS FOR THE CLASSROOM: VARIABLES THAT INFLUENCE ENGAGEMENT

The primary goal of our exercise is to improve classroom teaching. It would be useful for a teacher to know what a particular emotion looks like. For example, a teacher who knows that yawning is a release of nervous tension and not an indication of boredom have an immediate and obvious clue that a student needs help. (And the teacher knows not to get personally insulted by the yawn.) In the opposite direction, the teacher who wants to indicate positive emotions to the students would know how to do it because he or she would know what they look like.

To this end we took notice of some physiological indications of emotive arousal (flushed face, sweaty palms, muscle tension, etc.) and of body movement (twitching, sighing, laughing, etc.) but more work should be done here and these indications are not elaborated on in this paper.

We found that overall satisfaction in problem solving is directly related to the intensity of engagement with the problem. The engagement is influenced by several variables: the nature of the problem, the perceived usefulness of mathematics, the role of the observer, the use of mathematics rituals, and the testing situation. Each of these variables will be discussed along with their implications for the classroom.

NATURE OF THE PROBLEM

All solvers were more encouraged by harder problems than by ones marked "obvious" or ones perceived as easy. There had to be a sense of value of the problem, not that it must be directly applicable to daily life, but rather that one needed to think in order to understand the problem. If one could get the answer just by asking someone else or by looking it up then that made the problem artificial and was almost an affront to the solver.

Surprisingly, solvers felt threatened whenever they saw the words, "Clearly", "It is easy.", or "Obviously". Most felt that teachers should not say, "This is easy." and that textbooks should not indicate the easy exercises. Solvers sometimes worried that the problem looked so simple. They felt they were missing the point and that their solution was not elegant enough. One solver found three solutions by varying the constraints and then felt less humiliated.

One solver exhibited obvious arousal with eyes wide open, clear face and a slight laugh. "Hey, there's an infinite process..." Exploration didn't bear out infinite process and then there was "That was neat. What was the problem?" together with a clear drop of interest and rather emotionless settling again into the problem. The challange of the infinite process stimulated playing around in the "math-world."

The math-world is a mental out-of-body arena of intense concentration in which a person can play with ideas. Trivial problems do not make good play-mates. One solver's most satisfactory experience of problem-solving came after having spent a week on a problem only to have the professor tell the class that the problem was not solveable.

Solvers felt initial relief at seeing an easy problem but were quickly bored, disappointed or insulted. The classroom teacher must pay careful attention to the quality of problems offered and should not label them easy or difficult.

USEFULNESS OF MATHEMATICS

Doing mathematics is seductive but one must allow oneself to be seduced. Three different participants at three different sessions (all women) felt that going off and doing mathematics was a luxury. A teacher of older women said she had to convince her students that they were not squandering time while problem solving. Women are always productive. They even knit while watching TV. She got around her students' hesitancy by saying, "I'm going to show you some games to teach your kids and improve their math."

The notion of usefulness was mentioned by only three women but it is a construct that has been singled out as the most important attitudinal factor in decisions to take math classes (Sherman and Fennema, 1977.)

Usefulness was elaborated on at length by one solver waho was able to solve the assigned problem in a short time and with no intense engagement. The solver was disappointed and felt letdown. It was not clear if the following remarks would have been made had the solver been given a more engaging problem. I asked at the time but the solver was very agitated and insisted that another problem would have made no difference.

> "What would have beeen a meaningful problem? How come I'm not satisfied? I had an expectation about solving that problem that did not get fulfilled. It didn't make me happy. There were some moments of tension and some of excitement but not intense. It was entertaining like a grade C movie.

> "Math has no social relevance to me...I am willing to solve math problems, even ready but it feels completely disjoint from what interests me. I still love it (This solver has a Masters degree in math and is an active MD.) but its importance seems miniscule compared to world problems...beautiful but frivolous to use my mind in this way." (It would be useful for other people to do math but there were more pressing issues for this particular solver.)

Usefulness of mathematics in terms of careers or its sometimes theraputic value as a means of escape is an affective variable that may be easy for teachers to influence. Teachers can present information about the mathematics required by various careers as well as the mathematics courses that should be taken to keep options open in the future.

THE USE OF RITUALS

The use of formal routines that keep one's attention on task while providing a sort of restful interlude speaks directly to the classroom teacher. Students must have a comfort with notation not only becausd the notation itself sometimes points to the solution but because that comfort sustains concentration.

ROLE OF OBSERVER

Contrary to almost everyone's expectation, having someone observe while working the mathematics was positive. At first, some solvers felt less inclined to free associate with ideas in front of an observer who might have the problem already all figured out or the solver sometimes felt that the observer must be bored. Some solvers wanted to talk things over with their observer or would look up at the observer hoping for confirmation.

It turned out that the presence of the observer was an impetus to persistence in doing the problem. This is a very important point. Liking the problem was directly and positively related to the amount of time spent working on it. Almost everyone liked their problem more the longer they worked. Those that did not like their problem initially began to like it after all and to get interested in it. Without an observer, those solvers might have quit.

Being observed evoked other feelings. As noted earlier, the presence of an observer reduced the feeling of danger in isolation that lengthy immersion in the problem sometimes brought. There was a feeling of honor. "I felt honored that another peson was taking the time to observe me." Another feeling was intimacy. "It felt intimate to have someone commited to watch the workings of my mind."

While more emotion seemed to come from being watched, it was also important to be the watcher. Watching seemed to take away some of the secret charge of the observer's own problem-solving anxieties. The observer could recognize his or her own feelings in the other person and see how the feelings influenced their actions. Watching another person struggle with anxieties made the solver think, "Why don't they just get on with it."

One participant reported, "The most poignant part of the exercise was hearing the observer say what I'd done. I did not feel intimidated. I didn't get any of the bad response I expected. The observer demystified my emotional and intellectual engagement by simply listing what I did: 1, 2, 3, 4. This cut it down to size, gave it true proportion."

This exercise of being observer then reporter, then switching to being solver then recipient of report should be explored as a means of eliminating math anxieties in our students. The real key is the switching. This exposes and throws out the power of negative feelings while encouraging positive ones.

It should be noted that no one argued with their observer. A few points of clarification were made but there were no misinterpretations. It is possible that finer gradations or other categories of feelings can be made, but there was good correspondence within our vocabulary.

THE TESTING SITUATION

Concern about the nature of the problem carries over into the testing situation. One solver commented on the problems found on math tests. "A test is an almost random set of narrow problems where one thing must trigger another. It is not about figuring things out. Test questions do not show that math is a process." This solver had as a partner a professional research mathematician. The solver was not intimidated by being observed even though the problem was not solved because "The observer could hear that I have math training. He could see how my math mind works, how I assimilate information, manipulate, and use an asrsenal of strategies. This is so much different from taking a math test where I am not tested on how my mind works. On a math test, I could expect not to be able to show what I know. I would feel shame."

Part of almost any testing situation is a time constraint. Having only 15 or 30 minutes annoyed and inhibited these solvers. Some reported feeling "hemmed in...I do best by playing around...ordinarily would draw pictures and really understand...build up a pattern." Another felt pressure to categorize a solution method quickly. "Without a time constraint I probably would have been more impulsive...would have guessed and then worked backward. I felt forced to be more systematic, meticulous, more step-by-step and mechanical. I think I could have solved this in a shorter amount of time if there had been no time limit."

When the timing in itself counts, it is as though what the problem means in itself is not enough. Perhaps the discomfort of a time constraint forces one's attention to be divided between the math-world and present time. Not only are different methods of solution chosen at the onset of the process, but also the total immersion into the problem-world is not as possible or as deep.

CONCLUSION

It is important to state that a basic assumption of this experiment is that we professional teachers and mathematicians have at least the same feelings that students have. We may experience a difference in intensity (less anxiety, more confidence) or have other feelings in addition (sense of commitment) but overall how we respond gives an indication of how our students respond. A mathematics educator refused to participate in our exercise saying that it might be worthwhile for "personal growth" but that it would give no insight into how students feel. He believes that teacher feelings are completely different from student feelings.

But imagine your feelings if the Chair of your Math Department suddenly announced that you must take a test. If you have not taught a particular course in the past two years you must pass a test before you can teach it. What course are you scheduled to teach that you have not taught recently? What is your reaction to your Chair's announcement? You are not being tested on how well you review the materal during the semester or on how carefully you prepare your lessons. You are not being asked to share ideas with a colleague. You are being evaluated on questions someone else has chosen and already knows the answers to. I think your reaction to this thought-experiment may show that seasoned teachers can feel anxiety in a test situation similar to what their students feel in their test situations.

The act of knowing is not antiseptic; rather it is wrapped in feelings. It is the engagement of feelings. The primary goal of our work is to improve classroom teaching. This paper indicated only a few of the emotions inseparately connected within mathematical activity and specifically calls the classroom teacher's attention to the nature of the problems, the perceived usefulness of mathematics, the role of observer, the use of mathematics rituals and the testing situation.

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Appendix

MENTAL IMAGES

Shapes
dots
curves
numbers
symbols
colors
sizes
scenes
other

Movement
expanding, exploding
shrinking, collapsing
rotating, revolving
moving
other

neck
jaw
eyes
shoulders
back
stomach
adrenalin rush
blur of vision
sweaty palms, face
increased blood pressure
fight or flight syndrome
stomach butterflies
sense of euphoria
sense of paranoia
taste
tactile
auditory
smell
sensual
sigh
laughter
twitching
foot, finger tapping
flushed face
wiggles
ball (clutched, rolled)
other

nervous
dread
fearful
hostile
stuck
rigid, uncreative
frustrated
all alone, isolated
sense of urgency
sense of impending failure
desire for approval
feel in the minority
sense of holding back
don't know enough
desire to finish first
desire to finish last
desire not to be noticed
desire to be noticed
feel wasting time
feel stupid
vague depression
worry about making mistake
impulsiveness
feeling of boredom

relaxed
complacent
proficient
sense of familiarity
supported
encouraged
in-charge, in-control
comfortable
sense of enjoyment
sense of doing something useful
confident
elated
surprise
sense of persistence
sense of adequate energy
sense of elation
sense of significance
pride in accomplishment
feeling of having fun
sense of appropriateness
zest
vigor
sense of curiosity

WORKING GROUP B

THE PROBLEM OF RIGOR IN MATHEMATICS EDUCATION

GROUP LEADERS:
GILA HANNA
LARS JANSSON

 Lars Jansson & Gila Hanna

THE PROBLEM OF RIGOR IN MATHEMATICS EDUCATION

Funk and Wagnall's Standard Dictionary (1980) gives the following definitions for the term "rigor:"

1) The condition of being stiff or rigid;

2) stiffness of opinion or temper, harshness;

3) exactness without allowance or indulgence, inflexibility, strictness.

These dictionary definitions of "rigor" notwithstanding, the group did not seem to have a clear idea of what the term means, although it was evident that we wished to avoid its association with mortis. In order to focus our discussions we attempted to follow an outline which directed us to 1) the nature and function of rigor in mathematics, and 2) the place of rigor and proof in teaching.

As an exercise to be completed before the second session, each member of the group was asked to rank order four different proofs* of Pythagoras' theorem with respect to three criteria:

i) Which is the most rigorous?
ii) Which is the most convincing (to you)?
iii) Which one would you use to convince a nonmathematical friend of the truth of the theorem?

It turned out to be very difficult, if not impossible to reach any consensus on a rank ordering of these proofs in terms of how rigorous they were. This led to a discussion of what one means by the term (in the context of our deliberations the term "rigor" referred to rigorous proof). Rigorous proof is the procedure used in an axiomatic system to demonstrate the truth of a theorem in that system. The system should comprise:

1) a number of axioms
2) rules of inference
3) theorems (derived truths).

It was immediately recognized that this ideal can rarely be reached in practice with respect to major branches of mathematics in their entirety. Rigorous axiomatic presentations of small systems, e.g., games, were, however, recognized in subsequent discussion as being more easily attainable. We thought it important to speak not of absolute rigor as the property of activities within a well defined system, but of degrees of rigor within a system that is not completely defined. There were suggestions for definitions of "more or less rigorous:"

1. An argument is more or less rigorous to the degree to which it is free of unstated assumptions. The more it uses unstated assumptions the less rigorous it is.

2. A purported proof is rigorous if it is free of holes and it cannot be attacked, i.e., if nothing can be added to the chain of reasoning to improve it and if all of the analytical steps have been made explicit and are correct. It is "less" rigorous to the extent that these conditions are not met.

3. When the context of the proof is not analytical, e.g., proofs without words, the concept of more or less rigorous is not relevant. (Some group members viewed such a "proof" as only a schematic outline which could be expanded into a proof in various ways and of various degrees of rigor).

In the course of the deliberations we found out that

- Some of us are "unconcerned with rigor" in the teaching of mathematics--and unapologetically so.

- The authority of known mathematicians and of respectable textbooks or publications play a large part in our acceptance (although not in an absolute sense) of proofs, even in the absence of all of the analytical steps.

- A detailed and more rigorous proof may enhance the understanding of a theorem, but it also may hinder or contribute nothing to understanding.

- The degree of rigor desired seems to be a matter of taste and judgment depending on context and content. Demands for rigor rise and fall in history and depend in part on the function of the proof: ritual, validation, convincement. . . .

- The fact that mathematics is a social activity occurring in a social context and the need to communicate mathematics are very important to the notion of a desired degree of rigor.

- rigorous argument may exist in other disciplines--it is not peculiar to mathematics.

The group moved in the second session to a discussion of reasonable expectations of high school graduates regarding their knowledge of proof and rigor. Some expectations were:

- the realization that conclusions must be justified (and that this is part and parcel of mathematical activity);

- a knowledge of the role and function of axioms, definitions, theorems, proofs, and conjectures, and the ability to use these properly in a chain of reasoning;

- the ability to develop and sketch an argument/proof and the ability to defend or attack an argument/proof;

- some sense of the social conventions surrounding proof and rigor, e.g., the ability to distinguish between what constitutes a plausible argument and what constitutes a proof;

- we should be more concerned with rigorous thought and argumentation than with stylized written proof.

With regard to developing the above abilities and attitudes in students, some felt that

- mathematics which is exclusively content (as opposed to process) oriented is of limited value;

- that in order to develop the notions of proof and rigor a teacher may well have to rely on traditional content as a vehicle;

- a useful pedagogical technique is to

  - convince yourself
  - convince a friend
  - convince an enemy.

The final session focussed on what we could say to teachers' groups or curriculum committees regarding rigor. There was general agreement that teachers should:

1) emphasize the need for justification in drawing conclusions;

2) teach proof procedures in context rather than in abstract form;

3) provide students with opportunities to work on problems and situations which lead to observation of patterns, conjecture, justification, and looking back.

4) try to adjust the level of rigor (or of sophistication of the proof?) to the mathematical ability of the students.

Discussion on the content vehicle revealed that most mathematical topics were suitable for obtaining these four objectives. (There were a number of a number of pleas for geometry at the junior or senior high school level.)

A number of short readings were distributed and/or recommended during the sessions of the working group. Those that are available in other sources appear in the attached reading list. Post-conference comments on rigour are provided by David Wheeler, Ralph Staal, and Jörg Voigt.

*The proofs were

1) the standard proof given in Euclid
2) the Chinese "proof without words"
3) a proof using the inner product of vectors
4) the proof using the altitude to the hypotenuse and similar triangles.

RIGOR - A READING LIST

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POST-CONFERENCE COMMENTS ON RIGOR

by David Wheeler

Much of the discussion I found interesting, stimulating and helpful, yet I am left with a feeling that perhaps it was a pity that the focus of the group was on rigour rather than on, say, proof, mathematical reasoning, or some other more general conception. Rigour in mathematics seems such a specialised notion, far from what appears to me to be my central concerns about mathematics or mathematics teaching.

Probably the difficulty for me is that rigour in mathematics is essentially a technical matter. There is the formal apparatus of axiom, postulates, definitions and theorems, all embedded in a particular mode of deductive logic. Now I would grant that this apparatus has had two general consequences within mathematics: it has (1) encouraged some mathematicians to work on a clarification of the foundations (Peano is a good case in point) and (2) generated considerable activity in this century around the powerful concept of mathematical structures (Bourbaki and so on). Even so, the majority of professional mathematicians proceed on their ways ignoring the matter of rigour, and I am forced to wonder what possible application this technical stuff can have in the education of students, of novices, of people whose principal concern should be with knowing how to mathematize.

The pity of it is that the very special methods of ensuring (or approaching) mathematical rigour actually tend to reduce the attention educators give to rigour in its more general sense, that of "close reasoning". We can speak of rational arguments in any field as being more or less rigorous, and we sometimes refer to particular persons as "rigorous thinkers" (or not, as the case may be). This general appreciation of the value of rigour is very important, it seems to me. It gives a high valuation to such things as weighing evidence, being clear about one's assumptions, being careful about the validity of the steps in an argument, explicating the consequences of an argument even where these are not the ones hoped for, and so on. Some competence in this difficult art would serve any adult. I emphasize the word "art" to indicate that close reasoning is (in the present state of our knowledge, at least) something only a person can produce. The slight amount of evidence that computers can presently generate rigorous proofs in fact dismays me because it tells me that important ingredients of the process are being ignored.

It has often been claimed that exposure to mathematics helps students acquire general thinking skills. I believe that it could, but it hardly ever does. Mathematics is still largely taught, in spite of centuries of advice to the contrary, as a body of skills that can be imitated without understanding. Taught this way it

actually damages students' thinking powers (as can be seen in the substantial number of students who have become convinced that they are mathematically stupid). There is no doubt that mathematics could be used as a medium for encouraging careful thought. But how often in traditional classrooms does one hear teachers make interventions that promote attention and foster careful argument?

- Look at what you have done!
- Listen to what you are saying!
- Is she right? How do you know?
- Are not "this" and "that" contradictory?
- Would what you have said still be true if you substituted "this" for "that"?
- What have you forgotten?
- Can you convince John you are right?
- Do you need to use so much energy? Find a simpler way.
- Do not tell him! He can decide that for himself.

Mathematics is a very suitable medium to use in encouraging students to exercise reason since it relies very little on mature interpersonal experiences or sophisticated intellectual concepts, which students don't have, but a lot on immediate perceptions and fundamental mental operations, which they do. (How else could there be prodigies?) Once the habit of reasoning in mathematics lessons has been taught, arguments can be scrutinised and revised and made more rigorous. Eventually the students will come to see what a proof is. But this is a developmental process that takes a number of years. To offer the model of mathematical rigour enshrined in the axiomatic approach to school students is totally inappropriate.

by Jörg Voigt

. . . I enjoyed the working group and found the sessions quite interesting, especially because I was forced to think about the connections between rigor in mathematics and rigor in mathematics instruction. I agree with the report and will try to sum up my ideas of rigor.

I think that rigor in the presentation of mathematics should have little relevance to mathematics education, but rigor should be important for the discourse processes in the mathematics classroom. There rigor could be an implicit element of the discussions. Somewhere Hans Freudenthal wrote: "When does reasoning begin with the pupil? Before it is termed as proof or the like."

With regard to Vygotsky, Wittgenstein and others the development of mathematical thinking depends on the experiences gained by the pupils in the social interactions between the teacher and the pupils. One task of the teacher is to organize mathematics instruction in such a way that the processes of arguing interactively constituted are preliminaries of individual rigorous thinking. Surely, the teacher should have some knowledge of logic, but the problem is to see the lines of argumentation in the classroom processes and to organize them. The problem is the connection of the knowledge of logic with the practice of teaching in a specific context.

I have similar findings in mathematics classrooms to that of Thomas Russell in science classrooms (J. of Research in Science Teaching, 1983, v. 20, n.1): Often the dynamic of the social interaction replaces the rationality of argumentation. In classrooms the teacher's authority was established for the social organisation of teaching and learning, but it is at the same time a menace to the learning process.

If I had to work with mathematics teachers in this context, I would

- make them solve mathematical problems in little groups
- videotape the group work, and
- let the teachers reconstruct the lines of their argumentation.

In this case, the teachers could notice that it is important and very difficult to do mathematics rigorously with other persons.

I concede that mathematics instruction could and should be not an image of the ideal practice of reasoning in the discipline. But the teacher's authority should not be a substitute for the rigor of mathematical rationality.

by Ralph Staal

I found much of the discussion confusing, because although the chairman began by pointing out that "rigor" is a relative term, some participants continued to use it apparently in an absolute sense, as in "I don't believe in rigor at this level." This absoluteness could be an acceptable convention, except for the fact that it wasn't said what this absolute sense was. An analogy would be when someone says "I don't have any temperature in my office."

Another example of ambiguous terminology was the failure to distinguish between:

1. The degree of rigor of a proof that P implies Q

2. The degree of rigor of a proof of Q -- this proof using P, where P itself has been established on either more or less rigorous grounds.

The difference is that in (1) the degree of certainty of our knowing that P is true is irrelevant, whereas in (2) it isn't. This makes a great difference in talking about rigor in mathematics. Unfortunately, it takes a good deal of persistent effort to maintain the distinction.

With this much variation in meaning, it was not surprising that there was no consensus as to the ranking of four proofs of Pythagoras' theorem with respect to the three levels of rigor.

In our discussions, there seemed to be a strong undercurrent (hard to point to, but felt to be there) of wanting to show that the association of mathematics with rigor (presumably meaning "being very rigorous") was naive and was in need of correction. In my opinion, this point of view is often the result of "put-down through imperfections" by which one can show that there is no such thing as truth, or beauty, or objectivity, or justice, or virtually anything worth talking about. The study of foundations of mathematics does make one aware of the elusiveness of absolute or perfect rigor, but a thoughtful perspective on this matter nevertheless puts emphasis where emphasis is due, namely on the extremely (perhaps even uniquely) high degree of rigor with which mathematics can be pursued.

With this perspective as a guide, I can see no reason to modify the role of mathematics in education as stemming to a large degree from its association with a relatively high degree of rigor. (The work of Lakatos, so often misapplied, in my opinion, does not change this one bit--rather it shows how the search for greater rigor leads one to more and more rigorous definitions of concepts.)

WORKING GROUP C

MICROCOMPUTERS IN TEACHER EDUCATION

GROUP LEADER: CHARLES VERHILLE

WORKING GROUP C

MICROCOMPUTERS IN TEACHER EDUCATION

Content: Page

Leader:	Charles Verhille	University of New Brunswick
Participants:	Roland Eddy	Memorial University
	Sr. Rosalita Furey	Holy Heart High School
	Bill Higginson	Queen's University
	Joel Hillel	Concordia University
	Ross Finney	MIT
	Alan Schoenfeld	University of California

The main speakers at the meeting, Ross Finney and Alan Schoenfeld, each joined the working group for one of the sessions.

REPORT

Initially the group discussed personal experiences, direct and indirect, they had with teaching mathematics using computers and software. The following points gradually emerged from the conversation of the first evening.

Careful selection of software is necessary because:

a) there is a lot of expensive software that has little educational value;

b) of the current curriculum. At this early stage in utilizing computer as classroom teaching/learning tools, it must fit the existing curriculum.

Even good and powerful software does not necessarily lead to effective use. Teachers, both pre and in-service, need to devote considerable thought in preparing to use computers to help in the teaching of mathematics. Our discussion suggested that there is a growing body of evidence supporting the computer as a valuable teaching resource yet it is difficult to assess its potential during what is perceived to be an early stage in the development of the technology. Not only is the computer hardware undergoing continuous and rapid changes, but the development of software with exemplary features is slow. In a recent article published by members of the Shell Center for Mathematical Education at the University of Nottingham, a research study over three school terms in a secondary school indicated the following:

a) computer aided teaching will be successfully adopted if necessary resources are available, and

b) computers seem to be very versatile teaching aids and there are no grounds (at this time) for strongly recommending any particular style of use.

These suggestions as well as the developing nature of the entire field of computer use in education give negligable guidance and direction for microcomputer use in teacher education. The use of exemplary material in the average classroom with the average teacher was briefly considered. Several group members described sessions they had observed using the Geometric Supposer, a piece of software they considered exemplary. In each instance it was suggested that the scenario did not typify the average mathematics classroom with the average mathematics teacher. The developers and users of this software in these situations perceived that, because of its power and versatility, students could be successfully drawn into an inductive exploration or search for geometric truths, after which they would concern themselves with developing convincing deductive arguments (proof). They were behaving as geometers. Because of the features of the Supposers, in that one has a drawing and measuring tool which permits the operator to quickly and accurately produce, measure and alter geometric constructions, much of the drudgery and inaccuracy related to ruler and compass constructions is avoidable. Equally as powerful is the ability of this software to "remember" the current construction and repeatedly repeat it upon request. This potential permits geometric exploration and pedagogical approaches for teaching geometry that were previously imaginary. Unfortunately many of the seemingly best mathematically qualified, based on the amount of mathematics studied, mathematics teachers have never personally experienced learning mathematics in this way and thus fail to appreciate exciting new possibilities.

As a result of the initial general discussion we decided to begin by examining some of the software material available to the group. We ended up devoting the rest of the working group sessions to a discussion of the following software packages:

Apple Logo (Apple)
Algebra Arcade (Wadsworth)
Interpreting Graphs (Conduit and Sunburst)
Geometric Supposer - Triangles (Sunburst)
Calculus Student's Toolkit (Addison-Wesley)
Graphical Adventure (Saga Software)

Graphical Adventure is available only for Commadore 64's while the others were all Apple IIe packages (although some may be available for other microcomputers).

It was noted that to examine a software package as well as discuss it in some depth is deceptive. This working group found its time quickly spent in the process. The group did not deliberately proceed linearly through each package. We compared and contrasted features as the discussion proceeded. The software that invariably drew the greatest attention contained what the group considered to be powerful features. Invariably these necessitated a high, active participation rate with the operator in control. For example, the geometric supposers are able to draw, measure and repeat constructions only under the direction of the operator. Without these directions it will not do anything and the potential of this type of software can only be explored if the operator is able to interact with it to take advantage of these features.

During one of the sessions Ross Finney demonstrated the Calculus Student's Toolkit, a software package that he was involved in developing.

Sr. Rosalita Furey familiarized the group with the Graphical Adventure which seemed to have considerable potential for the secondary curriculum (particularly at $14.95). Unfortunately it is only available for the Commodore 64.

WORKING GROUP C

Appendix 1

SOFTWARE INFORMATION

1.	Apple Logo	Apple Canada	$ 150.00
2.	Algebra Arcade	Wadsworth Publishing Co. 8 Davis Drive Belmont, California 94002	34.45
3.	Interpreting Graphs	Conduit The University of Iowa Oakdale Campus Iowa City Iowa 52242	45.00 (US) 45.00 (US)
4.	Graphing Equation: (includes green globs)	Conduit	45.00 (US)
5.	Geometric Supposer (These are triangles, quadrilateral and circle versions as well as a pre-supposer)	Sunburst P.O. Box 3240 Station F Scarborough, Ont. M1W 9Z9	99.00 (US) 132.00 (CDN)
6.	Calculus Student's Toolkit	Addison-Wesley	?
7.	Graphical Adventure	Saga Software 418 Gowland Cres. Milton, Ontario L9T 4E4	14.95 (CDN)

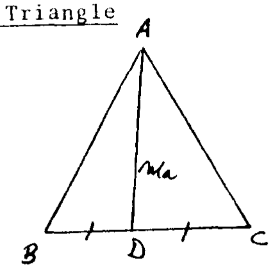
Appendix 2

Geometry via the Computer

Lesson X

by Roland Eddy

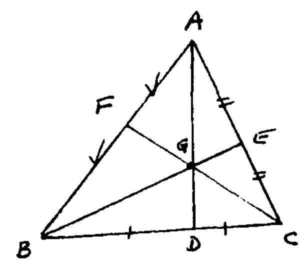
The Mediums of a Triangle



1. The medium AD bisects BC.

2. Calculate the areas of ABD and ADC.
Conclusion? (Equal)

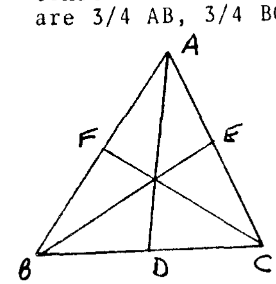
3. Construct several triangles and their three mediums.
Conclusion?

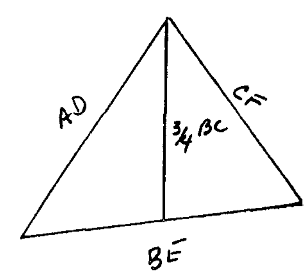


4. Measure AG, GD, etc. Conclusion?
(AG = 2/3 AD, etc.)

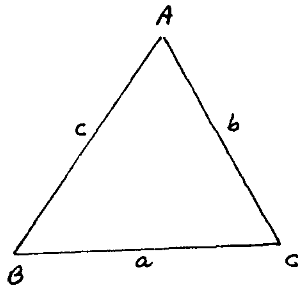
5. Calculate the areas of AGB, GBC, GCA. Conclusion?
(All equal)

6. Construct the triangle with sides AD, BE, CF and construct its mediums. Verify that their measures are 3/4 AB, 3/4 BC, 3/4 CA.





7.



Heron's formula:

$$\text{Area} = \sqrt{s(s-a)(s-b)(s-c)}, \quad s = \frac{a+b+c}{2}$$

Verify the corresponding formula

$$\text{Area} = 4/3\sqrt{s(s-m_a)(s-m_b)(s-m_c)}, \quad s = \frac{m_a+m_b+m_c}{2}$$

8. Verify: $m_a^2 + m_b^2 + m_c^2 = 3/4\,(a^2 + b^2 + c^2)$

9. Verify the inequality:

$m_a + m_b + m_c \leq 4R + r$, where R,r represent the circumradius and inradius respectively. When does equality occur?

WORKING GROUP C

Appendix 3

COMPUTERS IN MATHEMATICS EDUCATION

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WORKING GROUP D

THE ROLE OF THE MICROCOMPUTER IN DEVELOPING STATISTICAL THINKING.

GROUP LEADERS:
CLAUDE GAULIN
LIONEL MENDOZA

CANADIAN MATHEMATICS EDUCATION STUDY GROUP

JUNE 1986 MEETING (ST. JOHN'S)

Report of Working Group D

The role of the microcomputer in developing statistical thinking

by Claude Gaulin and Lionel Mendoza

The group was a follow-up to a working group in Vancouver in 1983, which had focussed its discussion on "the goal of developing statistical thinking for all" as well as on appropriate topics and methodology for a core curriculum. The report of the Vancouver working group can be found in the proceedings of the 1983 meeting of CMESG.

The objective of Working Group D in St. John's was to investigate the issue of how microcomputers could be used for developing statistical thinking. Among the aspects initially proposed for discussion were: software for teaching statistics,; graphical representations of statistical distributions; simulations of random experiments; and learning probabilistic and statistical concepts through programming. The preliminary discussion on the first evening enabled the group to determine the focus for the three three-hour sessions that followed.

The work and conclusions of the group can best be summarized by dividing it into three phases.

Phase 1

This phase raised the difficult question of what was meant by "statistical thinking". While no attempt was made to develop a formal definition, the group agreed that the core idea of statistical thinking was a comprehension of the nature of representations, distributions, and inferential statistic, as opposed to the ability to draw graphs or undertake statistical tests, per se. Also, at this stage it was decided to focus on the role of the computer as a teaching aid, and not as a computational aid (as epitomized by statistical packages). It also became apparent in the discussion that the members of the group were not aware of software specifically designed to develop statistical thinking.

Phase 2

In this phase the role of the computer was explored. The group mostly discussed how it could be used to VISUALIZE statistical ideas and processes.

(A) VISUALIZATION IN DESCRIPTIVE STATISTICS AND EXPLORATORY DATA ANALYSIS.

Utilizing the computer here involves displaying a variety of graphical representations (e.g. bar graphs, pie charts, stem-and-leaf plots) on the screen. A particularly effective use of visualization is having different data sets SIMULTANEOUSLY DISPLAYED on the screen, enabling students to interpret, discuss, and compare the data. Alternatively, displaying the same data in different ways develops an awarenes of the advantages and limitations of different displays and helps students to select the most convenient or best illustrative representation from among many possibilities.

ANIMATION can be an effective aid for descriptive statistics and exploratory data analysis. The ability of the computer to build up successive representations as the data is entered [either from pre-set data sets or student-collected data sets] gives students a visual image for comprehending the nature of the data.

(B) VISUALIZATION IN "INFERENTIAL STATISTICS". Animation can also be used in developing inferential statistics, an intuitive understanding of hypothesis testing, and the notion of confidence intervals. An example would be using the computer to select samples of a given size and building up the distribution obtained by repeated sampling. By varying sample size and the number of samples, students can obtain a feel for the nature of distributions and later on apply this to the distribution for a test statistic.

NOTE: Whether the computer is used to visualize ideas and processes in descriptive or inferential statistics, the group insisted that software should be INTERACTIVE, and not merely DEMONSTRATIVE. It should allow the user to ask questions and indicate displays that he or she would like to see. Thus, the interactive nature of the software requires a flexibility of choice, beyond that of merely allowing the user to choose from a limited selection of options. It is important to stress that it is the INTERACTIVE nature of the software and the DISCUSSION of ideas generated by the display that leads to statistical thinking.

Phase 3

During the last working session, the group discussed the structure of an introductory course in statistics for undergraduate students in which the microcomputer were to be fully integrated THROUGHOUT the course. The suggested components for such a course were:

1) Data "display" and interpretation
   [Computer displays and animation used]

2) Exploratory data analysis
   [Centrality, box plots,...]

3) Transformations of data
   [log, log normal,...]

4) Uncertainty
   [Exploratory games involving repetition]

5) Nonparametric statistics, sampling, etc.

A variety of themes occurred throughout the sessions, but time did not allow us to discuss them in depth. The following are some examples:

1) The use of computers to simulate random processes.

2) The role of probability in developing statistical thinking. This topic was raised at various times throughout the sessions. The group felt that much could be accomplished in developing statistical thinking without a detailed analysis of probabilistic concepts, per se. During these discussions a probability based game designed to develop statistical thinking was presented by Eric Muller. (See Appendix A).

3) The issue of decision making versus probabilistic thinking. There is a fundamental difference in the role of probability in the two situations. In statistical thinking a key aspect of probability is the role of repetition within the situation and it is 'assumed' that the situation can be replicated. However, in decision making, while some probabilistic information aids in the decision making process, it is usually not a repetitive situation.

4) The relationship between computer games/activities and the use of real objects. How do students relate computer 'generated' games/activities to similar real object games/activities? The group was concerned that students (particularly young students) might have difficulties effectively internalizing ideas developed in computer situations without experiences with real objects.

In conclusion, there is one point the group would like to make: the group felt that it would be interesting to do further work during the CMESG meeting at Kingston, and suggested a working group focussing on "Inferential Statistics for all High School Students". In particular, such a working group would explore, the following questions:

(i) How can the computer be used in conjunction with other traditional types of teaching aids?

(ii) What is the minimum amount of probability needed to study inferential statistics?

(iii) How can simulation be used in developing inferential statistics?

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APPENDIX A

Developing Statistical Thinking+

by

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In this supporting document for the Working Group "The role of the Micro-Computer in Developing Statistical Thinking" we consider an activity which has been used successfully with groups of students anywhere from elementary school to university. Although the activity does not involve the micro-computer the group spent a considerable amount of time trying to isolate the conponents of this activity which make it successful. Such components could then be structured in micro-computer simulation activities.

+Also submitted in modified form to the Ontario Mathematics Gazette.

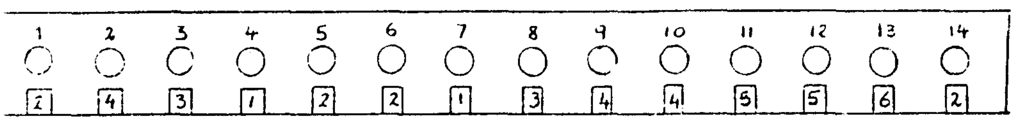
1. Activity to Develop Statistical Thinking

Materials

1. Board with r positions, marked 1 to r, for positioning coloured chips. The Board illustrated below has fourteen positions numbered above 1 to 14.

2. 2 regular six sided die -- it is useful to also have available pairs of the other four regular polyhedra (4, 8, 12 and 20 sided)

3. At least r/2 blue and r/2 red chips (or any other two colours)



Play

1. Two teams - teams of two students - work well as each student has a partner to discuss strategies. One team given red chips, other team given blue chips.

2. The two teams will alternate placing one of their chips in the places provided on the board. The aim is to have a chip in the position which corresponds to the sum on the faces of the two dice when they are rolled, eg., to have a chip in position 8 if a (five and three) are rolled. To start one of the two teams is selected, it places one of its chips in the position on the board which it believes is most likely to occur. The other team then places one of its chips in one of the (13) unoccupied places. This procedure alternates between the two teams until either (a) all positions on the board are filled or (b) one team no longer wishes to place any of its chips, then the other team may occupy all vacant positions.

3. The board is now set for n, (say 25) an odd number of rolls of the pair of dice. Each time the dice are rolled the team which has a chip on a position corresponding to the same on the dice records points (single points at level 1, points in the square below the position at level 2).

4. The team with the most points at the end of n (say 25) rolls wins that game.

5. All chips are now taken off the board and a new game may be started.

6. The objective for each team is to find a winning strategy, ie. a strategy for selecting the positions for their chips which will provide the best chance for winning.

The following three levels of play suggest a natural progression for statistical thinking. Some teams will not progress beyond level 1. One _must_ resist the temptation to provide solutions. This activity provides an ideal medium for exploration and one should only do the leading. We have always played with the following rules:

(i) Teams do not discuss their strategies with other teams.

(ii) When a team believes it has a strategy for winning it discusses it with me. I will not indicate whether the strategy is the best I know but I will change the team's opponents or materials to either

(a) expose the possibility of a better strategy

or

(b) reinforce the team's winning strategy.

The following three levels of play are suggested:

Level 1 (Estimating probabilities)

Objective: 1) Students to observe which outcomes, sum on the two dice, are possible and conclude that these outcomes are not equally likely

2) Students to quantify the uncertainty, ie. estimate the probabilities of each outcome

3) Students to develop the strategy of selecting those positions which maximize their probability of winning.

Procedure: The team whose chip is on the position with number equal to the sum on the two dice gets _one_ point. The team with the most points accumulated after n (25) odd rolls of the dice wins that game.

Note: To reinforce winning strategies supply the teams with one six and one eight-sided dice -- or a 12 and a 20-sided dice and a different board! To expose a non-optimal strategy change teams to play against a team with the optimal strategy.

Level 2 (Random variables and Expected values)

Objective: 1) Students to discover the concept of random variables.

2) Students to develop a winning strategy based on the concept of expectation value, ie. a set of positions such that the sum of products (of probability and points scored) is greater than that for the opposing team.

Procedure: The team's whose chip is on the position with number equal to the sum on the two dice gets the points indicated below the chip. The board illustrated above shows 2 points for a sum of six, 4 points for a sum of nine, etc. The team with the most points after 25 rolls of the dice wins the game. Follow the procedures outlined in level 1.

Note: I have a number of boards, each with a different sequence of points. By switching boards one can either reinforce an optimal strategy or expose one which is not optimal.

Level 3 (The effects of changing the number of rolls or trials)

Objective: Students experiment to show that as the number of trials is increased in a game the probability of winning the game with an optimal strategy is increased.

Procedure: The game is repeated 20 times for a fixed optimal stragegy and n rolls of the dice where
n = 1
then n = 3
then n = 5

From these results the students estimate the probability of winning in each case and plot these versus n. The probability of winning the game in 20 rolls should increase as n increases.

The reason for this is that it is an application of the binomial distribution with the following properties:

1) n identical trials, ie. n rolls of the pair of dice (for this game we choose n odd)

2) each trial results in one of two outcomes, ie. a loss if the total on the dice is not equal to one of the positions of the team's chip

3) probability of success, p, in a single trial remains the same from trial to trial, ie. the chips are not reset between rolls, probability of failure $q = 1-p$

4) trials are independent, ie. the result of one roll does not depend on that obtained in previous rolls.

Then probability of exactly x success is given by

$C(n,x)\ p^x\ q^{n-x}$

In this experiment we are interested in the probability of getting more than half of the points to win the game, ie.

$$\sum_{x=\lceil \frac{n}{2} \rceil}^{n} C(n,x) p^x q^{n-x}$$

where $\lceil \frac{n}{2} \rceil$ is the smallest integer greater than

(n odd by choice)

Students with a knowledge of the Binomial probability distribution can verify that their values are close to the theoretical ones, viz.,

$$n = 1 \text{ — } P = \sum_{x=1} C(1,1)\, p^1 q^0 = p$$

$$n = 3 \text{ — } P = \sum_{x=2}^{3} C(3,x)\, p^x q^{3-x} = 3p^2q + p^3$$

$$n = 5 \text{ — } P = \sum_{x=3}^{5} C(5,x)\, p^x q^{5-x} = 10p^3q^2 + 5p^4q + p^5$$

Positioning the chips in the most obvious position for a win, the starting team will have

$$p = \frac{21}{36}$$

giving for $n = 1$ - $P = 0.5833$

giving for $n = 3$ - $P = 0.6238$

giving for $n = 5$ - $P = 0.6534$

TOPIC GROUP P

NATURAL LANGUAGE AND MATHEMATICS IN HUMAN EVOLUTION.

BY HAROLD PADDOCK

Natural Language and Mathematics
in Human Evolution

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A paper presented on 11 June 1986 at the Tenth Annual Meeting of the Canadian Mathematics Education Study Group at Memorial University of Newfoundland 8-12 June 1986.

1. The Beginning

My main thesis[1] here today is that the later stages of our evolution, i.e., the distinctively human stages, have been mental rather than physical in nature.

We also note that there is potential danger in any evolutionary change for any species. The change may bring new opportunities or it may bring unexpected risks. Some species, such as the cockroaches, have played it safe by finding a nice niche and staying put in it for a very long time. We humans have been less "lucky" or less "sensible".

In his well-known and highly speculative book entitled The Origin of Consciousness in the Breakdown of the Bicameral Mind (1976) Julian Jaynes, a psychologist at Princeton, tries to show some of the gains and losses associated with the development of human language and human consciousness. For example, he claims that a schizophrenic-type of condition was associated with consciousness and language in pre-historical and early historical man. In particular, he claims that the experience of hearing "disembodied" voices was very common and led to the development of mysticism and religion, prophecy and poetry, as well as to such modern residue as hypnotism and mass "hysteria" (i.e., mass enthusiasm or mass ecstacy). Jaynes speculates that as language functions became localized in one hemisphere of the human brain, usually in the left hemisphere, schizophrenic-like consciousness became much less common in our species, and religion became institutionalized or fossilized because most of us could no longer hear the voices of the gods and angels, the devils and demons. Jaynes' bold attempt to explain our most recent evolution is very stimulating but it has been criticized for being too speculative. However, I would like to claim that we need to be even more bold and speculative if we are to understand the dangerous and critical nature of our most recent evolution.

Whereas Julian Jaynes attempted to link our purely human evolution to a left-right specialization in the human brain I will attempt to link it to a front-back specialization in the same brain. Like Jaynes, I want to link our cerebral development to the evolution of natural languages (the things we today call English, French, Chinese, etc.) but I also want to link it to our development of mathematical languages (the things we today call arithmetic, algebra, geometry, etc.). From a pragmatic point of view, we can regard natural languages as the tools we invented to control one another and mathematical languages as the tools we invented to control the rest of nature.[2]

As a species we have reached a unique point in the evolution of life on this earth. Because of the awesome power of mathematical languages, we have been able to create enough nuclear weapons to wipe out all (or nearly all?) forms of life on this planet.[3] Because of the equally awesome power of natural languages, one man in the U.S.A. or the Soviet Union can speak the few English or Russian words needed to begin the nuclear holocaust. Obviously, we have replaced Jaynes' individual schizophrenia of "primitive" man by the collective schizophrenia of "advanced" man.[4]

2. The Middle

When we compare ourselves with our closest primate cousins we are immediately struck by three major differences--two in our behaviour plus one in our brains. One major behavioural difference is that we have natural language, defined by the famous American linguist Noam Chomsky as a system that connects sound to meaning *via* syntax. Syntax is that wonderful human invention which allows us to talk or write forever despite a small vocabulary and an even smaller intelligence! Since early human language was spoken but not written we have no direct evidence about its nature.[5] Our oldest samples of writing reveal languages that are already highly developed. Moreover, along with the development of writing comes the development of early mathematics[6], which we can provisionally define as the language of quantification. It would appear, then, that our mathematical abilities emerged in parallel with our language abilities during the purely human stages of our evolution. With the introduction of proof into mathematics, attributed to the Greek known as Pythagoras (6th century B.C.) this specialized human language became the major tool of science and technology, the second major behavioural feature that distinguishes us from our primate cousins. If we look for a third major feature that might underlie and help explain the other two, we can find it in the distinctive frontal lobes of the human brain. It is those highly developed frontal lobes that give us our more prominent foreheads as compared to the receding, sloping foreheads of our primate cousins.

But what goes on in those frontal lobes of ours that makes us so different from all other primates, from all other mammals, from all other animals? Surprisingly enough, natural language functions are not all localized in the frontal lobes.[7] In fact, much of the human frontal lobes are made up of the so-called 'silent areas' of the cortex. These are areas "which, on stimulation, evoke neither sensory nor motor response" (Smith 1961: 193). Smith feels that the main function of the human frontal lobes is the integration of perceptions and knowledge,

particularly the time-integration of separate events that gives rise to our perceptions of cause and effect.[8] Smith admiringly quotes an 1824 entry in Emerson's journal: "Man is an animal that looks before and after." This remarkable insight of the then youthful Emerson explains the central paradox of human nature; that is, it explains why we are simultaneously the most rational *and* the most irrational of all creatures. When we compare ourselves to other mammals psychologically we are struck by our peculiar inability to enjoy the here and now. We are forever regretting the past and fearing the future. We note that our greatest buildings (temples, pyramids, cathedrals) used to be erected to those very regrets and worries, sins and hopes. We note too that the insatiable human sacrifices of the Aztecs were not motivated by ferocity but by fear: they were meant to keep nature operating in the future as it had done in the past. As our frontal lobes and (somewhat later?) our natural languages developed, our instincts were gradually replaced by learning and memory, by reasoning and faith. But this laid an intolerable burden of choice and responsibility on the individual. This must be the basis of our myths about our expulsion from the Garden of Eden, from a state of innocence and grace into a knowledge of good and evil. Never again could we be as "natural" in our behaviour as the other mammals seem to be.

Perhaps man's most heroic and rational response to this intolerable pressure was to invent mathematics. Natural languages already contained most of the raw materials needed for basic mathematics.[9] For example, in Modern English we can see the prototypes of set theory[10] in the words that linguists call determiners and quantifiers (Stockwell *et al*. 1973: 65-160). Such words are underlined in the examples given below:

<u>all</u> books -- the universal set (of books)

<u>no</u> books -- the empty set

<u>a</u> book, <u>the</u> book -- the unit set

<u>any</u> books -- a random subset

<u>some</u> books -- a non-random subset

etc.

The prototype of the finite/infinite distinction may be seen in our distinction between ("finite") COUNT nouns and ("infinite") MASS nouns. Examples are given in the table below:

COUNT[11]	MASS
a cup	some sugar
a shovel	some snow
an apple	some fruit
an egg	some butter

More important, perhaps, was the existance of logical connectives[12] in natural languages. In Modern English we find words such as the following (Kemeny _et al_. 1966: 12):

and for addition (conjunction)

_or_ for alternation (disjunction)

not for denial (negation)

_if_ ... (_then_) for dependency (conditional)

The most important of these logical connectives seems to be the _if_ (conditional) type. This is because the birth of "real" mathematics coincides with the explicit recognition of the methodology of proof, associated above with the sixth century B.C. Greek philosopher, mystic, and mathematician called Pythagoras. As E.T. Bell (1937: 20) has pointed out, "Before Pythagoras it had not been clearly realized that _proof_ must proceed from _assumptions_. Pythagoras, according to persistent tradition, was the first European to insist that the _axioms_, the _postulates_, be set down first in developing geometry and that the entire development thereafter shall proceed by applications of close deductive reasoning to the axioms." Pythagoras himself is not likely to have discovered that the square on the hypotenuse of a right-angled triangle equals the sum of the squares on the other two sides. This fact was apparently well known to the priests and land surveyors of Egypt and Babylon, both of which Pythagoras visited. His great contribution was to prove _why_ this fact _had_ to be true. The proof(s), using deductive reasoning, showed that this theorem had to be true for all right-angled triangles drawn on the surface of a plane. This was quite different from inductive reasoning based, for example, on measurements taken from a hundred specific triangles. Deductive proof guaranteed that not even the gods themselves could change this law of nature. Hence, it gave the Greeks a confident sense of security so that they, unlike the Aztecs, did not have to perform sacrifices in an attempt to preserve the laws of nature.

It was gradually realized in mathematics and the sciences that one did not have to start with postulates that conformed to one's sense perceptions or one's common sense. This led to developments such as geometries of spaces with more than three dimensions. This also allowed Einstein to assume that the velocity of light is constant for all observers, a postulate that violates common sense. But this counter-intuitive assumption allowed him to conclude that $E = mc^2$ _before_ experimental evidence was available to show the relationship between energy and matter in nuclear reactions. More recently, I have read of discontinuities in space called "strings" (Angier 1986). These may be relics left over from the Big Bang that are capable of bending light more radically than even the most massive collections of "solid" or "real" matter. Though we cannot observe such strings "directly", we (i.e., a few theoretical physicists) can describe them mathematically. Ultimately, then, our understanding of the universe, at either the macrocosmic or microcosmic extremes, fades away beyond our senses into the abstractions of mathematics. This means that mathematics defines the limits of our "knowable" universe.

Our provisional definition of mathematics above was the language of quantification. We can now amend that definition by calling it the language(s) of quantified iffiness.

3. The End

Let me try to recapituate. The development of the distinctive frontal lobes of the human brain and the concomitant development of natural language cut our species free from the control of instincts and forced it to rely on accumulated experience (i.e., memory) and on the uncertainties of inductive reasoning. Some human groups tried to solve the memory problem by developing writing. Some tried to solve the problem of inductive uncertainty by developing methods of deductive reasoning in logic and mathematics. The rapid advances made in European mathematics and science in the seventeenth and eighteenth centuries by men like Descartes, Newton, and Leibniz led to the remarkable optimism and self-confidence of Western Man in the eighteenth century. We managed to make it through the nineteenth century fairly safely, but the twentieth century destroyed our faith in both men and mathematics. On the human side we have seen two world wars and several attempts at genocide. We have also seen about a quarter of the human race suffering from acute starvation, chronic hunger, or crippling malnutrition. Between the two world wars science and mathematics also encounted their limitations. In 1927 Werner Heisenberg published his Principle of Indeterminacy for physics. In 1931 Kurt Gödel showed that mathematical systems can never be complete, that mathematics contains insoluble problems. As William Barrett wrote (1962: 39) "This means, in other words, that mathematics can never be turned over to a giant computing machine; it will always be unfinished, and therefore mathematicians--the human beings who construct mathematics--will always be in business." This good news I bring you!

Where, then, can we go from here? If, as I have claimed, natural languages and mathematical languages are the two most powerful tools, and therefore the most dangerous tools, that we have developed should we not approach the teaching of both of them with great care and caution? In particular, should we not be teaching something about the origins, the development, and the limitations of both natural and mathematical languages? Should we not be discussing the ethics of their uses and misuses in the history of our species? Is it not just as important to teach students about them as it is to teach students to use them. I have known many students who treated mathematics as a kind of black magic--"If you do this and this you'll get the right answer, but don't ask me why!" Would it not be better to teach primarily for understanding[13] even if it meant teaching less? Wouldn't less in fact be more in this case? Wouldn't the above suggestions solve some of the notorious problems of motivation in mathematics students, since it would make the whole subject less dry and more meaningful? Heaven knows that we have seen in this century some horrific results of blind obedience and unreflecting faith. We now know that enthusiasm and will are not enough to ensure the survival of the human race. If we do not pause to assess ourselves we may well stampede over the brink like a herd of buffalo.

But most of all we must learn humility again. We must relearn the joy of living within our limitations, of living here and now, of being part of nature again. After all, a star scientist is a much a product of nature as is a starfish! Let us not forget the nobility and grubbiness of our "struggle into light."[14] We imagine our remote primate ancestors attempting to stand upright on their hind legs so that they could better spot dangerous predators at a safe distance. Now we have become the most dangerous predators of all. Unless we can come to terms with our flaws we are finished.

* * * * * * * * * *

The orally presented version of this paper ended at this point. During the animated question period that followed I discovered that some members of the audience wanted me to speculate a little about our possible futures. Well, then, here goes!

I do believe that there is some slim hope for the human race. But it is a painful hope because it involves giving up some of our most treasured illusions. And, as we have seen in South Africa and elsewhere, people would sometimes rather die than surrender their illusions, along with the powers and privileges supported by such illusions. On a larger scale, we can observe the terrifying Star Wars illusion in the U.S.A., whereby millions are being misled into believing that their country can seal itself inside a safe cocoon on this tiny planet. There is in fact nothing to indicate that the Great (Space) Wall of America will provide better protection than did the Great (Stone) Wall of China. The real issue here is a psychological one--it is impossibly difficult for people to abandon their illusions of safety and superiority. People do indeed need myths as a source of motivation.

One lesson taught us by the twentieth century is that an astonishing quantity of human energy[15] is released by true belief. A former member of the Hitler Youth movement once said to me: "People just don't understand how beautiful it was to know that you were right and everybody else was wrong, that you were superior and everbody else inferior!" Conversely, a lack of faith reduces many of us to depression, inertia, and impotence. Even worse, we note that the energetic true-believer is often morally inferior to the lazy know-nothing. The great Irish poet W.B. Yeats summarized this painful paradox of modern man when he wrote that "The best lack all conviction while the worst/Are full of passionate intensity."[16] Our hope, then, must lie with people who can act without conviction, who can fight without faith, who can pray without God.

Such people will require a rare steadiness of purpose and a superior resistance to frustration.[17] This is because evolution generally proceeds not by abandoning the old for the new, but by building the new on top of the old.[18] How then are we going to accomodate the old mammals that lie behind our human frontal lobes? If we do NOT accomodate them, they are likely to destroy us. We must give them their due because without their evolutionary history we would not even exist. We must therefore learn to love and admire our bodies and our unconscious minds in the same "disinterested" way in which we so easily love the bodies and the unselfconscious minds of other animals, for we too are children of nature.

Nevertheless, our peculiar human consciousness in our inescapable fate. We cannot ever return to pre-consciousness. Our only hope is to go forward to to higher levels of consciousness.[19] We can get a better idea of where we might go only by learning more about where we have been. There is therefore a special responsibility laid on the shoulders of us who are describers and teachers of human languages, whether these languages be natural or mathematical. We must show our students that these languages are the most powerful and beautiful tools ever developed by the human mind. Our students should therefore learn to respect their power while admiring their beauty. Above all, both we and our students should try to improve these tools. Let us join T.S. Eliot in his mature concern "To purify the dialect of the tribe/And urge the mind to aftersight and foresight."[20]

FOOTNOTES

[1]This is a slightly revised version of the paper which was read at the conference. The revisions consist mainly of extended conclusions and additional footnotes.

[2]Of course, in controlling nature we also came to control one another even more, through the development of weapons, "predatory" economies, etc.

[3]In addition to worrying about our relatively sudden end in a nuclear war we can also worry about slower endings from nuclear pollution, chemical pollution, overpopulation, famine, etc.

[4]Compare the British psychiatrist R.D. Laing, who feels that schizophrenic behaviour is the sanest response to living in an insane world (Papalia and Olds 1985: 545-6).

[5]But see Hockett (1978) for a judicious weighing of the several types of indirect evidence.

[6]It is interesting to speculate on why mathematics developed so "early" in our history. One reason was no doubt the development of writing itself, which gave a new permanence and weightiness to language. Also, according to Guillaume (1984: 143) "Writing, more than speech, obliterates the turbulence of cogitation." If this is true, then writing would have led naturally to the reflectiveness, reasoning, and generally clearer thinking needed for mathematics. But perhaps more important was man's long history of precise hand-eye coordination, well recorded in his developing skills of tool-making. Even more intriguing is Hockett's hypothesis (1978: 295-301) that the primary medium of human prelanguage consisted of manual signs (gestures) rather than vocal sounds. If Hockett is correct, then this would help explain the "earliness" of mathematical development in our species. Hockett's theory is especially relevant for geometry, since a complex system of hard signals requires rapid and precise neuromuscular control of the hand as well as

equally rapid and precise visual perceptions of the resulting hand movements in space. Both these abilities no doubt unlie both writing and geometry.

[7]Besides the supplemental motor area, there are two main localizations of language in the human brain (usually stronger in the left hemisphere). The main speech <u>production</u> centre, called Broca's area, is in the (posterior inferior part of the) frontal lobe but the main speech <u>perception</u> centre, called Wernicke's area, is found in the (posterior) temporal and parietal lobes. This suggests that "language" perception might have preceded "speech" production in our evolution. In other words, it tends to support Hockett's speculation (1978: 295-301) that the primary medium of human prelanguage might have been sophisticated hand gestures rather than vocal sounds. In any case, the available evidence indicates that we achieved fine motor control over our hands well before we achieved similar control over such vocal organs as the lips, tongue, and larynx. Note too that we cannot teach apes to speak but we can teach them to use "prelanguage" that employs hand gestures. In addition, human beings who are deaf can communicate rapidly and fluently through the use of hand signal systems. Moreover, it has been demonstrated that apes can learn to use (at least part of) the Ameslan (American sign language) system that is commonly taught to the deaf in North America (Hockett 1978: 277-82).

[8]The crucial role of the frontal lobes for human behavior is demonstrated by the severe "side" effects of prefrontal lobotomies. These surgical operations (commonly carried out in the forties and fifties to relieve severe pain and some psychoses) often left patients "as apathetic shells of their former selves; some 5 percent developed convulsions; and more than 6 percent died" (Papalia and Olds 1985; 569).

[9]This claim has been advanced by several writers in the past. For example, the French theoretical linguist Gustave Guillaume (1883-1960) claimed that language "is the pre-science of science" and that "the loftiest speculations of science are built on the systematized representations" of language (Guillaume 1984: 146). Guillaume also makes several insightful comparisons between mathematics and natural language.

[10]See, for example, Kemeny <u>et al</u>. (1966).

[11]As a local dialectologist I note that (singular) COUNT nouns in Vernacular Newfoundland English are often preceded by <u>either</u> (or one of its "variants" such as <u>ai'r</u> or <u>a'r</u>) rather than

by a or an. Thus one commonly hears sentences such as: "Do you have either shovel with you?"

[12]All natural languages seem to be fairly equal in the subtleties of their LEXICAL and GRAMMATICAL distinctions but some may be superior in thier LOGICAL distinctions. To illustrate this ternary division we can break the English sentence "The cow drinks if she is thirsty" into nine linguistic forms (called morphemes by linguists). Of these nine, three are lexical (cow, drink, thirst); five are grammatical (the, -s on the verb, she, is, and -y on the adjective) and one is logical (if).

[13]Here I recall my own collision with geometry on entering high school. The teacher provided no introduction to the subject at all but began abruptly with the proof of a theorem. I was utterly lost for several days until I happened to read the excellent preface to our textbook. The result was that I "fell in love" with geometry and used to tutor other members of my own classes in that subject throughout my high school years.

[14]This phrase is from the English poet John Clare (1793-1864), whose own life epitomized the difficulties of this struggle. See Tom Dawe's (1983) poem of empathy dedicated to John Clare.

[15]Of all the pioneers of modern depth psychology it was probably Carl Gustav Jung (1875-1961) who had the best insights into this crucial problem of the "availability" of psychic energy. See, for example, the summary of Jung's theories in Woodworth and Sheehan (1964). The most pervasive mental problem of modern times is depression, a problem which can be seen as the inability to release one's psychic energy. This block is the mental equivalent of physical paralysis.

[16]From his poem entitled "The Second Coming".

[17]For example, every day of my life I want to malign, maim, or murder at least one other person. There is nothing unique about my feelings. Compare the Québec policeman Serge Lefebvre, who shot two of his fellow officers. He said that he turned of a life of crime "because he was frustrated with his job" (The Globe and Mail, Thursday, 10 July 1986, p. A8). It is certainly true that the increasing specialization, regulation, monotony, and mechanization of modern employment is a source of great frustration to many people. Barrett (1962) attributes such nihilistic urges to the feelings of powerlessness and hopelessness that have accompanied the general loss of faith experienced by modern Western Man. We note that the recent weakening of the church in the province of Québec has been accompanied by a rapid rise in the rate of suicide.

[18]See Chapter 12 of Homer W. Smith (1961) and especially p. 191.

[19]Note that the whole thrust of modern depth psychology (and the psychotherapies based on it) is towards higher levels of consciousness, which may allow us to transcend our personal problems or at least enable us to view them with a tolerable or livable degree of mental pain.

[20]From T.S. Eliot's poem "Little Gidding" in the Four Quartets. London: Faber and Faber, 1944, p. 54.

After this my daily fix of poetry, I find it possible to end this paper on an upbeat note, or at least on an upbeat footnote. The most hopeful sign to me is that we may now be starting to see ourselves as the protectors rather than the exploiters of our planet. For example, the defense capability of our space programs could be redirected to "dealing with threats from space" (Lemonick 1986) such as any dangerously large asteroid found to be on a collision course with planet earth.

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TOPIC GROUP Q

SECOND INTERNATIONAL MATHEMATICS STUDY: GENDER RELATED DIFFERENCES IN LEARNING - OUTCOMES.

BY ERIKA KUENDIGER

GENDER RELATED DIFFERENCES IN LEARNING OUT-COMES

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University of Windsor

A) Cognitive Learning Outcomes

In Northern America the topic "Gender and Mathematics" was discovered to be important during the early 70th. From the very beginning the question " To what extend do boys outperform girls in mathematical achievement ?" was and still is of particular importance to researchers and to the public as achievement often is looked upon as the one essential learning outcome.

By now an extensive body of research is available. Depending on the reseacher, results are summed up quite differently ,e.g.

- Bendow and Stanley (1983) come to the conclusion that by age 13 there is a significant difference in mathematical ability between the sexes, and that it is especially pronounced among high-scoring exceptionally gifted students, with boys outnumbering girls 13 to 1:
- according to Fennema and Carpenter (1982) very little sex related difference exist, if any; and
- summing up research carried out in nine countries, Schildkamp-Kuendiger (1982) concludes that sex - related differences in achievement were found to vary considerably both within and among countries.

The Second International Mathematics Study (SIMS) provides achievement results of students from twenty countries at the Population A level, that corresponds to grade eight in Canada. These results have been analyzed as to sex related differences using ifferent approaches (see Hanna & Kuendiger 1986 for further details).

Overall the data reveal that sex-related achievement differences

mostly do not occur. If they occur, they may be as well in favour of girls as of boys. Significant differences by country and subtests range between +5% to -7% only .

B) Attitudinal Learning Outcomes

In trying to explain sex-related achievement differences and course-taking behaviour, modells have been developed that stress the importance of the attitudinal aspects of the learning process, in particular the impact of genereal believes about the appropriateness of women being involved in mathematics (Eccles 1986, Kuendiger 1984).

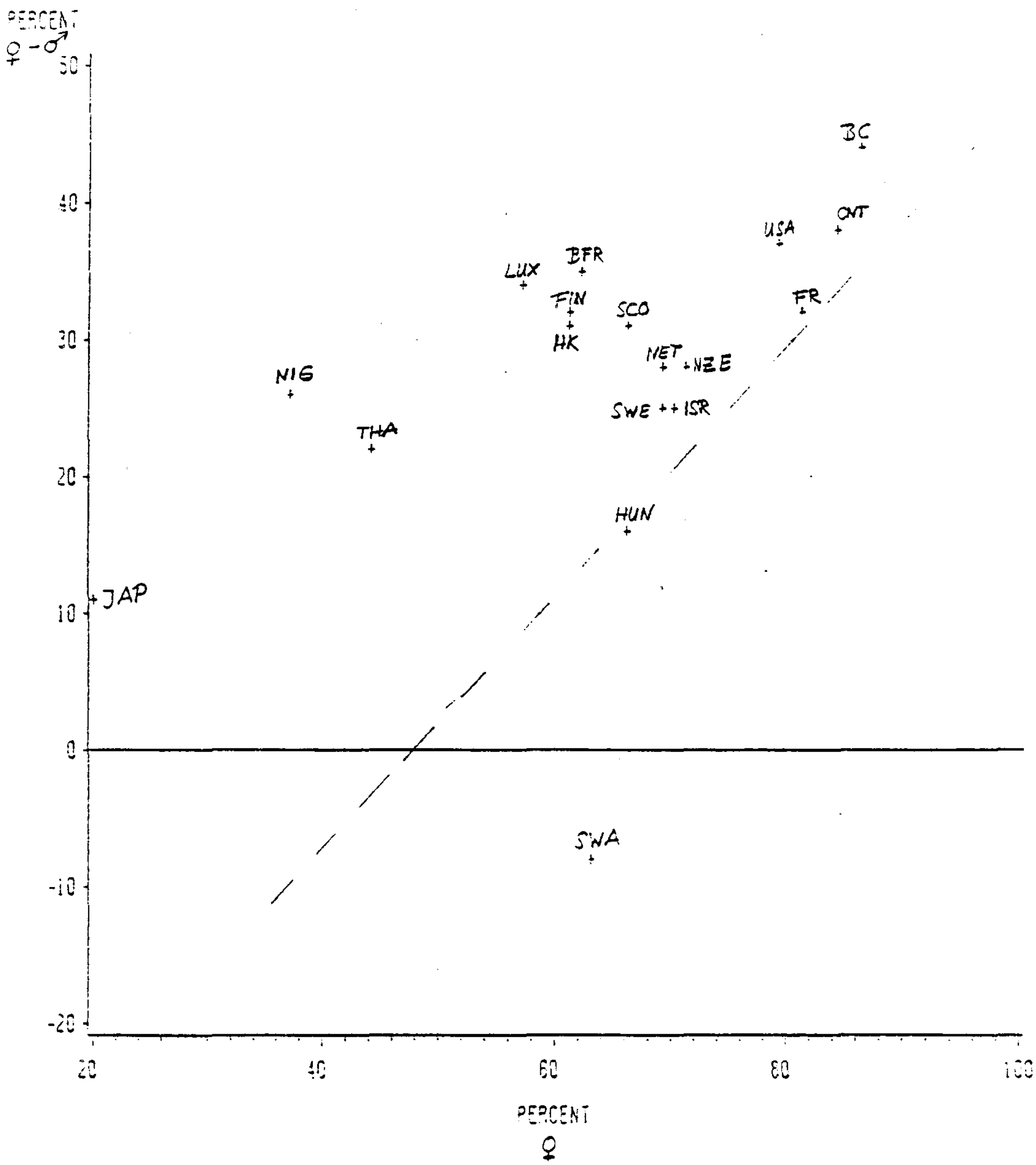
The SIMS contains a whole questionnaire focusing on students' attitudes towards mathematics. The scale "Gender Stereotyping" is directly related to the above mentioned aspect. The graphs below display the percentages of extrem responses for each of the four items by country. The percentages of female responses are plotted against the difference of female minus male percentages. It has to be noted that 3 of the 4 items are phrased negatively; for these items the categories "disagree" and "strongly disagree" have been considered; coraspondingly the categories "agree" and "strongly agree" have been used for the positively phrased item. In all graphs the line indicating extreme responses of 50% of the boys has been entered.

With the exception of Swaziland the graphs reveal some astonishing regularities: for all other counties the differences between extreme responses is 9% or more , with girls having the more extreme responses. Chi 2 - tests done for each item and and

MEN BETTER SCIENTISTS AND ENGINEERS

PERCENT DISAGREE AND STRONGLY DISAGREE

POP A



WOMAN NEEDS CAREER AS MUCH AS MAN

PERCENT AGREE AND STRONGLY AGREE

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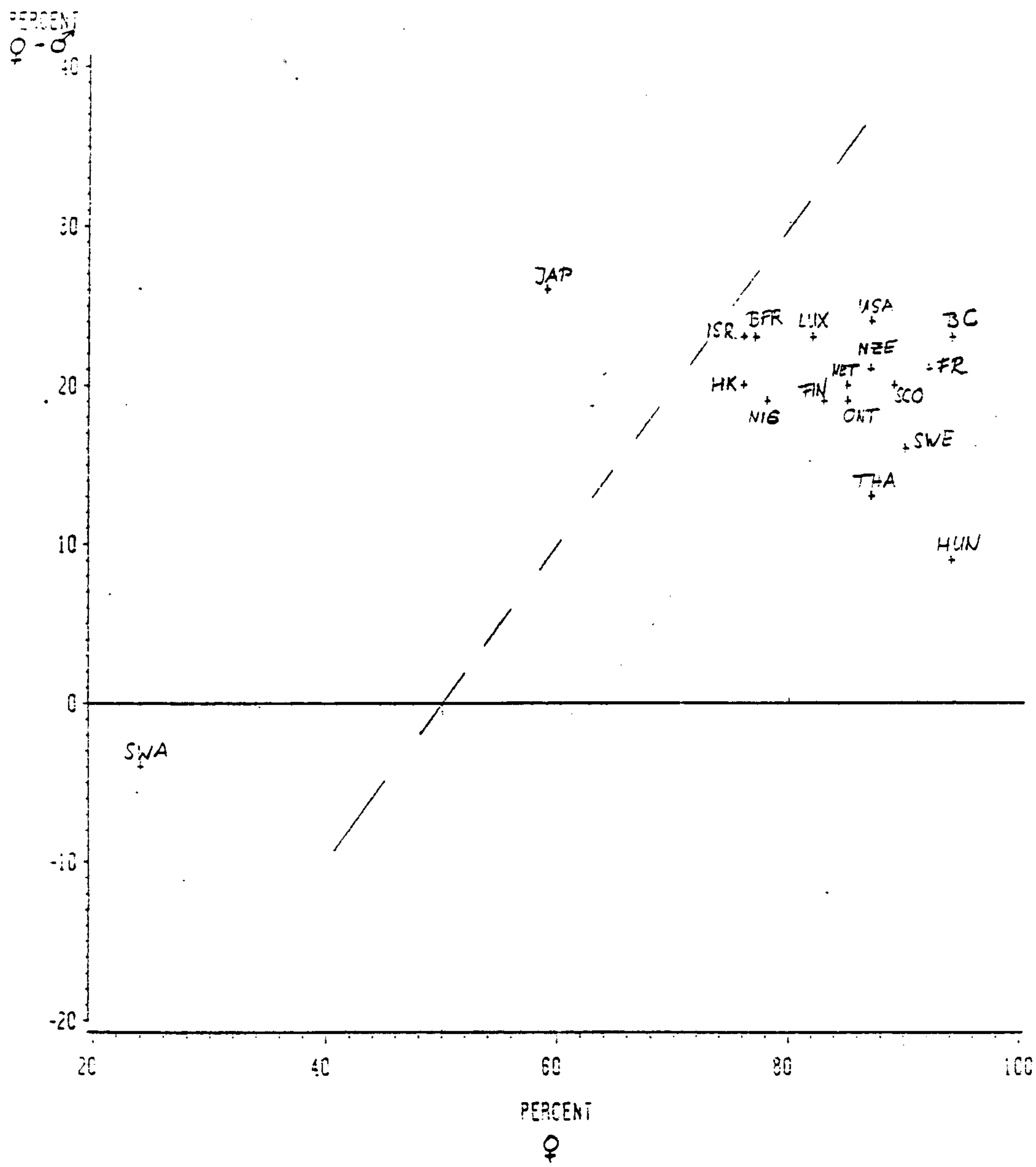
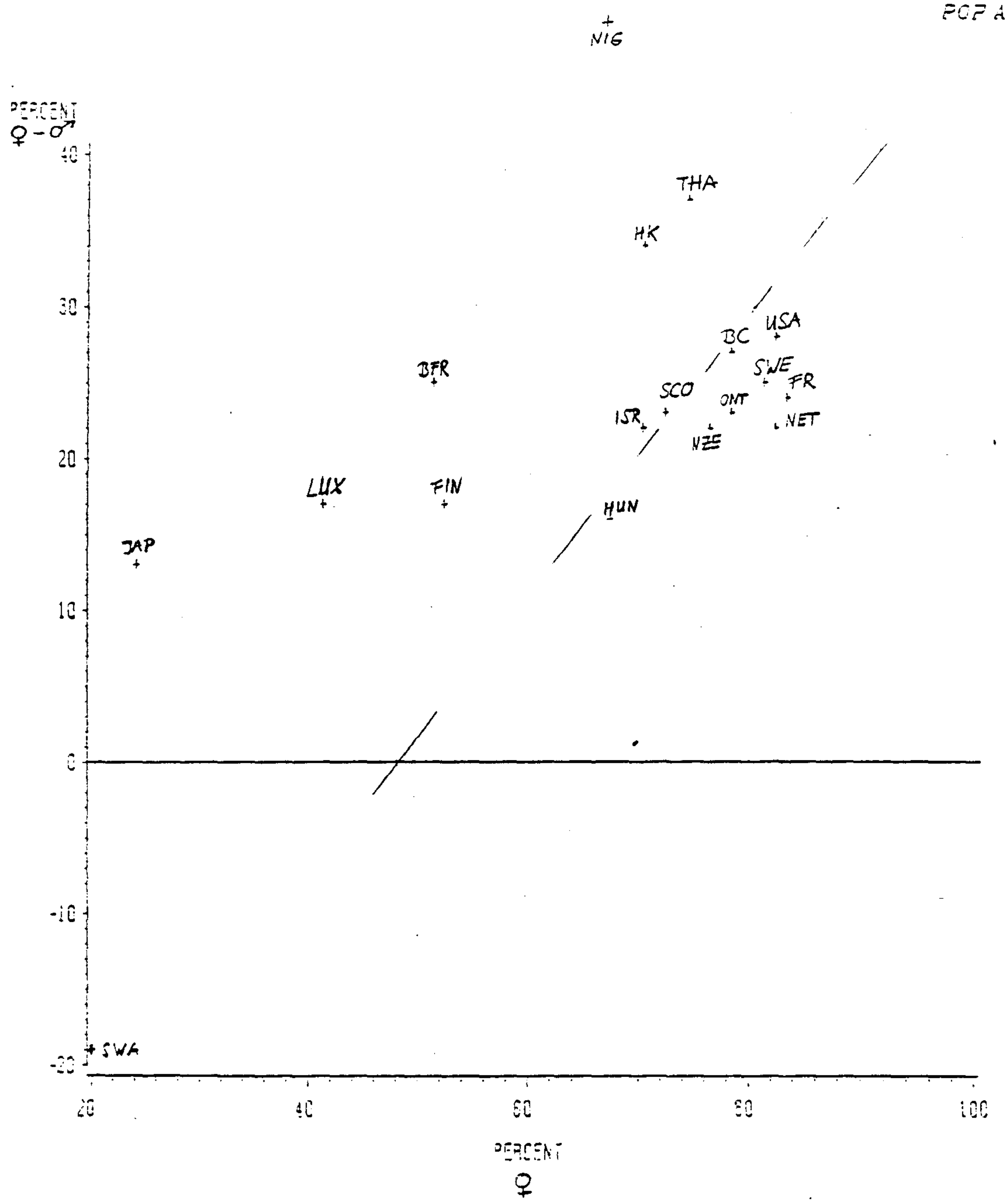


BOYS NEED TO KNOW MORE MATH

PERCENT DISAGREE AND STRONGLY DISAGREE



BOYS HAVE MORE NATURAL MATH ABILITY

PERCENT DISAGREE AND STRONGLY DISAGREE

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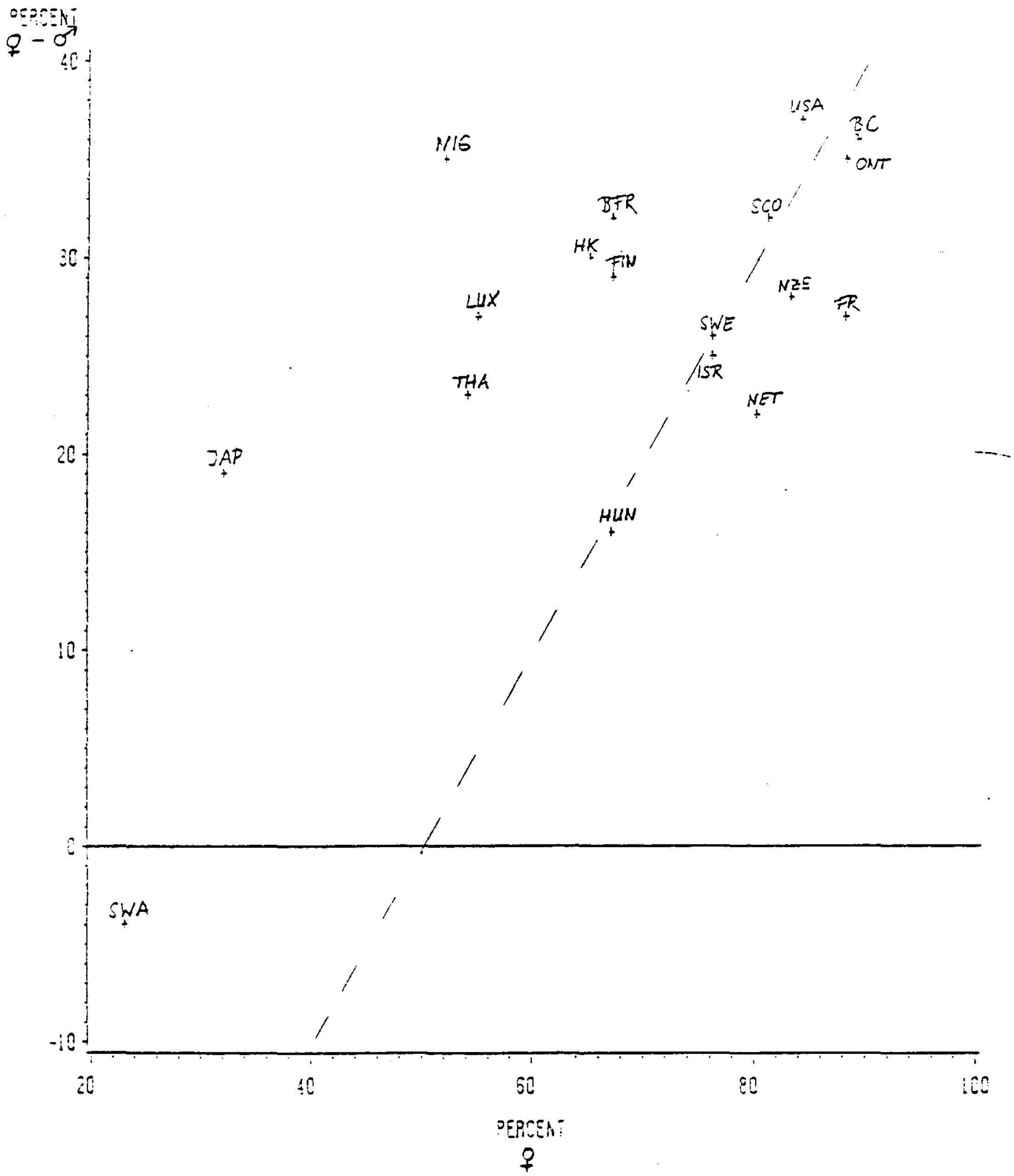

country seperately reveal a significant relationship between sex and respond pattern ($p < 0.001$).

Swaziland is the only country in which boys hold a more extreme position than girls. Moreover, only the answers to the items "boys have more natural ability in math" and "boys need more math than girls" are significantly related to sex ($p < 0.001$).

Future inspection of the attitude scales will reveal as to what degree regularities in the attitudinal learning outcomes appear within countries and/or between countries.

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