CANADIAN MATHEMATICS EDUCATION STUDY GROUP

GROUPE CANADIEN D'ETUDE EN DIDACTIQUE DES MATHEMATIQUES

PROCEEDINGS

1987 ANNUAL MEETING

QUEEN'S UNIVERSITY

KINGSTON, ONTARIO

May 29 — June 2, 1987

Edited by

Lionel P. Mendoza and Edgar R. Williams

Memorial University of Newfoundland

CANADIAN MATHEMATICS EDUCATION STUDY GROUP

GROUPE CANADIEN D'ETUDE EN DIDACTIQUE DES MATHEMATIQUES

PROCEEDINGS

1987 ANNUAL MEETING

QUEEN'S UNIVERSITY

KINGSTON, ONTARIO

May 29 - June 2, 1987

Edited by

Lionel P. Mendoza and Edgar R. Williams

Memorial University of Newfoundland

CMESG/GCEDM 1987 MEETING
Queen's University, Kingston
May 29 - June 2, 1987

Table of Contents

EDITOR'S FOREWORD

We wish to thank all contributors for their promptness in submitting manuscripts for inclusion in this Proceeding. We particularly wish to thank Yves Nievergelt of the University of Seattle and the Mathematical Association of America for permission to reprint the article *"The Chip with the College Education: the HP-28C"*, which appeared in the MAA Monthly, Vol. 94, No. 9, November, 1987. Dr. Herbert Wilf suggested reprinting this article, which he claimed summarized his lecture as accurately as he could have done. We are also indebted to Dr. Pearla Nesher, University of Haifa, for her excellent presentation of Lecture 2.

The 1987 CMESG/GCEDM Annual Meeting was held in conjunction with the Summer Meeting of the Canadian Mathematical Society and members of CMS had the opportunity to participate once again in our discussions. The quality of the main lectures and the wide ranging discussions in the working and topic groups made for a very productive and stimulating meeting.

Special thanks must be extended to the organizers at Queen's University, particularly Peter Taylor and Bill Higginson. The dinner arranged in the beautiful surroundings of the Royal Military College made for a delightful evening and the hospitality of Peter Taylor is appreciated.

We hope that these Proceedings serve as a reminder of what transpired in Kingston so that we can come together next time in Winnipeg fully prepared to meet whatever challenges put before us.

Lionel Mendoza
Edgar Williams
Co-Editors

March, 1988

ACKNOWLEDGEMENTS

The Canadian Mathematics Education Study Group wishes to acknowledge the continued assistance received from the Social Science and Humanities Research Council in support of the 1987 Annual Meeting. Without this support, this meeting would not be possible.

We also wish to thank Queen's University in Kingston for hosting this meeting and for providing all the necessary facilities to conduct the meeting in a very pleasant atmosphere.

We wish to thank all contributors for making this a very successful meeting and all those who attended, without whom, the effort would not be worthwhile.

LECTURE 1

THE CHIP WITH THE COLLEGE EDUCATION: the HP-28C

Herbert S. Wilf
The University of Pennsylvania

NOTE: The following article, reprinted with permission from the American Mathematics Monthly, Vol. 94, No. 9, November, 1987, pp. 895-902, is an accurate summary of Herb Wilf's lecture. Dr. Wilf suggested reprinting this article since it does give the main flavour of his lecture and was written, as we understand it, at the suggestion of Dr. Wilf. We reprint it here with thanks to the MAA and the author so that others who may not have access to it can be enlightened on this new development.

THE CHIP WITH THE COLLEGE EDUCATION: THE HP-28C

Yves Nievergelt
Graduate School of Business Administration,
Executive M.B.A. Program,
University of Washington, Seattle, WA 98195

Five years ago in the American Mathematics Monthly [15], the present editor of the Monthly augured a future in which students would have pocket calculators that could do symbolic calculus. Exactly five years later, *The Wall Street Journal* [1] announced the calculus calculator: the HP-28C. This hand-held machine deserves some attention - if it could walk into a standard lower-division mathematics course, it might well pass on its own. The following examples, which demonstrate the new capabilities of the HP-28C, provide a basis for the subsequent discussion of the potential of such super- calculators in the teaching of mathematics.

1. The power of the HP-28C

CALCULUS. First, watch how the HP-28C solves homework problems selected from various calculus texts.

Problem 1. Let f(x,y) := x ln xy; find $\frac{\partial b}{\partial x}$ (Fleming [**4**, p. 79, #1]).

To find $\partial f/\partial x$, enter the formula for f in the form '**X*LN(X*Y)**', specify the variable by entering '**X**', and press the differentiation key. The calculator answers

'**LN(X*Y) + X*(Y/(X*Y))**' .

To simplify this expression, select the **ALGEBRA** menu and then the **FORM** submenu; move the cursor onto the second * and execute the **COLCT** command. The machine collects similar terms and displays

'**LN(X*Y) + 1**' .

Problem 2. Find the Maclaurin polynomial of degree 3 for $\sqrt{1 + x}$ (Stein [**13**, p. 547, #10]).

To determine this Taylor polynomial, enter the formula '**√(1 + X)**', specify the variable with '**X**' and the degree with 3, and select the **TAYLR** command from the **ALGEBRA** menu. The HP-28C responds: '**1 + .5*X - .125X^2 + .0625*X^3**'.

Problem 3. Calculate $\int(ax^2 + bx + c)dx$ (Leithold [**10**, p. 376, #20]). To calculate this indefinite integral, enter the integrand, '**A*X^2 + B*X + C**', the variable of integration, '**X**', and the degree of the integrand, 2; then press the integration key. Now add your favorite constant to the display.

'**C*X + B/2*X^2 + A*2/2/3*X^3**' .

If desired, the **COLCT** command can simplify the redundant form **A*2/2/3**. This redundancy arises from the Maclaurin polynomial of the integrand, which the HP-28C integrates term by term. Although this procedure may seem unwieldy for mere polynomials, it also enables the calculator to tackle harder problems.

Problem 4. Find the Maclaurin series for $(2/\sqrt{\pi})\int_0^x e^{-t^2}\,dt$ (Hurley [**6**, p. 618, #31]).

As in problem 3, enter the integrand, '**2/√π*e^ - T^2**', the variable '**T**', and the degree of the desired Taylor polynomial, for example **3** (with a higher degree, the calculator runs out of memory space); then press the integration key. The HP-28C replies:

'**2/√π*T + 2/√π*(LN(e)*(-2)/2/3*T^3)**' .

To end this calculus quiz, let the calculator try a curve-sketching problem:

Problem 5. Graph the function $f(x) = e^{\sin x}$ (spivak [**12**, p. 326, #4b]). To sketch this curve, store the formula '**e^SIN(X)**', or '**EXP(SIN(X))**', into the **PLOT** menu, and execute the **DRAW** command. Within thirty seconds, the HP-28C traces the graph in exhibit 1a. Since the curve does not quite fit into the display, translate the center of the screen upward by 1.4; this will produce the graph in exhibit 1b. For a hard copy, enter the command **CLLCD DRAW PRLCD** and point the calculator toward its printer (with which it communicates by infrared beam).

(a)

e^SIN(X)

(c)

1/(X*√(2*π))*
exp(-.5*LN(X)^2)

(e)

Automatic scaling

(b)

e^SIN(X) pushed down

(d)

My program smile

(f)

Sample program REGR

EXHIBIT 1. These slightly enhanced, actual-sized copies from the HP-82240A printer are identical to the HP-28C displays, in both size and resolution. (a) and (b) graphs from Problem 5. (c) and (d) other examples of graphs. (e) and (f) scatter plots from problem 6.

STATISTICS. In addition to computing means, variances, correlations, and regressions, the HP-28C also distinguishes itself with two other novelties. First, it draws scatter plots.

Problem 6. Fit a least-squares line to the data (Freund [**5**, p. 352, #11.1]):

(5,16), (1,15), (7,19), (9,23), (2,14), (12,21).

"Always plot the data" [**5**, p. 367]. Therefore, enter the data with the **STAT** menu, revert to **PLOT** and execute the **DRWΣ** command. At first the screen shows the axes but no data, because the points lie outside its range. To correct this mismatch, press the **SLCΣ** key, which automatically fits the display onto the data set (but sends the axes away), as in exhibit 1e. To superimpose the least-squares line and bring the axes back into the picture, run the following sample program. which produces exhibit 1f:

<<SCLΣ (0,0) PMIN LR <<X PREDV>> STEQ CLLCD DRAW DRWΣ>> .

Besides drawing scatter plots, the HP-28C computes upper-tail probabilities, $\mathrm{upt}(x) := \int_x^\infty f(t)\,dt$, for normal, chi-square, t, and F random variables. For example, to compute the probability that a χ^2 random variable with 357 degrees of freedom takes a value bigger than 401.9, enter 357 and 401.9 and execute the **UTPC** command. The calculator gives .**050599...**, meaning that $P(\chi^2_{357} > 401.9) \simeq 0.0506$.

Combined with the HP-28C equation solver, upper-tail probabilities also give an easy solution to the following "percentile problem".

Problem 7. Determine the 99th percentile of the distribution χ^2_{357}.

To determine the value of x such that $P(\chi^2_{357} > x) = 1 - 0.99$, program this equation in the form **<<.01 357 X UTPC ->>** and invoke the **SOLVR**, the equation solver. After about a minute, the HP-28C displays **X: 422.08**... To appreciate this prowess, recall that this amounts to solving for x the equation

$$\int_x^\infty \frac{1}{\Gamma(357/2)\,2^{357/2}}\, t^{355/2} e^{-t/2}\,dt = 0.01 \ .$$

NUMERICAL ANALYSIS AND LINEAR ALGEBRA. In addition to its equation solver, the HP-28C offers further numerical routines that evaluate definite integrals, solve linear systems, and compute dot and cross products, determinants, operator norms, and inverses of real or complex matrices. Since similar features have been available for over five years on a predecessor, the HP-15C, a few illustrations will suffice to describe the speed and accuracy of the HP-28C.

Problem 8. Solve the following moderately ill-conditioned system, with condition number $\|A\|_1 \|A^{-1}\|_1 \approx 10^6$ (Burden and Faires, [**2**, p. 331, #5c]):

$$v + \frac{1}{2} w + \frac{1}{3} x + \frac{1}{4} y + \frac{1}{5} z = 1 .$$

$$\frac{1}{2} v + \frac{1}{3} w + \frac{1}{4} x + \frac{1}{5} y + \frac{1}{6} z = 1 .$$

$$\frac{1}{3} v + \frac{1}{4} w + \frac{1}{5} x + \frac{1}{6} y + \frac{1}{7} z = 1 .$$

$$\frac{1}{4} v + \frac{1}{5} w + \frac{1}{6} x + \frac{1}{7} y + \frac{1}{8} z = 1 .$$

$$\frac{1}{5} v + \frac{1}{6} w + \frac{1}{7} x + \frac{1}{8} y + \frac{1}{9} z = 1 .$$

After entering the right-hand vector, B, and the matrix of coefficients, A, simply press the division key, ÷. The calculator thinks for three seconds and displays its solution:

[5.00000076461 -120.000014446 630.000062793
-1120.00009538 630.000046863]

This result compares favorably to that of a large mainframe CDC CYBER 180/855 running IMSL (International Mathematical and Statistical Library), which blinked for just 0.01 second and printed:

5.000000002094 -120.000000038932 630.000000167638
-1120.000000253036 630.000000123768

Problem 9. Evaluate $P(X) = 8118X^4 - 11482X^3 + X^2 + 5741X - 2030$ for $X = 0.707107$ (Kulisch and Miranker [**9**, p. 12, #5]).

According to Kulisch and Miranker, $P(0.70107) = -1.91527325270\ldots \times 10^{-11}$. Using double precison (28 digits) the CYBER found $-1.91527325270819 \times 10^{-11}$. Working with 16 digits only, the HP-28C returned the single digit 0.

However, the HP-28C and the CYBER agreed on the answer to this last problem:

Problem 10. Find the eigenvectors of the following matrix (Johnson and Reiss [**8**, p. 104, #7]).

$$A = \begin{pmatrix} 6 & 4 & 4 & 1 \\ 4 & 6 & 1 & 4 \\ 4 & 1 & 6 & 4 \\ 1 & 4 & 4 & 6 \end{pmatrix}.$$

One possible solution (which takes advantage of the HP-28C built-in matrix multiplication and transportation) consists of programming Jacobi's method according to the algorithm in [**14**, pp. 341-342]. Three sweeps of Jacobi's method take only a minute and yield the following eigenvalues and eigenvectors: $\lambda_1 = -15$, $\lambda_2 = -1$, and $\lambda_3 = 5 = \lambda_4$, with

$$v_1 = \begin{pmatrix} .5 \\ .5 \\ .5 \\ .5 \end{pmatrix}, \quad v_2 = \begin{pmatrix} -.5 \\ .5 \\ .5 \\ -.5 \end{pmatrix}, \quad v_3 = \begin{pmatrix} -.155356107285 \\ .689829312165 \\ .689829312166 \\ .155356107285 \end{pmatrix},$$

$$v_4 = \begin{pmatrix} -.689829312165 \\ .155356107285 \\ -.155356107285 \\ .689829312166 \end{pmatrix}.$$

COMPUTER SCIENCE. Besides its symbolic and numerical capabilities, the HP-28C also provices bit-by-bit logical operators (**AND**, **OR**, **XOR**, **NOT**), register shifts, and hexadecimal, octal, and binary arithmetic, all on 64-bit words. This relatively large word-length makes the HP-28C well suited to one of computer scientists' favorite homework assignments: the simulation of one machine on another.

Exercise. The CDC CYBER mainframe computer operates with 60-bit words, in which it represents integers by "complement to 1." (Thus, the CYBER stores a positive integer as its binary expansion, but it represents a negative integer as the bit-by-bit logical complement of its absolute value.) Simulate the integer arithmetic of the CYBER on the HP-28C.

2. Supercalculators in the mathematics classroom.

FIRST ACADEMIC REACTIONS. Left to their own devices, four freshman familiarized themselves with the HP-28C in just two hours, with the help of the well-written *Getting Started Manual*. However, they felt that they needed a better understanding of the mathematics involved in order to use the calculator more intelligently. "Anyway", sighed one, "teachers won't allow it on tests, or will they?"

Nobody knows; in a recent informal poll, faculty showed mixed reactions: "What would I ask on tests now?" wondered one professor, obviously feeling threatened, while at another school, a colleague exclaimed: "It will save hours of calculations!" Because of such a divergence of opinions, the HP-28C and its successors will probably influence individual mathematics curricula in different ways, as does the use of different textbooks now.

Possibly, the HP-28C might enable students instantly to punch, read, and speak calculus. In extemely cook-book courses, students might do nothing but scan the HP-28C UNITS menu from "**A**" to "**tsp**" to find that one teaspoon equals **4.92892159375E - 6 m^3**.

The HP-28C may also allow users to leave the calculations to the machine, and to focus on ideas and strategies. For thinkers, including non-mathematicians, the availablity of supercalculators may increase the practial importance of theory. Indeed, this conjecture seems supported by the following informal survey.

THE EDUCATION OF THE UNDERGRADUATE MATHEMATICAL PRACTITIONER. What do employers look for in the mathematical education of a new graduate?

Peter Eriksen, who holds a bachelor's degree in mathematics and works for Boeing Military Airplane Co. near Seattle, also has a degree in *philosophy*, which he finds more helpful than mathematics. He considers most useful "the training from a particular philosophy professor, who insisted that we analyze problems logically. that we arrive at some answer, and that we write up our argument in a flawless style."

"Proficiency in undergraduate mathematics, experience in utilizing the mathematics library, attention to detail, and written communications skills," says Dr. Stephen P. Keller, who hires and supervises mathematicians at Boeing Compter Services Co. He illustrates the need for these intellectual abilities with the following example: "Suppose that you have to code a two-dimensional integration routine. Then you must understand something about the Riemann integral, be able to review the literature on you own, code your algorithm correctly, and document your work in a manner understandable to your colleagues." Unfortunately, Dr. Keeler has found that he cannot assume such an intellectual maturity from students with only a bachelor's degree in mathematics. He suggests one way in which supercalculators might help in education: "To emphasize the importance of details, give students a mathematical programming assignment [for instance as above] and insist that they get it absolutely right, be it on a LISP machine or on an HP-28C."

Aside from the aerospace industry, indications about the potentials of supercalculators in education may also come from elsewhere in the corporate world.

THE MATHEMATICAL EDUCATION OF THE EXECUTIVE. The Executive Master of Business Administration (EMBA) Program of the University of Washington offers a propitious environment for testing new ideas in the teaching of business calculus, including the use of fancy calculators. Immediately before entering the program, the participating senior executives attend a "business calculus" course designed to meet their needs on the job and in such EMBA courses as finance, microeconomics, and statistics. For this mathematics course, every executive must bring a powerful financial HP-12C (or a scientific HP-15C), which allows for more substantial case-studies, as in the following example.

Example 1. Consider a thirty-year Treasury bond purchased on 15 May 1984 for $9933.90 with $662.50 interest coupons every six months. The "yield rate" of this bond, r, is *by definition* given by the solution $v = 1/(1 + r/2)$ of the equation

$$10{,}000v^{60} + 662.50(v^{60} + v^{59} + \ldots + v^{2} + v) - 9933.90 = 0 .$$

Calculus shows that this equation has exactly one positive solution. While the calculators were computing the yield rate, one banker remarked that "the equation implies that you reinvest every coupon into a similar bond." Freed from the computations, the executive realized what the yield rate *means* and how to interpret it in business. Then the calculators gave the yield rate in the form of rationals on either side of a Dedekind cut or the starts of equivalent Cauchy sequences. The calculators also left time to explain those concepts.

Nevertheless, executives do not feel that supercalculators free them from mastering the basics, "I still need to understand my algebra thoroughly," says a company vice-president, "so that I can explain to myself what a formula means for my business." A health-services director adds that "we need much more graphical analysis, including the concepts of slope and area." Even a supercalculator would not help in the following assignment.

Example 2. Imagine that you sit on the board of directors of your local utility company. Discuss the advantages and disadvantages of setting the price equal to the marginal cost, instead of the average cost.

CONCLUSIONS. The HP-28C introduces one new element into the teaching of mathematics, namely awesome computing power at both a modest price and size, with admirable user-friendliness (all three characteristics compared to those of a CYBER, for instance). Students may thus purchase, carry, and utilize a power close to that of a main-frame as easily as they do textbooks. Still, in spite of the availability of this hand-held power, proficiency in certain basic skills remains essential to the students' ability to *apply* mathematics. Indeed, it appears that a new trend toward the use of the HP-28C and its successors would require that students understand the underlying concepts even better than before in order to decide what computations to perform, to interpret the results with lucidity [**11**, pp. 40-42], or even first to recognize that no calculator can address the issue at hand. In practice, the need for a deeper understanding of theory grows dramatically, as seen in two excerpts from the *The Wall Street Journal*:

> software defects have killed sailors, maimed patients, wounded corporations and threatened to cause the government-securities market to collapse [3].

> Morton Thiokol Inc., admitting that it never fully understood the working of the booster rocket blamed for the explosion of the space shuttle Challenger, said it made major changes [7].

Shortly after the Challenger disaster, a junior mathematics major at a university expressed the desire to work on the shuttle program but could not cope with the evaluation of $\int_{-1}^{1} |x| dx$. The faculty nevertheless decided to graduate the student. With pressure on the faculty to pass students weighed against the need to train students to detect software defects in supercalculators, mathematics instructors face a difficult choice. We may refrain from feeling partly responsible for mistested drugs and shuttle crashes, or we may insist that students (even students with supercalculators) be able to solve unfamiliar problems, detect errors in proofs and programs, and verify the validity of mathematical algorithms, models, and theories.

Acknowledgement. I thank Joyce D. Kehoe, Seattle writer, for her professional help in editing this review.

REFERENCES

1. Hewlett Unveils Calculator For Conceptual Algebra, The Wall Street Journal (12 January 1987) 18.

2. Richard L. Burden and J. Douglas Faires, Numerical Analysis, 3rd ed., Prindle, Weber & Schmidt, 1985.

3. Bob Davis, Costly bugs, The Wall Street Journal (28 January 1987) 1.

4. Wendell Fleming, Functions of Several Variables, 2nd ed., Springer-Verlag, 1982.

5. John E. Freund, Statistics, a First Course, 3rd. ed., Prentice-Hall, 1981.

6. James F. Hurley, Calculus, Wadsworth, 1987.

7. Frank E. James, Thiokol modifies shuttle-rocket testing, concedes uncertainty on booster design, The Wall Street Journal (27 January 1987) 58.

8. Lee W. Johnson and R. Dean Riess, Numerical Analysis, 2nd ed., Addison-Wesley, 1982.

9. U. W. Kulisch and W. L. Miranker, The arithmetic of the digital computer: a new approach, *SIAM Review*, 28 (March 1986) 1-40.

10. Louis Leithold, Calculus With Analytic Geometry, 5th ed., Harper & Row, 1986.

11. Elliot W. Montroll, On the dynamics and evolution of some socio-technical systems, *Bulletin of the A.M.S.*, 16 (January 1987) 1-46.

12. Michael Spivak, Calculus, 2nd ed., Publish or Perish, 1980.

13. Sherman K. Stein, Calculus and Analytic Geometry, 4th ed., McGraw-Hill, 1987.

14. Josef Stoer and Roland Bulirsch, Introduction to Numerical Analysis, Springer-Verlag, 1983.

15. Herbert S. Wilf, The disk with the college education, American Mathematics Monthly, 89 (January 1982) 4-8.

LECTURE 2

TOWARDS AN INSTRUCTIONAL THEORY: THE ROLE OF STUDENT'S MISCONCEPTIONS

Pearla Nesher
The University of Haifa

I. Introduction

During the past decade we have witnessed a new trend in cognitive research emphasizing expert systems. A great deal of effort has been dedicated to the study of experts' performance in various fields of knowledge. My presentation today deals with the question: what kind of expertise is needed for instruction? Researchers in the field agree that the process of learning necessarily combines three factors: The student, the teacher and the subject to be learned. In addition, it seems obvious that to teach a given subject matter we need at least two kinds of expertise: the subject matter expert who can knowledgeably handle the discipline to be learned, who can see the underlying conceptual structure to be learned with its full richness and insights; and there is also, obviously, the expert teacher whose expertise is in successfully bringing the student to know the given subject matter by various pedagogical techniques that makes him the expert in teaching. In this framework of experts' systems, what is, then, the role of the student? What does he contribute to the learning situation? And though it might seem absurd, I would like to suggest that the student's "expertise" is in making errors; that this is his contribution to the process of learning.

My talk consists of three main parts. First, I will focus on the contribution of performing errors to the process of learning. I will, then, demonstrate that errors do not occur randomly, but originate in a consistent conceptual framework based on earlier acquired knowledge. I will conclude by arguing that any future instructional theory will have to change its perspective from condemning errors into one that seeks them. A good instructional program will have to predict types of errors and purposely allow for them in the process of learning. But before we reach such an extreme conclusion let me build the argument and clarify what these "welcomed" errors are.

II.

In order to better understand the process of learning, I would like to make a digression here and learn something from the scientific progress. Science involves discovering truths about our universe, and does so by forming scientific theories. These theories then become the subject matter for learning. Philosophers worried for a long time about these truths. How can one be sure that he has reached truth and not falsehood? Are there clear criteria to distinguish truth from falsehood? These philosophical discussions can also enlighten our understanding.

It was C.S. Peirce, the American scientist and philosopher (1839-1914), who brought to our attention how we all act most of the time according to habits which are shaped by beliefs, (and from the history of science we know that there have been many false beliefs). But we do not regularly question these beliefs; they are established in the nature of our habitual actions. It is only when doubts about our beliefs are raised, that we stop to examine them and start an inquiry in order to appease our doubts and settle our opinion. Thus, in Peirce's view, this is not an arbitrary act of starting inquiry on a certain question, but rather an unavoidable act when some doubt arises. When do such doubts arise? It is when ones expectation is not fulfilled because it conflicts with some facts. On such occasions when one feels that something is wrong, only then, does a real question arise and an inquiry is initiated, an inquiry that should settle our opinions and fix our beliefs (Peirce, 1877).

A similar, though not identical view was strongly advocated by K. Popper (1963). In his book Conjectures and Refutations he argues against an idealistic and simplistic view of attaining truths in science. He claims that "Erroneous beliefs may have an astonishing power to survive, for thousands of years" (Popper, 1963, p. 8), and since he does not believe in formulating one method, leading us to the revelation of truth, he suggests changing the question about "sources of our knowledge" into a modified one - "How can we hope to detect error?" (Ibid. p. 25). If we are lucky enough to detect an error we are then in a position to improve our set of beliefs. Thus for Popper science should adopt the method of "critical search for error" (Ibid. p. 26), which has the power of modifying our earlier knowledge.

In the systems of these philosophers which I only touched upon here, there are several points relevant to learning in general that should be clearly stated:

1) Falsehood is adjunct to the notion of truth, or in the words of Russel: "Our theory of truth must be such to admit of its opposite, falsehood." (Russell, 1912, p. 70)

2) Though having a truth-value is a property of beliefs, it is established by many methods and it is independent of our beliefs whether it will ultimately become true or false (a point which I will take up again later).

3) We hold many beliefs that we are unaware of and which are part of our habits, yet, once such a belief clashes with some counter evidence or contradicting arguments, it becomes the focus of our attention and inquiry.

Is all this relevant to the child's learning? I believe it is. If I replace the terms "truth and false" with "right and wrong" or "correct and error" we will find ourselves in the realm of schools and instruction, in which unlike the philosophical realm, "being wrong", and "making errors" are negatively connotated. The system,

in fact, reinforces only "right" and "correct" performances and punishes "being wrong" and "making errors", by means of exams, marks etc, a central motive in our educational system.
I found it very refreshening when visiting a second grade class to hear the following unusual dialogue:

> Ronit (second grader with tears in her eyes): "I did it wrong" (referring to her geometrical drawing).
> "Never mind", said the teacher, "What did we say about making mistakes?"
> Ronit (without hesitating) answered: "We learn from our mistakes".
> "So", added the teacher, "Don't cry and don't be sad, because we learn from our mistakes".

The phrase "we learn from mistakes" was repeated over and over. The atmosphere in the classroom was pleasant and use of this phrase was the way the children admitted making errors on the given task. At this point I became curious and anxious to know what children really did learn from their mistakes. I will describe the task, and how the children knew when they made mistakes. Let us now observe a geometry lesson in which the students learned about the reflection transformation. The exercises consisted of a given shape and a given axis of reflection (see figure 1) which the children first had to hypothesize (or guess) and draw the reflected figure in the place where they thought it would fall, and then to fold the paper on the reflection axis and by puncturing with a pin on the original figure (the source) to see whether their drawing was right or wrong.

I would like to make it explicit that, from the child's point of view, he or she had to discover the "theory" of reflection. The teacher did not intend to serve as the authority for this knowledge, lecturing about the invariants of reflection, but instead supplied the child with a structured domain which his erroneous conceptions could be checked against. The line of dots created by the pin puncture served as ideal reality for this kind of reflection, and as feedback for the child's conjectures. In my view this resembles in a nutshell scientific inquiry in several important aspects.

Delighted to find such a supportive atmosphere in the classroom, I became interested in the epistemological question, what did the children really learn from their mistakes? When each child who made an error was asked to explain to me what was learned from his or her mistake I could not elicit a clear answer. Instead they repeated again and again that one learns from mistakes in a way that started to sound suspiciously like a parroting of the teacher's phrase. At this point it became clear to me that the teacher tolerated errors, but did not use them as a feedback mechanism for real learning on the basis of actual performance. I then drew on the blackboard three different errors:

The first one which I named Sharon's error, dealt with the property that a reflection is an opposite transformation, thus, what was right will become left in the reflection, and vice versa (see figure 2). The second error, I named for Dan, was dedicated to the size property, i.e., that lengths are invariant under the reflection (see figure 3). This was also the basis for the third error named after Joseph, that had to do with the distance from the reflection axis (see figure 4). I asked the children, whether one learns the same thing from each of the above errors? Should Sharon, Dan and Joseph learn the same thing? or, is there something specific to each error?.

At this point we turned from psychological support and tolerance or errors to discover the epistemological and cognitive value of errors in the process of learning. From these errors the child could learn the distinct properties of reflection, that he or she was not aware of before. (If they were aware, they would not commit this kind of error). Committing the error, however, revealed the incompleteness of their knowledge and enabled the teacher to contribute additional knowledge, or lead them to realize for themselves where were they wrong. The clash between their expectations, demonstrated by their drawings and the "reality" as was shown in the pin functure created a problem, uneasiness (up to tears), that they had now to settle. The solution to this problem in fact involved the process of learning a new property of the reflection transformation not known to them until then. As Popper (Ibid p. 222) wrote:

> "Yet science starts only with problems. Problems crop up especially when we are disappointed in our expectations, or when our theories involve us in difficulties, in contradictions; and these may arise either within a theory, or between two different theories, or as the result of a clash between our theories and our observations. Moreover, it is only through a problem that we become conscious of holding a theory. It is the problem which challenges us to learn; to advance our knowledge; to experiment and to observe" (Ibid p. 222).

I think that if we use the word "theory" in not too rigorous a manner, and substitute the word learning for science, then Popper's description is most pertinent to our issue.

III.

In the title of this presentation, I did not use the word "error" or "mistake" but rather "misconception". The notion of misconception denotes a line of thinking that causes a series of errors all resulting from an incorrect underlying premise, rather than sporadic, unconnected and non-systematic errors. It is not always easy to follow the child's line of thinking and reveal how systematic and consistent it is.

Most studies, therefore, report on classification or errors and their frequency, though this does not explain their source and therefore cannot be treated systematically. Or, when dealt with, it is on the basis of a mere surface structure analysis or errors, as in the case of "Buggy" (Brown and Burton, 1978; Brown and VanLehn, 1980), where we end up with a huge, unmanageable catalogue of errors. It seems that this lack of parsimony could be avoided is one looked into deeper levels of representation in which a meaning <u>system</u> evolves that controls the surface performance. When an erroneous principle is detected at this deeper level it can explain not a single, but a whole cluster of errors. We tend to call such an erroneous guiding rule <u>a misconception</u>.

I would like to describe now two detailed examples of misconceptions (out of many others) that demonstrate how errors do not occur at random but rather have their roots in erroneous principles. Moreover, these misconceptions were not created arbitrarily but rely on earlier <u>learned</u> meaning systems, and again, although seemingly absurd, they are actually derivations of our own previous instruction. These examples were chosen because they are each based on extensive research programs which deal with unveiling the students' misconceptions and focus on plausible explanations for their erroneous performance.

The first example is taken from a series of studies about the nature of errors made by elementary school children in comparing or ordering decimal numbers. In these studies at attempt was made to trace the sources of the student's systematic errors. The findings which emerge, following studies in England, France, Israel and USA (Leonard and Sackur-Grisvald, 1981; Nesher and Peled, 1984, Swan, 1983) show that in all these countries there is a distinct and common system of rules employed by those who fail in comparing decimals.

Consider for example the following tasks which were administered to children of grades 6, 7, 8, and 9. The subjects had to mark the larger number in the following pairs:

case I	0.4	vs.	0.234
case II	0.4	vs.	0.675

Jeremy marked in case I that 0.234 is larger than 0.4; and in case II he marked that 0.675 is the larger one. Does he or does he not know the order of decimal numbers? In our study in Israel the data was gathered in individual interviews, so that the children could explain their choices. This helped us understand their guiding principles. In both cases Jeremy said that the number with the <u>longer</u> number of digits (after the decimal point) is the larger number (in value). Jeremy had one guiding principle as to the order of decimals and, accordingly, in case I Jeremy was wrong while in case II he was right. Although his guiding principle was a mistaken one, he succeeded in correctly solving all the exercises similar to case II. It is also not hard to see that his guiding principle was one that served him well up to this point, having been imported from his knowledge of whole numbers where the longer numbers really are larger in value. And, unless something is done Jeremy's "success" or "failure" on certain tasks is going to depend on the actual pair of numbers given to him. This, of course, blurs the picture of his knowledge in any given test. Now imagine Ruth who decided in both cases I and II (in the above example) that 0.4 is the larger number, i.e., in each case she pointed to the shorter number as the larger one in value. Ruth gave the following explanation: "Tenths are bigger than thousandths, therefore, the shorter number that has only tenths is the larger one." Ruth does not differentiate between case I and case II either. She will be correct in all the cases similar to case I, but wrong in all cases which are similar to case II. We can understand this kind of erroneous reasoning in light of what is learned in fractions. Ruth has a partial knowledge of ordinary fractions and cannot integrate what she knows about them with the new chapter on decimal fractions and their notation. In particular she found it difficult to decide whether the number written as a decimal fraction is the numerator, or the denominator. She cannot coordinate the size of the parts with their number in the decimal notation.

It is interesting to note that about 35% of the sixth graders in Israel who completed the chapter on decimals acted like Jeremy and were, in fact, using the above mentioned rule which relies heavily on the knowledge of whole numbers, and about 34% of the Israeli sample of sixth grades made Ruth's type of mistake. Even more interesting, is the fact that while Jeremy's rule frequency declines in higher grades, Ruth's rule is more persistent and about 20% of the seventh and eighth graders still maintain Ruth's rule. (Nesher and Peled, 1984).

As I remarked before, these misconception are hard to detect. This is so because on some occasions the mistaken rule is disguised by a "correct" answer. [Or, the student may get the "right" answer for the wrong reasons.] Thus, for the student who holds a certain misconception not all the exercises consisting of pairs of decimal numbers will elicit an incorrect answer. For example, decimals with the same number of digits are compared as if they are whole number and, therefore, usually answered correctly. In fact this is also a method taught in schools: add zeros to the shorter number until it becomes as long as the longer one and then compare them.

An interesting question emerged: If the teacher is not aware of the cases that discriminate between various types of misconceptions and those cases that do not discriminate misconceptions at all, what is the probability that he or she will give a test (or any other set of exercises) that detects systematic errors. Irit Peled, my former student, in her Ph.D thesis dealt with precisely this question (Peled, 1986). She built a series of simulations that make it possible to evaluate quantitatively the probability of getting discriminating items on a test.

Let me return to the question of a discriminating item for a certain error: For example, given the following item "Which is the larger of the two decimals 0.4 and 0.234?" If the student answers 0.234 my may suspect that he holds Jeremy's misconception. But, if he answers 0.4, we cannot know whether he knows how to order decimals, or if he is holding Ruth's error, but happened to get lucky numbers and be correct on this particular item. Thus this item can discriminate and elicit those holding Jeremy's misconception, but cannot discriminate between those Ruth's misconception and experts (i.e. those who really know the domain). Along these lines, for the same task, the pair of numbers 0.4 and 0.675 can discriminate those holding Ruth's misconception, but cannot discriminate between those holding Jeremy's misconception and experts. Similarly comparing the numbers 0.456 and 0.895 cannot discriminate either Jeremy's, or Ruth's misconception (whether the child answers correctly or not).

So, if a teacher composes a test (or any other assignment) without looking intentionally for the discriminating items, there is little change that such items will be included. In Peled's simulations it was found that when pairs of numbers are randomly selected from all the possible pairs of numbers having at most three digits after the decimal point, the probability of getting an item that will discriminate Jeremy's error was 0.10, and Ruth's error 0.02. Thus both Jeremy and Ruth will succeed up to 90% on a test composed by their teacher, if she is not aware of this problem. It is not surprising, then, that teachers are usually satisfied with the performance of children holding Jeremy's or Ruth's misconceptions, and they should not be blamed. On the basis of one wrong item it is impossible to discover the nature of the student's misconception.

The teacher could of course increase the difficulty of the test by allowing only pairs of numbers with unequal lengths (up to three digits after the decimal point), which will raise the probability of getting discriminating items on the test, but will not insure correct diagnosing of a specific misconception (see Appendix B for a sample test). The probability is that on such a random test Jeremy will get 58% correct and Ruth - 48%. With awareness of the problem, the teacher can design a test to intentionally diagnose and discriminate the known misconceptions to a proportion and distribution already determined.

The teachers, however, are hardly aware of such analysis of misconceptions. Some of them listening to our report, could not believe the existence of Ruth's type of misconception at all, until they returned to their classes and found it for themselves.

Teachers do not generally build such knowledge into their instruction and evaluation of the student's performance. Thus, frequently the teacher completes the section of instruction on comparing decimals, gives a final test, and believes that the children know it perfectly well, not noticing that many of them still hold important misconceptions such as Jeremy's and Ruth's, as we and others found in our studies. In such a classroom it will be also very difficult for Jeremy or Ruth to give up their misconceptions since they are rewarded daily for their erroneous guiding principles by correctly answering non discriminating items.

Several lessons can be learned from these studies:

a) In designing the instruction of a new piece of knowledge it is not enough to analyze the procedures and their prerequisites which is, in many cases, done. We must know how this new knowledge is embedded in a larger meaning system that the child already holds and from which he derives his guiding principles.
b) It is crucial to know specifically how the already known procedures may interfere with material now being learned. In the case of decimal knowledge a fine analysis will show the similarity and dis-similarity between whole numbers and decimals, or between ordinary fractions and decimals. Some of the elements of earlier knowledge may assist in the learning of decimals, but some of them are doomed to interfere with the new learning, because of their semi-similarity (see Appendix A).

c) All the new elements, which resemble but differ from the old ones, should be clearly discriminated in the process of instruction, and the teacher should expect to find errors on these elements. Needless to say, although they elicit more erroneous answers, such elements should be presented to the children and not avoided.

My second example is taken from a series of studies by Fischbein et al (1985). In their study Fischbein's group claimed that in choosing the operation for a multiplicative word problems (let's say, choosing between multiplication and division) students tend to make specific kind of mistakes derived from their implicit intuitive models that they already have concerning multiplication. Thus, identification of the operation needed to solve a problem, does not take place directly but is mediated by an implicit, unconscious, and primitive intuitive model which imposes its own constraints on the search process. The primitive model for multiplication is assumed to be "repeated addition".

The data supporting their hypothesis is based on the following findings. Multiplication word problems, in which according to the context the multiplier was a decimal number (i.e., 15 x 0.75) yielded 57% success, while those consisting of a decimal number in the multiplicand (0.75 x 15) yielded 79% success. Fischbein's group attributed this to the fact that the intuitive model of multiplication as repeated addition does not allow for a non-integer number as a multiplier.

Similarly, in division when the numbers presented in the word problem were such that the students had to divide a smaller number by a larger one, they reversed the order and divided the larger one by the smaller, so that it would fit their previous notions of division. It also became apparent in the series of studies and students misconceive that "multiplication always makes bigger" (Bell et al., 1981, Hart, 1981). Fischbein's research paradigm was repeated several times with different populations always yielding the same results. (Greer and Mangan, 1984; Greer, 1985; Tirosh, Graeber and Glover, 1986; Zeldis-Avissar, 1985).

This set of misconceptions, again, is not easy to detect, without somebody isolating and comparing various variables in a controlled study. This is where research can directly effect school teaching. The probability of occurance of multiplication and division word problems that elicit such misconceptions in the textbooks is low. In the absence of items or problems purposely directed to detect misconceptions we are shooting in the dark. We are likely to put too much emphasis on trivial issues while overlooking serious misconceptions.

There is another lesson from these studies which is harder to implement. We can trace the sources of major misconceptions in prior learning. Most of them are over-generalizations of previously learned, limited knowledge which is now wrongly applied. Is it possible to teach in a manner that will encompass future applications? Probably not. If so, we need our beacons in the form of errors, that will mark for us the constraints and limitations of our knowledge.

IV.

So far, what I have said suggests that teachers should be more aware of the possible misconceptions and incorporate them into their instructional considerations. But this is not sufficient, and I would like to return to the example of the second graders working on the reflection transformation.

Let us suppose that in designing the pin puncture booklet the teacher was aware of the possible misconceptions and included all the discriminating items she could think of. However, another significant characteristic of this booklet was that it enabled the child to decide for himself whether he was right or wrong and in what respect was he wrong. This was possible because the rules by which the pin puncture behaved were dependent only on the mathematical reality and not on the learner's beliefs. The fact that the rules of mathematics and one's set of beliefs are independent allows for discrepancies between them. Therefore when the student held a false belief, or a false conjecture it clashed with "reality" as exemplified in their booklet. This kind of instructional device enabled the child to pursue his own inquiry and discover truths about the reflection transformation, and at the same time make errors resulting from his misconceptions, some of which were not anticipated by the teacher. He was working within what I call a Learning System to which I will devote the rest of my talk.

A Learning System (LS) is based on the following two components:

1) an articulation of the unit of knowledge to be taught based upon the expert's knowledge, which is referred to as the knowledge component of the system, and

2) an illustrative domain, homomorphic to the knowledge component and purposely selected to serve as the exemplification component.

Although "microworld" may seem a natural choice of term for a Learning System, I prefer to use a different term since "microworld" is sometimes identified with the exemplification component only, and sometimes with the entire Learning System. I, therefore, have introduced the term "Learning System" to ensure we understand that a microworld here encompasses both components. Various concrete materials employed in the past, such as Cuisenaire Rods, or Dienes' Blocks (Gattegno, 1962; Dienes, 1960) serve as illustrative aspects of Learning Systems. Moreover, I believe that the rapid progress of computers in the last decade, with their tremendous feedback power, will lead to the development of many more such Learning Systems.

The knowledge component in a Learning System is articulated, not by experts who are scientists in that field, but rather by those who can tailor the body of knowledge to the learner's particular constraints (age, ability, etc) and form the learning sequence. In order for the exemplification component to fulfil its role, it must be familiar to the learner. He should intuitively grasp the truths within this component. It is necessary that the learner while still ignorant about the piece of knowledge to be learned, be well acquainted with the exemplification so that he can predict results of his actions within that domain and easily detect unexpected outcomes. The familiar aspects of the Learning System provide an anchor from which to develop an understanding of the new concepts and new relations to be learned.

Familiarity, however, is not sufficient. The selection of the exemplification component should ensure that the relations and the operations among the objects be amenable to complete correspondence to the knowledge component to be taught. For example, in the case of teaching the reflection transformation, the exemplification by the pin puncture corresponded more to the knowledge component, rather than a mirror which enables reflection of only one half of the plain on the other (There were some other advantages as well which I will not go into here).

The gist of the Learning System is that we have a system with a component familiar to the child, from previous experience, which will be his stepping stone to learn new concepts and relationships, as defined by the expert in the knowledge component. A system becomes a Learning System, once the knowledge component and the exemplification component are tied together by a set of well defined correspondence (mapping) rules. These rules map the objects, relations and operations in one component to the objects, relations, and operations of the other component.

Functioning as a model, the exemplification component of a Learning System must fulfil the requirements described by Suppes (1974) i.e., it must be simple and abstract to a greater extent than the phenomena it intends to model, so that it can connect all the parts of the theory in a way that enables one to test the coherence and consistency of the entire system. This forms the basis for the child's ability to judge for himself the truth value of any given mathematical conjecture in a specified domain. It provides the learner with an environment within which he can continuously obtain comprehensible feedback on his actions, as was apparent from the second graders' behavior.

I believe that arriving at mathematical truths is the essence of what we do in teaching mathematics. This brings me back to the question I raised at the beginning of my talk about mathematical truths. This is a deep philosophical question that I will not delve into here, recalling instead, Russell's formulation on the correspondence theory of truth. Russell (1959/1912) clarifies that truth consists in some form of a correspondence between belief and fact. Thus, though the notion of truth is tied to an expressed thought or belief, by no means can it be determined only by it. An independent system of facts is needed toward which it is tested. This, however, is not the only theory of truth. In the same chapter Russell also mentions a theory of truth that consists of coherence. He writes that the mark of falsehood is the failure to cohere in the body of our beliefs.

How children arrive at truths is problematic. Clearly the child cannot reach conclusions about the truths of mathematics with such rigorous methods as those applied by a pure mathematician. While mathematicians can demonstrate the truth of a given sentence by proving its coherence within the entire mathematical system, young children cannot. If a young child is to gain some knowledge about truths in mathematics not based on authoritative sources, he should rely on the correspondence theory of truth rather than on the coherence theory. Thus, he should examine the correspondence between the belief and the state of events in the mathematical world. In our example this was between his conjecture where to draw the image of a reflection, and the result of his pin puncture as representing the mathematical reality.

But his approach is not without difficulties. Employing exemplifications as the source for verification commits one to introducing mathematics as an empirical science rather than a deductive one. On the other hand, I, believe that young children and even many not so young will be unable to reach mathematical truths merely by chains of deduction without first engaging in constructing and feeling intuitively the trust of these truths. Therefore, I think, that constructing a world in which the learner will be able to examine the truth of mathematical sentences via an independent state of events is the major task for any future theory of mathematical instruction. Such a world, which I have labelled a Learning System is the one in which all our knowledge about true conjectures as well as of misconceptions should be built in as its major constraints. Being limited by the System's constraints, the child will learn by experimentation and exploration the limitations and the constraints of the mathematical truths in question. On this basis can he later attend to the more rigorous demands of deductive proofs.

V.

In summary I would like to recapitulate several points touch on today. At the moment, unlike the promised of the title of this presentation, my remarks do not look like a theory at all, but rather they specify some assumptions that, in my view, will underlie any future instructional theory.

a) The learner should be able in the process of learning to test the limitations and constraints of a given piece of knowledge. This can be enhanced by developing learning environments functioning as feedback systems within which the learner is free to explore his beliefs and obtain specific feedback to his actions.

b) In cases where the learner receives unexpected feedback, if not condemned, he will be intrigued and highly motivated to pursue an inquiry.

c) The teacher cannot fully predict the effect of the student's earlier knowledge system in a new environment. Therefore, before he completes his instruction, he should provide opportunity for the student to manifest his misconceptions and then relate his instruction to these misconceptions.

d) Misconceptions are usually an outgrowth of already acquired system of concepts and beliefs applied wrongly in an extended domain. They should not be treated as terrible things to be uprooted, since, this may confuse the learner and shake his confidence in his previous knowledge. Instead, the new knowledge should be connected to the student's previous conceptual framework and be put in the right perspective.

e) Misconceptions are found not only behind an erroneous performance, but also lurking behind many cases of correct performance. Any instructional theory will have to shift its focus from erroneous performance to an understanding of the student's whole knowledge system from which he derives his guiding principles.

f) The diagnosing items that discriminate between one's proper concepts and his misconceptions are not necessarily the ones that we traditionally use in exercises and tests in schools. A special research effort should be made to construct diagnostic items that disclose the specific nature of the misconceptions.

I have tried to examine the instructional issues via the misconception angle. The examination consisted of more than the analysis of pedagogical problems; it had to penetrate epistemological questions concerning the truth and falsehood. Delving into questions of knowledge has traditionally been the prerogative of philosophy, particularly epistemology. Mental representation and the acquisition of knowledge, on the other hand, have been dealt with in the field of cognitive psychology. Obviously, each discipline adopts a different angle when dealing with the study of knowledge. While philosophers are concerned with the questions related to sources of knowledge, evidence and truth, cognitive scientists are mainly interested in questions related to the representation of knowledge within human memory and understanding the higher mental activities.

The educational questions are quite different. The agenda in education is to facilitate the acquisition and construction of knowledge by the younger members of society. While scholars of cognitive science and recently of artificial intelligence are interested mainly in the performance of experts who are already skilled in various domains, educators, on the contrary, are interested in naive learners, or novices and how they develop into experts. Sometimes I am afraid that the whole notion of 'expertise' is alien. My claim is that the road to the expert state is paved with errors and misconceptions. Each error might become a significant milestone in learning. Let these errors be welcomed.

References

Bell, A., Swan, M., & Taylor, G. (1981). Choice of operations in verbal problems with decimal numbers. Educational Studies in Mathematics, 12, 399-420.

Bell, A., Fischbein, E., and Greer, B. (1984). Choice of operation in verbal arithmetic problems: The effects of number size, problem structure and context. Educational Studies in Mathematics, 15, 129-147.

Brown, J.S., & Burton, R.R., (1978). Diagnostic models for procedural bugs in basic mathematical skills. Cognitive Science, 2, 155-192.

Brown, J.S., & van Lehn, K. (1980). Repair theory: a generative theory of bugs in procedural skills. Cognitive Science, 4, 379-426.

Dienes, Z.P. (1960). Building Up Mathematics, Humanities Press, N.Y.

Fischbein, E., Deri, M., Sainati, N., and Marino, M., S. (1985). The role of implicit models in solving verbal problems in multiplication and division. Journal for Research in Mathematics Education, 16, 3-18.

Gattengo, C. (1962). Modern Mathematics with Numbers in Colours. Lamport Gilbort Co. Ltd. Reading, England.

Greer, B. (1985). Understanding of arithmetical operations as models of situations. In J. Sloboda and D. Rogers (Eds.) Cognitive Processes in Mathematics. London, Oxford University Press.

Greer, B. and Mangan, C., (1984). Understanding multiplication and division. In T. Carpenter and J. Moser (Eds.) Proceedings of the Wisconsin Meeting of the PME-NA. University of Wisconsin, Madison, 27-32.

Greer, B. & Mangan, C. (1986). Choice of operations: From 10-year-olds to student teachers. In Proceedings of the Tenth International Conference of PME. London, pp. 25-31.

Hart, K. M. (Ed) (1981). Children Understanding of Mathematics. London: John Murray.

Leonard, F., & Sackur-Grisvald, C. (1981). Sur deux regles implicites utilisees dans la comparaison des nombres decimaux positifs. In Bulletin de l'APMEP. 327, 47-60.

Nesher, P. & Peled, I. (1984). The derivation of mal-rules in the process of learning. In Science Teaching in Israel: Origins, Development and Achievements. Pp. 325-336. The Israeli Center for Science Teaching, Jerusalem.

Peirce, C.S. (1958/1877). The fixation of belief. In P.P. Weiner (Ed) Selected Writings, Pp. 91-113.

Peled, I. (1986). Development of the decimal fraction concept. A Ph.D Thesis, submitted to the Senate of the Technion - Israel Institute of Technology, Haifa.

Popper, K.A. (1963) Conjectures and Refutations: The Growth of Scientific Knowledge, Harper & Row, New York.

Resnick, L.B. (1984). Beyond error analysis: The role of understanding in elementary school arithmetic. In H.N. Cheek (Ed.), Diagnostic and Prescriptive Mathematics: Issues, Ideas and Insights Kent, OH: Research Council for Diagnosis and Prescriptive Mathematics Research Monograph.

Resnick, L.B., & Nesher, P. (1983). Learning complex concepts: The case of decimal fractions. Paper presented at the Twenty-Fourth Annual Meeting of the Psychonomics Society, San Diego, CA.

Russell, B. (1912). The Problems of Philosophy. Oxford University Press.

Suppes, P. (1974) "The Semantics of Children Language," American Psychologist, Vol. 29, 2, 103-115.

Swan, M. (1983). The Meaning of Decimals. Shell Center for Mathematics Education. UK.

Tirosh, D., Graeber, A., & Glover, R. (1986). Pre-service Teachers Choice of Operation for Multiplication and Division Word Problems. In Proceedings of the Tenth International Conference of PME. London. pp. 57-62.

Zeldis-Avissar, T. (1985). Role of Intuitive Implicit Model in Solving Elementary Arithmetical Problems, M.A. Thesis, The University of Tel aviv.

Appendix A

A Random Comparing Decimal Test

(numbers up to three decimal digits)

.66	.154	Discriminating Jeremy's rule
.254	.045	Not discriminating
.122	.002	Not discriminating
.101	.067	Not discriminating
.885	.106	Not discriminating
.238	.433	Not discriminating
.233	.244	Not discriminating
.713	.838	Not discriminating
.245	.885	Not discriminating
.806	.702	Not discriminating

A Random Comparing Decimals Test

(Unequal lengths of numbers up to three decimal digits)

.15	.114	Discriminating Jeremy's rule
.185	.06	Discriminating Ned's rule (Not discussed here)
.51	.446	Discriminating Jeremy's rule
.31	.438	Discriminating Ruth's rule
.861	.33	Discriminating Ruth's rule
.606	.82	Discriminating Jeremy's rule
.72	.722	Discriminating Ruth's rule
.08	.822	Discriminating Ned's rule
.814	.46	Discriminating Ruth's rule
.404	.33	Discriminating Ruth's rule

APPENDIX B

Knowledge of Decimal Fractions: Identifying Place Value of Individual Digits

Elements of Decimal Knowledge	Corresponding Elements of Whole Number Knowledge	+ or -
A. Column Values:	A. Column Values:	
1. Correspond to column names	1. Correspond to column names	+
2. Decrease as move l to r	2. Decrease as move l to r	+
3. Each column is 10 times greater that column to r	3. Each column is 10 times greater than column to r	+
4. Decrease as move away from decimal point	4. Increase as move away from ones column (decimal point)	-
B. Column Names:	B. Column Names:	
1. End in <ths>	1. End in <s>	-
2. Start with tenths	2. Start with units	-
3. Naming sequence (tenths, hundredths...) moves l to r	3. Naming sequence (tens, hundreds...) moves r to l	-
4. Reading sequence is tenths, hundredths, thousandths	4. Reading sequence is thousands, hundreds, tens, ones	-
C. Role of Zero:	C. Role of Zero:	
1. Does not affect digits to its left	1. Does not affect digits to its left	-
2. Pushes digits to its right to next lower place value	2. Pushes digits to its left to next higher place value	-
D. Reading Rules:	D. Reading Rules:	
1. The number can be read either as a single quantity (tenths for one place, hundredths for two places, etc.) or as a composition (tenths plus hundreds etc.)	1. The number can be read as a single quantity and as a composition at the same time (e.g., seven hundred sixty two means seven hundred plus six tens plus two).	

Knowledge of Decimal Fractions: Identifying Place Value of Individual Digits

Elements of Fractional Decimal Knowledge	Corresponding Elements of Ordinary Fraction Knowledge	+ or -*
E. Fraction Values:	E. Fraction Values:	
1. Expresses a value between 0 and 1	1. Expresses a value between 0 and 1	+
2. The more parts a whole is divided into, the smaller is each part.	2. The more parts a whole is divided into, the smaller is each part.	+
3. There are infinite decimals between 0 and 1	3. There are infinite fractions between 0 and 1	+
F. Fraction Names:	F. Fraction Names	
1. The number of parts divided into is given implicitly by the column position	1. The number of parts divided into is given explicitly by the denominator	-
2. The number of parts included in the fractional quantity are the only numerals explicitly stated.	2. The number of parts included in the functional quantity are the numerator of the fraction.	-
3. The whole is divided only into powers of 10 parts	3. The whole is divided into any number of parts	-
4. The ending "-th" ("tenth') is typical for a fractional part	4. The ending "-th" ("fourth') is typical for a fractional part.	+

* Supports (+); contradicts (-)

LECTURE 3

THE LIMITS OF RATIONALITY

David Wheeler
Concordia University, Montreal

0. When I was invited to give an address to the Study Group, the title seemed to choose itself. I had in my mind traces of recent readings that had rubbed against each other and created a disturbance. The period of almost a year between the invitation and the delivery appeared to offer a fine opportunity to do what needed to be done to arrive at a fresh, structured, survey of the territory. As is usual with me, the opportunity somehow slipped by unseized. I bring only a few out-of-focus snapshots.

1. The phrase itself I take from the introduction to Herbert Simon's The Sciences of the Artificial. This particular occurrence of it has lodged with me, though the phrase - as against the context in which it is used - is unlikely to be original. Simon is talking about "artificial" phenomena which "are as they are only because of a system's being molded, by goals or purposes, to the environment in which it lives." (Simon, 1981, p. ix) How is it possible, he asks, to make empirical propositions about systems "that, given different circumstances, might be quite other than they are?" (ibid., p.x)

> My writing ... has sought to answer those questions by showing that the empirical content of the phenomena, the necessity that rises above the contingencies, stems from the inabilities of the behavioral system to adapt perfectly to its environment - from the limits of rationality, as I have called them. (ibid., p.x; my italics, D.W.)

Simon offers the image of an ant making its laborious way across rough ground. The track the ant makes is irregular and apparently unpredictable. Yet it is not a random walk for it takes the ant towards a particular goal. We can readily suppose that any very small animal starting at the same point and having the same destination may well follow a very similar path.

> An ant, viewed as a behaving system, is quite simple. The apparent complexity of its behavior over time is largely a reflection of the complexity of the environment in which it finds itself." (ibid., p. 64; author's italics)

Could we not hypothetically substitute the words "human being" for "ant"? Simon continues.

> A thinking human being is an adaptive system; man's goals define the interface between his inner and outer environments, including in the latter his memory store. To the extent that he is effectively adaptive, his behavior will reflect characteristics largely of the outer environment (in the light of his goals) and will reveal only a few limiting properties of the inner environment - of the physiological machinery that enables a person to think. (ibid., p. 66).

To show that there are only a few "intrinsic" cognitive characteristics and that "all else in thinking and problem solving is artificial" (ibid., p. 66), Simon analyses a familiar cryptarithmetic problem. He finds that solvers differ mainly in their solution strategies and suggests that efficient strategies could easily be taught to those subjects who do not spontaneously produce them. The "limits of rationality" are not to be found here but in the general weakness of human short-term memory, a weakness that makes it necessary for human beings to adopt compensatory strategies.

> Insofar as behavior is a function of learned technique rather than "innate" characteristics of the human information-processing system, our knowledge of behavior must be regarded as sociological in nature rather than psychological-than is, revealing what human beings in fact learn when they grow up in a particular social environment. (ibid., p. 76).

As always in reading anything by Simon, I get the sense of an immensely powerful intellect sailing on towards the magnetic rather than the true North. The clarity, however, is bracing, the ideas challenging to many of my presuppositions. I feel I am closer to grasping the nature and purpose of strategies in problem solving, for example; and the proposition that the complexity of behaviour arises from the complexity of the task and not the complexity of the organism working on the task becomes a hypothesis worth struggling to refute. But before I give in to the temptation to enlarge the first snapshot, let me change the slide.

2. A different and more alarming view of "the limits of rationality" is captured in the following sentence from Leon Brunschwicg's paper, "Dual aspects of the philosophy of mathematics":

> ... the preconceptions of an overly abstract and narrow definition transforms reason into a machine for fabricating irrationality. (Brunschwicg, 1971, p. 228)

Brunschwicg draws his theme from the Pythagoreans.

> When, by representing numbers by points, they showed that the successive addition of the odd numbers furnished the law for the formation of squared numbers, they were extracting evidence of a perfect harmony....between what is conceived in the mind and what is obvious to one's vision. (ibid., p. 225)

This "triumph of reason should have been decisive; it was immediately compromised by a twofold weakness in itself." (p. 226) On the one hand the Pythagoreans could not resist the temptation of push their luck, to go far too far. "Thus 5, the sum of the first even number, 2, and the first odd number, 3 (unity remained outside the series), would be the number for marriage because even is feminine and odd is masculine." (p. 226) And on the other hand, when the difficulty of incommensurability surfaced, the Pythagoreans turned their backs on rationality by banishing incommensurable magnitudes to a "beyond."

> They receive a command from their avenging gods to deliver to the fury of the tempest the sacrilegious member who had the audacity to divulge the mystery of incommensurability. (ibid., p. 227)

They implicitly - and the more dangerously because of the implicitness - decide that incommensurability will be "something that one does not dare to speak of" and so, Brunschwicg says, "the irrational threatens to obscure the whole philosophy of science." (p. 227)

From a rich and subtle paper I select another example.

> Pascal and Leibniz seem to be working together to force open the doors of mathematical infinity. But is this to be done by pushing beyond the normal resources of reason? Leibniz parts company with Pascal on this fundamental issue. He returns to the path of Cartesian analysis, while Descartes and Pascal find themselves united in their opposition to Leibniz's position that the deductive process is self-sufficient. The two of them have proclaimed the primacy of intuition, even though they otherwise give it a radically different meaning. (ibid., p. 232)

All three mathematicians reject the position that mathematics is a natural system reduced to its ultimate abstraction; for them "it is the fitting prelude to, and the relevant proof of, a spiritual doctrine wherein the truths of science and religion will lend each other mutual support." (ibid., p. 233) Not every mathematician, of course, chooses this same path.

Brunschwicg's general message is that there are fundamental characteristics of mathematical thought that underlie the disagreements among mathematicians about the sovereignty of reason, and that undercut all dogmatisms that would place the limits of reason "here" or "there." Fortunately for mathematics "the manner of investigation has no bearing on the value of a discovery." (p. 234) As to this, I can't be sure; meanwhile I retain that particular image of the Pythagorean machine, reason gone mad, spewing forth irrationalities. The image resonates unnervingly.

3. Less unnerving, but decidedly unsettling, is the drift of Dick Tahta's article, "In Calypso's arms", (For the Learning of Mathematics, 6, 1). Did mathematics originate in commerce of ritual?

> There was a time, for instance, when historians of mathematics would very confidently assert that mathematics began in the needs of highly organised social systems to calculate taxes and to deep inventories. In a less confident economic climate, there has begun to be some cautious speculations about other origins. (Tahta, 1986, p. 17)

We have no records to tell us unequivocally how mathematics began, and just as in other cases where we don't know the "facts", we construct "myths." Even the procedures and purposes of the high culture of Greek mathematics, about which we may feel we know a lot, remain essentially a matter for conjecture.

> For the purist, there is almost nothing that can be said about the early classical period with any certainty. We know the names of a handful of individual mathematicians ... (The) arithmetic tradition (of the Pythagoreans) is mainly interpreted from commentaries written <u>several centuries</u> later. (ibid., p. 18)

Tahta goes on:

> Such aspects have been mythologised to such an extent that it hardly seems relevant to question whether they describe what was the case. This is, however, to accept a view that "narrative" truth, or myth, is - in some situations - more important than historical truth; it is to accept willingly that myths grow by accretion, so that, for example, what people have thought about Greek mathematics may become part of the history of Greek mathematics. (p. 18)

When alternative myths are available, as they are for the origins of deductive geometry, say, which shall we choose? There is no real possibility of settling the question objectively. "It is, I claim, a question of preferred myth." (p. 21) Some myths may work better than others, especially for pedagogical purposes, and it is sensible to choose, openly and knowingly, those myths that are most powerful and helpful. Historians will naturally disapprove but the continuing reflective generation of the account mathematics gives of its own history is too important to be left solely to historians - or to mathematicians. Teaching is part of the mathematical enterprise and teachers can help decide what is to be considered significant at any one time. (ibid., p. 22)

It is, indeed, unsettling to suggest that reason cannot lead us to the unique right answers to questions about the nature of Pythagoreanism, the origins of deductive proof, the purpose of the arithmetisation of analysis ... or whatever. Well, we shall just have to be as brave as we require our students to be when we would prise them away from their treasured beliefs in the unique rightness of solutions to mathematical problems.

4. Consider the words of the title.

LIMIT → ULTIMATE
→ BOUNDARY
→ OBSTACLE

RATIONALITY → REASON
→ REASONABLENESS
→ RATIONALISATION

The alternatives seem to run from "high" to "low". This is particularly obvious in the second case. "Reason" cries out for a capital letter: for some it is the greatest of the mental powers, the characteristic that makes a human being human. "Reasonableness", on the other hand, is moderate and modest, a characteristic of the ordinary man, whether in the street or on the Clapham omnibus. "Rationalisation" is a low form of reason, a misprision of reason's power to grasp phenomena and make them comprehensible.

Rationality reminds us of the sober virtue of getting things "in proportion". Is it a coincidence that intelligence tests are full of questions of the form, "A is to B as C is to ?"? On the other hand, being rational may be no more than exhibiting common sense. It is this latter connection that supplies the essential social and consensual flavour. Rationality is an endowment of all human beings in the sense that everyone has the possibility of learning to be rational just as everyone is born able to acquire a spoken language, but the particular form of rationality (i.e. common sense) that a person acquires is determined by social and cultural factors as is the particular language that the person learns to speak.

5. David Bloor, in a speculative article contrasting Hamilton's and Peacock's views on the essence of algebra, talks of Hamilton's involvement with Idealism, which he learned mainly from Coleridge and Carlyle.

> Carlyle ... goes on to explain precisely how Idealism has a practical bearing ... By making matter dependent on mind, rather than something in its own right, Idealism removes the threat of a rival conception of Reality. (Bloor, 1981, p. 208)

In Carlyle's view, all conclusions of the Understanding have only a relative truth: "the Understanding is but one of our mental faculties. There is a <u>higher</u> faculty which transcends the Understanding and gives us contact with non-relative and non-dependent

Absolutes." (ibid., p. 209) This higher faculty is, of course, Reason which, in Carlyle's words, should

> "conquer ... all provinces of human thought, and everywhere reduce its vassal, Understanding, into fealty, the right and only useful relation for it."

This elevation of Reason to the level of the sacred (echoes of "which passeth all understanding"?) has powerful social and political implications, but I will not follow that track here. Bloor suggests that in relating algebra to our intuition of pure time, Hamilton was attempting to raise algebra to the level of the holy too.

> The essence of algebra was given a direct association with the Reason, with what was prior to and determined the form of experience. At the same time it was thereby put in close proximity to our insights into moral truths and their divine origin. In a word, Hamilton was irradiating algebra with spirit. (ibid., p. 216)

In the controversy between British mathematicians about the nature of algebra, Hamilton took neither the side of Frend, for whom algebra was universal arithmetic, nor the side of Peacock, for whom algebra was a symbolic system with arbitrary rules, but implied that "its essence was derived from the laws and constitution of the mind itself - and the most exalted part of the mind at that." (p. 217)

It may be arguable whether this last proposition necessarily belongs to Idealism or not, but the whole story (which I have not been able to offer here) suggests that attempts to give Reason an autonomous role, a position above all conflict, safe from refutation, only succeeds in embedding it the more firmly in a local, contingent, metaphysics.

6. In "Reflections on gender and science", Evelyn Fox Keller says:

> I argue that we cannot properly understand the development of modern science without attending to the role played by metaphors of gender in the formation of the particular set of values, aims, and goals embodies in the scientific enterprise. (Keller, 1985, p. 43)

At around the time of the foundation of the Royal Society, intellectual history could be described schematically in terms of two competing philosophies: hermetic and mechanical: "two visions of a "new science" that often competed even within the minds of individual thinkers." (p. 44)

> In the hermetic tradition, material nature was suffused with spirit; its understanding accordingly required the joint and integrated effort of heart, hand, and mind. By contrast, the mechanical philosopher sought to divorce matter from spirit, and hand and mind from heart. (ibid., p. 44)

The founding of the Royal Society in 1662 marked the victory of the mechanical philosophers and the defeat of the alchemists, stigmatised as anti-rationalists. The Baconian programme was adopted, and with it, the sexual metaphors in which it was expressed.

> A recurrent token of this is their Baconian use of "masculine" as an epithet for privileged, productive knowledge. As Thomas Sprat (1667) explained in his defense of the Royal Society, "the <u>Wit</u> that is founded on the <u>Arts</u> of men's hands is masculine and durable." In true Baconian idiom, Joseph Glanvill adds that the function of science is to discover "the ways of <u>captivating Nature</u>, and making her <u>subserve</u> our <u>purposes</u>." (Easlea, 1980, p. 214) (ibid., p. 54)

The last quotation suggests a clear association between scientific rationality and that act of rape. I am not sure one could wish that the hermetic alternative had entirely won, but the metaphors give an appalling indication of the social price that had to be paid for the establishment of modern science and certainly supply a motive for considering whether any of its damaging side-effects may be ameliorated. Three hundred and more years later, are we any wiser in our day?

7. The achievements of scientific rationality may seem so substantial that we choose to forget its tendency to tip over into irrationality. The process is more apparent in the human sciences where the danger of pushing rationality too far and forcing it to tip over is only too obvious. Or should be.

Pedagogy provides an illuminating example. It is a reasonable pedagogical principle to break up what is to be learned into manageable pieces; but this principle becomes an absurdity when <u>everything</u> presented to be learned is broken into separate pieces, each as small as possible, so that the totality cannot be perceived. It is a reasonable pedagogical principle to guide students in such a way that the do not fall into egregious error; but this principle tips over into foolishness when it becomes an attempt to prevent students from making any mistakes, denying them access to an important source of feedback. It seems to me a legitimate matter for rage and the gnashing of teeth when <u>teachers</u> (ha!) and <u>educators</u> (ha!ha!) close their minds to the irrationality of their actions. In my more pessimistic moments I fear that the educational system will <u>always</u> manage to pervert <u>any</u> rational principle in short order by pushing it further than it will stretch.

Of course, for many people, including a lot of teachers and educators, pedagogy has a dubious existence. They don't believe teaching is an activity one need be, or can be, scientific about. But teaching is not a transparent process for transporting something from place A to place B; it is not a catalyst, facilitating learning without influencing it. Consider how one may introduce students to, say, the solution of simple linear equations in algebra. The metaphor of the balance may suggest certain operations on an equation while making others, algebraically just as important, seem implausible. It is well known, that the "think of a number" approach and the "unravelling" technique it suggests work admirably for equations with a single appearance of other unknown but fail to give a lead to the solution of, say, $5x = 3x + 6$. On the other hand, the Dienes method of representing both sides of a linear equation with suitable pieces of wood gets around the particular limitation of the "think of a number" approach while introducing another obstacle: that of regarding two manifestly different amounts of wood as representing two equivalent algebraic expressions.

All pedagogical devices cast their imprint on the matter they are designed to teach. And in case one would be so naive as to suppose that this difficulty might be avoided by suppressing pedagogical devices altogether, let us remember that when we teach anything to someone who does not yet know it, we cannot proceed without the offering the person at least an implicit model of what is to be learned.

The need for pedagogy comes from another source too. There is an inevitable tension between engaging with mathematics in order to use it and engaging with it in order to teach it. The teacher and the mathematician do not have the same professional insights into mathematics; what is illuminating for one is not necessarily so for the other. The Hindu-arabic notation, when it reached Europe, played hell with the teaching of arithmetic, causing teachers to substitute "ciphering" for the counting and manipulation of beads and other objects. (Smith, 1900) Giving the number system a solid foundation in set theory was a liberation for mathematics and an aberration in the classroom. The HP 28C is a remarkable mathematical aid, but it is not the calculator that educators would like to have been able to design to sort out some of the difficulties for the learner of college mathematics. Indeed, what is best for mathematics and the mathematician is not always best for teachers and would-be mathematicians.

8. In coming to the end of this magic show, it seems appropriate to ask whether rationality is an instrument of human liberation or of human enslavement. To the extent that rationality is institutionalized and embedded in a specific culture, it has the power to be both. As Jules Henry puts it:

> Thus, the dialectic of man's effort to understand the universe has always decreed that he should be alternately pulled forward by what has made him homoinquisitor and held back by the fear that if he knew too much he would destroy himself, i.e. his culture. So it is that though language has been an instrument with which man might cleave open the universe and peer within, it has also been an iron matrix that bound his brain to ancient modes of thought. And thus it is that though man has poured what he knows into his culture patterns, they have also frozen round him and held him fast. (Henry, 1960, closing passage)

Henry, as always, stresses the negative side of the evolutionary dialectic. However difficult it may be to bring about certain shifts, nevertheless new knowledge can be constructed, language does gradually change, and cultural patterns are transformable. Past achievements are indeed a potential obstacle to future achievements. But that poses the challenge: to break the grip of past knowledge, fight the hegemony of language, and evade the restrictions of one's culture. One can't always win, but one won't always lose. These constraints are all inside us, in the mental schemata we have formed out of the experience of living in our world. As Bartlett reminds us, we have the power to "turn round upon our own schemata". (Bartlett, 1932, p. 301) That is what human consciousness is for.

References

Bartlett, F.C. (1932) Remembering: a study in experimental and social psychology. Cambridge: The University Press.

Bloor, David (1981) Hamilton and Peacock on the essence of algebra. In: H. Mehrtens, H. Bos and I. Schneider (eds.) Social history of nineteenth century mathematics. Boston: Birkhauser.

Brunschwicg, Leon (1971) Dual aspects of the philosophy of mathematics. In: F. LeLionnais (ed.) Great currents of mathematical thought. Volume 2. New York: Dover Publications. English trans. of the 1962 edition of Les grands courants de la pensee mathematique. Paris: Librairie Scientifique et Technique.

Henry, Jules (1960) A cross-cultural outline of education. Current Anthropology 1, 4.

Keller, Evelyn Fox (1985) Reflections on gender and science. New Haven: Yale University Press.

Simon, Herbert A. (1981) The sciences of the artificial. Second edition. Cambridge, Mass: The MIT Press.

Smith, David Eugene (1900) The teaching of elementary mathematics. New York: Macmillan.

Tahta, Dick (1986) In Calpyso's arms. For the Learning of Mathematics 6, 1.

PANEL

THE PAST AND FUTURE OF
CMESG/GCEDM

Moderator:
Bernard R. HODGSON
Université Laval

Le passé et le futur de GCEDM/CMESG

Bernard R. Hodgson
Université Laval

A l'occasion de cette rencontre *10e anniversaire* du Groupe canadien d'étude en didactique des mathématiques, le Comité exécutif du Groupe a pensé inscrire au programme une table ronde visant à faire un bilan des activités du GCEDM au cours des dix dernières années et à tracer des perspectives d'avenir. Quatre invités ont donc été appelés à présenter leur point de vue à ce sujet.

Le fait d'organiser une discussion comme celle-ci fait prendre conscience, si besoin en était, de la diversité et de la richesse des thèmes auxquels se rapportent les activités du GCEDM. Ces themes peuvent être abordés tant du point de vue du mathématicien que de celui du didacticien des mathématiques; à la fois en tant qu'enseignant et en tant que chercheur; en rapport avec l'enseignement aussi bien au niveau primaire ou secondaire que post-secondaire; soit comme universitaire, soit comme conseiller pédagogique à l'oeuvre dans les ecoles; etc. Cette diversité de points de vue relève de l'esprit même de notre Groupe et contribue a son caractère original.

C'est en tentant de refléter tant bien que mal une telle diversité que les quatre panelistes ont été choisis, chacun étant bien sûr libre de déterminer quels aspects des activités du Groupe il voulait souligner ainsi que le point de vue qu'il comptait adopter. Ces invités connaissent très bien les activités du Groupe pour y avoir participé depuis de nombreuses années, certains même depuis les tous débuts. Les textes qui suivent contiennent l'essentiel des commentaires qu'ils ont livrés lors de cette rencontre *10e anniversaire*.

Les invités ayant pris la parole lors de la table ronde étaient (dans l'ordre)

Tasoula BERGGREN	Department of Mathematics and Statistics Simon Fraser University
Charles VERHILLE	Faculty of Educaton University of New Brunswick
John POLAND	Department of Mathematics and Statistics Carleton University
William C. HIGGINSON	Faculty of Education Queen's University

PANEL PRESENTATION OF
Tasoula BERGGREN

Department of Mathematics and Statistics
Simon Fraser University

I would like to begin with a story from The Greeks told by H.D.F. Kitto.

> *"Xenophon tells an immortal story which can be retold here. An army of vicitorious Greeks were on their way home after a battle without a leader, paymaster or purpose. It was a group of people who wanted to go home but not through the whole length of Asia Minor. They had seen enough of it already. They decided to go North. They had hopes for reaching the Black Sea. They got a leader, Xenophon himself and held together week after week. They marched through unknown mountains and encounters with many natives, but they survived as an organized force. One day, after climbing to the top of a pass they all shouted:* **"Thalatta, Thallatta"** *the greek word of* **"sea."** *They were excited when they pointed North. A long nightmare was over. There was shimmering in the distance salt water; and where there was salt water Greek was understood. Their way home was open. As one of the* **The Ten Thousand** *said,* **"we can finish our journey like Odysseus, lying on our backs."**

The above reading was the story of a mercenary army concerning an incident in The March of The Ten Thousand. Xenophon described it in his book Kyrou Anabasis, which translates as Expedition of Cyrus. This story seems to me relevant to the history and future of CMESG. Like those Greeks we have been together for many years, we are a group of people on our way to better mathematics education.

For ten years we have gathered for the CMESG meetings, with their inspiring talks, workshops covering all aspects of mathematics education, panels, discussions about the talks, the past and the future. Like Xenophon's band we have chosen good leaders and we have loyal followers. Over the past ten years we have built a strong and dedicated group. A group of people consisting of those who have participated year after year by their presence, people who have put in time to administer and organize these meetings, and people who have shared their classroom work and research. Our members worked, produced and searched together.

But also what was good about the conference is that it presented to us ideas. The speakers conveyed to us exciting experiences and explorations and a fine search for better mathematics education. I generally like our meetings, I like their form, and I would like to see us continue and expand. CMESG is the kind of a conference which leaves me enthusiastic, refreshed and with renewed inspiration towards better mathematics education. I think, however, that we need to go beyond the stage of talk and translate all we have learned into action.

In the past we shared a lot of thoughts about various aspects of mathematics education and we shared thoughts about mathematics, its curriculum and the impact of high technology. I come to this meeting with concerns. We live in a time when technological changes -- from small calculators to microcomputers -- have a highly visible influence in the lives of our students. CMESG has long recognized the potential impact of high technology on mathematics education.

> In fact in 1982 Working Group A discussed **"The influence of Computer Science on Undergraduate Mathematics Education."** with Bernard Hodgson and Tony Thompson.
>
> In 1984 **"The Impact of the Computer on Undergraduate Mathematics"** was the subject in a panel discussion with Peter Taylor, John Poland, Keith Geddes, and George Davis.
>
> In 1985 the group led by Bernard Hodgson and Eric Muller participated in a workshop with a similar topic on **"The Impact of Symbolic Manipulation Software on the Teaching of Calculus."**

We all worked hard and thought deeply about the consequences of high technology for mathematics education. Yet, five years later, our group has no definite stand on the subject. We still have no definite conclusions about whether programmable calculators are acceptable in the classroom, or if computers are part of our curriculum.

Our invited speaker, Professor H. Wilf, five years ago in his article on **"Symbolic Manipulation and Algorithms in the Curriculum of first two years"** gave a list of topics that are often taught and that could be done on a little symbolic calculator of the future. Five years ago it was a 3" X 8" flat imaginary object. Today the dream of a symbolic calculator is a reality.

And now just as the victorious Greeks knew that where there was salt water Greek was understood and, therefore, the way home was open, we must recognize that where there is technology, mathematics is understood, and, therefore, the way to better mathematics education is open.

As a result of the 1982 Working Group A, discussions I mentioned before, Adler has already recommended that:

> "CMESG Study Group must try to find what computer knowledge our students should have, identify the mathematical ideas which generate this knowledge, debate whether this mathematics should be included in our curriculum, how and at what point."

Today, I would like to recommend that CMESG form a group, a group of volunteers who are willing to make recommendations on the changes in calculus, so that technology comes in. We need to identify specifically what goes out and what comes in. It is time to be ready for these big changes, and CMESG must play a leading part. I am sure that somewhere someone is already working to produce the right texts with the changes needed. I would like *us* to recommend the changes in the curriculum. CMESG should lead the way in the revision of Calculus. We have already done a lot of the work and now what is needed is to collect together all the opinions and take a specific stand on this subject.

For example, as a group we must know if we are going to continue teaching integration techniques. Is it fair to our students to spend their time memorizing? Perhaps, with computers doing the drudgery, we can ask: Is this the time to emphasize proofs? Can computers help our students understand the course material? How do we immerse our mathematics students in high technology?

I have assumed that much of our attention for university and college level should be focussed on the calculus, but I know some have said that calculus does not have to be gateway to university mathematics. For too many students, the critics say, it has been a gateway like that to **Dante's <u>Inferno</u>** with the words emblazoned on it **"Abandon Hope, all ye who enter here"**.

These critics argue that number theory or combinatorics, or some other area of finite mathematics would be more suitable. In my opinion, however, elimination of calculus may be risky to mathematics education. Sherlis and Shaw say that **"A mathematician's calculus course can serve as an excellent introduction to mathematical thinking"**. I claim that mathematical maturity also can be a direct consequence of calculus courses.

We all know that the physical sciences, computing science, and even economics all require their students to take calculus courses because they need it. For example, my own son, who is studying engineering, has told me how much he wished he had had a lot more calculus a lot earlier in his studies. Other disciplines such as business administration use calculus as a screening device. The verdict is in. Calculus is important for a student as early as she or he can get it. It follows that re-thinking its teaching in light of the new technology is a matter of great importance. Can we use computers and calculators to aid us? This group of volunteers which I recommend we form will choose the way to incorporate **high technology**. It will take calculus and reform its teaching. It will choose and plan curricula even planning model lectures if necessary.

CMESG can start conducting pilot projects of incorporating computers into the mathematics curriculum, encouraging teachers to experiment using computers and to compare results. We should be setting up a network with computer conferencing functions which allows members to send and receive data. Using electronic mail we can reach the mathematics teachers across Canada. CMESG must provide guidance and leadership.

Another and last point I have to make, which we can also do as a group is that we must raise the level of mathematics interest among our first and second year university students. Our best students come to us excited about their good scores in Euclid or AHSME. They are enthusiastic about mathematics competitions, and then what do they get in their first year of university? The Putnam examinations. Disaster!

So I recommend that we organize across Canada competitions or a paper on mathematics, even a group project on mathematics and its history for these undergraduate students. The level of the competitions must be somewhere between high school and Putnam examinations. The paper can be an expansion on ideas of theorems and problems from undergraduate mathematics, while the group project can be something similar in which students are working as a team. Such activities will give the opportunity to members of our group to work together, to gain support of many more members, increase the membership for CMESG and raise the level of our mathematics education. It will also bring together undergraduates, create a further inducement for our calculus students to do mathematics. It will demonstrate our commitment to encouraging excellence. It is time for CMESG to become the leading force in mathematics education.

I would like to thank Dr. J.L. Berggren, Ms. M. Fankboner, and Dr. H. Gerber of the Department of Mathematics and Statistics at Simon Fraser University for many helpful suggestions.

PANEL DISCUSSION OF
Charles Verhille

Faculty of Education
University of New Brunswick

When the Working Group on Methods Courses for Secondary School Teachers began the working group leaders, David Alexander and John Clark distributed the following letter from Mathematics Teaching by Chris Breen:

> I recently tried to imagine myself an anthropologist studying the now extinct society of South Africa in the mid 1980's. The only records that survive are some mathematical textbooks. What picture would I construct?

As a teacher education I immediately thought that this would be an interesting task to give either pre or in-service teachers to do. Over the years at CMESG gatherings I have gathered several gems like this and use them regularly to enrich my teaching as well as provide interesting environments for reflecting on mathematics teaching.

My first CMESG meeting was also here at Queen's in 1979. That meeting occurred shortly after my first acquaintance with Bill (Higginson) and David (Wheeler) who were both major speakers in Fredericton at a gathering of Maine and Maritime mathematicians. This is my third CMESG visit to Queen's in the elapsed nine years. From my first meeting I noticed that people became excited during our gatherings. Our meetings are professionally stimulating and the approach refreshing. On numerous occasions over the years, various people have suggested these as well as the following for their continued attachment to CMESG:

our small size
working groups
active participation
the people
the guest speakers

Because many of our group cherish these attributes, including myself, I am not about to suggest a future that would alter this image in any significant way. But I do believe that it is time for us to emerge above ground and take an active, visible role. In that regard, I would make four additional suggestions for future directions of the CMESG.

First, that the CMESG actively undergo a moderate increase in membership to give us a more representative national image. Currently we are not represented by several provinces. Also, a modest increase would provide the possibility for better financial stability.

Second, that we assume a lobbying role for mathematics education in Canada. We are possibly the only existing group that has a (or at least potential) national face that can legitimately address issues related to mathematics education.

Third, that this group has an opportunity, even a responsibility to offer its assistance to ICME 7 which is currently in the development stages for Laval.

Fourth and last, the CMESG occasionally makes quiet noises about publication. But other than the proceedings, nothing happens that is directly identified with the group. Certainly the group may very well play a catalyst role in this regard by stimulating individual or collaborative efforts. A publication group at an annual meeting structured on the same style as the working groups may be a workable format to try.

Thank you.

PANEL PRESENTATION OF
John Poland

Department of Mathematics and Statistics
Carleton University

Since CMESG meetings are a highlight of the year for me, I'm obviously not here to rock the boat. I've been to CMESG meetings which were a continuous high for me from breakfast to after midnight every day, bouncing ideas off people, people who care, like I do, about good teaching, what it is and what it is about. Rare people, who have thought deeply and carefully about many aspects of education. People with the same ideas as I and people with what seem to be very different ideas, people willing to test me ("maybe calculators and symbolic programming won't have <u>any</u> effect on the classroom situation" - "maybe it's better <u>not</u> to target special help to woman in math classes") and people willing to support me.

I teach in a university setting, a department of 35 mathematicians, with classes from 5 students to over 200. Most of my colleagues fit into and believe in the standard framework: respect for good research = original contributions to the subject matter of mathematics. They believe in objective criteria: objective in judging the value of research ("how important is this?", "how often do you publish?"); objective in evaluating textbooks ("does it cover the material?"); objective in evaluating their students: ("did they pass the final exam? Did they know the proofs and the methods of the course?"). And so good math teaching = clarity of exposition + coverage of the topic, very objective criteria. What then do we teach: the tools of math to the unwashed masses who will never really understand the glory of math (or what we do), and initiation to a few disciples. My colleagues believe in mathematical <u>talent</u>, something that no amount of hard work can compensate for. A university mathematician who cares about teaching can be a lonely figure in this milieu, not only isolated (with <u>very</u> few like-minded colleagues in the same department) in the quest for good teaching, but also attempting to grow and find self worth in a hostile medium of publish-research-mathematics-or perish. Of course, this individual may react in the very way one classically sees many woman and blacks react to their oppressive environments - producing the super, all-round mathematician who publishes excellent research, effective in administration, growing and interest in excellence in teaching and concerned with education issues. Even then, one's colleagues in the mathematics department say; "Yes, but imagine what research you could really do if you didn't waste your time on education issues". So once a year I get a chance to explode: to come to CMESG, rattle ideas off dozens of like-minded caring individuals, who know how badly mathematics can be and is taught, at all levels. And receive their support and well wishes.

Let me end my ranting and briefly turn to two concerns that I would like to see CMESG in future address. One is my desire to see CMESG promote good mathematics teaching more broadly in Canada. CMESG is an umbrella for its members, and this umbrella played its (minor) role, at least as a network of support, in Claude Gaulin's superb efforts to land the 1992 ICME meeting for Canada. My other desire is to see a modest extension of the time spent in working groups, to an extra 90 minutes, perhaps as at Memorial University on the initial evening of the meeting in 1986.

WORKING GROUP "A"

METHODS COURSES FOR SECONDARY TEACHER EDUCATION

David Alexander
University of Toronto

John Clark
Toronto Board of Education

CMESG 1987

REPORT OF WORKING GROUP "A"

Members of Workgroup "A":

Hugh Allen, Queen's University
Rick Blake, University of New Brunswick
Charlotte Danard, University of Rochester
Claude Gaulin, Université Laval
Harvey Gerber, Simon Fraser University
Bill Higginson, Queen's University
Lars Jansson, University of Manitoba
Tom Kieren, University of Alberta
Erika Kuendiger, University of Windsor
Bob McGee, Cabrini College
Lionel Mendoza, Memorial University
Barbara Rose, University of Rochester
Charles Verhille, University of New Brunswick
Leaders: David Alexander, University of Toronto
John Clark, Toronto Board of Education

Methods Course for Secondary Teacher Education

The group began by identifying issues related to the role of teachers of mathematics in the secondary schools. This led to a list of needs to be met by the combination of in-service and pre-service courses.

Some of the issues identified were:

- risk taking;
- critical thinking;
- examination of student learning with its relationship to diagnosis and remediation and "student talk";
- examination of personal beliefs both of students and of teachers;
- evaluation of program and of students with related assessment;
- the study of mathematics both as process (including problem-solving, use of technology, and values education) and as product (including a study of new content, a re-examination of previously studied content from a variety of perspectives, and the impact of technology);
- the features of methodology such as questioning techniques, planning and execution of teacher-centred and student-centred lessons;
- the awareness of curriculum change and the understanding of the process of curriculum implementation.

The group also identified the contextual forces which influence teacher education.

- pre-service/in-service structure
- practice teaching pattern
- background of participants in mathematics courses
- participants' instructional experiences
- tyranny of mathematics texts
- influence of associate teachers
- curriculum implementation philosophy of Ministry/Board
- influence of unions

We considered the differing needs of pre-service teachers and in-service teachers both as perceived by them, and as perceived by us and also the differing resources that each has:

pre-service: time to interact with individual students; opportunity to search for resources.

in-service: opportunity to try ideas with a class; opportunity to relate newly introduced theory to previous experiences.

This led to the sharing of the introductory activities employed in the pre-service mathematics education courses for secondary teachers offered by members of the group.

Such activities included:

- role playing
- mini-lessons to peers
- lessons to high school students
- diagnosis and remediation experiences
- simulated evaluation experiences
- assignments, tests, or simulated teaching experiences to raise awareness of weaknesses in content
- problem-solving activities
- activities to involve students in styles of questioning and basics of lesson planning

Further sharing of ideas related to more long-term strategies:

- journal writing
- modules produced by "editorial boards"
- technology as a classroom aid
- technology as a medium in the production of modules

Three specific problems were identified which require further study:

1. In most one-year pre-service programs for secondary teachers there is insufficient time to deal with all the issues in sufficient depth, yet there is no assurance that teachers will receive any further opportunities to extend their knowledge in a systematic way.

2. The role and status of associate teachers needs to be enhanced so that the benefits that their potential contribution to the development of pre-service and practising teachers is realized.

3. Practising teachers need opportunities to upgrade their knowledge and skills in relation to the use of technology, content, and process components.

The group recommends that future meetings include study groups which over a number of years would address:

1. in-service teacher education for elementary school and secondary school mathematics

2. mathematics education component of elementary teacher pre-service education

3. mathematics education component of secondary education (i.e. don't wait 10 years for another run at this topic)

We also recommend that a topic group next year might well focus on:

Writing in the classroom (We would like to hear more of Barbara Rose's experience with journal writing).

<u>Resources</u>:

Morris, R. (ed.), Studies in Mathematics Education. The Education of Secondary School Teachers of Mathematics, Vol. 4 UNESCO, 1985

Fullan, M. and Connelly, F.M., Teacher Education in Ontario: Current Practice and Options for the Future. A Position Paper written for Ontario Teacher Review. Ontario Ministry of Education, 1987

Johnson, D.A. and Rising, G.R. Guidelines for Teaching Mathematics (2nd edition), Wadsworth, 1972

Skemp, R.R. The Psychology of Learning Mathematics. Penguin, 1971

Farrell, M.A. and Farmer, W.A. Systematic Instruction in Mathematics for the Middle and High School Years. Addison-Wesley, 1980

Osborne, Alan (ed.) An In-Service Handbook for Mathematics Education, N.C.T.M., 1977

Corbitt, M.K. (ed.) The input of Computer Technology on School Mathematics: Report of an N.C.T.M. Conference, Mathematics Teacher, April 1985, pp. 243-250

Taylor, Ross (ed.) Professional Development for Teachers of Mathematics - A Handbook. N.C.T.M., 1986

WORKING GROUP B

THE PROBLEM OF FORMAL REASONING IN UNDERGRADUATE PROGRAMS

David Henderson
Cornell University

David Wheeler
Concordia University

Working Group Title: The Problem of Formal Reasoning
Starting Premise: Rigour and formalism are powerful tools of mathematics but are not the goal of mathematics. The goal of mathematics is understanding and meaning.

Working group B was attended by 18 people. There follows personal re-ports by 9 of these participants. Some quotes were read which got us started:

> In mathematics, as in any scientific research, we find two tendencies present. On the one hand, the tendency toward abstraction seeks to crystallize the logical relations inherent in the maze of material that is being studied, and to correlate the material in a systematic and orderly manner. On the other hand, the tendency toward intuitive understanding fosters a more immediate grasp of the objects one studies, a live rapport with them, so to speak, which stresses the concrete meaning of their relations. David Hilbert (from the introduction to the book Geometry and the Imagination by Hilbert and Cohn-Vossen).

> It is impossible to understand these definitions (of continuity) until you already know what continuity is. R.H. Bing (Elementary Point Set Topology, Slaught Memorial Paper #8).

> It is in the intuition that the ultima ratio of our faith in the truth of a theorem resides. ...The evidence leading to persuasion results from having a sufficiently clear understanding of each symbol involved, so that their combination convinces the reader. ...No elaborate axiomatic structure or refined conceptual machine is needed to judge the validity of a line of reasoning. Rene Thom ("Modern" Mathematics: An Educational and Philosophic Error?, American Scientist 59 (1971), 695-699.

During the working sessions we discussed and compared the formal and non-formal in mathematics. The table on the next page is a summary of this discussion and comparison and was prepared during our discussions.

NON-FORMAL	FORMAL
Intuitive	Abstract
Can present the whole	One meaning at a time
Expressing what is seen	Emphasizes or reduces to logic
Living	Mechanical
Multiple definitions	Logical consistency
Immediate and direct understanding	Reduction to an (axiomatic) basis Fear of inconsistency
Direct seeing	Connectional understanding
Analogue and pictorial	Digital and logical
Student is the producer and has more of a sense of power	Student can reproduce without understanding
Misconceptions are illuminated	Misconceptions are obscured
Interactive	Is bound to a specific formal context
Is imprecise and therefore shows the need to be careful	Is precise and therefore often illusion of safety
How mathematics is discovered	How mathematics is presented
Is difficult to apply in some areas	Extends intuition into other areas
Friend	Enemy
Trust	Fear
Intimate	Distant

It was suggested that these differences along with the emphasis in schools on the formal may be a major factor in the lack of participation of women in mathematics today.

FEAR, SAFETY AND DANGEROUS THINGS: REASONS FOR BELIEF

David Pimm, The Open University, England

Residues

As my colleague John Mason is fond of commenting, there is a dangerous myth going around that people learn from experience. He prefers to assert that the best that can be claimed is the possibility of learning from reflecting on experience. Pearla Nesher gave an evocative instance of a classroom where 'we learn from out mistakes' was apparently a shared belief, but pupils were unable to say what it was that they had learnt. There is a related (possibly apocryphal) story in the LOGO community about the nine-year-old who had learnt that the response "I'm debugging procedures" to the question "What are you doing" was very effective in causing an adult to pass on to bothering someone else.

Rather than attempt to give an account of (or even to try to account for) what happened in the Working Group on Formal Reasoning, I have chosen to offer some of the residues that I came away with for reflection and therefore possible learning.

Words

The first level of sediment is provided by individual words, evocative and potent, forming into clusters.

logical, rigorous, abstract, explicit

cleanse situation of extraneous elements (contaminants, hygiene)

generality, power, precision, ambiguity, clarity.

But it was also possible to listen behind the words that were being spoken to how they were being used to convey other, more subtle and covert meanings. It is possible to hear values expressed in the tone with which some of these words were being uttered. Such tone-rich talk is one of the feeble ways in which mathematicians communicate the values of part of the mathematical community; that is, the way mathematicians talk about what they do.

External Quotations

These were some of the thoughts of others outside the group that were offered either as support for claims or as starting points for discussion. 'Rigour is not the enemy of understanding' David Hilbert.

'If certainty is not to be found in maths, then where is it to be found' David Hilbert

'It is impossible to understand these definitions (of continuity) until you already know what continuity is' R.H. Bing.

'The Greek style of proof involves drawing the right picture and then saying the right things about it' David Fowler.

Internal Quotations

Things always are left out of proofs - frequently on the choice and judgement of presenter.

Formal proofs can fool students into thinking they can be safe.

Our culture makes certain ways of mathematising accessible and others inaccessible.

One powerful but dangerous practice of mathematicians, that of detaching the symbol from its referent and working solely with the symbols, is a semantic pathology.

As a student I experience a sense of power as a producer of one's own knowledge.

I study mathematics in order to learn about myself.

Some problems of formal reasoning

stability of conceptions.

working outwards from mathematicians' definitions rather than inwards from everyday language and experience.

fear of pictures. One answer to question of why formal reasoning - apply in places where no geometric intuition available.

ways to encourage students to develop discrimination about mathematical arguments.

SOME COMMENTS ON FORMAL REASONING

Ed Barbeau, University of Toronto, and Gila Hanna, OISE

That reasoning is a pedagogical issue at all bespeaks a conviction that mathematics is a dynamic rather than static process, in which student progress towards deeper levels of insight and skill. Thus, a classroom activity, including formal or informal reasoning, can be judged insofar as it enhances or retards greater understanding.

At the beginning is the naive idea, rooted in everyday experiences. To provide a base for further work, the idea must be clarified. This involves a degree of formalism. A language is created; symbols are defined; their rules of manipulation are specified; the scope of mathematical operations of delineated. Greater precision is afforded, so that the essential can be separated from the nonessential and greater generality is achieved.

However, a price has to be paid. Becoming removed from the original context, the student loses a sense of being connected with reality and becomes a symbol pusher. Experienced mathematicians have learned to handle this danger by acquiring the ability to make mental shifts in moving among levels of generality and formalism, and to build on specific examples, drawing on only those characteristics pertinent to a more general situation under study. They exploit symbolism and algorithms to work automatically and efficiently, and yet can intervene to monitor the accuracy and effectiveness of their work.

What are the issues to be kept in mind in teaching mathematics, and in particular developing reasoning power?

1. Formalism should not be seen as a side issue, but as an important implement for clarification, validation and understanding. When there is a felt need for justification, and when this can be provided to the appropriate degree of rigour, learning will be greatly enhanced.

2. It is not enough to provide mathematical experiences. It is reflecting on experience which leads to growth. As long as students see mathematics as a black box for instantaneous production of "answers", they will not develop the necessary patience to cope with their minds' erratic paths towards grasping what the mathematics is about. One goal of pedagogy should be to help pupils maintain a level of concentration to negotiate a line of reasoning.

3. Ironically for a discipline touted as precise, the student has to develop a tolerance for ambiguity. Pedantry can be the enemy of insight. Sometimes, an explanation is better given pictorially, loosely, by example or through an analogue; sometimes distinctions are better left blurred (e.g. the various roles of the minus sign, the use of f(x) as both the function and the value of the function at x), and sometimes a symbol varies its role in the discussion (e.g. the parameter which is now held constant, now allowed to vary). At the same time, when genuine confusion might develop, the student must become conscious of looseness and apply the necessary amount of rigour. It is this judgmental aspect of reasoning,

essential in mathematics education, that should be communicated to students.

Mastering mathematics is like mastering a musical piece. There are technical and conceptual problems which must be first handled, often by isolation from the larger task (learning the notation, analyzing the key and time structure, negotiating scales and arpeggios) before the final synthesis can occur (dynamics, rhythm, interpretation).

CONTINUUM FROM NON-FORMAL TO FORM REASONING

R.S.D. Thomas, University of Manitoba

At the non-formal extreme there is the situation, the actual content of which is sometimes unclear, in which a conclusion can be immediately seen. Such a conclusion depends upon the situation's being generic if the conclusion seen is to be regarded as a general one. At the formal extreme there are chains of reasoning where the connection between successive links is clear but the wood cannot be seen for the trees; there is no inkling of the conclusion in the hypotheses and no hint of the hypotheses in the conclusion. Between the extremes there are stages making a proof more careful, more symbolic, some rigorous, more lengthy. In some examples an informal proof cannot itself be made formal, but a new tack needs to be taken, e.g., when a result seems somewhat plain but must be proved by induction. However, in some examples, many intermediate stages do exist. Wherever a specific argument lies on this continuum, it can always be taken further; at no point is absolute certainty achieved. At no point should an instructor or the instructed by entirely uncritical.

INFORMAL GEOMETRY IS THE TRUE GEOMETRY

David W. Henderson, Cornell University

In the schools today formal geometry (with its postulates, definitions, theorems and proofs) is usually considered to be the apex or goal of learning geometry. Informal geometric topics and activities which do not fit into the formal structures are often given second class status and relegated to the domain of mere motivation or help for those who are not smart enough to learn the "real thing" - formal geometry. I am a mathematician and as a mathematician I wish to argue that this so-called informal geometry is closer to true mathematics than is formal geometry. I do not believe that formal structures are the apex or goal of learning mathematics. Rather, I believe, the goal is understanding - a seeing and construction of meaning. Formal structures are powerful tools in mathematics but they are not the goal. I don't blame teachers for giving formal geometry too much emphasis; mostly I blame my fellow mathematicians because we have done much to perpetuate the rumor that formal systems are an adequate description of the goal of mathematics.

As an example consider the notion of 'straight line'. I claim that this notion is not now and never could be entirely encompassed by a formal structure. I am talking here both about the notions of 'straight line' as used in everyday language and the notions as used by mathematicians. In fact, these various notions are closely interrelated through the felt idea of straightness that underlines them all. Ask any child who hasn't had formal geometry or any research geometer and they will tell you that "straight" means "not turning" or "without bends". (Of course the research geometer is likely to mumble something containing the formal notions of "Affine connection" or "covarient derivative", but if pressed for what that means he will admit that it is a formalization of "not turning".) Now "not turning" clearly has a different meaning from "shortest distance". So both the child and the research geometer have a natural question: Is a "non-turning" path always the "shortest" path? And, if so, Why? They then look for examples of "non-turning paths". (The child can do this by imagining and/or observing non-turning crawling bugs on spheres and around corners of rooms.) They can then convince themselves that the great circles are the straight lines on the sphere. (This is not something to assume, it is something to check; and it has meaning in the sense that a crawling bug on the sphere whose universe is the surface of the sphere will experience the great circles as straight.) It is then clear that going three-quarters of the way around the sphere on a great circle is a straight path but not the shortest path. (Going one-quarter of the way around in the opposite direction is shorter.) Thus a straight path is not always the shortest. (This can also be seen in situations where it is sometimes a shorter distance to go around a steep mountain rather than to go straight over the top.) But on the sphere it is true that every shortest path is straight. So the question becomes: Is the shortest path always straight? The research geometers have proved that this is true on any smooth (no creases or corners) surface which is complete (no edges or holes) and the basic ideas of their proof can easily be conveyed to high school students. But then the child might think about a bug crawling on a desk with a rectangular block on it and notice that there are two points on either side of the block such that the obvious straight path joining these points is not the shortest path and the shortest path is not straight. These explorations, whether by the child or the research geometer, are a good example of doing mathematics (or in this case, doing geometry) and they are not encompassed by any formal system. The mathematician will use formal systems to help in the explorations but the driving force and motivation and ultimate meaning comes from outside the system. It comes from a desire which the mathematician shares with the inquisitive child - the desire to explore the human ideas of "straightness" and "shortest distance".

So, should we teach formal structures? Definitely, yes. But not in geometry. The power of formal structures does not come through clearly in geometry - it would be better to look at the formal structure of a group with its various examples in geometric symmetry groups and number theory. The emphasis on formal structures in school geometry obscures the meaning of geometry and does not in the context in which it is used add any power.

REFLECTIONS ON FORMAL REASONING

Raffaella Borasi, University of Rochester

Through reasoning seems to be an essential part of mathematics (despite what students and high school curricula seem to believe!) it was very valuable for me to realise that there are fundamentally different <u>kinds</u> of mathematical reasoning. We identified two dimensions (at least) that could be considered to this regard: informal vs. formal, and non-rigorous versus rigorous (this last being rather a continuum that a dichotomy. It was quite a discovery for me to realize that these two dimensions are distinct and rather independent, and I think it will be worthwhile to explore a bit more, conceptually, what are the similarities, differences, interactions, between them (at the moment, I'm a bit confused to this regard). In particular, I wish I had some more examples of what mathematicians would consider as "acceptable informal proofs", to analyze.

Whatever the results of the prior exploration, through the discussions in the working group I have come to realize the almost total absence of informal reasoning in the math curriculum, and I think something should be done to change this situation. I don't buy the argument that students should learn first to deal with mathematics formally (even if that bears little meaning for them) and then, almost magically, they will leap into creative mathematical thinking, which involves reasoning to which they have not been trained in or even been exposed to. If formal and informal reasoning are different kinds, I think it follows that from the very beginning students should be exposed to both. I am looking forward to reading of any experience which has attempted to introduce math students (at any level) to informal reasoning.

I'd like to come back, once again, to the affective aspects involved in this discussion. During our working group discussion, I felt that many affective reasons (including our conceptions of the nature of mathematics as well as less "intellectual" emotions) were governing the behavior of many of us. In the insistence that formal proofs were to be requested from students, one could see (more or less explicitly expressed) the <u>fear</u> of pathological cases and unforeseen circumstances which could threaten the intuitive argument, the <u>search for</u> the <u>security</u> in teaching to the students only "right" things (how would we feel, then, if we had to teach history or literature, instead of mathematics?), a <u>distrust</u> in the students' ability to recognize the relative value (i.e. limitations) of their intuitive arguments and to benefit from mistakes. How much is an emphasis on formal arguments, then, just to cover for our insecurity as mathematicians and mathematics teachers, and an illusion that we are avoiding mistakes and working only with true things? Similarly, we may question if the students who do like working with formal proofs are also showing the same lack of courage, and thus if we are hurting them, too, by fooling them that what they are doing is what researcher mathematicians do, and that mathematics is a totally true and safe environment.

LOOKING BACK

Alberta Boswall, Concordia University

In asking myself the question "What happened?" I return to the first day and my own initial question - Why is it that students supposedly trained in some kind of logical reasoning (e.g. truth tables) very often fail to use this background in their college mathematics courses. One seems to be completely divorced from the other. Are we asking too much in expecting at least some understanding from the presentation of a reasonably reasonable logical process?

As for formal and non-formal, I think that these are labels we have been trained to pin on certain proofs that follow a prescribed pattern. Not only that but also that one is more valuable (mathematically speaking) than the other. There may not in fact be an easy distinction. As for formal and non-formal reasoning there may be even less distinction. Perhaps informal is almost intuitive and not as carefully presented. When one gives a detailed written or spoken expression of informal reasoning then one hopes that we have entered, if not a formal stage then at least a more formal one. If in a classroom we try to fill in the missing gaps in either informal and formal reasoning we run the risk of becoming boring, pedantic and repetitive. We begin to debate (if only with ourselves) issues which students regard as unimportant and irrelevant.

It occurred to me at one point that perhaps the teaching of mathematics involves a successive shattering of belief systems from one level to the next. My examples are: a smaller number is always subtracted from a larger number - or if $f'(x)=0$ then $f(x)$ has extreme values at solutions.

We should be helping students to gain power and confidence in their own reasoning abilities. Can we promote this? Can we present careful reasoning reasonably? Ought we to present occasionally reasoning which seems reasonable but which leads to a false conclusion? e.g. All triangles are isosceles.

If I may paraphrase a line from the paper Proportion: Interrelations and Meaning by Avery Solomon in For the Learning of Mathematics - perhaps not every proposition has to be carefully reasoned, but certainly some should.

Students I think, have the right to expect that all propositions, if true, can be supported by logical reasoning - either formal or non-formal.

A lot of things happened in this stimulating and even exciting group.

FORMAL REASONING AND LEARNING MATHEMATICS

Constance Smith, University of Rochester

My opening thought was how does "formal reasoning" relate to making mathematics your own as a student. I believe we reached that point in the sense that we are concerned with that issue but did not finish developing the relationship. Ideas that came out were that we should (could?) build from the students' intuition to informal reasoning to finally formalizing their own mathematics. The quote from Bing concerning the definition of continuity clarified some of these ideas for me.

Other important points:

- relation to creating doubts for students and then allowing them to try to resolve them
- correlation between philosophy's sign/signifier and signifier and mathematical symbolism
- importance of developing meaning at both ends of the process; we should not leave the students with the feeling that the result has only that meaning. We must relate our results to the mathematics and the world
- the influence of the students belief system and our impact on those systems.

SOME THOUGHTS ABOUT OUR DISCUSSIONS

Dan Novak, Ithaca College

The central core of the conference was, in my view, how to make learning mathematics meaningful for students. Our group focused on formal reasoning and out of an attempt to clarify and define the concept, it happened that ways of thinking and modes of understanding of the participants have shifted and changed. It was apparent, at least to me, that a definition or formal reasoning upon which everyone could agree, was not necessary any more. What I found happening was a process of changing my views about teaching. The non-formal way of presenting materials will become more frequent in my classes and I will try to look for other ways of presenting and looking at problems myself. (Shifts of creativity). One way of looking at problems will be also the Formal Way, but one needs to come back up to the fresh air of meaning of ideas. Otherwise creativity, joy and subsequently learning will die.

WORKING GROUP C

SMALL GROUP WORK IN THE CLASSROOM

Pat Rogers
York University

Daiyo Sawada
University of Alberta

SMALL GROUP WORK IN THE CLASSROOM

Participants: Tasoula Berggren, Dale Drost, Gary Flewelling, Olive Fullerton, Malcolm Griffin, Fernand Lemay, John Poland, Marilu Raman, Pat Rogers, Joan Routledge, Daiyo Sawada, Suzanne Seager, Avery Solomon, Peter Taylor, Lorna Wiggan.

Introduction

In traditional classrooms students often learn to view mathematics as a fait accompli, something that is given rather than created. They learn that it is a collection of rules and procedures, an environment where there is only one right answer to the teacher's questions and one which, for many, causes great anxiety. Students experience little control over their own learning and are not usually encouraged to share their ideas with each other or to work together towards a common solution. Research suggests that small group learning in which students work cooperatively leads to superior achievement in problem solving and higher thinking skills, to positive attitudes towards a subject area and to great motivation to learn. The purpose of the working group was to examine small group learning strategies in the light of recent research and to identify those strategies which might be used effectively at the post-secondary level. The interests and experience of the group members ranged from elementary school to the university level. A variety of books and articles were made available during the three sessions but there was insufficient time to examine them in detail; these are listed in the bibliography.

Session #1

Pair-interviewing was used at the beginning of the first session as a means of building community in our group. Participants were asked to find a partner they did not already know and to take turns interviewing each other for five minutes to exchange roles. After this exercise the whole group reconvened and partners introduced each other to the group.

The pairs functioned in a variety of ways. Some pairs adhered very closely to the instructions and this produced some excellent introductions. Others completely ignored the instructions and found that their session became a discussion, the result being that some introductions were more about the interviewer than the interviewee. One early introduction was so good that it provided a model for the later introductions. This created feelings of inadequacy for those who felt unable to live up to the example and a certain amount of discomfort and isolation for the "excellent student." Perhaps more care should have been taken to insist on adherence to the interview format. Nonetheless, there was general agreement that this was a useful strategy for groups of people who did not already know one another, that the provision of questions helped to break the ice and focus discussion and that, omitting the final introduction phase, this might be a good way to build community in a class.

For the remainder of this session, participants worked in small groups on two cooperative tasks.

The first task was a communication situation between two students in which one produced a written or oral message for the other in order to give information essential for that student to complete an assigned task. The task used here was adapted from Laborde (1986). Partner A of the pair received a card on which was drawn a simple geometrical figure with a supplementary line of a different colour. B had a sheet with a similar but incongruent figure, a different orientation and no supplementary line. Both partners were aware of the details of the situation as described. A's task was to describe the supplementary line without drawing so that B would be able to reproduce it perfectly. The purpose of this task was to focus, in the context of a mathematics problem, on the role of language in a cooperative activity.

The second activity was one of a series of small group cooperative geometry exercises developed by EQUALS to teach students cooperative skills. The exercise we used here was a complex spatial task in which each member of a group of four people had resources essential to the group's effort to solve the problem. Four congruent hexagons were divided in different ways into four pieces. The pieces were labelled A, B, C, and D, sorted by letter into four sets and clipped together. Groups of four were formed. Each group received an envelope containing the four clipped sets and were instructed as follows:

- No talking!
- Each member of the group gets one clipped set of shapes.
- No one may take a shape from anyone else but may offer a shape to someone who needs it.
- The group is done only when all four members have completed their hexagons (Erickson, 1986).

There were various responses to this task. Undoubtedly the most difficult and frustrating part of the exercise for everyone was observing the rule of silence. One person commented that his desire to talk was so great that it created enormous tension for him and interfered with his ability to concentrate on the task. Evidently, for some, talking is a way of lessening pressure when solving problems. On the other hand, it was also quite clear that the rule of silence facilitated our learning how poorly we cooperate and also highlighted the fact that cooperative skills do need to be learned. One participant was alienated by the artificial nature of the task and found himself wishing he was elsewhere. This was likened to the way many of our students feel about textbook word problems. One group so hated the imposed structure that the members either disregarded the rules completely or else conspired to find ways around them. Nonetheless, despite the hostility expressed, or perhaps even because of it, the experience of working together on this activity evidently succeeded in fostering very strong group feelings.

We concluded this session by formulating an agenda for discussion in the remaining sessions:

1. competition;
2. time (space, size);
3. communication of mathematical ideas;
4. learning styles/teaching styles;
5. evaluation;
6. individual learning/cooperative learning;
7. open-ended/specific-ended investigations;
8. "group" theory.

Session # 2

We decided to base our discussion of the agenda above on personal experiences with small group work. There was a rich variety of experience within the group and this is summarized below.

Small Group Work Within Lecture Class

There was a brief discussion of the use of small tutorial groups, and lab/problem sessions to provide students with contact with their lecturer and immediate feedback on their understanding of concepts introduced in lectures. John stressed the importance of the physical aspects of the room he has chosen for this purpose: high tables and stools so that the students have to lean forward and engage. However, in this kind of activity the students are often working alongside one another rather than together. By contrast, Pat described her use of "pairing" during lectures whereby students form pairs to work together on a problem, to investigate or discuss an idea that has just been presented, or to generate data for ensuing whole class discussion.

Group Projects

Many of us had assigned group projects to our students. In a large statistics course, Malcolm assigns projects to group of at most four students and has found that while there are obvious administrative advantages in terms of the decrease in time spent grading, there are also a number of problems: some groups waste all the time set aside for beginning the project in trying to set a date for the next meeting; others work independently on small parts of the project and then attempt, usually unsuccessfully, to glue it all together; there are also complaints of students' "piggybacking" and these are usually left to the students themselves to handle, often in an unsatisfactory manner.

These difficulties were familiar to other members of the group and raised a number of questions. What is the purpose of a group assignment? Is learning to work in a group so important that it should be mandatory or should students be allowed to choose to work alone? Marilu observed that, in one of her graduate courses where students were given the choice whether to engage in group work or to work alone, only two out of twenty students chose to work together. Do adult learners, or any other learners for that matter, naturally choose to work together? Is group work really an artificial setup, or do we simply lack the experience of working together? Has society imposed on us the notion that we are all individuals? Daiyo observed that it is precisely when he imposes group work on his students, or prescribes the size of the groups, that problems arise. Perhaps choice is the key.

Another important question: should the product of group work be evaluated and how? In the absence of any real training in collaborative skills is it fair to grade the outcome? Most proponents of cooperative learning argue that group grades should not be assigned if this is the only grade the project will receive. There are many creative ways of evaluating group work which combine elements of teacher-evaluation, peer-evaluation and self-evaluation (see, for example, Johnson, D.W., & Johnson, R.T., 1987). In support of this view, Gary offered his wife's success with evaluating group work in his way.

Interviews

Tasoula suggested that oral interviews might be a good way to evaluate the individual's contribution to a group project. Avery has used interviews as a means for evaluating his own teaching and determining the direction of future lectures and Olive uses interviews to get to know her teacher candidates at the beginning of each year. Interviews with pairs of students seem to be most successful when used for these purposes. Both Avery and Olive observed that these interviews have had an enormous beneficial influence on the atmosphere in the class afterwards.

Structuring Small Group Activity in the Classroom

A distinction needs to be drawn between the use of group work as a pedagogical device within the classroom and the use of group projects and assignments where the activity of the group takes place outside the classroom. Our collective experience of group work within the classroom ranged widely in kind and with respect to the amount of structure which was imposed.

1. At one end of the spectrum was Gary who prefers to have teachers in his inservice workshops move freely in and out of working in pairs as the need may be.

2. Tasoula described her experience with students working together in groups of four at the blackboard. (See also Davidson et al., 1986?)

3. Fernand described in detail his experiment with a class of thirty Grade 6 girls working on mathematical investigations in geometry. The children divided themselves into two groups; these groups in turn split in two and so on until the class was divided into pairs (see also Gorman, 1969). The names of the members of each group were recorded on a large sheet of paper. The main rule established was that the child must first work on a problem alone; when stuck she may then seek out her partner in her pair; if the pair needs help they seek out their partner pair forming a quartet and so on. Theoretically this might continue until the whole class is reconvened, however, it was found that there was usually a preference for a group of four. This format is reminiscent of a rule which teachers who work with LOGO often adopt. This rule, known as "Ask three before me," requires a student, or a pair of students, to consult with three other students, or pairs, before seeking assistance from the teacher.

4. Lorna briefly outlined the basic characteristics of cooperative learning:

   - Students work in small heterogeneous groups and sit closely together so that face to face interaction is facilitated;

   - Students work in positive interdependence - this means that each group works on one assignment, with one piece of paper and pencil, or one piece of chalk if working at the blackboard and they produce one product - they are constrained to depend on each other and work together;

   - There is high individual accountability through shared evaluation. Students evaluate themselves and know in advance what skills they will be expected to evaluate themselves on. Sometimes observers are assigned to help group members assess how they are improving their group sills.

   - Interpersonal and small group skills are taught - rules for working together are established, collaborative skills may be modelled by the teacher who may also choose to work on a distinct skill each day. Group work skills, as well as content, are processed.

The objectives of cooperative group learning are two-fold: there is an emphasis on the group process as well as on the content. Initially, there is more focus on developing the social skills and less on the content. But later, as the students become more skilled in working together, the stress shifts to the content. As students become more experienced with this way of working they become more independent and the concept of the teacher as the source of all knowledge becomes meaningless. In this classroom climate the teacher is a facilitator, a guide, rather than the "expert."

Some members of the group expressed discomfort with such a structured approach to teaching. However, as Olive pointed out there is a place for individual work, for group work, for competition, and for lectures. Really skilled teachers are able to choose the strategy that suits the objectives they want to achieve.

5. Peter described his work with an extremely heterogeneous group of Grade 13 calculus students with whom he developed a process which bears much resemblance to the JIGSAW strategy described below. At the end of the course, he provided the class with a set of problems of varying difficulty from which each student had to select one. Working independently, students had to solve their own problem, write it up and have it checked by Peter. In this way, each student became an expert in one of the problems. The class then had to do as many of the remaining problems as they could, and consult with the expert for each problem. The expert would judge the accuracy of the solution, giving a hint if the solution was incorrect and a check mark if correct. The final grade for this exercise was based on the number of check marks a student received. Most earned full marks.

6. The JIGSAW strategy is a variation on cooperative learning in which everybody gets the chance to be both expert and learner. For example, groups of four, called home groups, might be formed to investigate the properties of the straight line. The students in each home group are labelled A, B, C, and D and sorted by letter into other groups called expert groups. Each expert group might work together on one specific property of the straight line. The students then return to their home groups to teach each other what they know and to synthesize all they have learned together.

   This approach concerned Gary who worried that we might simply be replacing the single "expert" teacher by a battery of experts: "whereas, in the past, the teacher has cast the student in the passive role, now you take them into groups and have them cast in the passive role in smaller groups." There was some debate as to whether this strategy makes "learning mathematics less a process of discovery and more one of finding the page where the answer is. Most of the time we might want kids put in groups when there is some ambiguity as to the outcome and many answers, and it would be inappropriate to think of experts."

Peer Teaching

Joan cautioned the group on the need for careful teacher training if we are not to replace traditional methods with chaos in the classroom. She raised the issue of whether children have pedagogical skills and observed that peer teaching, which occurs when using cooperative learning techniques, is not a challenge for many students. Undoubtedly, there is a tremendous amount of unlearning that has to take place when ignorance has been pooled. However, many members of the group felt that the language developed in group work and in peer teaching is very important. Peer teaching affords students the opportunity to bring to the material the kind of organization that is so essential to full understanding and learning.

Towards the end of this session, Avery summed up the feelings of the group beautifully: "What we need to do is to isolate those things you can do with a small group that you actually could not do with a large group. Perhaps small group work might enable students to negotiate new meanings of mathematical concepts and to make mathematics part of their own individual understanding-meanings that are invoked from within. Our purpose is to put students in control of their own learning. We want them to have the experience of doing mathematics and thinking mathematically." Peter felt that it was crucial that teachers be provided with printed material that integrates content and style of teaching. This would seem to be a vitally important task for some members of our group to perform.

Daiyo concluded the session by listing certain distinctions which had arisen during the course of our discussions:

1. social process/content control;
2. spontaneity/control;
3. cooperation/competition;
4. the individual/the group;
5. the novice/the expert;
6. the means/the end;
7. life/death;
8. receiving/giving;
9. breaking/joining;
10. the group as obstacle/the group as facilitator.

Are these dichotomies or reciprocities? Are they problems? Or can we provide a more encouraging learning environment for our students by keeping these distinctions in mind?

Session # 3

The intent of session # 3 was to return to the agenda developed during session # 1. Group leaders therefore proposed that this be done by having small groups focus on a particular agenda item keeping in mind the distinctions developed in session # 2. People then self-selected themselves into groups of two or three which met for about an hour. The large group was then reconvened and discussion was led by each of the small groups in turn. Highlights are summarized below.

1. Evaluation [Fernand and Peter]

An analogy was made between works of art (paintings) which are purchased and displayed in a home and the painting which the homeowner may engage in herself. The works of art on display may indicate that art is thriving in the home but it may be dangerous to come to such a conclusion: The art may only indicate that the people living there have money. Indeed, if the paintings are only for show, then their presence foreshadows the death of art in the home rather than its vitality. Often the way mathematics is evaluated encourages, and sometimes forces, children to "display" their acquired goods in the manner of pieces of art which they relate to only for purposes of display. As with the home, such a classroom may witness more death than life in mathematics. Group work provides the teacher with an opportunity to observe and evaluate the alive part of the mathematics displayed.

2. Competition [Malcolm and Suzanne]

In the context of group work, competition is often minimized. Yet without competition stagnation and sloth sometimes become problems. Nevertheless, competition within a group can become a subversive influence leading to the destruction of the group. How can we have competition without subversion? One possibility is self-competition in which the student competes with the problem, not with others working on the problem. "To compete is to produce." By choosing the "unit of production" (for example, a pair of students) self-competition can be blended with cooperation.

3. Communication of Mathematical Ideas [Dale and Joan]

Usually communication as a process is associated with the learning of mathematics rather than with the content itself. Yet it is vitally important to get students talking about the content with each other as well as trying to help each other learn it. In this way communication should deal with the mathematical objectives per se as well as the processes for learning those objectives. Student/teacher communication can help to get students talking about the content.

Self-communication is significant in making mathematics a personal experience rather than a strictly objective experience with "something out there." Self-communication with mathematics can therefore involve communication as a process for dialoguing with mathematics as well as with other students. Since self-communication, from the perspective of others, is often associated with silence, there may indeed be a vital role of "silence" in the mathematics classroom. This kind of silence is quite different from the usual kind which exists as a product of the teacher who commands silence. A key question: How does one achieve a balance between self-communication and communication with others in group work?

4. Learning Styles/Teaching Styles [Lorna and Marilu]

Research consistently reveals an incredible variety of learning preferences. Do we as educators "use a wide enough range of methods to accommodate all these learning styles?" Can group work help to increase the sensitivity of teachers to these differences and provide alternate ways of taking them into consideration? What sort of teacher is "suited" to group work as a way of teaching? Are certain teaching styles more suited to the use of groups? There is a delicate balance between helping students by teaching to their preferred learning styles and enslaving students by only teaching to their preferred learning styles. Should there be as great a priority on guiding students to transcend their preferred learning styles as there is on catering to them? Perhaps a student's preference is simply a good starting point - "the art comes in knowing when to subvert preference."

5. Individual Learning/Group Learning [Gary, John and Tasoula]

There is the pervasive problem of the interests of the group interfering with the learning of the individual and vice versa. Is all group work done simply to enhance the learning of individuals? Does group work have value all of its own? If a group is a community, then does the group as community have value as a social entity? Certainly mathematics can be a very individual and personal activity, and many of the best students learn it largely on their own. Does this indicate that mathematics has little and perhaps nothing to do with community processes? Or do students often learn mathematics on their own because it is usually taught in the transmission mode (lecturing) rather than as a community project perhaps in the form of open-ended investigations? If educators were to loosen their reins on the classroom, students might spontaneously choose to work with others for some of the time in situations which naturally arise. In such situations, students might form a group when the need arises; when the need subsides, they may choose to work alone or perhaps with others. The problem of group versus individual may be nonexistent in this setting.

A Theme for 1988

As a way of bringing closure to the three days of discussion members were asked to suggest a recurrent theme which could serve as a summary as well as a focus of the "Feelings Group" for the conference in 1988. Although everyone was given the opportunity of suggesting a theme, cohesion quickly emerged around the idea of "naturalness" as first volunteered by Gary: "... a natural way of learning mathematics as opposed to how mathematics is taught and learned now which is quite unnatural, over-structured, always an approximation. I'm thinking of all the happy experiences you have had in the ways you are productive in the way you do mathematics. I'm going to call those the natural way."

Perhaps a significant aspect of naturalness in teaching is captured in the expression, "A guide on the side rather than a sage on the stage." As well as naturalness in the learning and teaching of mathematics there is naturalness in the content of mathematics. Some content is focussed upon to excess, not because of its inherent significance but simply because it occurs on tests. Some content is studied because it is mandated in the curriculum guidelines, but does that make it natural? In contrast, if a child were to pursue a topic in mathematics as a spin off to an open-ended investigation would her spontaneous pursuit be seen as unnatural?

The discussion came to a natural end with a spontaneous remark from Pat: "It is wonderful how a natural topic for next year arose naturally."

In keeping with the way our working group had operated throughout the three days of the meeting, we decided to open out discussion to other participants who might have been interested in contributing to the work of our group but had not been able to attend the sessions.

There were two main contributions from David Pimm of the Open University, England and John Clark of the Toronto Board of Education.

In England small group activity is common at the elementary level and is now being used more and more at the secondary level. One of the reasons David chooses group work is to remove himself as the focus of attention. In the Open University Summer Schools, he uses group work in investigation sessions with adult students. For these sessions he deliberately chooses pairs because he finds that domination by one person is frequently a problem in larger groups. With pairs, turn-taking is easy to set up. Initially he assigns students to pairs randomly, but then after monitoring the work of the pairs, he is more selective. He has also experimented with the JIGSAW method when working with students to explore the meanings in a text.

There was some discussion at this point of the JIGSAW method. Lorna has a sense that JIGSAW is more successful with older students. But John Clark cautioned that there are some problems having to do with the nature of knowledge and knowing with the assumptions behind JIGSAW.

John went on to observe that while group work has become commonplace in elementary schools in Canada, it is still rarely found elsewhere. He first started investigative work with high school students some ten years ago. He now uses investigations in his inservice workshops with teachers as an introduction to group work and has been surprised at the excitement this activity has generated amongst grade 13 teachers. He emphasized two important aspects in setting up successful investigative activity: giving each group good starting points and setting up a good system for report.

BIBLIOGRAPHY

Aronson, E., Blaney, N., Stephan, C., Sikes, J., & Snapp, M. (1978). The jigsaw classroom. Beverly Hills, CA: Sage.

Brandes, D., & Ginnis, P. (1986). A guide to student-centred learning. Oxford: Basil Blackwell.

Burns, M. (1981, September). Groups of four: Solving the management problem. Learning, 46-51.

Davidson, N. (1985). Small-group learning and teaching in mathematics: A selective review of the research. In R. Slavin, S. Sharan, S. Kagan, R. Hertz-Lazarowitz, C. Webb, & R. Schmuck (Eds.), Learning to cooperate, cooperating to learn. New York: Plenum.

Davidson, N., Agreen, L., & Davis, C. (1986). Small group learning in junior high school mathematics. School Science and Mathematics, 23-30.

Dees, R. (1985). How does working co-operatively help students in increasing problem-solving ability? Paper presented to Special Interest Group for Research in Mathematics Education of the American Educational Research Association, San Antonio, Texas.

Erickson, T. (1986). Off and running: The computer offline activities book. Berkeley, CA: EQUALS in Computer Technology, Lawrence Hall of Science, University of California.

Gorman, A. (1969). Teachers and learners: The interactive process of education. Boston, MA: Allyn & Bacon.

Hirst, K.E. (1986). Undergraduate investigations in mathematics. Educational Studies in Mathematics, 17, 373-387.

Hoyles, C. (1985). What is the point of group discussion in mathematics? Educational Studies in Mathematics, 16, 205-214.

Johnson, D.W., & Johnson, F.P. (1987). Joining together, group theory and group skills (3rd ed.). Englewood Cliffs, NJ: Prentice Hall.

Johnson, D.W., & Johnson, R.T. (1987). Learning together and alone: Cooperative, competitive, and individualistic learning (2nd ed.). Englewood Cliffs, NJ: Prentice Hall.

Kagan, S. (1985). Cooperative learning resources for teachers. Riverside, CA: University of California Department of Psychology.

Laborde, C. (1986). An example of social situations in mathematics teaching: communication situations between pupils. Response to the Working Group: Social Psychology of Mathematics, Tenth International Conference for the Psychology of Mathematics Education, London, England.

Sharan, S., & Sharan, Y. (1976). Small group teaching. Englewood Cliffs, NJ: Educational Technology Publications.

Slavin, R., Sharan, S., Kagan, S., Hertz-Lazarowitz, R., Webb, C., & Schmuck, R. (Eds.). (1985). Learning to cooperate, cooperating to learn. New York: Plenum.

Webb, C. (1985). Cooperative learning in mathematics and science. In R. Slavin, S. Sharan, S. Kagan, R. Hertz-Lazarowitz, C. Webb, & R. Schmuck (Eds.), Learning to cooperate, cooperating to learn. New York: Plenum.

TOPIC GROUP R

WORKING IN A REMEDIAL COLLEGE SITUATION

Annick Boisset
John Abbott College

Martin Hoffman
Queens College (CUNY)

Arthur Powell
Rutgers University (Newark)

This one-hour Topic Group began with a brief discussion of the context in which remediation in mathematics at the college level occurs. Acknowledging inherent differences in each college setting, the discussants agreed that students in their remedial courses tended to exhibit certain common characteristics, among which are that students

- have seen the material (several times) before.
- hold belief systems about mathematics that have been convoluted in time in ways not found in younger students.
- have negative attitudes about mathematics.
- have poor study habits.
- are not attuned to understanding in mathematics.
- exhibit low achievement levels based on standardized tests.

In spite of these difficulties, the discussants feel that progress can be realized through activities which promote mathematical thinking. This became the central theme for the topic group discussion.

Annick Boisset discussed a technique of wide applicability, called Reverse Problem Solving (RPS), she has developed for teaching problem solving skills. She demonstrated through several examples the pedagogical utility of RPS in which students are asked "to construct in their own words a problem statement to match a given worked out solution."[1] RPS contains both inductive and deductive components, promotes thinking, seems to alleviate fears for some students of attempting problems, and can be used effectively as a diagnostic instrument.

Arthur Powell discussed techniques relating to the affective domain, in particular the use of writing as a device for having students become more reflective about their mathematical activities. He outlined several forms that the writing might assume including free writing, focused writing, summary writing and journals.

The session ended with a brief discussion by Martin Hoffman of some of the difficulties that arise when attempting to evaluate student's progress. It was noted that standard evaluative instruments are often not able to measure the effect of techniques such as those mentioned in this discussion.

---

1 Boisset, Annick. "The Reverse Problem Solving Method: Diagnosing Underlying Causes of Inability." Paper presented at the National Council of Teachers of Mathematics 63rd Annual Meeting in San Antonio, Texas, April 1985.

The short, but lively, question and answer period that followed indicated an interest in the subject beyond the restricted group of CMESG/GCEDM members who teach remedial level college classes. It was felt by the discussants that since remediation is now a prominent part of mathematics instruction at primary and secondary levels, that consideration of remediation techniques will play a growing role in mathematics education courses.

TOPIC GROUP S

MOTIVATION THEORY IN PRE-SERVICE TEACHER EDUCATION

Erika Kuendiger
University of Windsor

When pre-service teachers start teaching mathematics for the first time, they very often experience a reality shock. Students do not seem to respond as positively to their teaching efforts as is expected by the pre-service teachers. To some extent this is true for all subjects, yet pre-service teachers find it particularly hard to motivate their students in mathematics, since many are not particularly interested in mathematics. Motivating students in this context does not mean getting short time attention, but rather to induce a long term involvement in mathematics.

If this experience occurs repeatedly during a pre-service teachers' first student teaching experiences, then they tend to explain it by either attributing the reason to the students and/or to the subject. Common explanations are: Mathematics is hard to understand; many students lack the ability to be successful, that's why they are not interested; or, mathematics is boring anyway, nothing is going to change this, and since mathematics is an important subject, we just have to teach it.

Obviously, the above explanations have an enormous impact on future teaching if they become part of a teacher's general belief system. In this case, there is a great chance that they form the basis for a self-fulling prophecy. If further teacher's efforts to motivate students are not rewarded immediately, he/she may give up easily. Each perceived failure will in return strengthen his/her belief that there is hardly any way to motivate those students that are not successful in the first place.

To avoid the development of the above beliefs, it is necessary to enable pre-service teachers to understand how some students are motivated in mathematics, while others are not. At the moment they recognize that motivation is the result of a learning process that starts at the moment a student has learned the first time about mathematics, it becomes clear that change cannot occur over a short period of time like two weeks of practice teaching.

Motivation theory, based on attribution, provides a basis for understanding how former and future achievement of a student are interlinked and how the motivational framework of a student develops. Moreover, this theory provides a basis that enables a teacher to understand his/her role in the development of a student's motivational framework. A recent summary of research results, geared for pre-service teachers, was done by Alderman et al. (1985). An application of a motivational process model focusing on mathematics and a more extensive discussion of the issue can be found in Kuendiger, (1987).

In mathematics education, the explanatory power of motivation theory has become particularly obvious in research projects geared understanding sex-related differences in both achievement and course-taking behaviour (see e.g. Eccles et al., 1985; Fennema, 1985; Schildkamp-Kuendiger, 1982). It seems that the importance of motivation and affect for the learning of mathematics in general has become more recognized. At the PME-XI 1987, for example, a whole series of papers focused on beliefs, attitudes and emotions (McLeod, 1987).

References

Alderman, M.K. and Cohen, M.W. (eds.) (1985): Motivation Theory and Practice for Preservice Teachers, ERIC, Teacher Education Monograph No. 4, CTE, Washington.

Eccles, J., et al. (1985): Self-Perception, Task Perceptions, Socializing Influences, and the Decision to Enroll in Mathematics, In: S.F. Chipman, et al. (eds.), Women and Mathematics: Balancing the Equation, Hillsdale, N.J., pp. 95-122.

Fennema, E. (ed.): Explaining Sex-Related Differences in Mathematics: Theoretical Models. In: Educational Studies in Mathematics, 16, pp. 303-320.

Kuendiger, E. (1987): Misconceptions about Motivation in Teaching Mathematics. In: J.D. Novak, (ed.) Proceedings of the Second International Seminar, Misconceptions and Educational Strategies in Science and Mathematics, Cornell University, Ithaca, NY, 269-276.

McLeod, D.B. (1987): Beliefs, Attitudes, and Emotions: Affective Factors in Mathematics Learning: In: J.C. Bergeron, et al. (eds.), Psychology of Mathematics Education, Proceedings of the 11th International Conference, Montreal, Vol. I, 170-183.

Schildkamp-Kuendiger, E. (ed.) (1982): An International Review on Gender and Mathematics, ERIC Clearinghouse for Science, Mathematics, and Environmental Education, Ohio.