CANADIAN MATHEMATICS EDUCATION STUDY GROUP

GROUPE CANADIEN D'ETUDE EN DIDACTIQUE DES MATHEMATIQUES

PROCEEDINGS

1988 ANNUAL MEETING

UNIVERSITY OF MANITOBA

WINNIPEG, MANITOBA

June 2 - 6, 1988

Edited by

Lionel Pereira-Mendoza

Memorial University of Newfoundland

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EDITOR'S FORWARD

I would like to thank all the contributors for the promptness in submitting their manuscripts for inclusion in these proceedings. Without their co-operation it would not have been possible to produce these proceedings so quickly.

A special thanks from all of us who attended the conference to those involved in the organization. In particular, our thanks are extended to Lars Jansson, through whose auspecies the meeting ran smoothly.

I hope that these proceedings will serve to generate much thoughtful discussion about the many significant issues raised at the conference.

Lionel Pereira-Mendoza

February, 1989

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Lecture I

MATHEMATICS EDUCATION AND TECHNOLOGY

Christine Keitel
Technische Universität Berlin

I. Introduction: Three Examples

This paper is devoted to the relationship of mathematics education and the social use of mathematics, i.e. the application of mathematics in and by society. I intend to focus on some aspects of the topic on a more theoretical level of consideration which, as far as I see, are still much neglected. As a consequence, in my view, much labour in the domain is spent in vain.

As an introduction to more general observations I will report three short school stories. They represent rather typical classroom situations, characteristic of the problems rooting in the relationship in question.

The first, borrowed from Alan Bishop, refers to an aboriginal student who had studied some mathematics at school." ...(he) told me that in his village in Papua, New Guinea, when there were disputes about areas of gardens the measure used was that of adding the length and the width (the gardens were roughly rectangular). For him, to multiply these was the 'white man's system' which he had learnt at school, but at home he would always add!" (Bishop 1988, p. 35)

The second case I met in an English comprehensive school, but it could have happened anywhere in Germany or France as well. It was a lesson under the heading of "ratio and proportion" and the teacher told me that she wanted to approach the mathematical concepts in a practical way. So she offered the following question: "Somebody is going to have his room painted. From the painter's samples he chooses an orange colour which is composed of two tins of red and one and a half tins of yellow per square meter. The walls of his room measure 48 square meters altogether. How many tins of red and yellow are needed to give the room the same orange as on the sample?" The problem seemed quite clear and pupils started to calculate using proportional relationships. But there was one boy who said: "My father is a painter and so I know that, if we just do it by calculating, the colour of the room would not look like the sample. We cannot calculate as we did, it is a wrong method!" In my imagination I saw a fascinating discussion starting about the use of simplifying mathematical models in social practice and their limited value in more complex problems (here the intensifying effect of the reflection of light), but the teacher answered: "Sorry my dear, we are about ratio and proportions."

The third story happened in a German school and is about a lesson given by a very ambitious teacher who emphasizes the practical value of mathematics education to the pupils. It is a problem of 'wasting energy' which she reports in a book on "Practical learning in mathematics and sciences". (Kämmerer 1988) In a previous lesson, the class had discussed the waste of energy. A vivid debate among students was going on when the mathematics teacher came in. The question was, whether an individual could contribute sensibly to saving energy, e.g. at home. Spontaneously the teacher had the idea that she could show how mathematics can help to make argumentation rational and to clear the debate. She set up the following task:

"Let us assume that the radius of a cooking pot is half of the radius of the electric plate, how much energy is wasted?

a) Give an estimation in percentage!

b) Give reasons for your estimations and check by calculating!"

Besides, the teacher had in mind to do some remedial work on the formula of area of circles and comparing areas of circles in relation to their radius, focusing on the difference of linear and quadratic relations.

First estimations were about 20% or 30%. Now she directed them to the second task to check the estimation by calculating. Students went into the school kitchen to measure some pots. With the teacher's help and insistence finally all pupils had a nice and surprising result: If we cook under the above conditions 75% of the energy will be wasted.

Several girls did not believe that the result was correct, in spite of their calculation. They did not trust the numbers: "In reality is not like this!" A normal mathematics lesson would end here. In this case, it was different. The teacher thought that the superiority of the mathematically established result could easily be proved by an experiment. One litre of water was brought to the boil in the smallest pot available, and then another litre in a big pot, and the times needed respectively were measured. However crude this experiment was, the result was most surprising for the teacher: there was no significant difference between the two pots; in contrary to the calculation the smaller pot even needed slightly less time. The girls reacted: "We knew from our experience in the kitchen that it does not matter very much".

What do these stories tell us? All of them deal with geometrical concepts, namely areas. From the first to the third, the view is narrowing from very general to specific problems of mathematical applications as reflected in school mathematics.

The first example shows a difference of the treatment of applicational problems in school and in society, which here apparently is due to different cultures effective in a mathematical education imported from one culture to a country with quite another one. Admittedly this is an extreme example, but the better it shows that even the simplest use of mathematical models in social reality is bound to and relying on a specific background of values and intentions: on the one hand an exact, however abstract rating of ground referring to predominant criteria of individually owned ground versus commons, to taxes, rents, prices; on the other to a concrete understanding of a garden as a useful piece of land, implying a manageable (rectangular) shape, and that characteristics such as shadow, humidity, quality of soil are much more important than the precise size. In Euro-American societies, such qualities might be expressed in the price per square meter, in Papua New Guinea they remain properties of a specific garden. For the comparison of gardens, the size, as approximately indicated by the sum of length and width, may be, after the comparison of more relevant properties, a largely sufficient criterion of description.

The second example may contribute to the recognition that society reproduces its culture in its youths, but not by making its youths understand its culture. Certainly the latter would not secure reproduction to the same degree. So it may be regarded if not as a deliberate, so as an effective end to this purpose that application problems provided by mathematics education do not really refer to social practice, but serve as more or less artificial embodiments of mathematical concepts of techniques without practical correspondence. Students come to understand that their practical knowledge may be irrelevant in school mathematics.

In the second example the teacher may simply have been unwilling to leave the old rut for a moment or she may have been aware of the constraints of her syllabus or the risks of a piece of reality she just was not too familiar with. In the third example on the other hand the teacher gets lost in the complications of the matter. She is not even aware of the reasons of the fiasco: the mathematisation of the problem mismatches reality by inadmissible simplification (by equating areas to energy), to a degree that spoils the result. She gives a faulty example of dealing with technology to her students and the result certainly is fateful for their understanding of applied mathematics.

If we overlook these examples we may find that the first of them suggests an explanation of the second and the third, it may make us feel like looking on the small visible top of a tremendous iceberg, when dealing with the social practice of mathematics in terms of conventional mathematics applications in the classroom. We may then stubbornly contend that the small top is the whole, and clinging to a mathematical programme is very helpful in that case - this is the second example. Or we may bravely acknowledge that there is something under the surface and get hold of one or another odd piece of context, and again miss the complexity of reality - this is the third example.

How can we resolve the dilemma? In my opinion, there cannot be any promising way without attempting to get a more comprehensive notion of the complexity of the iceberg, i.e. mathematics in social practice. Our first example suggests that by the very fact that social practice is involved, mathematical application is inextricably interwoven with the constituent components and holdings of a culture. The impact of culture is effective in both the conditions, values, intentions which direct us, and in the technologies which the culture has developed and keeps at hand for us. In Papua New Guinea the technology for measuring "area" is addition of lengths, and in our culture it is multiplication.

In the following I shall concentrate on the role of technology as it mediates the use of mathematics in social practice. This will allow of keeping closer to our topic (and nonetheless provide a notion of the pervading influence of cultural premises.)

II. What is technology?

Using a definition of the German sociologist N. Luhmann (Luhmann 1982), technology can be seen as "the science of the causal relations which underlay practical intentions, and which have to be acknowledged, if these intentions aim at success". Technology is therefore determined by *causality* (presupposing the aptness of chronological linear, regular order), *rationality* (following the scheme of means and ends) and *sociality* (as ends of actions are determined by social subjects).

The point of interference of mathematics within this complex is the causal-logical order as a prerequisite of any mathematical modelling, the mathematical model providing the means of acting and delineating a scheme for action. Action is rational if both the context of action and the adequate modelling follow the principle of causality and if it corresponds to the "practical intentions" which prompted the action. The ends of acting represent a second constituent of mathematical modelling, and by these the technologically determined structure is infused with social implications and significance. The context of action can be mathematics itself, in that case the role of a mathematical model may contribute to the technology of mathematics which in turn may generate more generally applicable mathematical techniques.

This statement about the relationship of mathematics and technology includes also a relation to school mathematics. If mathematics is a necessary and essential - although not the only - precondition of technology, then mathematics teaching and learning is a necessary prerequisite for everybody who wants to understand and reconstruct or develop technology - and to judge its use or abuse.

On the other hand one can only partly be introduced into understanding mathematical technology by only referring to mathematics itself, as the means-and-ends-relation stringently requires knowledge about both the objective and subjective context of interference as well. Hence an introduction into understanding and evaluating technology within mathematics education cannot be restricted to mathematical techniques or theorems, but must constantly refer to broad understanding of the subject of the context.

III. An Example of Technology: The Mechanical Clock

Let me illustrate these abstract considerations by an example of technology familiar to all of us: I draw on L. Mumford's fascinating interpretation of the invention of the mechanical clock (Mumford 1977): The invention of the mechanical clock represents a qualitatively new stage in the development of technology: all technological tools and instruments so far had served as extensions of man's natural abilities. The clock, however, is no longer a sort of artificial, multiplying prothesis but the mechanical clock is a machine which functions by itself. It is the first *autonomous* machine. Its construction is based on a particular perception of one aspect of nature, namely, time in relation to the movement of the planet system. This approach is generalized and condensed to a mathematical model, transformed into a technological structure, and as such installed outside its original limited realm of significance. Earlier human perceptions of time, which had grown out of both individual and collective experiences and remained bound and restricted to these, were now rivalled and ultimately substituted by this novel kind of perceiving time.

Its superiority consists in its regularity, uniform validity and presence, by all that it is an extraordinary tool of measuring. However, the notion of time loses its connection to and dependence on concrete sensory experience. From a physical notion time mutates to a part of social organization.

The consequences and implications of this change can hardly be overestimated:

- a new understanding of time develops: Time particularized by arbitrarily regular units comes to be viewed as a sum of such units. The association of time with periodical or circular movements is now replaced by the idea of time as an irreversible process. The Christian teleological view of time merges into concepts of progress and endless evolution;

- the mechanical clock extends the domain of quantification and measurability. Applying measure and number to time means measuring and quantifying all other areas, in particular those where time and space are related to one another. Measurability of time pushes forward the development of the natural sciences as (empirical) sciences of measurement (and hence objective sciences) and mathematics as the theory of measurement. Problems of constructing precise and accurate measuring instruments become a concern of mathematicians;

- the new concept of time encourages explorative approaches to reality by suggesting an interpretation of the world as a machine, as an extension of the model of the mechanical clock as a sum of autonomous subsystems which arbitrarily can be atomized and synthesized;

- the clock, used from the beginning as an instrument of *social order* and *social coordination*, changes the organisation of social life by allowing rigid, "objective" *determination*, *organization* and *control* of various social interactions.

Thus the mechanical clock changes the relation between mankind and reality far beyond its original domain of application. It initiated the creation of a second nature totally reconstructing the first, however, exclusively admitting objective, mathematical laws, devaluating the authority of individual and collective (subjective) experience or insight.

IV. Mathematics of Social Practice

The above example confirms the truism that technology, and hence mathematics, pervade present day society, and to a degree far beyond the plain presence of technical apparatus which surround us. So we easily accept the following statement as a commonplace: "The ultimate reason for teaching mathematics to students at all educational levels, is that mathematics is useful in practical and scientific enterprises in society" (Carss et al., p. 199).

As often with trivial wisdom, we do not really reflect it. Otherwise we might note that it is more of a conjuration than a justification. For it does not help to explain a contradiction which is with us since a very long time, namely

> no modern society can exist without mathematics, but the overwhelming majority in a modern society can and do live quite well *without doing* nearly any *mathematics*.

In fact, the hand-calculator is the recent culmination so far of a development by which, while reality is being structured more and more by mathematics, the average individual is more and more dispensed of using mathematics. The old objection to learning mathematics - again and again nourished by common practice and proved by empirical research - that in general one does not really need the mathematics learned at school - seems more justified today than ever.

It is true, of course, that an immeasurable amount of mathematical knowledge is available today and rapidly expanding, and there exist people who professionally use specific sections of this knowledge. However, the very scope of it and of eventual specialization put it beyond general education, and, as R. Fischer (Fischer 1984) points out, numerous agencies outside school fill in this gap, and they are much more prepared to provide special knowledge purposefully and effectively to those who need it. In fact, the need of high achievement in mathematics for all cannot be justified by this kind of argumentation.

So what about the contradiction of increasing *mathematisation* of modern society together with potential *dematheматisation* of its members (Chevallard 1988)?

Dematheматisation is enabled by the very existence of the products of our technologically structured environment: dematheматisation is *inherent* to these products as it is to technology.

If we turn back to an early technological achievement as represents the clock, we immediately see that the original mathematical considerations which resulted in the conceptualization of the clock and its eventual construction may be extremely far from the thoughts of an actual user of a watch, who does not wish to miss his train. And so may be all subsequent additions of mathematical and technical ingeniousity up to quartz-crystal clockwork and digital equipment. They all are incorporated in the actual instrument - and yet, for appropriate use, we must not have the slightest idea of them.

Thus it is an effect of technology to substitute our own imagination, mathematical knowledge and technical skill, and even more: it summarizes the best talents of generations of specialists before us. It is conceived to *replace* them.

Thus, the quintessential product, which is in our hands, makes the mathematics enclosed *implicit* mathematics. Mathematics continues to be effective, but without requiring respondent capabilities on the user's side. That is how demathematisation takes place. Whereas *explicit mathematics vanish* beyond the clouds in the summits of research and extreme specialization, *implicit mathematics* makes mathematics disappear from social practice.

From time to time lightenings from the heights of explicit mathematics illuminate the society. Scientific mathematics is the met with more respect, the more it is wrapped in mystery and the more astonishing technological achievements it apparently inspires. Contrarily, little attention is attracted to implicit mathematics - and that is also how school mathematics reacts to these phenomena.

V. Technology in School Mathematics

The pretention of usefulness of mathematics as the ultimate justification of school mathematics must be viewed in a historical context. The period of demathematisation of everyday social reality was preceded by a period of *trivialization* of mathematics (see also R. Fischer 1987). Trivialization resulted from both more general use of mathematics in social practice and the didactical progress of schools, which enabled mathematics to be taught at ever earlier stages; arithmetics e.g. shifted from university level to elementary school. Whereas the rank of mathematics as a formative discipline decreased, the demand of socially useful mathematical training became more and more pressing.

The argument of usefulness was readily adopted by educators, and in fact became the greatest asset in an argumentation in favour of general mathematics education. And it is still viewed as such by many educators, as shows the above quotation. In my opinion, this position bears great risks. It is not difficult to preview that the legitimacy of general mathematics education will again be placed on the agenda. Already at the present time informatics rivals with mathematics about social usefulness, already now the progress of demathematisation can hardly be overlooked.

The argument of usefulness could easily prove to be a deadlock: If the claim of usefulness is upheld, the phenomena of demathematisation cannot be acknowledged, for demathematisation makes the usefulness of traditional general mathematics education fictitious. Thus, by maintaining the idea of usefulness, the problems of mathematisation and demathematisation in modern society cannot become a matter of concern and appropriate response. And that is indeed what we observe in current mathematics education.

How is technology dealt with in present day school mathematics? Technology is no topic in mathematics education. There is no planned, purposeful treatment of ends and means of technology. Although mathematical techniques and technological constructs appear in school mathematics teaching in various ways, they serve quite different ends. They are mostly determined by:

- educational policies and their established claims ("social needs");

- by didactical-methodological intentions (artificial, dressed tasks as "problems", "embodiments" of mathematical structures, the introduction of modern technological equipment for various purposes);

- by pedagogical conceptions in which technologically determined situations or subject matter play different roles.

Among the latter, the following more influential tendencies expressly address technological phenomenon. But they are not the only ones and they may occur in various blends.

- There still is much impact of the old utilitarian conception of school mathematics which aims at a simple and direct correspondence of school tasks to those in social practice. By mere persistence a body of such tasks survived in many curricula far beyond any practical significance. By these no understanding of technology is intended nor achieved.

- In opposition to this approach other didactical intentions follows a compensatory intention. They also relate school mathematics rather exclusively to technologically determined social practice, however aiming at a backing of the individual against the pressure of society. There is integrated mathematics teaching and project work, but mathematics is only dealt with as a means of problem solving, only the instrumental aspect is recognized, and the theoretical level of mathematics, its systematic aspect, and that of technology are rarely attained: mathematics ends in "daily life" practice, as does technology.

- Another approach attempts to combine methodological or psychological demands, a social perspective, and (implicit) orientation towards mathematics and sciences as theory. Mathematical concepts and theoretical insights are developed through teaching units which start out from technologically determined situations.

We may roughly state an advancement - both historically and in the awareness of complexity - in the sequence of these approaches: The traditional instruction which mainly aims at practical skills is complemented by the goal of *understanding* a mechanism, and to ensure this, the conditions of cognitive development may in addition be observed. A claim of greater realism extends particularized tasks to larger sections of social practice, and finally the pedagogical intentions may also envisage to convey an introduction into the theories of mathematics and natural sciences.

These approaches either aim at mastering different activities related to technological phenomena (with varying ideological undertones), or they refer to these phenomena for other purposes, e.g. for learning mathematics. Accordingly, their concept of usefulness, rarely explicitly stated, varies from mere functioning in pre-determined social practice, to competence on a more general level of actively using mathematics and technology.

None of these approaches explicitly responds to the dilemma of mathematisation and demathematisation. As the dilemma at the same time is a social problem - taking part actively in the process of mathematisation being a matter of high standard professions - it can easily be left to separate consideration along with traditional problems of differentiation.

Already today we may state that arithmetical operations as processes within human brains have largely disappeared from vast areas of social practice. For the routine cases in specific jobs the necessary formulae are at hand and the operations themselves are executed by a machine. Under these circumstances, the maintenance of extensive arithmetics learning in mathematics education is either beyond justification ("you simply must know how to ..."), or founded in nostalgia ("at least for one time in his life, one should have done ..."), or merely a relic in the syllabuses, or a formative enrichment like art and music, which at best feed into leisure time occupations. Is that usefulness?

I would not criticize any such justification, if we could be sure, that by the respective organization of mathematics education we would not miss truly essential requirements. However, that I fear, is what occurs.

As we know, the increasing use of technology, that is to say of rational devices in social practice, has not fostered rational reconstruction of social processes. Instead, paradoxically, it rather created new mystification: Ever more processes disappear in the black boxes of technological instruments, and in them the processes become ever less reconstructible - in fact as far as to definitely defy intellectual control, as happens with very complex computer programs which involve a larger group of specialists (Weizenbaum 1967; Booß-Bavnbek 1988).

On a daily-life level, mystification is much more due to insufficient comprehension. And comprehension here not only refers to the acquaintance with a mechanical structure, as didactical approaches mostly suggest, but also to the understanding of the *significance* of its use.

VI. Another Example of Technology

What I wish to say by this can best be illustrated by another example. I propose to look at the economic instrument of the double-entrance book-keeping or calculation-model. It is, although not an autonomous, but a rather detached system which early spread beyond the field in which it emerged over all economic areas. Its efficacy is based on three characteristics: (Damerow et al. 1974).

- the entire calculation system of an enterprise can be organised according to one single, uniform reference unit and relative to one function. The reference unit is the capital invested;

- it allows to formalize all processes or occurrences within the economic system and thus to operate them in terms of mathematics;

- as a result of complete formalization of all processes the model is an excellent instrument of controlling and directing the enterprise.

The double-entrance book-keeping or calculation-model brought about two fundamental innovations, which initiated and fostered a development towards modern economic systems, but in turn only came to display the whole range of their possibilities and implications under the conditions of more developed economic systems. The one innovation was to treat all processes of trading - of both transformation and change of value within a system - detached of their concrete real properties exclusively and uniformly according to the rules of the calculation model; the other innovation was to separate labour structurally from capital: within the system labour is necessarily treated as costs, whereas added value and profits are allocated to the capital.

The development of this model can plausibly be explained by its origin: it really was a product of practice, not of theoretical construction: it originated from business activities of the big trading and banking houses in the Renaissance, in which all merchandise exchange was integrated within the banking activities and hence all goods tended to be treated as exchange value or capital value only, not in terms of concrete significance or utility.

Whereas the calculation-model was well in concordance with the practices and requirements of banking houses, at the same time it offered a completely new interpretation of the economic process of manufacturing and industrial production. By its transfer to these areas the economic model of capitalistic calculation became an enormous stimulus and driving force for development. The word "capitalistic" here proves not to emanate from philosophy or ideology but originally to denote appropriately a capital-centered technological construct for controlling and directing economic processes.

An important implication - if one may say so, the trick of the machine, - is the fact that the capital-orientation is built in, implicitly and eventually unnoticed. When applying the machine this orientation is adopted, consciously or not, with all its consequences and implications. The logic of the machine produces arguments and seemingly *objective constraints*, which on the basis of other premises might look quite different.

In all that works mathematics - at the core of the technological system. That may give an idea of what is implicit mathematics. Because of its enormous diffusiveness the calculation-model penetrates practically all fields of social practice related to money. And comparable to crystallisation in a liquid, which starting from one point expands over the whole surface, the calculation-model sets going systematization and formalization all over the area where it is applied. In industrial enterprises where for a long time production followed its own traditional patterns, "scientific management" and "system analysis" ultimately led to restructuring all production processes down to the most minute detail towards systematization and standardization, in order to bring them under the control of the calculation-model and its prescriptions.

The balance of trade is the center-piece of the technology of economics. It determines prices, taxes, wages everywhere. This is the most obvious way by which the technological system interferes directly with the daily life of all of us, and again and again challenges competent reaction. Does mathematics education in any way contribute to prepare students to this reality, so as to improve their chances to meet this challenge competently? I do not think so.

VII. School Mathematics and the Intercourse of Mathematics and Reality

I hope that the above example may have helped to substantiate my previous statement about implicit mathematics in technology. And it may have become clear that teaching mathematics as such - whether extensive arithmetics or advanced new mathematics, whether illustrated by tasks from social arithmetics and sciences, or not - is not the solution of the problem and does not even address it. The problem is *making implicit mathematics explicit*, and elucidating the significance of its application. That is to say: to address the intercourse of mathematics and reality.

Focus of all elements of technological construction is the mathematical model. In it mathematics and reality concur. It represents an intermediate level between mathematics and reality and hence requires translations and interpretations to different sides. Mathematics has, and real objects and contexts must have, specific properties which enable them to merge into instruments for rational and purposeful acting. So it is crucial to understand them in order to understand technology (Skovsmose 1987).

On the side of mathematics, it is its dual aspect: mathematics as means and as system (Fischer 1988). Mathematics as means, that are the instrumental, procedural, hence technological qualities of mathematics, whereas mathematics as system refers to its axiomatic order and systematic relations, which represent the prerequisites and basis of mathematical acting, whether in pure mathematics or in applications. For mathematics the coincidence of both the instrumental and the systematic aspect are constitutive, as mathematical concepts always are means and systems at the same time, and mathematical activity presupposes an awareness of that.

Contrarily, in the concretisation of a technological structure outside mathematics this balance tends to get lost: it is just for making do *without* the systematic background that the construction is undertaken. The mathematical process is encapsulated in the technological structure - the application is cut off from the requirements of explicit mathematical knowledge, without any regard for a-posteriori deduction or understanding.

On the side of a piece of reality and its context, which become object of technological structuring, the concurrence with a matching mathematical model presupposes a theoretical conceptualization of the part of reality in question. Theories about objects are intermediate steps of formalisation, by which relevant elements or properties of the object are selected, and their relation is determined.

Only in very simple and uninteresting cases we will find a direct and obvious one-to-one correspondence between model and object. That means that normally a model is not a model of reality as such, but a model of a conceptual system, created by a specific interpretation which is based on a more or less elaborated and more or less explicit theoretical framework. The establishing of conceptual systems takes place in different ways depending on the technology in question.

The mathematical model provides a morphism to this conceptual system and reveals possibilities to apply mathematical techniques: If the model is based on an exact, verified scientific theory of the object, the validity of the model can be verified by empirical data. If the model could be based on various contradictory theoretical approaches, it is necessary to analyze all premises and assumptions of these approaches.

Intentional orientation of the technological construction influences already the selection of the theories on the object, then the transforming of elements of reality into the conceptual systems. Purposes and interests determine the process directly as they cause the construction, and finally intentions determine the use of the structure which may be quite different from the original intention.

VIII. Conclusion

Let me try to draw a few conclusions from these peculiarities of technology. The most obvious insight, I think, is that we cannot hope to provide a serious and appropriate approach to the phenomena of technology in treating them in a by-the-way manner, by incidental glances from a "regular" mathematics syllabus. We cannot expect to convey understanding by using mathematical applications just for illustration of mathematical concepts.

Reconstructing technological instruments - not simply using them - requires, if not the same ingeniousity (for we know the results), but nearly as much understanding of the contexts and backgrounds, in both mathematics and reality, as did the original construction. Understanding of technology demands full consciousness of the connections, relations and processes on different levels of application - a meta-level of knowledge, consideration, and communication.

A major requirement to this end - and at the same time a major deficiency in present-day mathematics education - is a very strong emphasis on reasoning, interpreting, reflection and experimental attitude. With respect to mathematics this would in particular mean to stress the systematic aspect of mathematics, and to make the dual character of mathematics a subject of education. In fact one may state that the actual presence of applications in mathematics teaching tends to prevent insight into systematic structural relationships and connections rather than to promote them. (A typical example is the traditional social arithmetic teaching, where classical as well as modernized word problems, tasks, lead only to knowing rules, but not to theoretical insight). Technologically oriented mathematics education has to undertake particular efforts to overcome this immanent anti-systematical tendency of technology: only by theoretical mathematical concepts the duality of mathematics as means and system can be experienced, and duality is the germ of applicational potential, also future potential.

Again, interpretation, translation, valuation are the most important features of establishing and handling a mathematical model fitting a reality context and a specific activity impulse related to it, and thus they must also pertain to the field of social practice in view. A reflective attitude towards the subject in social practice not only implies to study the "raw material" in relation to respective theories about it, but before all to examine the intentional character of interferences on various stages.

Strengthening the reflective character of mathematics education (at the expense of a maximum amount of subject matter) would eventually meet with similar suggestions starting out from another point. I refer to considerations by R. Fischer (Fischer 1984) who compares the menace that the predominance of mathematics, science and technology represents for many of us, to the menace formerly emanating from an undomesticized nature. Fischer infers that similar to the liberation from the dominance of nature - by mathematics, sciences, and technology! - today a liberation from these blessings would seem to be necessary, liberation in the sense of reflected distance and self-determination. Thus entering into a better understanding of technology through mathematics education would not foster an evermore complete surrender to, but contribute to a freer, more critical and more self-conscious intercourse with technology.

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Lecture II

ALL ONE SYSTEM

Lynn Arthur Steen
St. Olaf College

I'm very glad to have the opportunity to come to Canada and participate, albeit rather late, at the end of your discussions. The informal chats I've had so far with people suggest to me that CMESG is a fairly special kind of group. I don't think there's anything in the United States that's comparable to it. If there were, maybe we wouldn't be in quite such a pickle as we are now. All of our similar meetings are big and huge, so you can't get anything done. CMESG seems like a group that has a certain direction and self-consciousness about what it wants to accomplish.

What I want to do this evening is to talk about efforts that are going on in the United States to try to revitalize mathematics education. The intent of my title, All One System, is to emphasize the principle driving spirit behind a lot of necessary changes. First, we have to look at mathematics from kindergarten through graduate school; dealing with just pre-college level work or just university work all by itself simply is not going to work because of the type of feedback loops involved. Second, I also intended to convey a sense--especially important in the United States--that we have to view mathematics education as education for all citizens and not simply for those who happen to grow up with a tradition of mathematics education. The challenge is to deal with these themes in a country as diverse and complicated as the United States.

Mathematical Sciences Education Board

Some of you may know that about four years ago the Conference Board of the Mathematical Sciences (CBMS), which represents the fourteen major U.S. professional societies of mathematics, recommended to the U.S. National Academy of Sciences that they set up a national board on mathematical sciences education. That board has been established; it's called the Mathematical Sciences Education Board. What I am going to do this evening is to spend the first part of my talk dealing with bureaucracy to give you a sense of who the players are, what they're dealing with, and why they're doing what they are doing. Then in the second part, I'll present some of the areas of consensus that have emerged from all this bureaucracy, which we hope in the next four or five years is going to have some impact on mathematics education in the United States.

Let me start by explaining what the National Academy of Sciences is and how it relates to the National Research Council. The most important thing about the Academy is that it is not a group of mathematicians. That's crucial because we can't change mathematics education if we are only working with mathematicians or mathematics educators. The National Academy of Sciences (NAS) is a private organization, chartered by Congress during the administration of Abraham Lincoln. It is an honorary body consisting of the nation's top scientists. They elect 40 or 50 new members every year and usually three or four or five are in mathematics; about that many die, so the membership of the National Academy stays pretty stable as the top several hundred scientists in the United States.

There is a similar National Academy of Engineering that is much younger --it was started after the Second World War--and even more recently an Institute of Medicine. The three organizations together operate what is called the National Research Council (NRC). The National Research Council is the operating agency of these three groups--The National Academy of Sciences, the National Academy of Engineering, and the Institute of Medicine. The President of the National Research Council (and of NAS) is Frank Press, a geophysicist, who was science advisor to former President Carter.

NRC is a large organization that commissions studies, principally, in everything ranging from highway safety to health and nutrition, from AIDS research to space stations. Any time that Congress wants to know something about science, they go to the National Research Council and say, "Will you please advise us about this?" NRC sets up boards and commissions to carry out this work. Sometimes NRC gets funding directly from Congress, but more normally they are supported through one of the agencies like the National Science Foundation. Apart from that, they are independent. They have a chance to say things independently--to speak for the scientific community.

The reason that the Conference Board of Mathematical Sciences decided to ask the National Academy of Sciences and the National Research Council to set up this board--rather than doing it themselves--is because CBMS felt that it was crucial to get the backing of scientists and engineers if they were going to make any changes in mathematics. Without that backing they knew they would fail. What the National Research Council did was to set up a very large board, the Mathematical Sciences Education Board (MSEB), chaired by Shirley Hill who is a former president of the National Council of Teachers of Mathematics, and professor of mathematics education at the University of Missouri. The board has two or three mathematicians who are members of the National Academy of Sciences (Andrew Gleason, David Blackwell, Isadore Singer, people like that...); it has half a dozen professional people with Ph.D.'s in mathematics education who train doctoral students in mathematics education; and it has about six or seven school teachers. In addition, MSEB has chief state school officers (these are the people who are either elected or appointed by governors to be the head of education in particular states). The Chief from New York went off the Board; now the Chiefs from Arizona and Illinois are on it. There are representatives from business, (a vice president of Citibank New York is there), the past president of the Parent Teacher Association, the National School Board Association... . It's a kind of Board whose members know how to get things done.

MSEB Agenda

Now, I want to use a chart (Figure 1) to show you what that Board is up to--to give you a sense of what they are trying to accomplish. I'm mostly going to talk about one little column in this chart, but first I want to show you the overview. MSEB is looking at issues that concern national

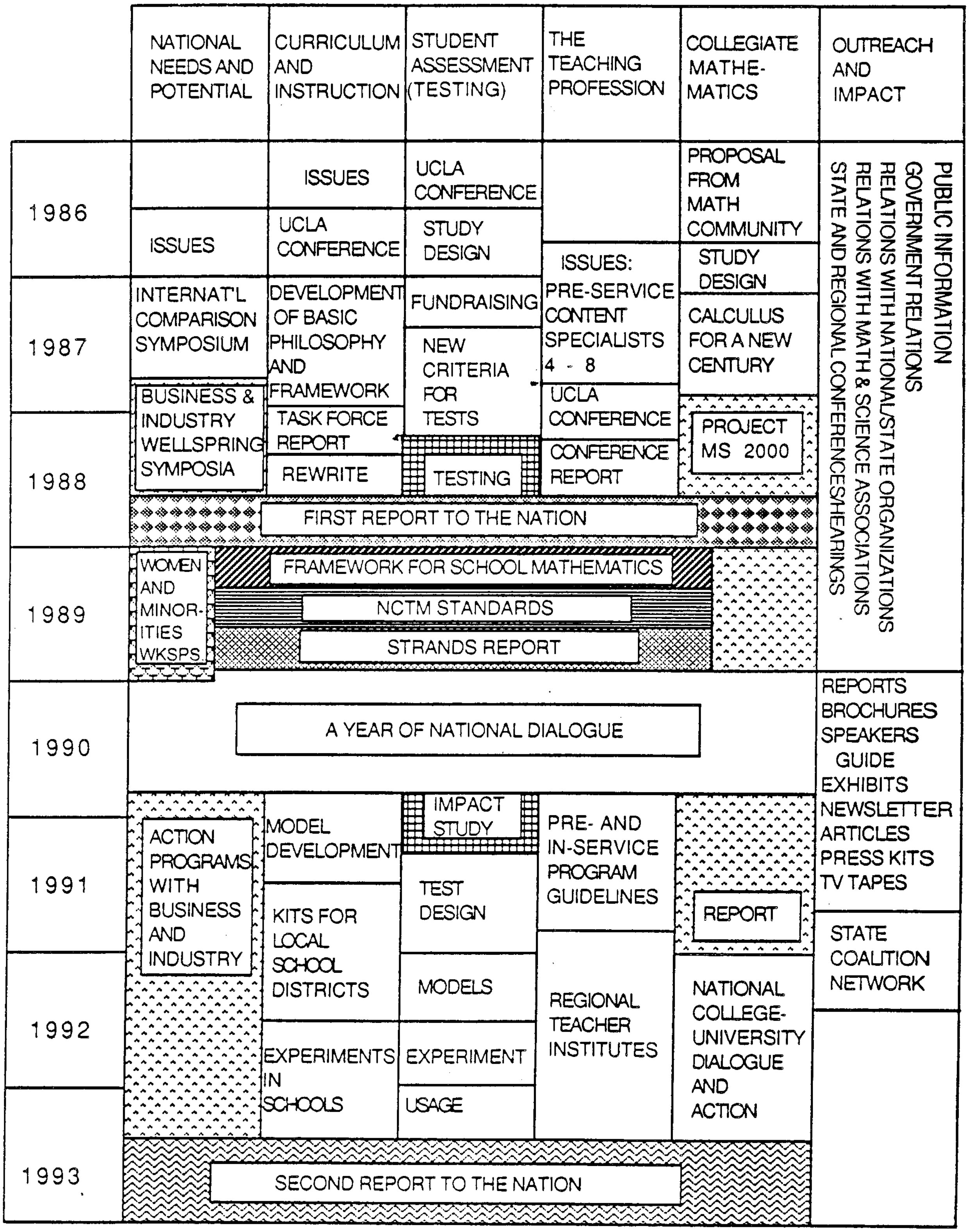

NATIONAL NEEDS AND POTENTIAL
CURRICULUM AND INSTRUCTION
STUDENT ASSESSMENT (TESTING)
THE TEACHING PROFESSION
COLLEGIATE MATHE-MATICS
OUTREACH AND IMPACT
1986
1987
1988
1989
1990
1991
1992
1993
ISSUES
UCLA CONFERENCE
STUDY DESIGN
PROPOSAL FROM MATH COMMUNITY
ISSUES
UCLA CONFERENCE
STUDY DESIGN
ISSUES: PRE-SERVICE CONTENT SPECIALISTS 4 - 8
STUDY DESIGN
INTERNAT'L COMPARISON SYMPOSIUM
DEVELOPMENT OF BASIC PHILOSOPHY AND FRAMEWORK
FUNDRAISING
CALCULUS FOR A NEW CENTURY
NEW CRITERIA FOR TESTS
BUSINESS & INDUSTRY WELLSPRING SYMPOSIA
TASK FORCE REPORT
UCLA CONFERENCE
PROJECT MS 2000
REWRITE
TESTING
CONFERENCE REPORT
FIRST REPORT TO THE NATION
PUBLIC INFORMATION
GOVERNMENT RELATIONS
RELATIONS WITH NATIONAL/STATE ORGANIZATIONS
RELATIONS WITH MATH & SCIENCE ASSOCIATIONS
STATE AND REGIONAL CONFERENCES/HEARINGS
WOMEN AND MINOR-ITIES WKSPS.
FRAMEWORK FOR SCHOOL MATHEMATICS
NCTM STANDARDS
STRANDS REPORT
A YEAR OF NATIONAL DIALOGUE
REPORTS
BROCHURES
SPEAKERS GUIDE
EXHIBITS
NEWSLETTER ARTICLES
PRESS KITS
TV TAPES
ACTION PROGRAMS WITH BUSINESS AND INDUSTRY
MODEL DEVELOPMENT
IMPACT STUDY
PRE- AND IN-SERVICE PROGRAM GUIDELINES
TEST DESIGN
REPORT
KITS FOR LOCAL SCHOOL DISTRICTS
STATE COALITION NETWORK
MODELS
REGIONAL TEACHER INSTITUTES
NATIONAL COLLEGE-UNIVERSITY DIALOGUE AND ACTION
EXPERIMENTS IN SCHOOLS
EXPERIMENT
USAGE
SECOND REPORT TO THE NATION

Figure 1

needs and potential, focusing on concern about international scores as typified by the fact that the U.S. did so poorly on the 1982 international assessment. There's also a major study going on about the needs of business and industry--that mathematics is a wellspring of innovation for business and industry; MSEB's goal is to get business and industry to look at the national impact of the status of mathematics education. Third, there is--in another column--plans for a series of conferences on women and minorities to try to do more to bring under-represented groups into full participation in mathematics.

The curriculum and instruction column principally deals with the pre-college curriculum. (There is a separate project under collegiate mathematics that looks at college-level issues.) There is a series of conferences, as well as a group that is developing what they call a curricular "framework," or more recently, a "philosophy and framework" document. It started out as a framework document, but then they decided they had to add some philosophy in front to make it comprehensible. This is not a curriculum in the sense of a scope and sequence report. It is intended, rather, to be an intellectual framework which can provide a good backdrop for the new Standards of School Mathematics that the National Council of Teachers of Mathematics (NCTM) is working on. I'm sure many of you here have probably seen the draft document on the NCTM Standards, which is listed in the middle of the curriculum column in Figure 1.

The assessment column represents an issue which many people think is the most important of the whole lot. Not surprisingly, the MSEB group responsible for assessment has had the most difficulty getting anything done so far. They are looking at the impact of standardized testing, at the question of what kind of skills are really being emphasized and rewarded on standardized tests. In the U.S., most mathematics testing is multiple choice testing--except for the New York State Regents' exams and the Advanced Placement exams for college entrance. Most mathematics tests require no writing--no creativity activity on the part of the students. However, in many states now governors and legislators are putting a lot of pressure on the school boards to measure students' performance based upon standardized tests. That means that the flow of money, the flow of resources, the incentives of teaching are going to be based to a large extent on the norms established by these tests.

At a meeting I attended in Washington about two months ago--there must have been about 200 people there from many different education-related careers--somebody from this task force asked the audience how many of them in the last three years had actually examined any national standardized test which was being administered in the United States. There were maybe four or five out of 200 who had actually done that. By and large, when a school board or a superintendent decides to buy a test and use it, nobody in that district looks at the test. They have no idea what they are buying. Then they put out those scores and the politicians use them to rank the schools and sometimes fire teachers. The American public does all sorts of things based on test scores, but they have no idea what is in these tests. It is a really big issue, but that is all I am going to say about it today because it is not one that I've been working with.

The MSEB agenda on the teaching profession has generated a great deal of movement in the U.S., not particularly because of MSEB but more because of a major report that the Carnegie Commission sponsored that I'm sure many of you have heard of. This report calls for radical restructuring of schools in the United States as well as in the way that teachers are prepared.

It includes such things as recommending that undergraduate education majors be abolished. In fact I think what they recommend is more the common practice in Canada where students take a subject matter degree first and then they enter a school of education. Is that common all over Canada? I think that is generally the case. But it is not the case in the United States. Virtually all persons who enter elementary school teaching go from high school into a university or college in which they take a degree in education. Typically an elementary school teacher will have just one course in mathematics. In high school they usually got a poor grade in geometry and took no more mathematics; then they take one more course in college which is an elementary discussion of numbers. That is typically all the mathematics our elementary teachers have. Although MAA and NCTM recommendations are for four courses, the average nationally is only one. The Carnegie Commission recommended that there be major changes at that level.

The Commission also recommended changes in terms of professionalism--the way that schools are organized--so that teachers would come in under a mentorship relationship and later on become lead teachers. These changes are intended to give teachers more autonomy with fewer rules and regulations from the administration. Because Shirley Hill, who is the Chair of MSEB, is a member of the Carnegie Board on Teaching, MSEB is working closely with this issue.

The column in Figure 1 on Collegiate Mathematics is what I plan to spend some of my time talking about. The "Calculus for a New Century" project is a big part of that effort. There's a big project called Mathematical Sciences in the Year 2000 (MS 2000) which is an effort to look at everything happening in colleges and universities, including teacher education programs, undergraduate degrees, masters degrees, doctoral degrees, etc., to see how these programs are working.

The column on the far right in Figure 1--Outreach and Impact--has to do with making sure that this board actually accomplishes something. This effort deals with public information, government relations, state organizations--with all the outreach that is needed to get things done. The MSEB emphasis on outreach brings me back to my first point: everybody involved in MSEB knows that they can't hope to make any progress if mathematics people only talk to the mathematics education people, because mathematics is just too big an operation.

Mathematical Sciences in the Year 2000

Now, as an example, I am going to show you the Issues Chart that the project Mathematical Sciences in the Year 2000 is working with (Figure 2). This, you remember, is just one of those five MSEB columns; other columns would have a similar kind of structure. MS 2000 is looking at undergraduate studies, graduate studies, professional development and research, continuing education, and other things that go on after people leave the formal educational system. They are looking at the question of national needs and potential. Are we producing people with the right qualities, the right kind of education for various national needs? What about the curriculum and instruction in different levels and programs?

Are the resources available to departments and universities adequate to meet the various needs?

You can see in Figure 2 all these interlocking connections. That's one of the main themes of this type of large national study. You have to orchestrate an examination of the system as a whole because there are so many feedback loops and so many unintended interactions.

I began by discussing the global picture (Figure 1) and then gave you one look at a column (Figure 2). Now I want to give a detailed look at the structure of one column in Figure 2 and then get on to some of the issues. Within the MS 2000 chart (Figure 2) there is a curriculum column; Figure 3, for example, shows what MS 2000 will be trying to look at in that area. Because of its structure, the National Research Council is not trying to duplicate work that is going on in other organizations. What they want to do is provide national leadership and try to pull people together. This is especially important because we have many other professional societies: the American Mathematical Society (AMS), the Mathematical Association of America (MAA), the National Council of Teachers of Mathematics (NCTM), the Society for Industrial and Applied Mathematics (SIAM), and many others. MS 2000 is going to those groups with concerns about the undergraduate and graduate curriculum and saying, in effect, "Let's pull you people together, gather the best thoughts that you have, and have some meetings to spread widely the best ideas of what is going on."

What MS 2000 is doing, in effect, is farming out the undergraduate work to the MAA, the master's degree study to SIAM--because the principal role of the Master's degree is to prepare people for entering jobs in industry--and the doctoral study to the AMS. In this strategy they are not asking for just a study of degrees which are of interest to those particular professional societies; rather, they are asking those societies to take on the responsibility of looking at the whole national scene.

The master's degree is a good example of the complexity, because half of the master's degrees given in the U.S. are for people whose principal occupation is high school teaching. Frequently they are earned through a succession of part-time study over a period of years. The other half of the degrees divide about equally among those who teach in the community

Topics Chart
Mathematical Sciences in the Year 2000

issues / levels of study	National Needs and Potential	Curriculum, Instruction, and Transitions	Workplace	Resources
School (K-12) Mathematics				
Undergraduate Mathematical Sciences	Support of other disciplines Education of school teachers Needs of business, industry, and government	Transition from high school to college Service courses Bachelor's programs Teacher's programs General education Continuing education	Transitions	Students Faculty Facilities and equipment Operational support Underrepresented groups
Graduate Mathematical Sciences	Support of other disciplines Education of college faculty members Education for research Needs of business, industry, and government	Transition from undergraduate to graduate school Service courses Master's programs PhD's programs Continuing education	Transitions	Students Faculty Facilities and equipment Operational support Graduate assistants Underrepresented groups
Professional Development and Research	Research Applications Needs of business, industry, and government	Postdoctoral study Continuing education	Transitions	Resources for faculty renewal Postdoctoral support Research support Underrepresented groups

Figure 2

Draft Outline for Cooperative Curriculum Project to Articulate and Disseminate the Current Best Thinking

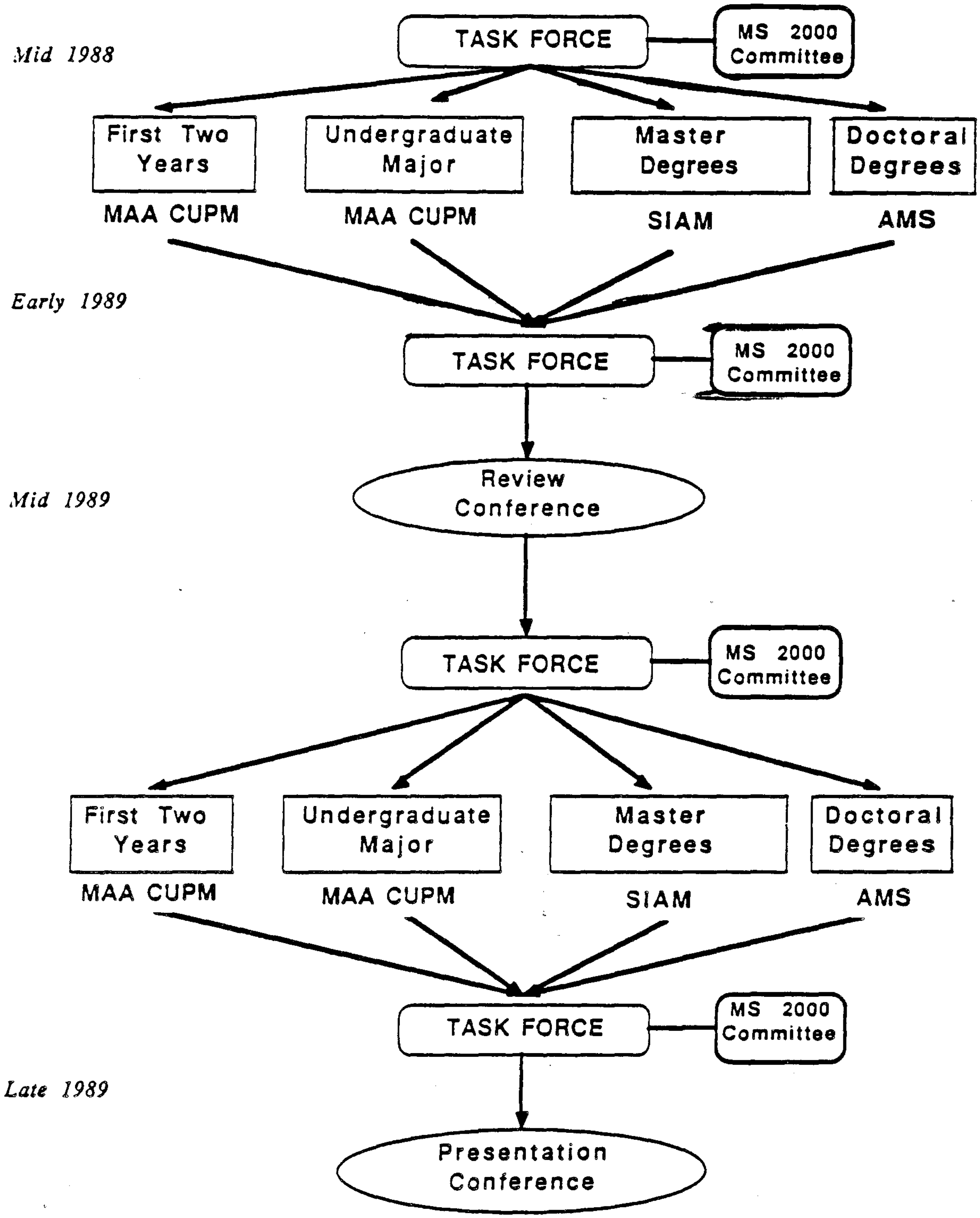


Figure 3

colleges, those who go into industry, and those who are working on a Ph.D. Now SIAM is an organization that is principally interested in the industry component, but they are being asked to pull people together who have knowledge about all these other areas so that they can put together a report on what is going on in all types of masters degrees.

When each report is ready, they will have a conference to pull it all together and discuss relations among the degree levels. After subsequent review and revision, they will arrange a big national conference similar to the "Calculus for a New Century" Conference. This is a typical mechanism that is used in every one of those columns in MSEB as a means of generating national focus. This strategy has already had certain significant effect, notably in calculus. By getting 600 people together to wrestle with calculus, putting out a report, and distributing it widely, you empower those people who want to make a change to go back to their local situation and start working on it. That's really what MSEB is all about.

U.S. Mathematics Degrees

Now the obvious question ought to be, "Why on earth would anybody set up such a huge bureaucracy to look at a subject that has been around for 3000 years and is likely to stay there even if you didn't have a bureaucracy?" One reason is that we are not sure it's going to stay there, at least in the United States. Figure 4 shows degree figures for doctoral degrees, masters degrees and bachelors degrees from 1970-86 in the U.S. Graphs like these began to alarm people. Now, in 1988, it is clear that they are bottoming out. But the numbers of degrees was declining so precipitously during the 1970's and early 1980's that people were really beginning to be alarmed. All of the studies that led up to the creation of MSEB actually started in the early 80's when the decline was still going on.

Decline in mathematics graduates is one indication of the dilemma facing the United States. There are other data that compare degrees to needs and it's clear from such comparisons that we are in deep trouble. (This data is only for mathematics--not computer science. If you add the computer science data, bachelor's degree totals would decline to about 1975 or 1976 and then they would start rising. The total would rise higher than the 30,000 figure in Figure 4: it went up to about 45,000. But now it's going back down again because the bachelor's degrees in computer science are dropping off more rapidly than the mathematics ones are rising.)

Figure 5 shows data for doctor's degrees in the U.S.--which is really what concerns people at the National Academy of Sciences. They weren't too worried about the bachelor's degree issue in mathematics, but they are very concerned about the doctor's degrees. And you can see from Figure 5, first of all, that the number of U.S. males declined precipitously; the number of foreign students stayed roughly constant but expanded a little bit in recent years; and the women, interestingly, have stayed essentially constant for more than a decade. It's uncanny that women Ph.D.'s have remained so constant. It may be because the increased emphasis on making sure that women do continue studying mathematics was exactly counteracted by the general decline in interest in mathematics. So the result came out constant. That is one plausible explanation. But the real problem is with U.S. males. You can see their data shows a monotone decrease and it still is going down. Graduate school enrollment data right now shows that Ph.D. production will continue to decline for a while. Although percentages are not shown in Figure 5, it is clear from the graph that this year more than 50% of the Ph.D.'s granted by U.S. universities will be to non-U.S. citizens.

(Again, this data does not include computer science. The phrase "mathematical sciences" in the United States does not normally include computer science. It includes statistics, operations research, applied

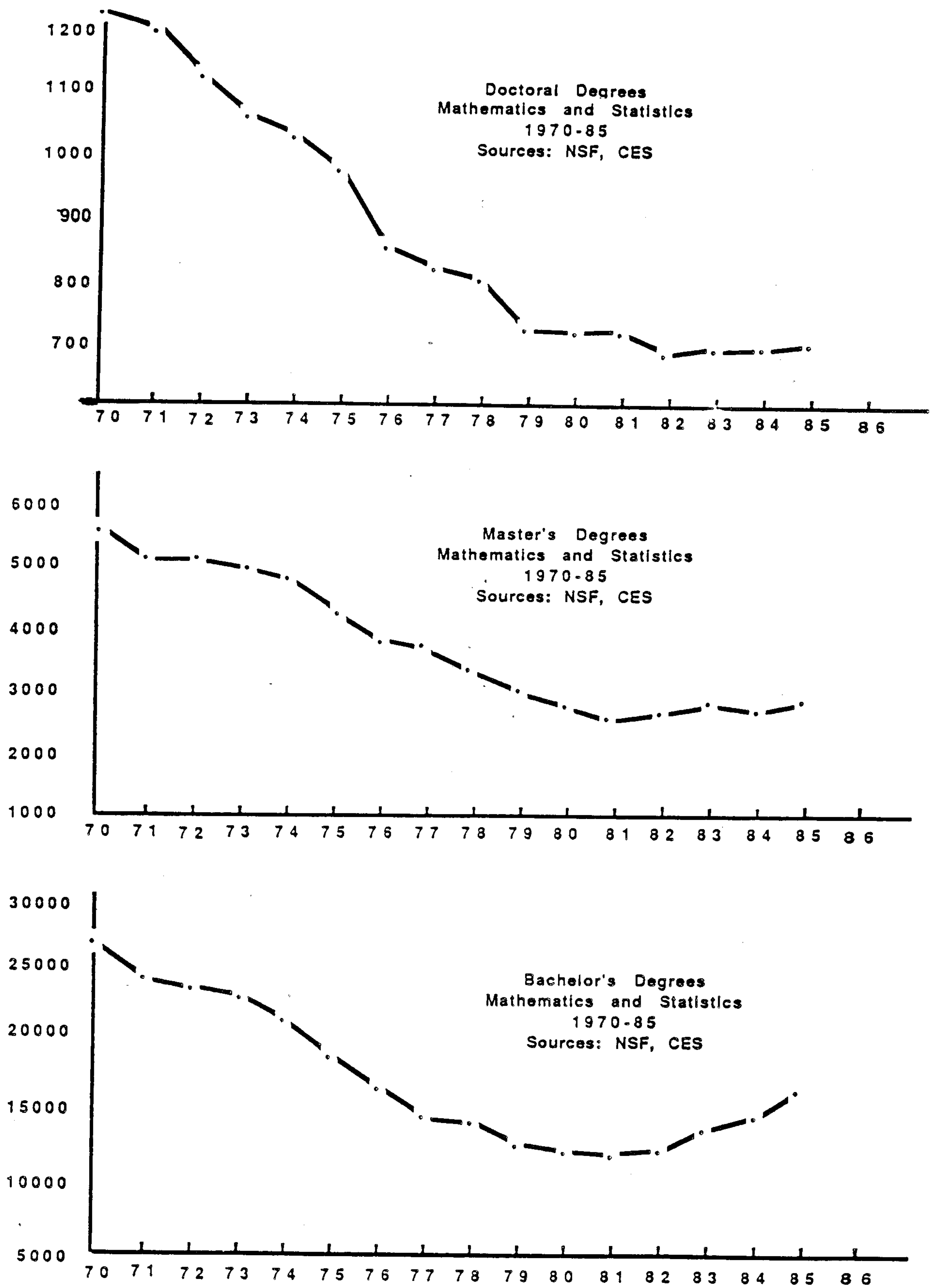
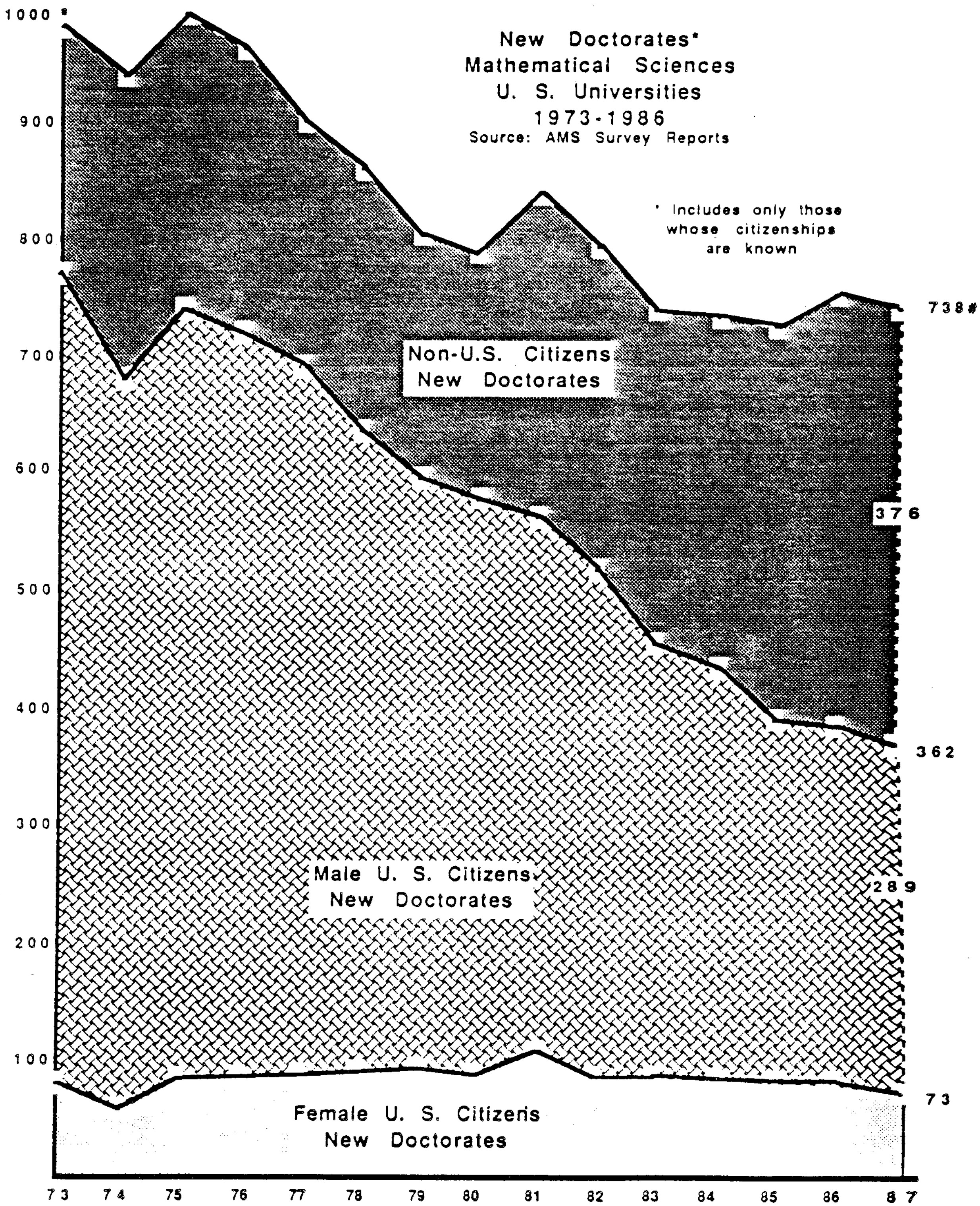

Doctoral Degrees
Mathematics and Statistics
1970-85
Sources: NSF, CES
1200
1100
1000
900
800
700
70 71 72 73 74 75 76 77 78 79 80 81 82 83 84 85 86
Master's Degrees
Mathematics and Statistics
1970-85
Sources: NSF, CES
6000
5000
4000
3000
2000
1000
70 71 72 73 74 75 76 77 78 79 80 81 82 83 84 85 86
Bachelor's Degrees
Mathematics and Statistics
1970-85
Sources: NSF, CES
30000
25000
20000
15000
10000
5000
70 71 72 73 74 75 76 77 78 79 80 81 82 83 84 85 86

Figure 4



In 1987, there were 40 new doctorates with unknown citizenship, while in 1986, there were only 2.

Figure 5

mathematics, but usually not computer science. If computer science is included, it is usually stated explicitly. U.S. computer science Ph.D.'s are about 220 to 250, total. I think the ratio is about 60% U.S. citizens and 40% foreign.)

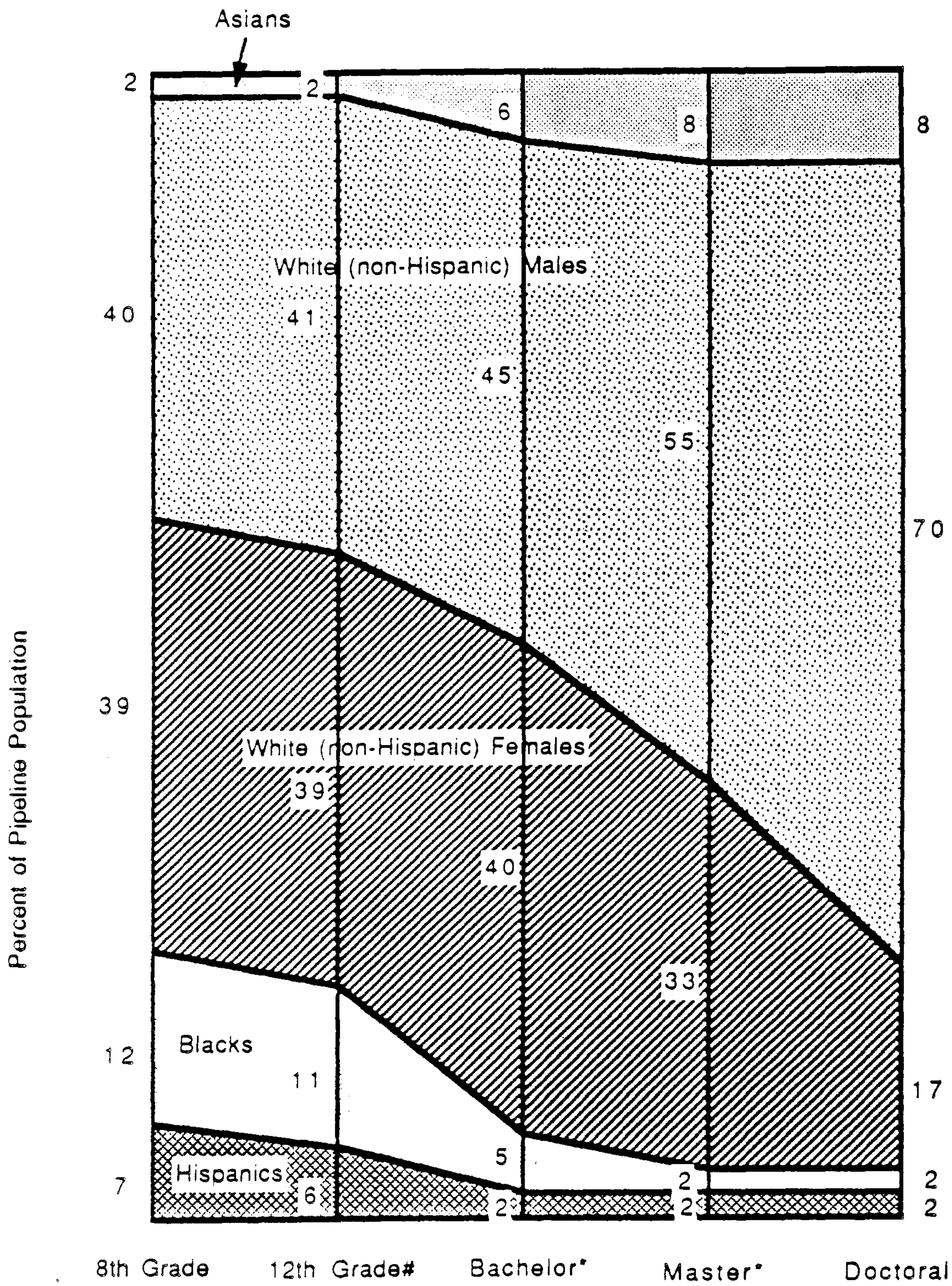
We are down now to about 380 U.S. citizen Ph.D. degrees. If this chart had been carried in the other direction, back to 1962/63/64, you could see that in that period the total was about 350, almost all U.S. citizens. So we are right back down to where we were 30 years ago.

Another type of evidence, which gets back to the point of my title about "All One System," is this: if you look at the total Black, Hispanic, and Native American population in the United States, between 1950 and the year 2000, this minority total, as a percentage of the whole, is rising from 13% back in 1950 to 40% in the year 2000. In public school systems, the percentage of minorities is already well over 50% in almost all the major cities. The ten largest public school systems in the United States are 70% Black and Hispanic.

With figures like that for the population pool as a whole, the kind of indication in Figure 6 about what is going on in mathematics is especially alarming. This figure takes a little bit of explanation to understand. It shows a percentage of the mathematics pipeline, which is why it all comes out constant at the top. In fact the numbers in the pipeline obviously go way down, but this figure is looking only at percentages, at roughly 4-year intervals. It indicates for each one of those levels (8th grade, 12th grade, bachelor's, and doctor's degrees) the percentage of students who are still in the mathematics pipeline classified as Black, Hispanic, White female, White male, and Asians. You can see an enormous drop off: nearly 2/3 of the population at 8th grade is female, Black, or Hispanic and that fraction drops to about 20% by the time you get to a doctor's degree. There is, of course, an increase in the Asian fraction. But our real concern is the under-representation of 50% or more of the population.

Women and minorities are simply being squeezed out by the way that mathematics is taught in schools. That's the other driving force behind MSEB. It is hard to look at this data and still say that we are all one country, that we are all one system. We have to figure out a way to make sure that members of our society who are not making their way through the pipeline will have a reasonable chance of doing so. A great deal of concern is based on the obvious fact that mathematics is increasingly important for jobs that have economic leverage. Many studies from various labor groups predict that because of the computer age, society is going to be divided between service occupations and information-based careers. To the extent that these predictions are right, what is happening in the United States is that the service-based occupations are becoming increasingly identified with the under-represented groups while the White, male, and Asian populations are increasingly dominating the high-tech, information-based society. That kind of division into two cultures is a very serious matter. It is why so many people are willing to work so hard on this particular issue at this particular time.

A Representation of U. S. Students in the Mathematics Pipeline
Estimates Made From Various Data Sources



High school record indicates possible choice of mathematics as a college major

* Degrees in mathematics

Figure 6

Undergraduate Mathematics Enrollments

Now that we've seen the big picture of degree patterns, I'd like to give you one example of relevant detail. Figure 7 shows what has been going on in university enrollments. This one figure shows about as well as anything what's happened to U.S. mathematics. I'll summarize it in one sentence: In the United States, mathematics in the colleges and universities is no different than mathematics in the high schools. For all practical purposes, there is no higher education in mathematics in the United States.

Figure 7 charts enrollments from a study carried out every five years by the Conference Board of the Mathematical Sciences (CBMS). Remedial enrollments refer to those courses covering Algebra I and II; they would at most go through the quadratic formula, but would not go into exponentials or trigonometry or any elementary function work at all. Pre-calculus enrollments represent the fourth year of high school mathematics, including trigonometry and elementary functions; it is the course from which one would be well prepared to go on to calculus. This category also counts finite mathematics, high school matrix algebra, elementary probability, and statistics as a pre-calculus course. Then there is calculus which is well defined, and then everything that is taught beyond calculus. The entire undergraduate major is in that little band up on the top.

Figure 7 shows the pattern of total enrollments in all institutions of higher education in the U.S. In the United States we have a large number of two year colleges. Something in the range of 40-50% of freshmen and sophomore students take their work in two-year colleges. Many of them don't go on any further, but sometimes they transfer into a four-year institution. So to get a proper perspective it is important to examine separately the two-year and four-year data. Figure 8 shows data from only those institutions which offer a four-year bachelor degree. It includes all four-year colleges and universities. The pattern is basically the same as in Figure 7, except for the bottom where remedial enrollment is much lower because most of those institutions have entrance requirements which prevent regular degree credit for courses below the level of a pre-calculus course for students who are enrolled in a bachelor's degree program.

Community colleges, on the other hand, bear the brunt of all that remedial work and so they really are nothing more than an extension of high school (see Figure 9). It's basically high school done over again until you get out of it. You have remedial enrollment, pre-calculus, and a thin band of calculus. Here the other mathematics enrollments which are above calculus on the graph are really below calculus in the mathematical level. This cluster is mostly mathematics for trades: for plumbing, for carpentry, for electronics technicians. In the community colleges the highest mathematics course is calculus, even though in this graph it happens to be the very middle of the chart. You can see again the division that was visible in the total: a huge chunk of remedial work,

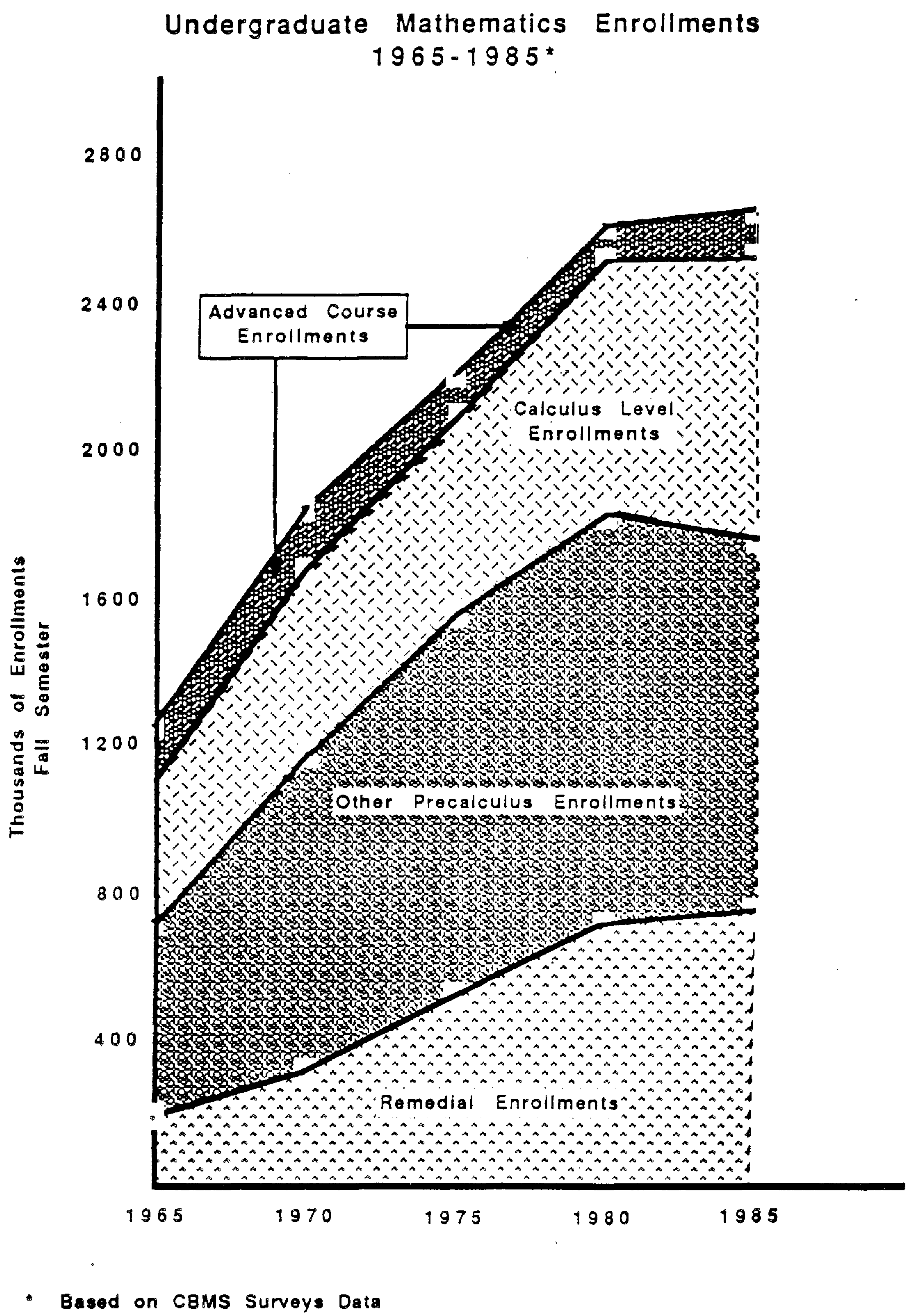

Undergraduate Mathematics Enrollments
1965-1985*
Thousands of Enrollments
Fall Semester
2800
2400
2000
1600
1200
800
400
1965
1970
1975
1980
1985
Advanced Course Enrollments
Calculus Level Enrollments
Other Precalculus Enrollments
Remedial Enrollments
* Based on CBMS Surveys Data

Figure 7

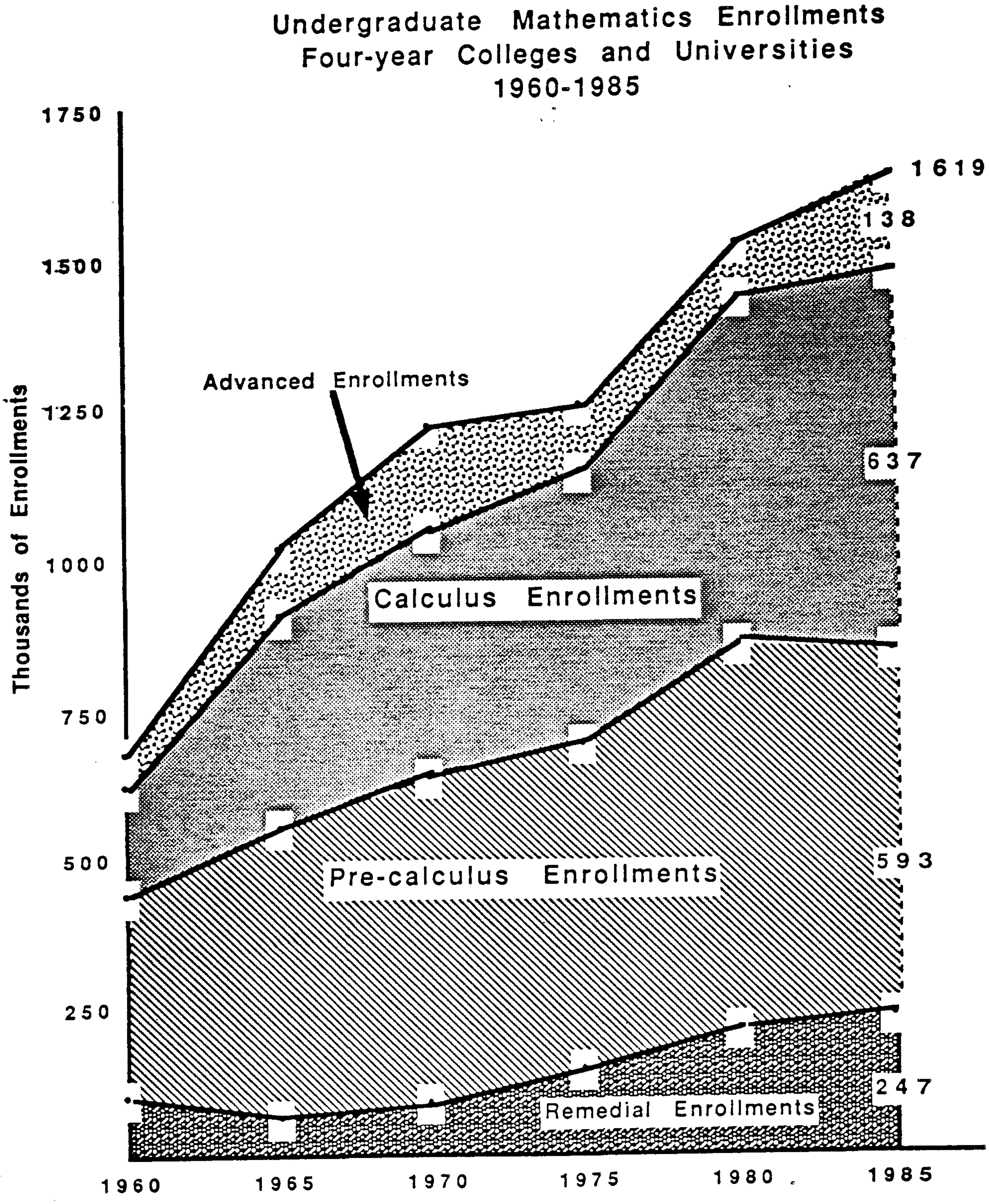

Undergraduate Mathematics Enrollments
Four-year Colleges and Universities
1960-1985
Thousands of Enrollments
1750
1500
1250
1000
750
500
250
1960
1965
1970
1975
1980
1985
1619
138
637
593
247
Advanced Enrollments
Calculus Enrollments
Pre-calculus Enrollments
Remedial Enrollments

Figure 8

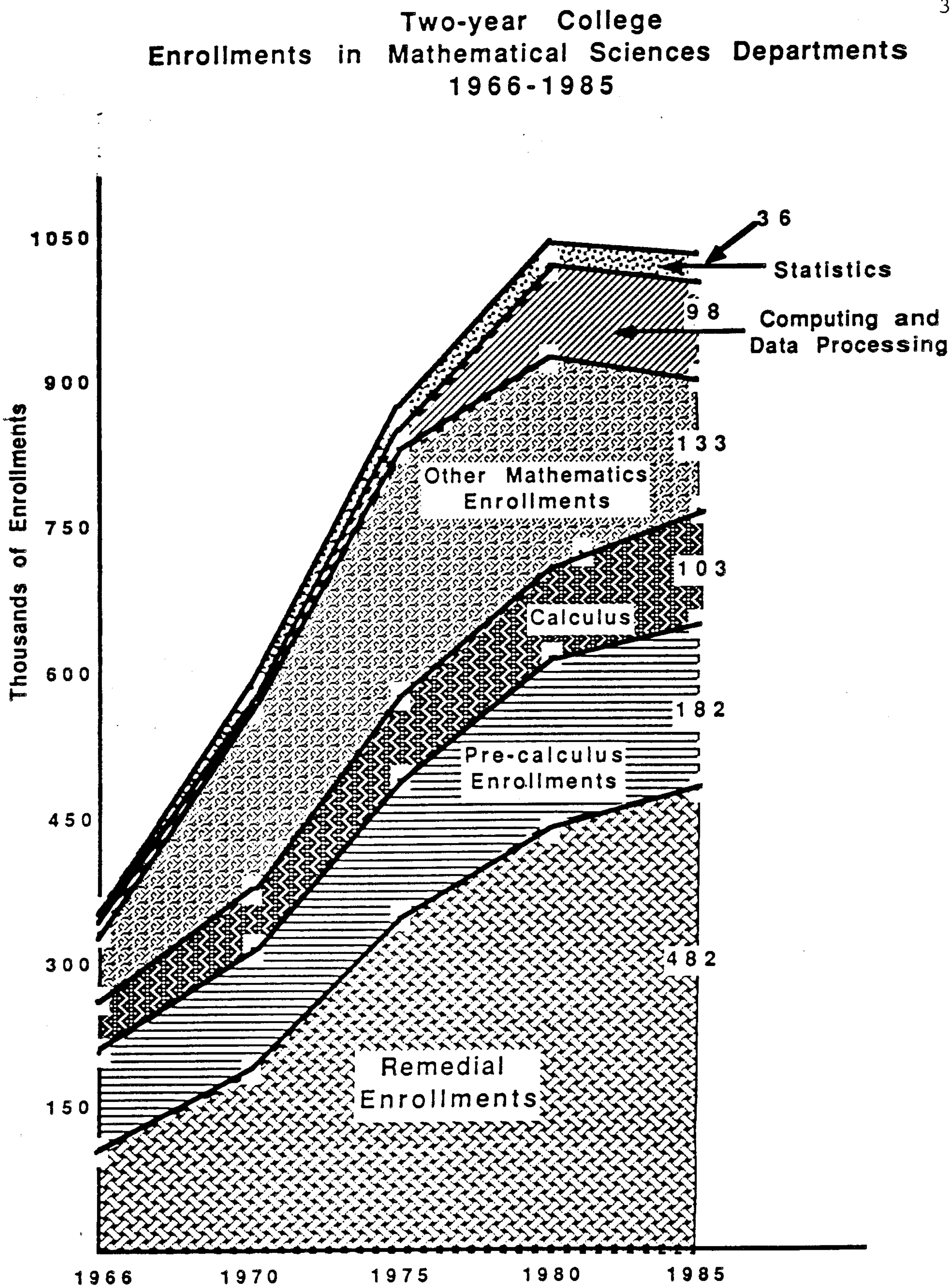

Two-year College
Enrollments in Mathematical Sciences Departments
1966-1985
Thousands of Enrollments
1050
900
750
600
450
300
150
1966
1970
1975
1980
1985
36
Statistics
98
Computing and Data Processing
133
Other Mathematics Enrollments
103
Calculus
182
Pre-calculus Enrollments
482
Remedial Enrollments

Figure 9

almost all of it carried by the two-year colleges. But a significant chunk is also carried by the universities.

The graphs provide a very strong sense that the academic level of college mathematics education is not appropriate to the needs of the nation. College enrollments are synchronized with the colleges' and universities' images of themselves, not with what the public image is. It's not well synchronized with funding, or with the faculty, or with anything else. Take a typical university situation where you have large numbers of remedial enrollments. Those universities are staffed by Ph.D. faculty who teach a variety of different things and do research. By and large, they are not interested in remedial teaching: they are almost universally not good at it. There is no sound academic argument for structuring an educational system that way. It would make much more sense for students to learn Algebra I when they are 13 years old and a freshman in high school than to have them doing it when they are 18 or 20 years old in a college environment. This obvious mismatch, coupled with the fact that ethnic distribution in the United States is changing, has led to a great deal of concern for figuring out some way to change the whole system.

Fundamental Issues

What I'd like to do now is to go through the tentative positions emerging from the NRC study and comment about these things one at a time. I am at the moment working for the Mathematical Sciences Education Board and the project MS 2000 on a report that pulls together issues growing out of all these different strands. I have selected from the draft of the report--and I emphasize that it is just a draft--key sentences that indicate major themes. In a way, you are the wrong audience to be addressing it to because the report is not being written for mathematicians. It's being written for politicians, educators, and others who are not in mathematics. So you will have to put yourselves in their mindset to figure out what's going on.

<u>Quality mathematics education for all students is essential to a healthy economy</u>. There are two words here that are really important. The most important one is "all." It is there to say that it is not good enough for the country to have quality mathematics education for the top 20%. That's not good national policy. We have to have appropriate and high-quality education for all students. It's clearly the case that we are not delivering that right now. The other important word, frankly, is "mathematics." Politically it's science or literacy that is often mentioned. People hardly ever mention mathematics by itself.

An example of this occurred about two or three weeks ago when the National Endowment for the Arts issued a major report recommending a curriculum in fine arts as a required component of schooling from kindergarten to grade 12. At every level and in every respect it would have involved more work in art than is currently going on in mathematics. They made a strong case for it and they got a lot of publicity from the report, largely because the people who are likely to be the movers and shakers in a community are the patrons of theatres--the people who support symphonies, who have money, who respond to culture.

Mathematics doesn't make the same kind of case. It is taken for granted. Everyone just assumes that because mathematics is in the schools, it will stay there forever. Well, once all the students come to class with a little machine in their pocket which does all the mathematics that they see in school, some politicians are going to say "Why do we need mathematics teachers?"

<u>Applications, computers, and new discoveries have totally transformed the nature of mathematics</u>. Remember I said earlier that in the MSEB frameworks for a mathematics curriculum, they decided to add a "philosophy" section to the front of it? That's because as professionals in mathematics we know--but outsiders do not--that the great increase in applications in mathematics, the role of computers, and a lot of new discoveries in mathematics itself is really changing the discipline. If you look at what's happening now at the curriculum level and what mathematics was like in 1950 or 1960, it's very different. There's more emphasis on geometry, on dynamical systems, on computer simulations--on a whole lot of things that were totally invisible then.

Now the public doesn't know that. The public's image of mathematics is that it began with Euclid and it ended with Newton. They believe that all important mathematics was laid down long ago, and if you just teach those principles then anybody who needs some mathematics will just find it in the right book--or pluck the fruit off the tree of knowledge--and put it to use. They do not think of mathematics as a growing discipline. This report is going to try to make that case--that mathematics is changing; that mathematics as a discipline is different now than it was before; and that all assumptions about what mathematics used to be have to be examined anew. They don't all have to be thrown out, but at least they can't be accepted unchallenged.

<u>Mathematics must become a pump rather than a filter in the personnel pipeline of American education</u>. This third statement in fact was a subtitle of the calculus conference. The phrase has an interesting history of word usage. Robert White, who was the keynote speaker at the October 1987 Calculus Colloquium in Washington, is the President of the National Academy of Engineering. He used that phrase in his talk and it was literally and truly his phrase. People like White have speech writers, so his staff people at NRC wrote an outline of what he should say. He added a lot of ideas of his own that were obviously related to personal anecdotes. There was in the notes prepared by his staff a phrase about mathematics as a filter: the problem with calculus is that it filters students out of careers and blocks their potential. White, the civil engineer, turned it around and put it this way.

Since then, in discussion with people it is interesting to see how this brings up an issue of great importance for the kinds of things you've been talking about. Namely, who is responsible for education? Is it the people who are being educated or the people who are doing the educating?

The filter metaphor suggests that it is the students who are responsible for getting educated, with the faculty setting exams and saying "Those of you who learn enough can pass. We'll certify you, and you can go out into society. Those who don't pass can try again, or go away."

The metaphor that White used--a pump--suggests a shift in the focus of responsibility. It says to the faculties and the universities, that they are responsible for the students they get no matter what they know or don't know about mathematics. Their job is to work very hard at trying to move them along. So they should push them a little bit ahead--be a pump--not just be a filter. This metaphor changes dramatically the whole way you think about what you are doing as a faculty member.

Mathematical literacy is essential as a foundation for democracy in a technological age. There's a lot of talk in the United States--I presume it's going on in Canada as well--about this cultural literacy book by E.D. Hirsch that purports to give you 5000 phrases that will make you an educated person. Have you ever seen that book? You should look at it. Actually my description is an unfair caricature. Critics have been panning Hirsch for his lists. I actually sat down and read the book two months ago and I must say it's not all that bad. He's got this infamous list in the back, but the first half of the book is an essay on communication among peoples--about how you can make a point, how you can persuade, how you can communicate. Hirsch's point is that literacy isn't just grammar and vocabulary, but it's the cultural significance of certain phrases that take on meaning because you know the history and the context and the culture. Then, of course, he says, "These are the phrases you ought to know." That's the place where people start arguing with him.

I don't think there is much dispute at all about his general point, which is that you can't communicate in society unless you know the rich significance of the words, terms, and images that are being used. What we're trying to say in this report is that what Hirsch is saying about the cultural ingredients that grow out of historical, political and literary language is equally true about mathematical language. If you are going to understand risks in society, or discussions about inflation rates, if you're going to understand arguments about national defense, about taxation, about all sorts of issues in the news, you have to know a reasonable amount about mathematics in order to have a literate sense of understanding of what's going on in those discussions.

Public acceptance of minimum standards contributes significantly to poor performance levels in mathematics education. This makes a public statement about the problem caused by all-too-many adults who are not scientists who say they were never good in mathematics in a way that is bragging about it, or who are proud of the fact that they couldn't do mathematics. Others who may not speak up just take mathematics for granted. When their children don't do well in mathematics--not so much the well-educated public--the average public, will excuse that. They'll say "That's understandable because I never could do it either." And so there is an acceptance in the public of lower standards in mathematics than there is in other disciplines. That public acceptance is clearly contributing to the actual low levels of performance. If we had greater public insistence on the fact that all children can learn mathematics then in fact more of them would. We let them off the hook too easily.

To meet tomorrow's needs, we must invest today in our nation's intellectual capital. As our first assertion expresses a question of clear national need, so this exhortation is again a political statement. Politicians everywhere are notorious for just looking at whatever will make an impact on the next election. A two-year or four-year time frame for some project is just not appropriate to education. If we are going to have a well-trained work force in the year 2000, we have to be in there working on it right now.

Weak preparation in mathematics contributes significantly to maintaining the economic weakness of Blacks, Hispanics, and Native Americans. If you look at salary comparisons between men and women, between the White population and the Black and Hispanic populations, and do a study that compares salaries to the kind of jobs people have as well as the kinds of jobs they are prepared for, the common denominator in the separation between high and low paying jobs tends to be jobs that require more and more mathematics versus those that do not. To the extent that the Blacks and Hispanics drop out of the mathematics pipeline very early, they close out the opportunity for getting involved in jobs that put them in positions of leadership. As a result, it sets their community back.

Another section of the report, which I didn't summarize here because it's more complicated, deals with the impact on women. The reason that's more complicated is because the drop-out for women in the United States is occurring essentially after the bachelor's degree. 47% percent of the bachelor's degrees in mathematics are women but only 17% of the doctoral degrees are. So there is a very different phenomenon going on--not the same kind of phenomenon that affects the minority populations.

Students retain only that mathematics which they learn by a process of internal construction and experience. This issue is quite important although I think we're going to have a very hard time selling it to the community. I, at least, have been persuaded by all the reading and discussions I've had about the way students learn mathematics. I've actually heard a lot of similar views at this session, so I think you probably know all this as well as I do.

The basic point is that lecturing is a failure for most students. The students that benefit from lectures would also benefit from probably anything else, and those that don't benefit from lectures aren't learning mathematics. What we have to do is to engage students to get them personally involved in what they are doing. There is a large group of researchers in the U.S. that has been working on what they call a "constructive methodology" in mathematics education. I know this work only as an outsider, but I've heard people lecturing on it and talked to many of them about it. Their arguments resonate with my personal experience and with that of a lot of other teachers. Students tend to really retain for a long period of time only things which they have been consistently engaged in by making it their own agenda and their own activity. They don't get that out of listening to lectures or doing routine exercises or homework problems. If a good lecture class with good homework works well, the best that can be said for it is that it will enable the student to earn a good grade on the exam at the end of the course, but it's not likely to last very long unless it is reinforced by something that engages the student.

It used to be the case that for a lot of mathematics students it was a physics or an engineering course that provided this engagement. When higher mathematics existed only for the purpose of serving physics and engineering, this may not have been quite as much of an issue because the mathematicians could present a tool and then three days later and forever afterward it would get used in these other courses. The students would engage it there and learn it.

Now of course we have students who are studying mathematics by itself, or who are studying it because they are going into law or biology or all sorts of things. Students are studying mathematics for lots of different reasons and there is absolutely no guarantee at all, in fact almost a certainty, that whatever particular thing we do in the mathematics class is not going to be amplified by these other subjects they're taking. As a result, they get none of that reinforcement anywhere else. So now the problem has come back to roost in the mathematics department. We must figure out a way to develop that engagement on our own.

There are a lot of studies--Alan Schoenfeld has been involved in some, but there are others as well, mostly of high school aged students--that show the great imperviousness to instruction of a priori ideas. They conduct studies on students who have a certain image about the way that something operates--a geometry problem, an arithmetic technique, or something in physics. The students will listen to a lecture, and do homework problems that show them another way to do it, yet they won't believe a word of it. They'll learn it all, but their intuition remains locked on their original assumption.

I think the most extreme example of that is from physics, but it's very similar to what we see in mathematics. Jack Lockhead of the University of Massachusetts at Amherst gave entering graduate students in physics a little test involving the physical performance of physical objects. One I remember was a spiral shell with a little bead in it that was being propelled outward by a blast of air from the middle. The question was to draw the path of the bead after it leaves its track. There were three or four questions depending on the velocity of the beads, whether it was going slow, fast, or very fast. Almost all the graduate students followed the curve, varying the curvature a little depending on whether it came out slow or fast. If it came out slow it would faithfully follow the spiral's curve; if it came out very fast it would only follow it loosely.

Without question, all these students could handle the mechanics of that problem in a mathematics or physics class. After all, they were entering graduate students in physics. But their formal knowledge--what they learned in their physics and mathematics classes--never penetrated their subconscious intuition about the way objects behave. Schoenfeld gives examples in geometry about the same kind of thing going on in mathematics. This statement is basically saying to mathematics professors, "You must find a new way to teach, because the old ways aren't working for today's students."

<u>Prospective teachers should learn their mathematics in a manner consistent with the style with which they'll be expected to teach</u>. This again speaks to university faculties, but the point is based on classroom observations of teachers at all levels in high schools, in middle schools, and even in elementary schools. The predominant method of teaching at all levels is instruction--that is, lecturing. The teacher presents something--it might be a short lecture if it's for a fifth grade class, or a little longer lecture if it's for a tenth grade class; but they always present something, they give examples, and then they sit the children down to work out the template. Now virtually all of the mathematics methods courses that those students have ever had when they were in universities showed them many other ways to teach. These methods courses gave them many examples about using hands-on methods, or getting students involved in projects. Usually new teachers do that for no more than six months when they get into the schools, and they go back to lecturing.

Now they're several plausible conjectures going on about why that happens. One is that the school system in its structure is so rigid and so bad that teachers just get forced down by the structure. In this case, nothing they learn from their mathematics methods course will have an effect. If the restructuring that the Carnegie Commission is recommending has an effect, this system may change a bit.

Most people don't seem to believe, however, that structure is the real issue. The reason they don't believe it is because of evidence from the community colleges and the four-year colleges where the faculty have a great deal of autonomy. There really are very few places where department heads or chairmen do anything that imposes on your teaching style. But in fact, everyone still lectures there as well.

So the general conclusion that keeps recurring is that teachers teach in fifth grade the way they learned mathematics: they learned it by being lectured at, so they lecture at their kids. The challenge is to break that cycle. What we've really got to do is encourage more diversity of teaching style in the universities. That will simultaneously engage students more so they'll learn mathematics better and make better teachers of them when they themselves go into schools and become their own teachers.

One final anecdote on that point is about manipulatives in elementary school. Everyone has been recommending that for about 30 or 40 years. There are all sorts of companies that make money selling kits and manipulatives to elementary schools. But the people who observe classrooms say they're not used. At least not very much. They're very rare.

California has a new structure for their objectives in mathematics and science and they're trying to impose this on the textbook industry. I'm sure you've heard some of the stories about that, since they wound up rejecting all of the textbooks the first time through on this. I heard recently that one of the things they've done is to require that for certain grades in elementary school, on certain pages, the textbooks must have blank pages. The reason for that is that the normal pattern of the elementary school teachers is to just turn the page and give it to the kids to work on. By putting blank pages in certain spots, it forces teachers to go to their teacher's guide to figure out what they're supposed to do. What the teacher's guide says is that this day you're supposed to use manipulatives--so close your books! That's how California is trying to get manipulatives into the classroom.

<u>What you test is what you get</u>. <u>No curriculum can be effective unless tests match curricular objectives</u>. We have to figure out a way to make sure that tests match the curriculum objectives, because no matter what you say on the curriculum, the teachers teach to the tests, so that's what you're going to get. Whatever is on the tests is what the students are going to spend most of their time working on.

If your curriculum objectives say that students are supposed to learn to communicate mathematics--to write it, to read it, to be able to talk and argue with each other--then you better be sure that your tests test whether they can write mathematics, whether they can read mathematics, whether they can engage in discussion with each other about mathematics. Now that's hard to do, but it's absolutely clear that if you don't do it, then you might as well throw out that curriculum objective because you're not going to get anywhere. Right now almost all of the standardized testing in the United States emphasizes lower-order skills that can be developed, learned, and forgotten very quickly.

<u>The United States must create a tradition of mathematics specialists for elementary schools</u>. This item is controversial. In fact it wouldn't surprise me if this one doesn't show up in the final report. It passed one level of review but it's still being debated and it may get thrown out at the next level. I should emphasize--because the text around it does--that "specialist" here is being used in a general, varied sense. There are many different ways in which you can employ specialists. What we want to do is to encourage a tradition in which all of those different ways are being used.

For example, you could have one teacher in an elementary school who is a mathematics specialist--who has special interest and training, who helps other teachers with lesson plans, and who takes on certain children who are either more advanced or who have great difficulty and develops special programs for them. That would be one model. In another model, like the fine arts specialist, the person is assigned to actually teach the mathematics lesson in certain classes. That's the model that I think is least likely to happen because it's the one that is most often resisted. Even though it's welcome and accepted in physical education and art, but in discussion we've had, it's not welcome in mathematics.

The most common model, and the one I think is the easiest and most natural thing to do, is paired teaching. In any school, if you take half the teachers who are the most interested and most confident about mathematics and you pair them with other teachers who generally are more interested in language arts, then in first, second, and third grade you have these teachers teaching in pairs so that they just trade classes for different subjects.

Paired teaching is done in a lot of schools already. If more schools did it, then we'd get better instruction. The problem now is that easily 70% or 80% of the children receive their elementary school mathematics instruction from teachers who are frightened of mathematics. That's what we want to break down, which is the point of that particular recommendation.

<u>The United States needs to reach consensus on new national standards for school mathematics</u>. This conclusion lays the groundwork for the new <u>NCTM Standards for School Mathematics</u> which are under development. It makes the case, first, that we have to have something that is new to reflect the changing nature of mathematics, and second, that we need a national consensus on what a mathematics curriculum should be. In the United States there is an incredible resistance with anything to do with a "national curriculum." Because of political concerns, any document that used a phrase that sounded anything like that would never get beyond a committee room. This issue goes right back to our constitution, to the Civil War, to everything dealing with states' rights: in the United States, the states are responsible for education.

The fact that there is an urgent national need for quality mathematics education has to be reconciled with state responsibility. In fact, because of the textbook industry, because people move a lot--the average school child is in three different school districts between kindergarten and graduation from high school--because the college entrance situation is pretty much uniform, we really have something close to national standards, but no agreement that there ought to be such a thing.

<u>Priorities for mathematics education must change to reflect the impact of computers on the way mathematics is used</u>. To most thoughtful mathematicians, this is sort of self-evident. It is the kind of issue for which Henry Pollak has been a forceful and long-time advocate. I'm sure most of you have seen the paper that he wrote about six years ago called "What is Still Fundamental and What is Not" which was submitted to the U.S. National Science Board Commission on pre-college education. The point here is not to use computers to teach mathematics, but to re-think the mathematics we're teaching in the presence of computers. Some subjects needs more emphasis, some need less emphasis. Some methods of teaching must change. All these things have to be worked on.

<u>Calculators should be used throughout elementary school, both in mathematics and in science</u>. This emphasis is very strongly supported by 80% of the mathematics education community, and very strongly opposed by 20%. It is very controversial among the public. Many of you may have

read that the Chicago public schools have issued calculators to every single child at every single grade level as part of their standard school equipment. Some publishers are now putting touch-panel vinyl calculators on the jackets of elementary school books. On the other hand, just three years ago only half of the students at the middle school level use calculators regularly. In the national assessment summary that is just coming out now there is a report that 90% of the children come from homes in which calculators are present and used regularly, but only 30% of them report that they are used in the schools.

When you talk to teachers about why they don't use calculators, the first thing they say is that it wouldn't be fair because not everybody has a calculator. Now that's not really their problem. The problem is that they don't want to use calculators. Dorothy Strong in Chicago called their bluff by convincing the school board to pay for calculators for every child in the city of Chicago, which, she said is the cost of two school lunches--which they pay for all the time.

<u>Secondary school mathematics must be made more relevant to the needs of students</u>. This item about secondary school mathematics is interesting. If you look at the elementary school curriculum, virtually everything that goes on--arithmetic, percentages, some simple geometry--is easily and directly within the experience of what children are going to need to do. If they are going to own a car or work for McDonalds, if they are going to shop or calculate taxes, they can see that what they're doing in school has some bearing on their life, or at least on that part of their life that they can foresee.

But that's not true of most parts of secondary school mathematics. Three-fourths of the secondary school curriculum is basically prerequisites to calculus, and the other part is a stalking horse for critical thinking in the guise of geometry. There's no real application of factoring trinomials that children are going to see, or of doing trigonometry. Even for those who are going to college, waiting until calculus is too long a deferral of reward for what they're working on.

So the emphasis here is to suggest that the secondary school curriculum is the one that really needs to be re-thought to make it more relevant, more immediately applicable. Throw out some stuff that can now be done by machine. Everybody is going to have Maple or Mathematica running on their machine soon, so a lot of routine algebra can be diminished a bit. Maybe some more interesting things can be done instead--for example, probability and statistics. Students can read about risks of smoking and automobile accidents and things of this sort that might have more relevance than the formal prerequisites for calculus.

<u>All students should study mathematics throughout every school year</u>. Here's another statement that may not survive the next levels of review. At the last meeting of MSEB there was a four-hour debate on this one item, about what we should say about how much mathematics students should study in high school. There were umpteen different proposals. This version is

the one that emerged from that discussion, but I've been through enough of these to know that six months from now they'll change it again.

The argument is an important and serious one. If you say anything less than this, if you say that three years is enough, or that two years is enough, then you provide legitimacy to drop out and that puts us right back in contradiction with the All One System idea. The reason that the under-represented groups--especially Blacks and Hispanics--are not able to take courses leading to specific careers in college is because they dropped out of high school mathematics. Saying to them, then, that it's all right to take only two or three years is wrong. On the other hand there is a great deal of hesitation about suggesting that every student --no matter whether they're going to college or not--should study mathematics all the way through high school. We do require that for English and social studies, but it is not required for mathematics.

In the United States the variation for mathematics requirements currently is from 0 to 3 years. About five years ago it would have been from 0 to 2 years, but recently a few states, as a result of all this turmoil, have increased their requirements from 2 to 3 years. But those states that have done that have also increased their dropout rate--not just from mathematics but from high school. Students will simply drop out. So it's a difficult issue to figure out just how one should say this.

The real point of this recommendation is to try to encourage schools to develop a structure in which there is a common curriculum. I've not stated that as one of my main points, but it is an underlying theme: make sure you don't trap students into dead-end programs. This happens far too often under the present system. We should, instead, let everyone go through a common curriculum, but go through it at different rates depending upon how rapidly they can learn the material and depending on their own maturation. Some students mature intellectually a lot earlier, and some will do it later. But if they're following the same curriculum, they will all make progress. Then you can legitimately say that even the slowest learners--the ones that haven't yet caught on to mathematics or haven't figured out why they're in school--ought to continue their study of mathematics, so that when they do figure it out, they will then have a platform that they can build on. Frequently this happens when they're thirty and they come back to a community college and want to get going again.

<u>The undergraduate mathematics major should provide a platform for careers in any field</u>. This is a similar statement but at a different level--about the undergraduate mathematics major. The issue here is to disabuse the notion that an undergraduate mathematics major is only for people who are going to be prospective mathematicians or mathematics teachers. In fact, mathematics is one of the most versatile undergraduate majors for <u>any</u> kind of career. The experience of the schools that have framed a major in those terms shows that doing this is the most effective way to increase the number of students going into graduate school in mathematics. The argument is stated here in terms of broad goals, but it is really an

argument whose hidden message is for the research mathematician who has blinders on and is only interested in creating more Ph.D. mathematicians.

The message is that the best way to do that is to make your undergraduate major into a broad major for everybody--because that will get more students to continue their studies in mathematics because of the intrinsic nature of mathematics. The appeal that mathematics has for certain students will wake some of them up by the time they are juniors or seniors in college. They'll say "Hey, you know, I might want to go to graduate school." If you lose them in the freshman year, you're never going to see them again. So this statement conveys a dual message. The undergraduate major should be a broad major, should serve all students for all careers, and in the process we will reverse those negative trends that I had on my opening figures.

Working Group A

TEACHER EDUCATION: WHAT IT COULD BE

Olive Fullerton
York University

Pat Rogers
York University

TEACHER EDUCATION: WHAT IT COULD BE

Participants:

Hugh Allen	Olive Fullerton
Christine Keitel	Lionel Pereira-Mendoza
Pat Rogers	Rick Scott

Introduction

> The process of communicating knowledge is the most important stage in making knowledge effective.
>
> \- ascribed to Bertolt Brecht

The purpose of this working group was to review the current status of teacher education in the Canadian context at both the pre-service and in-service levels. The intent was that participants would discuss what teacher education in mathematics is today, what educators would like it to be, and would make recommendations to help accomplish those objectives in the future. The interests and experiences of the group members centred around the elementary level and this became the focus of discussion in our three working sessions. We begin this report by outlining the content of those discussions and then we summarize our general areas of concern and the possible solutions and additional problems we considered. We conclude by listing our main recommendations.

Session #1

At the outset participants described in detail the models of teacher training in which they were involved. These included concurrent and consecutive pre-service training and in-service training. We explored the similarities among the models and discussed their strengths and weaknesses from the different perspectives of educators, teacher candidates, practising teachers and their students.

Session #2

Our second session was spent discussing significant common issues of concern in our individual programmes and the frequently inadequate preparation and experience of teacher candidates in those programmes.

Session #3

In this session we addressed the sometimes competing concerns of academic researchers and practitioners in the field. We decided that it was important to make any differences between the two groups explicit and that teacher training should ideally be a collaborative endeavour which values the unique contributions which each is able to make.

Areas of Concern

Five major areas of concern were identified during our three sessions:

1. Many elementary teachers are profoundly uncomfortable with mathematics and this discomfort is often generated in their students. The process can be a dangerously cyclical one.

2. Elementary teachers frequently try to teach mathematics by 'telling'.

3. There are no mathematics requirements for teacher candidates at the elementary level.

4. Current models of teacher training do not require a continuum of training or professional development for teachers - there is no requirement that practising teachers renew their certificates or that they upgrade their teaching methods or mathematics background. Neither is there any institutionalized support for teachers who are willing to go out on a limb and try new ideas in the classroom.

5. The time allocated to training in mathematics pedagogy is already extremely limited and leaves no time for adequately addressing the concerns raised above.

Possible solutions and additional problems

Potential solutions offered for discussion fell between two extremes of a total separation between the academic training of teachers and their practicum experience on the one hand and a blending of theory and practice which recognises and takes advantage of the differences between them on the other.

1. Feelings of self-confidence, success and love of the discipline must be engendered in all students of mathematics. Ways must be found to help teachers better understand how their attitudes towards mathematics are mirrored in their students' responses to mathematics.

2. Elementary teacher education must foster a constructivist view of mathematics. Students understand and retain only that mathematics which they learn by a process of internal construction and experience rather than by passive absorption. Prospective elementary teachers, then, should be taught mathematics in a manner consistent with the style in which they, in turn, are expected to teach it.

3. One sure way to increase the comfort level for teachers of elementary mathematics is to ensure that they have sufficient background in and success with mathematics themselves. Some participants felt that, at a minimum, elementary teacher candidates should be required to take at least one university mathematics course. However this also raised another concern. Where such courses are offered at present they are rarely taught in a manner consistent with a constructivist perspective. There is no point whatsoever in subjecting future

teachers to more mathematics if it is delivered to them in the classic lecture mode.

4. Participants supported the case for a "vanishing certificate". If teachers were required to be recertified or to retrain, this would provide them with support for reflection on their established practices and with information on recent research and thinking on effective pedagogy. It could also foster the notion of the teacher as researcher. As well, such retraining should provide opportunities for teachers to engage in peer coaching. This is an effective way of providing new and experienced teachers with the support necessary for experimentation.

5. Mathematics education must be afforded a more central role in elementary teacher education.

Recommendations

1. We opted for a collaborative model of teacher education in which the two views of the academic researcher and the practitioner are both valued and reconciled. In such a model the two groups would become partners, each with a different and essential contribution to make to the training of future elementary mathematics teachers.

2. It is essential that we develop ways for institutionalizing the investigative mode of doing mathematics both in primary/junior teacher education and in primary/junior classrooms.

3. Careful consideration has to be given to the question of requiring of new teachers more preparation in mathematics.

4. A continuum of retraining and professional development must be designed and established.

Working Group B

NATURAL LEARNING AND MATHEMATICS

Gary Flewelling
Wellington County Board of Education

Lesley Lee
Concordia University

Participants:

Tasoula Berrgren	Linda Brandau	Larry Copes
Sandy Dawson	Don Eastwood	Gary Flewelling
Jack Hope	Christine Keitel	Lesley Lee
Eric Muller	David Pimm	John Poland
Ewa Puchalska	Sol Sigurdson	David Wheeler

Introduction

The following 'advertisement' appeared in the preconference literature for Working Group B:

Working Group B: Natural Learning and Mathematics

Leaders: Gary Flewelling and Lesley Lee

By 'natural learning' we mean an approach to learning in which both the affective and cognitive aspects of the learner's activity are taken into account and brought into harmony. Mathematics is seen by many people as an abstract and impersonal activity, and it is often taught as a rule-bound set of mechanisms 'for finding objective answers to quantitative questions' - as if, indeed, all human feelings must necessarily be excluded. The group will continue the explorations begun in working groups at earlier Study Group meetings (on 'Mathematics and feelings' and 'Small group work in the classroom') into the ways that can be used to bring mathematics into a natural learning context. Issues that may be discussed, subject to the special interests of the group members are:

- active and interactive learning
- an enquiry/problem-centred approach to learning
- students' self-motivation and self-evaluation
- integrated vs compartmentalised learning
- student as recorder/as reporter/as teacher

Defining Natural Learning

Over the three-day period that Workgroup B met, no consensus was reached on a definition for the phrase **natural learning**.

David Wheeler offered that, "The word (natural) is not exactly meaningless; indeed, if anything, it means too much in that it hints at a host of potential properties but settles decisively on none." To prove David correct, the group discussed a host of likely properties of natural learning and saw to it that they settled decisively on none of the properties discussed.

The lack of consensus on a definition for natural learning, did not overly handicap proceedings and probably contributed in a major way to making the discussions lively and wide-ranging.

The lack of consensus on a definition didn't keep members of the group from offering their own definitions. Jack Hope for example, saw natural learning as, "the end point of a continuum representing the degree of control a learner has over his or her learning." Another member of the group defined natural learning as, "how you behave when no-one is watching you (not even yourself?)."

As John Poland remarked, "The image of natural learning most easily accepted by our workgroup was of a child stimulated by curiousity, exploring and learning; self-directed, with no external standards to be met. This image contrasts with the image of mathematics (traditional learning) as a body of knowledge, language and tools, external to the learner, rigidly logical and intellectually demanding."

Eric Muller contrasted the two images in the following way:

Natural Learning		Traditional Learning
1	self-directed	0
4	self-motivated	1
1	self-evaluated	1
4	self-appreciation	0
1	peer-directed	0
1	peer-motivated	0
1	peer-evaluated	0
1	peer-appreciation	0
1	teacher-directed	2
0	teacher-motivated	5
0	teacher-evaluated	5
0	teacher-appreciation	1
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15		15

The group spent its time looking at ways in which a natural learning approach could be used in the learning of mathematics, particularly in the light of the above view of mathematics.

Some of the observations from the three days of discussion are listed below under the headings of Natural Learning and the Classroom Environment, Some Thoughts for Students, Some Thoughts for Teachers and Natural Learning and Peer Interaction.

Natural Learning and the Classroom Environment

i) A frank and accepting climate is a prerequisite for natural learning. A climate where students need not feel that they have to preface most of their questions with, "I know this question sounds stupid, but ..."

ii) The classroom should be a place where feelings about ideas and experiences can be shared. A sense of community should be developed in the classroom.

iii) The classroom should be a place to think and investigate and not a place to memorize, accept and regurgitate.

iv) Materials should be available to provide food for conversation, which when digested may become food for thought. Materials are tools to think with.

v) With respect to the learning environment, David Wheeler states the following:

The temptation is to suppose, as various educators at various times have supposed, that the only problem to be solved is how to construct the right environment so that all the skills and knowledge we would like students to learn would be acquired in the 'natural' course of their interactions with these environments. But a romantic fallacy lurks here: the assumption that the learners will automatically want to acquire what these artificial environments have to offer and will therefore bend their energies to this acquistion. The evidence of several episodes of 'progressive education' is clear: getting the environment right is not enough. Or alternatively, it is too difficult, if not impossible, to get the environment right.

Some Thoughts for Students

i) Remember, math is something to be uncovered by yourself and not something to be covered by your teacher.

ii) Learn to suspend the self, that is, don't get in your way when problem solving.

iii) Compete with the problem at hand and not the students around you.

iv) Your brain is quite good enough to do math. To be ready to do math is to be ready to think.

v) Be curious.

vi) Be self-evaluative and self-reflective. Have an inner sense of what you know and don't know. Track your progress during problem solving and be able to answer the question, 'When have I done enough?' Think about what you are thinking about.

Some Thoughts for Teacher

i) "Don't fool your students into thinking they can be safe." (Raffaella Borasi, CMESG 1987)

ii) Your respect for the learner should be greater than your respect for the subject.

iii) Set a feast for the student's imagination. As Eric Muller points out, don't start your class with the equivalent of "put away your your imaginations and let's begin."

iv) Teach only subsets of the truth.

v) The Socratic Method should follow rather than lead.

vi) In the use of language, don't go too formal too fast.

vii) Be reinforcing, less critical, start slowly and accept first approximations especially early in a teaching episode.

viii) Teach for understanding.

ix) Model a willingness to be vulnerable.

x) Be the guide on the side rather than the sage on the stage.

xi) Be at ease and flexible in the classroom and not unduly constrained by the dictates of the curriculum or the needs of the next course.

xii) Teach math more the way math really is. What mathematicians do is a creative active artform. Give students the opportunity to act like mathematicians.

xiii) Consideration should be given to the relevance of the math being taught. Many students feel that math is important <u>and meaningless</u>.

xiv) Help the student to know when they know and know when they don't know.

xv) Don't play games, that is, things should mean what they mean. Be honest with your students. All cards should be on the table. There should be no hidden agendas.

xvi) With respect to student learning difficulties, consider the statement, 'I have met the enemy and the enemy is us.'

xvii) The environment within which natural learning takes place needs a structure. It is structured by goals, topics, problems, resources, etc.

xviii) The teacher provides the initial structure of the environment in which an investigation takes place and assists students in understanding the messages emanating from these investigations.

xix) Emphasize self-evaluation.

xx) Emphasize truth as it unfolds from pupil investigations over truth as presented in a lecture. Students retain only that math which they learn by a process of internal construction and experience.

Natural Learning and Peer Interaction

i) Verballizing helps clarify. It assists the thinking process.

ii) Work in small groups facilitates work in large groups.

iii) Peers can help you focus.

iv) Peer interaction is essential but used only when needed.

v) To have peers with similar interests, values or skills always available is to have the means of checking one's progress always available.

vi) Students should routinely be allowed to work in small groups.

Conclusion

David Wheeler sums things up rather nicely when he says, "Perhaps "natural" is best understood as signalling a protest--a protest against actions deemed to be manipulative, beliefs which are prejudicial, or conceptions which are overly formalised and esoteric. "Natural" may be antithetical to "alienating", that is, it tries to restore some human freedom that manipulation, prejudice or mystification, would remove. ... The questions implied by the phrase "natural mathematics" are whether it is possible for mathematics not to alienate people, not to mystify, not to cow." Most members of Workgroup B would answer 'yes' to these questions.

Working Group C

USING SOFTWARE FOR GEOMETRICAL INVESTIGATION

Alton Olson
University of Alberta

Participants:

Jim Beamer	William Korytowski
Helen Bonchonko	Al Olson
Claude Gaulin	Tom O'Shea
Joel Hillel	Charles Verhille
Tom Kieren	Ed Williams

This working group on geometry and computing considered the impact of computing on the teaching of geometry. In particular, the group experienced and analyzed the use of software tools such as the Geometry Series (WICAT Systems) and the Geometric Supposer (Sunburst Communications).

Initially, the task of this working group was bounded by three facets of geometric software usage: (a) process outcomes, (b) content outcomes, and, (c) research issues.

Process Outcomes

1. Collecting data
2. Conjecturing
3. Finding counterexamples or generalizations
4. Verifying conjectures
5. Proving statements

Content Outcomes

1. Clustered outcomes
   (Ex. What is invariant when a triangle is "sheared"?)
2. Transformation Geometry
3. Symmetry and Invariance
4. Constructions and conclusions resulting from them
5. Measurement
6. Geometric relationships of all kinds

Research Issues

1. How might such software change the geometry curriculum?

   . Integrating inductively based conjectures with conventional curriculum.
   . What teaching strategies are possible?
   . What strategies are optimal given the capabilities of the software?

2. How might such software be used as a research tool?

   . What student behaviors are now accessible for public scrutiny?

- What special behaviors now come to the fore (e.g. inductive reasoning)?

3. What research questions emerge as important in the face of such technology?

   - Content
   - Process skills
   - Attitudes

These proposed facets of software usage presume a familiarity with the programs. Unless this is so the efforts in a working group are necessarily directed toward providing the experiences in these software environments. At least that seemed to be the only path to take in our group, in view of the participants generally limited experiences with geometry software packages. With this development the questions that this group addressed changed over the work period from the facets described above to a final concern about the properties and capabilities of such geometry software environments that were considered absolutely essential for instructional purposes.

In particular, our group engaged in the following activities:

First day:

- Introduction to Geo Draw from Geometry One by WICAT Systems.
- Participants worked in pairs on this program and some began to note comparisons with the Supposer programs.

Second day:

- Introduction to and work with Proof Checker from Geometry Two by WICAT Systems.
- Discussion of the ultimate form of the product from this group.

Third day:

- Cursory work in small groups with Presupposer and two portions of the Supposer series by Sunburst Communications.
- Final group discussion.

In discussion, at an intermediate point of our work, the group identified six important domains of use of such geometry software. They were: (1) Conjecturing, (2) Exploring, (3) Proofmaking, (4) Informal arguments, (5) Spatial visualizations, and (6) Tutorial instruction.

In our final discussion, participants concentrated on what they would like to see in geometry software. Indirectly, this discussion addressed the domains of use noted above.

Useful Characteristics Noted in Final Discussion

1. Flexible and easy to use geometry drawing tools including:

   . dividing a line segment into n equal parts,
   . simple angle bisector,
   . easily accessible area measurement capability.

2. Visually communicated commands are crucial in order to get away from the demands of complex symbolism:

   . it should be possible to name angles with single letters or single numerals when appropriate,
   . it should be possible to label easily the intersections of lines, circles, etc.,
   . it should be possible to identify points, axes, directions of angles and lines, etc. with mouse/iconic menus or easy notations.
   . it should be possible to choose letters for points and locations for letters easily.

3. Transformation packages requires specific capabilities:

   . it should be possible to define and label transformations so that they could be composed easily,
   . it should be possible to ask if two lines are parallel, perpendicular, or if angles are congruent, etc.,
   . in more complex figures it should be possible to detach subfigures for transformation,
   . special transformation tools should be available to work with triangles, rectangles, parallelograms, etc. on an individual basis,
   . it should be possible to quiz the computer about transformations, e.g. are these two figures related by a rotation?

4. Input requirements of the computer should not distort the objectives of proof making:

   . presently the symbolism demands are too heavy to make use of proof checking capabilities,
   . good proof checking capabilities may have to await developments in artificial intelligence,
   . input requirements of the computer perpetuate the image of proof making as ritualistic and that is too high a price to pay,
   . creating beliefs in informal arguments, for which a computer might help, may be more important than building a pseudo-skill in proof making.

5. Enhancement of spatial visualization skills should receive more emphasis:

   . plane representations of 3-D figures must be done carefully and with attention paid to fidelity,
   . some high-end statistics packages and CAD/CAM software do a good job in this area - perhaps, mathematics education could take advantage of these software packages.

6. Finally, some other desirable characteristics of geometry software packages include:

   . easily controlled screen dumps to paper are essential,
   . generations of randomly sized and placed figures is important in promoting generalizations,
   . windows to record construction steps and computations are important,
   . in many of the tasks easy adjustments of the level of responses required would be desirable.

Appendix A

SCHOOL MATHEMATICS SOFTWARE SOURCES

James E. Beamer

The following list of suppliers can provide information on a variety of commercial software packages to support high school mathematics. Some of these have application to the university calculus program as well.

A more detailed description of many of the products can be found in the references. The NCTM volume: Developments in School Mathematics Education Around the World and the PDK volume: Exemplary Practice Series as well as the articles by Cathcart and Dickey which are extremely helpful sources.

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Steimnetz, Amber M. and Cunningham, Stan. "Factoring Trinomials: Trial and Error? Never!" Mathematics Teacher, Vol. 76, No. 1, Jan. 1983, pp. 28-30.

Tremblay, Jean-Paul and Bunt, Richard B. An Introduction to Computer Science--An Algorithmic Approach. McGraw Hill, 1979.

Yerushalmy, Michal. "The Geometric Supposer: Promoting Thinking and Learning." Exemplary Practice Series. Bloomington, IN: Phi Delta Kappa, pp. 137-141.

MATHEMATICS SOFTWARE SOURCES

muMath - 80 (Apple II Version)
Education Technology Centre
College of Education
University of South Carolina
Columbia, South Carolina 29208
(803) 777-3068 (approx. $40.00 U.S.)

muMath - 83 (MS-DOS, IBM compatible)
Soft Warehouse
3815 Harding Avenue, Ste. 505
Honolulu, Hawaii 96816
(808) 734-5801

Calculus - 87 (calculus and graphics for muMath 83)
Developed by Ralph Freese and David Stengenga
Department of Mathematics
University of Hawaii
R & D Software
549 Papalani Street
Kailua, HI 96734

Maple (VAX, IBM, MacIntosh, Cray...)
Watcom Products, Inc.
415 Phillips Street
Waterloo, ON
Canada
N2L 3X2
(519) 886-3700

Eureka: The Solver (IBM/MS-DOS)
Borland International
4585 Scotts Valley Drive
Scotts Valley, CA 95066

Reduce (IBM, VAX, Cray, and others)
The RAND Corporation
Attn: Dr. Anthony C. Hearn
1700 Main Street
P.O. Box 2138
Santa Monica, CA 90406-2138
(213) 393-0411

Spreadsheet Tools for Secondary School Mathematics
JEBCO (Apple II, IBM/MS-DOS)
126 Quill Crescent
Saskatoon, SK
Canada
S7K 4T9
(306) 242-6547

The Geometric Supposer Series
Apple II, Partial IBM
Sunburst Communications
P.O. Box 3240, Station F
Scarborough, ON
M1W 9Z9
(800) 247-6756 - Canada
(800) 431-1934 - U.S.A.
The Geometric preSupposer: Points and Lines
The Geometric Supposer: Triangles, Quadrilaterals, Circles
plus free loan video tapes and curriculum support materials
(not free)

IBM Software
IBM Geometry Series ($156 per part)
Geometry One: Foundation (with Geo Draw)
Geometry: Proofs and Extensions (with Proof Checker)
IBM Mathematics Tool Kit
IBM Corporation
Educational Systems, Department 8WH
Box 2150
Atlanta, GA 30035

Sharp Electronic Corporation
Sharp Plaza
Mahwah, New Jersey 07430-2134
Sharp EL 5200
Graphic Scientific Calculator

Working Group D

A STUDY OF THE REMEDIAL TEACHING OF MATHEMATICS

Martin Hoffman
SUNY

Arthur Powell
Rutgers University

Participants:

Ed Barbeau	Linda Davenport
Martin Hoffman	Sibongile Langwenya
Arthur Powell	Conrad Sigurdson.

Introduction

During the 1987 CMESG meeting, Topic Group R, "Working in a Remedial College Situation," briefly examined techniques relating to affective and cognitive domains as well as student and program evaluation. At that time, it was felt that these areas warranted further examination and, in response, a Working Group was organized with the following description:

> The broad intent of this working group is to examine how to assist secondary- and college-level students who have experienced difficulty in learning mathematics. Particular methods will be demonstrated and learning environments simulated. Presentations by participants will be encouraged. Development of criteria for judging the effectiveness of various methods will be a focus of discussions.

Among the questions to be considered are the following:

* What are the distinguishing cognitive and affective characteristics of secondary and college-level remedial students?
* How can awareness of these characteristics be used in developing curriculum and methodology?
* What external (institutional) considerations influence the choices of topics?
* What are appropriate uses of informal and formal methods?
* In what ways can manipulative materials and computers be utilized?
* What are effective uses of problem-solving activities?

Session #1

It was decided not to begin this session with a discussion of what is meant by remediation, who are remedial students, what are effective techniques, and the like, but to start immediately with an activity. It was anticipated that the shared experiences resulting from the participants' engagement in the activity would then lead to such a discussion.

The activity was of the type that had been presented in the 1987 Topic Group and has been used with remedial mathematics students in college. Though the content focused on greatest common divisor, the activity could be refocused on other areas found in a typical remedial curriculum. At the conclusion of the activity, participants wrote about their experiences and each read what they had written. The readings served three purposes:

(1) to detail the nature of the activity,
(2) to initiate a discussion of the use of writing as a learning tool, and
(3) to develop lists of characteristics of remedial students, constraints on the institutional setting, and useful activities.

A summary of the list referred to in item (3) above appears in Appendix A.

Session #2

This session began with presentations of activities thought to be effective in remedial situations. In the first presentation, Ed Barbeau illustrated an open-ended activity which promotes practice of specific arithmetic operations. The activity can be stated as follows: Choose any two numbers and then generate a sequence as follows: The next number in the sequence is determined by adding 1 to the last number and dividing the sum by the next-to-last number. This type of activity, unlike many remedial-level activities, contains elements of surprise and stimulates searches for generalizations, variations, and verifications.

Linda Davenport then presented a series of activities involving geometrically-based manipulative materials. She demonstrated how the activities can be used to generate both geometrical and arithmetical concepts while developing students' powers of imagery and verbalization.

Following the presentations, we discussed particular characteristics of the two activities and developed general criteria for determining appropriate remedial activities. In addition, we re-examined and elaborated the list of characteristics of remedial students and constraints of institutional settings.

Session #3

The final session began with an activity which seemed only to involve searching for patterns in an addition table. The activity, however, provided an example in which one could engage students of varying levels of sophistication and which had the potential to move into areas typically found in remedial curricula. It was shown, for instance, that this activity could lead students to develop meaningful procedures for adding and subtracting signed numbers.

Another activity which provides students opportunities for creating meaning in mathematics is writing. We examined two ways in which writing can be used to learn. First, we wrote about our reflections on activities in which we engaged and discussed both the content of our writings and the effect of the process. It was noted that this type of informal writing encouraged a quality of reflection which is not always reached through discussions. Second, we discussed different ways that writing, can be used in a remedial setting. One technique discussed was journal writing, where students make note of and explore their reflections on the mathematics that they are learning. Another technique examined was "multiple-entry" logs. The multiple-entries referred to are (1) an item selected by the student for discussion, (2) commentary on the item, and (3) follow-up commentary. A distinguishing feature of these logs is that the follow-up commentary tends to promote deep reflective thinking in students about both the mathematical item and their initial reflections on it.

These writing activities were seen to encourage students to reflect on their learning and their affective responses. Attending to their affective responses to mathematics and their learning was viewed to be as important as focusing on particular content. This view was also incorporated into our list of criteria of activities appropriate for remedial situations. At the conclusion of the final session, open questions, suggested by the three days of deliberations, were raised for further consideration. They included the following:

1. How valid are our characterizations of remedial students and the institutional constraints?

2. What is the role of investigative activities? Can they be of primary importance in implementing a curriculum?

3. How should remedial curricula be assessed?

4. In what ways do the suggested activities affect institutional constraints and the behaviors of students?

5. What are effective ways of educating pre- and in-service teachers on the needs of remedial students?

The following appendices consist of a list of characteristics of remedial situations (Appendix A), three summary statements by Working Group participants (Appendix B), and two bibliographies used in the Working Group (Appendix C).

APPENDIX A

Characteristics of Remedial Situations

Compiled by Linda Davenport

Characteristics of remedial students:

- lack understanding of what they are doing
- want prescriptions
- have negative beliefs about self as learners of mathematics
- don't know how to study productively
- evidence convoluted thinking
- lack appropriate imagery
- passive

Institutional considerations:

- time constraints
- lack of priority for remediation
- negative orientation (e.g., "If they don' know it by now...")
- focus on individualistic learning
- little explicit promotion of thinking skills, reflection, autonomy

Useful features or criteria for selecting/developing appropriate activities:

- open-ended
- work on imagery, perception
- elicits active involvement
- minimal verbal skills to get into the activity
- work on verbalizing perceptions and actions
- multiple agenda, embedded skills
- element of surprise
- potential for generating variations
- embodiment of mathematical concepts/structure
- enriches understanding of mathematical concepts/ structures
- writing to support the reflect process

Appendix B

Summary Statements

Ed Barbeau

One can isolate various aims of a remediation course:

(a) to provide a student with a perception of mathematics, its nature and use, which will act as a basis for efficient learning later on;
(b) to address specific deficiencies in knowledge and skills;
(c) to help a student deploy resources most effectively toward completion of assignments and passing of tests and examinations.

A remedial course should not be a rehash of previously learned material. Since one can assume previous exposure, one might renew the ideas into other patterns which might advance (a) and (b), by providing a fresh look at material and alternative strategies for solution of problems.

It may be a mistake to break the course into isolated packets of techniques since it may be an inability to get an integrated view of mathematics which undermines a student's ability to make sense of and perform its procedures.

Rather, one might look for a few crucial examples which involve necessary techniques and which can be pursued in some detail for what insight can be provided about the mathematical enterprise in general. Mathematics can be seen as a language for manipulating ideas through the use of symbolism, algorithm and reasoning.

A choice of material is necessary, but the goal should be student autonomy, so that in future he may be better able to express ideas, consult texts, make up and do exercises, write solutions and prepare for examinations.

Sibongile Langwenya

Back home we do not have a group of students identified as remedial students (i.e., among the full-time university students). So I joined this group to see if there is a possibility that we do have such students but possibly overlook the remedial teaching they need thus teaching to a high failure rate.

Also, I wanted to see if there are any possibilities of using informal teaching methods over at higher levels of educations and the range of possible informal methods.

Through discussions in this group, I realized that one contributing factor to the increasing failure rate among our first year students is the lack of remedial teaching. It is always assumed that students at such and such a level should know this and that and if they do not know it, hard luck. I can't go back and teach high school material.

Also, our students tend to concentrate on the know how (i.e., memorise formulas) not on the conceptual understanding. Through these discussions I realised some possible ways of attracting students interest to conceptual understanding and possible ways of improving students' skills, not only to allow them to know or learn one approach to a problem.

I realised the importance of what the teacher says in creating awareness among the students.

I also realised the importance of writing in mathematics. Most of the time the focus is on whether the students can solve such and such a problem and not on description, interpretation and calculation from mathematical patterns.

Conrad Sigurdson

The group leader presented a pattern on the chalkboard and elicited responses from the group in a non-verbal manner. He requested responses by pointing at individuals. I felt early that there had to be a pattern. I searched through the different basic operations of +, /, -, x, modular arithmetic, etc. Finally I felt that we might be dealing with HCF or LCM - I tested it. My first test suggested LCM, although I really knew it should be HCF. There were only six people, including the 2 group leaders. I felt that had there been many more, I would have been embarrassed to respond and to be wrong. As it was, I guessed without fear.

I think that this kind of approach would be motivating for students in mathematics. It could also be used for a novel practice exercise for LCM, HCF, and other concepts as well, e.g., multiplying, dividing, etc. I am not actually in Math Ed at the present; however I did find the above activity, and others, very refreshing, interacting with students in a diagnostic and remedial way.

I found the process of writing my ideas worthwhile. It forces one to reflect on what one does. "You don't learn by doing, you learn by thinking about what you're doing."

There is value in discussing your ideas and those of others. It helps to clarify your thoughts and it gives you other ideas. It is very useful to get other people's ideas in this interactive way. There are relatively few people who are concerned with these problems and issues, so it is very revitalizing to get a chance to talk with them.

One issue revolved around what to do about students who do not have the necessary prerequisite skills when they are students in your course.

After the Working Group searched and probed, a few suggestions and ideas did occur to me:

1. Diagnose first of all.
2. Students should shoulder some responsibility for prerequisite skills.
3. Remedial classes.
4. The institution must be involved with the problem.
5. Set up math centres.

I don't think anyone of us alone would have come up with so many good suggestions in such a short time. I felt good about this interaction.

The suggestions presented on operations with integers and was also very helpful to me. It was a clear and very direct reminder that students can find patterns, make rules, and if given the chance, be involved.

Journal writing or report writing for students I do believe would be very helpful. It forces them to reflect on what they do. You learn by reflecting on what you do.

I felt good about having my own ideas assessed and reviewed by colleagues; it encourages me to think more about what and how I do things.

<u>Some concerns</u>:

1. How do we get ideas at a conference like this reflected in the classroom?
2. There are so many topics we didn't discuss, or discuss thoroughly, e.g., geometry, problem solving, estimation, diagnosis, evaluation!

Appendix C

Bibliographies

Writing and Mathematics Learning Bibliography

Writing to Learn Mathematics

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Topic Group A

CULTURE AND MATHEMATICS EDUCATION

Patrick B. Scott
University of New Mexico

It is both an honor and a privilege to be invited to make a presentation to a Topic Group on Culture and Mathematics Education for the Canadian Mathematics Education Study Group (CMESG). As it appears to be a very personable group I will deliver my remarks in a very personal way.

I was recently sitting in my office contemplating the weight of my assignment before CMESG when a colleague dropped by. I related something of my task to her and lamented the difficulty in trying to decide how to begin. She offered that she had just read a new book by Rudy Rucker (1987) called Mind Tools that had a definition of mathematics that might be a good starting point.

Rucker stated that "There is no such thing as Chinese Mathematics or American Mathematics: mathematics is the same for everyone. Mathematics consists of concepts imposed on us from without. The ideas of mathematics reflect certain facts about the world as human beings experience it." He went on to suggest that those "concepts imposed on us" are number, space, logic, infinity and information.

If we embrace Rucker's point of view then there is little point in talking about mathematics and culture. However, it would appear that it was just this position that Alan Bishop (1987) was challenging when he suggested that "this view confuses the 'universality of truth' of mathematical ideas with the cultural basis of that knowledge." Bishop also asserted that mathematics, rather than consisting of "concepts imposed from without" is activities that humans engage in: counting, locating, measuring, designing, playing and explaining. This view would seem to agree with that of Philip Davis (1987) who suggested in the first issue of the Newsletter of the Humanistic Mathematics Network that we live in a "mathematized world": "By adding the suffix 'ized', I want to emphasize that it is humans who, consciously or unconsciously, are putting the mathematizations into place and who are affected by them."

Having accepted that Mathematics is a human, and therefore cultural activity, we can consider why there has been a growing interest in the relation between culture and mathematics. I would like to suggest that there are at least four major reasons why the cultural aspects of mathematics and mathematics education have been receiving increased attention:

- Curiosity
- Cultural Reaffirmation
- Bringing the World into the Classroom
- Improving Teaching and Learning

Curiosity

My own curiosity concerning culture and mathematics began while serving as a Peace Corps Volunteer (PCV) in a Primary School Mathematics Project in Jamaica. Over fifty PCVs were struggling with ways to help introduce a "New Math" curriculum to mostly rural primary school teachers. Someone got a copy of Gay and Cole's New Mathematics in an Old Culture and many of us read it with much interest. We were intrigued by their suggestion that for the Kpelle students in Liberia "the instruction he receives in school dismays and confuses him." We felt that Jamaican students were often dismayed and confused by their imported/adapted math curriculum, but we did not seem to be able to apply much knowledge about culture to the learning of mathematics.

A few years after the Jamaican experience Claudia Zaslavsky's Africa Counts appeared with its rich examples of African uses of mathematics. Perhaps it could have given us some help in Jamaica.

Curiosity has led me in the past few years to three more studies of culture and mathematics: Pinxten's The Anthropology of Space, Lancy's Cross-Cultural Studies in Cognition and Mathematics, and Orr's Twice as Less. Pinxten said that "Navajo space appears to be founded on ... movement, volumeness/planeness, dimensions". I tried to apply some of Pinxten's ideas to my work with Navajo students in New Mexico, but can report no real breakthroughs. I tried to replicate with four indigenous groups in Guatemala some of Lancy's work in New Guinea. The Piagetian tasks were tried first. The results did not make any sense to me. I do not know if it was inadequate training of field workers or the general inappropriateness of the Piagetian tasks for the intended population.

A consideration of Orr's book led me to further reflection on the mathematical learning of minorities in the United States. Although for most minorities (except the Orientals), the general situation is dismal, there are at least three notable exceptions, each taking into account the culture of the students with respect to their mathematics education: Navajo children in the Rock Point bilingual school, Black students at Cal-Berkeley and Hispanic students at Garfield High in East Los Angeles.

The Navajo children at Rock Point Community School in Arizona experience a rigorous bilingual education program. Rossier and Holm reported that in first and second grade their math achievement test scores are dismal. By fourth grade they are well above national norms, and even do relatively better in problem solving than in arithmetic operations.

Uri Theisman was puzzled about the high failure rates of Black students in first year Calculus at Berkeley, at the same time that the success rate of Chinese students was very high. Friends and colleagues suggested that it was because the Black students came from deprived educational and economic backgrounds, and did not study enough. Theisman found the situation to be much different. Most Black students at Berkeley came from solid middle or upper middle class families, had good academic records at good high schools, and studied very hard. Despite that, their grades in Calculus were inversely proportional to their Scholastic Aptitude Test (SAT) scores. So he decided to look at the culture of the mathematics study of Blacks and Chinese. To simplify, he found that the academic lives of the Chinese were very integrated with the social lives. They studied together, had lively discussions of concepts and problems. The Black students had academic lives completely separated from their social lives. Although they studied hard, they studied completely alone.

Programs have been designed that have encouraged group study and the achievement of Black students in those programs has been very impressive.

The popular movie "Stand and Deliver", based on fact, has shown how Hispanic students when given the proper conditions and motivation can achieve incredibly well in Advanced Placement Calculus in high school. The teacher's timely use of Spanish and his understanding of the students culture, he himself is of Bolivian descent, must be of considerable help.

Cultural Reaffirmation

Perhaps the two persons who have most stressed the importance of culture and mathematics in bringing about a cultural reaffirmation have been Ubiritan D'Ambrosio of Brazil and Paulus Gerdus of Mozambique.

In his book, Socio-Cultural Bases for Mathematics Education, D'Ambrosio has suggested that "an individual who manages perfectly well numbers, operation, geometric forms and notions, when facing a completely new and formal approach to the same facts and needs creates a psychological blockage which grows as a barrier between the different modes of numerical and geometric thought." Bridges must be established between the Ethnomathematics of the individual and traditional school mathematics.

Gerdus has insisted that a "cultural-mathematical-reaffirmation" is possible. As he puts it: "It is necessary to encourage an understanding that our peoples have been capable of developing mathematics in the past." He looks for examples of "frozen" or "hidden" mathematics in building, weaving and other artisan productions that he uses in mathematics courses for future teachers.

My own recent experiences with a "cultural-mathematical-reaffirmation" have been with Mayans in Guatemala. I have had the pleasure of working with Guatemala's National Bilingual Education Program (PRONEBI) which serves preprimary and primary children of the four principal indigenous language groups. While in a meeting with preprimary teachers who were discussing the revision of the preprimary mathematics curriculum, I became concerned that the concept of zero was being introduced too early. I pointed out that the concept of zero was such a difficult one that it was not well understood in Europe at a time when their Mayan ancestors were using it fully. They responded that I had just given them an important reason for introducing it early with Mayan children.

Another Guatemalan experience has been the introduction of two microcomputers in a little primary school in a Mayan village far into the Guatemalan Highlands. The teacher most responsible for the computers insists that because the children use their Mayan language, Quiché, to talk about Logo and the computers, and teach other children about them, they have learned that modern technology is something they can use with their own language. Another teacher in the school reports that she has finally been able to interest the Parents Committee in pressuring local authorities for a new school building, because they "need to protect their computers".

Bringing the World into the Classroom

Claudia Zaslavsky stresses that bringing mathematics from other cultures into the classrooms of all children can help foster more understanding of the world and an appreciation of mathematical contributions from other lands, as well as helping children to understand mathematics.

Marcia Ascher at Ithaca College has developed a course on Ethnomathematics for university students. She apparently limits her definition of Ethnomathematics to the mathematics used by nonliterate peoples. D'Ambrosio and Gerdus use the term in a much broader sense.

To Conclude

My own search for the connection between mathematics and culture, and application of those connections to my teaching have not been completely successful. The learning of mathematics is a complicated process that appears to be very much influenced by cultural factors. I trust we will soon see the day when all teachers have sufficient cultural understanding and sensitivity to incorporate Ethnomathematics into the curriculum.

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Topic Group B

THE USE OF COMPUTERS IN UNDERGRADUATE MATHEMATICS SERVICE COURSES

Eric R. Muller
Brock University

Introduction

In the past three years the International Commission on Mathematical Instruction has undertaken two separate studies. These were "The Influence of Computers and Informatics on Mathematics and its Teaching", Strasbourg 1985 [1 & 2] and "Mathematics as a Service Subject", Udine 1987 [3 & 4]. My participation in both of these studies gave new impetus to the further development of the work which we had started at Brock in the 1970's [5]. I am now convinced that the influence of computers on undergraduate mathematics will be far reaching, and since Canadian mathematics departments have a large service course component, additional pressures will come from departments whose students are required to take mathematics. This presentation explores how the Mathematics Department at Brock is responding to this changing education environment.

Service Courses

The number of students taking mathematics courses as a requirement of a discipline other than mathematics has grown surprisingly large in Canadian universities. In a recent survey [4] we estimate that, on the average, 81% of course enrolments in mathematics departments are not mathematics majors. When a mathematics course is offered only for non-mathematics majors the course tends to have a very different atmosphere than that of a course of mathematics majors. Generally mathematics faculty find service courses less interesting to teach and they also experience additional constraints and external pressures. The students in these courses often view mathematics as an additional, low priority burden and approach mathematics with little enthusiasm. The mathematics background of these students tends to be heterogeneous. The reasons why students are required to take mathematics courses can be very different - the courses may be used to screen students for disciplines which have more applicants than places, they may be required to develop knowledge of very specific mathematical methods, while others may require a course to develop mathematical maturity with little concern of the curriculum.

Computers

In the last three years there have been significant software developments designed to support the teaching of undergraduate mathematics. This software is easy to use, requires no previous computer experience, is usually menu driven, is aimed at the multiuser environment rather than the single user large system research environment, and is developed for the popular microcomputer technology.

In the area of statistics and operations research one finds "student versions" of the large research packages or some texts are now available with supporting software. In the area of calculus, linear algebra etc. the symbolic manipulation software is now accessible to undergraduate mathematics education in Mumath (University of Hawaii) and MAPLE (University of Waterloo). One must not neglect the important breakthrough in calculator technology which, on different machines, now offers statistics, graphics or limited symbolic manipulation capabilities.

When the twain meet

Why introduce the use of computers into service courses? I believe that recent software developments provide a rich and different learning environment which can

(a) increase the student's confidence in understanding mathematics by using the system to do the repetitive arithmetic and algebraic computations allowing for concentration on the mathematical concepts,

(b) show that mathematics has relevance and application by providing more realistic problems involving a sense of reasonableness, estimation, etc.,

(c) motivate the exploration of mathematical concepts by the creative use of multiple examples.

There are different levels at which the computer can impact on a course. In the service courses we have used it to

(i) (a) generate individualized sets of data so that each student essentially solves different problems,

(b) generate solutions for these problems,

(ii) provide student laboratories where each student accesses a piece of software (no knowledge of programming required),

(iii) motivate change in course content,

(iv) motivate the development of new courses.

Examples of each of these will appear in the following discussion.

As expected the impact of computers has, to date, been most felt in the Statistics and Operations Research courses. Clearly both these areas provide an opportunity where individual sets of data for each student are pedagogically useful. Students can compare their data to those of others for the same experiment, and observe that the samples can look very different even though they are from the same population. Samples from the whole class can be compared and the statistical concepts intuitively developed. Programs are available or can be written to produce the statistical analyses for the marking of assignments. A similar situation exists in Operations Research where one can introduce large realistic problems early and explore solutions as a motivator for studying the methods by which these solutions are obtained. We have observed a substantial change in the approach to teaching these courses, for example, formulation of problems play a greater role and large, more realistic problems have become the norm for term work. Students have responded very positively to these changes. In both these areas we use software specifically designed for teaching on micros, which is a major improvement from the early days when we used research packages on a mainframe.

We are in the process of introducing the Computer Algebra System MAPLE in the first year Applied Calculus course. This year a section of 100 students (out of 500+ in the course) attended a required computer laboratory. Due to the large enrolment and lack of computer facilities, laboratories were structured in very much the same way as in an experimental science. In other words students did not have general access to the software. Laboratories of two types were structured, the first type was aimed at a particular Calculus concept whereas the second type of lab was more exploratory in nature. An example of each is provided in the Appendices. Students have responded positively to the whole experience but there were many frustrating moments caused by the lack of "user friendliness" of the software and the slowness of the computer system when taxed by a lab full of students. We have much to learn in this area and meetings such as the one to be held in Colby College, Maine, U.S.A. [6] will help us through this exploratory stage.

The influence of the computer has also been felt in the development of new courses for computer science students. One such course in Discrete Mathematics has been offered for a number of years and is described in the proceedings of the Strasbourg ICMI meeting [2].

To conclude, we have yet to integrate with any effectiveness the use of computers within

(a) the lecture presentations

(b) tests and examinations, i.e., overall student evaluations.

These are very serious criticisms of the process. How can the student understand the importance of the work he/she performs with the aid of computers if much of both the course presentation and the course evaluation is based on paper and pencil work?

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Appendix 1
(Example of Concept Development Lab)
- 1 hour lab -
MAPLE LEAF A

A property of all polynomials (and many other functions) is that when we magnify a small section of their graph it looks like a straight line.

MAPLE SYRUP 1: Use MAPLE to develop a polynomial which cuts the x-axis many times in the interval [1,2]. For example:

```
> pl:=(x-1.01)*(x-1.22)*(x-1.53)*(x-1.95);
```

Use MAPLE to plot the polynomial in smaller and smaller intervals, for example:

```
> plot (pl, x = 1..2);
> plot (pl, x = 1.. 1.2);
> plot (pl, x = 1...1.001);
```

etc. What do you conclude?

Therefore, in a sufficiently small interval, polynomials behave like straight lines. The slope of that straight line, called the secant line, is a measure of how the polynomial is changing in that interval. We get a different straight line if we zero in on another small interval, for example:

```
> plot (pl, x = 1.2..1.21);
> plot (pl, x = 1.2..1.2001);
```

etc.

In general, given a polynomial, p(x), the slope of the secant line defined by the two points (x, p(x)) and (x+h, p(x+h)) is given by

$$\frac{\text{change in } p}{\text{change in } x} = \frac{p(x+h) - p(x)}{x+h - x} = \frac{p(x+h) - p(x)}{h}.$$

What information about the polynomial can we extract from the slope of its secant line?

MAPLE SYRUP 2: Consider the polynomial

```
> p2:= (x+7)*(x+3)*(x-5)*(x-8);
```

and find the slope of its secant line between the points x and x + h

```
> q2:= subs (x = x+h, p2);
> s2:= (q2 - p2)/h;
```

Do you agree that s2 is the slope of the secant line? MAPLE will multiply out the factors and do some simplifying algebra

```
>s2:= normal (s2);
```

Now we choose points x and x+h close together--a distance of h = 0.01 for example

```
>t2:= subs (h = 0.01, s2);
```

and plot both the polynomial and its secant line on the same graph

```
> plot ({p2, t2}, x = -7.2..8.2);
```

Identify the polynomial from the two graphs (we know where it cuts the x-axis!) Explore what happens to the polynomial when the slope of the secant line is:

(a) zero
(b) positive
(c) negative

Can you suggest some general principle?

What happens to the graph of the slope of the secant line when we add a constant to the polynomial? Can you guess?

MAPLE SYRUP 3:

```
> p3:= p2 + 7;
> plot ({p2, p3}, x = -7.2..8.2);
```

Do you recognize which polynomial has which graph? Now repeat MAPLE SYRUP 2 and obtain the slope of the secant line to p3.

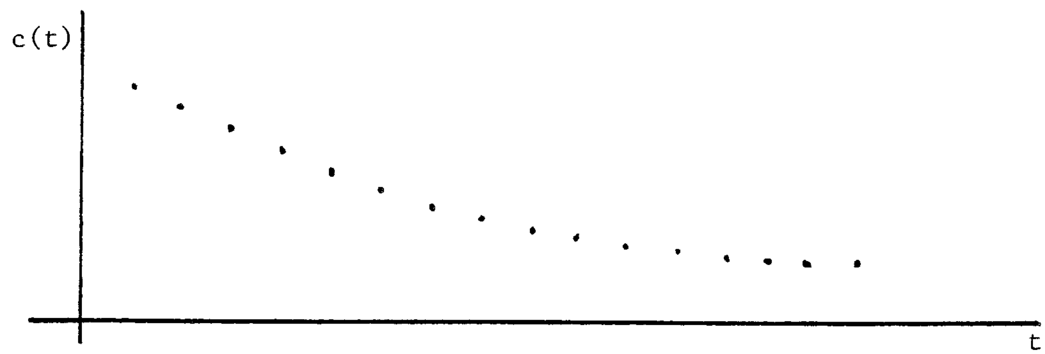
```
> q3:= subs (x = x+h, p3);
> s3:= normal ((q3 - p3)/h));
> t3:= subs (h = 0.01, s3);
> plot ({t2, t3}, x = -7.2..8.2);
```

What do you conclude? Does this work for any constant?

Appendix 2
(Example of Exploratory Lab)
- 2 hour laboratory -

MAPLE LEAF B

The concentration of a single dose of a drug administered directly into the blood stream is monitored. Experiments show that this concentration slowly decreases with time as the drug is eliminated from the blood stream. A typical graph is



Analysis of the experimental data shows that:

The percentage change in concentration between the time t and a subsequent time t+h is approximately proportional to the difference in time h (A)

that is,

$$\frac{c(t+h) - c(t)}{c(t)} \times 100 = -kh \times 100. \qquad \text{(B)}$$

In the right hand side a negative sign is used since c(t+h) is smaller that c(t)--the drug concentration is decreasing--and k is a positive constant of proportionality. It is important to note that since the dimensions of both sides of the equation must be the same kh must have no dimension, and since h is time, k has dimension 1/time. Note that an equality is used even though the data shows that this is approximately true. (Are you satisfied that B is the mathematical formulation of A?)

The right hand side of equation (B) is independent of t, thus the percentage change in concentration is independent of t, the time when the first measurement is taken. (It only depends on, h, the length of the time interval between measurements). Systems which show this property of independence on when measurements are first taken are called <u>memoryless</u>.

MAPLE SYRUP 1. Use MAPLE to plot the following four sets of data and determine which, if any, have the memoryless property.

(i) (0,25.0), (1,23.8), (3,21.5), (4,20.5), (6,18.5)
(ii) (0,25.0), (1,18.8), (3,12.5), (4,10.7), (6,8.3)

(iii) (0,25.0), (1,12.5), (3,2.5), (4,1.5), (6,0.7)
(iv) (0,25.0), (1,16.8), (3,7.5), (4,5.0), (6,2.3).

Recall MAPLE allows definition of ordered pairs and plots

```
> a1:=[0,25,1,23.8,......];
> b1:=[                   ];
etc.
> plot ({a1, b1, c1, d1});
```

Equation (B) can also be written in the form

$$\frac{c(t+h) - c(t)}{h} = -k\, c(t) \tag{C}$$

where the left hand side is the average rate of change in concentration over the interval h. Thus another interpretation provided by the experimental data is that:

at any given time, the average rate of change of concentration of the drug in the blood is proportional to the concentration at that time. (D)

(Are you satisfied that (C) is the mathematical formulation of (D)?)

The solutions of equation (C) are now explored in two different ways:-

(i) h is fixed and the concentrations c(t+h), c(t+2h)...c(t+nh) are determined,
(ii) the limit as $h \rightarrow 0$ is taken in equation (C) and the solutions of the corresponding equation explored.

MAPLE SYRUP 2: (In the spirit of calculus visualize h small)
From equation (C)

$$c(t+h) = c(t)\ (1-kh) \tag{E}$$

The average rate of change in concentration between the times (t+h) and (t + 2h) is from statement (D)

$$\frac{c(t+2h) - c(t+h)}{h} = -\, kc(t+h)$$

Therefore $c(t+2h) = c(t+h)\ (1 - kh)$

$$= c(t)\ (1-kh)^2 \text{ from Eqn. (E)}$$

(i) Repeat the above for c(t+3h) and c(t+4h).
What can you infer about c(t+nh)?

(ii) Given that $k = 0.25$ and $c(0) = 20$, use MAPLE to compute the concentrations in the time interval 0 to 10. Plot these values to satisfy yourself that the graph is similar to the one presented on the first page.

We now return to equation (C) and take the limit $h \rightarrow 0$. The left hand side is the definition of the derivative, so (C) becomes

$$\frac{d}{dt} c(t) = -k\, c(t) \tag{F}$$

This is called a DIFFERENTIAL EQUATION.

MAPLE SYRUP 3: Use MAPLE to solve the differential equation (F).

```
> del:=diff(c(t),t) = -k * c(t);
> dsolve (del,c(t) );
```

Read statements (A) and (D) again, what function models this information?

Find the solution which at time $t = 0$ has a value of 20, i.e. $c(0) = 20$. Assume $k = 0.25$ and use MAPLE to plot the graphs of the concentrations obtained in MAPLE LEAFS 2 and 3.

Repeat MAPLE SYRUP 3 for different initial concentrations $c(0)$ and different elimination constants k. What can you conclude if k is increased?

Topic Group C

A MODEL FOR THE STUDY OF MATHEMATICAL PROBLEM SOLVING

Eric D. MacPherson
University of Manitoba

Acknowledgements

This paper was first presented at the 1988 Canadian Mathematics Education Study Group meeting in Winnipeg.

I am grateful to several participants for pointing out, in addition to ordinary errata, several places where it needed to be written more clearly or expanded. I have taken most of their advice in this revision.

I am particularly grateful to Tom Kieren, Jack Hope, Tom O'Shea, Robin Connor, and Tony Riffel. I may not have met entirely the questions raised by some of them, but what appears here is at least better on account of all of them.

Part I - The Big Picture.

Introduction

There has been no lack of interest in mathematical problem solving among mathematics educators or psychologists for many decades. Even so, it was probably inevitable that some contemporary analysts have chosen to study the phenomenon de novo. That has led to the reworking of some plots that are well-tilled and, more important, the elision of more subtle components of the phenomenon that, by now, we might better regard as understood.

When that happens in any science, the classical solution is to pay increased attention to the models that guide the formulation of research hypotheses. The present paper is in that tradition.

Models, of course, can never be true or false, only more or less useful in refining our language, accounting somehow or other for both extant research and natural language, and exposing questions worthy of attention. For this as for any model, the reader must judge whether or not it has those qualities.

Definition

First of all, we are concerned with mathematical problem solving, not the social activity called 'problem solving' that now attracts at least equal attention. They do not share much more than their labels. As R. J. Sternberg pointed out recently in the Kappan, social problem solving is characterized by two initial hurdles; knowing that one has a problem and deciding what the problem is. Resolving those two matters is generally more difficult than finding a solution. In mathematics, it is generally all too evident that there is a problem, and most often it is clear what the problem is. The two are therefore qualitatively different from the beginning, and they continue to diverge. In mathematics, there are widely accepted conventions for deciding whether or not a solution is correct. In social problem solving, it is difficult to even recognize a solution, let alone judge whether or not it is correct.

It would be interesting, on some other occasion, to explore the results of confounding the two as in, for example, the now classical lifeboat problem. But our focus here is on mathematical problem solving.

We have always had problems distinguishing between what is problem solving and what is 'just' the performance of an algorithm. They cannot be distinguished on content grounds. For most sixth graders, 32 x 15 would be given with the expectation that the child will now perform an algorithm. For a second grader recently introduced to the meaning of the symbol 'x', the same expression would likely be taken to embody a problem, and a rather difficult one.

Since the performance of algorithms and the solving of problems cannot be distinguished on grounds of content, for all but hard behaviorists the distinction must lie in the performer's mind. And, over the years, that locus of the distinction has driven me to shift the line ever closer to the performance of algorithms.

I now consider any performance in which a person's first impulse is to engage some ritual to be the performance of an algorithm. Otherwise the person is problem solving. To make the distinction sharp, suppose a student is asked to find 6 1/2 x 8. A student who recognizes the question as being of a known 'type' and immediately engages any one of several possible algorithms, including finding 6 x 8, 1/2 of 8, and adding, is performing an algorithm. One who ponders for a moment and then says or thinks something like, "Well, I can take 6 eights and 1/2 of 8 and add" is problem solving.

The two flagrant violations of the canons of operationalism in the above definition are, of course, deliberate. It is negative and it makes reference to mental states that are unobservable. Those, however, are not the flaws that they may seem to be. We owe a lot to David Willer who, so far as I know, was the first to point out that we use both nominal definitions, those we think with, and operational definitions, those used in experimental research. The latter are necessary, but are always intended to be surrogates for nominal ones.

What is given above is a nominal definition. I have the conventional respect for operational definitions in all research, including research on problem solving, and would frame the closest possible operational surrogate for what is said above in any research study.

The Model

I find it preferable to suppose that people learn to solve mathematical problems. That is, that they do not come hard-wired with schemas for the purpose. That may not be so, but there are two grounds for acting as though it is. First of all, mathematics is a human invention, and a particularly arbitrary one. We can argue persuasively for inborn propensities to throw rocks, nurse babies, or speak a language, but it is difficult to see how a sensitivity to polynomials could have played a role in natural selection. If that is so, then it is futile to spend much time using, say, factor analysis to search for the innate components of mathematical ability. They probably do not exist.

This does not, of course, preclude the more likely premise that the mind has evolved so as to have both specific schemas for dealing with more common environmental threats and opportunities and some non-specific ways of processing random information.

Second, and here we owe something to Russian psychologists, even if children have some relevant hardwired capacities it is most prudent to suppose that they are minimal in their effects. It is better that a teacher suppose that there is some way to lead a child to learn than to suppose that there are innate limits to his or her ability.

It is therefore assumed here that all mathematical problem solving behavior is learned. It may engage intellectual capacities that evolved for other purposes, but the links to those capacities are taken to be sufficiently remote that we have more to gain by paying attention to the arbitrary structures of mathematics and ways of teaching problem solving within those arbitrary structures than to hypothetical mathematical abilities.

I find it useful to think of the phenomenon along three dimensions. First, we must pay attention to what is learned. Anecdotal references to what it is people learn when they learn to solve mathematical problems go back, like just about everything else, to Ionia, but modern attention to the question seems to begin with Poincaré. Since then, categories and models have gone off in several directions. In my opinion, the two most productive directions have been one through George Polya and Oscar Schaaf and another through Glenadine Gibb and Henry Van Engen. A small group of us who once cohabited at the University of British Columbia, now completely dispersed, brought those two lines of thought together.

My analysis along the first dimension owes a great deal to Leo Rousseau, Tom Bates, Gail Spitler, Gerry Weinstein, and Rick Blake, even if some of them might now have refined the model in their own ways.

The analysis along this dimension, as along the others, is sharply non-reductionist. I posit that there are two quite different kinds of mathematical problem solving, sufficiently different that they may be learned and used independently and interact differently with both the factors along the other dimensions and with teaching methods. I suppose that both kinds of mathematical problem solving are used, to some extent, by everyone from pre-schoolers to Ph.D. mathematicians, and that differences in people's ability to engage one or the other kind of problem solving are differences of degree, not of kind.

The categories and interrelations of this first dimension of the model are given in part II of this paper.

The second dimension of the model attempts to partition some qualities of learners that may interact with the learning of mathematical problem solving.

For some years I preferred not to think about this dimension and, for the reasons given above, continue to view the exercise with suspicion. But Krutetskii changed my mind. I am uneasy with his conclusion that some of the qualities he identifies may be innate and all-or-nothing, but the qualities he identifies are so compelling that he led me, finally, to take this dimension seriously. Once I did, I found that some of our accumulated research and a goodly portion of our natural language can be accommodated to a revisionist version of Krutetskii's list.

Like Krutetskii (and just about everyone else), I have excised I.Q. as a variable, but probably for a reason different from his. I.Q. is clearly a surrogate for something going on in the mind, but it is not at all clear what. It seems that whatever I.Q. taps is sufficiently remote from mathematical performances that little purpose is served by using it to differentiate students for instruction.

At the other extreme, for reasons, given earlier, I am not impressed with the potential of attempts to identify unique innate factors of mathematical proficiency. I am, of course, biased. I have said that, and have explained why. But I do not know of any research to date that gives me any reason for considering changing my mind.

And that leaves the middle territory. Are there any propensities of learners more specific than I.Q., quite possibly learned, that are identifiable and worth taking into account as we attempt to enhance one or the other kind of problem solving ability?

Until recently I would have concluded that, all in all, there are not. I now think perhaps there are, and Krutetskii is to blame. Those who are familiar with his work will recognize his influence in the choice of variables for this and the third dimension in part II of this paper. Further, having crossed the Styx, I saw no reason for continuing to exclude one or two other variables that look promising.

The third dimension concerns the context of learning to solve mathematical problems. We all swim in the evidence that the context of learning matters, but most often mathematics educators' formal attention to this dimension has been restricted to thinking of it as a clastic farrago of nuisance variables to be controlled in experiments.

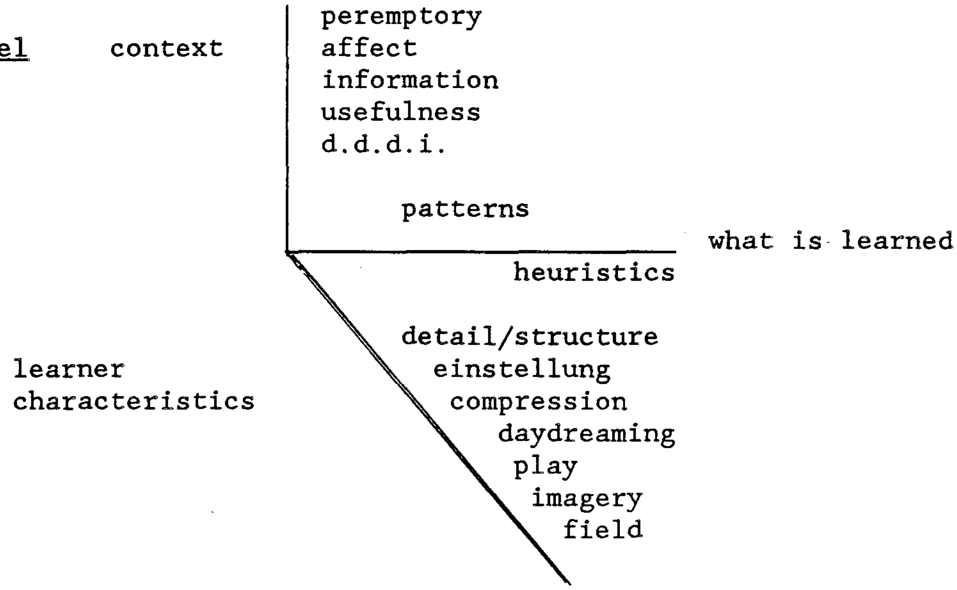
It may repay some explicit attention. As is the case with all unexamined models, because this dimension has often been left implicit it has ended up being inhabited by dubious folklore and rules of thumb. For example, a good number of researchers have asked whether or not it matters what mathematics the teacher knows, but those attempts may well have confounded the variable with others. I don't think we know whether or not they have. That's what a useful model is for. It does not answer any questions, but it may lead us to ask the right ones.

Outside of mathematics education, there is a massive body of research and commentary addressed to this dimension, but I can find very little common ground in the categories that have been proposed or used implicitly. I have therefore constructed my own, somewhat under the influence of Philip Jackson, Bruce Joyce, Jack Stevens, S.B. Sarason, M. Apple, and J. Schwab, but would not be astonished to be shown that someone has published a similar, identical, or better list.

Some detail concerning this dimension is also given in part II.

Part II - A Detailed Examination of the Dimensions.

The Model



The First Dimension

This model assumes that two different kinds of behavior are called mathematical problem solving.

For the first, it is supposed that all of us store away, somehow, a set of abstractions from past experiences that we will call patterns.

When we become aware that we are confronted with a problem, it seems that we first use some set of cues, probably most often verbal, to search for a pattern that matches what we perceive. If we find what seems to be a suitable pattern, whether or not others would judge us to have found the correct one, we say in effect, "Aha, it's one of those". At that point we have solved the problem, even if our solution is incorrect. We proceed to slot given numbers into the pattern and crank out an answer.

Informal references to this sort of problem solving go back at least as far as Aristotle, but Gibb and Van Engen seem to have been the first to analyze it thoroughly and publish both a statement of its significance and a protocol for its pedagogy.

Gibb dwelt, as I will here, on the simplest patterns that are learned. But patterns for more complex patterns inhabit all of applied mathematics. We expect engineers and dentists to react to many highly complex situations by saying, "Aha, it's one of those", and we become alarmed when they do not.

My explorations with a variety of subjects and subject matter suggest that one's first search for a suitable pattern is conducted unconsciously and remarkably quickly, and that if that search is not successful, one either quits or immediately engages the second kind of problem solving behavior.

I have rarely encountered anyone who fails to find a pattern who does not, after at most four seconds, exhibit some flick of the eye, head, or hand that indicates that the search is over. Subjects interrupted before emitting that signal give confused accounts of what they have been doing. Subjects interrupted after it has been emitted give evidence that they have begun the second kind of problem solving.

For both present purposes and for teaching, it is necessary to have some way of representing patterns. At all levels we seem to have a choice between algebraic and diagrammatic representations. Both are shown in the following figure.

For the second kind of problem solving, people accumulate a very different set of procedures, algorithms if you wish, which are intended to impose structure on problems for which no pattern seems to be available.

Earlier commentary goes back at least to Poincaré, but George Polya seems to have been the first to take steps in the direction of providing a list of what those procedures, which he called heuristics, are. Oscar Schaaf refined and extended Polya's list, and we at the University of British Columbia made further changes and additions.

That is not to say that the bulk of mathematics teachers would subscribe to the list of heuristics given here. Some might cavil at its level of generality, holding that a functional list must be more or less general. For example, anyone who studies number theory learns that a speculation as to whether or not there is a number with specified qualities may often be tested by assuming that there is a least such number. This is certainly a 'number theory heuristic'. The problem is that if functional heuristics must be that specific there are hundreds of them, and there is nothing for it but to spend the years required to learn them one at a time.

At the other extreme, folks like Poincaré and Koestler have suggested that there may be some very general functional heuristics. For example, 'ponder the problem, go to sleep, and wake up with an insight'. This proposed heuristic is not recommended for use in examinations, but it may sometimes work.

The question of what the correct heuristics are entails a return to a position set out at the beginning of this paper. I doubt that the human mind comes pre-wired with any heuristics. I believe that they are learned. And if that is the case, they are learned en passent as one solves problems under the guidance of a teacher.

For example, I would say that the teacher who suggests taking the face of a clock seriously, asks what we can learn about this new kind of number system, and proposes that students explore what adding and subtracting will be like in it, is, by inference, teaching them to use the heuristics of analogy and preservation from the following list. The teacher who presents a parallelogram, the speculation that its area might be found by multiplying the lengths of a pair of adjacent sides, and then suggests testing that speculation by making the angle at an acute vertex vanishingly small, is training his or her students in another heuristic from the list, cases-extremes.

My premise is that, within the costraints imposed by the nature of mathematics, we decide what heuristics we will teach and teach them, largely implicitly.

The question, then, is not whether or not there are heuristics. There are, and it is taken for granted throughout mathematics that there are. Much of the training of graduate students in mathematics is based on the implicit premise that some set of heuristics can be learned. Otherwise there would be little point to their regimen of solving novel problems.

The question is whether or not we can go beyond the faith that they exist and choose a set that is short enough to give us some hope of teaching all of them explicitly in reasonable time and that are demonstrably effective in enhancing problem solving ability.

The fact that his list seemed to fit rather well with some things that effective teachers have been doing for centuries and that it was of the right length to satisfy the first of the above criteria, probably motivated Polya to begin with, and certainly motivated some others of us to explore it further. There is reasonable evidence, as in Lionel Mendoza's doctoral dissertation, that these heuristics are teachable at the school level, but further evidence is required. The crucial tests of the model remain those of showing that its heuristics are both teachable and applicable.

It might be supposed that the selection of a list of heuristics is not so arbitrary as is suggested above. Surely it would be easy to establish whether or not people have and use, say, the heuristics suggested here. In fact it is. In her unpublished master's thesis, L. Dinsmore found that many school students do use many of them. But that is where models are dangerous. They create both categories and data, and one could therefore find evidence that people use the heuristics in any reasonable list. That kind of research is therefore not as persuasive as we might hope. The sharp test remains that of finding out whether or not a set of heuristics are teachable and then transferable to novel problems.

Two things must be added about the distinction between type I and type II problem solving. First, the distinction is between kinds of problem solving, not problems. No matter what our intentions as we deliver them, all problems may be solved one way or the other by different solvers.

Consider the problem, "Jack bought six cigars. He gave the clerk five dollars and received eighty cents in change. What did each cigar cost?" Students who have worked with what will be called the residue pattern below may well sense that pattern and write 6 x [ ] + 80 = 500.

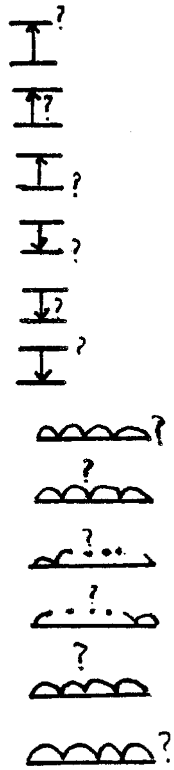
Students who do not sense that pattern can solve the problem using the heuristic that will be called analysis. The problem is informally decomposed, and each part reveals a pattern. In one part, 'something' plus eighty equals five hundred. In the other part, six times 'something' is the price of the cigars. The two parts may then, hopefully, be combined.

Second, it is common for experienced solvers to detect a pattern but to simulate type two problem solving so as to be credited with 'brilliance'. I have found that this phenomenon is first observable in the secondary grades, and is increasingly common later.

It is interesting to note that some able students irritate their teachers by using heuristics where they are intended to use patterns and that, particularly at more advanced levels, mathematics professors prefer that their students use heuristics while professors of engineering commonly prefer that their students have and use patterns. Those differing preferences may account for some occasional tension between the two in the delivery of courses.

	Algebraic Representations
1	a + b = [ ]
2	c + [ ] = d
3	[ ] + e = f
4	g - h = [ ]
5	i - [ ] = j
6	[ ] - k = m
7	a x b = [ ]
8	c x [ ] = d
9	[ ] x e = f
10	g ÷ h = [ ]
11	i ÷ [ ] = j
12	[ ] ÷ k = m
13	a x b + c = d
14	___ a:b ___
15	

Diagrammatic Representations



. continuations
. into engineering
. and sciences

from actions

to static situations

to actions

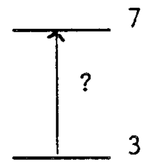
Low
1. Analysis
2. Templation
3. Smoothing
4. Cases

Middle
5. Deduction
   - Hypothetical
   - Direct
6. Inverse Deduction
7. Invariation
   - Fixing
   - Exclusion
8. Symmetry
9. Preservation

High
10. Analogy
11. Variation
12. Extension

Some Comments on this Dimension

1 - 12. The first twelve patterns are based on Gibb's work. A primary student sees, for example, 3 blocks being put in an empty box, some further blocks being snuck in, and later counts that there are 7 blocks in the box. He or she comes to sense the pattern and, probably under some guidance, learns to represent the pattern as 3 + [ ] = 7 or as on the diagram to the right.



The 11 following patterns are sensed and then represented in similar ways, whether the actions are deliberate, as in a classroom, or accidental, as in the real world.

We tend to suppose that these are almost trivial patterns, and that a sense of each of them will always be established in the primary grades. That is not so. It is quite common for twelve-year-olds to lack one or more of them and to use one or more heuristics to struggle with problems that embody them.

Glenadine Gibb was the first to create text material designed to train children explicitly to detect the first twelve and the fourteenth pattern. Max Beberman created similar material around the fifteenth.

It is worth noting that Gibb's text material was rewritten after it left her and became a good deal more rigid than she intended. As she well knew, solvers have a propensity to use patterns other than those intended. Consider, for example, the problem, "Sue gave away two of her boyfriends and had three left..." We might intend that the solver sense [ ] - 2 = 3, but many would write 2 + 3 = [ ] or 3 + 2 = [ ].

The fact that either of the latter representations leads more directly to arithmetic is not the point. The pattern of the problem had to be detected before it could be transformed. One explanation is suggested in the model. Solvers first learn to 'see' unique patterns in actions but they are soon required to impose those patterns on static situations; to imagine actions that might have led to what is seen. There are always several possible choices. Varying patterns can thereby become attached to one static situation and soon to one another.

Piagetians have an alternative explanation, and there are other possibilities. All of them are eminently testable, but intricate and lengthy observations would be required.

7 - 12 Gibb also noted that the multiplicative-divisive patterns tend to collapse into three composite patterns. This is not the place to elaborate on that compression.

13. The Residue Pattern. This pattern has been analyzed, by Bates and Rousseau among others, as 'The Division Algorithm'. It is not an algorithm

and it is not, in the main, about division so it is renamed here. As we shall see, it can serve as a bridge between patterns and heuristics.

14. Ratio. The idea is that a rule, say 2:3, can control variations in two quantities. Traditionally, only those quantities were recorded. The ratios were left implicit. Gibb was one of the first to see how much more powerful this pattern becomes when ratios are used explicitly, and was instrumental in disseminating the new pedagogy that resulted. Unfortunately, she used fractions as her explicit notation, creating confusion elsewhere.

15. Felix Kline first called attention to the significance of the function pattern, but his attention was premature. Max Beberman, Dave Page, and Bob Davis disseminated the idea when the time was ripe, and it is now a commonplace.

The function pattern can subsume everything that follows, but those who apply mathematics certainly use more specific patterns than that. Hundreds of them. For examples, pick up any engineering text. But the objective is exactly the same as in the elementary school examples given above; that, confronted with a problem, an engineer will say, "Aha, it's one of those."

We turn now to the heuristics.

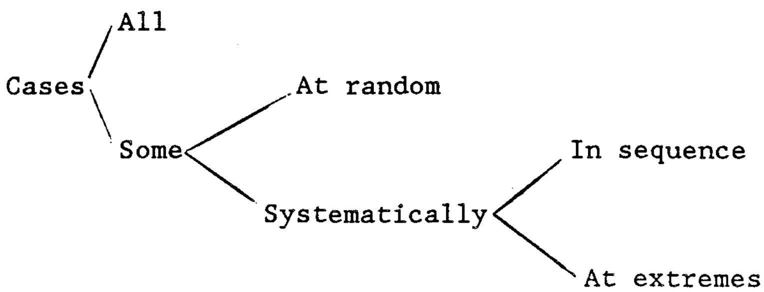
Analysis refers to the heuristics of attempting to partition a problem into parts, with the hope that a known pattern may be found in one or another part or else the solver will at least be able to focus on the nub of the problem. For example, if asked to find the outside surface area of a paper cup, most of us would detach the base, notice that it is a disc, and discard it. The nub of the problem is finding the area of the lateral surface.

Templation refers to the heuristic of carrying a variety of known patterns across a problem, hoping that something will be seen to fit. Templation is therefore a deliberate exercise of what happens unconsciously when one first perceives any problem.

The model suggests that these two heuristics are the most closely linked to residue and function patterns. Where a solver does not detect one or both of those patterns, these heuristics may be used, if only ephemerally, to provide an alternative solution.

Smoothing refers to those things we do to strip a problem of what are, hopefully, irrelevancies. Mathematical ditches do not have crenulated edges and mathematical tanks have no welded seams or dents. Smoothing is a major component of all real world problem solving but is also regularly called for in textbook problems, first by way of irrelevant names of protagonists and objects and later by way of irrelevant data. Although the case is rarely made, it seems that we have made a virtue of providing irrelevant data precisely so as to prepare solvers for the real world.

The next heuristic is the examination of cases. It is a powerful and commonly used heuristic. There are four ways in which it is used, shown below.



On occasion, it is possible to examine all possible cases. [Which of the factors of 60 are themselves composite numbers?]. More often, we can examine only some cases. We might do so at random [Before having a theorem, what is peculiar about the sum of the measures of the angles of a triangle?] but in some cases we do so systematically. We might examine cases in some sequence [Can you generalize what you found for triangles to other polygons?] or by testing some hypothesis at extremes [Try x = 0, or x very large, or make this segment vanishingly small], the origin of the odd saw, 'the exception proves the rule'.

Deduction. We now move to heuristics that are more open-ended. Occasionally, we engage the heuristic of free-style deduction to see what unexpected things may follow from what is known. And sometimes what is 'known' is hypothetical, as in proofs by contradiction.

Inverse Deduction. Probably the oldest known heuristic. The idea is to ask what we would need to know in order to show that some speculation is true, and to then backtrack, hopefully, to a list of things that are all known.

Invariation. This heuristic has long been known and used, but Oscar Schaaf seems to have been the first to treat it explicitly. The idea is to exclude or fix some component of a problem and then, hopefully, reintroduce it after a solution to the invaried problem has been found. [Could we add fractions if we made the denominators the same? Can we solve a quadratic if we fix the leading coefficient at 1? Could we solve a general cubic if we excluded the squared term?]

Symmetry. This heuristic is used quite often in geometry [So if I draw a median, the areas...] but less often than it should be in other parts of mathematics. [Consider one of Krutetskii's problems, with numbers deleted. S travels from A to B at r kph. Leaving at the same time, T arrives 10 minutes sooner than S. If T were to leave 5 minutes later, where would they meet?]

Preservation. A popular higher level heuristic in the last century. When we are extending anything (a number system, for example), begin by trying to preserve as much as possible of what was known about the simpler system. [If we insist on keeping the distributive principle, how must we multiply integers? If we delete the fifth postulate, what theorems of geometry survive?]

There is a way in which all heuristics are ways of creating new problems. In analysis, for example, each part of the original configuration poses a new problem, hopefully solvable. But the three high level heuristics focus exclusively on the generation of new problems. They are a driving force behind the creation of new mathematics.

In the heuristic of analogy, having noted (often imprecise) gross similarities between mathematical domains, we ask how questions in one domain might be asked and answered in another. [At an elementary level, having been introduced to clock arithmetics, we ask how the ordinary arithmetical operations of addition, subtraction, multiplication, and division should now be performed. The previously noted heuristic of preservation might guide our explorations. At a more advanced level, we would include exponents and ask the same questions concerning complex numbers. In abstract algebra, we might shift from groups to rings and ask what corresponds to subgroups and abelian groups.]

In variation, we deliberately vary the constraints on some component of a configuration and ask what happens to the known properties of the original configuration. [What happens if we allow a decimal point in a divisor? What happens to algebra if complex arguments and/or solutions are allowed? What happens to everything we know about rectangles if we allow a vertex angle to become acute?]

In extension, we generalize a situation. In the extreme it is clear what sort of behavior this heuristic refers to, but as is noted above, there is no sharp line between it and variation. [Can we think as we did before to find a procedure for squaring trinomials? Or to cube an expression? Or to find a general formula for expanding the nth power of an expression? Is there a formula for the roots of a general cubic? Or quartic? Or quintic? To create integers we ran numbers 'down' past 0. Can we run numbers off sideways? And then up and down? Or into n dimensions? Is there a theorem of Pythagoras in three dimensions? What happens to knots in four dimensional spaces?]

The Second Dimension

In the late 1500s, the astronomers of southern Europe were more than ready for a revival of Aristarchus' heliocentric model of the solar system. But, as Giordano Bruno and Galileo were to learn, it was not prudent to propound it too openly. The proponent ought to be a contemporary, but it might be safer to ascribe the idea to Copernicus, off on the remote Baltic. It helped that his continued and convoluted use of Ptolemaic epicycles made his manuscript almost unreadable, then and now.

On a smaller scale, a parallel syndrome may account for Krutetskii's influence in redirecting North American attention to the qualities of learners. We have protected ourselves from a lot of nonsense by insisting that brave new variables be translated into objective tests, but some of us may have occasionally missed the point. It is not enough that we have a new test. A new test must still be a surrogate for a nominal variable in a well-thought-out model. Otherwise it soon joins the considerable detritus that has accumulated in what is euphemistically called 'the literature'. For that reason, anyone in the western world who offers a list of learner qualities and adds that they may be instrumental in learning to solve problems in mathematics may be in little danger of being barbecued but had best be prepared for psychologists' guffaws. As one of their gospels says, "Of the making of tests there is no end".

Better to ascribe it all to Krutetskii, as I do here, even if we must sift through a bit of traditional marxist doctrine on the way to his keen insights. He makes a good Copernicus.

This dimension lists seven learner characteristics that are specific enough to seem useful in a model for the study of mathematical problem solving. Most of them can be found, in some form, in Krutetskii. But in fairness to him, it should be noted that he seems to have identified them for a somewhat different purpose. Without distinguishing patterns and heuristics, he makes the case made here that problem solving ability is learned. "Anyone", he says, "can become a mathematician." But he has been driven by his experience to conclude that great mathematicians are born, and that they may have innate all-or-nothing qualities. His list devolves from his attempt to identify what those qualities are.

Without, here, going into the question of whether or not it was wise to do so, numbers of us on this continent spent some years promoting, usually informally, the development of the sorts of characteristics that Krutetskii made explicit for us. That experience leads me to suppose that the qualities he found are as much learned behavior as more specific problem solving skills and that their distribution is also a matter of degree.

If that is so, then Krutetskii's list is of more general interest than he may have supposed. Qualities that may be evident in only a few students who are constrained to work through a rigid and demanding curriculum may be nascent in more students than he supposed. There may be some irony in the fact that my position seems to lie closer than Krutetskii's to the doctrine he enunciates in his introduction and conclusion.

1. Detail/Structure.

As we watch solvers try to come to grips with mathematical problems we are struck, once it is pointed out to us, by the fact that some seem to focus on the particular data given and others seem to focus on the mathematical relationships in which the data are embedded. There does not seem to be much middle ground.

The former may be better able to recall the specific numbers given than the latter. Further, it is arguable which sort of solver would do best on some classical quizzes or some standardized tests of 'problem solving'. But it is certain which sort of focus would make a person a better problem solver.

It seems as though the propensity to seek out mathematical structure should be an advantage for both kinds of problem solving, but a continued search for structure should be particularly germane to building a set of patterns for type I problem solving.

Even if the propensity to focus on mathematical structure rather than the data proves to be an immutable trait, which I very much doubt, it is important that we know more about it. It could easily become the most important component of learning style for partitioning mathematics students. But if it proves to be teachable, the implications are even greater. So far as I know, no one has seriously tested whether or not it is and, if so, how we might best teach it explicitly. But I have not searched the recent literature thoroughly. True to my practice if not my profession, I am waiting for a suitable graduate student to do it.

2. Einstellung.

Krutetskii does not use this word, but his references to flexibility seem to be a close extension of earlier work on einstellung by M. Wertheimer, A.S. Luchins, and Martin Scheerer. The latter psychologists were attempting a reductionist analysis of general problem solving, whereas Krutetskii carried the notion into the domain of mathematical problem solving.

The idea is that once some solvers perceive a problem to have some particular pattern, they persist in attempts to fill out that first perceived pattern no matter what difficulties they subsequently encounter. Others soon consider the possibility that there may be more productive ways of looking at a problem and engage either templation or some higher level heuristic for the purpose.

We have considerable incidental evidence that einstellung and the propensity to override it is a cogent intellectual quality. For example, the Miller Analogies, an I.Q. test that applicants must write as part of the admission criteria for many graduate programs, seems to be primarily a test of two things; vocabulary and einstellung.

The propensity to override einstellung comes rather close to some nominal senses of having mathematical talent. It therefore seems that we should place a high priority on finding out whether or not it is predictive of one or the other sort of problem solving ability, whether or not it has been confounded with field independence, and whether or not it is teachable.

3. Compression.

Krutetskii notes that, in solving a sequence of similar problems, some solvers tolerate or even take pleasure in working through all the steps, even when an observer would have strong reason to suspect that he or she has no need to. Good problem solvers, he claims, are characterized by the compression of routines. It is not clear how much of his conclusion derives from impatience with ritual-bound solvers and how much from evidence that they are, in fact, poor problem solvers.

We are into tricky territory here. Because group I.Q. tests tend to be speeded, this propensity may well be mixed up with our operational measures of I.Q., but that may not be all that is involved. Numbers of us, including some teachers, seem to take satisfaction in lengthy rituals that 'work out pat, just as we knew they would' both in and outside of mathematics. Second, all that work that Jerome Kagan did on impulsive and reflective problem solving styles may be involved. Like Scheerer, Kagan's primary concern was not mathematical problem solving. It is worth finding out what the relationships are between impulsive/reflective behavior, compression, and problem solving ability.

In any case, it seems that if this variable has any relevance for problem solving, it is for type I problem solving.

4. Daydreaming.

This variable is not in Krutetskii's list. For good reason, one might suppose, because the circumstances of his studies made it well-nigh impossible for anyone to do it.

Formal interest in daydreaming goes back to near the beginning of this century, culminating in some provocative work by Benjamin Bloom. But interest in this phenomenon, among other things, died in the Second World War.

As Jackson suggested some years ago, it is due for a revival. The premise is that we all daydream, all the time. In two ways. We may, for example, begin by wondering whatever led us to select a particular section meeting at a convention, and end up rehearsing the evenings' activities.

But there is another sort of daydreaming. It is likely impossible to learn cognitive material without regularly taking sidetrips to connect what is being heard or seen or read to past learnings, and the quality of one's learning may well depend on the quality of those excursions. The purported advantages of various individualized forms of instruction may indeed rest on the fact that in mass instruction those excursions compete with the primary flow of information.

It is not at all certain how learners manage that conflict in mass instruction. Effective teachers seem to evolve ways of helping learners do it. But because mathematics is so strongly recursive, it seems that this variable might be crucial in accounting for some learning. It therefore deserves more attention than it has received. A brief search of the literature soon exposes the reason why it has not; it is difficult to get at the variable in adults, and more so in young children.

5. Play

Serious business.

The impulse to play seems to run deep in we mammals, and most of all in us. Long ago it was decided that it must serve some function, and it is now a platitude that it provides an ordinarily safe arena within which both the young and the mature can rehearse mental and physical skills that may have ultimate survival value.

More than anyone else I know of, Jack Stephens persisted with that argument. He argued that the Gods punish cultures for grim utilitarianism as sternly as they do individuals for hubris. Sooner or later they lack the resources to meet some new challenge. For that reason, he argued, natural selection has created school curricula comprising literature, mathematics, history, geography, nature study, art, music, and physical education; the decorative and playful artifacts of civilization. The bulk of the school curriculum in surviving cultures is decorative and playful, not because they are decadent, but because that is why they have survived. Only those cultures will have the resources to meet novel threats.

Stephen's case accounts for our propensity to find some way to turn virtually everything into a game. England's wars may, after all, have been won on public school playgrounds.

While scientists seem loath to make it explicitly, they continually make the same point in less direct ways. They protest that 'targeted research' is killing science; what is needed is more free-style exploration; more play.

Over and against a later section on evident usefulness, then, success at becoming a mathematical problem solver may depend in part upon mathematics capturing the propensity to play. And the two may not be antithetical at all. We play at what may, someday, be serious business.

So, after all, 'Friday afternoon mathematics' may serve an important purpose. It seems important to know whether or not that is so, particularly in the long run, and if it is, what sorts of things lead people to play with mathematics.

6. Spatial Imagery.

There seems to be little doubt but that the ability to create, hold, and manipulate mental images of spatial configurations is important for at least some problem solving.

Krutetskii considers it to be a variable of secondary importance, and withholds judgement as to whether it is innate or learned. It has attracted considerable attention on this continent because some researchers have shown that what they tested by some instruments may be sex-related. That fact does not, of course, resolve the question of what, if anything, is innate.

It is not so clear as has sometimes been supposed exactly what we are talking about when we refer to spatial imagery or visualization. At the simplest level, one must be able to 'see' parts of a drawing that are obscured, as in all those logically flawed 'stack of blocks' problems. For other purposes, one must be able to hold a configuration in the mind and pay attention to parts of it.

For practical purposes and often for test questions, one must be able to imagine either rotating pictured objects or viewing them from imagined directions. And for yet other purposes, one must be able to use present perceptions to project what would happen if ...

As with most of our attempts to find out what is 'really going on' in the mind, we will probably end up being able to choose from along a spectrum between a marginally useful 'general spatial imagery' variable and 134 or so narrowly defined components of spacial ability. Those who use the general variable will likely take it to be sort of innate and those who work near the other end of the spectrum will more likely hold that the skills involved are probably teachable.

7. Field Dependence/Independence.

Since 1954, Witkin et al. have explored this seemingly stable characteristic of cognitive style. In its most general form, it refers to the propensity to be compromised by a problem's ambience. Blake's excellent work seems to be the main serious study of its possible relationship to mathematical problem solving. His results are provocative, for both the likely significance of the variable and its possible interactions with other variables.

This variable may be even more important in 'real' problem solving than Blake found it to be, but if it is we are not likely to notice that it is in the classroom, where problems are generally stripped of their ambience.

<u>The Third Dimension</u>

Anyone's first impulse is that the context of learning is important and must be taken into account, somehow, if we are to understand how students come to learn how to solve mathematical problems.

But, given some intervening musing, a researcher's second impulse is almost certain to be that it is a rat's nest, and that it might be just as well to control all that and focus on something simpler. There are peers, teachers, parents, texts, workbooks, computers, the various media, and a large number of physical qualities of the environment to consider, each with multifaceted influences; and the massive possible interactions between all of them.

It seems unlikely that we will ever have a comprehensive model within which we can account for all of these influences. But since we won't, there is less need to apologize for having selected a few variables that seem particularly cogent and whose interactions may be worthy of study.

That does, however, raise a problem. Some geography texts notwithstanding, it is not sufficient that a model simply categorize things. The categories in a model are intended to guide us to asking productive questions, and questions that do not get asked on account of the categories and relationships of the model are supposed to be less likely to be productive. Since those given here are far from exhaustive, graduate students in particular who might consider working from this model should pay close attention to the danger that some important variable might have been ignored.

I have selected five likely-looking variables, somewhat under the influence of Jack Stephens and Philip Jackson.

1. The Degree to which the Environment is Peremptory.

Take reading. Most children learn to read within a remarkably narrow zone centered on the age of six, remarkable because most other arbitrary accomplishments arrive across significantly broader spans of time. There must be a reason.

Considering how arbitrary and complex the task is and the few centuries during which we have placed any sort of premium on reading, it seems most unlikely that the reason has to do with any sort of hard-wired schemas.

It seems more reasonable to suppose that the reason is that, where reading is concerned, the environment is highly peremptory. Parents link, rather often, going to school and learning to read. The almost mandatory sequitur for every adult who asks "How old are you ?" is, "Oh, you'll soon be ..." Just about everything in the environment says 'read me', and says, "When you go to school ..." So they go to school, and they learn to read.

Our common experience and a quite massive body of research from across psychology, sociology, political science, and education all suggest that when the environment is to that degree peremptory, we tend to do what we are 'told'. We give credence to the significance of this variable whenever we seek to modify "everyone's" attitudes towards women, the handicapped, ethnic minorities, homosexuals, the aged, smokers, and premiers. The premise seems to be that if a critical mass of influences create a sense of the peremptory, everyone's behavior will change.

And yet, things do not always work that way. Some of the nice and nasty things children learn seem to be mastered in the face of indifference or active opposition from most or all of the same quarters. It is easy enough to speculate as to what some of the reasons may be, but those reasons tend to be adduced post hoc.

All this is not so remote from mathematical problem solving as it may seem to be. It is a powerful force, even if we do not know where all the levers are. The degree to which learners persist with one or another part of the curriculum may well depend on their sense of the peremptory, and Krutetskii's observation that persistence is a quality of good problem solvers may not, and probably does not, stem from any innate propensity to persist with mathematical problems. It is more likely that, somehow or other, persistent problem solvers have caught a sense of the peremptory where problems solving is concerned.

If that is so, we had better find out how it works, where and when it works, and where the levers are.

2. Affect.

It is common lore that many of us look back to some adult or friend who made us what we are, not so much by what he or she taught us as by caring; by loving us. Jack Stephens suggests that that person's motives may not be altruistic; that he or she might be driven and, in the adult world, would be indistinguishable from a crashing bore. But the effect would be the same.

Yet institutional education is designed as though the common lore is not true. Powerful sanctions mandate that we all act, most of the time, as though it is not. It is therefore worth thinking through what the implications are for the selection and training of teachers in general and for the teaching of mathematics in particular if we suppose that it is.

The first step is to try to discover whether or not it is true. We have good reason to be careful with this kind of folklore. It is also common lore that many important scientific discoveries have been accidental; someone's nose dripped into a Petri dish, some raw rubber spilled on a stove, a laboratory dog pissed on the floor and flies were uncharacteristically attracted to it.

And, in the sense in which they have been passed along, none of these legends are true. R.S. Root-Berstein tracked down every such story he could find. He found that many of them are patently false and that the residue have been distorted to create a myth of serendipity.

Good scientists do draw inferences from unexpected events, but that is not the same thing as stumbling over great discoveries. We seem to want to believe that those arrogant scientists actually bumble along, stumbling over their most important discoveries, about as badly as we want to believe that others care about us. In short, many of us may exaggerate the influence of "old-what's-his-name".

The theories that can be posed around this variable are eminently testable, even if some of the research designs might be unconventional. If the folklore is supported, it is hard to imagine results that could have greater impact.

3. The Quality of Information Available.

In addition to old-whats-his-name, in my case a reputed alcoholic named Tom McCaughy, I look back fondly to three unusual Christmas presents in the Vancouver slums, "The Wonders of Science Simplified", "Half Hours with Great Scientists" and a massive Webster's Dictionary. I memorized the first two and wore the third to a frazzle.

Did they matter in shaping me ? I don't know. I do know that we account for much of what we do well and badly in education in terms of the quality of information we make available to learners, and that for 'quality' we have often read 'quantity'. We spend a massive amount on the implications of that premise.

But our attempts to finger just what information in this smorgasbord does matter have not been all that productive. At the elementary level, what mathematics teachers know does not seem to matter at all. At the secondary level and beyond it seems to matter a little. The quality of textbooks and computer programs matters a little, but not a lot. As do more remote sources like libraries. Perhaps it is the cumulative effect of all of them that matters and we have not been asking the question in a sufficiently general way. That is the conventional explanation. But that may not be the case at all. Perhaps, as the above personal reference was intended to suggest, the mistake has been to read 'quantity' for 'quality'. Perhaps we have not been asking our questions in a sufficiently specific way. If the religious fundamentalists are right about conversion, perhaps one high quality communication, delivered at the right moment, can burn its way into the soul and shape the way everything that follows is perceived.

Again they may be spurious, but autobiographical accounts lend credence to this speculation. And if it is true, it has huge implications for the teaching of mathematics. Perhaps much of what we do in mathematics classrooms only sets the stage for those occasional moments when a student may catch something that shapes the rest of his or her life.

If that is true, we had better find out what is caught. It may, for example, be what is reported in another of Krutetskii's astute observations. He holds that good problem solvers are not satisfied with 'a' solution, that they have a strong sense of elegance. They will work and rework the solution to a problem until it looks right.

Just because mathematical elegance is hard to define ought not to lead us to dismiss it too quickly. It may or may not be about the same thing as analogous qualities of art, music, poetry, architecture, anatomy, and weapons systems, but there are good reasons for supposing that it is.

All sorts of evidence suggests that we come wired up with some sort of hunger for the elegant, and that many of us reify it in what we do, no matter what that may be. We speak of an artist with a front end loader, and not just as a metaphor. Many of us are prepared to argue that it is the same thing, no matter what the field of endeavor.

If Krutetskii's observation and the above dialectic are both correct, the propensity to become a good mathematical problem solver may be due, in part, to someone or something having somehow forged a link between mathematics and the general thirst for elegance. We had then best ask how folks catch an appreciation of mathematical elegance.

So far, introspection is about all we have to go on, but natural language is never a bad place to begin. So I asked a few folks what 'elegance' is and how they found out, unfortunately long before deciding that the variable has a place in this model.

If my recollections are not now warped, they support the notion that, somewhere down the line, one or a few high quality communications may be crucial.

So perhaps our shotgun approach to making quality information available is misguided or at least incomplete. Perhaps the right book, the right words, or even the right problem, at the right moment, may be more crucial.

Whether or not elegance is a key component of it, we need to know more about how the quality and quantity of information available interacts with the learning of mathematical problem solving skills.

4. Evident Usefulness.

Until recently I attributed the perennial student question, "What is all this goodfer" to lack of interest rather than to any real desire for an answer. Deep down, I sensed that students want to play with mathematics. If I had thought of it when I was a classroom teacher I would have anticipated one of our student teachers who, this past year, replied, "It's all tied in with next year's sex education program".

I suspect that part of my reluctance to answer the question can be traced to my anticipation of the rejoinder, "But I have no intention of doing any of those things". I knew how the rest of the dialectic went, but felt weary at the prospect of arguing it through, and implicitly assumed that nothing but a direct and simple answer would do.

I, at least, became accustomed to avoiding the question but a case can be made for taking it seriously. Ours is a world in which one learns A in order to learn B in order to do C in order to get D, and mathematics falls at the extreme of interposing letters between A and C.

I think, perhaps, I erred both in misunderstanding the question and in underestimating the instrumental power of the right sort of answer. I recall teaching a third year course in abstract algebra. After the first lecture, three chemistry majors cornered me and said, "Don't misunderstand us, but we were sent here to take this course. Why are we here?" About a week later, I gave them an answer to their question. About six years later, I understood their opening phrase.

By that time, I had paid more attention to the reasons why students do things and had come to the conclusion that their reasons for working hard at some things and not working hard at others are about as complex as ours, and that the simplistic answers they sometimes give us are given for reasons similar to ours. In other words, I think that from the early years students are more often motivated by evident usefulness than we may suppose, even when the causal links are subtle, and that we might be surprised to discover at how early an age many of them are able to cope with rationales that use most of the letters of the alphabet.

All that may not be so, but it is certainly worth finding out whether or not it is. It is a natural for a package of masters' theses.

5. Denial/Delay/Distraction/Interruption

In his 'man from Mars' study of classrooms, Jackson found the four qualities listed above to be the most cogent ones in classroom instruction. Students come to cope with them to varying degrees, and their abilities to cope with them are almost certainly strongly related to their achievement.

And not necessarily in negative ways. Anyone who has completed a major in computer science at my university has more than a passing acquaintance with all four, and their survival might well be linked to having built up some early tolerance. Learning to wait for the pencil sharpener may not be a bad preparation for waiting for a free terminal.

Yet these four qualities of group instruction remain virtually ignored in even reductionists' analyses of classroom learning. Consider, for example, B. O. Smith's monumental analyses of tapes of lessons. One would suppose that students are rarely told "You can't do that now", or "You get to think about that next year", or "That's the end of that. Pack up for your next class", or are distracted by discussions of other matters elsewhere in the room. Yet that is what happens in even the best of real classrooms; all the time.

It is not certain what those qualities of classroom instruction have to do with the qualities of learners and the learning of mathematical problem solving skills. One might suppose that they are considerably involved.

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Topic Group D

MATHEMATICAL EXPOSITION AND WRITING

Ed Barbeau
University of Toronto

A high school student once told me that he tutored younger students in mathematics. I asked him what, in his opinion, was the most serious impediment faced by his clients. His response was striking: "They can't talk about what they are doing."

This is indeed a severe impediment. But is this addressed in an essential way in the teaching of mathematics? In class, mathematical techniques are presented in standard ways and the focus is on "getting" exercises and problems mathematically, so that we lose sight of and fail to address explicitly the real difficulties of expressing and organizing mathematical ideas, even when we think they are understood. This deficiency eventually catches up with the student when he discovers that he is without a voice and has no appreciation of the relative significance of the mathematical concepts he has been taught.

I would like to suggest three ways in which we might unlink mathematical exposition from other issues which would otherwise crowd it out, such as the trauma of trying to solve a problem or apply a technique.

The first of these is to focus on what the <u>problem</u> is rather than what the solution is. In fact, we may not even care to get a full solution as far as most of the students are concerned. Let me illustrate with two examples:

1. Let c be a positive integer. The sequence (f_n) is defined as follows: $f_1 = 1$, $f_2 = c$, $f_{n+1} = 2f_n - f_{n-1} + 2$ $(n \geq 2)$. Show that, for each subscript k, there is a subscript r such that $f_k f_{k+1} = f_r$.

Is this a problem you would give to a class? Most teachers would instinctively answer, no. It certainly looks like a monstrosity. Yet one does not have to venture very far in mathematics before subscripted notation and parameters are encountered. So let us press ahead, but with a modified end in view: to understand what the result is rather than to establish it.

First of all, c is supposed to be a positive integer, so why not take one and give it a dry run? Try $c = 1$.

Now cope with the subscripted notation. We need the idea that we are going to list a sequence of numbers: f_1 is the first, f_2 is the second, and so on. The first two numbers of our list are given; each one following depends on its two predecessors - can we describe in words how we are to generate the terms of the sequence ("from twice the last term, subtract the one before and then add 2")? Having gone through this, we can now prepare some examples:

1, 1, 3, 7, 13, 21, 31, 43, 57, 73, 91, 111, 133, 157, 183, ...

1, 2, 5, 10, 17, 26, 37, 50, 65, 82, 101, 122, 145, 170, 197, ...

1, 4, 9, 16, 25, 36, 49, 64, 81, 100, 121, 144, 169, 196, ...

1, 5, 11, 19, 29, 41, 55, 71, 89, 109, 131, 155, 181, 209, ...

1, 6, 13, 22, 33, 46, 61, 78, 97, 118, 141, 166, 193, 222, ...

Preparing these examples should not just be an inert process of substituting into the recursion. Are there any short cuts to getting the numbers? Are there any patterns? Describe what they are? (The 1, 4, 9, ... sequence should be particularly transparent.)

Now we can address the problem. What are we supposed to observe? Can we say it in plain English ("the product of two consecutive terms in the sequence occurs later in the sequence")? Can we verify the result for any particular values of c? Do you believe the result: try to falsify it.

By this time, probably a good hour has been spent with the group and, for most of the participants, one might be happy to stop here, and leave the more eager among them to pursue the matter further. Already a lot of mathematics has been covered. But most importantly, we are taking pleasure in the journey rather than giving all the emphasis to the destination.

The next example goes the other way and asks the group to make up the problem.

2. $3^2 + 4^2 = 5^2$

$$10^2 + 11^2 + 12^2 = 13^2 + 14^2$$

$$21^2 + 22^2 + 23^2 + 24^2 = 25^2 + 26^2 + 27^2$$

$$36^2 + 37^2 + 38^2 + 39^2 + 40^2 = 41^2 + 42^2 + 43^2 + 44^2$$

What is the pattern? Can we construct another instance? Each equation deals with consecutive integers? How can we describe the smallest? the largest? the two beside the equals sign? Do we have to add squares in order to check the equations, or can we do it some other way (eg. by factoring differences of squares)? How can we formulate in mathematical terms a problem to solve based on these equations?

Unlike the first example, the solution is much more easy to obtain once the problem has been properly formulated. In fact, there are several ways to proceed and one can well enquire how deeply mired in mathematical notation we have to become. Try it yourself.

The second way to focus on exposition is to make exposition the purpose of the homework assignment rather than the getting of the problem. In this situation, the teacher chooses a single problem of appropriate difficulty upon which she is prepared to spend up to an hour of class time. The purpose of the classroom session is to try to ensure that the problem with its solution is understood by all. The homework assignment is to write up the solution. Here is an example:

3. A number of houses are along one side of a straight street; in each, there lives a boy. Where should the boys meet so that the sum of the distances they walk is a minimum? In mathematical terms, we are given n real numbers a_i and are asked to determine a value of x which minimizes.

$$\sum_{i=1}^{n} |x - a_i|$$

We need not give the mathematical formulation. Rather, let us line up a half dozen or so students along the blackboard and get the class to discuss where the likely meeting point should be. The answer is not immediately obvious, but should become clear after some time of discussion.

Another problem appeared on this year Descartes' competition:

4. Let a_1, a_2, ..., a_n be the numbers 1, 2, 3, ..., n written in any order. Prove that

$$\sum_{i=1}^{n} |a_i - i|$$

is always even.

The quick solution uses the fact that $|a_i - i| \equiv a_i - i$ modulo 2. However, as a marker for this problem, I can attest that very few of the thousands who wrote this contest did the problem this way. A fair number of the candidates tried to pursue an argument along the following lines: consider the parity of the pair (i, a_i). There are four possibilities: (i) i and a_i are even; (ii) i and a_i are odd; (iii) i is even and a_i is odd; (iv) i is odd and a_i is even. Possibilities (i) and (ii) contribute an even term to the sum, so the whole issue turns on showing that there are exactly as many pairs satisfying (iii) as (iv). This may be a good approach on which to base a classroom discussion.

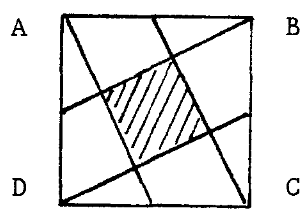
Classroom discussions on a single problem may well start to address the expository difficulties that might arise in writing up the solution, and how these might be handled. Some questions that might arise are:

. do we really need a contradiction argument, or do we just have a direct argument in disguise?

. do we really need to use subscripted notation, or can we make the notation simpler?

The final way of focussing on exposition which I would like to discuss depends on a quality of teaching I call "footwork". It is desirable in teaching mathematics not to be put into a position of answering questions the students have no interest in or need of asking. As a result, there are many small points that one will normally take for granted since they are accepted by the student. However, occasionally one of these points may cause some consternation and at this time the wise teacher will pounce and try to draw out an explanation. Let me give a few examples from my experiences with groups of high school students.

5. A charming problem is to determine the area of the shaded region in the diagram below, given that the square ABCD has side length 1. Each side of the shaded region is determined by a line through a vertex and a midpoint of a side of the square.



There is a simple solution to the problem, but you will find that many students get into some deep water chasing similar and congruent triangles. Normally, it is taken for granted that the shaded region is square. However, one day, there was a student who questioned this fact. So I said: "Who would like to convince X that the shaded region is square?" The group was stopped in its tracks. They discovered that this was not such an easy question to dismiss. After some time of trying to identify equal sides and angles, I asked: "How do you really know it is square?" Eventually, we started to hone in on the symmetry of the situation, which suggested a transformation argument? Can you give a quick justification for this fact?

6. An old chestnut is to show that the product of four consecutive positive integers can never be square. A little experimentation suggests that we get a square if we add 1 to such a product; indeed, $x(x + 1)(x + 2)(x + 3) + 1 = (x^2 + 3x + 1)^2$. Once this is established, one concludes triumphantly: "The product is not a square, since two nonzero squares cannot differ by 1." One day, a student in the group said: "I don't see that." So I put it to the group: "Explain to Y that two nonzero squares cannot differ by 1." The students got into a lot of trouble on this one. It was pointed out that if you take

consecutive squares in order, the differences increase steadily. But I complained that they had not told me what this has to do with nonconsecutive squares. I tried to get them to see that the explanations they were trying to provide were no more transparent than the thing to be proved. Here a little notation helps cut through the fog: we want to examine the solutions of $x^2 - y^2 = 1$. With the factorization of the left side as a difference of squares, we can now arrive at a quick argument.

7. Another fact of the "everybody knows ..." variety is that the longest chord of a triangle is equal to the longest side. But how would you convince the doubter in twenty-five words or less that this is so? Remember that a simple fact deserves an explanation no more weighty than the fact itself.

8. The following problem was given to two groups - a class of experienced teachers of mathematics and the high school students who were candidates for a place on the Canadian team in the International Mathematical Olympiad:

   a, b, p, q, r, s are positive integers for which

   $$\frac{p}{q} < \frac{a}{b} < \frac{r}{s}$$

   and $qr - ps = 1$.
   Prove that $b \geq q + s$.

You might want to try this. Almost nobody in either group brought it home. Let me give a popular argument: Since $r/s = qr/qs = (ps + 1)/qs$, we have the inequality

$$\frac{ps}{qs} < \frac{a}{b} < \frac{ps+1}{qs} \quad .$$

Now, "everybody knows" that any fraction between two consecutive fractions with a given denominator has a denominator bigger than the given denominator. Hence $b \geq qs \geq q + s$.

Since we have actually ended up with a stronger result than desired, we might be inclined to smell a rat. And indeed, the result appealed to is false; can you provide a counterexample?

However, in this connection, there are two small facts which I will leave you to establish in a clean way:

(i) Is it always true that $qs \geq q + s$ for positive integers q and r? Not if one of them is 1. If both exceed 1, show that this is true and indicate when equality occurs.

(ii) A modified version of the "everybody knows" statement <u>is</u> valid:

$$1 < \frac{x}{y} < \frac{u+1}{u} \qquad \text{implies } y > u$$

where x, y, u are positive integers. Is there a quick proof?

Topic Group E

THE PERRY DEVELOPMENT SCHEME AND ITS IMPLICATIONS FOR TEACHING MATHEMATICS

Larry Copes
Institute for Studies in Educational Mathematics

A few weeks ago I gave a homework assignment to students in my course called Computing for the Liberal Arts: "Discuss the thesis 'Programs in Artificial Intelligence should not be developed.'" As experienced teachers you need know nothing about Artificial Intelligence to guess the result. One student put it very well: "I looked in five books and none of them told me whether Artificial Intelligence was good or bad."

That attitude reminds me of the student in an Abstract Algebra course who articulated what many of her peers never had, at least to me. She came into my office, deliberately took out a notebook and pencil, looked up at me expectantly, and said, "Now, for once and for all, please tell me the steps for proving something."

Why are so many students uncomfortable if the teacher isn't lecturing? Why do some students seem to refuse to reason logically? Why do students have so much difficulty solving problems? Why do students seem to leave their creativity at home? Why do some students get upset when I answer questions with other questions?

As William Perry, to whom I'll be referring today, says, "When bright people act stupid, we are in the presence of very powerful forces."

Background

In the early 1950's, as head of Harvard University's Bureau of Study Counsel, Perry began asking similar questions. To find out something about those "powerful forces," he began a series of open-ended interviews with Harvard and Radcliffe freshmen, trying to categorize the "personality types" of students. Then, since the interviews were so much fun for all of those involved, he and his colleagues invited the students back at the end of each academic year until they graduated. As they did this for several classes of students through the 1950's and 1960's, they encountered more complexity than they had anticipated. In fact, students changed "personality types" rather radically. It was not until the researchers began looking at the ways in which the students changed that they saw patterns.

What is now known as the "Perry Scheme" consists of nine "positions" from which students successively view their worlds, and of transitions between consecutive positions. Oh, none of the students travelled through all nine positions in their years at Radcliffe or Harvard; but by considering overlaps, Perry's judges could trace out a journey that the students probably took over a much longer period of time: Some students also did some backtracking on that trip, but none bypassed any of the positions.

For simplicity in this introduction, we'll consider these nine positions in four groups. A written synopsis of the entire scheme is appended. I'll describe the four categories, and then discuss some of the general principles and specific applications to teaching mathematics.

The Perry Scheme

Students in the positions Perry calls Dualism see everything as clear-cut. There is right and wrong, good and bad, we and they. It is as if, to use Perry's metaphor, the Right Answers are written on tablets in the sky. There are Authorities - perhaps teachers, ministers, rabbis, even parents - who can read those tablets. Being intellectually right, or correct, is the same as being morally good, and being wrong is wrong, evil. The student's identity is as a person trying to learn to read the tablets, to become an Authority. Those who disagree with Truth are Bad People.

Does this sound somewhat primitive? Perhaps it is. But many, if not most, of our students use this lens to view at least some areas of their experience. And perhaps we do, too. Most of my students, anyway, look at mathematics dualistically. After all, God put the odd-numbered answers in the back of the book, and the teacher had better know the answers to the others or else he's incompetent. Class discussions? They're a waste of time; if the other students had anything to offer, they would be Authorities, not students. If a professor doesn't answer my questions, she must be a fake.

Note how the intellectual, ethical, and identity lenses work together: one's identity is in trying to be morally good by being intellectually correct. For me, one of the most enticing aspects of the scheme is how it recognizes that growth in one of these areas affects development in the others.

Anyone concerned with social survival, as most adolescents are, learns fairly early in life that one doesn't make friends by expressing the opinion that anyone who is different faces eternal damnation. So at least in some areas of their lives, students begin to realize that there are clouds between the Authorities and the tablets, so that some things are not so clear cut after all. In fact, the legitimacy of diversity is quite liberating - so much that students swing to a position almost the opposite of Dualism. "Everybody has a right to his own opinion," chant the students in the positions that Perry labels **Multiplicity**. "All opinions are equally good." The ideal teacher is the one in blue jeans sitting cross-legged on the desk, running a discussion in which everyone says Something Meaningful, and letting the students grade themselves. "What's cool with you is cool with me."

There's Dualism present in Multiplicity, however, just as each position embraces and extends the previous ones. Multiplists still have moral good and bad, even though they may deny it. To them, it is morally bad to believe in anything smacking of intellectual correctness or "standards." As Perry puts it, "Where nobody knows, anything goes."

Of course, we see this "laid back" attitude in pre-college students these days. And most college students keep it in many areas of their lives. But do any of our students get even this far with respect to mathematics? Rarely, I think. How many of them can, therefore, move on to think in ways we want them to think? It's not that dualistic and multiplistic students can't reason logically, or be creative - they just don't see any appropriateness in their doing these things.

And how is it that we want them to think? Well, multiplists are fond of saying "everything's relative" to mean "it doesn't really matter," without claiming that anything is actually related to anything else. We, on the other hand, tend to believe that things are related. Opinions are related to data, and even data themselves are related to context. (I have a tee shirt that says "Scientists should express the opinions on which their facts are based.") In the positions that Perry calls **Contextual Relativism**, we see that there are some opinions to which nobody has a right - at least an intellectual right - because they're not supported by data, or because they're internally inconsistent. There's no intellectual basis for racism or sexism, for example. But we also see that there are conflicting opinions that are equally valid, that have equally good arguments. We might disagree with the premises of Christianity or Islam (or both), but they are - and probably always will be - equally valid theologies. Euclidean and non-Euclidean geometries have different theorems - "truths" - but is one right and the others wrong?

Moreover, a contextual relativistic lens gives a person the moral and intellectual power to consider all things from a variety of perspectives: to be able to view herself from outside herself, to be able to see ethical questions as special cases of intellectual questions, to be able to look at a mathematical situation in different ways while trying to analyze it. Again, the identity is here, since one's identity is now as an active learner, as a peer with the authorities - now uncapitalized, seen as more experienced analyzers.

David Pimm pointed out to me many years ago that these first three categories are represented by three cricket umpires. The dualistic umpire says "I calls 'em like they are." The multiplistic umpire says "I calls 'em like I sees 'em." And the contextually relativistic umpire says "Till I calls 'em they ain't anything!"

But I promised you four categories. Do we really think that the ethical is a special case of the intellectual? Perry found that students could not live long with the realization that conflicting philosophies, careers, or lifestyles are equally valid for them. A few retreated, bitter and angry, to dualism or multiplicity. "Here are the answers," said someone, offering them a Cause to support - a bible, an -ism. Most, however, plunged onward, into what Perry calls **Commitment**. This is Commitment with a capital C, not a commitment to go to the party Friday night, but rather Commitment born out of Contextual Relativism, with the often painful realization that different Commitments would be equally valid. It's often Commitment to a career, chosen out of several fascinating possibilities, or Commitment to a particular person, realizing that there might be several persons who are equally "Mr. Right." One's identity is, at first, in that Commitment. "I am a mathematics teacher." But as the number of Commitments increases, career Commitments begin to conflict with relationship Commitments and philosophical Commitments, and one's identity is found in the style one uses to balance those intellectual and ethical

Commitments. Underlying all this is the realization, within Contextual Relativism, that these firmly held Commitments may have to be re-examined and even changed someday.

That, briefly, is the Perry Scheme. You can find out about the many subtleties of ebbs and flows by reading Perry's book, **Forms of Intellectual and Ethical Development in the College Years: A Scheme** (Holt, Rinehart, and Winston, 1970). If you want a feeling for the vast amount of further research on the Scheme, you might want to join the "Perry Network" of researchers and practitioners of the Scheme. For information, contact Bill Moore, CADI, 1670 Prince Avenue, Athens, GA 30606, USA.

For now, let me stress several ideas. First, Perry is describing an <u>is</u>, not an <u>ought</u>: he makes no claim that this is what <u>should</u> happen to students. Also, as the title of the book suggests, the scheme describes the <u>form</u>, not the <u>content</u>, of student developmental positions. One might be strongly committed to philosophies of the Left or the Right. Finally, lest we be dualistic about the Perry Scheme, I must say that one can be viewing different parts of her world from different positions; she might be relativistic with respect to relationships, multiplistic with respect to her career, and dualistic with respect to mathematics. Moreover, of course, the Perry Scheme provides just one of many metaphors we might find helpful in interpreting <u>our</u> experiences with students.

<u>Implications for Mathematics Teaching: Why?</u>

Before discussing some concrete ideas of how the Perry Scheme might affect our teaching, we should look at some general principles. These might be introduced by examining first a very simple application. That is summarized by the idea of "meet students where they are." When we hear about learning styles we often hear "no one style is better than the others; you need experiences to communicate with students of different styles." Our simplest application of the Perry Scheme is based on the idea that "everyone has a right to his own Perry position." To dualistic students, provide the Truth; to multiplists, provide the diversity; and make sure that the relativists think contextually.

This approach appeared satisfactory to me until I found from one of those learning style "inventories" that I was equally strong, or perhaps equally weak, in all categories. Naturally, I then got to thinking that we should consider persons who could learn in many different ways better off than those who were limited by just one major learning style. But is one Perry position better than another? If so, of course, we as persons committed within Contextual Relativism - we are, aren't we? - would suspect that these positions are better than dualistic or multiplistic positions. But why? We as mathematics educators can think about this bias in two ways: locally and globally.

Locally, we must ask the question, "Is mathematics dualistic?" I personally chose to major in mathematics because it seemed to be an island of sanity in a sea of Relativism. In his talk Friday, Pat Scott quoted

Rucker about the universality of mathematics, "imposed from without" and independent of culture.

But Dorothy Buerk tells of putting two lists of words on a chalkboard and asking mathematicians to describe the difference between the lists. They said that the more relativistic words described doing mathematics, and the more dualistic list described how we teach mathematics. Pat himself seemed to prefer quoting Alan Bishop in saying that mathematics is an activity. When doing mathematics we generate data, find patterns, make conjectures, try to prove them, try to find counterexamples, refine conjectures, try to prove the new conjectures or find counterexamples, etc. There are no absolute formulas for engaging in this process. And even when we are focused on solving a particular problem in a particular context, when the answer should be as absolute as possible, Alan Shoenfeld points out that we need a "metacognition" to examine our own thinking. (See **Mathematical Problem Solving**, Academic Press, 1985). Remember, only in Relativism can a student do this kind of thinking about her own thinking. Hence we can argue that to do mathematics well one needs to be able to think relativistically.

On a more global level, one needs to be in more complex positions to handle the complexity of the world better, to be good citizens in a democracy, and so on. Certainly a great many educational philosophers (not to mention our own college catalogs) give us ample reason to change the "is" to an "ought." We can make a Commitment - within Relativism, of course - to the Perry Scheme as describing goals for our work - an operational definition, one might say, of liberal education.

Remember, this is Commitment within Contextual Relativism. There are counterarguments. After all, many people can and do live very happy lives as dualists. We are saying that we think it good that students give up this security for the rest of their lives, to experience a great deal of unhappiness. Once when I was consulting with a university faculty a professor told our group about a philosophy student who proudly showed his family his much-agonized-over credo, which had even been published in the school newspaper. Because his set of beliefs differed from his parents', they rejected him, and as a result he committed suicide. Development on the Perry Scheme is not a guarantee of eternal, or even temporal, bliss.

Implications for Mathematics Teaching: Principles

If we nevertheless accept the Scheme as providing us with some goals to work toward, we can draw on general development theory for four guiding principles.

First, we must realize that we have to change a traditional concept of ourselves as teachers. I know that when students begin to discuss personal problems in my office I tend to back off. "Leave that to the Counselling Center," I want to say. "I deal only with the intellect." But Perry has taught us that the intellect cannot be separated from the ethical and the identity. And the connection goes beyond the claim of a student who explained to me that she hadn't done her homework because she had been trying to decide whether or not to have an abortion. The way she had come to her decision - whether she had weighed evidence or just followed the advice of an Authority, whether she had considered it entirely as an intellectual question to be analyzed or had recognized the necessity for a "leap of faith" - was affected by the development encouraged in her courses. And the response that she got to her decision outside the classroom affected the way she could function in her role as a student. We don't have to be psychologists to be good listeners, to give support to other human beings.

A more technical principle arose out of investigations of moral development on Lawrence Kohlberg's scheme of moral development. According to the "Plus One Phenomenon," persons in an environment that is consistent with their view of the world do not change. (See, for example, E. Turiel, "An experimental test of the sequentiality of developmental stages in the child's moral judgment," **Journal of Personality and Social Psychology** 3: 611-618, 1966). Nor does development take place when they are in an environment that is radically different from the way they view the world; they do not even move in the direction of that environment. An environment must be one step beyond a student for a person to change. For example, the relativistic environment of many college courses does nothing for dualistic students except frustrate them. As uncomfortable as we may be with students who think that all opinions are equally good, we must recognize that such Multiplicity is a prerequisite for Relativism.

But that environment must have some other features too, as Nevitt Sanford had found. His "challenge and support" model describes how students need support for the way they are before they can take the risks of accepting a challenge to change. (See **Self and Society**, Artherton Press, 1966). I think of when our daughter Lynn was two years old. Since she was now a "big kid," one Saturday we bought her a lunch box to take to her day care in place of some rather grubby paper bags. She was very proud of that lunch box and anxious for Monday to come. That Sunday evening we had dinner guests, including an infant accompanied by a baby bottle. Reminded of baby bottles, Lynn wanted one, and she enjoyed sucking water from one we found in the cupboard. Monday morning, the bottle was still lying around, and Lynn decided that it should go with her to day care. Now, try to envision the scene, as the two year old, profile enhanced by a diaper under the pants, toddled across the living room floor, the challenge of the lunch box in one hand and the supporting baby bottle clutched tightly in the other.

We tend to think of growth as a good thing, but we forget that our students not only look at their worlds from a particular position; in that perspective they have embedded their identity: their hopes and dreams and fears. Asking them to abandon that position and take up a different one is asking them to restructure their dreams and fears, their very selves. We must realize that there will be grief for the self that used to be....

A fourth principle comes from the question of how to handle variety of student views among the enrollees in a course. Most of our students are dualistic or multiplistic, especially with respect to mathematics. But how do we challenge and support both? One way I find effective is to change the course as it progresses. Instead of deciding on a particular teaching method, structure, and kind of exam for a course, I plan each of these to increase in complexity as the course progresses, not going beyond challenging multiplists with Relativism.

We can now apply these three principles to a few concrete examples for which most of us have been waiting eagerly.

<u>Implications for Mathematics Teaching: Examples</u>

The specific ideas that follow are summarized on an appended chart. They deal with several aspects of teaching: structuring courses, assignments and exams, teaching methods, evaluation, and personal interactions with students. We have time here only to hit some high points; you can consult the Perry Network to find many other ideas about implementing the Perry Scheme in teaching.

Dualists like courses with a lot of structure, while multiplists prefer a great deal of diversity. One way to support both kinds of students fairly well, while challenging the dualists to move into multiplicity, is to embed diversity within a great deal of structure, especially at the beginning of the course. "By Friday at the beginning of this class period, please write a 3-page description of the two different approaches to exponential function taken by the text and by the class."

Note that the assignment does not ask for comparison and contrast, which are relativistic activities. Dualists prefer exercises, tasks; multiplists are fond of lists and essays expressing those opinions to which everybody has a right. The assignment above challenges dualists and supports multiplists and is appropriate for early in a course. Later I might provide more multiplicity in structure, by giving more choice in assignments. I might even try to challenge multiplicity by a more relativistic assignment such as "Compare and contrast the concepts of function as a rule, as a set of ordered pairs, and as a selector. In what contexts might these different concepts be useful?"

When students are asked about teaching methods, dualists generally are happier with lectures; they see no reason for the class to "share ignorance" through discussion. Multiplists like discussions, as long as critique is not expected. So a teacher might plan more lectures for the beginning of a course and more discussions for later. But discussions can be dualistic, with the teacher playing the role of an Authority, and offer flexibility needed to respond to different students as needed. And, of course, lectures need not be dualistic - "Today I am going to show you three different mathematical models for this situation" - although I personally find it difficult to remain multiplistic, and I soon lapse into discussing, as Christine Keitel has suggested we should, the strengths and weaknesses of the various models: "On the one hand...and on the other hand...."

Dualistic students, and even some multiplists, find this relativistic behavior incomprehensible, and would share the frustration of former U.S. President Harry Truman, who said "Give me a one-handed economist." In fact, Bill Perry tells of a situation in which a mathematics professor at a small religious college told his class of three methods for solving a problem and refused to say which was "best." A dualistic student had the professor taken to task by the Board of Directors for being "ungodly."

When it comes to the question of evaluating students, dualists expect a carefully-defined grading scheme, while multiplists prefer choices at least and to do it themselves if possible. After all, without standards, grading is arbitrary. The only possible basis for a teacher's giving grades is the quantity of work done, so "extra credit" assignments make sense. I satisfy both dualists and multiplists by laying out a grading scheme in detail at the beginning of the course, but giving the students the option of arranging their own system if they think that the given one won't let them demonstrate their learning.

Student evaluation of faculty members is tricky since standard "course evaluation" forms ignore the fact that students are viewing the course from different positions. For example, a standard question on such forms is "The professor was well-prepared for class: strongly agree, agree, disagree, strongly disagree." A dualist would probably mark "strongly agree" to a fact-oriented course, while a multiplist might mark "strongly disagree" since the professor was intolerant of diversity. The reverse might hold if the course emphasized relativistic thinking. The dualist would "strongly disagree" that the professor was prepared to read the tablets in the sky, and the relativist would "strongly agree" that the teacher was prepared to support several different viewpoints with data. Course evaluation instruments that do not show "where the student is coming from" provide virtually no information about how well the course went.

In individual interactions, the teacher attuned to development on the Perry Scheme will try to adjust to student positions at the time of the interchange. My standard response to the dualistic question "Why don't you give us the answers?" is something like "I like you and I think that you're a wonderful person, but I don't want to live with you for the rest of your life. You need to learn to sort out answers for yourself." The phrase "sort out answers" might be heard multiplistically or relativistically, depending on the student. If I know the student's needs fairly well, I try to offer appropriate challenge or support.

Conclusion

Life is much easier for dualists than for relativists. Life was not nearly so complex when I started college, when mathematical theorems were absolutely true, as it became when I began to encounter mathematical research. Similarly, teaching mathematics was much simpler when, as a colleague once said, the only obstacle between me and perfect teaching was my imperfect memory for "the best way" to explain each mathematical concept and procedure.

The trade-offs for complexity and uncertainty, of course, are many. They include such practical matters as being able to solve problems better, to do mathematics, to "cope with a complex world." And they include more metaphysical advantages, such as awe at the world around us and learning about ourselves from really hearing our students.

I find the Perry Scheme one of the more valuable lenses through which I can view my world of mathematics and education.

Appendix

A Synopsis of the Perry Scheme*

* From William G. Perry, Jr., "Cognitive and ethical growth: The making of meaning." In Chickering, A. (ed.), The Modern American College. Jossey-Bass. 1981.

Dualism modified	Position 1	Authorities know, and if we work hard, read every word, and learn Right Answers, all will be well.
	Transition	But what about those Others I hear about? And different opinions? And uncertainties? Some of our own Authorities disagree with each other or don't seem to know, and some give us problems instead of Answers.
	Position 2	True Authorities must be Right, the others are frauds. We remain Right. Others must be different and Wrong. Good Authorities give us problems so we can learn to find the Right Answer by our own independent thought.
	Transition	But even Good Authorities admit they don't know all the answers yet!
	Position 3	Then some uncertainties and different opinions are real and legitimate temporarily, even for Authorities. They're working on them to get to the Truth.
	Transition	But there are so many things they don't know the Answers to! And they won't for a long time.
	Position 4a	Where Authorities don't know the Right Answers, everyone has a right to his own opinion; no one is wrong!
Relativism discovered	Transition	But some of my friends ask me to support my opinions with facts and reasons.
	(and/or)	
	Transition	Then what right have They to grade us? About what?
	Position 4b	In some courses Authorities are not asking for the Right Answer; They want us to think about things in a certain way, supporting opinion with data. That's what they grade us on.
	Transition	But this "way" seems to work in most courses, and even outside them.
	Position 5	Then all thinking must be like this, even for Them. Everything is relative but not equally valid. You have to understand how each context works. Theories are not Truth but metaphors to interpret data with. You have to think about your thinking.
Commitments in Relativism developed	Transition	But if everything is relative, am I relative too? How can I know I'm making the Right Choice?
	Position 6	I see I'm going to have to make my own decisions in an uncertain world with no one to tell me I'm Right.
	Transition	I'm lost if I don't. When I decide on my career (or marriage or values) everything will straighten out.
	Position 7	Well, I've made my first Commitment!
	Transition	Why didn't that settle everything?
	Position 8	I've made several commitments. I've got to balance them--how many, how deep? How certain, how tentative?
	Transition	Things are getting contradictory. I can't make logical sense out of life's dilemmas.
	Position 9	This is how life will be. I must be wholehearted while tentative, fight for my values yet respect others, believe my deepest values right yet be ready to learn. I see that I shall be retracing this whole journey over and over--but, I hope, more wisely.

Topic Group F

THE POWER OF MATHEMATICAL AUTOBIOGRAPHY

Linda Brandau
University of Calgary

> This is the great private problem of man: death as the loss of the self. But what is this self? It is the sum of everything we remember. Thus, what terrifies us about death is not the loss of the future but the loss of the past. Forgetting is a form of death ever present within life... But forgetting is also the great problem of politics... A nation which loses awareness of its past gradually loses its self... (Kundera, 1980, p. 234-5)

> For Jung the autobiographic process is not after the fact but a part and a manifestation of the living, and not only a part but, in its symbolic recall and completeness, the whole of the living. (Olney, 1972, p. 40)

> ...the fact of thinking in terms of stories does not isolate human beings as something separate from the starfish and the sea anemones, the coconut palms and the primroses. Rather, if the world be connected, if I am at all fundamentally right in what I am saying, then thinking in terms of stories must be shared by all mind or minds... (Bateson, 1979, p. 13)

Introduction

As the title suggests, this paper deals with autobiography, the story of one's life. But it deals with story in a broader context, one suggested by Kundera and Jung in their references to self and by Bateson in his reference to story. For Kundera, self becomes autobiographic by being the sum of our memories. For Jung, self and autobiography are synonymous, self is always in the process of becoming and this process IS autobiographic. And story is the essence of not only how we become but how we think, according to Bateson. So autobiography will be used in several ways in this paper --- as the writing of and about one's life, in the literary meaning of story, as a way of thinking, as construction of and examination of self.

The power of autobiography, for education in general and for mathematics education in particular, lies in story, not only that it is the essence of self but because stories disturb us out of our forgetfulness.

Let me expand on the idea of forgetfulness first. Two of its forms are commonly labelled objectivity and subjectivity. Objectivity is a world view that is concerned with being distanced and detached from the personal. It forgets the individual by focusing to a great extent on groups. Thus research is concerned with studying many, many people -- the idea that statistically we need large "samples" to gain knowledge about individuals. But by focusing on groups, objectivity makes assumptions about sameness and difference. It assumes that all people in a large sample can be considered to be the same. It is a view that strives for better and better methods to find THE variables that will provide THE answer to our educational ills. It is desire for fact, verifiable truth.

On the other hand a subjective view of the world is one concerned with attachment, the personal, and feelings. In doing so, it focuses on the individual, so much so that it can forget that individuals belong to groups, or cultures. A subjective view makes assumptions about difference. That is, in its extreme, subjectivity considers all people to be individual and hence different, and therefore can become synonymous with solipsism. In this extreme focus on self, individualism, and acknowledgement of individuals as different, the assumption can be made that everything is opinion. Instead of there being one answer, there now are an infinite number of answers, or truths, as many as there are individual people; it is almost as if "anything goes".

Each of these views however is reductionist, and for that reason enormously seductive. It would be comforting if there was THE answer to our educational problems or if everyone's opinion was as good as the next person's. And because each of these views is so seductive, it becomes easy to slip into one of them, to stay there, to forget either the individual or the group. But what do we want to forget by adopting one of these views? That the world is more complex than either of them. That we live in a world of contradiction, paradox, and ambiguity. And so we need to be disturbed out of our complacency and habitualness, out of subjectivity or objectivity. And it is story that disturbs us, pulls and pushes us out of these two forms of forgetfulness. How does it do so? What does disturb mean?

If we are stuck in objectivity, we get detached from people, become indifferent to them. Although it is acknowledged, in this view, that story can be important, it is only in the sense of being anecdote, examples to support theories. Then after the examples are given, story is forgotten. One consequence is that our theories become more and more abstract, and in time no longer relevant to the people we intended them for in the first place. Story then is needed, not merely as example, but to return us to the centrality of the personal, to be heard in its personalness, in how it shows connectedness between people, not as a distillation of objective events and facts, not as one example of thousands, not as anomaly.

If we are stuck in subjectivity, in individualness, we can speak only of the self, hear only the self; we forget that individual stories have commonalities. Now we need to be disturbed in the sense of listening and looking for patterns, relationships, and connections, not only in our own story of self but in hearing themes common to all people. Story again shakes us out of complacency; this time by forcing us to acknowledge how much individuals are alike in their differences.

Thus, the ultimate powerfulness of autobiography is that we gain new insights or knowledge. By pulling us out of objectivity or subjectivity a powerful story brings us to a new edge of awareness and pushes us over it. We acknowledge lived experience as complex, ambiguous, and contradictory. We are disturbed out of our desire for one truth or for an infinite number of truths.

I begin with two stories, the first told about Mahatma Gandhi. Apparently he often would leave heads of state waiting while he would perform his chores -- washing clothes, cleaning the house, or even the toilets. Gandhi's plan, his Ashram, was based on the idea that everyone, including himself and his wife, needed to clean their own messes. Gandhi knew that once people forgot what it was like to clean up a mess, even a toilet, it would become too easy to create a mess. Gandhi knew that people needed to be kept close to the memory, lest it get forgotten. My point is that it is easy to get detached, to relegate our messes to someone else. Like the objectivist who desires one truth, or the subjectivist who desires an infinite number of truths, research becomes "relegated" when either view is adopted. But life is messy, educational life particularly so. And yet we need to keep close to the messiness so that it doesn't get forgotten.

The other story involves the Vietnam war and the May 8, 1988 segment of the television program "60 Minutes" which concerned a university class centered on remembrance and learning. Most of the students in this class were 19-20 years old, about the age of many of the men who fought in Vietnam. And many of the students had fathers, or other family members who were killed in the war. Much of the class involved Vietnam veterans speaking to the students, telling their personal war stories. What was especially compelling and poignant were the student reactions, many of whom had heard of the war, but knew nothing more about it than facts or events.

Here, unlike the Gandhi story, one cannot actually redo the experience. But one can relive it through retelling. And what is retold is important, the personal experience, not abstracts about "some mythical soldier". But it is not only what is retold, but how it is heard. If the objectivist hears only facts like the battalion numbers, dates of attack, and so on, then story has not disturbed. And if the subjectivist hears only stories of individual selves, not the commonalities, again story has failed to disturb.

The purpose of this paper then is to pull us out of our forgetfulness through the use of story, especially autobiography.

Story

Although I am using the term autobiography in several senses, as writing about one's self and one's life experiences, as story in the literary sense, as a way of thinking à la Bateson, and as construction of self à la Jung, they all focus on different meanings of story. So ideas central to story need to be examined. I will first focus on the word story in the sense that Bateson (1979) speaks of:

> A story is a little knot or complex of that species of connectedness which we call relevance. First, connection between A and B by virtue of their being components in the same story.
>
> And then connectedness between people in that all think in terms of stories...
>
> What happens when, for example, I go to a Freudian psychoanalyst? I walk into and create something which we call a context that is at least symbolically [as a piece of the world of ideas] limited and isolated by closing the door...
>
> But I come with stories - not just a supply of stories to deliver to the analyst but stories built into my very being. The patterns and sequences of childhood experience are built into me. Father did so and so; my aunt did such and such; and what they did was outside my skin. But whatever it was that I learned, my learning happened within my experiential sequence of what those important others - my aunt, my father - did.

Bateson's idea of connectedness is important, first in terms of relevance. Elements in a story, such as character and plot, are connected not only by being in the same story, but because they evoke commonalities in each other. They connect through interacting, juxtaposing meaning upon each other, thus becoming relevant to each other.

But another kind of relevance is important too, one where the story itself is relevant when it connects to the reader or listener. Now relevance means the evokation of some memory, some common experience, thought, emotion, an event, a feeling, a person similar to a character, a response that was similar to our own, and so on. A powerful story attaches to us through these evokations -- the deeper and more important the memory is to us, the more powerful the story.

Bateson also speaks of the connectedness between people, with the heart of his argument being that we all think in terms of stories. Is this another way of saying that we all think in terms of particulars? Perhaps, but it's more than that also. We think in patterns, plots, characters, beginnings, endings. But not only do we think in terms of story, our lives are stories. As Bateson says, stories "are built into our very being". And now Bateson intersects with Jung, who links story to self. That is, story, life, and self are constructed. Each has a plot, characters, beginnings, and endings. Events, which are in themselves stories, get woven into larger and larger stories. We are story -- weaving, constructing, living -- all at the same time, a continual process, the self becoming.

And this process involves memory. In construction of self, to Jung, we take outer events and transform them into inner experiences, which are what become indelibly engraved, become our memories. These inner experiences are stories. And memory then becomes the recall of some story. When we forget, we forget not merely facts and events but stories.

Kundera links forgetting with the loss of self, where self is the sum of all we remember. I bring Kundera into this discussion about story because of the literary nature of his work, a standard way we think of story. Yet it is significant that his work is difficult to categorize. His books involve story yet transcend common definitions of what a novel is. His writing, especially exemplified by The unbearable lightness of being, is a confrontation of the political, philosophical, metaphysical, and historical. Story is at the core but is also used as the forum for such confrontation. But he not only uses story. His style and form of writing is one that allows him as the writer to step out of story to reflect about the characters, their philosophy, their psyches, and so on.

He says, "...I invent stories, confront one with another, and by this means I ask questions. The stupidity of people comes from having an answer for everything. The wisdom of the novel comes from having a question for everything...it seems to me that all over the world people nowadays prefer to judge rather than to understand, to answer rather than ask, so that the voice of the novel can hardly be heard over the noisy foolishness of human certainties." (p. 237, 1980)

Thus Kundera speaks to the theme of this paper. Story pulls us out of the desire for certainty, for facts, provable truths. It does so by raising questions, rather than providing answers, and in doing so, helps us acknowledge life as contradictory, ambiguous, and paradoxical.

In education we need to retrieve story. By placing it at the centre of our work, not at the periphery, we confront educational dilemmas directly. When stories are viewed as merely anecdotes, it is too convenient to forget how they disturb us, too simple to dismiss educational complexity, too simple to slip into the desire for certainty.

And so I share Marilyn's story, part of it at least. Marilyn was a student of mine when I taught at a small college in New Hampshire. She was in her mid thirties, working on a degree in social work. What was preventing her from finishing was that she needed a mathematics course, and she was deathly afraid of math.

> I have been in and out of school for the past fourteen years, and all the way through, I have made it a point to avoid math classes. Suddenly, I found myself at the end of my educational road, and a giant math book was standing in front of me holding up a stop sign. The message was very clear, I had to stop, I couldn't pass "go", nor could I collect my diploma. I had a choice to make between three options. The first option was that I could take a computer course. So I signed up for one. However, I found myself lost after the first class. Also, I found that the only thing worse than my math phobia, was my computer

> phobia. So, once again I cured the phobia by dropping the course. At this point in time, I had two options left available for me to choose from. The first being that I could always drop out, and attempt it at a later point in time. I almost settled for this option, however, a friend told me that she took a math course that was actually fun, even though she too has a math phobia.
>
> So I mustered up enough courage, went to the registrar's office, and filled out an add slip. I walked over to the classroom as slow as possible, hoping that I would either lose the slip, or that you would say "I'm sorry, but the class is full", that way I would have an excuse for not taking a math course. However, you didn't tell me that the class was full, nor did I lose the add slip. So there I was, sitting in a math class, and I was still alive, I didn't break out with a rash, which I was sure I would do, since I was allergic to math. What I did feel though was very nervous, anxious, and afraid. At that point in time, I was sure that there was no way that I could pass the course. And I had run out of options, and I wasn't sure what my next step would/should be.
>
> I can still remember the first class, we did the "horse trade" problem. And it was really interesting, and before I knew it the class was over --- and I survived. I must admit though, when you started talking about what we would be working on throughout the semester, I panicked again. When I panic, it seems as if my heart races faster, I get a large knot in my stomach, and before long I end up with an intense headache. This has been a consistent chain of events for as long as I can remember. So, at the end of class I once again considered trying to find a way out. The only problem was, that there was no way out, I had to follow through with it. However, I did carry a drop slip with me for three weeks, just in case. It was sort of a security blanket for me.
>
> As time went by, I found myself becoming more comfortable in your class. And after a while, I pulled the wrinkled drop slip out of my pocket, and reluctantly threw it away. I realized that I was in a math class, and I was still alive, so maybe I wasn't allergic to math after all...

In Marilyn's story we hear not only the words but what is underneath the words, the emotion, pain, anxiety, shifts in feelings. What does this story tell us about Marilyn? How does it attach to us?

Marilyn used metaphor in her story, in particular she saw herself as "having mathematics", much as one has a disease. And she perceived this one as fatal. Her pattern of behavior surrounding mathematics was one of avoidance, at all costs, perhaps understandable if she saw herself as able to die from this disease. However, by coming into my class, she began to break her pattern, began to work on "curing" herself. But even in her willingness to do so lay the need to cling to her drop slip for many weeks, a disbelief that such a fatal disease could be overcome. And so she needed to allow for the possibility of relapse back into her disease, using retreat and avoidance as her standard treatment.

What we also hear is conflict and struggle in Marilyn, between a desire to cure her phobia or disease, and her fear of the phobia itself. Perhaps a turning point was that first class when Marilyn's desire won this struggle. She stated that she considered finding a way out but that there was none, a significant statement since she could have dropped the class, avoiding her confrontation with her mathematical demons once more. But she did not. She even took another math class with me the next semester.

As a mathematics educator what is most disturbing in this story is the strength of Marilyn's mathematical fear. It was strong enough at the beginning of the course to make her physically ill whenever she even thought of opening the mathematics textbook. And as she tells us, it was strong enough to control her life, her career.

Stories such as hers keep reminding us of the need to listen, to hear people's fear. Her story is powerful because while we listen, it evokes memories of fear in us -- thoughts of what our own demons are, of what we are so afraid that the course of our lives has been altered.

But as mathematics educators not only do we need to hear these stories over and over, we need to commit ourselves to action. Certainly one action is to dismiss these fears, to listen to the story but then forget it, become detached from it. Another action is to create math anxiety clinics, send people to get cured. But this action seems like another dismissal, another way of detaching ourselves from the person by sending him or her to someone else for help. The action I prefer is stay close to these disturbances, to weave them into the fabric of my university teaching and of my research.

Mathematical Autobiography

I now turn to what autobiography has to offer mathematics education, in terms of people's stories disturbing us out of our forgetfulness, but also in terms of self. There are two main parts to this discussion, one dealing with my university teaching of pre-service elementary education students and the other concerns my work with one of these students in particular.

A way of describing my teaching could be summarized by the phrase "examination of self", a phrase that refers not only to the students but to myself as professor. All facets of my methods courses seem to be autobiographical in a variety of ways.

First, at the core of my teaching style, is a focus on disturbing the student. This focus permeates all aspects of what is done in class and in out-of-class assignments. Class work centers on activities, ones that students can either actually do with children, or modify. Let me describe a class from my fourth year methods course that focuses on teaching mathematics in grades K, 1, and 2. In this class the topic focus is equality and the theme centers on teaching mathematics with meaning.

The first question posed to the students is: "what is the meaning of meaning?" We toss ideas back and forth, the students and myself, with me writing down all ideas at the overhead projector. Responses usually include ideas about teaching for understanding rather than rote memory and children needing to see relevance in what they are learning.

We then touch on equality in an introductory way, a topic on which they have done some reading, especially in terms of children's perceptions of the equal sign. The purpose then of the class is to delve into the concept of equality in a meaningful way.

The first activity involves small group work, with the students discussing ideas for acting out the concept of equality for an addition fact like $2 + 3 = 5$. Each group then presents their ideas for how this could be done with children. Disagreement often centers on questioning the need for only five children to act this out or for a total of ten children. Discussion leads into equality as balancing and/or joining. We then bring out balancing scales and relate equality to balancing -- the idea of a seesaw. This leads into discussion of the algebraic concept $a + b = c + b$, one most of them memorized in high school as "if you add something to one side of an equation, you have to add it to the other". By demonstrating this concept with a scale, it begins to have meaning for them. And they recognize that concepts young children play with are eventually connected to more abstract mathematical ideas.

A more philosophical sense of equality is explored next. That is, if there are two chairs in front of me, in what sense can I say that they are equal? Even two chairs that seem exactly the same may not be "equal", one may have different scratches on it. Or if we say that one apple equals another, how are they equal? Does a Macintosh apple equal a Delicious apple in taste?

Equality then extends into everyday situations involving commutativity. That is, if I put on my shoes first and then my socks, is this equal to putting on my socks first and then my shoes? What does the "same result" mean? Is it visualness? Or is it the fact that I now have on my shoes and socks? Another situation a student posed was: Is eating my dinner first and then dessert equal to eating my dessert first and then dinner? What does equality mean now?

Geometric equality is considered through the use of the geoboard, the Mira, and pattern blocks. Students explore equality by exploring the limits and boundaries of each material. Discussion now moves into congruence, equality of area, volume, length, and so on.

At the end of the class, we speak of many meanings of equality, a concept that now seems much broader and more complex to the students than merely the equal sign. As one student commented: "The meaning of equality seemed clear before, but now I have many more questions than answers." To me, this meant that I had succeeded in disturbing her. I had pulled her out of the objective search for certainty into doubt and contradiction.

How does this happen? The tasks are varied, intentionally so, to open up many possibilities for the meaning of equality. The tasks involve the students actively -- either through exploration of concrete materials or the exploration of ideas. Through this focus on personal involvement, students are not only examining mathematical materials or ideas, but themselves. They begin the tasks with certain ideas about equality, for example, but these ideas get challenged. The tasks require them to think in different ways -- to critically examine their concept, to broaden it. The point is that it is through their active, personal involvement that this examination of self happens; it is not through me lecturing about different ways of viewing equality.

An important aspect to the social interaction that occurs between students in the small groups, and then between them and myself in whole class discussion, is that they (and myself I should add) are pulled out of objectivity, the desire for one truth, and subjectivity, the desire for an infinite number of truths. The debate, argument, and discussion that occurs involves critical thinking -- listening to different interpretations or ideas and deciding what makes sense and what doesn't. Yet such critical thinking is focused not on trying to find THE truth but to distinguish between many equally sensible truths.

What promotes this focus on critical thinking is the style of questioning, one which is open-ended. They get interpreted in many ways. But if more than one truth can exist, does this mean that ALL truths are equally good? That is, if we are pulled out of objectivity, do we then slip into subjectivity? It is important then to be clear that more than one truth does not mean chaos, an infinite number of truths. So students are pulled into a perspective of mathematics as complex and interpretative, yet not chaotic. Ideas and responses to questions need to make "sense", need to be examined critically. Hence, the view that even the world of mathematics is contradictory, ambiguous, paradoxical.

Other experiences in my courses also allow for such disturbance, for examination of self. The papers students write for assignments are not what they are accustomed to writing. They require personal involvement, digging into their minds and emotions to discover and uncover new insights into children and/or themselves. Students have told me that they are rarely asked to write such papers; educational institutions usually want them to delve into facts, read books by "experts", be detached, be objective. They have been so well taught that they see a paper written in the first person as invalid and not scholarly. And yet I do not give them the impression that, again "anything goes". This is difficult for students to understand. That is, if I am allowing for what they think of as "opinion papers", then how can they be graded? How can one person's opinion be worth an A and another's a B or C? Again they need to acknowledge that even so-called opinions can be examined in a critical manner.

This method of teaching is not easy for students to cope with. The majority of them have viewed mathematics as "one right answer, one right method" for many, many years. An examination of themselves tends to throw many of them into confusion. They are pushed over the edge into new awarenesses, yet some are overwhelmed, not excited. Creating such disturbances in students' concepts of mathematics, creates fundamental disturbances in self, in what has been believed for years. They may be shaken out of complacency and habitualness, but often contradictoriness and complexity are too much to handle.

As a professor, it is incumbent on me to help students with this process. One way is to discuss the troubling nature of the process openly in class, for them to share their frustrations with each other and me. Acknowledging the difficulty of altering one's fundamental perspective is important so that students don't believe that change occurs overnight.

Another way of dealing with this process is through the dialogue journal. As the name suggests, the purpose is for conversation, in this case between the student and myself. And part of this conversation is support and encouragement from me.

But I have discovered that the journal process is also an occasion I use to "disturb". My comments, or questions back to the student are meant to poke, probe, get them to rethink an issue or see another perspective -- in short, to pull them out of themselves. But this process influences me too. Student thoughts, feelings, and comments pull me into new perspectives, shake me out of complacency.

Another use of the journal is for students to work mathematics problems, to examine themselves both cognitively and emotionally. What we use is the book <u>Thinking mathematically</u> by John Mason, Leone Burton and Kaye Stacey, a book which contains problems very different from what students think of as mathematics. Problems in this book cannot be completed in only a few minutes; many of them may not even have answers.

The importance of propelling students back into being a mathematics learner again is that they not only gain empathy for the children they will be teaching one day, but find many new insights for and about themselves. It is yet another opportunity for them to do an examination of self, but different from what occurs in the class. Now it is private, or only between the student and myself, and written.

The process of writing, not only about the problems being solved but about the emotions or memories evoked, is crucial for examination of self. Writing is very different from speaking, especially because now verbalization can be captured for eternity. Thoughts and/or feelings float away in speech. Writing forces us to commit words to paper, to make a conscious decision to do so, to organize our thoughts, to capture thoughts and feelings and find words for them. And in the process of finding the "right" words, we are exploring self, even formulating self. That is, sometimes we don't know what we think or feel until we search for the words to express those thoughts and feelings. And then seeing those words on paper gives meaning to what may have been obscure before.

The dialogue journal, especially with the added focus of working mathematical problems, has helped students formulate their own personal problem solving process, part of which is coming to understand how what one does today may be influenced by past experience. Students begin to see connections between mathematics and their lives, ones they may never have seen before. And it is the process of writing that evokes such connections, especially if the writing proceeds in a free-flow manner.

Another important aspect to keeping this journal over the entire methods course is that a chronological record of both cognitive and emotional process is obtained. Thoughts and feelings are now both inside AND outside of ourselves, on the table so-to-speak, for examination and re-examination. Now students can not only reflect about mathematical experience, but have something that seems "concrete" to form the basis of reflection. They can go back tomorrow, next week, next month, next year and examine what was written. They can trace their process, search for patterns.

At the end of the course, the last entry in the dialogue journal is a reflective essay, one where students go back through their own journal and think about what they have learned about their own problem solving process. I share one student's (Jim) final journal entry.

April 6, 1988

At this stage in the journal I would like to think that I now approach math problems differently. I use the word "approach" very carefully and consciously here. Prior to reading Mason's book I used to think that math problems were somehow written in stone in terms of interpretation. <u>Thinking mathematically</u> made me think about this approach. That is, I was forced to consider that problems may be interpreted more than one way.
And what are the implications of this for the student (like me) who has been taught math in a very traditional manner? Well, at first glance it would suggest that this liberal attitude (or approach) is a good thing. That is, the student is freed from trying to figure out what the question is "after". The student is not being dictated to by the question -- in reality the teacher -- anymore. He/she is now free to put his/her own meaning into the question and from there solve it. In other words, solving math problems changes from a one-way process (question → student) to a two way process (question ↔ student). If I may, it democratizes mathematics for the student.

However, there is a catch! It is that in the more general world where there is freedom and democracy, there are also responsibilities and costs. What, one might ask, are these new costs? They are that the student is now required to put more of himself/herself into the question-answer process. No longer can the student sit back and say "I can't get it! I just don't know what <u>they</u> want." By approaching math in this new fashion (i.e. personalizing or democratizing it) students must take a more active role. And it is a role that many students may be reluctant to play because it requires risk taking, and the taking of risks, although offering the possibility of benefit, is potentially costly as well. That is, the student's input to the math question -- his/her interpretation -- reveals a personal side of that person; a side that he/she may not want exposed, especially if he/she feels that he/she did not somehow answer the question "right".

I know that several times (at least) as I was doing the problems in <u>Thinking mathematically</u>, I would come across a certain question and I would avoid it by simply going on to the next question. What was going on in my mind when this would happen? It seems to me that I would view the question (problem) and right away make a snap decision as to whether or not I could answer it (i.e. get the "right" answer). In other words, for me in the early problems of the book,

> I was saying that math problems were only worth attempting if there was a reasonably good chance that I would get the right answer. The problems that I skipped in the book were usually passed over for that reason. Towards the end of the book, however, I think that I was willing to try problems even if I didn't like the looks of them initially. Sometimes, as in the case of the "Black Friday" problem, I became very frustrated and somehow felt that all my efforts were a waste of time. Again, I found myself thinking in the old ways. Why? Well, maybe it was because I was only reading the question and not allowing for my unique interpretation to take place. (⟷)
>
> The "Furniture" question is interesting in that the solution was that there may not be a solution. My training in math (in elementary and high school) was that there is <u>always</u> a solution to math problems. This training got in the way of me being able to say, finally, that "I don't believe it can be done". Mason's book, perhaps, has taken me forward a little in stepping backward. By this I mean I am able to look at problems now the way (maybe) I should have been taught when I was a kid. Then problems with no answers were, of course, "trick" questions. Excuse the term, but <u>Thinking mathematically</u> has deprogrammed me in terms of the way I have been trained to solve problems. For example, I don't recall teachers putting much emphasis on diagrams then and manipulatives were definitely not encouraged in any math I did at that time. For this reason "booklets", "paper bands", and "multifacets" were quite enjoyable for me. Enjoyment -- fun -- satisfaction. These are the things that <u>Thinking mathematically</u> has helped me experience in math, finally.

This summation tells a great deal about Jim and his process of reformulation of self. We first hear his discovery of mathematics as interpretive, a feeling of freedom at last. And yet we also hear his discovery of the ambiguous and paradoxical nature of such freedom. That is, freedom also brings responsibility for the student, a responsibility that can seem worse than the shackles of being bonded to "what the question is after". So having the freedom to interpret and work mathematical problems in different ways seems powerful on one hand, and yet we question: who has the power now, Jim or the problem?

As Jim explores this ambiguity, he speaks first of avoidance of certain problems, of the hold they have on him, perhaps similar to the hold that Marilyn's mathematical demons had on her. For Jim, the hold is related to getting right answers, to feeling able to do so, to the fear of risking the solution to a problem because he may not be successful. Yet he saw his process change, saw himself become willing to attempt problems.

We see Jim in the midst of a reconstruction of self -- from one who entered with a narrow perspective of mathematical problems to becoming one with a broader perspective not only of problem solving but of his abilities in mathematics. This search for a reformulated mathematical identity permeates Jim's work, not only in terms of the dialogue journal. The following piece of narrative is from some writing (also April 1988) he has done in terms of exploring his mathematical past -- experiences in elementary and high school, relationships and feelings about teachers, parents, his brother.

> ... In Shadows on the grass, the struggle of Blixen against nature in Africa for some twenty years is an experience that I can relate to easily. Not just because I know how seductive and hard that part of the world can be on a person emotionally, but also because I understand how disappointing life can be when you work and work and in the end it all seems to have been for nothing...
>
> And how are my struggles like those of Blixen? I believe that I can look back over the years and see that I have been trying to establish my own sense of identity, to create for myself, my own sense of power. I honestly believe that the experiences that I had as a student in the elementary classroom were typical examples of the struggle that people have for identity. It was a power struggle where the teacher's objective was to "turn out" students, all pretty much identical. My objective, on the other hand, was to become my own person -- to become free. In my earlier writings in this autobiographical process, I discussed how I was so envious of my brother Jack who could quickly complete homework and then go out and have a good time. I wrote that I recalled one incident where I had to do homework rather than go out and play with friends. That was what I really thought when I wrote it, but now I think that the story does not totally reflect my behaviour during those years. The truth is that I was quite happy to occupy my time alone, even when I had no homework to do. And what a provocative statement that is in itself! Children who do not want to play with others their own age (i.e. those who would rather be alone) are considered to be weird by peers and perhaps in need of observation by teachers. But isn't it strange that once you go back into the classroom you are expected to sit alone, obediently, and quietly at your desk there -- and work. School work was not to be a team or co-operative effort in the schools I attended. Math was to be learned individually. I assume that my teachers believed that children could not learn math through working on problems with their peers in groups. Maybe they were afraid that by allowing the students to work

this way they would lose some of their power and authority in the classroom.

When I was practice teaching, I taught my grade five class in much the same way I was taught at that grade level. I did this because I was afraid, more than anything else, of not being able to control the students. By control, I mean being able to maintain order so that some learning could take place. My reasoning was that the teacher had to be the focus of the class at all times. Is this another way of saying that I did not want to risk losing whatever power I had at that time? I'm still not sure. Whatever I was doing though had predictable results in the classroom. That is, a few students performed in such a way that they could be called "star" students -- a term that I used in reference to my brother Jack and other "smart" kids during my elementary school years. Now that I think again of those practice teaching days, I am struck by how much I thought of the "star" students of the class when I was lesson planning. Oh, how easy it was to tailor my lessons towards the abilities of Thomas, Sara, and Stuart. It was easier to do. That is the answer! By focussing on those children I was, of course, paying less attention to the others -- the ones who were more like myself in terms of performance in school. So who has the power really anyway? Did I have it then? Or, did Stuart, Sara, and Thomas have control of the class through me?

...Of course, as a student in elementary school, I was totally under the influence of my parents and teachers whose means of control were found in the methods that were used to teach math as well as other subjects. It was at this stage that I learned to dislike mathematics. I was taught through endless pages of drills that doing math meant getting the correct answer. Nothing else was of any importance except getting it quickly. Those few students, like my brother Jack, who were quick and accurate in math were held up as models to the rest of us. Nothing was ever said, of course, but the unspoken message was very clear: "See how well Jack does! You should try to be that good."

Given this type of attitude towards me by my teachers, I think I understand why I went off to Africa. This was one thing that Jack had not done. He may have been excellent -- and he was -- in calculus, chemistry, and physics but he had not been a CUSO volunteer. This was an area where I could at least say that I had done something worthwhile (school, we were always taught, was worthwhile). At last, here was an area where I could make my parents proud of me. Where I had come in second place in

> school, with mathematics being the most bothersome indicator of ability, now I would be noticed as well. I use the word bothersome above because there seemed to be a feeling then that if you were good in math (i.e. got high marks) then you were good in all subject areas...

What is heard loudly in this portion of Jim's story, is his struggle with power, control, freedom, and authority. We hear of his search for identity which in elementary school turned into a power struggle with the teachers; his recognition of a teacher's fear of losing power and authority -- one acknowledged in himself as well. Again we hear, as in his journal, of the ambiguous nature of power in the classroom, his recognition that perhaps in trying to control the students so much, they ended up controlling him instead. And then we turn to power in relation to mathematics, how the endless pages of drill and practice held him, controlled his view of self, i.e. because he could not do these problems quickly he was not good in math. Finally there is a sense of his using power -- going off to Africa to do something that Jack had not done.

And as we listen to these words we hear more. We also hear his struggle to give voice to self-truths and inner feelings. There are places in this portion of his work that are virgin territory for Jim -- the first time he has set some of these awarenesses to words -- his acknowledgment of liking to be alone, of being considered strange for doing so; his fear of not being able to control the students in his practicum and hence its connection to teaching the way he was taught; his insecurities with mathematics and his abilities to teach it; his heading for Africa as a way of getting his parents to acknowledge him.

There is a great deal more to be said about Jim's work of course. But I will stop here to share a recent conversation with him. I asked him to come over to listen to this paper and then to talk about the journal process and his other autobiographical writing in terms of what has been important to him and what he feels this work has to offer mathematics education.

He spoke of the autobiographical process as torture, especially the one involved in his narrative writings about past mathematical experiences. In remembering the past, it was painful to bring old memories to the surface. And yet he felt this was needed to make this new beginning into a teaching career. This attitude is different for Jim. He used to think that all he needed to do to make a fresh start was forget the past and throw away old emotional baggage. And yet he now acknowledges it as part of the self, a part that often needs examination and exploration, not to be thrown away but to learn from it. This idea circled us back to Kundera, to his view that losing awareness of the past is loss of self, and that when we think of discarding our past we are trying to forget who we are, trying to deny the many sides of self.

To close, I share the story that involves insights that came to me after my talk with Jim. There was a time during the conversation when we were speaking of the dialogue journal process. Jim spoke of it being a journey into himself, a challenge to face the sensitive issues evoked when seeing a math problem. That is, the process returned him to his elementary school days, to the thought "I'm stuck, therefore I'm dumb in math". So by confronting his own insecurities, the process helped him think differently, not only of himself (that he's not so dumb in math after all), but about mathematics; that he now sees math problems differently, will hopefully avoid teaching math the way he was taught; that math is more than "number crunching"; that he now thinks about what math is (something he never did before).

As these comments were spoken, I kept saying to Jim, after each one or two, "well so what's important about that for math education?" At that time, I saw my technique as pushing to get more and more ideas. But Jim was bothered, had this strange look on his face so I stopped and we chatted. What bothered him was the "so what" question. He saw it as a way of dismissing his responses and then finally said in frustration: "You want an answer from me -- I can't give you one!"

My point is that the theme of this paper has been on story pushing us into acknowledgement of the world as complex, contradictory, and ambiguous. But there I was being pulled back into objectivity -- desiring THE truth, wanting to give you OUR answer as to the power of mathematical autobiography. The astonishment I felt when I realized I had shifted into objectivity is significant. I don't like seeing this side of myself -- the one who desires the simplicity of THE truth. So I have circled back to the thesis of this paper - that what is most powerful about all the forms of autobiography mentioned is that they provide us with a mirror to see ourselves, not only the aspects we want to see however, but ones we need to see. We are provided with a way of reflecting, of acknowledging the self as many co-existing sides, ones that are ambiguous, contradictory, and paradoxical.

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[Acknowledgement: Many thanks to friend and colleague David Jardine for his insightful ideas].

Topic Group G

DOES IT MATTER WHAT OR HOW WE TEACH MATHEMATICS IN SCHOOL?

Thomas O'Shea
Simon Fraser University

Summary of Presentation

A number of incidents have recently focused my attention on the question of whether what we do in teaching mathematics at the public school level has any long-term effects on people's perceptions and understanding of mathematics. Among these was contact with an Indonesian engineer in Java who attributed his present high level of problem-solving skills to his Grade 5 teacher who used concrete materials to develop mathematical concepts. I have also found that cocktail party conversations eventually get around to personal confessions about school experiences in mathematics when people discover that my area of interest is mathematics education.

One of my graduate students is using Flanagan's (1954) critical incident technique to study the interactions of students and telephone tutors in a distance education institution. This led me to consider the critical incidents related to mathematics in my own life. I recall little of my school mathematics experiences although I believe the instruction was at least adequate. On the other hand, I can pinpoint exactly when my interest in mathematics was sparked during the year when I was preparing to become a teacher. That incident was critical to my future career.

Kogelman and Warren (1978), in describing the phenomenon of mathematics anxiety, give some details of their students' recollections of their mathematics experiences. Some students described classroom situations, while others recalled teachers' comments and behaviours. In all cases, the end result was a negative attitude toward mathematics. At the other end of the spectrum, Albers and Alexanderson (1985) interviewed 25 active mathematicians to discover why they chose or were excited by mathematics. Of the 25, only 4 chose to talk about a particular mathematics teacher or experience in their pre-university schooling.

Why is it that even among mathematicians, few people can cite positive school experiences with mathematics? Perhaps the reason lies in the nature of schooling as we know it. Goodlad (1984), for example, found that about 70% of class time was teacher talk, mainly telling. Less than 3% of the time was devoted to "praise, abrasive comments, expressions of joy or humor, or somewhat unbridled outbursts such as "wow" or "great" (p. 230)." I find one of Goodlad's suggestions particularly attractive: "... Mere refinement of conventional [teaching] practice is not sufficient. We will only begin to get evidence of the potential power of pedagogy when we dare to risk and support markedly deviant classroom procedures (p. 249)."

My basic question is, "Does anything we do to teach mathematics in public school make a difference in the long term?" Does it really matter what the curriculum is or how we teach it? If we interviewed school graduates twenty years from now, what would they say about their school experience in mathematics? What does the general population now think? From the math anxiety literature, and from isolated mathematicians we know a little of peoples' experiences at the extremes. But what of the great majority of people who study mathematics for 11 or 12 years and then drop it? Or even go on to university to take further math courses as part of their professional requirements?

Have there been studies in mathematics education that have attempted to address this problem? Would the effort be worthwhile? It seems to me that the potential payoff could be great. If we give serious attention to the things that seem to matter in people's lives, we may be able to develop learning activities that not only are effective in immediate learning but will also leave lasting impressions. What of teaching style? We may find that deviant teachers (in the Goodlad sense) were the ones that really made a difference. Perhaps we should aim at developing teachers who can digress from prepared lesson plans when the time is right, or who can laugh when describing how Kepler spent twenty years on calculating planetary orbits just as logarithms were invented, or cry in describing how Einstein vainly tried to prevent the dropping of the atomic bomb on Hiroshima.

Summary of Discussion

The eight members of the discussion group were asked for their ideas about whether and how a research project might be designed to investigate people's mathematics experiences in school. Support ranged from indifference to dubious endorsement. The group foresaw many problems that would have to be overcome, such as that of selective recall by the informants. Others suggested that critical incidents may merely reflect the accumulation of knowledge that precedes the incident. My summation at the time was that "there does not exactly seem to be a groundswell of support for the proposition."

On the other hand, several people suggested further readings. Among those were the relevant portions of the Cockcroft report from Britain and the Coleman report in Canada. In a private conversation, Christine Keitel suggested Peter Galbraith's article entitled **Society's Image of Mathematics** based on Australian data, forthcoming in Educational Studies in Mathematics, and Gila Leder's presentation at ICME VI **Images of Mathematics in Society**, also from Australia.

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Ad Hoc Presentation

A TALK ON THE CONFERENCE *WOMEN DO MATH*

by

Tasoula Berrgren
Simon Fraser University

Originally I come from the island of Cyprus, an island famous today for its sunny beaches, good wines and tourism. In the ancient world tourists came to Cyprus as well, but they were pilgrims coming to worship at the shrine of Aphrodite. In fact it was onto an eerie, black-sand beach on the West coast of Cyprus that Aphrodite, the Greek goddess of love and beauty, emerged from the sea. Or, as Homer sang in his Odyssey,

> *"Laughter-loving Aphrodite went to Cyprus, to Paphos, where is her realm and fragrant altar."*

Of course all this is a myth, but it was a powerful myth, with a great effect on history.

> *"Women do not understand mathematics"*

is also a myth, but the effects of this myth have been disastrous.

Mathematics is not born into us, it is not the consequence of gender, heredity and descent, but mathematics is the growth of knowledge in which much depends on motivation, external encouragement and other circumstances. In the learning of mathematics women have been victims of circumstances, which have led them to believe that they are no good with numbers. Brothers and sisters, parents and teachers are partially responsible for all the fears women have towards mathematics.

And women themselves know that lack of mathematics generally holds them back in their careers. It is important to look at the employment numbers and examine the quality of the job structure for women. There are reasons to be concerned about female employment. Even jobs held by women in high-technology are mostly clerical and they do not earn high incomes. Women are under-represented in certain disciplines such as engineering and sciences. So women need to be well-prepared and well-equipped in order to achieve a much-needed change. A key element is education beyond the normal schooling years. If women are to enjoy jobs in science and technology, it is essential for them to focus their education on courses in mathematical sciences.

We all know this is not happening, and will not happen until there is a major and radical shift in attitudes. Here are some measures that must be taken now:

First of all I and my colleagues, the mathematics educators at the university level, need to convince, to train, to help, and to encourage our mathematics teachers to make mathematics available to more students, not only to the select few, those who seem to understand and learn mathematics without effort.

Here is a true story:

> "In high school I was convinced that I was no good in mathematics", said my nephew. "I never got an A and my teacher did not like me. When I went to the university

> I was scared to take mathematics. At the end of my first term I got four A's but I almost failed math. The second term was the same, four A's and a bare pass in math. The third term my math professor called me to her office to talk to me. She asked: 'How can you get A's in Physics and then almost fail math?' I explained to her that I was NEVER good in math."

She actually offered to help Paul and she arranged regular meetings with him. The third term he was the top math student in the class. This is a true story which happened to a hard-working rather than to a brilliant young boy during his university years. The high school was in Cyprus. The university was Donetsk in the Soviet Union. Paul is now an engineer, with specialization in robotics and industrial automation, and the story shows, I think, that a good career in what is the highest of high-tech is open even to those who are not math whizzes, provided their teachers are willing to go the extra mile to help.

The second measure is to encourage young women to work harder with their math and to stay with their math courses throughout high school.

We all witnessed in our lives comments like these:

- *"I never liked math"*
- *"I was never good at math"*
- *"Math was my worst subject"*

and so on. I recall my younger son telling me when he was in the second grade, *"Mom, everyone in my class except me hates math."* The boys, though, will often - because of cultural expectations - stay in math and eventually come to terms with it, because they are expected to. No one expects the girls to do so, so the result has been a much higher rate of dropping out of math by the end of high school for girls than for boys.

Fear and negative feelings of frustration are harmful to our math students in general and, moreover, they are harmful to our society. Women especially lose opportunities as a result of this fear, and so our world is denied their talent and their contributions to science and technology.

In the past, often, women tailored their objectives to topics that could be achieved easily and quickly so they can start their families. Now, with the parental leave, day care, and other efforts to provide support to women, they are at last able to apply themselves to all areas of work. This is the time for women to examine, explore, do research, generate ideas, convey knowledge, expand and develop.

The majority of research right now, in fact more than 85% is done by male scientists. But, along with men, we want women to assist in this scientific development.

These are the thoughts which urged me to organize the conference I call

WOMEN DO MATH

I wanted to spread the word, to tell young women in the 9th and 10th grades that mathematics is beautiful, and I wanted to urge them to stay with Mathematics, because this is the subject which will open the doors for them to careers in Science. Moreover, I wanted them to know that we, the women, can also contribute to scientific research and, who knows, perhaps this time with refreshed values.

On November 27, 1987 the top of Burnaby mountain was in full fall colors, and the sun was at its best. The girls were pouring into the campus and they kept coming along with their parents, teachers and counselors. It was important to encourage parents to come since a positive attitude towards mathematics at home is very important in deciding whether a girl will stay in the subject or not. A high-spirited atmosphere was all over. Everyone looked excited, interested, enthusiastic and happy. They were visiting an institution which had been there for almost one quarter of a century. It had graduated women in science before any of them had been born. For some of them this was their first visit. Nested on this mountain the university was on a working day and it was flooded with students, and the wide blue sky, the bright sun made the trees, the mountains, the waters, the views from above majestic. This was the morning picture which the girls experienced that beautiful day.

<u>Registration</u>

Registration took place at the S.F.U. theatre. Classical music and slides showing scenes around the university occupied the free time. The morning talks started with the chairman of the department of Mathematics and Statistics giving the opening remarks and then three talks given by women in science opened the girls' eyes to career opportunities.

Louise Routledge, an instructor of Mathematics at B.C.I.T., had a list of 82 professions and the math requirements for many of them. She indicated exactly which math courses were required for these professions and in addition she encouraged the girls to take math and stay with math. She told them her own story how a mathematics teacher changed her whole life by encouraging her to overcome the dislike of math she had developed when she was small in school. Now she teaches mathematics and statistics, and is encouraging other girls to overcome their dislike of math.

The next talk was by Radmilla Ionides who is an electrical engineer and works for B.C. Hydro. She explained to the girls what she is doing in her job and stressed the point that her career did not prevent her from raising her family.

Lily Yen is an honors math student at S.F.U. Some of you know her. She gave the last morning talk. She explained to the girls the different math competitions and how much fun it is to take part in these competitions. For Lily solving problems is really lots of fun. She herself took part in the International Mathematics Olympiad. Because of this she had opportunities to travel to Europe. Also she went to Hungary as an exchange student.

Workshops.

Since the title of the conference was "Women Do Math", with equal emphasis on all three words, we arranged a wide variety of workshops that would get all the girls involved in mathematical activities. The objective of the workshops was to see the beauty and importance of mathematics as well as to have fun doing it. After the lectures the girls attended the workshops which were organised by Dr. Bernice Kastner. Each workshop was led by faculty members - including several women - who had several undergraduate or graduate students as helpers. The workshops were excellent, and they not only educated the girls, but they also made our predominately male faculty aware that there were a lot of girls out there who could be good mathematics students.

Here are the titles of the different workshops

1. Experiments in Probability
2. Computer Graphing: It's Easy and Fun
3. Seeing is Believing - Or is it?
4. Folding Paper to Study Geometry
5. In Quality, Sampling is Job #1!
6. Be a Mathemagician
7. Image Reconstruction to See Inside the Brain
8. The Art of Counting
9. Patterns in Numbers
10. Rubber Sheet Geometry
11. Symmetry in Nature
12. Symmetry

Now I will describe some of them.

1. **Experiments in Probability by Dr. R. Routledge and Dr. T. Swarzt.**

The first half of this workshop was an interactive computer session in which the computer played the role of a huckster talking the girls into a doubling-up strategy. This strategy involves betting **"double or nothing"** after any losing bet. Naive gamblers think that they can avoid a gambling loss by betting **"double or nothing"** until they eventually recover their losses. The interactive computer session showed them that they could, if they were unlucky, lose their entire fortune before winning such a bet, and showed them that they could not expect a net gain in using the strategy of doubling-up.

The second half involved a simulation of the **Buffon needle** experiment, in which tooth picks were dropped on a lined sheet of cardboard. Results were collected and used to estimate π. Also they discussed a practical application to forestry where a picture of logs was lined up and by measuring the logs crossing the lines (provided that they were all of the same length) the girls tried to give a rough estimate of their numbers. Also they discussed some other ways of calculating π.

2. **Computer graphing: it's Easy and Fun by Mr. A. Ozgener.**

Some knowledge of computers and some knowledge of linear and quadratic equations was very useful for this one, but also it was good for all those who were interested to learn about computers. At first we showed the girls how to graph linear equations and how to find the slope of a line. Quickly we talked to them about the graphs of trigonometric functions and we graphed them on the computer. Last we played **"Algebra Arcade"**. It teaches the student how to guess the graph of linear and simple quadratic functions. A record of points was kept as the girls played the game. This was the only workshop where there was some competition.

3. **Seeing is Believing - or is it? by Dr. M. Dubiel.**

There were five parts to it. At first the girls were given geometrical puzzles like the **Pythagorean puzzle**, where a square is cut into parts and then the girls were asked to rearrange the parts to make a rectangle. They were given other figures as well and they were ask to cut them up in a specific way and rearrange their parts to achieve a new form. Some of these results were very surprising and unexpected.

Next the girls were introduced to the four colour problem and they were asked to colour some maps and try use as few colors as possible.

Another activity, in fact the most exciting one, was the making of **Moebius bands** and experimenting with them. They had to answer questions in advance like what would happen if they colour one side of the band (which of course colored the whole band), cut along the middle or cut along one-third from the edge. They were also encouraged to work with bands of more twists.

In addition to all these this workshop there were Eschers drawings where the girls were encouraged to study and explain the optical illusions and their paradoxical geometries.

As a last activity they had to cut an eight by eight square of course in a specific way and then rearrange the pieces to make a five by thirteen rectangle. The girls were asked to explain the paradox of 64 being equal to 65.

4. **Folding paper to study geometry by Dr. B. Kastner.**

Here the girls were provided with paper triangles of all shapes and sizes and also long strips of paper.

At first they classified the triangles into equilateral, isosceles and scalene and furthermore into all angles acute, one right angle and one obtuse angle.

Then by folding triangles they produced an angle bisector, the perpendicular bisector of a side, a median and answered the provided questions.

Next the girls tried to find out what happens if a triangle is folded so that the three angles meet at a point on the base with no space left between the sides of the adjacent angles?

Last they tied knots with the long strips of paper to produce a pentagon and they interlaced two strips to produce a hexagon.

5. **In Quality Sampling is Job One! By Mrs B. Dwyer.**

At this workshop they scooped balls of a predominate colour out of a sampling bowl to demonstrate a technique called Sequential Sampling. This is the same as taking 10 or 15 items off an assembly line to check for defects. After they collected enough data they used the graphs they were provided to add up the results and make decisions about sampling. They could tell if the product was "in control" that is only producing the allowable number of defective items. For example under-weight filling of cans. If they are underweight your company could be charged with defrauding the public, if over-weight you are going to lose money.

This technique is one of many in use in factories where large quantities of small items are produced every hour. Samples of 10 or 20 (or whatever is convenient to test) are taken off the line at regular intervals, every half hour for example, and the quality is checked. The advantage of the system is that you can take a series of samples and make decisions about the number of defective items in a short time.

6. **Be a mathemagician by Dr. A. Freedman.**

Here the girls saw several tricks like **"The Lost Digit"**, **"The Card Trick"** and **"The Memory Trick"**.

7. **Image reconstruction by P.E.T. to see inside the brain, by Dr. R. Harrop.**

Positron Emission Tomography (P.E.T.) is one of the most technically advanced procedures in medical imaging; filtered back projection is a standard method for image reconstruction. The students used the concepts of projection and back projection to **"construct"** an **"image"** of an **"object"** and they were shown slides of brain images obtained by P.E.T. methods.

The objectives of this workshop was to help the students see mathematics especially geometry in application in a field where interdisciplinary work like math, computing, physics, chemistry, medicine, is of great importance. They met with women who were co-op students and graduate students from mathematics and computing science who have worked and are working in this field.

8. **The Art of counting by Dr. K. Heinrich.**

This workshop was mostly a problem solving session with problems taken from combinatorics.

9. **Patterns in numbers by Dr. J. L. Berggren.**

The object of this workshop was to learn some ways of doing mental arithmetic. Beginning with the squares of the numbers from 1 to 9 the girls learned how to use patterns in numbers and geometrical diagrams to figure out a rule for computing mentally products like 37 x 89. The basic tool is the identity $(a+b)^2 = a^2 + 2ab + b^2$.

10. **Rubber Sheet Geometry by Dr. D. Kay and Dr. L. Palmer.**

The students sketched a given figure on a balloon and then blew up the balloon. They also made the same figure with a pipe cleaner and they tried to cast the shadow of the figure and compare it to the one on the balloon to study projective geometry.

11. **Symmetry in Nature by Joan Sharp.**

This workshop attracted the attention of many girls. It made them aware of the different symmetries found in nature. At first they defined symmetry and classified objects accordingly. As an exercise they were supplied with a collection of natural objects and they were asked to study their shapes and their symmetries. This workshop gave the girls the opportunity to discuss symmetry among themselves and with the workshop leaders. As a consequence of this came an awareness and appreciation of shapes in nature, and the role mathematics could play in helping them analyze nature.

12. **Symmetry in Solids by Tasoula Berggren.**

This workshop was organised and led by the students of the honors calculus course. Some patterns and other necessary materials were supplied and the girls were asked to construct the solids by themselves. They discussed reflections, points, axes and planes of symmetry. They talked about rotational symmetries as well. After cubes and pyramids, tetrahedrons and other solids were constructed the girls were asked to study the different symmetries in these solids, figure out **Euler's formula** and answer a number of questions. For example they had to find the planes, axes and points of symmetry in a cube.

After the workshops we all gathered for lunch and after lunch there were lessons how to juggle bean bags. There was a message along with these lessons: **Just like the juggling needs concentration, practice and hard work so does mathematics**. With concentration, practice and hard work we can learn to do what seemed impossible at first. This was a great hit with many of the girls and I think it made the point about learning, but it also provided a chance for them to get to know each other. Quite frankly we believe that students who have made friends while at S.F.U. will be more likely to decide to go there to university.

Immediately following the juggling was an activity that we called **"Circles"**; they were small discussion groups. They gave us the time to be alone and talk with each other. They were directed mostly by undergraduate women and other women in science and technology, mainly at S.F.U.

Following these groups were more talks.

Jean Cook from Foreintec is a mathematician who is working in business. She talked about her work and how mathematics is used in the forest industry and forest research every day. Since a lot of the girls had relatives who worked in the industry they very much enjoyed Jean's entertaining talk, illustrated with slides and punctuated by jokes.

The Dean of Science, Dr. John Webster, introduced our next speaker. The high point of the day came then, when Dr. Geraldine Kenney-Wallace, Director of the Science Council of Canada and Professor of Chemistry at the University of Toronto, addressed the girls. She encouraged them to take risks in their lives and aim to reach the top. She talked to them about her research, about lasers and science in general. Her talk was interesting and challenging.

After the closing remarks we all gathered for a reception, where the girls had more opportunities to talk with professional people and ask questions. The numerous letters which we received later on is the indication of the great time everybody had and the enrichment which took place on November 27, 1987. A couple weeks later, in fact, my husband and I were leaving Canada Place, down at the Harborfront, and a woman came up to me and said she had been at the Conference with her daughter, and how much it meant to her daughter to have been at the conference.

I strongly believe that this is the time we need to help attract women into the sciences and engineering. I think conferences aimed at girls in the 9th and 10th grades can help, but obviously the problem is too big for just one approach to be the solution. In this regard, the Committee on the Status of Women in Ontario Universities has issued a report with some hints on how to attract more women to sign up for science courses. Here are some of the measures they recommend.

1. Offer academic "sampler" sessions to provide those entering or returning to university with a preview of lectures, tutorials or laboratories;

2. Set up kiosks in shopping centres featuring promotional videos, brochures and demonstrations aimed at attracting women to non-traditional fields of study;

3. Schedule classes to take into account the needs of students with family responsibilities;

4. Hold special orientation sessions for women entering "male-dominated" fields to inform them about support programs, services and other campus resources;

5. Make concerted efforts to appoint women to highly visible administrative posts such as deans and department chairs;

6. Develop strategies to "humanize" science by integrating social and ethical issues and the history of women in science into the curriculum and ensuring that diverse problem-solving approaches and learning styles are accepted.

There are clearly some controversial suggestions here, but there is also much that is good in these ideas, and if we are to solve what all admit is a major problem in our society today we must be willing, as Dr. Kenney-Wallace told the girls at S.F.U., to take some risks and to try some new ideas. I believe the S.F.U.'s <u>Women Do Math</u> conference is one of the new ideas that offers real promise of helping solve the problem.

Previous Proceedings

The following are a list of previous proceedings available through ERIC.

Proceedings of the 1980 Annual Meeting	-	ED 204120
Proceedings of the 1981 Annual Meeting	-	ED 234988
Proceedings of the 1982 Annual Meeting	-	ED 234989
Proceedings of the 1983 Annual Meeting	-	ED 243653
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Proceedings of the 1985 Annual Meeting	-	ED 277573
Proceedings of the 1987 Annual Meeting	-	ED 295842