CANADIAN MATHEMATICS EDUCATION STUDY GROUP

GROUPE CANADIAN D'ETUDE EN DIDACTIQUE DES MATHEMATIQUES

PROCEEDINGS

1989 ANNUAL MEETING

BROCK UNIVERSITY

ST. CATHARINES, ONTARIO

May 27-31, 1989

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CMESG/GCEDM Annual Meeting
Brock University
St. Catharines
May 27 - 31, 1989

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EDITORS' FORWARD

We would like to thank all the contributors for submitting their manuscripts for inclusion in these proceedings. Without their co-operation it would not have been possible to produce the proceedings so quickly.

Special thanks go from all of us who attended the conference to the organizers, and particularly to Eric Muller, who worked tirelessly before and during the meeting to ensure smooth sailing.

I hope these proceedings will help generate continued discussion on the many major issues raised during the conference.

Lionel Pereira-Mendoza
Martyn Quigley

April 10, 1990

ACKNOWLEDGEMENTS

The Canadian Mathematics Study Group wishes to acknowledge the continued assistance of the Social Sciences and Humanities Research Council for their support. Without this support the 1989 meeting and proceedings would not have been possible.

We would also like to thank Brock University for hosting the meeting and providing excellent facilities to conduct the meeting.

Finally, we would like to thank the contributors and participants, who helped make the meeting a valuable educational experience.

Lecture One

Teaching Mathematical Proof

Relevance and Complexity of a Social Approach

N. Balacheff

IRPEACS-CNRS

Introduction

What a mathematical proof consists of seems clear to all mathematics teachers and mathematics educators. That is: "A careful sequence of steps with each step following logically from an assumed or previously proved statement and from previous steps" (NCTM, 1989, p.144). This description is almost the same all over the world, and it is very close to what a logician would formulate, perhaps more formally. Comments made on mathematical proof as a content to be taught emphasize two points: first they stress that it has nothing to do with empirical or experimental verification, second they call attention to the move from concrete to abstract. Here is an example of such comments:

> "It is a completely new way of thinking for high school students. Their previous experience both in and out of school has taught them to accept informal and empirical arguments as sufficient. Students should come to understand that although such arguments are useful, they do not constitute a proof." (NCTM, 1989, p.145).

We can say that the definition of mathematical proof, as an outcome of these official texts is mathematically acceptable, but there is a long way from this definition to the image built in practice along the teaching interaction. More or less, teaching mathematical proof is understood as teaching how to formulate deductive reasoning: "Pour les professeurs, une démonstration, c'est très nettement l'exposé formel déductif d'un raisonnement logique" (Braconne, 1987, p.187).

The construction of this reasoning, and its possible relationships with other kind of reasoning, is hidden by that over emphasis on its "clear" formulation. That conception is so strong that some teachers can come to an evaluation of a mathematical proof just considering the surface level of the discourse. For example, in her requirement for teachers comments on a sample of students formulations, Braconne reports[1] that:

> "Les professeurs ont réagi aux longueur inutiles du texte de Bertrand, au désordre dans la solution de Karine, au fait que le texte d'Elodie ne suive pas le raisonnement déductif, etc. Toutefois, sept professeurs n'ont pas remarqué que, dans le texte de Bertrand, c'est la réciproque du théorème nécessaire à la démonstration qui était cité au premier paragraphe, et huit n'ont pas signalé que le texte de Laurent contenait la même erreur [...] Donc pour l'élève, et pour nous, les notes ne reflètent pas le fait que le professeur se soit apperçu de l'erreur ou non." (Braconne, 1987, p.99)

A report on proof frames of elementary preservice teachers shows a similar behaviour:

> "Many students who correctly accept a general-proof verification did not reject a false proof verification; they were influenced by the appearance of the argument - the ritualistic aspects of the proof - rather than the correctness of the argument. [...] Such students

[1] The interviewees were 13 French Mathematics teachers.

appear to rely on a syntactic-level deductive frame in which a verification of a statement is evaluated according to ritualistic, surface features."

(Martin & Harel, 1989)

Thus, mathematical proof appears ultimately as a kind of rhetoric specific to the mathematical classroom, it is not surprising then that it appears as such to the eyes of students. The nature of mathematical proof as a tool to establish a mathematical statement is to some extent hidden by the emphasis on the linguistic dimension. What does not appear in the school context is that the mathematical proof is a tool for mathematicians for both establishing the validity of some statement, as well as a tool for communication with other mathematicians. Also, it is often forgotten that what constitutes the present consensus about rigor has not been created ex nihilo, but that it is the product of an historical and a social process within the community of mathematicians[2]. As Manin[3] recalls, ultimately "a proof becomes a proof after the social act of 'accepting it as a proof'."

There is another reason for considering so strongly the social dimension of mathematics teaching and learning. For as we recognized that learning is a personal process, we should also consider that its outcome is likely to be firstly a private knowledge: The students' conceptions. But that conflicts with two constraints specific to the teaching, which has to guarantee the socialization of students' conceptions for the following reasons:

- Mathematics is a social knowledge. Students should make their own the knowledge that exists outside the classroom. It has a social status in society, or in smaller social groups under whose control it is used. For example, the community of mathematicians or that of engineers can be taken as a social reference.

- The mathematics class exists as a community. The teacher has to obtain a certain homogeneity in the meaning of the knowledge constructed by students, and she or he has to ensure its coherence. Otherwise, the functioning of the class will hardly be possible. Because of the constructivist hypothesis we consider, the use of authority is not desirable. Thus, the homogenization can only be the result of a negotiation or of other specific social interactions such as the one Brousseau (1986) has described in the frame of his *théorie des situations didactiques*.

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2 The essay of I. Lakatos (1976) on the dialectic of proofs and refutations gives a good insight of this historical process.

3 Manin quoted by Hanna (1983).

Social Interaction and Situations for Validation

What is now clear is that as long as students rely on the teacher to decide on the validity of a mathematical outcome of their activity, the word 'proof' will not make sense for them as we expect it to do. In such a context they are likely to behave mainly to please their teacher, just as one of the British students interviewed by Galbraith (1979) told his interviewer: "To prove something in maths means that you have worked it out and it proves how good you are at working questions out and understanding them."

But it is not sufficient to propose a problem to the mathematics classroom and to tell the students that they have the responsibility of solving it. There is no reason for them, a priori, to consider that the problem is their problem and to feel committed to solving it; they can still think that they have to do so in order to please the teacher and thus their behaviour will not be significant.

Before going ahead, let us consider a short story told by Sir Karl Popper, which will throw a relevant light on what we want to suggest:

> "If somebody asked me, 'are you sure that the piece in your hand is a tenpenny piece?' I should *perhaps glance at it again* and say 'yes'. But should a lot depend on the truth of my judgement, I think I should take the trouble to go into the next bank and ask the teller to look closely at the piece; and if the life of a man depended on it, I should even try to get to the Chief Cashier of the Bank of England and ask him to certify the genuineness of the piece."
>
> (Popper, 1979, p.78).

And then Popper adds that "the 'certainty' of a belief is not so much a matter of its intensity, but of the *situation:* Of our expectation of its possible consequences." (ibid.)

Along the same line, I would like to suggest that if students do not engage in any proving processes, it is not so much because they are not able to do so, but rather that they do not see any reason. Even if they engage such a process, its level depends heavily on the way students understand the situation. Following a principle of *economy of logic* they are likely to bring into play no more logic than what is necessary for practical needs (Bourdieu, 1980, p.145).

Then the true meaning of the outcomes of students proving processes is to be traced in the characteristics of the situation in which they are involved.

In situations in which they have to decide a common[4] solution to a given problem, students have to construct a common language and to agree on a common system to

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4 By 'common', we mean here a solution supported by the whole classroom, or smaller groups of students as is usually the case.

decide of the validity of the solution they propose. The essential role of the social dimension, mainly in situations for communication, provoking a move from "doing" to "telling how to do", and their importance in the construction of meaning have been put in evidence by Brousseau in his *théorie des situations didactiques* (Brousseau, 1986). Here we would like to recall what this author wrote about the situations for validation:

> The situations for validation "will bring together two players who confront each other regarding a subject of study composed on the one side of messages and descriptions produced by the pupils and on the other side of the a-didactic milieu used as referent for these messages. The two players are alternately a 'proposer' and an 'opposer'; they exchange assertions, proofs and demonstrations concerning this pair 'milieu/message'. This pair is a new apparatus, the 'milieu' of the situation for validation. It can appear as a problem accompanied by the attempt at solving it, like a situation and its model, or like a reality and its description...
>
> While informer and informed have dissymmetric relations with the game (one knows something that the other does not know), the proposer and the opposer must be in symmetrical positions, both regards the information and means of action about the game and the messages which are at their disposal, and as regards their reciprocal relations, the means of sanctioning each other and the objectives vis-à-vis the pair milieu/message."
>
> (Brousseau, 1986, p.158).

We should realize that in such situations, behaviours that are more social than mathematical, would probably appear. For example, because of self-esteem, some students might refuse to recognize that they are wrong, or others might refuse to accept that their opponents are right.

Thus, to sum up, to provoke students proving behaviours we should design situations in such a way that students come to realize that there is a risk attached to uncertainty, and thus that there is an interest in finding a good solution. In order to obtain a significant scientific debate among students, we should provide them with a situation promoting contradiction, but also promoting acceptance. Otherwise systematic rejection could become an efficient defensive strategy. In other words, the situation should allow the recognition of a risk linked to the rejection of a true assertion, or to the acceptance of a false one.

Following these principles we have designed teaching situations as experimental settings in order to study students behaviours in such contexts, and the nature of these behaviours in relation to the characteristics of these situations. A priori, we thought that genuine mathematical proving processes will be observed, a deep analysis of our experiments shown that things are a bit more complex than what is usually acknowledged by innovative practice resting on social interaction.

In the following section we will report, in some detail, on one of these experiments.

A Case Study[5]: The Perimeter of a Triangle

A first principle we wanted to satisfy in designing the experiment was to obtain the devolution[6] of the responsibility for the validity of the problem's solution from the teacher to the students. For that purpose we have chosen a context of communication: We told the students that they will have to write a message for other students, of the same grade, in order to allow them to solve a given problem. In such a situation the criteria for success are left to be decided by students according to their own means for the evaluation of the efficiency and the reliability of the message they have produced. We thought that this setting would be sufficient to ensure that students will consider that they have the responsibility for the truth of their solution, and that they will not refer to the teacher expectation.

In such a situation there is usually some tensions because of the different individual motivation and commitment. For this reason, we think that it is not desirable to ask the students to work individually, but on the other hand it is not desirable to ask for a collective production form the whole class insofar as some students might feel that they are not concerned, leaving the job to the others. So, we decided to constitute small teams of three to four students working together, telling them that the final solution will be one of the ones proposed by the teams, or a modification of it. To promote collective work, each team must propose only one solution, and during the debate for the choice of the class solution the team will be asked to express its position through the voice of a chosen representative. That constraint obliges students to be explicit and to discuss a priori the correctness and appropriateness of what they want to be said. We think that the quality of the debate will rest on the motivation of each team, its willingness to have its message chosen, but also its commitment to the success of the class as a whole.

The mathematical problem we chosen was the following:

> *Write for other students, a message allowing them to come to know the perimeter of any triangle a piece of which is missing. To do it, your colleagues will have at their disposal only the paper on which is drawn a triangle and the same instruments as you (rules, etc.).*

Together with this text a triangle such as the following (fig. 1) was given to the students. All the teams in the classroom had the same materials.

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5 The case study reported here has been made possible because of the close relationships established by academics and teachers within a research group of the *IREM de Lyon*. It is a small part of a four year project which had allowed us to collect a large amount of data. The complete report is available from *IREM de Lyon, Université Clause Bernard, Lyon*.

6 Devolution: "A delegating of authority or duties to a subroutine or substitute" (The American Heritage Dictionary of the English Language, 1979).

The study we made before this experiment (Balacheff, 1988, pp 321-360), allowed us to think that all the students will be able to enter the problem-solving process, with quite different solutions. This diversity was expected to be the source of interesting debates. We know that some students, and thus some teams, will miss the fact that the solution must work for a general case and not only for the triangle given as an example. But we were sure that this will be pointed out during the debate, and then that it will be taken in consideration, even with more strength than if the teacher had warned about it a priori.

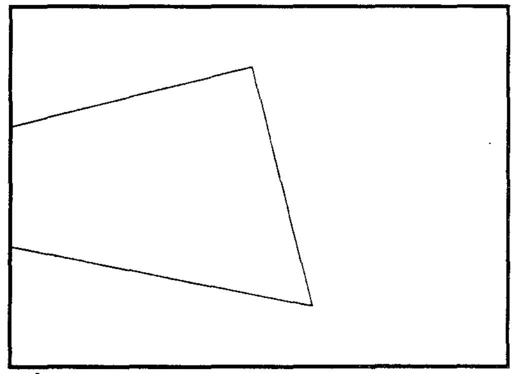

Figure 1

The role of the teacher was to present the situation, then not to intervene in any case up to the time when all the teams have proposed a solution; then the teacher's position will be to regulate the debate and to give the floor to the teams' representatives. The end of the sequence will come from a general agreement on the fact that one of the solutions, or a new one obtained as a result of the interactions, is accepted. The debate was organised in the following way: The messages were written on a large sheet of paper and then they were displayed on a wall of the classroom. Each team had to analyze the messages and their representatives had to tell the class their criticisms and suggestions. These criticisms had to be accepted by the team which was the author of the message discussed. In case of an agreement of the class on a false solution, the teacher was allowed to propose to the teams a new triangle invoking that such a triangle might be considered by the receptors. (Such material had been prepared taking into account what we knew from the first study). On the other hand if more than one message was acceptable with no clear decision from the class then the teacher was supposed to organise a vote to make the choice, asking the students to tell the reasons for their choice.

I will not report here in detail on the analysis of this experiment. A complete report is available in Balacheff (1988, pp 465-562). I will here focus here on the outcomes relevant to my present purpose, as they are related to the observations which have been made in two different classrooms.

The First Experiment

The first experiment was carried out with students of the eighth grade (13 to 14 years old). The teacher was a member of the research team, which meant that we were in a good position to assert that the project was well known to her. The observations lasted for two sessions of 1:30 hours. After the first one we felt really happy with what had happened. The second phase raised a feeling of some difficulties ... beyond these feelings only the close analysis of the data gathered, led us to discover the existence of the parameters which have played a critical role in the teacher decisions, and thus in the students behaviours:

- First, a constraint of time, which made the teacher intervene in order to ensure that the whole process would keep within the limits imposed by the general school timetable in which the experiment took place.

- Second, the teacher's willingness to guarantee an acceptable end in her own eyes. There was a huge tension between this willingness and the willingness of not breaking the contract of "non intervention". This tension is the indication of what we would like to call in the future: *the teacher epistemological responsibility.*

Because of these two constraints, the decisions the teacher made tended to oppose the devolution of the problem. In particular, to guarantee that the problem solving phase would not be too long, the teacher invited students to propose a solution as soon as she thought that it was mature enough, but with no information about the real feeling of these students. Also, some teacher's interventions aimed at calling the attention of students to the word "any" (in the sentence "any triangle"), and doing so she did not think that it was a mathematical intervention, insofar as she thought that it was only due to the students' lack of carefulness. But all these interventions led students to a feeling of dependence and the idea of a possible responsibility of the teacher for the validity of their answer.

A significant phenomenon, is that the teacher (as well as the observers) did not realize what a continuous contact she kept with the students, making about one intervention every minute over an 80 minutes period. The content of these intervention could have been light, as: "Are you O.K.?", or more important as: "Are you sure you have carefully read the statement of the task? ". All together we have counted, within these interventions, 129 different items. We see this phenomenon as an indicator of the intensity of the relationship between the teacher and the students in a situation that we thought to be quasi-isolated from the teacher before we did a close analysis of the records.

The same constraints were an obstacle to the functioning of the second phase. After a first exchange of critiques by the teams' representatives, the teacher intervened because she thought that nothing positive will come out of the engaged process – at least within the time available. The teacher then tried to facilitate the progress in the discovery of a solution, calling explicitly for ideas and suggestions to start from them and go further. Actually, it was quite clear from her attitudes that not all the ideas were of the same value. The students' behaviours were deeply transformed by these interventions. They got confused and they were no longer committed to any real discovery of a solution.

The teacher thought that she had kept the spirit of the sequence, the basic frame being: search for a solution, critics, new ideas and suggestions to go ahead. But only the superficial aspects of the intended sequence were still there; its meaning for the students were fundamentally changed. They did not enter a true mathematical activity, as expected, but just a new school game not so different, beyond the new and exiting social setting, from the ordinary one.

The Second Experiment

We learned a lot from this first experiment, and we thought that it would be worthwhile to make a second one. We decided to keep the same general framework, but to overcome the obstacles we came to be faced with, we chose the three following modifications:

(i) To observe a tenth grade classroom in order to be sure that no mathematical difficulty will disturb the phenomenon we wanted to observe. Also, at this level students have already been introduced to mathematical proof. The situation could be an opportunity to evidence its power as a means for proving...

(ii) To open the time, that means that we decided to leave open when the end of the experiment will end up. We thought that three or four sequences of about one hour each would be sufficient.

(iii) To ask the teacher not to intervene, as strictly as possible, during the first phase (the initial problem-solving phase), and then to act just as a chairperson and as the collective memory[7] of the class during the second phase (the debate).

The first phase[8] did not present any special peculiarity. The teacher did not intervene at all, leaving students free to decide that they had a solution to propose. Four teams among the five reached a solution, the fifth one which was clearly close to surrender, finally proposed a "contribution" to the collective effort, as a response to the teacher demand.

During the second phase, she also followed the specifications we decided together. Then ...

More than a scientific debate, that is, proposing proofs or counterexamples, the data show that students entered a discussion with some mathematical content in it, but which mainly consisted of an exchange of arguments *pro et contra* not necessarily connected the one to the others. They argued about the different proposed solutions, but they did not prove mathematically.

The situation for communication has really been taken into account as such by students, as their remarks on the proposed messages show. The main critics are related to the fact

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7 To be the "memory" of the class means to take a record of what is said, in particular by writing students' decisions on the blackboard.

8 This phase took about one hour.

that this message must be understandable and usable by its receptors. But the problem of the validity of the proposed solution is not really considered. In that sense we can say that the situation does not realize a situation for validation. For a clear distinction between "arguing" and "proving" in mathematics[9], we refer to the distinction as formulated by Moeschler:

> "Un discours argumentatif n'est pas un discours apportant à proprement parler des preuves, ni un discours fonctionnant sur les principes de la déduction logique. En d'autres termes, argumenter ne revient pas à démontrer la vérité d'une assertion, ni à indiquer le caractère logiquement valide d'un raisonnement [...] Un discours argumentatif, et c'est là une hypothèse de départ importante, se place toujours par rapport à un contre-discours effectif ou virtuel. L'argumentation est à ce titre indissociable de la polémique." (Moeschler, 1985, p.46-47).

In that sense, what we have observed is first of all an exchange of arguments about the simplicity of the solution ... or of its complexity. The context of a communication with other students has favoured the feeling of the relevance of critics in that register. But what leads us to suggest that this debate is more an argumentation than a scientific debate, in the Moeschler sense, is the frequent lack of logical relationships between arguments. Even more, some students can pass in the same argumentation from one position to another completely contradictory. These arguments can have nothing to do with mathematics, or even with what is required by the situation ... and it could be the same for the objections opposed to an argument. Finally, the involvement of some of the teams in the game, I mean the fact that they are eager to win, had favoured the appearance of polemics: The strongest opponents to the "too complex" message are the authors of the "too simple", and conversely.

After a first period of debate the messages had been accepted, provided that some modifications were made, but their validity has not been really discussed. So, the teacher proposed a new triangle, in order to challenge the messages. This triangle was such that the wrong solutions will obviously fail. The debate following this checking phase, shows how strongly students are more involved in an argumentation than in a scientific debate. Finally, one solution being accepted as the solution of the class, the teacher asked students whether they were sure of that solution. They answer: "Yes, because we have done it in a lot of cases." So, it is even not sufficient to directly address the question of the validity. Note, that when later on the teacher asked the students about a possible mathematical proof of their solution, they gave one showing that technically it was possible to them.

9 We do not refer necessarily to formal proof, or mathematical proof in the classical sense.

Discussion

Efficiency Versus Rigour

Even if we are able to set up a situation whose characteristics promote content specific students' interaction, we cannot take for granted that they will engage a "mathematical debate", and finally that they will produce a mathematical proof.

A peculiarity of mathematics is the kind of knowledge it aims at producing. Its main concern is with concepts specific to its internal development. There is evidence that Egyptians used intellectual tools in practical situations for which we have now mathematical descriptions, but the birth of mathematical proof is essentially the result of the willingness of some philosophers to reject mere observation and pragmatism, to break off perception (the *monde sensible*), to base knowledge and truth on Reason. That actually is an evolution, or a revolution, of mathematics as a tool towards mathematics as an object by itself, and as a consequence a change of focus from "efficiency" towards "rigor".

It is a rupture of the same kind which happens between "practical geometry" (where students draw and observe) and "deductive geometry" (where students have to establish theorems deductively). Also in numerical activities, like the one reported by Lampert (1988), the same rupture happens when students no longer have to find some pattern out of the observation of numbers, but that they have to establish numerical properties in their "full" generality (using letters and elementary algebra).

Here we have to realize that most of the time students do not act as a *theoretician* but as a *practical man*. Their job is to give a solution to the problem the teacher has given to them, a solution that will be acceptable with respect to the classroom situation. In such a context the most important thing is to be effective. The problem of the practical man is *to be efficient* not *to be rigorous*. It is to produce a solution, not to produce knowledge. Thus the problem solver does not feel the need to call for more logic than is necessary for practice.

That means that beyond the social characteristics of the teaching situation, we must analyse the nature of the target it aims at. If students see the target as "doing", more than "knowing", then their debate will focus more on efficiency and reliability, than on rigor and certainty. Thus again argumentative behaviours could be viewed as being more "economic" than proving mathematically, while providing students with a feeling good enough about the fact that they have completed the task.

Social Interaction Revisited

Social interaction, while solving a problem, can favour the appearance of students' proving processes. Insofar as students are committed in finding a common solution to a given problem, they have to come to an agreement on the acceptable ways to justify and to explain their choices. But what we have shown is that proving processes are not the only processes likely to appear in such social situations, and that in some circumstances they could even be almost completely replaced by other types of interactional behaviours. Our point is that in some circumstances social interaction might become an obstacle, when students are eager to succeed, or when they are not able to coordinate their different points of view, or when they are not able to overcome their conflict on a scientific basis[10]. In particular these situations can favour naive empiricism, or they can justify the use of crucial experiment in order to obtain an agreement instead of proofs at a higher level (Balacheff, 1988).

Perhaps some people might suggest that a better didactical engineering could allow us to overcome these difficulties; indeed much progress can be made in this direction and more research is needed. But we would like to suggest that "argumentative behaviours" (i) are always potentially present in human interaction, (ii) that they are genuine epistemological obstacles[11] to the learning of mathematical proof. By "argumentative behaviours" we mean behaviours by which somebody tries to obtain from somebody else the agreement on the validity of a given assertion, by means of various arguments or representations (Oléron, 1984). In that sense, argumentation is likely to appear in any social interaction aiming at establishing the truth or falsehood of something. But we do consider that argumentation and mathematical proof are not of the same nature: The aim of argumentation is to obtain the agreement of the partner in the interaction, but not in the first place to establish the truth of some statement. As a social behaviour it is an open process, in other words it allows the use of any kind of means; whereas, for mathematical proofs, we have to fit the requirement for the use of a knowledge taken in a common body of knowledge on which people (mathematicians) agree. As outcomes of argumentation, problems' solutions are proposed but nothing is ever definitive (Perelman, 1970, p.41).

Insofar as students are concerned, we have observed that argumentative behaviours play a major role, pushing to the backside other behaviours like the one we were aiming at. Clearly enough, that could be explained by the fact that such behaviours pertain to the genesis of the child development in logic: Very early, children experience the efficiency

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10 I mean, content specific.

11 The notion of "epistemological obstacle" has been coined by Bachelard (1938), and then pushed on the forefront of the didactical scene by Brousseau (1983). It refers to a genuine piece of knowledge which resists to the construction of the new one, but such that the overcoming of this resistance is part of a full understanding of the new knowledge.

of argumentation in social interactions with other children, or with adults (in particular with parents). Then, it is quite natural that these behaviours appear first when what is in debate is the validity of some production, even a mathematical one.

So, what might be questioned is perhaps not so much the students' rationality as a whole, but the relationships between the rationale of their behaviours and the characteristics of the situation in which they are involved. Not surprisingly, students refer first to the kind of interaction they are already familiar with. Argumentation has its own domain of validity and of operationality, as all of us know.

So, in order to successfully teach mathematical proof, the major problem appears to be that of *negotiating* the acceptance by the students of new rules, but not necessarily to obtain that they reject argumentation insofar as it is perhaps well adapted to other contexts. .Mathematical proof should be learned "against" argumentation, bringing students to the awareness of the specificity of mathematical proof and of its efficiency to solve the kind of problem we have to solve in mathematics.

Here negotiation is the key process, for the following reasons:

- First, because the teaching situation cannot be delivered "open" to the students, otherwise many of them will not understand the point and they will get lost. The following quotation from Cooney makes it clear:

> "Maybe not all of them but at least some of them felt 'I am not going to participate in this class because you [referring to the teacher] are just wasting my time'. It is so ironic because if I was doing the type of thing they wanted to do, they would be turning around in their seats and talking. So it's a no-win situation."
>
> (Cooney, 1985, p.332).

- Second, because of the rules to be followed, the true aim of the teacher cannot be stated explicitly. If the rules for the interaction are explicitly stated, then some students will try to escape them or to discuss them just as many people do with law. Also because interacting mathematically might then become "mastering a few clever techniques" which may turn into objects to be taught, just as teaching "problem solving" has often become teaching quasi-algorithmic procedures (Schoenfeld, 1985).

The solution is somewhere else, in the study and the better understanding of the phenomena related to the didactical contract, the condition of its negotiation, which is almost essentially implicit, and the nature of its outcomes: the devolution of the learning responsibility to the students. We cannot expect ready-to-wear teaching situations, but it is reasonable to think that the development of research will make available some knowledge which will enable teachers to face the difficult didactical problem of the management of the life of this original society: The mathematics classroom.

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Lecture 2

Geometry is Alive and Well!

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Geometry is Alive and Well[1]

The reports of my death are greatly exaggerated.
Mark Twain, 1897 (cable to Associated Press
from London, upon reading of his death.)

It is a widely held opinion that geometry is dead. At the Fourth International Congress on Mathematical Education held in Berkeley in 1980, a lively debate on the topic featured J. Dieudonné, B. Grünbaum, and R. Osserman — all well recognized research mathematicians with deep interests in and strong opinions about geometric questions and the teaching of geometry. In his address, Osserman noted "... to speak of the 'death of geometry' at the post-secondary or any other level is clearly an exaggeration, [though] it nevertheless reflects a reality."[2]

The evidence of the death of geometry as a vital part of the body of mathematics seemed convincing:

— The small role of geometry in the high school curriculum: rarely required, and typically a one year (or shorter) course.
— The insignificant role of geometry in College and University curricula: if offered at all, limited to a course for prospective teachers, or specialized courses (projective geometry, differential geometry).
— The dearth of research papers, conferences, and symposia devoted to geometry.
— The small number of geometry texts at the college level, and absence of any new texts.

Historically, a knowledge of geometry was considered the mark of an educated person. However, in recent times, a reverse kind of snobbery has occurred: a lack of knowledge about, and disinterest in geometric questions is a common profile of the mathematical research community. The view towards geometry is generally a mixture of one or more of the following beliefs:

1. Euclidean geometry, like Latin, is GOOD FOR YOU. It should be studied (in high school) for historical appreciation and to build character. The geometric content is not expected to lead (mathematically) anywhere.

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[1] An earlier version of this address was given at the Conference on Learning and Teaching Geometry, June 1987, at Syracuse University, New York.

[2] All quotations in this paper from ICME IV may be found in The Two-Year College Mathematics Journal, 12 (September 1981) 226-246, which contains the addresses given by Dieudonné, Grünbaum, and Osserman.

2. Euclidean geometry is where students learn logic, the axiomatic method, and deductive proof. The geometric content of the course is secondary to these aims. (This course could be titled GEOMETRY AS A MILITARY DISCIPLINE.)

3. Geometry provides some interesting low-level recreational problems to solve, but there aren't any important unanswered questions. Mathematicians who claim to do research in geometry are not considered as serious in their interests.

> Most mathematicians are totally unaware of the fact that the elementary, intuitive approach to geometry continues (and will continue) to generate mathematically profound and interesting problems and results. (B. Grünbaum)

4. The content of geometry has been integrated into (absorbed by) almost all of higher mathematics — linear algebra, analysis, algebraic geometry, topology, group theory, etc. — so there is no need to teach it apart from these.

> ...mathematicians have been extremely appreciative of the benefits of the geometric language, to such an extent that very soon they proceeded to generalize it to parts of mathematics which looked very far removed from Geometry.
> (J. Dieudonné)

This last view was met with a memorable rejoinder by Osserman, who summarized Dieudonné's position as follows:

> Geometry is alive and well and living in Paris under an assumed name.

Even in recent years, there has always been a small core of mathematicians who have done considerable research in geometry despite the prevailing mathematical fashion. H.S.M. Coxeter might be considered the "dean" of such researchers. In an interview in 1979 for The Two -Year College Mathematics Journal with David Logothetti, he gave testimony to his enduring interest in and excitement about geometry, and his belief in its vitality. The interview closes with a question by the interviewer, and Coxeter's reply:

> L. If I or my colleague Jean Pedersen start rhapsodizing about geometry, the reaction that we frequently get is, "Oh well, that's a dead subject; everything is known." What is your reaction to that reaction?
>
> C. Oh, I think geometry is developing as fast as any other kind of mathematics; it's just that people [research mathematicians] are not looking at it.

In his closing remarks at the 1980 ICME, Osserman echoed similar sentiments:

> ...geometry...has gone through a period of neglect, while the arbiters of mathematical taste and values were generally of the Bourbaki persuasion. On the other hand, ... that period is already drawing to a close. ... I would predict that with no effort on any of our

> parts, we will witness a rebirth of geometry in the coming years, as the pendulum swings back from the extreme devotion to structure, abstraction, and generality.

Today we witness a renewed interest in teaching and learning geometry. In 1987, the NCTM yearbook and an international conference in Syracuse, New York were devoted to the topic. The newly announced NCTM standards (1989) address the need to strengthen geometric content in the K-12 curriculum, and an article by Marjorie Senechal in a collection of position papers (to be published in 1990 by the National Research Council) on the mathematical content in the K-12 curriculum, identifies shape as a major content strand at all levels of learning. These are timely events, since there is convincing evidence that points to a renaissance in geometry. There is strong interest in geometric figures in the plane and 3-space — exploration of their properties, their interrelationships and enumeration of their types. In what follows, I want to convince you that reports of the death of geometry (in 1980, and even more so today) are greatly exaggerated. The remarks by Coxeter and Osserman in 1980 were prophetic — for whether or not the official teachers and researchers in the mathematical community choose to lead (or even join) in this renaissance, it is happening.

The Evidence

Activity outside mainstream mathematics

While mathematicians were neglecting (or ignoring) geometry, its importance grew in many other fields. Those areas in which geometry has always been central — art, architecture, design and engineering — make direct use of geometry to create and build forms which satisfy aesthetic desires and structural needs. The three-dimensional Euclidean world which we inhabit demands answers to complex geometric questions, and manufacturers, craftsmen, architects and engineers have not waited for the mathematical community to provide answers — they always have and still continue to solve geometry problems, sometimes in an ad hoc and ingenious manner. Renewed interest in geometry related to structure is evidenced in the recent publication of several books concerned with the geometry of spatial forms,and the topics of incidence and symmetry in design (see, for example, [Baglivo and Graver], [Blackwell], [Gasson]). One especially active site of research into structure and form is the University of Montreal, and its associated "Structural Topology" group, which seeks to have investigators from many disciplines contribute to the common search for a better understanding of and solutions to geometric problems.

Many other fields have found geometry a rich source of ideas for creating models to understand complex forms, relations, and processes which cannot be viewed directly. Historically, artists and artisans as well as mathematicians have been interested in polyhedra (Leonardo da Vinci and Albrecht Dürer, as well as Johannes Kepler and Leonhard Euler to name but a few), but today, it is not likely that students or their teachers even know why a soccer ball has hexagon and pentagon faces, or why it must have exactly 12 pentagon faces. Polyhedra, sometimes viewed by mathematicians merely as pretty ornaments, rather than a rich source for study, are indispensable as models in diverse fields. The idea of ball and stick polyhedra models to represent molecules gained wide acceptance by the late nineteenth century. This modelling of chemical structure (the balls representing atoms, the sticks the bonds between atoms) has been one of the most productive ideas of modern chemistry. Tetrahedron is the name of an international journal of organic chemistry, signifying the importance of the model which considers carbon atoms to be situated at the centers of tetrahedra. Inorganic chemistry as well has recently developed simple and successful polyhedral models; an international journal in that discipline is named Polyhedron.

Some of the most exquisite polyhedra can be found in nature as crystals. But the inner atomic structure of crystals is also highly geometric — it is modeled by a vast lattice of atoms which can be viewed as packed polyhedra, and has been the subject of intense investigation in recent times by crystallographers, chemists, mathematicians and physicists. In biology, polyhedra serve as useful models for the structure of viruses which often (surprisingly) have icosahedral symmetry. The investigation of how information is carried by viruses, and how viruses self-replicate has led to the study of repeating patterns on polyhedra, and to questions on polyhedral packing. Soap bubble froth has been used to study aggregates of polyhedra which model biological structures. Difficult questions concerning packing of spheres are of interest to those who model chemical (atomic) structures and biological processes; these same studies have important applications in algebraic coding theory.

Another active area of geometry research which has recently emerged involves dynamic polyhedral models — here investigators might attempt to model the growth of a rigid plant stem through the division of packed polyhedral cells, or model the functioning of a robot mechanism. An extremely readable and well illustrated overview of the rich topic of polyhedra — history, properties, occurrences in nature and man-made design, importance as a modelling device, activities, questions — can be found in the book *Shaping Space*.

Symmetry is a concept that encompasses very diverse fields; here geometry also plays a central role. Symmetry is not only a powerful tool for creating or analysing beautiful designs in the plane or space by means of Euclidean and affine transformations; it is also a profound idea that gives an approach to understanding many of nature's structures and processes. Recently there have been several conferences, articles, and books devoted to symmetry and its many manifestations and applications. A large and varied collection of articles on symmetry, by authors representing many disciplines and countries, is contained in the collection *Symmetry: Unifying Human Understanding*; a sequel volume has just been published. A newly recognized 'type' of symmetry, that of "self-similarity", has revealed not only beautiful graphic images of dynamic processes, but offers a new view of forms and dynamic systems that were previously viewed as random or unpredictable in shape or behaviour. (See, for example, [Gleick], [Mandelbrot], [Barnsley].)

Activity within the mathematical community

Two measures of the vitality of activity in a mathematical field are the output of research articles and the number lectures, seminars and conferences devoted to the topic.

In recent years, the number of pages in *Mathematical Reviews* devoted to reviews of articles on geometry has grown dramatically. Indeed, the category 51, simply titled "Geometry", now has 14 subtitles (51A - 51N), and category 52, "Convex sets and related geometric topics" has become a catchall for the large number of papers on geometric topics for which a separate category has not yet been designated. (Differential Geometry and Topology have their own category numbers.) This increase in publication reflects not only a proliferation of articles, but also the establishment of several new journals devoted primarily to research in geometry. In 1989 alone, two new journals, *Combinatorial Geometry* and *Symmetry* were launched.

Two new areas of research activity in which the publication of papers has been especially prolific are signalled by the titles of recently published books: *Tilings and Patterns*, and *Computational Geometry*. Artisans of all cultures have designed decorative patterns and geometric tilings, and many popular recreational problems concern tilings of geometric figures. Yet mathematicians B. Grünbaum and G.C. Shephard found when they set out to write a work on "visual geometry":

> Perhaps our biggest surprise when we started collecting material for the present work was that so little about tilings and patterns is known. We thought, naïvely as it turned out, that the two millennia of development of plane geometry would leave little room for new ideas. Not only were we unable to find anywhere a meaningful definition of pattern, but we also discovered that some of the most exciting developments in this area (such as the phenomenon of aperiodicity for tilings) are not more than twenty years old.
>
> (p.vii, Tilings and Patterns)

Their book brings together the work of many who have investigated tilings, sets out definitions and classification schemes, and, most importantly, indicates many avenues for further investigation.

The title "Computational Geometry" is simultaneously suggestive and ambiguous — I doubt that agreement could easily be reached on what it is and what it is not. The authors Preparata and Shamos indicate in their introduction that several contexts have been clothed with that title, but make clear that the essence of computational geometry is the design of efficient algorithms (for computers) to solve geometric problems. Classically, the restrictive tools of compass and straightedge and the algorithms of Euclidean constructions were used to solve geometry problems. With Descartes and later Gauss, algebraic and analytic tools could be employed to solve geometry problems, and in addition, the question of what constructions were feasible could be discussed. Today's researchers may use computers as restrictive tools and so the problems as well as the methods of solution must be recast:

> One fundamental feature of this discipline is the realization that classical characterizations of geometric objects are frequently not amenable to the design of efficient algorithms. To obviate this inadequacy, it is necessary to identify the useful concepts and to establish their properties which are conducive to efficient computations. In a nutshell, computational geometry must reshape — whenever necessary- -the classical discipline into its computational incarnation. (p.6, Computational Geometry)

A few of the concerns are the development of new coordinate systems to encode geometric information, the creation of very accurate data bases for geometric objects, and the visual (screen) representation of geometric objects in 2, 3, and higher dimensions. The emphasis on computation has even changed the way in which many geometry questions are asked. Instead of asking "How many different types of polyhedra are there with n vertices?", the researcher asks "How can the computer determine whether two given polyhedra are of the same type?" and "What is the complexity of the best algorithm to do so?"

Conference activity on geometry is decidedly on the upswing, with the participants representing many areas of mathematics and other disciplines. Here are just a few special conferences largely concerned with geometry held during 1984-87:

"Shaping Space", an interdisciplinary conference on polyhedra, Smith College, April 1984.
"International Congress on M.C. Escher", Rome, April 1985.
"Eugene Strens Memorial Conference on Recreational and Intuitive Mathematics", University of Calgary, July 1986.
Special semester devoted to the Geometry of Rigid Structures, CRM, University of Montreal, January-May 1987.
"Computer-aided geometric reasoning", INRIA, Sophia Antipolis, France, June 1987
"SIAM Conference on Applied Geometry", Albany, July 1987.

In the last two years, the number of such special conferences on geometric topics has risen dramatically, and in addition, at the National MAA and AMS meetings the number of lectures, minicourses, and special sessions reflects the growing interest and diversity of research in geometry. Here is a list of items on the program of just one such meeting, the AMS-MAA meeting held August 7-10 in Boulder, Colorado:

Colloquium Lectures:	"Geometry, Groups, and Self-Similar Tilings", William P. Thurston
Special Session:	"Mathematical Questions in Computational Geometry"
Minicourses:	"Chaotic Dynamical Systems", Robert L. Devaney
	"Group Theory Through Art", Thomas Brylawski
Invited Addresses:	"The dynamics of billiards in polygons", Howard A. Masur
Jean E. Taylor:	"Crystals, in equilibrium and otherwise"
Progress in Mathematics Lecture:	"Liquid Crystals", Haim Brezis

The impact of technology

Perhaps the greatest single impetus to renewed activity in geometry has been the availability and proliferation of technological tools. This has created a two-way interaction involving geometric activity and technology.

On the one hand, the design and implementation of computers and other high-powered research, design, and diagnostic tools require a high level of understanding of traditional geometry and the solution of many new geometric problems. For example, computer-aided design (CAD) and manufacturing (CAM) (imaging and robotics), communications (networks and coding), and diagnostic imaging (computer-aided scanning devices) are areas in which geometry plays a central role. On the other hand, technological tools can also be utilized to investigate and even prove geometric statements.. The ability to make and test conjectures in geometry (or any subject) is greatly enhanced by looking at a large number of specific cases. Complicated geometric forms can be shown rapidly in many aspects on a computer screen, changed and modified effortlessly, and data recorded and compared. Plausible conjectures based on such experimental data can be subjected to traditional methods of proof, or in some cases, proved by computer programs. As high-powered "eyes", technological devices can reveal the inner geometry of crystals, plant cells, viruses, and even chemical molecules, making it possible to test the veracity of accepted models and provide challenging new geometry problems to solve.

Titles of several of the sessions at the meetings held in France and in Albany in the summer of 1987 (listed below) will indicate some of the areas in which there is strong interest and active research:

> Image processing; Surfaces; Mathematical Methods and Design
> Packing and Tiling; Mesh Generation; Graphics; Computational Geometry; Robotics; Solids; Modelling for Manufacturing;
> Automatic Theorem-proving; Computer-aided design; Applications to Rigidity of Structures; Applications to Scene Analysis and
> Polytopial Realization; Algebraic, Topological and Combinatorial Aids to Geometric Computation.

The availability and use of technology, especially microcomputers, has also begun to affect the teaching of geometry at all levels. Exploratory activities with LOGO ("turtle geometry"), computer-aided Euclidean constructions ("The Geometric Supposer", "The Geometric Constructor", "Cabri") , and transformations using computer graphics can enrich the teaching and learning of geometry in elementary and secondary school. To construct a computer program which produces an image on a computer screen — the first task of computer graphics — requires a good knowledge of geometry, and affords an excellent opportunity to teach some traditional college geometry in a new light. A recent text, Projective Geometry and its Applications to Computer Graphics, develops the geometric machinery necessary to understand the representation and transformation of geometric objects in order to produce a screen image. Along the way, the main theorems of projective geometry are proved analytically. The strong purpose of the book linking the subject to computer graphics makes a compelling case for learning the geometry. On page 1, the authors make clear that a knowledge of Euclidean geometry is assumed:

> The primary purpose of this [first] chapter is to introduce projective geometry and discuss it in relation to Euclidean geometry. The reasons for doing this are twofold. First, Euclidean geometry is well-known and is a good foundation for the discussion of a "new" geometry. Second, the geometry of real objects is Euclidean, while the geometry of imaging an object is projective; hence the study of computer graphics naturally involves both geometries.

Controversy

A subject can be declared moribund only when people cease to ask questions and never challenge assumptions or methodology. Controversy is a certain measure of health in research. We are accustomed to announcements of new theories, new interpretations, and public squabbles among scientists as they seek to explain nature's phenomena — revision of old tenets, and even simultaneous acceptance of competing but equally convenient theories is not unusual. But controversy in geometry? That has not happened since the reluctant acceptance in the nineteenth century of non-euclidean geometries as consistent systems apart from Euclidean geometry. In fact, perhaps more so than in any other branch of mathematics, the view of geometry has been one of orthodoxy, ruled by the views of F. Klein's Erlangen program, in which geometry is primarily the study of invariants of transformation groups, or by the influence of 20th century seekers of complete axiomatic systems, perfecting the original Euclid. The narrowness of these confines is being challenged by many.

Among those most vocal is Grünbaum, whose provocative piece "The Emperor's New Clothes: Full Regalia, G string, or Nothing?" decries the arrogance of those mathematicians who will only analyze geometric figures from the standpoint of symmetry groups, and who declare decorative art as "wrong", or a "mistake" if it doesn't fit that scheme. The plea is made to look for other ways to understand and analyse; to look to the motives and methods of the creators of the works. As if to underscore this very point, in the last couple of years scientists have seen nature mock the orthodox geometric model of internal crystal structure, which postulates a periodic repetition of cells, and hence forbids the occurrence of crystals with five-fold (pentagonal) symmetry. Yet imaging technology has revealed that such "crystals" do exist, and now mathematicians, physicists, and crystallographers are scrambling to try to explain how this can occur (see [Steinhart] and [Jaric]). Adding a bit of extra irony, these "quasicrystals" appear to have lattice patterns related to aperiodic tilings discovered by Roger Penrose — about which the symmetry group theory gives absolutely no information, since no symmetry leaves these patterns invariant. This incident also illustrates the fact that so-called "recreational" mathematics (as Penrose's tilings were viewed) is largely a matter of fashion — now researchers are making "serious" attempts at understanding aperiodic tilings. (See [Gardner] and [Grünbaum and Shephard, Chapter 10].)

Open questions

By now it should be apparent that there are more unanswered than answered questions in geometry — even geometry in the Euclidean plane and Euclidean 3-space. It may seem, from the applications and illustrations that I have given, that most are extremely technical in nature, and are difficult to formulate and understand, much less to solve. Of course, many are, but many are deceptively simple to state, and point to how little we really do know about the geometric structure of the space we inhabit. Many are amenable to experimental investigation by students and amateurs — they will yield (at least partially) to patient enumeration, or to ingenious insight rather than to what may be inappropriate and complex mathematical structure and theory.

The subject of packings and tilings is rich with such unanswered questions. Many can be found in *Tilings and Patterns*; I would like to point out just a few which are easy to state.

1. *Describe all of the convex pentagons which can tile the plane.*

Although congruent regular pentagons cannot tile the plane (fill it completely, without gaps or overlaps), there are many pentagons which can be used as paving blocks to tile the plane. But the list of such pentagons has not been proved to be complete. The problem was thought to have been solved by a mathematician in 1918, and again in 1968 by another mathematician, yet each was wrong. After Martin Gardner discussed the problem in his Mathematical Games column in *Scientific American* in July 1975, several new types of pentagon tiles were discovered by amateurs. In addition, in 1976, a high school summer class in Australia discovered all but one type of equilateral pentagon that tiles the plane. (See [Schattschneider: 1978, 1981, 1985].)

2. *If a tile can fill the plane by half-turns only, must there exist a periodic tiling of the plane by that tile?*

Tiles that can fill the plane in a periodic manner using only half-turns were characterized by J. H. Conway; analysing and creating tiles using his criterion is an enjoyable exercise. The question above, however, has not yet been answered. (See [Schattschneider, 1980].)

3. *Does there exist a single tile that can fill the plane only aperiodically?*

The first sets of aperiodic tiles (tiles that can fill the plane only with tilings having no translation symmetry) contained many differently shaped tiles; R. Penrose is credited with discovering the first such set containing only two different shapes. Other sets of two tiles which tile only aperiodically have since been discovered, but still a single tile that does so (or a proof that no such single tile can exist) has not been discovered. (See [Gardner], [Grünbaum and Shephard].).

4. *Which tetrahedra pack space?*

Dissecting simple forms that pack space, such as boxes and prisms, into congruent (non-regular) tetrahedra gives some answers to the question. But the list is far from complete. (See [Senechal].) Related to this question is the more general one: For a given n, describe all convex polyhedra having n faces which also pack space.

5. *Is there an upper bound on the number of faces of a convex polyhedron that packs space?*

It is known that no convex polygon having more than six sides can tile the plane. Although it seems plausible to believe that there cannot be a convex polyhedron which has a great number of faces and also packs space, no one has yet proved it. Amazingly, a convex polyhedron has been found that has 38 faces and packs space. (See [Danzer et al] (the answer to the question posed in the title of that article has been shown to be "no"); see also [Grünbaum and Shephard].)

Conclusion

I hope that the evidence has convinced you that, indeed, the reports of the death of geometry are greatly exaggerated — the reporters have not kept abreast of the many exciting developments which are contributing to its rebirth. The news needs to be spread — colleagues and students need to be made aware of the vitality of geometry.

What can teachers do to help bridge the gap between what is happening on the research frontier and what is learned in the classroom? Osserman ended his address by offering this advice:

> We can initiate and revitalize courses in which students become familiar and comfortable with geometric insights and methods. Perhaps most important and difficult of all is to develop courses where the fragile but vital ability to invoke geometric intuition will be fostered and nurtured. (R. Osserman, ICME IV, 1980)

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Working Group A

Using Computers for Investigative with Elementary Teachers

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Thomas Kieran

All three sessions were devoted to discussions around a LOGO-based software called "Les deux tortues" presented by B. Côté[1] and a set of mathematical activities that it allows. Although the working group had been planned to focus on using computers in the context of elementary teacher training activities, we ended up spending most of our time looking at the mathematical activities presented, and discussing the role of computers in mathematical learning.

The system presented is the result of an effort to build a bridge between computer activities with the LOGO turtle and middle school mathematics curriculum. It is essentially based on two sets of ideas:

1. *Construction and exploration:*

Here the turtle is not used in the context of learning programming. A set of commands are provided that are used in direct mode to produce effects. So we have construction activities that deal with creating objects, usually geometric figures. The notion of procedure is used as a tool to create a bridge between the concrete world of actions and the symbolic level of descriptions of actions as sequences of instructions. The turtle belongs to both worlds. It is a "real" object that we can identify with, that we can simulate with our body or a paper clip. It is also a geometric object, a point that has an orientation. Construction has to do with going from one level to the other by simulating what needs to be done and describing our own actions in terms of instructions, or starting with instructions and simulating them in order to understand why they do what they do.

Questions arise naturally in the context of construction activities. Is it possible to do...? Are there other possibilities? What are all the possibilities? What will happen if...? Can we make a prediction of what will happen if we use such a number, or change that instruction...? The activity of formulating such questions and trying to generate an answer is what we mean by exploration. Construction has to do with "doing". Exploration has to do with "understanding". We have to explain or justify why something is like this or why it is impossible. It is a world of induction and deduction, where we try to establish what is true, what is false.

[1] This project is funded through a research contract between UQAM and APO Québec research centre on uses of computers in education. Hélène Kayler, Lise Paquin and Tamara Lemerise have been involved in the first stage of the project.

2. *Working on paper:*

Working with the turtle at the computer will often result in an "interactive" way of functioning, where the student tries things until it works. This empirical mode of functioning is an important aspect of concrete intelligence and is certainly present in the mathematician's toolbox. But from the point of view of logico-mathematical development, it has to be put under the control of a more "reflexive" mode that works within explicit representations of problems with several kinds of reasoning. (These modes correspond roughly to what Hillel and Kieran (1987) call "visual" and "analytical".)

So we need to set up situations where the interactive mode does not work and the student has to switch to a reflexive mode. We also need to help students extract the mathematical knowledge that is interwoven in their interactive functioning (what Vergnaud (1982) calls "theorems in action"). The notion of turtle is actually a very powerful tool to build that bridge, provided 1) that these activities are overtly identified as mathematical, and 2) that the computer is used as an essential reference point but within a larger space that includes some work on paper. Working on paper means that you have to set up a representation of how the turtle works, that you can test afterwards with the computer. You can also use the computer to gather information that you write down in order to analyse a phenomena and try to understand it. It helps us keep in mind that the goal of all this is to learn mathematics and not particularly to get a computer to work.

Based on these ideas, we have redefined the basic turtle commands in order to facilitate work on square paper, with a metric ruler, protractor and compass. We have also added a set of commands that allow exploration of specific topics like fractions, polygons, integer operations, perimeter and area, motion geometry, variable ... Moreover, because of the central role of the notion of turtle, it was important to shape the basic commands in order to facilitate the understanding of its different aspects. So we ended up with *two turtles*, each with its basic commands and the possibility of working in a fraction or a decimal mode.

The *square turtle* is a simplification of the LOGO turtle that evolves on a square grid. It goes forward in terms of number of squares and turns a fourth of a turn, which allows only four possible orientations. It can also move along any diagonal. The *angular turtle* evolves on a blank plane. It goes forward in terms of centimetres and turns in a fraction of a turn that the user can set up. If we type TOURCOMPLET 360, the turtle turns in degrees. If we type TOURCOMPLET 12, it turns in twelfth of a turn. So we can simulate the square turtle on square paper and the angular turtle with a ruler and a protractor.

The *first session* of the group was spent looking at the basic commands of the two turtles and getting acquainted with the notions of construction and exploration in this context. Much time was spent around an exploration activity related to the command CYCLE, that asks for a sequence of instructions and repeats it until either the turtle comes back to its initial state (in which case it prints the number of repetitions) or it finds that it goes indefinitely away from it (in which case it prints VERS L'INFINI...).

Using this command with the square turtle, we can ask what are the possible answers of the machine and how we can predict them. It is an interesting problem in the sense that:

1) although an empirical approach can help find the possibilities (which are 1, 2, 4 and ∞), it does not help to find out why they are the only ones ; one has to identify what is the relevant factor and try to formulate a prediction rule;

2) one can develop a gradual understanding of the situation; that is, understand some cases before formulating the general rule;

3) the general solution comes from breaking all possibilities into a few categories and solving the problem for each of them. Although usually not obvious, the solution is quite accessible to 5th and 6th grade children (and their teachers) since there are only four possible orientations for the square turtle.

The *second session* was spent mainly discussing fractions and decimals. There are three representations of fractions in the system: turn, length and ratio. The notion of fraction is already involved in relationship with the command TOURCOMPLET. To understand it, we can fold a paper circle to separate it into 8 or 12 equal pieces and use it as a protractor. This makes the link with the traditional "pie or pizza" approach to fractions. We can also type instructions like D (right) 1/3 DE TOUR, that work directly in terms of fraction of a turn. In this context, no special distinction needs to be made between fractions that are smaller or larger than unity. The group discussed different ways to build the operations and some interesting situations like TOURCOMPLET 1/4.

We can also have fractions as lengths. One can make the square turtle move out of its grid by going forward fractions of squares. Using the ruler with the angular turtle, one comes naturally to want to express centimetres and millimetres, which is done with decimal numbers. The command POTEAU helps compare lengths and so create activities where one goes from fractions to decimals and vice versa. The command FUSÉE uses a fraction to specify the path of a rocket. It is basically the slope interpretation where the numerator is associated with the vertical component of a move and the denominator with the horizontal one. This creates activities on equivalence and order that promote the development of qualitative reasoning on fractions as ratios.

The discussion went around the notion of *microworld*. Is it a useful concept? Does it cover almost any software that is not based on direct teaching? In this case, we can talk about the system including several microworlds; that is, commands that create activities around a well defined topic. We can also think of domains of knowledge, for instance fractions, as microworlds. This is a way to see knowledge as a dynamic entity made out of a network of elements integrating formal and concrete aspects in a way that has to function. From this point of view, each learner has to build his own network.

The *last session* started with trying to characterize construction as *object formation* and exploration as *relating variables*. Although the activities with the two turtles start as construction of figures, the notion of construction gets eventually a larger sense. In order to build a figure, one has to choose the right command, with the appropriate number (build an instruction), formulate a sequence of instructions (build a procedure). Through the exploration activities, one has to manipulate objects like numerical and algebraic expressions, to build geometrical transformations like translation or rotation, to formulate rules ...

Exploration activities are generally based on some classification of objects. We have a set of commands or characteristics of commands, that can be put into categories, and a set of results, that can also be put into categories. What needs to be done is to formulate the relationship between variations in the command side and variations in the turtle side. For example, we might ask what will happen if we tell the square turtle to turn of a number larger than 4. All the possible turning instructions can be divided into four categories according to their end result and the question is which object (instruction) belongs to which category (orientation). We have the same thing with CYCLE where on one side we have procedures and on the other side number of repetitions (or ∞); with FUSâE, where on one side we have fractions, and on the other side the same or different paths, or above, below or equal to the middle path, or general ordering in terms of steepness of slope. We could have on one side possible items of addition and subtraction of integers (classified in terms of $^{+}+^{+}$, $^{+}+^{-}$, ... , $^{-}-^{-}$) and on the other side the interpretation in terms of turtle move. Or we can have on one side the regular polygons divided into normal and stars, and on the other side the sequences of instructions that generate one or the other.

The session ended with a general discussion on a question raised throughout all workshop by S. Dawson: do mathematical activities defined around computers induce a reduced view of mathematics, in particular, and the real world in general?

Much debate throughout the three sessions focused on the supposed neutrality of the computer, a question centrally addressed by C. A. Bowers in his recent book *The cultural dimensions of educational computing*.

> "The question has to do with whether the technology is neutral: that is, neutral in terms of accurately representing, at the level of the software program, the domains of the real world in which people live. If the answer to this question is that it is not neutral, the critically important question of how the technology alters the learning process must be addressed." (Bowers, p. 24)

In particular, computers foster a digital, dichotomous, context-less, ultra rational form of world view, which though extremely productive in many ways, is also at the foundation of many misunderstandings about the world. To paraphrase Gregory Bateson, if we separate an object from its context we are likely to misunderstand it. Yet computer educators perpetuate the view that the computer is culturally neutral, that it is simply a 'dumb' machine.

But this overlooks the fact that "...the classroom strengthens certain cultural orientations by communicating them to the young and weakens others by not communicating them." (Bowers, p.6)

Bowers goes on to say:

> "By interpreting rationality, progress, and efficiency in terms of technological achievements, this mind-set has developed the hubris that leads to viewing the ecological crisis as requiring a further technological fix rather than the recognition that our most fundamental patterns of thinking may be faulty". (Bowers, p.8)

Much debate throughout the three sessions focused on the supposed neutrality of the computer and of Logo.

The conclusion which Bowers draws, noted below, was hotly debated:

> "Thus the machine that the student interacts with cuts out of the communication process (the reduction phenomenon) tacit-heuristic forms of knowledge that underlie commonsense experience. While the technology amplifies the sense of objectivity, it reduces the awareness that the data represent an interpretation influenced by the conceptual categories and perspective of the person who "collected" the data or information. The technology also reduces the recognition that language, and thus the foundations of thought itself, is metaphorical in nature. The binary logic that so strongly amplifies the sense of objective facts and data-based thinking serves, at the same time, to reduce the importance of meaning, ambiguity, and perspective. Finally, the sense of history, as well as the cultural relativism of both the student's and the software writer's interpretative frameworks, is also out of focus. As a symbol-processing technology, the computer selects and amplifies certain aspects of language... " (Bowers, pp 33-34)

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Working Group B

Computers in the Undergraduate Mathematics Curriculum

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Session I

At Brock University, the Department of Mathematics and Statistics has established an undergraduate computer laboratory as an instructional aid in teaching various undergraduate mathematics courses, particularly introductory calculus courses. This laboratory contains thirty MacIntosh-SE desktop computers linked to an overhead video display unit. Working Group B was able to take advantage of this facility for some hands on experience.

During the first session, held in the laboratory, Eric Muller of Brock University presented a brief overview of the lab set-up and how it is utilized. The symbolic manipulation program MAPLE, developed at the University of Waterloo, is the computer environment in which sessions are conducted. Eric indicated that although MAPLE was not developed primarily for educational use, it is being used by a number of universities in the teaching of undergraduate mathematics. The availability of other software designed for specific educational use was mentioned.

The first session continued with a demonstration of some of the capabilities of MAPLE by Stan Devitt. Participants were given the opportunity to experience the considerable power of MAPLE as a calculator. The ability of the system to carry out routine as well as complex calculations was demonstrated. As a result, participants gained some appreciation of the capabilities of MAPLE as an instructional aid and this resulted in a discussion of some of the implications of this technology for teaching.

Stan Devitt indicated that the primary objective of current efforts to incorporate computer algebra systems (CAS) such as MAPLE in undergraduate mathematics instruction is to build an environment in which all so-called paper and pencil calculations can, with appropriate commands, be carried out on a computer screen. He suggested that in order to reach this objective, it will be necessary to design special routines so that students can easily utilize the full power of the system. For example, special routines, perhaps fairly advanced in nature, are necessary in order for students to realize the full potential of CAS as an aid in problem solving in areas like Linear Algebra and Number Theory.

Generally, during this session, participants had the opportunity to play around with the system and become familiar with some of its capabilities and potential problems. Even trivial problems such as how to change an expression after entering it were evident.

After coffee break, Stan Devitt gave a demonstration lesson using MAPLE. He indicated how the software evolved and introduced some of the commands, such as those for finding summations, evaluating definite integrals, and for performing numerical integration using Simpson's and the Trapezoidal Rule. Also, the use of computer graphics in estimating the area under a curve was demonstrated.

Towards the end of the session, several issues were raised by participants relative to possible implications of this technology on the mathematics curriculum. In particular, questions dealing with the evaluation of student learning and how to incorporate computer algebra systems such as MAPLE in the mathematics curriculum were discussed. The need to address such issues in a meaningful way was emphasized. The need to know what has worked well to date in the use of CAS and the need to identify some of the problems not just the advantages was emphasized.

Session II

At the beginning of the second session, Stan Devitt provided the group with some anecdotal experiences resulting from his own attempts to incorporate CAS in undergraduate mathematics courses. He pointed out that even though CAS have been around for some time, to date such programs have had very little evident impact on undergraduate teaching. One of the first available CAS programs was MACSYMA, developed at MIT and available on mainframes about 1980. MAPLE and other CAS programs were subsequently developed in an attempt to reduce the large amounts of computer memory that such programs require, and thus make the capabilities of CAS available to a much wider audience.

In 1986, the Sloan Foundation provided funding to eight institutions to establish computer laboratories using computer algebra systems. Included were the University of Waterloo and the University of Saskatchewan, both of which are using MAPLE. Other institutions are using different systems, such as Mu-Math at the University of Hawaii. These projects are now underway and workbooks have been produced. In fact, participants of this working group each received a copy of "Calculus Workbook; Problems and Solutions", compiled by Stan Devitt for the project now underway at the University of Saskatchewan.

The collective experience of the institutions funded by Sloan was reviewed at a conference held at Colby College in the summer of 1988. It was a disappointment to some that several of the projects were just getting underway after the initial eighteen month start-up period. Also, institutions reported varying experiences. For example, the reaction of students using CAS was not as positive as expected. Some students reported that they experienced more difficulty using CAS than with traditional instruction. On the other hand, most faculty members involved in these projects indicated that they would not consider teaching undergraduate mathematics without using CAS. In summary then, there appeared to be moderate disappointment with the extent to which progress had been made in implementing CAS into the undergraduate mathematics curriculum of the participating institutions, and some disappointment at the initial reaction of students exposed to CAS in their courses.

By way of elaboration on the above, Stan Devitt explained that at the University of Waterloo, where the MAPLE project has been underway for the past eight or nine years, it is not generally being used by faculty members in their teaching. Also, students at the University of Waterloo indicated that they were under a lot of pressure to get through their assigned work and the use of CAS meant additional work and material to cover.

At other universities, however, there was a more positive reaction. At Dennison, all students enrolled in undergraduate mathematics courses receive instruction in a computer laboratory environment. Also, at Brock University, all faculty members in the Department of Mathematics are involved in computer labs. However, at the University of Saskatchewan, with 30 members in the Department, only three members were seriously investigating the potential of CAS.

One explanation for the apparent lack of interest on the part of some faculty members is the fact that most are busy people and are not willing to invest large amounts of their limited time unless there is some evidence that the result will be worthwhile. Clearly, some faculty remain unconvinced that the result is worth the effort, and it is clear that much more thought and effort will be required before CAS can become widely accepted.

The above summarizes some of the comments of Stan Devitt at the beginning of the second session. Eric Muller then gave an overview of the Brock experience. He indicated that the original objective was to develop over a three year period, computer labs for all service courses offered in the Department. In the first year, VAX MAPLE was used by 100 out of 110 students enrolled in such courses, with students meeting in compulsory lab groups of 15. At the beginning of the second year, 30 Macintosh-SEs were purchased and used in the laboratory, with approximately 600 students now using CAS in the computer laboratory.

At the end of each year, a questionnaire was administered to participating students dealing with their attitudes toward the use of CAS. There were some obvious differences in the responses of the first group (1988) compared with those of the second group (1989). For example, 47 percent of the students in 1988 rated CAS as a good learning aid while 16 percent rated it poor. In 1989, the corresponding percentages are 11 and 67. Similar results were reported on such measures as confidence to do mathematics and enjoyment of mathematics. The course in which these students were enrolled was a traditional calculus course with applications.

In attempting to explain such results, it has been suggested that "better" students perform at a higher than normal level using CAS while weaker CAS students perform below normal.

Eric Muller then described how the computer lab at Brock was set up. He reviewed some of the practical considerations that received attention. For example, who is responsible for each lab session and who should be present in the lab with the students? At Brock, it was the practice to have one faculty member and one senior student (familiar with MAPLE) associated with each lab session. Each week, students would receive prior to the lab session a sheet of questions. A total of 29 lab sessions of one hour duration were scheduled over the 40 hour period per week available with about 28 students per session. There was a network server for each 10 machines in the lab (a total of 30 machines in the lab).

It was evident that using CAS resulted in changes to the style of teaching. There were more question and answer sessions than traditionally. However, in the lab setting, many of the questions were of a technical nature having to do with how to use the system to solve problems. There was open access to the terminal room during the semester and at the beginning of the year some introductory sessions outside of class time were scheduled to familiarize students with the system.

It was also evident that students at Brock preferred using the Macintosh to the VAX. However, one complaint, especially in multi-sectioned courses, was that some of the weekly assignments could be completed without the use of the computer and hence students did not see the need for the computer lab. This type of problem, however, seems to be one that could be solved if all faculty members teaching a course could agree on the nature of assigned work.

With respect to the attitudes of students using MAPLE relative to those of students in sections of a course not using MAPLE, it was reported that at Saskatchewan the drop-out rate in the MAPLE sections was higher. One explanation offered for this was that MAPLE students were left on their own more so than the others and the consequent lack of feedback when needed may have caused students to quit rather than persevere. In fact, the reaction of students left in the lab on their own was often very negative.

Some participants, as a result of the above discussion, questioned what possible good was resulting from this effort to incorporate CAS in the teaching of undergraduate mathematics. Did the costs justify the results? Is the use of computer/calculator technology being driven by a stick or a carrot? It was suggested that before many questions could be answered, there was the need for research on the impact of the technology in the classroom, and the only way to do this was via controlled experiments rather than anecdotal reporting of experiences.

Some of the drawbacks of the MAPLE system were mentioned. For example, the lack of a good graphing package and the fact that the user interface is not one that is very user-friendly. It was speculated that some of these problems would be addressed in future developments of the program. For example, a menu driven interface would improve matters considerably. One suggestion was that there could be developed an educational version of MAPLE to complement the scientific version. This led to a discussion of the pros and cons of MAPLE as opposed to a discussion of the pros and cons of symbolic algebra systems in general.

Session III

At the beginning of the third session, the group convened once again in the computer laboratory at Brock. Various reference materials were distributed. The session continued with a typical in-class CAS demonstration by Stan Devitt on limits and continuity.

The Group then reconvened for a group discussion. Eric Muller described the nature of an applied calculus course offered as a service course at Brock to non-math majors. A brief outline of the course was presented: functions, special functions, limits, continuity, differentiation, anti-differentiation, definite integrals, differential equations, probability distributions, and partial differentiation. In response to a question, Eric indicated that integration was not introduced as a limit of a sum, to which the question Why? was posed. This line of discussion raised the following questions: When is CAS a tool to help concept development? and When is it a tool just to compute? Where does one learn when to use an algorithm? This resulted in some discussion about the type of student being taught, that is math versus non-math students.

Perhaps the most interesting question posed was this: If the computer can draw pictures and compute derivatives, etc., why would a student have to learn any of this? How do we as mathematics educators deal with this question? Is there any attempt to try and show students that there are things in mathematics that the computer cannot do? The suggestion was that we need to give good examples to students that illustrate when it is (a) stupid, (b) hopeless and (c) inappropriate to use the computer. Perhaps good thoughtful examples to address the above questions would indicate to students why theory is so important in mathematics.

The end result of this question was: How do we teach intelligent uses of the computer? and Why is it important that we teach intelligent uses of the computer? The point was made that certainly the domain of computation in college courses is different than in the past or at least it should be. The discussion ended with some comments on potential dangers of using CAS in the teaching of undergraduate mathematics or a least a realization that if used inappropriately, certain undesirable outcomes may result. Again, the issue of the apparent negative attitudes of those students whom, we might assume, stand to benefit most from using CAS was raised. Also, the need for extra time perhaps to use CAS effectively.

In summary then, a synopsis of the activities of Working Group B is as follows:

1. A discussion of a philosophy of teaching mathematics using Computer Algebra Systems.

2. An overview of CAS in general, providing an awareness of the current state of the art and what efforts are underway to integrate CAS into the teaching of undergraduate mathematics.

3. An opportunity to experience in a laboratory setting how CAS can be used in a teaching situation with an actual demonstration of a lesson in introductory calculus.

4. An opportunity to become familiar with a Calculus Workbook incorporating CAS, produced at the University of Waterloo.

5. An overview of several projects at other universities that have been initiated since 1986 with the assistance of grants from the Sloan Foundation.

6. An overview of the Brock University experience of using CAS in the teaching of undergraduate calculus courses.

7. An indication of some of the problems associated with the implementation of CAS in undergraduate teaching, including the attitudes of students and faculty.

8. A look at what is likely to happen in the future. For example, the conclusion that the implementation of CAS requires a great deal of effort and planning for little evident initial payoff.

9. The opportunity to obtain a number of articles on CAS for retention and further use.

In conclusion, it is obvious that Working Group B accomplished much in a short time. However, it is also clear that as many questions were raised as were answered. It seems that before we can integrate CAS generally into the teaching of undergraduate mathematics, there is a need for much more thought, discussion, and investigation. There is no doubt that the availability of CAS has the potential to change dramatically how we teach and what we teach. It has the potential to remove much of what we might call the drudgery of elementary mathematics. However, care must be taken in the design of CAS based curricula that we do not replace one form of drudgery with another form that may be perceived by students to be equally distasteful.

There was a clear indication that CAS has a great deal of potential but at the same time that it can never be used to teach some of the fundamental understandings that are required of one whom we might classify as a mathematically literate person. Perhaps one of the important benefits of using CAS in undergraduate teaching is to make available to instructors more time to concentrate on some of the essential ideas and concepts of mathematics than is available at present.

The need for the development of good research in this whole area was also evident. Controlled experiments on the effects of CAS on mathematics learning and retention seems to be called for before we jump on any bandwagon. The need for major curriculum reform efforts appear warranted and perhaps this should happen in any event. The past practice of permitting textbook writers to essentially determine the curriculum in calculus and other undergraduate mathematics courses, need not continue. It is possible with desktop publishing and sophisticated word and text processing capabilities for individual departments to produce their own curriculum materials and not depend on increasingly expensive and perhaps inadequate commercially produced textbooks.

In summary, this session proved to be interesting, informative and timely. Special thanks go to Stan Devitt for sharing his considerable experience with the group and to Eric Muller for superb local arrangements at Brock University, including of course, the use of the computer lab which made the session more than a speculative discussion group.

Working Group C

Natural Language and Mathematical Language

David Pimm

The Open University

Participants

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Brian Calvert	David Pimm
François Conne	John Poland
Don Eastman	Joan Routledge
George Gadanidis	Constance Smith
Bill Higginson	Ralph Stahl
Gary Lessard	Lorna Wiggan

> Science begins with the world we have to live in ... From there, it moves towards the imagination: it becomes a mental construct, a model of a possible way of interpreting experience. The further it goes in this direction, the more it tends to speak the language of mathematics, which is really one of the languages of the imagination, along with literature and music.
>
> (Northrop Frye, *The Educated Imagination*)

The descriptive advertisement for the three sessions went as follows.

> The group will examine aspects of communicative and other functions of language used in the service of mathematics and mathematicians. It will have a partly historical, partly linguistic and partly mathematical focus, exploring some of the means by which mathematical ideas are expressed and ways by which neophytes are encouraged to increase their command of the mathematics register.
>
> Further possible topics for discussion include the notions of metaphor and metonymy and their uses in mathematics as means for the creative extension of the expressive potential of language for the invention and control of mathematical notions.

I started the first session by attempting to share some of my current worries and concerns with the rest of the group. The first was the myth of learning by experience and the relation of language to that experience (see Pimm, 1986, in reply to Liebeck, 1986): in particular, the passive role often attributed to language in merely describing or representing experience, rather than being either a constituent component of the experience or the experience itself.

The second was an over-narrow conception of meaning in mathematics in terms of reference rather than connections in both form and content, and meaning in this restricted sense being claimed to be the most important, indeed only goal of mathematics teaching. In England, at least, an increasingly common dogma is if in doubt at any stage in anything mathematical, then told to go back to the 'meaning' (often the concrete) from which everything is presumed to stem. Valerie Walkerdine (1988) has recently drawn attention to the implausibility of such an account in the case of the teaching of place value. She offers a much more telling if complex account, one that intimately implicates the teacher's language and positioning within classroom activity. "Signifiers do not cover fixed 'meanings' any more than objects have only one set of physical properties or function" (Walkerdine, op cit., p. 30).

In an article entitled *On Notation*, Dick Tahta has claimed (1985, p. 49) that:

> We do not pay enough attention to the actual techniques involved in helping people gain facility in the handling of mathematical symbols. ... In some contexts, what is required - eventually - is a fluency with mathematical symbols that is independent of any awareness of current 'external' meaning. In linguistic jargon, 'signifiers' can sometimes gain more meaning from their connection with other signifiers than from what is being signified.
>
> Linguists have called the movement 'along the chain of signifiers' *metonymic* whereas 'the descent to the signified' is *metaphoric*.

The third concern I mentioned was one recently raised by Tahta (at the 1989 ATM Easter conference) of the current trend towards only stressing how we (or pupils) *differ* from one another, rather than what we have in common. How can we endeavour to develop ways of working together in relation to the learning of mathematics? One particular fear Tahta expressed was of the loss of consensus and commonality as a result of overemphasis on individual differences, with resulting isolation and lack of community. (I'm sure you will appreciate the political background of these concerns - in particular, following a decade of Thatcherism and the attempted wholesale destruction of collectivism at any level, whether inside education or outside it.)

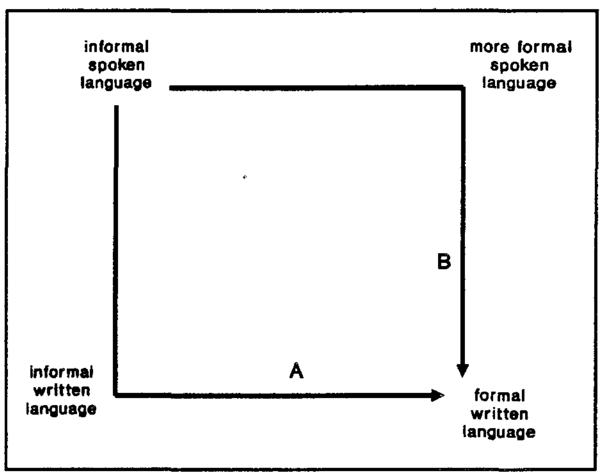
Spoken language is one of the things that we share in common to a marked extent. It is socially acquired by considerable individual effort and little overt teaching. Language exists as a cultural repository, but also as a magnificent resource into which we can tap. A language both reflects and shapes the conceptual framework of its users. We can ask how thought is constituted in terms of and in relation to a system of signs, which by definition are social.

One way of describing the relation between mathematics and a natural language such as English is in terms of the linguistic notion of register. Linguist Michael Halliday (1975, p. 65, my emphasis) specifies this notion as 'a set of meanings that is appropriate to a particular *function* of language, together with the words and structures which express these meanings'. One function to which a language can be put is the expression of mathematical ideas and meanings, and to that end a mathematical register will develop.

Thus, while providing pupils with opportunities to gain access to the resources implicit in natural language can be seen as a common aim of all teachers (one interpretation of the 'language across the curriculum' idea), a particular aim of teachers of mathematics should be to provide their pupils with some means of making use of the mathematics register for their own purposes. To that end, a mathematics teacher needs knowledge about the language forms and structures that comprise aspects of that register. Part of learning mathematics is gaining control over the mathematics register so as to be able to talk like, and more subtly, to mean like a mathematician.

For these sessions, we were mostly in the realm of the signifier, and tried to explore to what extent signifiers can be used relatively autonomously from the signified they are taken to represent. In the second session, we worked on two classroom excerpts on videotape: Anne Tyson with a class on base five arithmetic and Irene Jones with a class working on a geometric poster (both from the Open University videotape PM644 *Secondary Mathematics: Classroom Practice*). In both cases, the pupils and adults were clearly engaged in a discussion - but about what? Where were the referents for what they were discussing - to what is the language pointing?

There are a number of different characteristics and functions of spoken and written language. One use of written language is to externalise thought in a relatively stable and permanent form, so it may be reflected upon by the writer, as well as providing access to it for others. One characteristic of written language is the need for it to be self-contained and able to stand on its own, with all the references internal to the formulation, unlike spoken language which can be employed to communicate successfully when full of 'thises', 'its' and 'over theres' due to other factors in the communicative situation.



One difficulty facing all teachers is how to encourage movement in their pupils from the predominantly informal spoken language with which they are all pretty fluent (see Brown, 1982), to the formal written language that is frequently perceived to be the hallmark of mathematical activity. There seem to me to be two ways that can be tried. The first (and I think far more common) is to encourage pupils to write down their informal utterances and then work on making the written language more self-sufficient (Route A in the diagram), for example by use of brackets and other written devices to convey similar information to that which is conveyed orally by stress or intonation.

A second route to greater control over the formal written mathematical language (shown as B in the diagram) might be to work on the formality and self-sufficiency of the spoken language prior to its being written down. In order for this to be feasible, constraints need to be made on the communicative situation in order to remove those features that allow spoken language to be merely one part of the communication.

Such situations often have some of the attributes of a game, and provided the pupils take on the proposed activity as worthy of engaging with, then those pupils have the possibility of rehearsing more formal spoken language skills. One such scenario is described by Jaworski (1985), where the focus of mathematical attention is a complex geometric poster. Pupils are invited to come and out and 'say what they have seen' to the rest of the class, under the constraints of 'no pointing and no touching'. These help to focus the challenge onto the language being used to 'point' at the picture. The situation is an artificial one: in 'real' life, one can often point and this is completely adequate for effective communication. However, if the artificiality is accepted by the pupils, natural learning can take place that would otherwise not have been so readily available. There is an interesting .paradox here, one of how quite artificial teaching can give rise to natural learning under certain circumstances.

A second instance of such an approach comes from the contexts of 'investigations', when pupils are invited to report back to the class what they have done and found out. Because of the more formal nature of the language situation (particularly if rehearsal is encouraged), this can lead to more formal, 'public' speech and structured reflection on the language to be used. Thus, the demands of the situation alter the requirements of the language to be used. Reporting back can place some quite sophisticated linguistic demands on the pupils in terms of communicative competence - that is, knowing how to use language to communicate in certain circumstances: here, it includes how to choose what to say, taking into account what you know and what you believe your audience knows. A further example of these demands at work can be seen in the study by Balacheff (1988) on thirteen- year-old pupils' notions of proof, where he asked them in pairs to write down their claims about a mathematical situation to tell another pair what they had found out. By providing them with some plausible justification for them writing a message, he was able to gain access to their proficiency in this matter.

Educational linguist Michael Stubbs writes (1980, p. 115): "A general principle in teaching any kind of communicative competence, spoken or written, is that the speaking, listening, writing or reading should have some genuine communicative purpose". Pupils learning mathematics in school in part are attempting to acquire communicative competence in the mathematics register, and classroom activities can be usefully examined from this perspective in order to see what opportunities they are offering pupils for learning. Teachers cannot make pupils learn - at best, they can provide well-thought out situations which provide opportunities for pupils to engage with mathematical ideas and language.

For the third session, a couple of dynamic mental geometry activities were offered (see Beeney *et al.*, 1982, for further school examples), including the pole/polar construction between a point outside a circle and the two tangents to the circle passing through it. What happens when the point moves inside the circle?

In conclusion, the following quotation from the Second World Conference on Islamic Education (1980) was offered, which was their justification for the compulsory teaching of mathematics in school.

> The objective [of teaching mathematics] is to make the students implicitly able to formulate and understand abstractions and be steeped in the area of symbols. It is good training for the mind, so that they [students] may move from the concrete to the abstract, from sense experience to ideation, and from matter-of-factness to symbolisation. It makes them prepare for a much better understanding of how the Universe, which appears to be concrete and matter-of-fact, is actually *ayatullah*: signs of God - a symbol of reality.

Items which stood out for me during the discussion

A discussion of Helen Keller and her realisation by means of associating the running of water over one hand with a pattern being repeated tapped into her other of the *possibility* of symbolisation (the juxtaposition being essential in the creation of a sign - and the notion of sign itself) and her subsequent rapid 'linguistic' progress by demanding the symbols for many objects or phenomena. Valerie Walkerdine, in *The Mastery of Reason*, asks a fundamental question which has particular salience for mathematics teaching: "How do children come to read the myriad of arbitrary signifiers - the words, gestures, objects, etc. - with which they are surrounded, such that their arbitrariness is banished and they appear to have the meaning that is conventional?" This called to mind how we tend to project our understanding onto the symbols which can then trigger those meanings subsequently. We read the meanings into the symbols, and yet the projection can be so strong that we forget that the external manifestation is only the signifier and not the sign.

Being aware of structure is one part of being a mathematician. Algebraic manipulation can allow some new property to be apprehended that was not 'visible' before - the transformation was not made on the meaning, but only on the symbols - and that can be very powerful. "The sign $\sqrt{-1}$ represents an unthinkable non-thing. And yet it can be used very well in finding theorems." Johann Lambert, in a letter to Immanuel Kant.

Where are we to look for meaning? Self-reference *is* reference. Mathematics is at least as much in the relationships as in the objects, but we tend to see (and look for) the objects. Relationships are invisible objects to visualise. Caleb Gattegno, writing in his book *The Generation of Wealth* (p. 139), claimed:

> My studies indicate that "mathematization" is a special awareness, *an awareness of the dynamics of relationships*. To act as a mathematician, in other words, is always to be aware of certain dynamics present in the relationships being contemplated. (It is precisely because the essence of mathematics is relationships that mathematics is suitable to express many sciences.) Thus, it is the task of education in mathematics to help students reach

> the awareness that they can be aware of relationships and their dynamics. In geometry, the focus is on the relationships and dynamics of images; in algebra, on dynamics per se.

Mathematics has a problem with reference so it tends to reify its discourse in order to meet the naive desire for reference. "The questions 'What is length?', 'What is meaning?', 'What is the number one?', etc. produce in us a mental cramp. We feel that we can't point to anything in reply to them and yet ought to point to something. (We are up against one of the great sources of philosophical bewilderment: a substantive makes us look for a thing that corresponds to it.)" Ludwig Wittgenstein, *The Blue and Brown Books*.

'I can count faster than I can skip.'

There is an important difference between *wanting* to follow and *having* to follow the teacher. What is the teacher's role and responsibilities in attempting to create meaning for her students? Is it a pretence for the teacher not to be an authority? Who is the custodian of truth in a mathematics classroom?

Finally, two quotations about symbols:

> Civilisation advances by extending the number of important operations we can perform without thinking about them.
>
> (Alfred Whitehead, *Science in the Modern World*)

> Underlying the notations of mathematics there are verbal components; so the mastery of the spoken language means that it is possible to base mathematics on language.
>
> (Caleb Gattegno, *The Awareness of Mathematization*)

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Working Group D

Research Strategies for Pupils' Conceptions in Mathematics

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The report contains four parts and three appendices:

A) A summary of the discussions (C. Janvier and R. Borasi)
B) Crucial questions raised (R. Borasi)
C) Short descriptions of some presentations (R. Borasi)
D) A bibliography collected by Claude Janvier, annotations for 3 provided by R. B. Appendix A, B and C

A) Summary of the Discussions

Our discussions on conceptions had taken place in a constructivist theory perspective: a theory in which individuals are actively involved when learning. Activity is two-fold: the individual decides to enter into the process of learning and he/she has to integrate into his/her past knowledge the new elements making the resulting knowledge a personal construct. Conceptions are important to consider in mathematics instruction because they influence such mental activities.

The discussions have shown that we could distinguish (for each individual) **cognitive conceptions** from **belief systems**. **Cognitive conceptions** could be considered as elements triggering the action in mathematics reasoning beyond or underneath a set of mathematical concepts. **Belief systems** can be regarded as a set of judgments that control the action of the individual in the sense that they determine his/her willingness to engage, to remain engaged and define ways of engaging (continuity, multiplication, circle...).

Note: Even though such a distinction was discussed and commented upon, **belief systems** and **cognitive conceptions** are not always distinguished in the summary. Firstly, the group has not analysed and described their difference. Secondly, it appeared all along the discussions that, most surprisingly, participants could argue their points and agree having in mind one concept or the other. We note then that the term conception is general enough so as it can convey the idea of beliefs.

Among the **belief systems**, it has seemed relevant to distinguish:

the one about the self
the one about mathematics
the one about school

Combinations of these such as those listed below are important:

self and mathematics,
mathematics in a school setting
school and self,
self and school mathematics.

In the group discussions, we have further enriched the following points: the beliefs about the self and the ones about mathematics (interpreted as of what mathematics is and what doing math is), the ones about school (its implicit role). We have also discussed about the self and mathematics (personal judgments and contrasting them in math and other disciplines), mathematics in a school setting, school and self (general history of success or failures in the regular school program and the expectations derived), self and school mathematics (specific success or failures in mathematics and the expectations derived).

It was proposed to consider conceptions as mental constructs induced by the observers (self-observation included) on the basis on specific behaviour (action and discourse) on the part of the subject. As a result, it is no easy matter to identify and describe a conception. It is important to distinguish between the **individual** conceptions and the **more general** categories that can link several individual conceptions. The latter are more abstract in nature. For instance, mathematics viewed as a set of rules to obey may be concretized differently in each individual. Equally of importance is the fact that conceptions are difficult to imagine without a theory that organizes the observations made with or on a specific individual.

Conceptions in the teacher-student relations

During the group discussions, it became clear that when considering conceptions relevant to mathematics education, teachers conceptions, students conceptions and the relations between these two categories should be considered.

The following three paragraphs are a personal version of the exchanges of ideas. (C.J.)

It has been suggested that when we envisage the teaching-learning relations between the teachers and the students, we must consider **STUDENT COGNITIVE CONCEPTION (SCC)** and **STUDENT BELIEF SYSTEMS (SBS)** not only per se but also as they are an integral part of the **TEACHER BELIEF SYSTEM (TBS).**

It could be interesting to denote the teacher's version of SCC and SBS as SCC' and SBS'. This part of the **TEACHER BELIEF SYSTEM** also controls the action of the teacher as he/she interacts with students in classroom situations.

If the change of TBS becomes a concern, then one must minimally consider in addition a new variable the **TEACHERS TRAINERS BELIEF SYSTEMS**. Note the importance of this new category.

Are there right or wrong conceptions? Changing them!

The dichotomy right or wrong appears to be incorrect. In fact, the word functional describes more clearly what an appropriate conception is since one can evaluate a conception only in relation with the effect it has in achieving a set purpose. The essential factor is the fact that no value should be attached to a conception in absolute terms.

For example, if we take conceptions about what mathematics is, there should be some room for an informal kind of mathematics that would be distinguishable from "official mathematics". The idea becomes much more to focus our attentions on the mathematics activities such as reasoning, generalizing, formulating hypotheses... If one imagines that official mathematics results from an understanding between mathematicians, mathematics educators and mathematics teachers, one needs that informal mathematics by accepted as valuable by learners. **This is partly what has to be changed.**

If conceptions need to be changed, it must not be forgotten that teachers and students stay actors within the **school framework** constituting a system. And it is clear that taking into account the students' belief systems in the organisation of mathematics teaching would have to produce results within the actual school system. Perhaps, **assessment in schools** should be adjusted.

Changing the students' conceptions required that first of all they **become known** to the teachers or the researchers. **How** can we determine conceptions? More, **from the actions** then from the dialogue? But anyhow, how much do we need to know about students' specific and individual conceptions since similar past experiences will produce similar conceptions?

Should the students become aware of their own conceptions as a starting point for changing them? In the process of change in students' conceptions, should the teacher expect specific conceptions as goals? Should he/she consider replacement or adding something stronger? It would mean a **certain discontinuity** among conceptions: one being underivable from the others.

At any rate, the working group has agreed that conceptions cannot be directly taught, but rather developed or formed (implicitly or explicitly) in the individuals on the basis of experiences. Individuals are partially aware of their conceptions in the sense that they can only make a partial explicit account of them when solicited.

Acting on the conceptions cannot be achieved without taking into account the ways they develop. As a consequence, we cannot hope to change conceptions only by talking people into them or by teaching them directly. Individuals must be confronted with relevant or meaningful experiences.

It means that changing a belief system consists perhaps in introducing the seed for a new conception to emerge and that, as a result, the subject will be faced with a multiplicity of conceptions "available". This will imply on the part of the subject some abilities to **discriminate and choose how and when to resort to them.** Then the notion of **context awareness** appears to be of prime importance.

During the last sessions , we turn to the questions asked in the description of the working group work appearing in the announcement.

Difficulties involved in research and otherwise

Finding out a belief or a conception in children is time consuming and many teachers are not willing to envisage that it can be worthwhile. On the other hand, as we have said previously conceptions belong to a theory that is the mental framework enabling the researcher and the teacher to detect them. Many have claimed that the presence of a particular conception cannot be assessed if one has not been prepared mentally to notice it and, even then, the fact that a conception is effectively active remains a hypothesis.

Moreover, it is never sure whether a conception does belong to a more general conceptual system, a fact that would be more important for its pedagogical consequences. Also, conceptions are constantly changing and what can be really observed is not the presence of a conception but mainly the movement of conceptions, and the sudden action of one particular conception while the others are likely to be activated but not in action at a particular moment. In fact, we are back to the notion of an efficient model which requires the recourse to an appropriate conception among others.

The formulation or discovery of new conceptions by researchers does not seem to bring about unanimity in the group. On the one hand, some members of the group believe that the formulation of prior hypotheses and the relationships discovered between the previously analysed variables will lead **necessarily** to the conceptions that are involved in the more or less explicit a priori analysis. Others took more optimistic stands. Even though they agree that there is a discontinuity between the previously selected variables and the new variables, some people are able to reach the level of creativity needed for the discovery of a conception.

Are the conceptions personal or do they belong to a category of students?

Changing them

The conflict seems to be the "natural" technique. It involves that the teacher should introduce some facts or events that will clash with or contradict the conception held by the student(s). This method clearly depends on the capacity of the student(s) to be receptive to contradiction. Several examples were provided of students supporting contradictory positions. For instance, a few cases were reported of students believing that a specific fact could be false in arithmetic and true in algebra. In other words, mathematics for many is governed by a "special" logic (or by an absence of logic) which makes the contradiction that the teacher can see or appreciate strictly out of reach of the children.

As far as changing the conceptions is concerned, the "necessary but not sufficient reason" principle was very often mentioned. This was the case for having the students talk about the contradicting fact which is often either neglected or accepted with special sorts of reasoning. This was also the case for the reflection made possible via the use a daily journal. Even the list of key words that are slowly arrived at does not guarantee that the contradiction will be assumed. It is clear that the process requires two phases or stages: first the actions (and done meaningfully) and then the rejection often helped by the contradiction.

Reflection leads to awareness and then the chances that they will use their will to do it is magnified. One needs a motivation to deal with the contradiction. One often accepts things as they are and one doesn't mind since changing would be too costly for several reasons. In fact, there are always many things any individual doesn't understand. Consequently, there is nothing surprising in the fact that the contradiction is not the powerful tool to resolve issues as we would like it to be.

The interviews can be nice (a fruitful and efficient tool) because the students observe themselves. The actions during interviews are more meaningful and some participants think that the contradictions are thus more efficiently made explicit. However, it is not easy for the teachers to make the right moves and conduct interviews adroitly.

As far as the research goes, the word constraint is more appropriate in the circumstances than the vocable difficulty because it reflects the fact that there will always be a limit to the capacity of any research tool. Consequently, one should try to use a research approach that will maximize the outcomes in view to the objectives that are far from being unique.

Personal conclusions (C.J.)

The whole session was a real challenge and very fruitful. It is easily noticeable that the questions specific to research issues were less debated than the more fundamental problems. Thanks to the contributions of everyone, great steps were made in the understanding of the intricate network of conceptions of the many actors in the system.

B) Crucial Questions Raised in the Discussions (R. Borasi)

The questions/issues raised seem to cluster around three fundamental themes/topics:

(a) Determining and studying conceptions
(Whether they are teachers' conceptions or students' conceptions):

- How are conceptions determined:
  - through **verbal** reports of the subject?
  - through observation and interpretation of the subject's **action**?
  - what combination of the two?

- How can we take into account the researcher's frame in "interpreting" conceptions?

- How much do you need to know about **specific students'** conceptions? (Yet at the same time we may want to be aware of the *motivational* value that a teacher's research on his/her students' conceptions may have, **independent of results**, just because it shows the students that the teacher cares for them).

- Connection between "getting at" conceptions and "acting on them" (can we really do one and not the other?).

(b) Studying how conceptions are developed
(mainly for students)

- How does (past) teaching influence the development of certain conceptions?

- Are there crucial times/events/contents which can affect students' conceptions?

(c) "Changing" conceptions

- Can we talk of right/wrong conceptions? (or rather: dysfunctional? unrealistic? inappropriate?). Thus, can we really talk of "changing" conceptions?

- How can we "change" conceptions?

- How can we assess a change of conception?

C) Short Descriptions of Some Presentations (R. Borasi)

About teachers and students' conceptions:

- **J. Bergeron, N. Herscovics and J. Dionne:**
  Description of a course for in-service teachers, consisting essentially of a re-examination of basic math concepts (such as NUMBER) and geared at changing the teachers' conceptions of maths and teaching mathematics.

  (*Research strategies* used to assess change in teachers' conceptions (JD): triangulation of:

(a) how the teacher graded (and justified) a set of students math tests
(b) questionnaire, asking teacher to rank and assign a weight, to the three views of mathematics: traditional (stress: algorithms); formalistic (stress: rigour); constructivist (stress: process)
(c) individual interview, also discussing previous tasks)

- **S. Brown and T. Cooney** (reported by R. Borasi):
  In-depth study of 4 math teachers' belief systems (of math, teaching, teaching math, etc).
  (*Research strategies*: classroom observations + ethnographic interviews, initiated through the teacher's discussion of several "episodes", transcribed; the teacher read the transcript and marked significant statements, and later categorized and labelled those).

- **Erika Kuendinger:**
  Study on teachers' conceptions of themselves as math teachers.
  (*Research strategies*: combination of:
  (a)learning history of the teacher (w.r.t. math)
  (b)questionnaire
  (c)classroom observations (to validate responses on questionnaire))

- **Linda Davenport:**
  An intervention study for students, but also addressing the necessity of dealing with the teachers' conceptions at the same time.
  (*Research strategies*:
  FOR STUDENTS: an open-ended math test and interviews addressing essentially their conception of specific math concepts — ex: asking to explain and draw what 1/2 means.
  FOR TEACHERS: questionnaire (by P. Ernest — see excerpt in Appendix A) addressing explicitly the teachers' conceptions of mathematics, learning math., teaching math and self w.r.t. math).

- **Arthur Powell:**
  Using writing (more specifically, dialogue journals) to help students' learning of mathematics (including a movement towards less dysfunctional conceptions of math.).
  (*Research strategies*:
  Analysis of what the students write (guided by questions, see Appendix B).
  NOTE: to help the students being more reflective and personal in their writing they had:
  •a peer and the teacher responding to their journal
  •a list of "processes involved in thinking mathematically" (see Appendix C) they were supposed to refer to).

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APPENDIX A

QUESTIONNAIRE ON THE TEACHING OF MATHEMATICS

Scale I: Attitude Towards Teaching Mathematics

a.	My knowledge of mathematical concepts is sound enough to teach basic math.	YES!	yes	??	no	NO!
b.	I am very enthusiastic about teaching math to students.	YES!	yes	??	no	NO!
c.	I am confident about my ability to teach math.	YES!	yes	??	no	NO!

Scale II: View of Mathematics

a.	Someone who is good at mathematics never makes a mistake.	YES!	yes	??	no	NO!
b.	Math consists of a set of fixed, everlasting truths.	YES!	yes	??	no	NO!
c.	Math is always changing and growing.	YES!	yes	??	no	NO!

Scale III: View of Teaching Mathematics

a.	If students learn the concepts of math then the basic skills will follow.	YES!	yes	??	no	NO!
b.	Students should be expected to use only those methods that their math books or teachers use.	YES!	yes	??	no	NO!
c.	Students should learn and discover many ideas in mathematics for themselves.	YES!	yes	??	no	NO!

Scale IV: View of Learning Mathematics

a.	In learning math, each student builds up knowledge in his or her own way.	YES!	yes	??	no	NO!
b.	Learning math is mainly remembering rules.	YES!	yes	??	no	NO!
c.	Most errors students make are due to carelessness.	YES!	yes	??	no	NO!

From the work of Paul Ernest

APPENDIX B

Professor Arthur Powell
Developmental Mathematics I

About Journals

You are asked to keep a journal on 8½" x 11" sheets of loose-leaf paper. Generally, one or two sheets will be sufficient for a week's worth of journal writing. Neither your syntax nor grammar will be a concern or checked; my only concern and interest is what you say, not how you say it. You are asked to make, at least, one journal entry for each meeting that we have, and, as a rule of thumb, you need not spend more than five to ten minutes writing each entry. Each week, the latest journal entries will be collected and returned with comments.

The focus of your journal entries should be on *your learning* of mathematics or on the *mathematics* of the course. That is, your reflections should be on what you do, feel, discover, or invent. Within this context, you may write on any topic or issue you choose. To stimulate your thoughts and reflections, here are some questions and suggestions.

1. What did you learn from the class activity and discussion or the assignment?
2. What questions do you have about the work you are doing or not able to do?
3. Describe any discoveries you make about mathematics (patterns, relationships, procedures, and so on) or yourself.
4. Describe the process you undertook to solve a problem.
5. What attributes, patterns, or relationships have you found?
6. How do you feel about your work, discoveries, the class or the assignment?
7. What confused you today? What did you especially like? What did you not especially like?
8. Describe any computational procedure you invent.

APPENDIX C

PROCESSES INVOLVED IN THINKING MATHEMATICALLY (OR HABITS OF THE MIND)

1. Posing problems and questions
2. Exploring a question systematically
3. Generating examples
4. Specializing
5. Generalizing
6. Devising symbols and notations
7. Making observations
8. Recording observations
9. Identifying patterns, relationships, and attributes
10. Formulating conjectures (inductively and deductively)
11. Testing conjectures
12. Justifying conjectures
13. Communicating with an audience
14. Writing to explore one's thoughts
15. Writing to inform an audience
16. Using appropriate techniques to solve a problem
17. Using technical language meaningfully
18. Devising methods, ways of solving problems
19. Struggling to be clear
20. Revising one's views
21. Making connections between equivalent statements or expressions, transformations
22. Making comparisons
23. Being skeptical, searching for counterexamples
24. Reflecting on experiences
25. Suspending judgement
26. Sleeping on a problem
27. Suspending temporarily work on a problem and returning to it later
28. Listening actively to peers

Submitted by Arthur Powell

Topic Group A

Implementation of an Apple Centre for Innovation and Year 1 Mathematics Results

W. George Cathcart

University of Alberta

Recent surveys (Petruk, 1985; Hubert, 1988) have shown an exponential increase in the number of computers in schools during the past decade. Hubert (1988) estimated nearly 27 000 computers in Alberta schools at the end of 1987. This translates into about a 1:15 computer to student ratio or an average of about 100 minutes per week of computer access for each child in Alberta. The actual time a child spends at a computer, of course, varies significantly from this theoretical average. Questions remain. If children had continuous access to a computer all day, every day, what could they do? What would they learn? Would their thinking patterns change? How would the school program change?

The Proposal

In an attempt to at least partially answer the broad and open questions stated above, a proposal was submitted to the Apple Canada Education Foundation (ACEF) for the establishment of an Apple Centre for Innovation (ACI) in a third grade classroom. The proposal called for the installation of 1 complete Apple II GS microcomputer workstation for each child in the classroom. The plan was to network the computers and printers and ultimately to incorporate a file server. With respect to the curriculum, the plan was to develop materials that would uniquely integrate the computer into the language arts and mathematics programs.

Implementation

Hardware

The proposal was approved by the ACEF early in the summer of 1987. Thirteen[1] complete workstations were set up on temporary furniture ready for the 26 grade 2/3's first day of classes in September.

Plans for new functional furniture were completed and the furniture ordered. The design consisted of an octagonal desk-like cabinet with rectangular wings emanating from every second side of the octagon to form a 4-student workstation as illustrated in Figure 1. The wings housed the keyboard on a pull out shelf below the table top. Two small shelf-like compartments and a longer one along one side of the rectangular wing provided storage for disk drives and the CPU respectively. Only the monitor sat on top of the wing. The octagonal area in the centre could be used for individual and group work.

Electricians rewired the classroom so that there were no floor or post outlets. AppleTalk cables were also strung through the walls to completely remove all wiring from places where it could be accidentally pulled or tripped over. AppleTalk was also extended to the school office and library at this time.

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[1] In year 1 the computer to student ratio was 1:2. In year 2 (1988-89) the ratio was increased to 1:1.

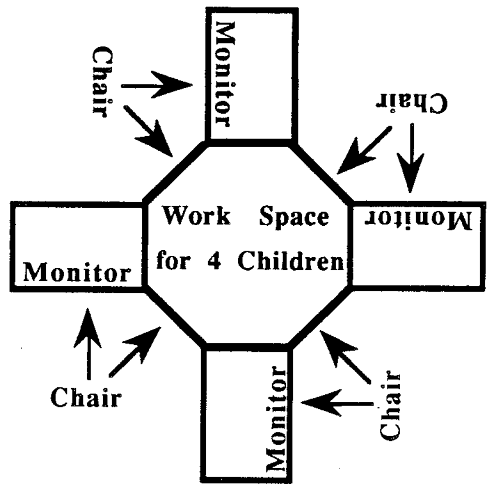

Figure 1. Octagonal 4-Student Workstation

Program

During the summer, the project director (grade 3 teacher) spent many hours preparing language arts materials which would incorporate the use of the computer so that there would be experimental materials in place for the beginning of the school term.

It seemed apparent that if grade 3 children were going to make productive use of the materials, efficient keyboarding skills would need to be developed. A professor in business education at the University of Alberta agreed to teach keyboarding to the 4 grade 2 and 22 grade 3 students in the ACI classroom. She taught a 30 minute keyboarding lesson 4 days each week for 3 months. Afterwards periodic keyboarding review lessons (approximately once a week) were conducted for the remainder of the year.

In language arts the computer was used as a tool in creative writing, responding to reading comprehension exercises, theme and book studies, research reporting, and for a variety of data base activities.

In mathematics the computer was used primarily as a means of providing practice during the first year of the project. Courseware included MAC 3 (Houghton Mifflin), MECC (Minnesota Educational Computing Consortium), graphing activities from National Geographic's Project Zoo, and some practice and problem solving software written by the project director.

Year 1 Mathematics Results

Data is available on the keyboarding and language arts component of the program as well as student and parent attitudes towards the project. The focus of this paper, however, is the performance of the ACI students in mathematics during year 1 of the project.

Instruments

In order to monitor achievement in mathematics a test based on mid to late grade 3 material was developed by the researcher. Part A of the test consisted of 30 open-ended questions. Part B contained 20 multiple-choice questions adapted from released items used by Alberta Education. There were 48 basic fact items in multiple choice format in Part C; 12 facts for each of the 4 operations. Students were given 1 minute to do each section. Parts A and B were not timed.

In addition to the 4 basic facts scores, the test yielded scores on the 5 strands in the Alberta curriculum, number, operations, and properties (25 items), numeration (12 items), graphing (2 items), measurement (6 items), and geometry (5 items). These 50 items formed what will be referred to as the concepts portion of the test. In addition 7 items from these strands were considered to be problem solving and were scored as a sixth strand.

The mathematics test was administered in September 1987 and again in late May 1988 in 2 sittings, usually before and after recess.

The Kuder-Richardson reliability for Parts A and B was 0.78 using pre-test scores and 0.81 using post-test scores.

Control Classes

For comparison purposes, 2 control classes were also given the mathematics test during the same week as the experimental class. One control class was in a neighbouring school, the other was in a very different part of the city. Table 1 shows the age and IQ scores for the 3 classes. There were no statistically significant differences among the 3 groups on the first 3 variables in table 1. There was, however, a significant difference among the classes on non-verbal IQ.

Table 1
Mean Age and IQ Scores

	Control 1	Control 2	Experimental
Age (months)	105.85	109.55	107.71
IQ			
Verbal	114.95	99.91	106.67
Quantitative	113.65	104.23	112.62
Non-verbal	105.45	97.95	112.48

Pre-Test to Post-Test Gains

It was expected that a significant positive gain in mathematics would be made over the course of one school year. A one-way analysis of variance with repeated measures for each class confirmed, in general, this hypothesis but there were some interesting exceptions. The actual (raw) gains made by each group on the mathematics measures are included in Table 2.

The gains made by the experimental class were all statistically significant. The 2 control classes, however, had a total of 7 non-significant gains. All but one of these were in the concepts portion of the test. Both control classes failed to register significant gains in graphing and geometry, the 2 strands with the least number of test items. Control group 2 did not reach the level of statistical significance ($p \leq 0.05$) on measurement and control class 1 did not reach that level on problem solving. Control 1 also failed to reach significance on the addition section of the basic facts test. Table 2 also contains a summary of the one-way analysis of variance with repeated measures.

Relative to the 2 control classes, the experimental group improved its rank from pre-test to post-test on 9 of the 13 scales, maintained its rank (highest) on 3 of the measures and declined in rank (highest to middle) on the numeration subscale.

Comparison of Classes

There were no significant pre-test differences (one-way ANOVA) among the groups on the major scales (concepts, facts, total score). There were, however, significant differences on 2 of the subscales of the concepts test (numeration and geometry) and on the subtraction section of the basic facts test.

The major analysis involved a two-way ANOVA (groups (3) by repeated measures (2)). A summary of this analysis is included in Table 3.

Table 2
Means, standard deviations, gains, and anova summary for each group

CLASS/VARIABLE	MEANS PRE	MEANS POST	GAIN	ST.DEV. PRE	ST.DEV. POST	F
CONTROL 1						
CONCEPTS						
Number, Operations, and Properties	12.40	17.70	5.30	3.66	4.01	55.54***
Numeration	6.15	8.95	2.80	1.60	1.47	39.62***
Graphing	0.75	1.25	0.50	0.79	0.64	4.13
Measurement	2.25	3.50	1.25	1.12	1.19	23.06***
Geometry	3.70	3.65	-0.05	0.92	0.93	0.03
Problem Solving	4.00	4.55	0.55	1.52	1.50	2.81
TOTAL CONCEPTS	25.25	35.05	9.80	6.00	6.36	59.71***
BASIC FACTS						
Addition	10.45	10.90	0.45	2.50	1.74	1.98
Subtraction	7.95	9.05	1.10	3.15	2.59	5.62*
Multiplication	3.60	5.85	2.25	2.78	3.10	8.83**
Division	1.20	5.55	4.35	1.58	3.47	20.75***
TOTAL FACTS	23.20	31.35	8.15	6.60	8.57	22.61***
total score	48.45	66.40	17.95	10.48	12.98	84.86***
CONTROL 2						
CONCEPTS						
Number, Operations, and Properties	11.09	15.73	4.64	4.24	4.63	37.46***
Numeration	5.91	6.96	1.05	2.62	2.15	5.00*
Graphing	0.64	0.77	0.13	0.73	0.69	0.59
Measurement	2.36	2.64	0.28	1.22	1.40	0.68
Geometry	2.36	2.59	0.23	1.40	1.40	0.63
Problem Solving	3.05	3.59	0.54	1.68	1.65	5.01*
TOTAL CONCEPTS	22.36	28.68	6.32	8.42	7.83	42.03***
BASIC FACTS						
Addition	10.00	11.59	1.59	2.29	1.01	12.53**
Subtraction	5.91	10.14	4.23	2.33	2.23	65.59***
Multiplication	2.91	5.14	2.23	1.54	2.34	16.87***
Division	1.77	5.41	3.64	1.51	3.08	27.63***
TOTAL FACTS	20.59	32.09	11.50	5.10	6.74	103.65***
total score	42.96	60.77	17.81	11.29	12.31	143.05***
EXPERIMENTAL						
CONCEPTS						
Number, Operations, and Properties	9.76	18.67	8.91	4.77	3.43	144.92***
Numeration	7.43	8.62	1.19	1.81	1.83	16.89***
Graphing	0.33	0.95	0.62	0.48	0.67	14.70**
Measurement	2.05	3.14	1.09	0.97	0.91	36.48***
Geometry	2.76	3.67	0.91	1.09	1.11	21.75***
Problem Solving	3.38	4.95	1.57	1.75	1.32	20.28***
TOTAL CONCEPTS	22.81	35.05	12.24	8.56	5.97	120.09***
BASIC FACTS						
Addition	10.62	11.86	1.24	2.54	0.48	6.32*
Subtraction	8.38	10.71	2.33	3.46	2.05	12.66**
Multiplication	3.10	7.10	4.00	2.59	3.83	39.53***
Division	1.62	6.38	4.76	2.04	4.56	31.77***
TOTAL FACTS	23.71	36.05	12.34	8.84	9.17	119.49***
TOTAL SCORE	46.52	71.10	24.58	15.96	12.66	223.39***

* $p \le 0.05$ ** $p \le 0.01$ *** $p \le 0.001$

Table 3
Anova summary (Group by Repeated Measures)

Scale	Source	df	F	P
CONCEPTS				
Number, Operations, and Properties	Group (G)	2,62	0.98	0.38
	Measures (M)	1,63	216.66	0.00
	G X M	2,63	9.66	0.00
Numeration	G	2,62	4.71	0.01
	M	1,63	50.31	0.00
	G X M	2,63	5.65	0.01
Graphing	G	2,62	3.08	0.05
	M	1,63	13.62	0.00
	G X M	2,63	1.64	0.20
Measurement	G	2,62	0.84	0.44
	M	1,63	31.96	0.00
	G X M	2,63	3.86	0.03
Geometry	G	2,62	7.57	0.00
	M	1,63	5.80	0.02
	G X M	2,63	3.51	0.03
Problem Solving	G	2,62	2.89	0.06
	M	1,63	24.95	0.00
	G X M	2,63	3.68	0.03
TOTAL CONCEPTS	G	2,62	2.57	0.09
	M	1,63	214.23	0.00
	G X M	2,63	7.08	0.00
BASIC FACTS				
Addition	G	2,62	0.68	0.51
	M	1,63	19.27	0.00
	G X M	2,63	1.84	0.17
Subtraction	G	2,62	2.31	0.11
	M	1,63	63.38	0.00
	G X M	2,63	8.04	0.00
Multiplication	G	2,62	1.13	0.33
	M	1,63	57.56	0.00
	G X M	2,63	2.49	0.09
Division	G	2,62	0.43	0.65
	M	1,63	78.57	0.00
	G X M	2,63	0.47	0.63
TOTAL FACTS	G	2,62	1.45	0.24
	M	1,63	191.45	0.00
	G X M	2,63	2.76	0.07
TOTAL SCORE	G	2,62	1.93	0.15
	M	1,63	423.71	0.00
	G X M	2,63	5.20	0.01

Testing Effects

The two-way ANOVA produced significant effects due to the testing on all 13 measures. In all cases the post-test composite mean was significantly higher than the pre-test composite mean.

Group (treatment) Effects

The study was primarily interested in differences among the groups. There were significant main effects due to treatment on 3 of the 13 scales; numeration, graphing, and geometry. All of these were components of the concepts test. There were no significant group effects on the basic facts test.

A Scheffe post-hoc pairwise comparison of unweighted main effects on the numeration subscale found a significant difference between control class 2 and the experimental class ($p \leq 0.02$). The experimental class had a significantly higher composite mean than control group 2.

The Scheffe comparison of unweighted main effects due to treatment on the graphing subtest found no significant differences among the groups. The difference between the experimental group and control 1 came the closest to reaching significance ($p = 0.08$). On the geometry subtest the significant main effects were primarily due to a significant difference between the 2 control groups although the difference between the experimental group and control class 2 was close to significance ($p = 0.06$).

Interaction Effects

There were significant interaction effects on 8 of the 13 scales used in the study. The graphs in Figure 2 picture the interaction for the 3 major scales (concepts, facts, total score). Figure 3 contains a graph of the interaction on the 3 subscales of the concepts test which had significant main effects due to treatment.

The interaction on the facts test was not significant. The interaction on the concepts test and the total mathematics score seems to be due to the steeper slope (greater gain) of the experimental group. On 2 of the 3 concepts subscales where there were significant treatment effects there were also significant interaction effects. On the numeration subtest the interaction seems to be due to the greater gain of control group 1 and on the geometry subscale it seems to be a greater gain by the experimental class that caused the interaction.

Summary and Discussion

On mathematics concepts, basic facts, and on the total score, the experimental class (extensive use of the microcomputer) made greater gains than 2 control classes (incidental computer use). In fact the experimental class made greater gains on 9 of the 13 measures used in the study.

There were statistically significant differences among the groups (based on a two-way ANOVA with repeated measures) on 3 of the 13 scales used in the study. These were the numeration, graphing, and geometry subtests of the mathematics concepts test. On the first 2 subtests the experimental group significantly outperformed one but not both control groups. On the geometry subtest, the difference was due to the difference between the 2 control groups.

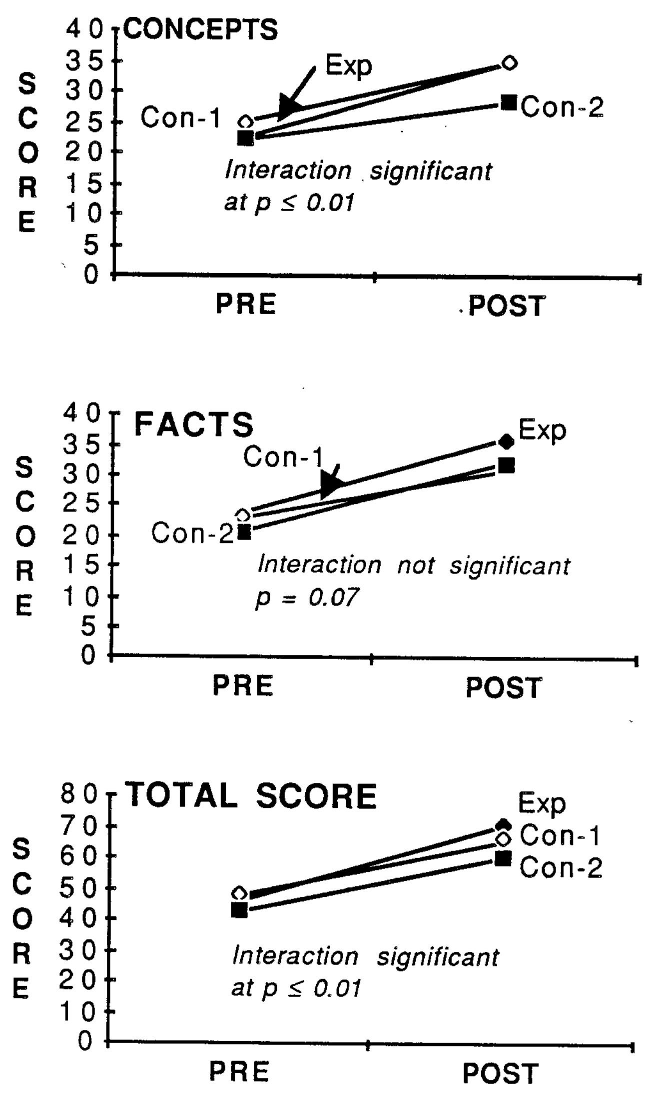

Figure 2. Interaction Effects on Major Scales

The results of this study lend some, although not strong, support to the thesis that supplementary computer experiences enhances mathematical skills. Given the nature of the treatment (major emphasis on the language arts and a lesser emphasis on mathematics), the results are not surprising. If the same time and energy could have been given to mathematics as to language arts, the results may have been more definitive.

Computer use was carefully controlled, the teacher factor was minimally controlled, but there are many variables such as teaching style, school philosophy, use of manipulatives, and others which were not controlled in this study. These certainly could have a bearing on the results.

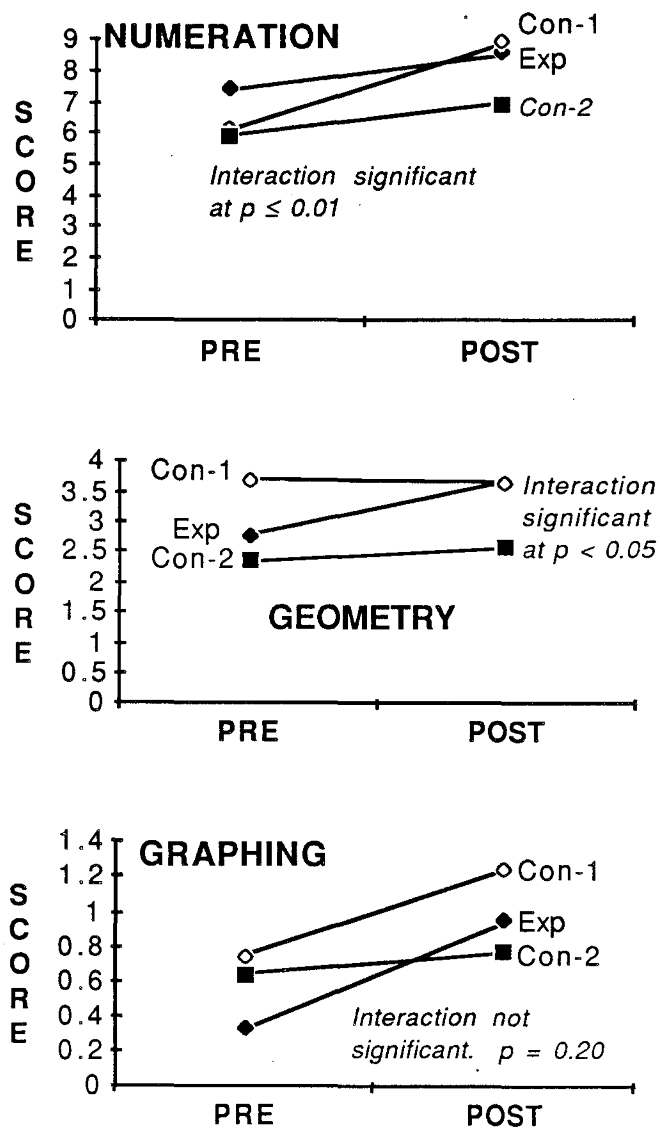

Figure 3. Interaction Effects on Subscales which had Significant Treatment Effects

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Petruk, M. W. (1985). *Microcomputers in Alberta Schools* - 1985. Edmonton, AB: Alberta Education, Media and Technology Branch.

Topic Group B

A Model to Describe the Construction of Mathematical Concepts from an Epistemological Perspective,

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In L. Pereira-Mendoza and M. Quigley (Eds.) Proceedings of the 1989 Annual Meeting of the Canadian Mathematics Education Study Group, St. John's, Nfld; CMESG/GCEDM.

Introduction[1]

For the last fifteen years, we have witnessed extensive discussions on the need to define what is meant by "understanding". Ephraim Fischbein (1978) stressed the importance of **intuition** for the understanding of mathematics and Richard Skemp (1976) provided an early model which distinguished between **instrumental understanding** ("rules without reason") and **relational understanding** ("knowing what to do and why"). Using Skemp's model and combining it with Bruner's distinction between analytic thinking and intuitive thinking (Bruner, 1960), Byers and Herscovics (1977) suggested the tetrahedral model of understanding which identified four complementary modes of understanding: **instrumental, relational, intuitive, and formal**. Later on Skemp (1979) extended his model to three modes of understanding (instrumental, relational, and logical) each one subject to two levels of thinking (intuitive and reflective) and, three years later, he added a fourth mode, that of symbolic understanding (Skemp, 1982). A more extensive survey can be found in **Models of Understanding** (Herscovics & Bergeron, 1983).

The reasons for finding better answers to the question "What does it mean to understand mathematics?" are not purely aesthetic and academic, they are also very practical. Without some answer to this question, one can hardly expect to train teachers to "teach for understanding". The training of teachers in the analysis of mathematical concepts through the use of models of understanding was attempted with a class of practising primary school teachers (Bergeron, Herscovics, & Dionne, 1981). Results proved to be most promising since these teachers ended up de-emphasizing the value of the written answer and instead assigned equal importance to the thinking processes underlying these answers (Herscovics, Bergeron & Nantais-Martin, 1981).

The early models of understanding were heavily oriented towards **problem solving** and proved inadequate to describe the comprehension involved in **concept formation** (Bergeron & Herscovics, 1981). Thus, a new model identifying **four levels** of understanding in the construction of mathematical concepts (**intuitive understanding, initial conceptualization, abstraction, and formalization**) was suggested (Herscovics & Bergeron, 1981). In the early eighties, this initial model was constantly improved in the sense of providing clearer criteria for the different levels of understanding (Herscovics & Bergeron, 1982, 1983, 1984). By 1982 we had characterized our second level of understanding as "**procedural understanding**" instead of "**initial conceptualization**", and by 1983 we were distinguishing between "abstraction" in the psychological sense (detachment from the concrete) and "mathematical abstraction" (the construction of mathematical invariants). In 1984 we adjusted our definition of "procedural

[1] Research funded by the Quebec Ministry of Education (F.C.A.R. EQ-2923)
Based on the analysis of number developed in this paper, we undertook an international study to assess the kindergartners' knowledge of natural number. Some highlights of this study appear in a companion paper "The kindergartners' construction of natural numbers: an international study".

understanding" to include both the acquisition of mathematical procedures as well as the ability to use these appropriately.

The continued attempt to provide an epistemological analysis of the various conceptual schemata taught at the elementary level has proved to be the whetstone on which we have refined our evolving model. Of course, we are using the term "epistemological" very broadly in the sense of "growth of knowledge" but also within a pedagogical context which acknowledges the impact of instruction. On one hand, our model of understanding provides us with a new perspective raising new questions such as "What kind of knowledge could be considered as evidence of intuitive understanding?". On the other hand, the research results force us to refine our initial model.

The objective of this paper is to present our two-tiered model of understanding and to illustrate how it can be used to describe the understanding of a fundamental mathematical concept such as natural number.

The Understanding of Preliminary Physical Concepts

Back in 1983, we described intuitive understanding by pointing out that

> For most of the (arithmetical) notions taught, one can find some pre-concepts which can be viewed as embryonic to the conceptual schema whose construction is intended...There is not yet any (numerical) quantification, maybe at most some simple (visual) estimation. These are situations which lead to what Ginsburg (1977) describes as 'informal knowledge'.
>
> (Herscovics & Bergeron, 1983, p. 77)

The above characterization of intuitive understanding served us well, since it forced us to search for appropriate situations in the child's experience that could be used as starting points for each intended concept. The acquisition of new knowledge would thereby be endowed with meaning and relevance. This last year, we achieved some kind of breakthrough when, in our analysis of the number scheme we decided to apply our existing model to the two notions we consider as pre-concepts of the number concept.

We have identified the notion of **plurality**, that is, the distinction between one and several, and the notion of **position** of an element in an ordered set, as two physical concept preliminary to the concept of number. We can then define 'number' teleologically, that is in terms of its initial uses and functions, as a **measure of plurality** and as a **measure of position.**

Applying our existing model to the notion of plurality and to the notion of position meant we had to find non-numerical criteria which might be interpreted as representing intuitive understanding, procedural understanding, and logico-physical abstraction of these two concepts. We would not attempt to find a fourth level of understanding, that of formalization, since, in effect, the construction of the number concept could be viewed as the mathematization of plurality and position. We have been successful in identifying the needed criteria and in converting these into tasks which have been used to assess the kindergartners' understanding of plurality and position. We provide here a brief summary of the criteria and tasks used to evaluate each level of understanding.

Regarding the **intuitive understanding** of plurality, quite early in our work we had designed tasks involving discrete sets that children could compare on the basis of visual estimation in order to decide which one had **more**, which one had **less**, where there were **many**, where there were **few**, or if one set had **as many** objects as another one. More recently, we developed tasks in which children used visual estimation to decide if an object was **before** (or in front), **after** (or behind) another one, if two objects were **together** (or at the same time), whether an object was **between** two other ones. The ability to estimate these notions visually could be considered as evidence of intuitive understanding of plurality and position since neither needed to be determined with any precision, rough approximations proving to be sufficient.

To identify a level of **procedural understanding** of plurality and position one had to find logico-physical procedures that were non-numerical, in which no counting was involved, but which provided precision to the notions introduced at the intuitive level. Procedures based on one-to-one correspondences answered this requirement since they provided accuracy and reliability to questions regarding plurality and order. Our investigations have shown that by the time children complete kindergarten, most of them can use one-to-one correspondences to generate sets that are larger, or smaller, or equal, or that have one more element, than a given set. They can also generate ordered sets subject to positional constraints such as before, after, at the same time.

Abstraction in the logico-physical sense was also easy to identify. We used as criterion the children's ability to perceive the invariance of plurality or position under various surface or figural transformations. The logico-physical processes which enable them to overcome the misleading information they obtain from their visual perception provides them with more stable conceptions of plurality and position. The **abstraction of plurality** was assessed through tasks in which sets of objects laid out randomly were rotated and displaced within the same space, dispersed, and contracted. Two tasks dealt with the visual impact of the elongation of a row, the first task involved a single row, and in the second task, one row was stretched while another one was kept fixed (Piaget's conservation of plurality). The invariance of plurality with respect to the visual perception of the elements was tested by hiding some of the objects. The **abstraction of position** was evaluated by assessing the invariance of position with respect to the elongation of a row, with respect to the visibility of all the objects in a row, and with respect to conservation of position when one of two parallel rows was translated. The abstraction of position was also assessed by verifying if the child was aware that the position of an element changed when one of the preceding objects was removed.

As can be seen from the above outline, it is quite possible to identify criteria that will clearly describe three levels of understanding of preliminary physical concepts. These three levels replace advantageously the level of understanding which in our previous model we described as "intuitive understanding of a mathematical concept" since they enable us to provide a full blown epistemological analysis of the preliminary concepts rather than view them as merely the initial embryonic stage in the construction of the intended mathematical concept. Of course, a model of understanding applied to physical notions needs to be distinguished from a model applied to mathematical ones. For instance, the procedural understanding evidenced by the use of a 1:1 correspondence between two sets of objects can be considered as a logico-physical procedure whereas the 1:1 correspondence between objects and the number-word sequence (counting) is of a logico-mathematical nature. A similar distinction applies to the construction of invariants. These comments provide us with the following description of the levels of understanding of physical concepts:

Intuitive understanding refers to a global perception of the notion at hand; it results from a type of thinking based essentially on visual perception; it provides rough non-numerical approximations.

Procedural understanding refers to the acquisition of logico-physical procedures which the learners can relate to their intuitive knowledge and use appropriately.

Logico-physical abstraction refers to the construction of logico-physical invariants (as in the case of the various conservations of plurality and position), or the reversibility and composition of logico-physical transformations (e.g. taking away is viewed as the inverse of adding to; a sequence of increments can be reduced to fewer steps through composition), or as generalization (e.g. perceiving the commutativity of the physical union of any two sets).

The Understanding of the Emerging Mathematical Concepts

We distinguish mathematical concepts from physical concepts when explicit mathematical procedures and invariants are involved. We then can identify three distinct constituent parts of understanding: procedural understanding, logico-mathematical abstraction, and formalization. Once again, we illustrate this with the number concept. In our opinion, the number concept is present only when enumeration (counting) is involved. Of course, knowledge of the number-word sequence by itself does not imply numerical knowledge. However, it is an essential pre-requisite to counting. Fuson, Richards & Briars (1982) has described different skills in the child's handling of the number-word sequence (reciting from one, reciting on from a given number, reciting backwards, etc.).

The **procedural understanding** of number involves explicit counting procedures. Since we defined number as a measure of plurality and of position, we had to design various tasks in which all the counting procedures could be used. For instance, asking children to count up a pile of chips "as far as they could go" would assess their mastery of the counting-from-one procedure and their numerical range. Asking them to generate a set of a given cardinality or to identify an object of a given position would assess their ability to count and stop at a given number. A task which might favour the counting-on procedure was developed (cf. Steffe et al, 1983) a row of thirteen chips was glued to a cardboard and the first six were hidden in front of the children. They were reminded how many were hidden and then asked: How many there were altogether? Could they find the ninth chip? Could they find the position of an indicated chip? Another task which might favour counting backwards also involved a row of twelve chips, some of which were hidden: with six chips hidden and the tenth chip pointed out. The children would then be asked: How many are hidden? With three chips hidden and the tenth chip identified, they were asked to find the seventh chip and afterwards, to find the position of an indicated chip. Finally, even more sophisticated tasks were selected, tasks that would involve double counting forwards or backwards. For instance, children might be asked to count out loud five number words from a given number, or to find how many number words are between two given ones. As can be seen, many tasks can be designed to evaluate procedural understanding.

In view of our definition of number as a measure of plurality and of position, the logico-mathematical abstraction of number must reflect both the invariance of plurality and the invariance of its measure, leading to the **abstraction of cardinal number**. It must also reflect both the invariance of position and the invariance of the measure of position, leading to the **abstraction of ordinal number**.

Over twenty years ago, Piaget's collaborator Pierre Gréco (1962) felt the need to distinguish between plurality and the measure of plurality. He modified the original conservation task involving two equal rows of chips by asking the children to count one of the rows before stretching the other one; he then asked how many chips were in the elongated row while screening it from view. Those who could answer the question were said to **conserve quotity**. Gréco found that many five-year-olds claimed that there were seven chips in each row but that the elongated row had more. Thus, these children **conserved quotity without conserving plurality**. For these children, to conserve quotity simply meant that they could maintain the numerical label associated with the elongated row, but their count was not yet a measure of plurality, since they thought that the plurality had changed. It is only when both plurality and quotity are conserved, when both invariances are perceived, that number becomes a measure of plurality. At that stage, one can claim that the child has achieved a logico-mathematical abstraction of cardinal number. Of course, the Piaget and the Gréco tasks are not the only ones by which abstraction of cardinal number can be assessed. These involve a specific type of transformation. All the other tasks previously used to assess the invariance of plurality can also be used here by modifying them to include enumeration.

An entirely analogous approach can be used to describe the logico-mathematical abstraction of ordinal number. Similar to the notion of quotity, one can introduce its parallel in the context of position. We define **ordity** as the ability to maintain the numerical label associated with the position of an element in an ordered set subject to various transformations such as elongation, translation, hiding part of a row. And of course, there are children who perceive the invariance of ordity without perceiving the invariance of position. Only when both are present can one claim to have achieved a logico-mathematical abstraction of ordinal number.

By the **formalization** of number, we mean the gradual development of various mathematical notations. When asked to send a message indicating how many objects are in front of them, children will represent each one by a drawing and later on by a tally mark. Once they learn to write their numerals, they may write the sequence 1, 2, 3, 4, 5, 6, 7 to represent the cardinality of a set of seven objects, thereby indicating their need to rely on a 1:1 correspondence between the objects and the numerals; by the end of kindergarten, most of them can use the numeral '7' with its intended cardinal meaning.

In fact, many of them can write down numbers exceeding nine. Of course, this does not imply any awareness of place value notation. Nevertheless, it indicates that they perceive the concatenation of two digits globally (e.g. '12' no longer means 'one and two' but 'twelve'). However, even the understanding of positional notation grows gradually: from mere juxtaposition (numerals are written next to each other without regard to relative position), through a chronological stage (the order of production prevails over the relative position), to a final conventional level.

The above discussion of number suggests the following description of the understanding of mathematical concepts:

Procedural understanding refers to the acquisition of explicit logico-mathematical procedures which the learner can relate to the underlying preliminary physical concepts and use appropriately.

Logico-mathematical abstraction refers to the construction of logico-mathematical invariants together with the relevant logico-physical invariants (as in the abstraction of cardinal number and ordinal number), or the reversibility and composition of logico-mathematical transformations and operations (e.g. subtraction viewed as the inverse of addition; strings of additions reduced to fewer operations through composition), or as generalization (e.g. commutativity of addition perceived as a property applying to all pairs of natural numbers).

Formalization refers to its usual interpretations, that of axiomatization and formal mathematical proof which, at the elementary level, could be viewed as discovering axioms and finding logical mathematical justifications respectively. But two additional meanings are assigned to formalization, that of enclosing a mathematical notion into a formal definition, and that of using mathematical symbolization for notions for which prior procedural understanding or abstraction already exist to some degree.

As can be seen from the first two definitions above, the understanding of a mathematical concept must rest on the understanding of the preliminary physical concepts. We thus end up with a **two-tiered model of understanding**. However, this does not imply that the understanding of a mathematical concept needs to await the prior three levels of understanding of the preliminary physical concepts. For instance, our research shows that kindergartners master counting procedures and the formalization of number well before they perceive all the invariances of plurality and position. Nevertheless, due to the very definition of logico-mathematical abstraction, this component part of understanding cannot occur without the prior logico-physical abstraction of the preliminary physical concepts. The non-linearity of our model is expressed by the various arrows in the following diagram:

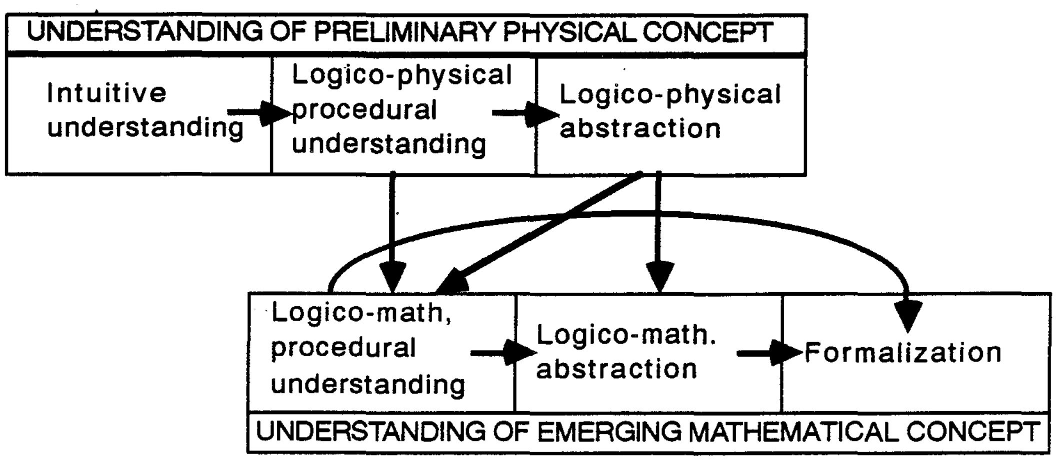

Figure 1. The two-tiered model of understanding

Two further important changes in the model need to be brought out. The first one pertains to our definition of 'formalization'. Whereas in our earlier models we required prior abstraction in order to recognize formalization as comprehension, we have now loosened this restriction to include procedural understanding. For instance, when sending a numerical message, the child may write out the whole sequence of digits and this can be considered as a formalization of the counting procedure. The other change is more general. We have avoided using the word 'level' to describe the understanding of mathematical concepts and replaced it with the expression 'constituent part' in order to prevent an overly hierarchical interpretation.

The following tables summarize the criteria used to assess the child's understanding of natural number:

Table 1. Understanding of preliminary physical concepts of number

INTUITIVE UNDERSTANDING	PROCEDURAL UNDERSTANDING (LOGICO-PHYSICAL)	ABSTRACTION (LOGICO-PHYSICAL)
Plurality: Visual determination of more, less, many, few, as many	1-1 correspondence used to generate a set that has more, less, as many as, one more than a given set	Invariance of a single set wrt dispersion, displacement within a given space, rotation, elongation, the non-visibility of some of its elements. Invariance of plurality in Piagetian test
Position: Visual determination of before, after, between, at the same time, first, last	1-1 correspondence used to generate an ordered set subject to positional constraints (before, after, at the same time)	Invariance of position of an object in a single row when the row is elongated, when some of its elements are visible. Invariance of position of two corresponding objects when one row is moved forward. Variability of the position of an object in a row when the first element of the row is removed.

Table 2. The understanding of number

PROCEDURAL UNDERSTANDING (LOGICO-MATHEMATICAL)	ABSTRACTION (LOGICO-MATHEMATICAL)	FORMALIZATION
Counting from 1; from 1 and stopping at a given number; from a given number M; from a given number M and stopping at a given number N>M; backward recitation from a given number; from a given number N and stopping at a given number M<N **Double-counting** Recitation of the N number-words following a given number-word; Recitation from A to B>A, keeping track of how many number-words are pronounced; Backwards recitation of the N number-words preceding a given one; Backwards recitation from B to A, keeping track of how many number-words are pronounced.	**Cardinal number:** uniqueness of cardinality; invariance of card of a row wrt the direction of the count; perception of the invariance of **plurality** and **quotity** of a single set wrt dispersion, wrt elongation, wrt the non-visibility of some elements; perception of the invariance of **plurality** and **quotity** of two equal rows when one of them is elongated; synthesis of counting-on and cardinality. **Ordinal number:** perception of the variability of **position** and **ordity** of an object when the first object of the row is removed; perception of the invariance of **position** and **ordity** of a single set wrt elongation, or the non-visibility of some of its elements; perception of the invariance of **position** and **ordity** of two corresponding objects when one row is moved forward.	Ability to recognise a numeral and generate a corresponding set of objects or identify an object of corresponding rank; ability to represent the cardinality of a set: - by drawing an equivalents set of pictures of the objects - by putting down an equivalent set of tally marks - by writing out the equivalent sequence of numerals - by writing a numeral as the cardinal of the set; ability to write the rank of an object in a given row; **Positional notation** (for those who can recognize or write two-digit numbers) - as **juxtaposition** - **chronologically** - **conventionally**

By way of conclusion

The construction of a fundamental concept in mathematics involves many different ideas that need to be related to each other into some kind of cognitive grid. As opposed to the acquisition of isolated parcels of knowledge, the development of fundamental mathematical concepts involves linking together several different notions into some organic whole forming some kind of cognitive matrix. But all this knowledge needs to be significant and relevant. This can be achieved only when it can be related to problem-situations, that is, situations in which this knowledge provides answers to some perceived problem. For instance, what would be the point in learning the sequence of the number words unless these were used to answer questions about cardinality and rank? We use the expression **'conceptual scheme'** to convey both the idea of a cognitive grid or cognitive matrix, as well as the relevant problem-situations.

As can be inferred from the theoretical part of our paper, our intention has been to study the learner's construction of a conceptual scheme and not just a part of it. It is with this objective in mind that we have developed our models of understanding. These models were to provide a frame of reference in which we could follow each learner's construction. In this sense, our models can be called 'epistemological'. Of course, this type of work actualizes what is meant by a constructivist approach to mathematics education. For instance, since the acquisition of fundamental concepts taught in primary school mathematics require two or three years, the conceptual analyses obtained by using our models provide the teachers with an overview of a given conceptual scheme. Without diminishing the importance of mathematical procedures, our models situate these in a broader context and emphasize the thinking processes involved. We thus realize a Lakatosian (Lakatos, 1976) perspective in the context of concept formation.
Our latest model of understanding suggests a basic structure that distinguishes between a first tier dealing with preliminary physical concepts and a second tier involving the emerging mathematical concept. This distinction is somewhat analogous to the one Piaget makes between 'simple' abstraction (or 'physical' abstraction) based on the properties of objects, and 'reflective' abstraction (or 'logico-mathematical' abstraction) that is based on the coordination of actions or operations. This distinction can be justified as long as actions and operations are in the mental domain. However, one cannot justify it as readily when the actions and operations are carried out on concrete objects. In fact, Piaget has acknowledged this when he suggested two forms of reflective abstraction:

> We will speak in this case of "pseudo-empirical abstractions" since the information is based on the objects; however, the information regarding their properties results from the subject's actions on these objects. And this initial form of reflective abstraction plays a fundamental psycho-genetic role in all logico-mathematical learning, as long as the subject requires concrete manipulations in order to understand certain structures that might be considered too 'abstract'. (Piaget, 1974, p.84, our translation)

The existence of two tiers in our model takes into account the subject's action on his or her physical environment. The two forms of reflective abstraction are comparable to the two aspects of understanding in our model: Piaget's pseudo-empirical abstraction is equivalent to our logico-physical abstraction, his logico-mathematical abstraction is the same as the one we mention in our second tier.

Our new model has several pedagogical implications. It links up explicitly the children's mathematics to their physical world and thus strongly suggests using the latter as a starting point in the construction of their mathematical concepts. One cannot over-emphasize the importance of this approach, for Ginsburg's work (1977) has brought to light the gap that may exist in the children's mind between their school mathematics and their 'informal' mathematics, that is, those acquired outside of school. The informal knowledge that Ginsburg identifies as System 1 corresponds to what we call 'preliminary physical concepts'. Hence, with its two tiers, our model encompasses the two forms of knowledge.

Other implications of a more practical nature involve applications to instruction and evaluation. While our model of understanding is definitely not a model of instruction, nevertheless, its use for the analysis of a conceptual scheme brings out several aspects of understanding that are often neglected. For instance, few teachers or textbooks assign to the ordinal aspect of number the importance it deserves. Moreover, logico-mathematical procedures are often introduced prematurely, thus neglecting the prior development of logico-physical procedures. Activities that may provide the children with the possibility of achieving some degree of abstraction, at both tiers, are usually ignored. Following the analysis of a conceptual scheme, teachers can develop tasks related to every aspect of understanding of a given concept. They could thus present to the children a far broader range of activities whose complementarity adds up to a much richer cognitive environment.

Such analyses also provide a frame of reference for the evaluation of a child's knowledge. They enable the teacher to assess the shortcomings in his or her background. For instance, a child who cannot recite the number words backwards will not be able to deduce the new rank of an object in a row following the removal of one of the preceding objects. They also enable the teacher to verify if appropriate linkages have been made, if different aspects of a conceptual scheme have been integrated. For example, the child who can count-on from a given rank in a row of chips, but cannot tell the cardinality of the row, has not yet achieved a synthesis of the counting-on procedure and the notion of cardinality.

We do not claim that this model will be suitable to describe the understanding of all mathematical concepts. Up to now we have applied it successfully to the analysis of the addition of small numbers (Herscovics & Bergeron, 1989) and early multiplication (Nantais & Herscovics, 1989). Héraud (1987, 1989) has applied it to length and the measure of length, to surface and the measure of surface (area). Dionne and Boukhssimi (1989) have applied it to algebraic concepts: to physical point and algebraic point (coordinates); to the physical notion of steepness and to the measure of steepness (slope); to physical straight line and to linear equation.

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Topic Group B

The Kindergartners' Construction of Natural Numbers: An International Study

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Introduction[1]

In a companion paper, "A model to describe the construction of mathematical concepts from an epistemological perspective", we presented a two-tiered model that could be used to follow the evolution of the child's understanding. The first tier dealt with the physical pre-concepts of the notion involved and consisted of three distinct levels: intuitive understanding, logico-physico procedural understanding, and logico-physical abstraction. The second tier described the comprehension of the emerging mathematical concept and involved three component parts: logico-mathematico procedural understanding, logico-mathematical abstraction, and formalization. This model was then applied to identify criteria that might be used to characterize each of the six aspects involved in the understanding of natural number.

Initially, each of the six components of understanding of natural number was the subject of an assessment study. Several of these studies have been reported. Two papers have dealt with numerical procedures (Bergeron, A., Herscovics, N., & Bergeron, J. C., 1986 Herscovics, N., Bergeron, J. C. & Bergeron, A., 1986a). Results on different tasks dealing with logico-physical abstraction of plurality and logico-mathematical abstraction of number have also been reported (Herscovics, N., Bergeron, J. C. & Bergeron, A, 1986b). The kindergartner's symbolization of numbers has been studied and discussed (Bergeron, J. C., Herscovics, N. & Bergeron, A., 1986). Following these assessment studies which were carried out with different groups of kindergartners, we experimented all the different tasks on the same children in four case studies (Herscovics, N., Bergeron, J. C. & Bergeron, A., 1987; Bergeron, A., Herscovics, N., & Bergeron, J. C., 1987).

However, to identify some general tendencies in the children's construction of number, a few case studies were not sufficient. This is why we extended our study to a larger group of kindergartners. We first experimented the tasks on the preliminary physical pre-concepts with a group of 30 Montréal kindergartners (Bergeron & Herscovics, in press). The following year we were ready to investigate both levels of our two-tier model with another group of French speaking Montréal children. And to determine if the cognitive structures observed here were comparable to those of other urban children from a different culture but with the same language or from the same culture but speaking another tongue, kindergartners from Paris, France, and Cambridge, Mass., were also assessed.

The samples used in our study involved 29 Parisian kindergartners of average age 5:8 whose school was situated in a lower socio-economic neighbourhood (lower middle class and working class); 30 kindergartners of average age 5:10 whose school was located in a lower socio-economic neighbourhood in Cambridge, Mass.; 14 of these children were

1 Research funded by the Quebec Ministry of Education (F.C.A.R. EQ-2923)

in regular classes whereas 16 of them were following an activity oriented program for early childhood based on Mary Baratta-Lorton's **Mathematics Their Way** (1976); 32 kindergartners of average age 6:2 from 4 different schools in the Montréal area, two being situated in higher socio-economic suburbs and two located in lower socio-economic neighbourhoods. For the overall project, which dealt with all the different aspects of understanding number, three to four individual interviews lasting on average 30 minutes were carried out with average children selected by the school authorities. Here are some highlights of this international study.

Enumeration skills

Pre-requisite to any mastery of the enumeration procedures is the child's memorization of the number word sequence. However, prior research has shown that a majority of kindergartners perform better on the enumeration of a large set of objects than on the mere recitation of the number-word sequence (Bergeron, A. et al. 1986). Thus in order to assess the extent of their knowledge of the number-word sequence, a set of 76 chips was provided with instructions to "Count as far as you can". The following table indicates the distribution of their enumeration skills.

Table 1. Enumeration skills

City	N	0-9	10-19	20-29	30-39	40-49	50-59	60-69	70+	Ave.
Cambridge										
Regular classes	14	0	0	3	3	0	1	3	4	53.7
Lorton classes	16	0	0	0	0	2	1	1	12	70.8
Totals	30	0	0	3	3	2	2	4	16	62.8
Paris	29	1	10	6	5	1	3	0	3	32.4
Montréal										
Higher Soc. Ec.	16	0	4	1	4	1	0	4	2	45.1
Lower Soc. Ec.	16	1	3	4	4	0	1	1	2	37.3
Totals	32	1	7	5	8	1	1	5	4	41.2

What is striking at a first glance is the similarity between the Parisian and Montréal samples, this, in spite of the fact that the French children were six months younger than the Canadian ones. But even more striking is the shift in the distribution of the Cambridge children. Not a single child is in the 0—19 range, in contrast with the 37% and 25% in the other two cities. Moreover, half the Cambridge children can count beyond 70, compared to 10.3% and 12.5% in the other two cities.

The distributions provide another interesting fact. It seems that for the Cambridge *regular* classes, as well as for the Parisian children and the two Montréal groups, the number 39 constitutes a temporary limit point: 42.9%, 75.8%, 56,3% and 75.0% respectively are within the 0—39 range. Perhaps this might indicate that these children have not yet learned the sequence of multiples of ten. That two decades, from 20 to 29, and 30 to 39, are sufficient for the generalization of the decade structure, seems evident from the fact that when the children learn their multiples of ten, their range jumps up to the sixties and seventies. Few of them remain in the 40 to 59 range.

A greater frequency of the Parisian children in the 50—59 range might be explained by a lack of knowledge of the multiples of ten beyond 50. One might conjecture that the 5 Montréal children in the 60—69 range (16.7%) have difficulties with 70 since in French, the tens pattern changes (..., cinquante, soixante, soixante-dix, ...). However, the data does not bear this out, since in the regular Cambridge classes, 3 out of 14 children (21.4%) are in the same range.

Understanding counting—on

Fuson, Richards & Briars (1982) report that when the number word sequence becomes a breakable chain, children can start **counting-up** (reciting-up) from a given number and that this skill translates into a cardinal operation, that of **counting-on** in the context of addition (p.52). In our study, we have experimented numerical tasks requiring counting-on in non-additive situations involving both cardinal and ordinal contexts.

Our results indicate that 84 of the 91 kindergartners could recite up from a given number and that most of them did not even need a running start. Comparing the performance in the three cities shows that nearly all (90%) the Cambridge children can start at 12, that about two thirds of the Montréal children (68.8%) , and about half of the Parisian sample (48.3%) can also do so. However, when asked to recite up starting from 6, 100% of the Cambridge children, 93.8% of the Montréal ones, and 75.9% of the Parisian ones succeeded. These differences can easily be explained by the emphasis on counting found in the Cambridge school and by the age difference of the Parisian children who were six months younger than the Montréal ones.

Having assessed the children's reciting-up skills, some special tasks were designed to determine their spontaneous use in the solution of cardinal and ordinal problems. Initially, these tasks were similar to the one used by Steffe, von Glasersfeld, Richards and Cobb (1983). Each child was presented with a row of 13 chips glued to a cardboard, the interviewer stating:

Here is a cardboard with some chips. Look, I'm putting it in this bag (while inserting it in a partially opaque plastic bag)

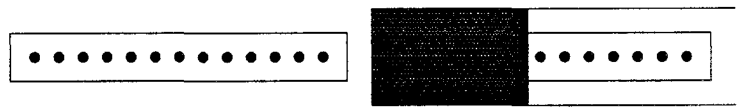

Figure 1: Inserting row in bag

Look, six chips are hidden here (indicating the opaque part)

Can you tell me how many chips are in the whole bag?

The results indicate that with the exception of the Lorton classes, the predominant procedure used was that of figural counting (counting first the hidden objects by pointing at each imagined unit and then continuing the count with the visible part): 50% of the children used it (50.0%, 48.3%, 43.8% and 56.3% respectively in the usual order of presentation).

Following this cardinal task, the same material was used for an ordinal task that required locating the chip corresponding to a given rank. The interviewer asked:

Remember, there are six chips that you can't see. Here is the first one (pointing out the one on the extreme left of the hidden part)
Can you put this little arrow next to the ninth chip?

The data show that once again, with the exception of the Lorton classes, figural counting is the most common procedure: 78.6%, 65.5%, 62.5% and 75.5% respectively in the usual order of presentation. Although most children can recite-up, the use of the counting-on procedure is relatively low, except for the Lorton classes. Less than a third of the children who possess the reciting-up skill think of using it in the above tasks, 21.4%, 4.5%, 31.3% and 28.6% respectively for the cardinal task and 21.4%, 27.3%, 31.3% and 28.6% respectively for the ordinal tasks.

These results bring into question the meaning of counting-on for most of these children. To investigate their interpretation, a simple task in which they were asked to count-on was proposed. The interviewer presented them with 11 chips glued to a cardboard. This cardboard was then inserted in a partially opaque plastic bag so that 4 chips would no longer be visible:

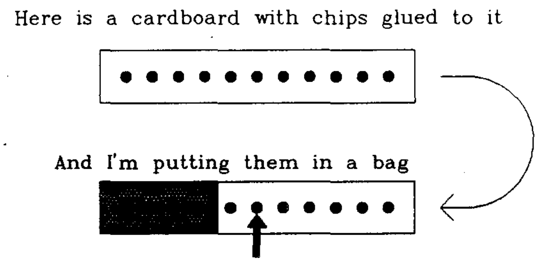

Figure 2: Counting-on a partially hidden row

Look, there are some hidden chips. When I counted them, I started from here (pointing to the first hidden chip on the left) **and when I got here** (putting a small arrow next to the sixth chip) **this was the sixth. Can you continue counting from here on, from the sixth one?**
When the counting was completed:
Can you tell me how many chips are in the whole bag?

The results show that out of 87 subjects who could count-on (compared with 82 who could recite up), only 33 of them (37.9%) could tell how many. Thus a full 60 % could not! Of course, this brings into question the children's interpretation of the counting-on procedure. It is evident that they have not yet been able to achieve a synthesis of the counting-on procedure and their cardinality scheme. The surprisingly poor performance on this task might be explained in terms of three conjectures: (1) Perhaps it is the non-visibility of some of the objects that affects the children's capacity to relate the counting-on procedure with the cardinality of the set; (2) Perhaps it is their need to still establish a one-to-one correspondence between the number-words and the objects; (3) There might be a gap in the children's integration of the cardinal and ordinal aspects of number. Further details on the procedural understanding of number appear in Bergeron & Herscovics (1989).

Logico-mathematical abstraction of cardinal number

Several different tasks were used to assess the children's logico-mathematical abstraction of cardinal number. We provide here details of the two most difficult tasks as well as an overview of the results obtained for the six criteria.

Piagetian tasks. One of the tasks used to assess the invariance of cardinality was the classical Piagetian test on the conservation of plurality and the Gréco modification mentioned earlier. The following table shows the results obtained:

Table 2. Success rates on Piagetian tasks

City	Invariance of plurality	Invariance of quotity	Invariance of both
Cambridge			
Reg. classes (n=14)	8 (57.1%)	12 (85.7%)	8 (57.1%)
Lorton cl. (n=16)	16 (100%)	16 (100%)	16 (100%)
Totals	24 (80.0%)	28 (93.3%)	24 (80.0%)
Paris (n=29)	7 (24.1%)	21 (72.4%)	7 (24.1%)
Montréal			
Hi soc.ec. (n=16)	13 (81.3%)	14 (87.5%)	12 (75.0%)
Low soc.ec. (n=16)	8 (50.0%)	12 (75.0%)	7 (43.8%)
Totals	21 (65.6%)	26 (81.3%)	19 (59.4%)

Results indicate a maximal rate of success among the children following the Barrata-Lorton program. On the invariance of plurality, the sample from the regular Cambridge classes compares with the sample from the two Montréal lower socio-economic neighbourhoods. The sample of Parisian children achieves a much lower rate (24.1%). This can be attributed in part to their younger age. However, this result is fairly consistent with their earlier performance on the elongation of a single row, for their success rate there was about 25% lower than the lowest results obtained in Montréal.

Invariance with respect to the visibility of the objects. In another set of tasks dealing with the invariance of cardinality, children were given in the first interview a row of 11 chips glued on a piece of cardboard. They were told: "Here is a large cardboard with little chips glued to it. Look, I'm putting the cardboard in a bag (the interviewer inserting the cardboard in a transparent bag). Good, are all the chips in the bag?". Following confirmation: "Look, I'm putting a plastic strip in the bag (the interviewer inserting a plastic strip with an opaque part large enough to cover three chips). And now, are there more chips in the bag, less chips, or the same number as before?". Usually in the second interview, this task was repeated but the children were asked to count up the number of chips before they were inserted in the bag. The following table shows the results obtained:

Table 3. Success rates on partially hidden row

City	Invariance of plurality	Invariance of quotity	Invariance of both
Cambridge			
Reg. classes (n=14)	1 (7.1%)	10 (71.4%)	1 (7.1%)
Lorton cl. (n=16)	5 (31.3%)	14 (87.5%)	5 (31.3%)
Totals	6 (20.0%)	24 (80.0%)	6 (20.0%)
Paris (n=29)	8 (27.6%)	6 (20.7%)	1 (3.4%)
Montréal			
Hi soc.ec. (n=16)	3 (18.8%)	13 (81.3%)	2 (12.5%)
Low soc.ec. (n=16)	0	12 (75.0%)	0
Totals	3 (9.4%)	26 (78.1%)	2 (6.3%)

Whereas the results on the invariance of quotity are similar in Cambridge and in Montréal, their discrepancy with those obtained in Paris is hard to explain. But it is the uniformly low results on the invariance of plurality that are most astonishing. They indicate that among most kindergartners, including those in the Lorton program, the visibility of the objects is still primordial. This is not a question of the permanence of the objects since it is acquired well before the age of five. Nor is it a question of the enumerability of the partially hidden set, as evidenced by the invariance of quotity. Visibility of the objects affects these children's conception of plurality.

In order to have an overview of the children's understanding of cardinal number, the results (in percents) obtained on the various tasks are summarized in the following table, invariance of cardinality signifying the invariance of both plurality and quotity:

Table 4. Hierarchy of criteria for cardinality

Invariance	Cambridge		Paris	Montréal	
	Lorton classes	Regular classes		Lower income	Higher income
Uniqueness of card.	93.8	100.	96.6	100.	93.8
Inv.wrt direction of count	100.	92.9	86.2	81.3	100.
Inv.wrt elongation of row	93.8	71.4	48.3	75.0	81.3
Inv.wrt dispersion of set	93.8	57.1	65.5	62.5	87.5
Inv.wrt Piagetian tests	100.	57.1	24.1	43.8	75.0
Inv.wrt visibility of objects	31.3	7.1	3.4	0.	12.5

What is most striking about this table is that apart from the Parisian results obtained on tasks involving the elongation of a set, the basic hierarchy is similar in the three samples. By and large, the uniqueness of the cardinality of a set and the invariance with respect to the direction of the count seem to be achieved in this age group. The Cambridge and Montréal results on the elongation of a row and on the dispersion of a set are similar in the two regular classes (71.4%) and the two lower income classes (75.0%). Compared with the dispersion of a set, the Piagetian tests are more difficult for both Parisian and Montréal children. The invariance with respect to the visibility of the objects has the lowest rate of success in all groups.

Also remarkable is a comparison of the success rates in the three middle columns. Again, if the odd results obtained in Paris on the elongations tasks are ignored, very similar rates are found among the Cambridge children from the regular classes, the Parisian children (who also come from a lower middle class and working class area), and the two Montréal classes situated in comparable neighbourhoods.

Logico-mathematical abstraction of ordinal number

Four different criteria were used to assess the logico-mathematical abstraction of ordinal number. We present here the details on the tasks dealing with the variability of ordinality and its invariance with respect to the visibility of the objects and with respect to translation. We also provide an overview of the results obtained for the four criteria.

Variability of the rank. In order to determine if children perceived the variability of the rank of an object with respect to the number of elements preceding it, we used the following task. A set of 8 little cars of different colours were aligned in a row.

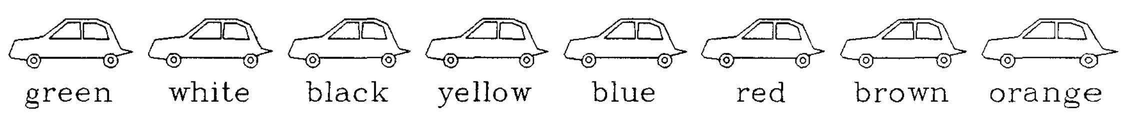

Figure 3: Variability of rank

Once a common vocabulary was established using the word "number" in an ordinal sense (Herscovics & Bergeron,1988), we told the following little story: "The parade is now stopped because the green car broke down. The tow truck is coming to get it (the interviewer removing the green car), and it won't come back in the parade. Now look at the little blue car. Do you think that the blue car still has the same number as before in the parade or do you think that its number has changed?"

In the second interview, the variability of rank was investigated by repeating a similar question but with an important addition. As soon as the parade of cars was laid out in front of the children (in the same order as before), they were asked "Can you tell me the number of the brown car?". Once the children had found (by counting) that it was in seventh position, they were again told that the green car (the first one) had broken down, and the interviewer then removed it. At this point they were asked: "Now, without counting, can you tell me the number of the brown car?" while screening the parade with both hands or the forearm to prevent the possibility of counting. The following table shows the results obtained :

Table 5. Success rates on variability of rank

City	perceived change of blue car's position	were able to find new rank of brown car	succeeded both tasks
Cambridge			
Regular classes (n=14)	12 (85.7%)	5 (35.7%)	4 (28.6%)
Lorton classes (n=16)	14 (87.5%)	13 (81.3%)	12 (75.0%)
Totals (n=30)	26 (86.7%)	18 (60.0%)	16 (53.3%)
Paris (n=29)	25 (86.2%)	16 (55.2%)	15 (51.7%)
Montréal			
High soc-econ (n=16)	14 (87.5%)	14 (87.5%)	13 (81.3%)
Low soc-econ (n=16)	15 (93.8%)	10 (62.5%)	10 (62.5%)
Totals (n=32)	29 (90.6%)	24 (75.0%)	23 (71.9%)

Column 1 shows that without any counting, most children perceived that the position of the blue car had changed. However, the success rates shown in the second column vary widely. A low of 35.7% is obtained in the regular Cambridge classes. Results found in the Parisian group and the two Montréal classes in lower socio-economic neighbourhoods are comparable (55.2% and 62.5% respectively) as well as for the Lorton classes and the two other Montréal classes (81.3% vs 87.5%). These comments apply to the third column, when both tasks are considered.

These results are somewhat surprising. Except for the Parisian children, it is difficult to explain the poorer results obtained in finding the new rank of the brown car. Among the French kindergartners only 18 of them (62%) could recite the number-word sequence backwards from at least 6. Thus that only 55% succeeded in identifying the brown car's new rank (6) is reasonable. But for the Cambridge and Montréal samples, absolutely all children were able to recite backwards. Clearly, this indicates that the cognitive problem at hand is much deeper than that of mastering recitation skills. Indeed, the very integration of cardinality and ordinality is at stake here since by removing the head car, the number of cars preceding the brown car is reduced by one and this should induce a corresponding change in the perception of ordinality.

Invariance with respect to the visibility of the objects. A task introduced during the first interview dealt with the invariance of position when part of the set is hidden. A row of 9 little trucks was drawn on a cardboard, each truck coloured differently. The children were told: "Look, here is a parade of trucks. Can you show me the white truck?" (in sixth position). After it was duly pointed out, the interviewer announced "The parade must now go under a tunnel" and then proceeded to slide the cardboard under the tunnel in such a way that part of the first truck was still visible but three trucks were hidden. The children were then asked: "Do you think that the white truck has kept the same number in the parade or do you think that it now has a different number?"

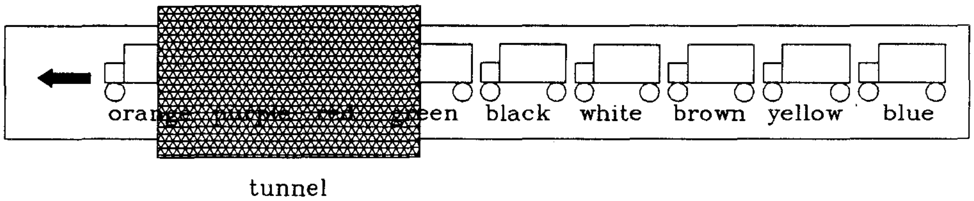

Figure 4: Trucks in tunnel

The above task was repeated in the second interview with an important variation. The children were now asked to find the rank of the white truck. After they had counted to determine its rank (sixth), the parade was moved forward into the tunnel and the interviewer asked again "Now, can you tell me the number of the white truck in the parade?". The following table provides the data obtained with this task assessing the invariance of position and of ordity:

Table 6. Success rates on partially hidden row of trucks

City	Invariance of position	Invariance of ordity	Invariance of both
Cambridge			
Regular classes (n=14)	4 (28.6%)	6 (42.9%)	3 (21.4%)
Lorton classes (n=16)	11 (68.8%)	12 (75.0%)	8 (50.0%)
Totals (n=30)	15 (50.0%)	18 (60.0%)	11 (36.7%)
Paris (n=29)	6 (20.7%)	10 (34.5%)	5 (17.2%)
Montréal			
High soc-econ (n=16)	10 (62.5%)	12 (75.0%)	9 (56.3%)
Low soc-econ (n=16)	5 (31.3%)	7 (43.8%)	3 (18.8%)
Totals (n=32)	15 (46.9%)	19 (59.4%)	12 (37.5%)

An examination of the results in the first column indicates that the number of children who think that the white truck has kept the same position in the row is comparable in the regular Cambridge classes, the Parisian children, and the Montréal kindergartners from the two schools situated in a lower socio-economic neighbourhood. The results of the other Montréal children compare with those of the Lorton classes. On the invariance of ordity alone as well as on the invariance of ordinality based on both position and ordity, again the results regroup themselves in two comparable sets, the Lorton classes and those of the Montréal children from the higher socio-economic areas in one set, and the other three samples in the other set.

Invariance with respect to translation. The last set of tasks we developed in our assessment of ordinal number were somewhat similar to the Piagetian conservation tests for plurality and quotity , for they involved the comparison of two parallel rows. The interviewer aligned a row of 9 identical cars, and asked the children "Would you make a parade just like mine and next to it?" while handing over another 9 cars. Then using a blue coloured sheet of paper (the river) and a small piece of cardboard to represent a ferry she explained: "The parades must cross the river on a little ferry boat. But the ferry can only carry two cars at a time, one car from each parade. When we are ready, we take one car in my parade (putting her lead car on the ferry), and one car from your parade" (asking the children to put their lead car on the ferry). The ferry then crossed the river with the two cars, unloaded them, and came back for two more:

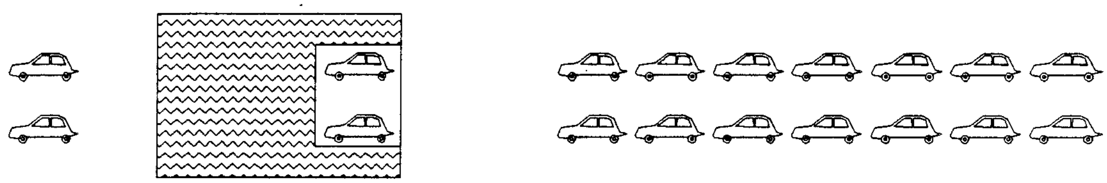

Figure 5: Two parades crossing a river

The cars were then put back in their initial position and the children were told: "Now I'm putting this little arrow on this car (the seventh car in the interviewer's row). Can you put this other arrow on the car in your parade which has the same number as mine?"

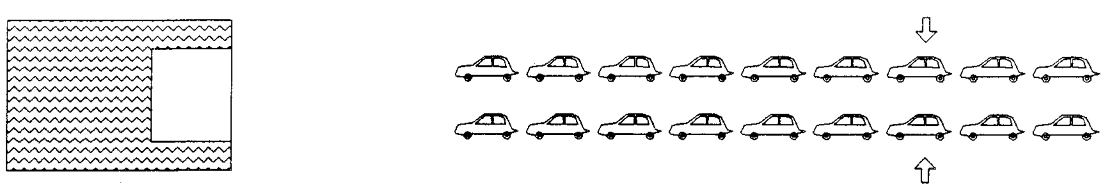

Figure 6: Two cars having the same rank

Once this was done, the interviewer announced "Now look, the parades move on" while moving the child's parade a small distance and moving her own parade somewhat further by the length of two cars:

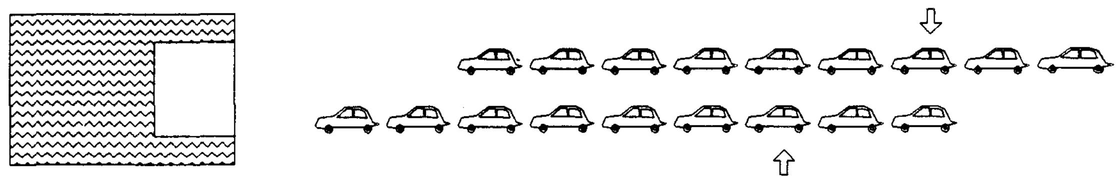

Figure 7: "Will the two cars cross at the same time?"

The children were then asked: "Do you think that the two cars with the arrows will cross the river at the same time?" Following their answer, they were asked to show the interviewer how the two parades were to cross the river in order to verify that they were aware that the cars had to be ferried in pairs. Following the above task, the invariance of ordity was immediately assessed. The next table provides data on the invariance of position and on the invariance of ordity:

Table 7. Success rates on translation task

City	Invariance of position	Invariance of ordity	Invariance of both
Cambridge			
Regular classes (n=14)	2 (14.3%)	9 (64.3%)	2 (14.3%)
Lorton classes (n=16)	4 (25.0%)	12 (75.0%)	3 (18.8%)
Totals (n=30)	6 (20.0%)	21 (70.0%)	5 (16.7%)
Paris (n=29)	2 (6.9%)	18 (62.1%)	2 (6.9%)
Montréal			
High soc-econ (n=16)	1 (6.3%)	11 (68.8%)	1 (6.3%)
Low soc-econ (n=16)	3 (18.8%)	10 (62.5%)	2 (12.5%)
Totals (n=32)	4 (12.5%)	21 (65.6%)	3 (9.4%)

The results on the invariance of ordity with respect to translation vary but little between the groups. The very low results on the invariance of position induce a very low rate of success on the invariance of ordinality as shown in the third column. Quite clearly, even the children in the Lorton classes and in the Montréal classes in higher socio-economic neighbourhoods do not manage to overcome the visual effect of the translation of one of the rows.

In order to have an overview of the children's understanding of ordinal number, the results (in percents) obtained on the various tasks are summarized in the following table, variability and invariance of ordinality signifying the variability and invariance of both position and ordity:

Table 8. Hierarchy of criteria for ordinality

	Cambridge		**Paris**	**Montréal**	
	Lorton classes	Regular classes		Lower soc-ec	Higher soc-ec
Variab. of ordin. no	75.0	28.6	51.7	62.5	81.3
Inv.wrt elongation	100.	64.3	27.6	56.3	68.8
Inv.wrt visibility	50.0	21.4	17.2	18.8	56.3
Inv.wrt translation	18.8	14.3	6.9	12.5	6.3

Again, as with cardinality, there are marked distinctions in the performance of the Lorton classes as compared with the regular Cambridge classes and important differences between the two Montréal samples.

As was the case with cardinal number, the similarities are quite striking. We find essentially the same hierarchy in the last three columns. For the two Cambridge groups, the success rates on variability and invariance with respect to elongation are inverted when compared with the other groups. Aside from this difference, the general hierarchy is the same. As mentioned earlier, the low performance of the regular Cambridge classes on the variability of ordinal number is somewhat surprising. The Parisian children's poor performance on the invariance of ordinal number with respect to elongation is similar to their poor performance on the other comparable elongation tasks dealing with the invariance of cardinal number. Regarding the comparison of the Lorton classes with the Montréal children in the schools located in higher socio-economic suburbs, the similarity is still quite strong.

By way of conclusion

A most important conclusion implied by the international study on the construction of natural number is that kindergartners in Western urban environments evolve similar cognitive structures, despite some cultural differences, despite linguistic differences. This is evidenced by the hierarchies we uncovered among the different criteria used to assess each component part of understanding. It is particularly true for the three groups that were comparable in terms of classroom activities and socio-economic background. But this remains true for the other two groups, those Cambridge classes using activities based on the Baratta-Lorton program and those Montréal children belonging to classes in schools situated in wealthier neighbourhoods. Although these last two groups had markedly higher success rates on the various tasks, the hierarchy of the success rates was essentially the same as for the other three groups.

In terms of logico-mathematico procedural understanding, Table 1 shows that, with the exception of the Lorton classes, the other four groups were still developing their counting skills and that in each group the number 39 constitutes a temporary limit point. Again, in each of these four groups the predominant procedure used to solve cardinal and ordinal problems in the context of a partially hidden row was figural counting, while less than a third of the children who could recite up from 6 used counting-on.

Regarding the children's abstraction of cardinality, Table 4 shows that for the regular Cambridge. classes, the Parisian classes and the Montréal pupils in the lower income group, not only is the hierarchy of the criteria essentially the same, but the success rates are also comparable if we except the Parisian results on the tasks involving elongation. For the Montréal children in the higher income group, if we ignore a difference of 6% due to a difference of one child out of 16, the first two criteria are met by all, the next two criteria meet with comparable success rates (81.3% and 87.5%). And thus, the hierarchy obtained is essentially the same as for the three previous groups. For the Lorton group, since they all meet the first five criteria, one cannot order them. Nevertheless, even they have not achieved the invariance of cardinality with respect to the visibility of the objects. Regarding the abstraction of ordinality, Table 8 shows that, except for the inversion of the first two criteria for the Cambridge classes, all five groups indicate the same hierarchy.

Our critical analysis of the children's performance may have obscured the fact that these kindergartners possessed a surprisingly extensive knowledge of number. For instance, nearly all could recite up from either 6 or 12, as well as recite backwards. Fuson et al (1982) have shown that such children are dealing with the number word sequence as a bi-directional breakable chain. While they did not use counting-on in order to solve problems in which some of the chips in a row were hidden, nevertheless they showed great ingenuity in inventing a new procedure, that of figural counting. In terms of abstraction too, we did not anticipate that nearly all these pupils could perceive the uniqueness of the cardinality of a set and its invariance with respect to the direction of the count. But their knowledge was far more extensive than reported in this paper, for almost all could leave a numerical message indicating the number of objects in front of them. About half the Parisian children and almost all the North American ones could read and write numbers beyond 10.

This international study has also brought to light the existence of four unexpected cognitive obstacles. The first one relates to the task of counting-on from the sixth chip on a partially hidden row (see Fig. 2). A full 62% of the children who could count-on were unable to answer the question "How many?". Even the Lorton group did not succeed much better (50%). Three possible conjectures that might explain this problem were suggested in the analysis of the data. A second cognitive obstacle involves the children's perception of the variability of the rank of an object in a row (see Fig 3). Table 5 shows that in three of the five groups, many children still had problems in determining the new rank of a car following the removal of the lead car.

If the first two obstacles might show evidence of a lack of integration of specific counting procedures (counting-on and counting backwards) into the children's notions of cardinality and ordinality, the last two obstacles seem to be more of a developmental nature. For instance, in all five groups, the visibility of the objects affects the kindergartners' perception of the invariance of cardinality (see Table 3) and to a lesser extent the invariance of ordinality (see Table 6). A similar interference due to the visual apprehension of the objects seems to be present in the task involving the translation of a row of cars (see Figure 7).

Comparing the different results obtained from the two Montréal groups, it is clear that the overall success rate of the children from the classes in the higher socio-economic suburbs is greater. We are but mentioning the difference here without any pretence at a scientific investigation since the objective of our study was to assess the children's understanding. While we did not control for the quality of teachers, nevertheless it is well known that quite often, better schools manage to attract better teachers. Moreover, since officially there is no mathematics program for Quebec kindergartens, teachers are free to choose their classroom activities, with the result that these may differ both qualitatively and quantitatively from school to school. Finally, it is also well known that children in the wealthier neighbourhoods are more likely to experience at home a richer variety of educational activities. These are all variables that would have to be taken into account in any investigation of the effect of the pupils' socio-economic background. However, in our study, the reason for choosing schools in different socio-economic neighbourhoods was to provide us with a wider variety of subjects.

A comparison of the different results obtained from the two Cambridge groups indicates much higher success rates for the children following the Baratta-Lorton program. A closer look at this program shows that it makes extensive use of concrete material, games and rhythmic body movements, thus touching upon some aspects of the preliminary physical concepts. Regarding the logico-mathematico procedural understanding of number, it goes well beyond simple enumeration from 1, and teaches explicitly counting-on and counting backwards, procedures that are then used primarily in cardinal tasks. In some of the kindergarten classes, the early arithmetic may even include addition and subtraction of small numbers. Children are also taught numerals and the conventional symbolization of the operations. While some of the tasks involved the invariance of cardinal number with respect to the partition of a set, most of the activities related to procedural understanding and to formalization. But these activities seem to have had a marked impact on the children's logico-mathematical abstraction. Table 4 shows that on the last four tasks dealing with the invariance of cardinality, the Lorton group scored much higher than the regular one. This pattern holds for the first three tasks on ordinality listed in Table 8

The results obtained by the Lorton group have serious pedagogical implications. They bring into question the various government policies specifying that no mathematical program ought to be assigned to the kindergarten level. These policies stem from a laudable desire to allow these children time to play and develop without any curriculum pressure. However, without confining them into a rigid mathematical program, one can envisage many numerical activities allowing them to play and develop their mathematical thinking. But for many kindergarten teachers, these activities are limited to simple counting tasks. As our conceptual analysis and our international study have shown, kindergartners possess intellectual abilities that far exceed those needed to master such simple procedural skills. In fact, our work suggests that many different numerical activities could be developed that would enable the child to progress along the different component parts of the understanding of number.

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Topic Group C

Multicultural Influences in Mathematics Education

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I know I've been asked to speak about multicultural influences in mathematics education, generally, but what I would like to do in this topic group is focus specifically upon cultural influences among Black, Hispanic, and Native American students. These are three populations of students which historically have been underachieving and underparticipating in the area of mathematics and often cultural influences are associated with their low levels of achievement and participation. Over the past year I have been working with a project designed to increase the levels of achievement and participation for Black, Hispanic, and Native American students and to look at cultural influences on their achievement and participation. Consequently, in this presentation I would like to discuss this project and our observations regarding the role of multicultural influences.

A review of the literature on the mathematics achievement and participation of Black, Hispanic, and Native American students contains a great deal of data documenting their low levels of achievement and participation. There are few studies which look empirically at possible causes of this low achievement and participation. Until recently, the only major studies addressing this issue were a clinical study of the mathematical understanding of Hispanic algebra students focusing on the relationships between language proficiency and mathematical understanding (Gerace & Mestre, 1982a, 1982b) and an examination of the role of Black English on the mathematical understanding of Black high school students (Orr, 1987). More recently, a series of chapters devoted to linguistic and cultural influences on mathematics achievement has been published (Cocking & Mestre, 1988), although these tend to address opportunities to learn in specific multicultural teaching contexts as well as the role of language in learning rather than multicultural influences on mathematical thinking itself. Incidentally, I did not feel that Orr made a convincing case for the effects of Black English on mathematics learning, although her book is rich with examples of student work demonstrating their difficulties with particular mathematics concepts.

In our project, we were interested in looking at the relationship between culture and mathematics achievement. We did believe that Black, Hispanic, and Native American students were underachieving and underparticipating because they were not being provided with the kinds of opportunities that would help them construct meaningful mathematical knowledge. This seemed to fit with the research findings that these populations of students had little conceptual understanding of many topics in the mathematics curriculum. We also believed that the reason that these students had little conceptual understanding was that they were not being provided with opportunities to explore, discuss, and socially negotiate meaningful mathematical knowledge. Finally, we believed that since mathematical knowledge is constructed in a cultural context, there might be some cultural influences on the mathematical thinking of these students which we hoped would emerge during the course of the project.

I might add that I think the approach we are taking is a rather novel one considering the frameworks that are often used to examine the underachievement of Black, Hispanic, and Native American students generally. During the 1960's, efforts to explain the low levels of achievement of these students focused on issues of cultural disadvantage, followed by, in the 1970's, issues of cultural difference. Now, in the 1980's, issues of effective instruction for students at risk seems to be the framework for much of this research. I think it's interesting that the cultural disadvantage and cultural deficit frameworks were really not all that different. This can be seen if you look at some of the so-called cultural differences discussed in the literature of that period. For instance, in an article discussing the relationship between culture and school achievement, a chart entitled "Contrasting Values and their Effects on Mexican Americans" suggests that the chicano student "frequently lacks enthusiasm and self-confidence", "works more effectively in groups; usually noisy", and "apathetic in school; often embarrassed by deficiency in English and few successful experiences; may become a dropout" (Instructor, 1972). Oddly enough, the title of the article is "Building on Backgrounds". In the current framework of effective instruction, researchers are advocating greater academic learning times with learning broken down into smaller pieces. In this framework, cultural issues are ignored completely.

Our project involved examining changes in mathematical thinking of Black, Hispanic, and Native American middle school and high school students as they progressed through a *Visual Mathematics* curriculum (Bennett & Foreman, 1989) as opposed to a more traditional textbook-based curriculum. This *Visual Mathematics* curriculum, built around *Math and the Mind's Eye* activities developed through an NSF grant (Bennett, Maier, & Nelson; 1987), is highly student-centred, allows for student exploration and discussion, and encourages students to construct and share their personal visions of fundamental concepts in mathematics. We felt that this kind of mathematics instruction would provide opportunities for underachieving Black, Hispanic, and Native American students to construct and negotiate mathematical meaning as well as allow any culturally distinct mathematical views to emerge through their personal visions.

We planned to collect information on attitudes towards mathematics, using the Fennema-Sherman Mathematics Attitude Scales, and achievement in mathematics using both a standardized test and an open-ended mathematics test focusing on mathematical concepts and problem solving, for students using the *Visual Mathematics* curriculum and students using the traditional curriculum. In the open-ended test, for example, we asked students to explain their idea of multiplication and to draw a picture of multiplication. We also planned to develop case studies of the mathematical thinking of students in each of those instructional settings through the use of clinical interviews. We were especially hoping to be able to explore some cultural issues in these interview settings.

This is only the first year of our project and I have to say there have been some difficulties, the primary one being that the implementation of the *Visual Mathematics* curriculum has been quite challenging for many of the teachers. These teachers took a 3-credit course in *Math and the Mind's Eye* before the project began and, during the academic year, we have been meeting once a month for an all-day session designed to provide support as they implement the curriculum. I believe that the challenge lies in the fact that teachers are not accustomed to student-centred instruction. In the *Visual Mathematics* curriculum, the teacher is largely the problem-poser and the facilitator of discussion, and it is often impossible to know exactly where the lesson is going to go. This uncertainty seems to make teachers new to the approach a bit uncomfortable. Also, as students are encouraged to express their vision of mathematical concepts under discussion, teachers are often called upon to facilitate discussions about representations they may have not seen before. This also makes teachers a bit uncomfortable. Finally, we encourage teachers not to "show and tell", but let students try to figure things out for themselves (with the help of some probing questions from the teacher and lots of class discussion). Teachers seem a bit uncomfortable letting students go with their ideas and sometimes seem to want to show them "the right way". All in all, it is a big change for most teachers. I will say that when the *Visual Mathematics* curriculum is working well it can be very exciting for both teacher and students. We have seen it happen in some classrooms. Fortunately, our teachers have had enough of these exciting experiences with the curriculum to keep going and most claim that they could not go back to teaching from a textbook.

I would like to comment on what we have been seeing in our student interviews thus far, as this has been our primary vehicle for exploring cultural influences. The students we have been interviewing have shown very little conceptual understanding of mathematics topics usually included in the elementary school curriculum. Their facility with basic mathematical procedures has been quite limited and they seemed to have little in the way of visual models to help them solve the problems they were asked. We have seen some changes in students who have been using the *Visual Mathematics* curriculum. They seem more apt to say things like "This is how I see it" or "This is how I think about it". We are seeing a great deal of diversity in the approaches used by these students although I would have a hard time categorizing those approaches by cultural background. The only example I can cite that might even remotely suggest a cultural influence is when a young Native American girl drew several pictures of a measuring cup to help her think about adding fractions. I should say the questions we were asking were rather traditional and had a computational focus, for example, asking students to think about $1/2 + 1/3$. However, our intent was to explore how they were thinking about the problem and we did probe for any contexts in which the problem might be made meaningful.

Looking back on our beginning efforts to explore cultural influences on mathematical thinking, I think there are several issues which made our effort particularly difficult. One is that we were asking questions about school mathematics in the school context. Had we moved to a more culturally-relevant out-of-school context and asked questions about mathematical applications in that context, perhaps we would have found cultural influences. However, I think a larger problem lies in the area of defining "culture". Although these students were from several ethnic backgrounds, their affiliation with their ethnic culture varied tremendously. All of these students were born in the United States and were to some extent participating in its mainstream culture. Some students seemed uncomfortable with their ethnic identity and, for example, corrected us with an Anglified version of their name when we used its correct pronunciation. For all these students, I think there were numerous cultural influences such as those associated with television, contemporary films, rock music, and the many influences associated with the peer culture of middle school or high school.

As I have tried to think about perhaps better ways to uncover cultural influences on mathematical thinking, the work that I have found most helpful is a chapter in a recent *Review of Research in Education.* The chapter, entitled "Culture and Mathematics Learning" (Stigler & Baranes, 1988), provides a thoughtful overview of what is known about the role of culture in mathematics learning. The authors review cross-cultural research conducted both in and out of schools with both children and adults. In a discussion of the role of culture in mathematics learning, they suggest that:

> As children develop, they incorporate representations and procedures into their cognitive systems, a process that occurs in the context of socially constructed activities. Mathematical skills that the child learns in school are not logically constructed on the basis of abstract cognitive structures, but rather are forged out of a combination of previously acquired (or inherited) knowledge and skills, new cultural input. Thus, culture functions not as an independent variable that merely can promote or retard the development of mathematical abilities, but rather as a constitutive part of mathematical knowledge itself... In short, we are claiming that culture-specific representations of number do not merely influence the development of mathematical knowledge, but in fact remain part and parcel of that knowledge.

This view is similar, I believe, to that of D'Ambrosio in his several discussions of the concept of ethnomathematics (e.g. D'Ambrosio, 1985). If it is true that culture becomes and *remains* "part and parcel" of socially constructed mathematical knowledge, shouldn't it be possible to examine the influences of culture in any mathematical context? If it is true that we need to return to a culturally appropriate context in order to identify those cultural influences, what would that context be for many of the Black, Hispanic, and Native American students attending schools in some of the country's largest urban areas? These, I think, are some very intriguing questions which need to be answered if we are to more fully understand multicultural influences on mathematics education.

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Topic Group D

Teacher Assessment Practices in High School Calculus

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Introduction[1]

The purpose of the research was to explore the meaning of grades assigned by different teachers of the same Grade 13 mathematics course, and to formulate possible explanations of any differences in meaning found to exist among the teachers. In Ontario, the high school curriculum is constrained by provincial guidelines, which specify minimum course content and length. Local school boards are responsible for implementing the guidelines in their schools, with differences consequently possible in topic emphasis, grading methods, and quality of achievement expected for a given mark.

The project began in the spring of 1986, with selection of a mathematics course. In making this choice, due regard was given to an ongoing transition from Grade 13 courses to Ontario Academic Courses (OACs). A comparison of Ministry Guidelines for Grade 13 and OAC mathematics courses (Ontario Ministry of Education, 1972, 1985) revealed that the overlap was substantial for Grade 13 and OAC Calculus. Choice of calculus also meant that the results of the survey of calculus examinations by Alexander (1987) would complement and inform the results of this study.

The choice of calculus was made in consultation with a five-member Advisory Committee for the project, consisting of:

- Dr. David Alexander, Faculty of Education, University of Toronto, and The Ontario Ministry of Education.
- Dr. Edward Barbeau, Department of Mathematics, University of Toronto.
- Dr. Gila Hanna, Department of Measurement, Evaluation and Computer Applications, The Ontario Institute for Studies in Education.
- Mr. George McNabb, Mathematics Teacher, Sudbury Board of Education, representing The Ontario Association of Mathematics Educators.
- Mr. John Scott, Mathematics Consultant, Toronto Board of Education.

The design of the study called for the recruitment of 20 teachers. Practical considerations limited choice to teachers working in the urban core of Southern Ontario. In the end, the participants were 17 teachers from 17 different schools in 13 different boards. (The teachers were guaranteed anonymity, so their names must remain confidential.) All participants held an undergraduate degree in mathematics, had at least five years experience teaching senior mathematics, and had taught the calculus course at least three times. Some descriptive information on the classes of the 17 teachers is provided in Table 1.

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[1] This is a short version of a long report under the title "Teacher Assessment Practices in a Senior High School Mathematics Course." A copy of the long report is available from the authors on request. This research was conducted with the support of the Ontario Ministry of Education, through Transfer Grant 52-1028 to the Ontario Institute for Studies in Education.

Several types of data were collected. The 17 teachers were asked to:

- complete a log throughout the Spring 1987 Semester for one Grade 13 Calculus class, recording the activities undertaken each class period and the time devoted to each activity;
- list on the log the homework and seat-work assigned each day;
- report the criteria used to arrive at student grades for the course, including tests, quizzes, examinations, and other factors (e.g., participation, attendance), along with the relative weights of each; and
- mark a common set of 20 final examination papers obtained from a class not involved in the study.

In the material that follows, the use of classroom time is considered first. Successively thereafter, attention is turned to the content of the teaching and testing, testing policy and practice, and, finally, the exam-marking study.

Use of Class Time

In their daily logs, most teachers provided descriptions that indicated the kinds, order, duration and focus of the teaching/learning activities undertaken during each class period. Four teachers, however, provided more information about topics covered in a class than about activities undertaken, and a fifth delegated the task of completing the log to different students, thus providing an uneven record. The logs of these five teachers were excluded from further consideration in this part of the study, leaving the data for 12 teachers.

There was considerable variation among the 12 teachers in number of class periods and length of courses. The number of class periods, ranged from 80 to 109 (Mean 86). Total class time for the calculus credit ranged from 96 to 114 hours (median 105). The scheduled length of class periods varied from 40 to 80 minutes, although on a given day the time actually spent in class might have been less than what was scheduled for any number of reasons.

Six categories of class activities were defined from terms used in the logs:

- Administration: taking attendance, making announcements.
- Direct teaching: presentations, demonstrations and discussions focusing on new material.
- Student practice: seat-work and board-work pertaining to new material (including handouts, assignments and orally presented problems considered in class), with opportunity for individualized instruction.
- Homework: tasks assigned for independent completion, either in class time or outside.

Table 1. Some Characteristics of the Schools and Classes

Teacher Number	School Size	Class Size	Teaching Hours	Number of Tests
1	1400	17	108	10
2	950	30	103	8
3	300	13	107	7
4	1200	15	97	6
5	1100	18	96	6
6	1200	28	105	7
7	900	22	106	6
8	1900	19	105	10
9	1900	16	103	7
10	1000	25	106	10
11	2050	24	105	13
12	1800	31	103	10
13	1500	27	110	9
14	1250	21	105	9
15	1300	13	104	8
16	1400	26	103	3
17	950	27	114	9

School size: Rounded to the nearest 50.
Teaching Hours: Including time for examinations and tests.
Number of Tests: Not including mid-course and final examinations, if either was administered.

- Review: class time used (i) to cover previously learned material, including prerequisite knowledge acquired in other courses (e.g., algebra), and content previously covered in the course, (ii) to prepare for tests and exams, and (iii) to mark or review tests, quizzes and exams.
- Assessment: quizzes, class tests and exams administered in class time.

Analysis of the time spent on each type of activity as a percentage of total time produced the following results: (a) Administration - 0 to 5 percent (median 1%); (b) Direct Teaching - 17 to 52 percent (median 26%); (c) Student Practice - 8 to 47 percent (median 29%); (d) Homework - 11 to 43 percent (median 18%); (e) Review - 4 to 14 percent (median 11%); and (f) Assessment - 8 to 16 (median 10%). Clearly, the teachers differed substantially in their use of class time.

In a search for patterns in these data, coefficients of correlation (over teachers) were computed. A negative correlation was found between total logged time (in hours) and the percentage of time spent on direct instruction. This suggests that teachers with greater amounts of class time tend to do less direct teaching. This interpretation is corroborated by the findings that percentage of time for direct instruction correlated negatively with percentage of time for homework and student practice, whereas the percentages of time for homework and student practice were each positively correlated with total time. The largest percentage of time, overall, was devoted to student practice. Negative correlations between the percentage of time for student practice and the percentages of time for review and for assessment suggest that teachers who place relatively high emphasis on practice in their teaching of calculus place a relatively low emphasis on review and assessment activities.

Content of Assignments and Tests

The information about content came from the daily logs of homework and seat-work assignments that were maintained by the teachers, and from the quizzes, term tests and exams (plus marking schemes) submitted by the teachers. For this and the remaining parts of the report, information has been included from all 17 teachers.

A comment is in order at the outset of this section lest our results be taken as implicitly critical of the teachers who participated in the study. The Guideline for the Ontario Grade 13 Calculus Course (Ontario Ministry of Education, 1972) mandates broad content areas, but not relative importance. Thus, we were not investigating whether some teachers exercise better or worse judgment as to what should be in the curriculum. Instead, we were investigating differences in the judgments made by qualified and experienced teachers.

A scheme was devised for categorizing the content of the course. The starting point was the 1972 Grade 13 Calculus Guideline (Ontario Ministry of Education, 1972) and the contents of two Ministry approved texts for the course. When a satisfactory version of the category system had been produced, two students, both about to graduate from a B.Sc./B.Ed. program in mathematics education and both experienced in practice teaching the calculus course, reviewed and revised the system, and then applied it to questions on teacher-produced handouts, quizzes, tests and exams.

The category system includes 126 topics. In applying this scheme, the two students achieved an inter-rater agreement of 88%. To simplify reporting, the 126-topic scheme was collapsed into 14 categories; at this level, inter-rater agreement was 97%. The 14 Content Categories were themselves classified into six Content Groups: I - the basic skills of calculus (limits, sequences and series, differentiation, and integration); II - proofs of basic theorems; III - applications of differentiation skills, also referred to as differentiation graphing (slope and equation of a tangent, curve sketching); IV - applications of integration skills, also referred to as integration graphing (area between curves, volume of revolution); V - situational problems (motion problems, related rates, maxima and minima); and VI - optional topics (complex numbers, polar coordinates), optional in that a school might decide to deal with these topics in one of the other senior mathematics courses - Algebra or Relations and Functions.

Content of Assignments

Table 2 is a record of the percentage of homework and seat-work questions assigned during the course by each teacher, the questions having been classified according to the six content groups. In addition, the total number of question on which the percentages are based is given for each teacher. Differences among teachers in total number of questions assigned was great. The median was 1037 questions, but the range was from 460 to 1622 questions.

Several results in Table 2 stand out. First, the emphasis on basic skills (Content Group I) was high for all teachers, ranging from 41% of questions assigned to 76%. Second, the greater the emphasis on basic skills, the more almost everything else was de-emphasized. Third, little attention was paid to questions involving proofs and first principles, although increased emphasis on these issues is mandated in the new Ontario curriculum.

Table 2. Percentage of Assigned Questions by Content Group and Total Number of Assigned Questions

Teacher	Content Group						Total
	I	II	III	IV	V	VI	Number
1	49		12	5	13	20	1622
2	56		11	5	25	3	836
3	65		13	5	12	4	1498
4	53		21	9	17		456
5	74		8	8	6	5	1003
6	64		10	8	18		460
7	66	1	9	4	20		917
8	76	2	11	1	11		1388
9	49		14	11	14	12	1395
10	41		24	15	18	2	1341
11	57	2	17	5	19		967
12	70		10	5	14		1083
13	46		14	11	21	8	970
14	54		19	5	23		1037
15	66		10	12	12		1343
16	50	2	13	5	18	12	850
17	54		18	8	19		1274

Note: The percentages for a row may not sum to 100 due to rounding error.

Note: See text for a description of the content groups.

Content of Tests

All questions used by teachers in quizzes, classroom tests, and exams were categorized. From teacher-supplied marking schemes and weighting systems, the relative (percentage) weight of every question in the calculation of final grades was determined. These relative weights were summed to yield the percentage of marks toward the final grade that were allocated by each teacher to questions in each of the six Content Groups (Table 3). The percentages in Table 3 indicate a relatively heavy emphasis in testing on basic skills (Content Group I). Emphasis on Content Group II (proofs of basic theorems) was relatively low. The teachers varied considerably in the degree to which they emphasized each content group, but this variation is especially noticeable for Group VI (optional topics).

Effect of exemptions. A study was made of the effects of exemptions from final exams on the content of the assessments of student achievement. Four teachers followed a policy, mandated by the board or the school, of exempting students with a high term mark (typically 65% or more) from the final exam. In one of these four classes, the final marks of the exempted students were based on assessments of substantially different content than the final marks of non-exempted students.

Table 3. Percentage of Test and Examination Marks by Content Group

Teacher	Content Group					
	I	II	III	IV	V	VI
1	35	2	14	12	20	17
2	50	3	13	3	20	11
3	51	7	16	4	15	6
4	33	7	18	12	30	
5	43	2	13	16	20	6
6	50	3	13	13	21	
7	46	2	18	8	26	
8	58	1	18		23	
9	37	3	14	11	22	14
10	33	2	23	12	27	2
11	35	6	27	8	24	
12	48	8	15	8	22	
13	26	4	19	15	28	9
14	33	5	24	7	31	
15	48	6	14	14	18	
16	38	4	15	6	25	13
17	34	10	15	14	25	

Note: The percentages for a row may not sum to 100 due to rounding error.

Note: See text for a description of the content groups.

Effect of discarding test results. One teacher divided the semester into five segments, referred to as terms. Each term contained up to six short quizzes and one test. The tests for Terms 3 and 5 were considered to be the mid-course and final exam respectively. Students were allowed to drop the test and quiz results for one of Terms 1, 2, or 4 from the calculation of their final grades. Dropping the test and quiz results for Term 4 produced final marks based on an assessment of somewhat different content than dropping the test and quiz results for either Term 1 or Term 2.

The foregoing results point to problems with the practices of exemptions and selectively discarding test results. The expectation of many consumers of high school grades is that they reflect achievement of the same curriculum. By exempting some students from final exams or discarding some term results from the calculation of final grades, with different results discarded for different students, marks within the same class will reflect achievement of different kinds. Unbeknownst to consumers, differences among such marks are uninterpretable.

Comparing the Contents of Tests and Assignments

A comparison of corresponding percentages in Tables 2 and 3 reveals a general tendency for teachers to do less testing than assigning of content in Group I (basic skills), slightly more testing than assigning of the content in Groups II and III (proofs and differentiation graphing), and considerably more testing than assigning of the content in Groups IV and V (integration graphing and situational problems). At least some of this pattern must be due to differences in the relative size of questions for basic skills compared to that for the other content groups. (Ten differentiation exercises may require less time to complete than one applications problem.) The greater emphasis on basic skills (Group I content) in assignments than in tests may also reflect the belief that practice makes perfect, not the belief that basic skills are especially important. Moreover, the greater emphasis on Group II content in testing than in assignment may mean that proofs are considered important, but are dealt with by class instruction and demonstration rather than by assigned exercises.

Grading Practices

The data collected about testing and grading practices were used to study the grading processes that were used and the actual grades that were assigned.

The Process

The 17 teachers were found to use 22 different grading systems. More than one system was in use by each of the four teachers who followed an exemptions policy - the grading system for a student of these teachers depended on whether the student had been exempted from the final examination. Also, the teacher who set aside some of a student's marks in calculating the final grade employed at least two different systems. (Refer to the description of this method given in the previous section of the report.)

Two grading criteria were used almost exclusively: tests, usually administered at the end of units of work, and examinations, administered near the mid-point of the course or the end or both. The number of tests ranged from three to 13, the number of examinations from one to two. For seven of the 17 teachers, tests and exams represented 100% of the students' grade. For the other 10 teachers, the weights for tests ranged from 30% to 80% of the final grade. The additional criteria used by these teachers included quizzes (six teachers, weight ranging from 2% to 20%), assignments (six teachers, weight ranging from 3% to 6%), and a subjective mark for participation (four teachers, weight ranging from 5% to 20%). Six of the teachers gave no mid-course exam, while the four who followed exemption policies had no final exam for the exempted students. The remaining grading systems included both mid-course and final exams. (We use the designations first-half and second-half of the course rather than first-term and second-term to avoid possible confusion over the meaning of term and semester. All our data were collected in semestered schools during the Spring Semester. The break between first-half and second-half of the course occurred about mid-April.)

The teachers combined the different test and examination marks into final grades in several different ways. The marks for a test or exam were either (a) weighted according to the number of marks in each (simple summation) or (b) re-weighted to make the weights of each test or exam equal or (c) re-weighted to reflect the teacher's perception of the relative importance of the topics covered by each test or exam. Similarly, the halves of the semester were either weighted equally or unequally. Five teachers weighted each half equally (at least for some students). The other eleven teachers weighted the first-half of the course less (about 30%) than the second-half (about 70%).

The time spent on testing activities (including exams) varied enormously, ranging from nine hours for one teacher to more than 17 hours for another. On average over the 17 teachers, 8.8 hours (range 3 to 15.2 hours) were spent in writing 8 tests (range 3 to 13 tests), not including mid-course and final examinations. The lengths of tests varied from 25 to 75 minutes. The total number of test questions administered during the semester averaged 104, and ranged from 53 to 180.

The cycles of teaching and testing throughout the course were examined. Nine of the teachers seemed to have more regular cycles of teaching and testing than the others. All teachers tested at more-or-less regular intervals throughout the first-half of the course, but the testing patterns for eight teachers became erratic in the second half. The teachers who followed more regular teach-test cycles also gave a greater number of tests on average (10 compared to 7).

Another factor that was considered is the number of different topics covered in a test. The taxonomy of 14 content categories was used to describe test coverage. The number of categories covered ranged from one to eight per test over all the tests given by the teachers. The average (for a teacher) of the number of categories per test ranged from

2 to 4.7. All but two teachers gave at least one test covering only one content category. For five teachers, the final exam included content categories that had not been covered in class tests. A possible reason is that the final examinations used by these teachers were set by other calculus teachers. When one teacher in a school sets a common final exam, that individual requires a certain amount of prescience to set questions that the students taught by other teachers have had an opportunity to learn. A class that proceeds more slowly than expected, for example, is likely to be disadvantaged by the exam.

The Grades

The average mid-course and final grades for each class and the difference between the two were calculated. The distinction between mid-course and final grades is important in Ontario for Grade 13 courses offered in the Spring Semester. Early in April, Ontario universities begin their admissions process. For the courses a student is taking in April, the school submits interim grades (normally the mid-course grades in semestered schools) to the Ontario University Applications Centre, and students receive a conditional acceptance or rejection based on these grades. A concern expressed by several teachers in the study is that students become less motivated to work once the interim grades have been submitted.

Averaged over all 17 teachers, the final grade was 67, six points less than the average mid-course grade of 73. For every teacher, the mean final grade was either lower than or at best equal to the mean mid-course grade. The range of mean final grades was 54 to 75, that of mean mid-course grades, 61 to 79. Although there is a high correlation between the two sets of grades for a teacher, the difference between a teacher's mean mid-course and mean final grades was as much as 15%.

A comparison of term marks with final exam marks turned up two results of interest: in the 13 classes in which all students wrote a final exam, the final exam marks were lower than the term marks by about 12 percentage points (58% compared to 70%). Also, final exams tended to discriminate more than term marks. For the 13 classes with no exemption policy, the standard deviation of final exam marks was about 50% larger than that of term marks.

Despite all the differences found in grading processes and in the grades themselves, no clear indication was found in the data provided by the 17 teacher-participants that the observed differences in grading process were related to the observed differences in grades.

Responses to the Common Marking Task

A study was made of the extent and nature of the variation among the 17 teachers in the standards they applied in marking a set of examination papers. A final calculus examination, administered in June 1987 to students in a school not otherwise involved in the study, yielded a set of 20 papers that spanned a range of quality. The 17 teachers were given the 20 papers, and each was asked to prepare a marking guide and then mark the papers against it. In addition, the teachers were asked to provide written comments, should they care to make any, about the examination and performance of the students.

The Examination

The exam contained 11 questions, several of which contained three or more sub-questions. In total, the examination consisted of 37 sub-questions. The content of each question can be described briefly as follows:

1. Find the point on a quadratic function where the tangent has a specified slope.
2. Obtain the derivative with respect to x for each of 13 different functions of x - six logarithmic or exponential functions, four polynomial functions, and three trigonometric functions.
3. Find integrals of nine functions - three trigonometric functions, four logarithmic or exponential functions, and two polynomial functions.
4. Integrate using the method of parts.
5. Find the limits of three polynomials.
6. Solve a problem involving a) acceleration, b) velocity, and c) the position of the particle in motion after a specified amount of time has elapsed.
7. Find the area enclosed between two trigonometric functions of the same variable over a specified range of the variable.
8. For a cubic function, a) find the coordinates of all maximum and minimum points of the function, b) find the coordinates of all points of inflection, and (c) sketch the function.
9. Find the rate at which the distance between two moving objects is increasing or decreasing, given information about the direction and rate of motion of the two objects.
10. Prove that the formula (given) for the volume of a sphere can be obtained as a volume of revolution.
11. Find the radius and height of a cylinder, such that the cylinder will have a given volume and an unspecified but minimum surface area.

The exam was strongly weighted toward the testing of basic skills. According to the scheme for categorizing homework and test questions, the basic skills topics (Group I) contained most of the exam questions (26 sub-questions). (The total number of sub-questions for all other content groups combined was only 11.)

The Marking Guides

The marking guides prepared by the teachers indicate the maximum number of marks to be awarded for responses to each sub-question. There was considerable variation among teachers in the total number of marks allocated for perfect performance. The smallest of the maximum marks was 78, the largest 144, and the median 116. The teachers also differed in their allocations of marks to individual sub-questions. For example, wholly satisfactory performance of Question 7 was rewarded with as many as 12 marks by two teachers, and as few as 4 marks by one teacher.

What accounts for differences such as this? For the most part, they seem to stem from differences in the number of steps or stages to an answer that are awarded marks. Another difference was in the use of bonus marks and deductions. Several marking guides indicated bonus marks for good form and for stating the answer in a complete English sentence. Several others indicated deductions for failing to include the constant of integration in answers or for failing to specify units in answers to questions involving measured quantities. These bonuses and deductions, when used, were either one mark or one-half mark.

Despite the obvious disparities found among the teachers' marking schemes, the teachers were in general agreement as to the order of importance of the examination questions and sub-questions. A coefficient of correlation was computed for each pair of teachers between the maximum marks allocated to the questions and sub-questions of the examination. All the intercorrelations were substantial, ranging from 0.76 to 0.95, with a median of 0.89. Clearly, the teachers possessed very similar views of the relative importance of the questions and sub-questions of the examination.

This does not mean that the teachers thought the exam was particularly good, at least as judged by coverage of the calculus course described in the Guideline (Ontario Ministry of Education, 1972). Several teachers objected to the strong emphasis in the exam on integration. Three teachers noted the lack of coverage of polar coordinates and complex variables. Two teachers pointed to the coverage in the exam of trigonometric functions, with one feeling it was inadequate and another thinking it was overemphasized. It was observed by two teachers that volumes of revolution, trigonometric limits and differentials were given short shrift. And three teachers objected to the preponderance of skill-type questions, and the lack of questions involving problem-solving. Note that volume of revolution, polar coordinates, and complex numbers are optional topics. (We did not suggest that the exam was a model for all teachers to emulate; it was only a means to the end of studying differences in marking behaviour. In fact, for present purposes we eliminated the section of multiple-choice questions that appeared in the exam as originally administered.)

In a draft document entitled A Handbook for the Examination Component of Evaluation in the OAC - Calculus (Ontario Ministry of Education, 1987), attention is paid to the number of marks awarded for arithmetic and algebraic simplification in answers to OAC calculus examination questions. An analysis was made of the marking guides in an attempt to assess the extent of differences among them in the proportions of marks awarded for arithmetic, algebraic simplification, and other skills and knowledge (from earlier grades) compared to the calculus skills and knowledge to be acquired in the course. (This analysis was possible for 13 of the 17 guides; four guides indicated only total numbers of marks per sub-question.) The percentages of marks for calculus as opposed to other kinds of mathematical knowledge and skill ranged from 60 to 76, with a median percentage of 66. Thus, there was some variation, but not a lot, in the extent to which knowledge and skills peripheral or prerequisite to calculus were rewarded.

Total Student Marks

The comparability of the total marks assigned each paper by the 17 teachers was assessed by computing for each pair of teachers a coefficient of correlation between the total marks assigned the 20 papers. These coefficients were uniformly high, ranging from 0.81 to 0.97, with a median of 0.92. Obviously, there is close agreement among the teachers in the relative orders into which they placed the 20 papers.

Grading achievement in calculus and other subjects involves more than rank-ordering a group of students. Determinations of fail and pass and honours are usually required. How well, then, did the teachers agree as to which papers represented failing performance, which represented passing performance, and, of the passes, which represented honours? To address this question, the total mark a teacher assigned a paper was converted into a percentage of the total mark given in the marking guide. Here we find evidence of inconsistency in standards. Three teachers assigned no paper a mark in the honours range, and one teacher assigned failing marks to seven papers. On the other hand, seven teachers assigned no paper a failing mark, and one teacher assigned percentage marks of 80 or more to 10 papers. Variation in standards is apparent, despite the fact that the teachers ranked the papers for quality in very much the same way.

The teachers offered comments, several of which are relevant here. For example, the stiffest of the markers directed comments at student performance: the solutions were poorly developed, diagrams were missing, and the responses lacked clear, concise statements. These might be described as errors of form in the student responses. (Although the marking guide of this teacher indicated five marks for the first question, no student was awarded more than three. The apparent reason for this was the failure by all 20 students to include all the steps listed in the teacher's model answer. Thus, for example, no mark was awarded for finding the y-coordinate of the answer if the determination of this coordinate had not been made explicit, even when the student's answer did contain the correct coordinate.) Another of the hard marking teachers also noted the errors of form as a problem with student answers, but so did two teachers who were in the middle of the group as regards severity of marking. The fact that three other hard-marking teachers did not mention form of answer as a problem, suggests that this factor does not fully explain the source of severe marking standards.

In fact, no type of comment appears to distinguish the hard from the easy markers. The easiest marker described the marking exercise as boring. This teacher also described the exam questions as being all of the skill and recall type, and as not requiring higher level thinking skills. It is not apparent that adopting this point of view should cause one to be an easy marker, although another easy marker also commented on lack of problem solving questions on the exam. So too, however, did a teacher in the middle of the group for marking severity. Perhaps more in line with what might be expected, given the severity of his/her marking, was a teacher's registration of disappointment in the students' problem solving abilities. Other comments were made to the effect that the exam was too easy, was of uneven difficulty, with questions being either very easy or very difficult, was too long, was "too tricky by half", and was nicely balanced between straight-forward and challenging questions.

One factor, however, may distinguish hard from easy marking teachers. Ten of the teachers followed one textbook (published by Gage) and six others followed another textbook (published by Holt). (One teacher used a set of notes, and followed no published book.) The teachers who used the Gage text were, on average, relatively easy markers, whereas the teachers who used the Holt text were, on average, relatively hard markers.

In a final attempt to understand differences among teachers in marking standards, a study was made of the marks assigned by three teachers - the hardest and easiest markers, and a teacher at the centre - to three students - a high, middle and low scorer. It was found that these teachers differed relatively little in the percentages of marks each awarded for performance of the 26 basic skills sub-questions. Against this standard, however, the corresponding results for the other sub-questions are dramatically different. For example, one of the students was awarded about half the marks allocated by the easiest marking teacher for performance of the other-than-basic skills sub-questions, but the other two teachers assigned only one-fourth the marks they had allocated for performance of the same sub-questions. These results suggest that the main source of the difference among these teachers lies in their marking of the exam questions that test other-than-basic differentiation and integration skills.

Summary

The analysis of use of classroom time showed relatively substantial differences among the teachers in their allocation of class time to different categories of activities - administration, direct teaching, review, homework, practice, and assessment. Moreover, those with more class time available expended a smaller percentage of time on direct instruction, and allocated greater percentage to homework and practice.

Substantial differences were also found among teachers in content emphasis.

- Teachers varied widely in the number of questions assigned as homework - from under 500 to more than 1600. A large part of this variation was accounted for by differences in the number of questions on basic skills.

- Teachers varied considerably in the extent to which they emphasized different topics in their assignment of questions. For example, the number of questions on basics ranged from 40% to 75% of the total number of assigned questions.
- The emphasis on basic skills was less in the tests than the assignments, whereas the emphasis on other content groups was greater in the tests than the assignments.

Some of the within-teacher differences between assignment and testing emphases are probably intentional, as when a teacher decides to test only at the top of a small hierarchy of skills or knowledge, ignoring the prerequisite skills and knowledge that had been included in assignments. Conversely, a teacher may teach a difficult concept and choose not to test it because most students failed to grasp it. Whether or not discrepancies between teaching and testing constitute a problem to be corrected is a matter not addressed in the present study. All we have done here is provide evidence that such discrepancies as these exist.

From the analyses of the grading practices of the 17 teachers, it was learned that examinations and term tests were the two main determinants of student grades. All students in all classes wrote a minimum of one examination, as required by provincial policy. However, the nature of this examination varied. For the four classes following an exemption policy, the majority of students took their only exam on material learned in the first half of the semester. For six other classes, the only exam was a final exam based on the entire semester's work. In the remaining seven classes, both a mid-course exam and a final exam were required. The time that students from different classes spent in an examination situation ranged from 2 to 4.5 hours. The final examination mark was weighted from 15% to 40% of the student's final grade and, when a mid-course exam was administered, the resulting mark was weighted 9% to 30% of the final grade. In some cases, the calculus content topics that were tested during the semester were not emphasized to the same extent on the final exam. This might be attributed to the fact that, while the setting of term tests was usually the teacher's responsibility, the final examination was the mathematics department's, and not necessarily the participating teacher's, responsibility.

Term testing was found to vary in the following respects: (i) number of tests (ranging from 3 to 13), (ii) number of items comprising the tests (from 53 to 180), (iii) amount of classroom time used for test taking (from 3 to 15.2 hours), and (iv) schedule of tests (sporadic or regular). Term tests were weighted from 30% to 80% of the final grade.

It is evident from this study that students taking Grade 13 Calculus in the Spring 1987 Semester from the 17 teachers in this study did not demonstrate their achievement in calculus through a common process of assessment and grading. It is reasonable to question whether or not it would be beneficial for students to have experienced similar grading processes, and to have been judged according to similar standards on similar criteria of achievement.

The empirical study of the marking process revealed the following:

- The presence of substantial agreement among the 17 teachers as to the relative importance of the examination questions.
- Substantial agreement among the teachers as to the relative quality of the 20 student papers that were marked.
- Substantial disagreement among the teachers as to the absolute quality of the 20 student papers.
- The marking standards of teachers varied, to a limited extent at least, as a function of the textbook being used.

These results pose a challenge for the Ministry of Education in a jurisdiction where there is no external mechanism - no common, province-wide examination - for aligning standards of calculus achievement. This challenge has not been lost on critics of education in Ontario, and it has not been ignored by the Ontario Ministry. A Handbook for the Examination Component of Evaluation in the OAC - Calculus (Ontario Ministry of Education, 1987) was developed for the purpose of fostering a greater degree of consistency in calculus examinations across the province. The handbook addresses several problems found in the present study - (i) the practice of granting exemptions from final examinations and the variation in value of final examinations, (ii) the emphasis on basic skills to the virtual exclusion in teaching and testing of problem solving, and (iii) the wide differences in amount of testing and other assessment activities. But the results of the present study suggest that consistency in assessment will be increased only when other steps are taken as well.

These steps include the following: increase the consistency of what is taught; increase the consistency with which those examination questions that test other-than-basic calculus skills are marked; increase the consistency with which displays of other-than-calculus knowledge and skills are marked; have more than one teacher independently mark every student exam paper, and set the exam mark equal to the average of the several marks; and ensure that exams are sufficiently long and numerous so that all content is covered and so that the impact on a student's grade of performance on any one question or type of question is minimized.

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Topic Group E

ONTARIO IAEP RESULTS

Dennis Raphael

Ontario Ministry of Education

Dr. Raphael discussed the results of 13-year-old Ontario students on the 1988 International Assessment of Educational Progress. The results pertain to both Anglophone and Francophone student achievement in relation to achievement in the Canadian provinces and other countries.

The presentation was based on a paper read at the Annual Meeting of the American Educational Research Association in March 1989. The paper is available through ERIC. ED 306259

Previous Proceeding

The following is the list of previous proceedings available through ERIC.

Proceedings of the 1980 Annual Meeting	-	ED 204120
Proceedings of the 1981 Annual Meeting	-	ED 234988
Proceedings of the 1982 Annual Meeting	-	ED 234989
Proceedings of the 1983 Annual Meeting	-	ED 243653
Proceedings of the 1984 Annual Meeting	-	ED 257640
Proceedings of the 1985 Annual Meeting	-	ED 277573
Proceedings of the 1987 Annual Meeting	-	ED 295842
Proceedings of the 1988 Annual Meeting	-	ED 306259