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EDITOR'S FORWARD

I should like to thank all the contributors for submitting their manuscripts for inclusion in these proceedings. Without their co-operation it would not have been possible to produce the proceedings.

Special thanks must go from all of us who attended the conference to the organizers, and particularly to Sandy Dawson, Tasoula Berggren and their team of helpers who did so much before and during the meeting to make it such an enjoyable and profitable event.

An innovation at this year's meeting was the taping of some of the sessions. This has permitted the inclusion of the questions following the lectures and the subsequent follow-up discussions. Whilst some judicious editing seemed appropriate, I do not believe I have misrepresented anyone; my sincere apologies if I have been guilty of a *faux pas*.

I hope these proceedings will help generate continued discussion on the many major issues raised during the conference.

Martyn Quigley

April 14, 1991

ACKNOWLEDGEMENTS

The Canadian Mathematics Study Group wishes to acknowledge the continued assistance of the Social Sciences and Humanities Research Council. Without this support neither the 1989 meeting nor the production of the proceedings would have been possible.

We would also like to thank Simon Fraser University for hosting the meeting and providing excellent facilities to conduct the meeting.

Finally, we would like to thank the contributors and participants, who helped make the meeting a valuable educational experience.

Lecture One

Values in Mathematics Education

U. D'Ambrosio

UNICAMP

Values in Mathematics Education[1]

I am so pleased to be here and I feel that it's a great honour to be invited to come and talk with you and explore some ideas and discuss them a little bit and of course it is very important for me to get your reactions to my ideas. The arrangement of this conference is such that we have the opportunity of learning so much because I'm sure tomorrow you will come with ideas that we will work on many times.

I will be talking about values in mathematics education, but it will be more than that. It will be about education in general, and you'll see some reference to mathematics along the way, so please don't be concerned that there may be less mathematics than the title suggests. In the end I think we'll look into mathematics in a broad way and this is what I have been trying to do during these last few years.

If we are to talk about values in mathematics; that is difficult indeed. Everyone that does mathematics knows how important it is, and even those who do not do mathematics, they recognise that it is important and society supports this reasonably well. If you want to be a mathematician, it is a good profession and is well respected. You can find jobs as a mathematician and you can derive lots of excitement in doing mathematics. It is something enduring, one of the most beautiful things that you can do anywhere in the world and if you move from one country to another, the mathematics goes from one place to another absolutely the same. If you are a professor in Basil you could be a professor in China; you can be a professor anywhere. If you do mathematics you are doing something which is easy to justify as important. There was a big article in *The Economist* (of London) saying that we cannot be a citizen of the Twentieth Century without knowing mathematics. So everybody says mathematics is important.

But what are the values in mathematics education? This seems a little more complicated. Which takes precedence? Education, to satisfy my perception of the values of mathematics though I put mathematics in my perception of the values of education; and this is probably the main issue which I want to discuss.

An alternative title for my talk could be *Karlsruhe 15 Years After*. As you know at Karlsruhe in 1976 we had ICME3—the International Congress of Mathematics Education—and in this congress there was a very special feature. The congress was different than any other before and any other after. The congress was a process and we were invited to be mainly speakers or leaders of groups with the mission of writing a book for UNESCO. The book came out called *New Trends in Mathematics Education IV* and each one of us invited to write a chapter in this book started to get together about two years before the conference and the group of leaders of each of the themes of the congress used to meet every four, five, six months and this is how Tom and I got

1 **Editor's Note:** This lecture, the question period and the follow-up discussion (and also the question period and follow-up for Lecture 2) were transcribed from tape recordings made during the sessions and it was necessary to do some editing. I apologise most sincerely if in so doing I have inadvertently misrepresented anyone.

together several times because we both were responsible for one chapter of this book, that is for one major session of the congress.

In my session, which came out as Chapter Nine, the question was *Why Teach Mathematics?* So when I talk about values in mathematics education I have to look at this question, why teach mathematics? What are the values in teaching mathematics? This question arises very naturally and in fact this was one of the major sessions in the congress of Karlsruhe. As I said when we were given this task of writing a chapter for the book, the chapter should be written before the congress, sent to several people worldwide, and discussed in the congress. Then we would get back together to rewrite this chapter according to the reaction of the participants of the group, and in this way the chapter in *New Trends in Mathematics Education* was developed. Probably to us the most elaborate and most costly book written in recent years in mathematics education because the process, the entire process, took about two years. What did I do as the one responsible for this chapter, *Why Teach Mathematics?* I started by writing to many mathematicians, teachers, professionals, students and I sent out about four or five hundred questionnaires getting their reaction, which they duly sent back. From the returned questionnaires I prepared new questionnaires, sending to other people and getting comments, and some people answered with papers almost. Some people took great care to answer very carefully and this helped me so much in preparing the paper. Thus I got from everywhere in the world, from different parts of society, the feelings of why should we teach mathematics. Why is it important to have mathematics as a major subject in the school system? Out of all these appeals some of the basic values came as, well, because it's useful. Many people say that it is useful—and the general feeling we seem to have is that we teach mathematics because it's useful. If you go and question teachers, they will say it's useful. Many people say this is the reason mathematics is in the curriculum.

People say mathematics helps to develop clear and precise thinking, that we mathematics people are thinking more precisely. Others say mathematics is of great aesthetical value, that it is beautiful. Others say mathematics is a cultural asset. Still others say it provides an important instrument for our critical view of modern society.

Of course, I cannot question a certain usefulness in a number of topics that we teach in school, but I would claim that many of things that we teach in school are absolutely useless. There are many mathematical things that we have seen in our lives only as students and as teachers. Someone who does not teach for many years probably will have never seen many of the things that he used to teach, because we never find these in life. Some will say that mathematics will prepare or develop clear and precise thinking, but there are so many mathematicians that act so stupidly, and there are so many lawyers who say "I was always a failure in mathematics" and "I was so poor in mathematics" that are so astute in society. There is no correlation at all and if there was a subject in school that did not develop clear thinking, then that subject should not be in the school. So everything in the school helps to develop clear thinking, and mathematics is one of these subjects. I think Freudenthal was the one who emphasized this very much, that there is a preposterous way of mathematicians saying that mathematics is what teaches us to be clear in our minds. Lots of people have good minds and are completely naïve in mathematics. So only this would not justify teaching mathematics.

How about its aesthetic value, is it beautiful? Beauty is something very relative and to look at beauty in the modern movements in art and through its history, and in different cultures, usually the big movements in art are a break with standards, a break with existing paradigms. Usually creativity breaks with paradigms and so things that break completely out of any mathematics aesthetics or mathematical rule of thought can be beautiful because they please so many people. To say that something is beautiful and that that beauty is understood the same way by everybody—this is unsustainable. It's beautiful for some; not so beautiful for others and not at all for yet others.

Some say mathematics is a cultural asset. A cultural asset of whom? Of the central European powers in the Mediterranean? I cannot say that this is a cultural asset of the Incas. The mathematics that we teach in our school: this is Greek. All the heroes that we are used to talk about are heroes from foreign countries. As our children in Brazil or in Mexico or in Africa or even in China the heroes that they know when we talk about mathematics—our Euclid, our Archimedes, or if you will, Newton, Euler. How can you say that this is a cultural asset for all the other cultures? In fact the mathematics of today is a Eurocentric form of knowledge which has been imposed on the entire world in the process of conquest and colonisation. (This is another discussion I will not pursue now, but you can imagine where I would go if I could continue for one hour about this.) So let's not talk more about mathematics as a cultural asset because it is a very weak argument.

"Mathematics provides an important component for a political view of modern society". Indeed. Probably this is the very important thing that comes out when we talk about values and probably is the least regarded as important. This is because our society is based on data; we always justify actions, political actions, with data, with numbers. The government defines the problems saying we have to increase such and such to so much percent and they give us statistics and so forth. When you start to talk with people about the important issues affecting society that translate into political terms, they always come in with arguments using mathematics as the final word. Concluding my manipulation of statistics; we do terrible things in society. I come from a country where this is very, very important. Our inflation rate given by the government, by word of the president, is zero. Our inflation rate given by some organs of the government is three percent, and the workers union do their calculation and say it is thirty percent. So we have zero percent, three percent or thirty percent and you can manipulate these data according to your political design. This is true all over the world—I can give multiple examples of this. Thus the manipulation of data became something very characteristic of modern society. To take a critical view of data, through mathematics, is something very, very important and it is a value of mathematics which should be taught in the school for everyone.

I can summarise these five views in a few words.

1 A utilitarian value,

2 An ethical value, (that's if you seek correctly you develop a concept of truth, you develop a concept of precision which leads to an ethical value for mathematics)

3 An aesthetical value

4 A cultural value (maybe)
5 A socio-political value.

What we currently see in our school systems is a highest—it's almost total—emphasis given to 1, and when we reach 5 it is practically nil. You look at the curricula, you look at the books, you look at the standards and you find there is an over emphasis on the utilitarian value of mathematics. They always say it is important, it will help you to solve problems, real problems. All these are things that give you the emphasis on the utilitarian value—mathematics is useful. This is why it is justified to be a major and such an important part in all the school systems all over the world.

As for the ethical value, there are some teachers that still demand rigorous thinking, but this going, becoming less and less important. There are no more demonstrations, no more geometry, no more proofs; a little proof, just a little bit, but the emphasis is on the utilitarian value.

Aesthetical? Drawing, using ruler and compass to do beautiful things, the analysis of ornaments, these are practically gone.

Cultural? We don't talk about the history of mathematics, this is not part of any teaching. People mainly don't refer to other cultures. Nothing. So the cultural issue has also very, very little attention paid to it.

And socio-political, practically nil. No one discusses a newspaper piece and tries to analyze the graphs and the statistics in the newspaper and draw some mathematics out of this. It is too easy for the students to think because these are numbers, these are precise, that this is truth. But you have to look into this in a critical way. You must ask *What's the meaning of these numbers?* This is not given.

So my proposal is to restore a balance; to restore a balance by looking into the dimensions of mathematics, and their place in education. How do these dimensions of mathematics get into education? That is, how did mathematics find its way into education, and by finding its way into education how do its dimensions fit into education? This idea of looking at the dimensions of mathematics and understanding a little better the essence of mathematics, leads us to the values of mathematics as a form of knowledge and then enables us to see the insertion of mathematics into education. This is a definition of mathematics education. It is a part of education; it is a part of mathematics. I have looked at mathematics education as this question of the dimensions of mathematics when they are brought into the educational system.

When we talk about mathematics education as a discipline, we have to look into the mathematics which is the subject of our teaching and try to understand its philosophy, its history, the building up of mathematical knowledge and all the values of mathematics as such; and at the same time, separately, we have to look into what education has to do with society as a whole? How does this relate to children, to the future, to building up of society and how does mathematics play a role in this?

This discussion about mathematics education begins with Plato (the first good source about mathematics education). Plato is very clear about why we should teach mathematics. He said there is a very utilitarian value, the Egyptians used to teach this to all the children. The children perform so well, so we Greeks should do the same thing for all our children—that is the utilitarian value.

Mathematics is very important as a tool in selecting the elite. It is very interesting that the most important parts of mathematics education, that is mathematics inserted in the educational system, come in the book called *The Republic*, the book of politics. So it is a political discourse: when we talk about education—education is a political discourse.

In this looking into education, we find again the idea of the Ptolemaic and Copernican metaphor, if I may call it this. We live sort of in the Ptolemaic age. Our sun: everything gravitates around our subject. We make school, education as a whole, and make children, the teachers; they all gravitate around this. As if they take their energy, their motivation out of this. So mathematics is the sun; it is the source of energy for all the educational system for children. A child that does not get this energy is dead. It fails in the school, is a failure in society, is put aside by many, many, many sectors of society.

We should go to a Copernican era where the thought was that all our energy, all our life comes for our future as children, and make education and mathematics around the people; in the essence, mathematics is a product of the people. People produce mathematics. So the energy comes from the people and I symbolize the people with children as the people of the future, and we will have more and more mathematics out of this energy *if* these are not treated as mere satellites of a body of knowledge. So the focus, that should be the children.

Thus, I replace my initial question, the Karlsruhe question, "Why teach mathematics?" by another question; subordinated this question becomes "Why education?". So let us think a little bit and not talk that much about mathematics, but talk about education as a whole. Why do we have education? I think this is a very important question. We are putting lots of our resources, human resources, natural resources, all sorts of resources into education. There are some people who are advocating the end of educational systems, of some radical thinking—away with schools! I think we have to keep schools but we have to look carefully why we keep schools, why do we put so much resource into our schools? A good amount of the energy of society goes to schools. It should be much more, we should increase the resource for education. Why? Well, we are a privileged species in the animal kingdom. We are basically animals who are in some differentiation from other species. As animals, we have all the animal needs. We have needs for survival, we have needs for continuation and preservation of the species and many others. But other than these pure animal needs, we have some human needs and this is what distinguishes us from the other species. We want *to explain*, no other animal wants to explain. We want *to understand* and we want *to transcend*. The struggle for survival is very important: survival and preservation of the species. But we have something that no other animal has; this call for transcendence.

We want to transcend—but transcend what? Transcend our own existence. We want to know how we came to this world. We want to know what we are doing here, how do we fit in this world, and we want to know where we go. We are not satisfied with just knowing this; we want some way to conquer this. We want to go beyond this.

If you look at the history of mankind, all the cultures, *all* the cultures, there is this drive towards transcendence. We want to go beyond our existence. But this drive to go beyond our existence makes man the animal who wants as an animal to survive and at the same time to transcend, and this how art was developed. This is how religious thinking was developed and in trying to transcend, you try to explain, to understand what's going on with yourself, with others, and you try to explain and to understand and in this way you create forms of knowledge.

Learning basic things for survival, is something every animal does. We know that molluscs, they have some learning processes. I saw a few days ago a beautiful movie of Cousteau showing how an octopus learns and learns very fast, it has a big brain—an octopus is an invertebrate, a very simple form of life. He learns very fast, very bright, very diligent in learning. But the octopus is not one to transcend. Life for him is the end. He does not want to explain anything or to understand anything. *We* want to. To look for the other, for the other species, is a way to preserve the species or fight for their position as animals, but to look for the other, in our species, is to build up society.

So in education we have to satisfy animal needs but for this you don't need schools. When we start to need some reflective thinking, when we try to satisfy human needs and when we start helping people to live in society, then we need schools. In the search for survival and in the search for transcendence; how to deal with the other in a non-animal way, this is how schools can help us very much.

I think to live in society, basically I can summarise as developing the respect or the capability of respecting others with their differences and to have solidarity with others and their needs. To satisfy your animal needs, you learn how to satisfy your hunger and lots of other animal needs. You learn how to do it without anyone asking you to reflect about it. To satisfy the human needs, to try to explain, to understand, to survive you make, even develop your own ways of explaining. But to live in society you first have to be immersed in the society. You have to be *in* society, to be able to respect others with their differences, because if you expect everyone to be like you, you are not respecting others. You have to respect others with their differences and at same time you have to have solidarity with others and their needs.

Well, I think this is a function of schools. This is why we should have schools, if we have schools that satisfy this, very good. Out of this, of course, you start to develop ways of transcending because you have solidarity with others, you are developing working systems. You help others because you have solidarity with their needs. You respect others with their differences, so together you search for explanations and understanding.

I have to confess that I have my utopia. I dream of my utopia and I am dreaming of a beautiful society. My society would be really beautiful. A very strong part of my activities is with peace groups, working for disarmament and similar kinds of activities. What I work for is this: a society where you respect others with their differences without changing others to be like you, because this is not love. You have solidarity with others with their needs—this is love. One development of this is the way the teacher in the classroom, he exerts capabilities of researcher and this trend is really growing. I claim it is a trend in mathematical education, nowadays. Probably the largest number of new projects that I have seen in a single area is in this area: the teacher as a researcher. That's a different role for the teacher in the classroom.

When we look at this situation, what does the teacher, when he gets into his practice, give us? First we have to shout over there, at others or children. We have pressure from parents, from schools, administrators and all this and we have great pressure from the discipline. Indeed, the most important pressure that we have in our practice in the schools is to bring a certain amount of the discipline subject matter to the child. So there is a sort of domination, a dominating force, bringing the discipline to the child. The teacher sometimes acts this way, trying to put in the head of the child some contents of the discipline. You have seen variants of this picture in many, many places I'm sure and this is an image that I think is close to what happens in the classroom. You force subject matter, force subjects, force topics into the child.

Maybe this is a very Ptolemaic approach. Maybe to shift to a Copernican approach would be a better way. Both put ingredients in the cauldron to produce new knowledge. You notice that between this kettle and this kettle there's a difference. Where is the handle and where is the spout? Where are these two things? Well, in my drawing they are in this dimension because they do not belong to one or the other. Both must be partners in the search for new knowledge. Each one brings the ingredients that he has, that he knows. This kind of bringing of new knowledge out of the school system is probably the most important function of the school. New knowledge means new ways of explaining. New? But new for whom? New for those involved in the process. New for child, maybe not new for other people but it's new for the child. It may be new for the teacher if the teacher is humble enough to pay attention to the child, to listen to the child. He may learn lots of things from the child.

I have experience in giving talks about peace and disarmament. I usually do this in high schools, in elementary schools, studying about the nuclear threat and other such things. Talking after the address, asking for comments and contributions from the people listening, children ask much, much better questions and even have better knowledge about those subjects than the teachers. The teachers were sort of ignoring some of the basic facts of the nuclear race but the students knew about this. So if the teacher listens a little bit to the children then they may learn a lot even in subjects that they feel expert. Many students contribute proposals for the solution of mathematical problems that never occur to the teacher. We know that.

So what's feeding this, what is the energy for this, what is the fire that keeps this boiling? It is the real world. By regarding our reality and by looking into reality and by drawing energy from reality, and put in the ingredients that both of us—myself and the student—know, we build new knowledge. In a sense, this is the way we colleagues do it when we have a seminar. Why not extend this to the children? We can make the children something much more important than they are used to being treated in the school system. They are treated as passive beings. What kind of new knowledge can we derive out of this? What kind of new knowledge are we looking for? Basically we'll be faced with problems of finding new ways or techniques of understanding, of explaining, of managing reality. This is the big drive, we want to explain things and children also want to explain. They look for explanations, if you give them a phenomena, they are curious about it. You could give them a situation, they want to manage that situation and games for example are so important in doing this. So they have to be challenged, and to be challenged is to develop new knowledge, it is to find new ways or techniques of understanding, of explaining, of managing reality (of course, in their own social culture and emotional context).

If you do a written problem you can do it by giving a problem on the blackboard. I go to the market to buy so many bananas and I pay so much, how much do I get as change? Well this is absolutely artificial. I have to work in the social culture and emotional context of the child; a child must be attached to the situation that we are discussing. In some way there must be something that motivates the children very highly, either worries them or it is something that they are curious about or they are anxious to know about. This is what I call, the social cultural and emotional context.

Well, now I have a game of words. You know the root for *techniques* is the Greek tekhne (τεξνη), and *tics* derives from this. Understanding, explaining, in Greek is *mathema*, so we have the *tics* of *mathema*—mathematics. The social culture is what I call *ethno*, so if you want to know what ethnomathematics is about, it is the development of new ways, or techniques, of understanding, of explaining and managing reality in specific social cultural and emotional contexts. This is what I call the programme, Ethnomathematics.

What we are doing in the school is just to transmit one of the millions of *tics* of *mathema*. The *tics* of *mathema* that were so familiar to the Greeks and became familiar to the Romans and became familiar to the French and the English and which were imported to Latin America, to Africa and other countries, were imported to the entire world. This is one of the *tics* of *mathema*, one that comes from the Greeks. It is very efficient, no doubt about this. The ethno- or social cultural and emotional context of **that** *tics* of *mathema* has to be generated in very special situations. The programme that I call Ethnomathematics is a programme in history because I'm trying to understand and to find how sound is this proposal to look into knowledge as a whole.

Some ways of understanding and explaining this are ways that lead to religion, other ways lead to art, but religion, art, mathematics all have the same origin. If you go to some cultures today you cannot distinguish mathematics and religion. If you go to the geometry of the Middle East, it is called now sacred geometry because you cannot distinguish geometry and religion. If you go to some components of the history of mathematics, the *tics* of *mathema* we now use in the historical lines of this always get close to art, to religion, to some practical things. Always searching for a way of understanding, explaining, managing reality in a certain context. So this is why this programme is a programme in history also, history and philosophy of mathematics—that's the building up of mathematical knowledge that I want to understand. I want to understand this throughout history, I want to understand this in the making, I want to see how mathematicians do mathematics and that fits this proposal. I want to see how children develop their own mathematics in their own environment before they come to school—it fits this model. I want to see how people in the rural areas do this. I want to see how the Indians in the Amazon do this—they all fit this model. Some of the things that they do you may not recognize as mathematics as such. You don't find measurements, you don't find numbers, because the socio-cultural and emotional context is a different one from ours. But they do these things in search of explanation and everything tries to explain, and behind any mathematical subject, any mathematical topic, if you go back in time, you find this kind of historical description.

This is the research programme. It is now going on and I have a few results, a few examples, but this is a major project that of course is so difficult to touch different cultures other people are working on this project, Ethnomathematics. As you know, ethnomathematics is going on in several different places. The most recent contribution is by a Canadian, Jerome Turner of Saint Francis Xavier University. He has done his PhD study in Bhutan. It is a beautiful work within this theoretical framework of ethnomathematics. There are some beautiful bridges between this and the Niels Bohr theory, the complementarity bridge. That's very nice. So this is a very ambitious programme because it touches all aspects of cognition and history and it has a clear implication for education.

How would this work in practice? Surely not with the current concept of curriculum, based on three components; objectives, contents and methods in solidarity each one with the other. Sometimes I like to think of curriculum in a three dimensional cartesian scheme, with axes of objectives, contents and methods (see Figure 1). In this model, a point in the three dimensional space is characterised by the three "coordinates" objectives, contents and methods. I call this the *curricular moment*. Of course, if you touch one of them you have to touch the others.

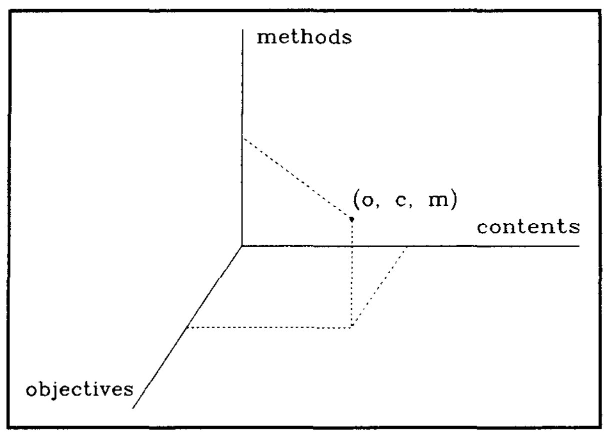

Figure 1

When I make analogies of some curricular failures it is exactly because this is not taken into account. We change content but don't look at the objectives and don't change the methods. For example, the disaster of modern mathematics to a great extent I can explain with this model of traditional curriculum. As another example, go into computers now, if we see the same objectives and the same contents but changing computer methods, it would be a disaster. The moment you have computers, the moment you have calculators, of course, the content has to change accordingly and objectives have to change accordingly. So everything is with the solidarity in this. This is one of the ways I used to be critical of the traditional teacher training program when you have classes of methodology and you have classes of content. It's impossible to separate them; once you are teaching contents you have to teach methodology altogether and objectives.

So this is what I feel that should be left. This is the traditional curriculum and I don't think that this leads anywhere. This will lead to more disaster in mathematics education. Wait to see the results of the third international study; they will be worse than the second and the fourth will be even worse. But the children are reacting to this. There is no way out, children are consciously reacting against mathematics teaching and I have enough evidence for this.

The new concept of curriculum that is most suitable for what I am proposing is this: the teacher manages the process, he is part of a group. If you have 20 students and one teacher, you'll have in effect 21 individuals in the process. One of them is managing, just like a maestro in the orchestra. The maestro does not go and teach the instruments; most of the time he does not play as well as the players. But the maestro is capable of managing the process because he's more experienced, has much broader knowledge, and this is the role of the teacher, the new role of the teacher.

What is the purpose of schools? We cannot go to school just to learn cumulative contents or to accumulate knowledge. First of all, there is so much knowledge that there is no time in school to give all the knowledge that is important. This is why they invented, in a very important revolution in thought, the encyclopedia. Now we have modern encyclopedias: why? because the knowledge is there. I don't have to learn all that is in there. The knowledge is printed and available and it just fine to stay there and I can go and I take what I need. It's as silly as to say I can operate a telephone only after I have memorized the entire telephone book. Why, the telephone book is there! If I need the telephone I go and I read the number. To operate the telephone well then I have to be the instrumentation. I have to know how to operate a telephone and how to read the numbers in the book, but I don't have to be taught all the contents even if I have to be instrumental to go to the contents.

What is going on with instrumentation? The child goes to school fortunately knowing how to speak, otherwise the teacher trying to teach the child how to speak would be a disaster. They know how to speak, they know how to listen, but this can be improved. Both speaking, listening can be improved. The Romans enacted with this with the trivium. They improved listening, they improved speaking, they improved the capability of discussing. This is important. Children have to learn how to read, how to write, maybe how to count, but most of the things they can learn from each other.

This is what I mean by instrumentation, maybe a little more sophisticated than programming. If you are in a modern society you'll need to help children to program, to deal with the computer, how to retrieve information, not because this knowledge is important in itself, what is the importance of reading if you don't read? You have to do something with the instrumentation that you receive, and this is socialisation. There is no substitute for this. The school is what provides the capability of a person working with another, and at this moment this is what provides the capability of the one respecting the other who is different. The other who does not know as much, he has a need to know more; this is the moment to teach how to help others to fulfil their needs. This is the moment to help the child respect the other; because the other is different, it does not matter. We are all different and this is the component of socialisation. Working through projects, group work, discussions, seminars, they will start to feel together that kettle with the ingredients that each one has and everybody knows something to put there. Let's make soup and everyone brings what he has in his home. So they put everything together and if there is something that no one knows, together they will go to the

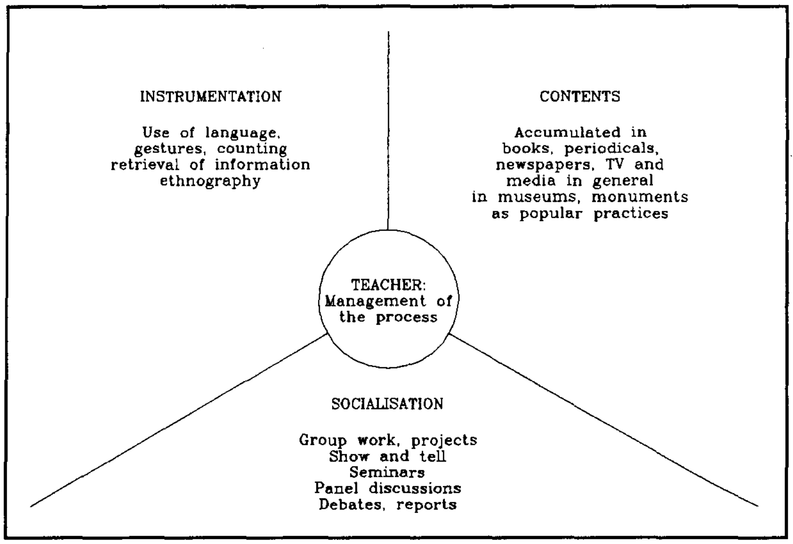

Figure 2

accumulated knowledge, and the teacher is the one who manages this process. So this is the school of my dream.

It is not altogether a dream because I have used this system in some experimental projects. I have worked with this kind of curriculum and it was a big success. If you look at the project 2061 of the AAAS, it is very close to this. We have to give up the old way and move into something more dynamic like this. This kind of curriculum is very dynamic because the one helps the other. If I use the telephone number after looking at the telephone book five times, the sixth time I don't have to go back. This becomes part of *my* knowledge, it's part of the component that I have, of the ingredients that I have, to prepare that soup in the same kettle, in the same cauldron. I don't have to go back and remove all this. I acquired knowledge taken out of a source and then it's incorporated in you.

So this is a concept of curriculum that may in a sense provide the right environment for a different concept of school; according and following a different concept of education and as I said absolutely biased by my utopia. But if you are an educator without a utopia you should think twice. Thank you very much.

Question Period

I was disappointed in your definition of ethnomathematics after having set us up very nicely with "meeting our basic physical needs", and "our need to understand", and also "our need to help others in society". You talked about the techniques of the first two, but not the techniques of "helping others in society". Instead you just left this to the social-cultural-emotional context of the student. I was not really satisfied with that.

The implications are very clear. If you work within a social context and you have tried to explain, to understand, then the reason for you to explain and to understand is to transcend and in order to transcend, you are, in a sense, looking for the others. Of course, it's oversimplified and reduced, but this idea of transcendence is in the basis of everything. As you look, if you discover a sort of moment of your life, you discover the other. First, you discover yourself, but the moment you discover the other, in order of transcending, the first step towards transcending is the search for the common other because everyone is looking for the other. He is trying to communicate with the other and in bringing this concept of the common other you put transcendence in solidarity and this is why you have people working say, in a religious system, in an arts system, in a knowledge system. So you pool resources together in order to transcend.

Of course, I oversimplify sometimes, but could go on. But I'm very glad with your disappointment because this is the essence of all my thinking, of my way of life. I want the school to look for those two things. If I fail to pass this on, I have to improve my lectures.

The question that occurs to me with your model is how you might test or evaluate the students; if you are going to apply it to older children or university students—it must be a problem.

Well, this is really a practical problem that you have to cope with in the context of your work. Someone will ask "How do you put this into practice, must we change the school system?" If you work with big ideas it is not possible to put them into practice immediately. But the way you exert your practice, and this is why I draw that picture of theory and practice, must be according to your reality. There is no point me getting into a school system and saying "Now let's change the curriculum and structure and let's all work this way". Where are my teachers to do this? So I have to compromise with the real situation guided by my theoretical framework, which is based on my utopia. Sometimes I do things that I don't like, but I have to do them this way because the boundary conditions do not allow me to do better. But the focus must be present.

This means that every step in your practice, you are worth a little bit more and you improve your theory in the sense of *how to improve my practice more according to my utopia*. So it's a process; this is a good reason why Project 2061 gets its name. Some people say well in 2061 no one of us will be there. Well, it does not matter but if we are focusing on these sorts of things, in the process we do lots of new things and lots of improvements.

So what we achieve are small parts of this improvement and how to cope with the evaluation? I think we have to reduce evaluation to its real value. Evaluation in the school system is a disaster. If you want to change evaluation by something that's feasible, it's possible now to change evaluation by sorts of games like you have several kinds of Olympiads, for mathematics, for physics, for basketball, for literature, for poetry and so on. Each one will choose where they feel more competent. This is a proposal of Teresa Amabile of Boston University for a new form of evaluation that does not produce trauma, because each one looks for where he wants to be evaluated in the way he wants to show how good he is.

That some would never choose mathematics is a fact of life. Some people will never learn mathematics, never get interested in mathematics, as others will never play basketball, as others will never play piano. Why do we have this fixation that everybody has to learn mathematics? I think this is shocking and foolish. We know lots of people who know nothing of mathematics and are so successful in life.

I wonder if you are familiar with the school set up by A.S. Neill in England called Summerhill? To what extent does your utopia correspond to Summerhill?

Well I liked that and probably this is the background of the development of these ideas. It is something that I remembered when I read it, I liked the idea and was so enthusiastic about it. It has not been built up, of course. The history of our lives brings into our minds lots of experiences, and I learned much from Neill. No doubt about this; from him and Rudolf Steiner, many things that I like so much.

Discussion of Lecture One

You talked early in your lecture about restoring a balance in mathematics education, and I felt that you were fairly brief about this. You had listed five main areas that your respondents had highlighted about why we teach mathematics. When you are talking about balance are you thinking of aiming towards something like twenty, twenty, twenty... for the five? You were quite disparaging about some of them having any value at all.

Not of having no value at all but they are not all being recognised as being valuable, for example, critical skills. The tools that mathematics gives which lead to critical thinking about society's problems, this is not regarded of any importance in the current school practice, and I think this is very important. All five, I think, have their importance; the question is that in the current school practice, everybody thinks only mathematics is useful, useful, useful, without stressing critical values and without exercising this in the school.

So, what I was suggesting is that we should have in the school system a balance, not twenty, twenty, twenty... but a balance in the sense of bringing all these components into the school practice; it's possible. For example, if you introduce some historical remarks then you bring the cultural issue into play, when you introduce some things from art and literature you may introduce aesthetical values. By looking into newspapers, for example, when you are dealing with things like statistics and graphs, analyze the article, then try to see what's behind the idea of what's in *that* graph, or talk about salary and many other things.

You can go a little further and discuss the *meanings* of this graph or whatever, this is critical thinking. So you can bring everything together in your classroom environment. I never thought—and I hope if I give this impression that you correct me—that some of those five have no value at all. All of them should be in the school system, all of them, but right now we don't have all of them, we have mostly the usefulness, I think almost exclusively the usefulness.

Ubi, in Brazil, do your kids sing a song about a teapot? Like here kids have a little ditty about a teapot:

"I'm a little teapot short and stout,
where is my handle,
where is my spout..."

now that's our version of the teapot, (remember you were talking about where is the handle, where is the spout, and who is doing the pouring?). And you had both the teacher and the student bringing the ingredients, to put into this brew. Well, that part of the image was fine but then you had as a source of energy, to make this whole thing bubble, to motivate it, the real world and that was the part that I took exception to. Is it perhaps better that it be the child's world, because that child's world may be quite a fanciful world? It might not be related to those things that are practical or truly useful or real.

What I wanted to convey with the teapot was that you are dealing with a real situation. I would give you a good example we have in Brazil, you all know our currency reform. We had a change from the former currency, three years ago it was cruzeiro, then there was a change from cruzeiro to cruzado. It was called *the cruzado plan*. This was a very carefully studied monetary change, they studied this for several months and took care of all the details. For example, if you had today a debt in the former currency, what happened with your debt the next day or one week after or one month after the change-over day? What happens with a payment that you have to make? What happens with the prices? All these were detailed in the law that introduced this new currency.

Well, a student of mine was teaching there in high school. He went to the school prepared to do some work in the classroom and when he was going to the school to teach, he heard on the radio that this new plan was introduced, and when he arrived at school, everybody was talking about this new plan. So he could not lecture on what he had prepared, so they all started to talk about the new plan and each one brought what they knew. Because it was such a big thing each student brought forth, from his partial reading, one aspect of it and they all were talking about these. Some of the kids had formulas that they used to convert interest and they were rather complicated so the teacher, in this case, helped them to move towards understanding the formulas. So this is what I mean, everyone contributed to *that pot* and what was feeding this boiling was the real world, a real situation. So this is the sense that I wanted to give to that, for example, when you go after a soccer game or after a basketball game, or whatever, the students probably would be very excited about the results of the game. So the real situation, the real world, is the game that happened and this can be the motivation for everyone in this situation. This is my image of the energy drawn from the real world.

*But that's **a** source of motivation...*

Yes, *a* source to get the group interested in something. Of course, the idea of *real world* may be made up, or you may have some new idea or invention or say the men went to the moon and so on, some may even draw on an imaginary situation. But let there be interest in everyone, everyone's motivated by this interest; this is what I need for this fire.

You were talking yesterday about utopias. We have seen some utopias in the history of mankind; a recent one in education was the new mathematics. Another one I lived in was socialism. This was a very humanistic utopia of the nineteenth century and I may assure you that it's not easy to live in that utopia. Many people have been imprisoned because they didn't like it. We're supposed to go into the classroom with your utopia and some teachers don't like it—what are they going to do? We could put the children out of the door if they don't like it. The problem with utopias is that they don't take into account the laws that govern reality. A school system is an institution and it is governed by some laws. I don't know how these laws come into being, it's perhaps the laws that govern society and our thinking of the role of school in society. If the utopia does not take into account the reality, the laws of the schools, it will simply fail. I didn't think about that until the end of your talk yesterday, when you said that if you introduce that into the *school and you can make some arrangement. If it's confronted with reality we can change it a little bit, and compromise and do something like that.*

We do compromise in our schools in Poland, now. After the introduction of the new mathematics in the seventies; we had a very big building programme, and we built some beautiful buildings but it turned out that it was impossible to learn in them, and so we started to take off some bricks from the buildings, and they are ruined now. If this is what you have said then your utopia will come to ruins after a while and it's impossible to change anything that is part of that. You have to wait until one generation dies and come up with another one to build anything again. I don't know whether it's so good to have utopia and not be realistic.

I used the words *my utopia*, and my utopia, and I stress this, is to respect others' differences. There are some people whose utopia is to abolish everyone that is different. This is not *my* utopia. Of course, if there are many teachers who with this utopia try to eliminate everyone that is different, this would be the hell of a world. I hope it's not like this; in my utopia it is not to be like this. Utopia is what guides your practice and utopia is what is behind your ideology. You may not like some of the ideology, the ideology may perturb us very much, and the utopia which drives the ideology may be a perverted one from my viewpoint.

Politics is the capability of being or performing or working within your utopia in the face of a real situation, with the reality. So we have to combine our utopia with what is possible, what is real, what is feasible. This is why some people can survive under so much pressure, others are immediately killed. Some of them are able to keep their utopia and to survive; others are not able to do this. How can we live without a utopia? How we can live accepting whatever orders are imposed on us? Sometimes that *is* what you have to do, this is the way you have to behave, and we may do this for political reasons, to survive. But we keep our utopia inside us otherwise I don't know how we can attain our full dignity as human beings. I know it's very difficult, there are so many difficulties, so many problems. This is with regard to politics. With regard to modern mathematics, this is no utopia at all, I will not use *utopia* for modern mathematics. It would be insane for someone to have as utopia a "modern mathematical" world. This is just one instrument, maybe too rigid, but, well I don't know, maybe some people believe in it.

But I think we should know more about reality, the laws that govern reality. I think this is what changed sociology, at the turn of the nineteenth century, from an ideology to a science. It stopped talking about what society should be and started asking questions about what society is. And I think that what made the present mathematics education into a science is that we have started asking ourselves questions about what education really is, what learning is, not what it should be. We should really know more about what the reality of education is and perhaps adjust out utopias according to the laws that govern reality and not duck them.

Well I have to disagree with this. In the name of ideology we may do lots of terrible things. The absolute principles of ideology tell us that something is wrong, something is right so let us eliminate the wrong. I don't think this is possible; we cannot give up the dream we have. This dream may be not pleasant for others; this is why *my* dream is to let everyone respect the others and if everyone does this, I think the world can work. But if I find anything that says "this is absolute, this is perfect", and mathematics education may lead to this, it's right or wrong, it's correct or not; there are manacles associated with this and in my utopia I try to avoid this. I am afraid of this.

But mathematics education theory is not about being right or wrong, not that. It is a theory about how an institution functions, how a child functions, about learning, not about something being right or wrong. I am not talking about absolute truth because theories are not about truth.

This is the message that mathematics carries: something is right or wrong, correct or not, zero or one.

Under some assumptions!

Well, you see assumptions are dangerous things; if I do not fit the assumptions then I am an outcast.

The reality of the mathematics of any teacher is political expediency; nobody ever, that I know of, thinks in terms of black and white, right or wrong, ones or twos. They think only in terms of what is expedient thing to do, and there are many shades of grey. So...

Yes, this in my understanding is the right approach. But if you carry mathematics education to a very formal stage, and do not allow questions like "let's see what is going on in society" in a more tolerant way, you may get into these manacled situations. I think it is very dangerous for society as a whole.

Ubi, I think part of that reality includes a whole set of social issues, such as poverty, nuclear arms, AIDS, drugs, labour strikes, that can be the grist for mathematical applications and I presume that within your view there is a place for these in a mathematics classroom. I would like you to comment on that and I would like you to comment on what are the implications for a teacher having to deal with these issues in a context of political and religious views that may perhaps put pressure on the teacher.

Yes, these kinds of problems are affecting the minds of the students, and they are geared towards this kind of tolerance. Just to give you an example of the AIDS case. We have been treating AIDS as an epidemic and using the equations that are useful for epidemics, the Lotka-Volterra system of equations. The kind of results that come out of the application of these common, traditional ways of dealing with epidemics, would give us a picture of the evolution of AIDS as a certain curve, I think it would be an exponential curve, or something like that. People were working on the presupposition that the epidemic of AIDS would follow this kind of behaviour, and this was mathematically absolutely correct. You are applying a differential equation that has been used in all sorts of epidemics, and they have been successful in dealing with these epidemics. The result was that all the planning for treatment, all the planning to deal with AIDS was following this kind of behaviour.

The results were not positive; they were always in trouble with the way they were dealing with AIDS. Then a group of mathematicians, I think it was in Lausanne last year, had an idea. They looked at the epidemic in a completely different way; they forgot the established equations that were the "right equations" to deal with the problem, and they started to look at the problem in a fresh way. They started looking at individuals and they had different parameters, and these parameters took into account the individual behaviour of people affected by AIDS and their family relationships, and friendships; all sorts of factors and parameters that never were considered before. They came up with a different equation and an entirely different approach to treat AIDS as an epidemic.

My point in giving this example is that if they had insisted on the formulas which were proven correct in a situation which was apparently similar, probably they would have persisted in doing wrong things about the treatment of AIDS. So they had to consider a fresh, new approach to this disease and derive other equations and prove the derivation of these other equations. So AIDS cannot be classified with other epidemics, it is another thing.

So this kind of openness, to use new tools to approach new problems, this usually is not the kind of thing we transmit in traditional mathematics education. In the traditional mathematics education, the message that we give is a message of precision, it is a message that everything is correct—if you apply *this*, your result is *that*. This idea of cause and effect, the idea of law, when applied to human beings; this is very, very complicated. You cannot have cause and effect in dealing with society, with human beings. The message drawn from the way we teach mathematics in our educational system is that for every cause there is an effect, and if you identify enough variables, you will know what will happen the next day, you know what will happen to your fellow human beings, you know what will happen in your society. This kind of very deterministic approach to unknown situations is the message that mathematics gives, and I think this is a wrong message as we are much, much more complex (as human beings). There are some factors that we absolutely do not know, and human beings have many different behaviours. The idea of cause and effect does not work. So "laws" that deal with human beings are very dangerous.

At the end of your lecture yesterday, you contrasted what you called the traditional curriculum approach and an approach which was more in line with your vision of things. But you did more than that, you actually presented them in very different ways, and I want to challenge you on that, and question if it was fair. You presented the traditional approach in terms of Cartesian axes and just a point in the plane, and then the other approach had a much more richly textured set-up. Well, I can reverse this and I can put your approach on the Cartesian axes and just call one dimension socialisation and the other one instrumentation. So I wonder if you haven't simplified the picture a little bit too much? I think what triggered that off was that I was trying to think about the Standards that we are looking at today and I was asking myself where does it fit in that picture; is it more kind of a traditional curriculum approach to things or more in line with your new approach? I think actually it is still fairly traditional in some things and yet it is a very richly textured approach to the issues of objectives, content and methodology.

We agree that the components are objectives, content and methods and the arrangement in a Cartesian way is what you don't like?

I don't question the components, I just think by presenting them in two different ways we see the power of representation here because one kind of simplified something quite a bit, and I think perhaps not fairly.

What I want to stress by giving that Cartesian representation is that this is the way it is dealt with.

*But **is** it dealt with that way?*

This is the way it is traditionally dealt with; you have improvements in dealing with the curriculum, and you can work with the content, objectives and methods in a better way. Of course! I want to emphasise the way it is usually done. You have variants of this traditional curriculum; and I have not the slightest intention of saying that the next one that comes along is the solution. The solution may be within a modification of this Cartesian approach. But the way I see the curriculum dealt with now in many cases—there are exceptions of course—many times you look into the content, you look into the objectives, and you look into the methods with the argument that they are not even considered as one affecting the other. So my approach to the traditional curriculum is that it would not be so bad if you consider that every time you touch the content you touch the objectives, you touch the methods. But it is not even treated this way; because sometimes you touch the objectives without touching the methods or the content. For example, I repeat the example of modern mathematics and the use of calculators. You do not consider the content, the objectives and the methods together. If you have solidarity in there, the traditional approach is, I would say, in a favourable situation. But of course, I think you can be as Cartesian and rigid with one model as you are with the other. My idea is that that is very dynamic—I tried to give the feeling of dynamics.

But I'm saying that the traditional approach could also be dynamic...

Of course!

...and by putting it on an axis you get a one-dimensional component and diminish other aspects.

It seems to me when you are generating a solution to a problem that certainly you must have some aspirations, some guidance to where you're going but on average I think that should receive 5 percent of the effort, and the reality should receive 95 percent of the effort, but I agree that both of them are necessary.

Then we come to your comment on using AIDS as a reality to make use of in the teaching of mathematics. I think one of the characteristics of mathematics is that it is a reflective process (when done in the common environment) where you're not in a panic situation, and I think that although a lot of people would say Sir Isaac Newton was affected by reality, I think he, in some sense, was the most unreal kind of person who lived a very isolated life and sat is his little room at Cambridge and generated some marvellous mathematics. Now I'm a little worried that a focus on reality really detracts from one of the essentials, something fundamental about mathematical activities, and that is that it has this sense of detachment and perspective and remove. When you bring the example of AIDS up, in fact it sends me into shock and it doesn't get me into a mathematical frame of mind.

This example I brought up to show that you have to be flexible in what kind of mathematics you are using. It is possible to get into trouble with some fixed sense of mathematical approach; if you use some approach that has been used several times before. For example you can train students and hope them to transfer what they learned in one situation to another, to the extreme of them believing that if they apply *that* to the other situation the result will be mathematically correct, mathematically perfect. My example was to show that you cannot apply the same tools to different situations and that the real world is *very* complex; but even so I never give up the real world. I struggle to change what in this real world does not fit my utopia. I think that if someone gives up this, he is led, and I don't think it is the aspiration of human beings to be led; we want to lead, we want to have some say in the future according to our utopia. Some of us according to this utopia may be given extreme power, we should be careful about this and not let it happen because all the distortion that comes to our minds is because of this. Someone that is given too much power can try and impose his utopia on all the others. This is not my utopia, but it may be for someone, some people. This we should avoid.

On the other hand we should not give up understanding our reality and knowing where we are standing. When Isaac Newton was sitting in the room reflecting, he was not isolated from reality. Actually in the anecdote of the apple falling he was under a tree, that's the real world! He had a utopia, we know that from his personal history. What happens in dealing with the real world and producing all these results? This is guided by something; we want to reach something. If we give only five percent on this, 95 percent of our lives is unproductive. I react against this. I may do something that I don't like, but I cannot lose consciousness that I am doing this by being forced. We must never give up our utopia. That is my view.

We must not forget that when Isaac Newton was at Cambridge doing his work on the calculus the reason he went home was because his university was closed down because of the ravages of the Plague. You are talking about AIDS and that sort of thing, but Newton was in a situation pretty much the same. I don't remember reading anything Newton wrote about the spread of the Plague. It wasn't the ***real*** *world that was really his major concern. The* ***physical*** *world certainly was.*

Newton writes in one of his less-known works on geometry that the justification of science is what we use of it in society. Science is justified by what society makes of it. It is written very clearly in the geometry of Newton, so he was concerned with the view of society. He was not a dreamer disconnected from society.

I'm an little confused about this real world and utopia. You mentioned five things, and at the top was usefulness and at the bottom was something that I thought was usefulness too. I don't actually know where you stand on the utilitarian aspects of mathematics.

Well the point is this, in a way usefulness comes in all of them of course, but all that is *stressed* in the curriculum, problems, books and so forth is that mathematics will be useful for you to solve some kinds of problems. The idea of real problem solving or of modelling tells you that if you go to some place, if you don't know mathematics you won't be able to do this kind of problem. So there is a sort of simple utilitarian view, and this is stressed. If you look at all the problems that are formulated in the books, you find problems like "you leave school and on your way home you had to buy something", well you need mathematics to do that; this is what is mostly stressed. Another dimension of utilitarianism is, for example, to make a critical analysis of society this is *not* stressed. This is where I stand you see, opposed to the intensity of stress on utilitarianism and what is utilitarian. Everything is utilitarian!

Just a comment: it seems to me that it is useful in developing the GST in Canada to have ways of computing the tax in all cases and it's a nice thing to be able to say that because I know something about percentages I can see that because 90 percent of the manufacturing in Canada goes on in Ontario, we must pay extra tax in Alberta because 90 percent of it is going to have to be transported to Alberta. We are using mathematics to criticise, if you will. Is the tax fair? It's another thing to say here is a set of equations that allows us to tell what the tax ought to be in a certain circumstance, that's usefulness, but also I can value mathematics as a point of view. I don't know whether that would be actually correct or not, but it seems to me that the distinction is likely between a kind of utility at one end, which is how do you do something with it, and at the other one is "can I use it as a point of view".

I'd like to change the focus. In your picture of the pot, with the teacher and the student both pouring ingredients into the pot, I agree very much with you that the role of the teacher should not be the source of the knowledge to transmit or pour into the child. On the other hand surely the teacher is more like the expert and the child a novice, but the relationship doesn't have to be the expert telling the knowledge, maybe it should be the expert apprenticing the novice to learn more about these sorts of things. I think there has to be a little bit more of a role for the teacher than just one more person in the classroom.

Yes, you probably contribute more to that pot. There are students who have more experience and more knowledge and they will probably even help other students to put something there. The image of the pot is that the student should not be passive in this building up of knowledge. The student is a partner in this building up of knowledge; if he does not bring former knowledge that he knows which the teacher does not, then he really isn't a partner in the process.

I have a comment on the Ptolemaic and the Copernican thing. Recently I was reading Roger Penrose's new book, The Emperor's New Mind, and he talks about "superb" theories and "useful" theories, he talks about how useful the Ptolemaic theory was even though it was completely wrong, and that people used it for two thousand years, and navigated perfectly well. It seems to me that in schools we have that same kind of problem. We have a nationally centred or institutionally centred view of things, we have adjusted that from time to time and maybe that would be what the Standards in one sense is, a kind of Ptolemaic adjustment rather than a revolution, but it seems very interesting that nevertheless, you can do useful things under wrong assumptions.

Yes, we have lots of flexibility—that's what makes a human being a quite remarkable being. We can survive mistakes and errors. In schools we have new ideas, new generations coming, we have changed even within this system, the same way as Ptolemaic astronomy. Yes, the entire world was circumnavigated, and they did not use Copernicus; the manual of astronomy was Ptolemaic. It worked, fortunately.

About the aesthetic nature, the aesthetic part of what you were talking about yesterday? The beauty of mathematics, the precision of mathematics—things that we look upon positively that don't carry with kids at all—I'm just wondering how that manifests itself in the child; is it in an activity that they find compelling or sustaining or interesting or fun?

We have noticed that children feel very attracted to drawings and pictures and dealing with geometric forms and there are even some methods they have, for example, for dealing with sequences, numbers and patterns: it's so attractive for children. We have many games now that I tried in schools, using patterns and things, and drawings and these have been less and less used in our teaching of mathematics. There was a time when students used rulers and compasses to do pictures and drawings. For example, when you do geometry by using ruler and compass and you start by drawing the triangle and then you use the compass, you draw three sides and then you draw the height of the triangle and bisector of an angle. All this, it's fun, it's attractive for children.

After doing this, when we start to talk about theorems and geometry as formal geometry (Euclidean geometry) and when we start to talk about proofs and how to prove that one triangle is similar to another, if you have some experience of doing this with ruler and compass it helps very much your understanding of the steps of the proof. So you are quite familiar with the geometry that's based on the idea of some sort of perfection that's behind the ideal of Greek geometry. So you develop in the child in the elementary school this idea of beauty of this perfection, all the sense of beauty that comes out of this, in the drawings, the decorations, the patterns, and this has a value in itself and at the same time helps to build up the entire structure of the mathematics that we want to transmit.

I just wonder though if kids can appreciate that underlying perfection. Are they reacting to that or something else? Is it something more immediate?

What we have seen is that they get excited about, for example, the kaleidoscope. The idea of symmetry, they get excited about this. We have been doing some work with kites, building kites and so forth, and the immediate reaction of the child is to do it symmetrically. So symmetry comes very, very spontaneously to a child, at least this is what we have observed. Geometry, elementary geometry not deductive geometry, drawing on symmetry has practically disappeared from the curriculum. We don't see much of this done. But I think this is a beautiful thing to do, you can get hours and hours of child excitement just drawing pictures that keep symmetry, and they do this very, very easily.

> (Addressing the previous questioner) *Was the question more towards the elegance thing, is there a mathematically elegant way of doing things as supposed to a less elegant way?*

> *I'm thinking that we keep pushing the precision, beauty and elegance as the aesthetical aspect of mathematics. But that doesn't wash with the bulk of the kids. It has to be something else and I'm thinking it has to do with the way in which it engages them? The way in which they can engage themselves, do neat stuff? "I'm doing things here that I've never done before, look into the results, what if I do that?" Are we adding an extra aesthetic value then? I'm thinking that we downgrade the aesthetic unnecessarily, it's there, it's just that sometimes we pursue the two-dollar form of it rather than the five-cent form of it that we should be doing day by day.*

Yesterday you made a few provocative statements about mathematics as a body of knowledge inherited from the Greeks and, later, Europeans and so on, and you mentioned not only the dominance of that view but even the imperialism, if I may say so, of such a view all over the world. You sort of meant, or sort of implied that there could be a pluralist view of mathematics. I would like you to elaborate a little on that, because, if I may play the role of the Devil's Advocate, I might say is there not one mathematics, the one that has built up through history? What do you mean when you say when there might be several mathematics and what does this imply in practice in terms of education?

It is just like language, there are different languages and if you speak one language at home it does not mean that you cannot speak two or three other languages. There are different kinds of mathematics, there are different manifestations. The number system of the Aztecs for example, was a number system that used the idea that a quantity, a number always carried the idea of *what* you were talking about. When you say a number, you say a number *and* what you are talking about; an object, an animal or some kind of thing. So the idea of a number as an abstract thing without an object, this does not appear in the Aztec system. Then we have the modern kind of number system that carries an idea of abstraction, that the numbers are abstract things. Of course, the Aztec system would be very local, you cannot transfer the way you deal with numbers associated with objects to other kinds of cultures, and this is why the Aztec number system would not be suitable to use in Japan for example. But the number which is abstract *can* be transferred to others, so in a sense it has an advantage over the other, and this is why this system, in the big expansion of Europe, was imported and adapted to just about every other part of the world and so it became the dominant system.

Well, the fact that it is the dominant system makes it much more interesting for us to learn and to use. It would be of no use at all to learn the Aztec system if you want to be in the international action, if you want to reach to the world. So we have to learn this system and of course, the system will continue in our schools. But suppose a child comes from home where the environment was dealing with the Aztec kind of number system. The moment the child comes to school, he must use our modern system. The moment you say that *that* system is of no use, stop using *that*, and start using *this*, then you are provoking a sort of reaction in the child because the child is being violated in its cultural past, and this I think should be avoided. This does not mean that you have to continue to use your first system for your whole life because everyone knows this will not be useful. It is the same thing as to speak a language which is limited, a local language; you have to learn a language which will help you to communicate with many others. This is why we are all talking in English, but the fact that I speak English does not forbid my speaking Portuguese in my environment, does not give me shame about speaking Portuguese, does not give me the sense that Portuguese is a useless language. This is very important for me; all my background, my culture, is attached to Portuguese, but for now I am speaking English. So it is the same thing with mathematics; you perform mathematics according to your background. This is why some old people still count in their former language; this is very frequent, when people start to count they use their former language instead of using their new language.

Lecture Two

Remarks On Understanding in Mathematics

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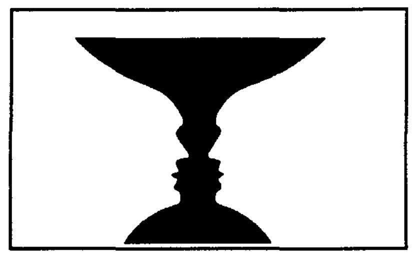
REMARKS ON UNDERSTANDING IN MATHEMATICS

Introduction

The present paper is composed of two parts. The first part is concerned with some general problems of the meaning of the concept of understanding. It is proposed to conceive of understanding as an act (of grasping the meaning), and not as a process or a way of knowing. Also, the notion of meaning is discussed. Relations between the notions of understanding and epistemological obstacle are found; it is argued that understanding as an act and overcoming an epistemological obstacle are complementary concepts. A categorization of acts of understanding is presented, inspired by the philosophy of John Locke, the psychology of education of John Dewey and the UDGS model of learning mathematics (Hoyles 1986, 1987). A method for elaborating an epistemological study of mathematical concepts, inspired by the philosophical hermeneutics of Ricoeur (1989), is suggested. This method is then tried, in the second part of the paper, on the example of understanding the concept of limit of numerical sequence, which the author has studied from the point of view of epistemological obstacles elsewhere (Sierpinska 1985a, 1987a, 1987b).

I. What Does It Mean "To Understand"? General Considerations

The notion of understanding intruded into my research rather suddenly, in a kind of gestaltist illumination. Of course, it was always there, but as a common term, tacitly admitted as a clear and unproblematic everyday concept: an epistemological obstacle was an obstacle to understanding. It was in the "background". And then, one day, it became the "figure".



A Gestalt Illumination

1. Questions concerning the meaning of the notion of understanding.

Understanding is, in fact, a common word in mathematics teaching. And a very important one, too. "Do you understand?" asks a teacher hundred times a day. "No, I don't" thinks a student to himself almost as many times. The main goal of elaborating teaching designs, projects, new textbooks is to promote a better understanding in students. Sometimes understanding is the goal of learning (as in "From doing to understanding", by C. Hoyles 1987). Sometimes it is its pre-condition (as in "Learning without understanding" by M. Hejný 1988). Some research is concerned with difficulties and obstacles to understanding.

But understanding has also become an object of study in mathematics education (Byers 1985, Gagatsis 1984, 1985, Duval 1984, Hejný 1988, Herscovics 1980, 1989a, 1989b, 1989c, Krygowska 1969-1970, Skemp 1978, Sierpinska 1990).

Taken as an object of philosophical study, understanding is no more this well-known, unproblematic notion we use everyday. From Locke and Hume to Dilthey, Husserl, Bergson, Dewey, Gadamer, Heidegger, Ricoeur and Heisenberg, philosophical views and contexts have varied enormously. Neither has the notion of understanding the same well defined meaning for everyone in mathematics education. Skemp's famous paper "Relational understanding and instrumental understanding" (1978) was, in fact, a call for clarification of this notion in this field. The challenge has been taken up by, especially, Herscovics and Bergeron, and we are much further now than we were in 1978. However, are we all sure of answers to the following questions?

Q1. Is understanding an act, an emotional experience, an intellectual process or a way of knowing?
Q2. What are the relations between understanding and
a. knowing; b. conceiving; c. explaining; d. sense; e. meaning; f. epistemological obstacle; g. insight?
Q3. Are there levels, degrees or rather kinds of understanding?
Q4. Are: "understanding a concept", "understanding a text", "understanding a human activity and its products" different concepts or are they just special cases of the general concept of understanding?
Q5. What are the conditions for understanding as an act to occur?
Q6. What are the steps of understanding as a process?
Q7. How come that we understand?
Q8. Can understanding be measured and how?

I have mentioned some papers in which the word "understanding" is explicitly used. This word may mean different things in different papers. On the other hand, there are papers in which the word "understanding" is not used but might be used. These are, for example, articles on mathematical thinking like those published in Volume 3, Number 1 of the Journal of Mathematical Behaviour, 1980 and many papers and books referred to there (especially the works of Matz, Davis, and Minsky referred to in Davis 1980). They deal mainly with question Q7.

In the sequel I shall reflect on questions Q1-Q3, only. This reflection will serve as a starting point for proposing a method for epistemological analyses of mathematical concepts.

2. Is understanding an act, an emotional experience, an intellectual process or a way of knowing?

Understanding can probably be regarded as an act as well as a process and the decision between the two is not a matter of argument but of focus. We have all experienced those sudden illuminations when the solution to what we thought an unsolvable problem appeared clearly and plainly before our astonished eyes. Reports of such experiences in famous mathematicians, scientists and poets can be found in Hadamard's "Essay on the psychology of invention in the mathematical field" (Hadamard 1964). But for Hadamard these stories were just proofs of the importance of unconscious work for the mathematical (and other) inventions. Poincaré was suddenly illuminated by the solution of his problem when he did not consciously think about it. But he had spent a lot of time on unsuccessful attempts, analyzing the problem and trying different solutions. If we focus our attention on this long period of, first, conscious, and then unconscious mental work then the idea we make for ourselves of understanding is that it is a process which can be crowned, eventually, by a moment experienced as an "illumination".

In psychology one of the problems is how we can understand what we are told—how we understand the information which is communicated to us in our interactions with other people or by media. Understanding a sentence in the mother tongue is always very quick (if we admit that recognizing the sentence as incomprehensible to us is also a kind of understanding)—so it may be regarded as an act even in a child. The rapidity of understanding is not a discriminating property. The problem is not there. What is important is the quality of understanding, its "level". And this level changes with the growth of knowledge, the complexity and richness of its structure. As we focus on the changing level of understanding, we think of understanding as a process and not as an act:

> The continuous development of cumulated knowledge, stored in our memory system, strongly influences the way how new information is assimilated. It also strongly discriminates between the coding of information in a child's memory and the coding of the same information by an adult. In a child, new concepts must be built up in empty spaces. The initial stage of construction of a data bank is necessarily linked with huge amounts of information memorized mechanically. *Understanding is achieved slowly*, along with the accumulation of properties of objects, examples and development of concepts concerning relations between classes of concepts. At the beginning, concepts in the memory are generally only partially defined and weekly related to other stored information. In later years of age, when the resources of information are rich and organized in a data bank built on an elaborate system of criss-crossing connections, the character of learning changes. New concepts can be assimilated mainly on the basis of analogies with what is already known. The main problem lies in incorporating the new concept into the existing structure. When the relation is established, all the previous experience is automatically included into a fuller interpretation and understanding of new situations.
>
> (Lindsay & Norman 1984, p. 438).

However, the quality of understanding need not grow with age; understanding does not depend solely on the richness of accumulated experience, information and highly elaborate structuralisation of the "data bank". Lindsay and Norman speak about a mechanism which explains the existence of epistemological obstacles in our ways of knowing:

> Very seldom does an adult meet with something completely new, unrelated to his or her conceptual structure (...) Even if the received information is in obvious contradiction with the previous experience, his or her conceptual structure which constructed such a complicated system of interrelations, stands against any revision. And thus an adult prefers to reject inconsistent information or change its meaning rather than rebuild the system of his or her convictions.
>
> (Lindsay & Norman 1984, p. 439)

Understanding is an act in Gestaltpsychologie where the influence of idealistic conceptions of Husserl and Bergson can be clearly felt. Understanding is thus an act of mind which consists in a direct perception of the "essence des choses". This act "is not prepared by a preparatory analysis of relevant relations between elements of a problem situation. These relations are perceived directly, like the sensory properties of objects". (Tichomirov 1976, p. 45).

In the frame of Husserl's theory of the intention of meaning (Bedeutungs-intention), mental acts of understanding a sign are directed towards some object; this object is called the *meaning* of the sign. Meaning is an ideal object, i.e., belonging neither to the physical nor to the mental reality. (The existence of ideal objects is justified as follows: take, for example, number 4. True as well as false statements can be made about this number. E.g., $4 = 2^2$ is a true statement. Truth is conformity with reality. Therefore, if something true can be said about the number 4 then it must belong to some reality. This reality is neither physical nor mental. Therefore it must be some third kind of reality, let us call it: domain of ideal or abstract objects) (Mała Encyklopedia Logiki 1988, p. 233).

Neopositivism in philosophy and behaviorism in psychology define understanding as a kind of reaction to stimulus (so it is rather an act than a process). In understanding concepts, this stimulus is the name of the concept. The word "meaning" has no designation in any reality (even ideal reality) because it is not a name even though its grammatical form gives this impression. It cannot be considered outside of expressions like "x has a meaning" or "x and y have the same meaning". The last expression means that x and y stimulate the same behaviour (Mała Encyklopedia Logiki 1988, p. 234).

Understanding as an experience has been considered by Dilthey not in the context of understanding concepts but human activity and its products, i.e., in the context of theories of humanistic interpretation. Humanistic interpretation is attribution of sense to human activity (and its products). According to Dilthey, this attribution of sense is made by means of an experience called "understanding" (Verstehen). This sense is a value, which, in its turn, is the goal to be attained by the activity. Dilthey conceives of understanding as purely intuitive and even preconceptual: it is not based on establishing logical connections between the given phenomenon and its sense but rather in grasping the phenomenon and its sense together directly. (Filozofia a Nauka p. 265, 408-411, Dilthey 1970, Krasnodębski 1986, p. 75)

Were we concerned only with understanding mathematical concepts, we might disregard Dilthey's point of view and forget about his theories. But as we are interested in understanding mathematics in the context of classroom interactions between teacher and students, where the student has to grasp not only the meaning of concepts, but the sense of the teacher's and his/her own activities as well, understanding as an experience is not all to be neglected.

For Dewey (1988, first published in 1910), "to grasp the meaning", to understand", "to identify a thing in a situation where it is important", all mean the same. All these expressions define "the fundamental <u>moments</u> of our intellectual lives" (Dewey 1988, p. 152). Therefore, Dewey seems to be conceiving of understanding as an act. However, this is not the intuitive and preconceptual act of Dilthey, at least not all acts of understanding are of this kind (see p 38). In its more sophisticated forms, understanding is a result of a thinking process, and, in fact, the goal of all knowledge: "All knowledge, all science endeavours to grasp the meaning of objects and events and this process always consists in stripping them of their individual character as events and discovering that they are parts of a bigger whole that explains, clarifies and interprets them, thus providing them with meaning" (pp. 152-153). This way, explanation, which Dilthey opposed to understanding, becomes a means for understanding.

Understanding and explaining are even deeper reconciled in the conception of interpretation (of discourse or text) presented by Ricoeur:

> For the purposes of a didactical presentation of this dialectic of explaining and understanding as phases of a specific process, I propose to describe this dialectic as the passage, first, from understanding to explaining, and then from explaining to comprehending. At the beginning of this process, understanding is a naive grasping of the meaning of the text as a whole. For the second time, as comprehending, it is an elaborate way of understanding, based on explanatory procedures. At the beginning, understanding is making a guess; at the end it becomes consistent with the notion of 'appropriation,' which we characterized above as a reaction to a kind of distance, strangeness, which are the results of the full objectivization of the text. This way explaining appears as a mediation between two phases of understanding. Explanation, considered outside of a concrete process of interpretation is but an abstraction, a product of methodology
>
> (Ricoeur 1989, pp. 160-161)

So the process of understanding starts with a guess which we further try to justify and validate. In the course of validation the guess may be improved, changed or rejected. The new guess is then subjected to justification and validation. The spiral process continues until the thing to be understood is considered as appropriated.

However, in this complex dialectic, understanding is again an act. On the other hand, explaining is a process: "I assume that while, in the process of explaining, we develop a series of statements and meanings, in the course of understanding, a chain of partial meanings are related and composed into a whole in a single act of synthesis." (Ricoeur 1989, p. 157)

Ricoeur's model is concerned with the interpretation of literary texts. However, this is not seen from the excerpts we have quoted above. The specificity lies in the way Ricoeur conceives of procedures of explanation (Ricoeur 1989, pp. 161-179). If we let the procedures of explanation be a variable in the model, it generalizes to a model of understanding any text. It is probably not as easy to generalize from "text" to any "discourse" (verbal or written) because, while, in the interpretation of a text, the validation of the guess is made on the basis of the same material (the text is reread and analyzed), in the spoken discourse the validation of the guess is developed in the course of a dialogue in which new pieces of discourse are introduced and have to be understood. It would be even more difficult to modify this model so that it comprises understanding (mathematical) concepts, because understanding of a concept is not reached through reading a single text, normally. It demands being involved in certain activities, problem situations, dialogues and discussions, and interpretation of many different texts. Let us keep, then, of the model of Ricoeur, just the general idea of the dialectic between understanding and explaining, which starts with a guess and develops through consecutive validations and modifications of the guess. Presented this way, Ricoeur's model strikes us with its similarity to the Lakatosian model of development of mathematics through a chain of proofs and refutations (Lakatos 1984).

I propose, then, to regard understanding as an act, but an act involved in a process of interpretation, this interpretation being a developing dialectic between more and more elaborate guesses and validations of these guesses.

3. What are the relations between understanding and knowing?

Skemp (1978) defines "understanding" by "knowing". "Instrumental understanding" means "to know how", and "relational understanding"—"to know not only how but also why". In the article, "instrumental" and "relational" are qualifications not only of understanding but also of thinking, mathematics, teaching and learning. They are used as names of styles. Can we speak of styles of understanding if we conceive of it as an act? The words "understanding" and "knowing" are used very much together in literature. Do they mean the same?

Under the title "An essay concerning human understanding" John Locke discusses the notion of knowledge, its different "sorts" and "degrees".

Dewey (1988) distinguishes between two kinds of understanding, and says that in many languages they are expressed by different groups of words: "some denote direct appropriation or grasping of meaning, other—a roundabout understanding of meaning; for example: *gnònai* and *eidènai* in Greek, *noscere* and *scire* in Latin, *kennen* and *wissen* in German, *connaître* and *savoir* in French; in English the corresponding expressions are: *to be acquainted with* and *to know of* or *about*. Our intellectual life consists in a particular interaction of these two kinds of understanding." (p. 154)

Thus, in spite of conceiving understanding as an act, Dewey defines kinds of understanding by ways of knowing, like Skemp. How could we explain this?

Perhaps understanding is an act; but this act brings about a new way (or style) of knowing. Understanding as an act appears in expressions like: "Oh, I understand now!", or "Oh, I see!". Understanding as a way of knowing (manière de connaître)—in, e.g., "I understand it this way: ...".

If we stick to conceiving understanding as an act, we may say that Skemp classifies acts of understanding according to the styles of knowing they produce. And Dewey classifies acts of understanding into direct (which he further calls apprehensions) and indirect (comprehensions - those that have to be consciously prepared). These different kinds of acts of understanding lead to different ways of knowing: *gnònai, noscere, kennen, connaître,* to know; or: *eídènai, scire, wissen, savoir,* know that.

4. What are the relations between "understanding", "sense" and "meaning"?

"Sense" is often used as a synonym of "meaning", but let us consider the following two sentences:

(a) "I know what it means now".
(b) "It makes sense to me now".

For the boy in Skemp's article (1978), multiplying length by breadth to get the area of a rectangle was obviously a sensible activity, although he was unable to grasp the meaning of the rule. He knew *why* he was multiplying: because in doing so he got all his answers right, and this certainly is a highly valued goal of an activity. He might have also used multiplication "because the teacher said so". To satisfy the teacher is another goal worth of effort in a young student's life.

On the other hand, knowing the meaning of a procedure does not imply its making sense for us. The main difference between sentences (a) and (b) is that (a) refers to something objective (the meaning), and (b) tells us about a subjective feeling of the speaker.

Perhaps we should not be satisfied as teachers with our students "understanding" their tasks only in this subjective sense, but certainly all we ask the students to do should make sense to them.

But "sense" may also have an objective meaning; for example, when we ask: "In what sense are you using this word?" The explanation is normally given by an example of a more common use of this word: a sentence in which this word is used.

The sentence gives *sense* to the word, by placing it in its structure which defines the function of the word.

The structure of the sentence *is the sense*, in which the word is used. But the sentence also *refers* to something, denotes something, states something that can be true or false in some reality. And it is the sense of the sentence together with its reference that make the meaning of the word.

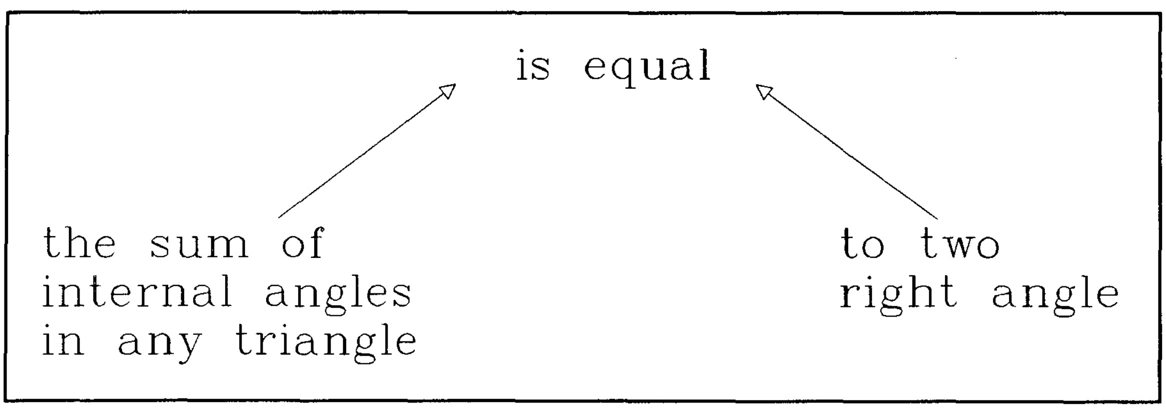
While sense is considered within the language, reference transcends it and forces us to decide upon ontological questions.

The principle that the meaning of names should never be considered outside of sentences, as well as the distinction between sense (Sinn) and denotation or reference (Bedeutung) are attributed to Frege (1967; cf. Ricoeur 1989, p. 89). His ideas have been developed and formalized in logical semantics by Church (Mała Encyklopedia Logiki, p. 233), but the above presentation of the duality of sense and reference in meaning is based on Ricoeur's interpretation of "Über Sinn und Bedeutung" on the ground of philosophical hermeneutics (Ricoeur 1989, pp. 89-91).

Ricoeur defines the sense of a sentence as an answer to the question: "What does the sentence say?" Reference tells us: what is the sentence about. Let us consider the following sentence, to better see the difference between the sense and the reference:

"The sum of internal angles in any triangle is equal to two right angles."

The structure of the sentence is as follows:



Sense: The sentence states the equality of two objects.

Reference: The sentence is true in the following reality: ideal objects - triangles, defined in a theory called Euclidean geometry (to be distinguished from non-Euclidean geometries). A triangle is ... . An angle is ... . etc.

5. An idea of a method for epistemological studies of mathematical concepts.

Ricoeur's considerations have a methodological value: the distinction between sense and reference is directly linked with the way he discriminates between semiotics and semantics. They can be also inspiring in finding a method for elaborating epistemological analyses of mathematical concepts.

Suppose we start from the informal language of mathematics. Let us find words and expressions used in defining, describing, working with the concept we are analyzing. Let us then find sentences which are the senses in which these words and expressions are used. Then let us seek the references of these sentences, and then - relations among all these senses and references.

This will lead us to a description of the meaning of the concept in question (on a certain level, depending upon the degree of analysis we have made). Understanding the concept will then be conceived as the act of grasping this meaning.

This act will probably be an act of generalization and synthesis of meanings related to particular elements of the "structure" of the concept (the "structure" being the net of senses of the sentences we have considered).

What are these acts? Are they always syntheses and generalizations? Or, maybe, there are some other kinds of acts of understanding. We shall deal with these questions in section 7.

6. What are the relations between the notions of understanding and epistemological obstacle?

All our understanding is based on our previous beliefs, pre-judgements, pre-conceptions, convictions, unconscious schemes of thought. Claiming that we can do without these or are able to get rid of them is an epistemological day-dream (cf. Descartes, Dilthey, Husserl). However, discovering that our understanding was erroneous, we use ugly names for the same kinds of things, and call them: myths, prejudice, misconceptions, preconceived opinions, intellectual habits. All these are ways of knowing.

We know things in a certain way. But the moment we discover there is something wrong with this knowledge (i.e., become aware of an epistemological obstacle), we understand something and we start knowing in a new way. This new way of knowing may, in its turn, start functioning as an epistemological obstacle in a different situation. Not all, perhaps, but some acts of understanding are acts of overcoming an epistemological obstacle. And some acts of understanding may turn out to have been acts of acquiring a new epistemological obstacle.

A description of acts of understanding a mathematical concept would thus contain the list of epistemological obstacles related to that concept, and, moreover, provide us with a fuller information about its meaning.

In many cases, overcoming an epistemological obstacle and understanding are just two ways of speaking about the same thing. The first is "negative", and the other "positive".

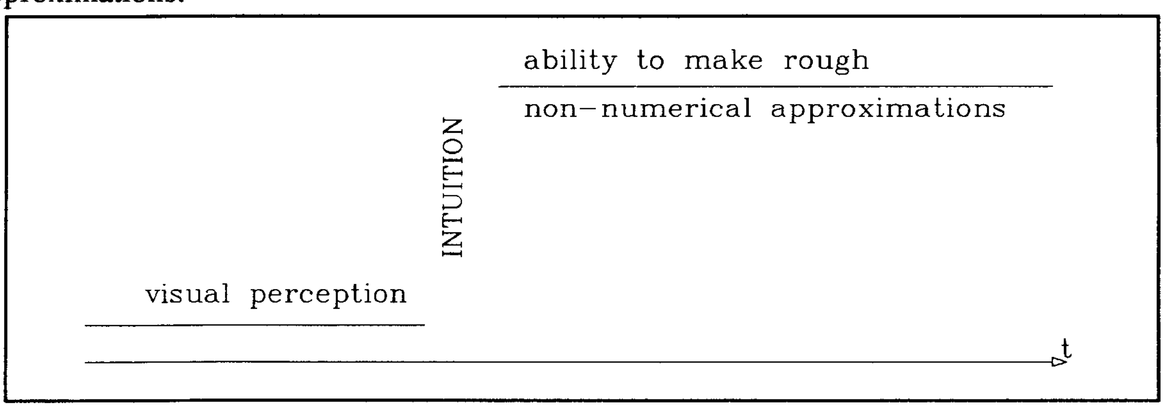
Everything depends upon the point of view of the observer. Epistemological obstacles are looking backwards, focusing attention on what was wrong, insufficient in our ways of knowing. Understanding is looking forward to the new ways of knowing. We do not know what is really going on in the head of a student in the crucial moment but if we take the perspective of his or her past knowledge we see him or her overcoming an obstacle, and if we take the perspective of the future knowledge, we see him or her understanding. We cannot take the two perspectives at the same time. Still, neither can be neglected if the picture is to be complete. This looks very much like complementarity in Niels Bohr's sense (cf. Heisenberg 1989, p. 38, Otte 1990): overcoming an epistemological obstacle and understanding are two complementary pictures of the unknown reality of the important qualitative changes in the human mind.

This suggests a postulate for epistemological analyses of mathematical concepts: they should contain both the "positive" and the "negative" picture, the epistemological obstacles and the conditions of understanding.

7. Are there degrees, levels or kinds of understanding?

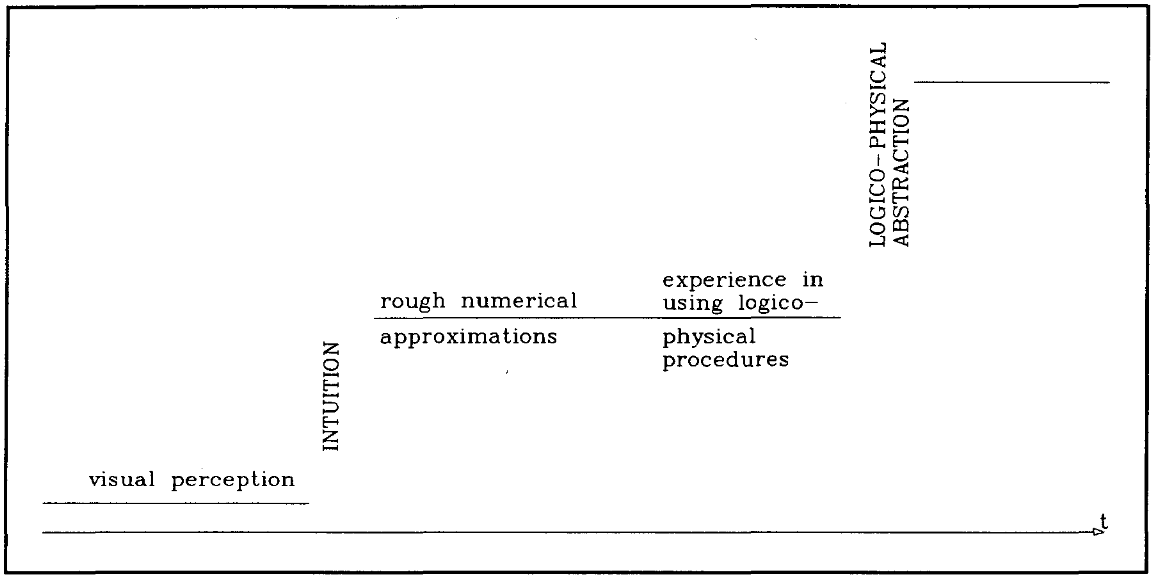
The Herscovics-Bergeron model for understanding mathematical concepts (1982b) distinguishes three "levels". Two of these levels can be regarded as categories of acts of understanding. The third seems to be rather a kind of knowledge. The two categories of acts of understanding are: intuition and logico-physical abstraction.

Intuition or "intuitive understanding", as the authors call it, of number is defined as being a "global perception of the notion at hand" which arises from "a type of thinking based essentially on visual perception" and results in an ability to make rough non-numerical approximations.



Acts of understanding which constitute the category of logico-physical abstraction consist in becoming aware of the logico-physical invariants (like, e.g., conservation of plurality and position) or of the reversibility and composition of the logico-physical transformations, or in generalization (perceiving the commutativity of the physical union of two sets).

These are all acts of understanding and not ways of knowing. However, the reason why they have been divided into such two categories does not lie in the specificity of these acts themselves but in the levels of knowledge from which these acts sprang up. Visual perception is what suffices to give birth to "intuition"; rich experience and complex mental operations are required to produce the awareness of logico-physical invariants, reversibility and associativity of the logico-physical transformations, not to mention the generalization. This "rich experience" is named "procedural understanding" and constitutes the third level in the discussed model. Therefore what is classified here, in fact, are levels of the children's mathematical knowledge, and not their acts of understanding.



In his "Essay concerning human understanding", John Locke speaks about "degrees" of knowledge. There are three degrees, two of which resemble Descartes' types. They are: the "intuitive knowledge" (immediate perception of agreement or disagreement between ideas); "demonstrative" or "rational" knowledge (when the mind perceives the agreement or disagreement between ideas not immediately but by the intervention of other ideas, i.e., proofs); sensitive knowledge (knowledge of the existence of particular external objects). Intuitive knowledge is irresistible and certain. Rational knowledge is acquired with pain and attention.

Although Locke speaks about "perceptions" which are acts, this again is a classification according to levels of intellectual effort that is needed to produce such a perception.

But Locke speaks also of "sorts of knowledge" and this resembles more a classification of acts of understanding.

For Locke, "knowledge" is perception of "connexion and agreement or disagreement and repugnancy" of any of our "ideas". He distinguishes four "sorts" of this "agreement and disagreement". The first he calls "identity or diversity", because "it is the first act of the mind to perceive its ideas and to perceive their difference and that one is not the other", like in: "blue is not yellow". This act of knowing might be called **identifying ideas & discriminating between ideas.** It might also be useful to distinguish these as two different sorts of acts of understanding.

The second Locke's sort of knowledge is "relation" or "perception of relation between two ideas", like in: "two triangles upon equal bases between two parallels are equal". This is important, Locke says, because "without relations between distinct ideas there would be no positive knowledge". Let us call this sort: **finding relations between ideas.**

The third sort of knowledge might be called: **discovering properties** of a complex idea: "co-existence or necessary connection", in Locke's terminology. This appears in saying, e.g., that "gold is fixed", or "gold is resistant to fire", or "iron is susceptible of magnetical impressions".

The fourth sort of knowledge is concerned with "the actual real existence agreeing with any idea", like in "God is". let us call this: **finding relations with reality**.

Locke's distinctions remind us of models comprising generalizations and discriminations mentioned, e.g., in Dewey (1988), and developed in mathematics education by Hoyles (1986).

According to Dewey (and in this he is a forerunner of Piaget), concepts are not abstracted from sensory impressions; the child does not develop the concept of "dog" by abstracting from characteristics such as colour, size, shape etc., but starts from **identifying** one dog it has seen, heard, touched. Then it tries to transfer its **experience** with this single object onto other objects, anticipating some characteristic ways of behaving (this, in fact, is **generalization**). Cats become "small dogs", horses - "big dogs". Then comes the **discrimination**, distinction between properties characteristic of dogs and non-characteristic of dogs. **Synthesis** does not consist in mechanical cumulation of properties but in the "application to explaining new cases with the help of a discovery made in one case" (op. cit., pp. 164-165).

Experiencing, identifying, generalizing, discriminating, synthesizing, applying are, according to Dewey, the crucial moments of concept formation, but, besides perhaps experiencing and applying, they look like good candidates for the important acts of understanding.

In Hoyles (1986) a model for learning mathematics is presented which is very similar in spirit and terms to Dewey's: "**using** - where a concept/s is used as a tool for functional purposes to achieve particular goals; **discriminating** - where the different parts of the structure of the concept(s) used as a tool are progressively made explicit; **generalising** - where the range of applications of the concept(s) used as a tool is consciously extended; **synthesising** - where the range of application of the concept(s) used as a tool is consciously integrated with other contexts of application" (Hoyles 1986, p. 113).

8. Categories of acts of understanding

Let us synthesize Locke's, Dewey's and Hoyles' ideas and try a categorization of acts of understanding a mathematical concept:

Identification of objects that belong to the denotation of the (or a) concept (related to the concept in question), or: identification of a term as having a scientific status; this act consists in a sudden perception of something as the "figure" in the gestaltist experiments.

Discrimination occurs between two objects, properties, ideas that were confused before.

Generalization consists in becoming aware of the non-essentiality of some assumption or of the possibility to extend the range of applications.

Synthesis is grasping relations between two or more properties, facts, objects and organizing them into a consistent whole.

Of course, the necessary condition of all these acts to occur is experiencing, using and applying: "If we behave passively towards objects, they remain hidden in the confused blotch which absorbs them all." (Dewey 1988, p. 159)

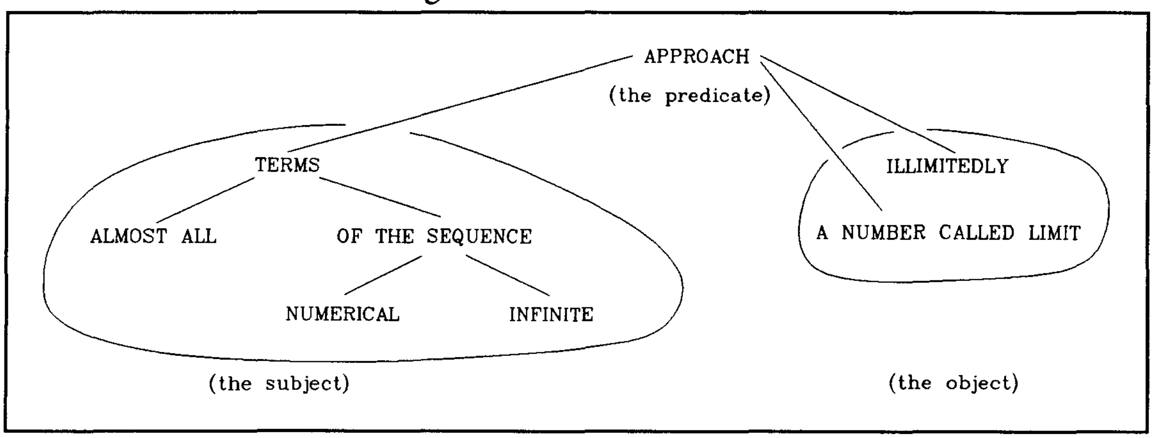
II. What does it mean to understand the concept of convergent numerical sequence? —an example of an epistemological analysis of a mathematical concept.

Of the many sentences that can be formulated about a convergent sequence, let us choose this one:

> "Almost all terms of the numerical infinite sequence approach illimitedly a number called its limit."

This sounds a bit artificial but has the advantage of comprising the elements of the definition of convergent sequences.

Let us first consider the logical structure of this sentence:



This structure defines the **sense** of the sentence. It says that something approaches something in a certain way.

The predicate states something general about the subject which points to something individual. "The subject (...) identifies one and only one subject. The predicate, on the contrary, points to some quality or class of objects, or type of relation or type of activity." (Ricoeur 1989, p. 78). In the case of our sentence the verb, APPROACH, seems to point to an activity; however, understanding of the concept of limit will lead to perceiving that, in fact, it defines a type of relation. But, in saying this, we enter the area of the REFERENCE of the sentence.

Let us then consider the question: what is the sentence about?

1. The subject: TERMS OF INFINITE NUMERICAL SEQUENCE

The subject refers to the world of infinite sequences. Hence, the first step towards understanding the notion of convergent numerical sequences (CNS) must be to discover the world of infinite sequences. The first infinite numerical sequence encountered by the child is most often the sequence of natural numbers. Becoming aware that one can count on and on forever is probably the first act of understanding of what an infinite sequence can be. Later, other infinite sequences come as well: odd and even numbers, numbers divisible by three, etc. When the students start converting vulgar fractions to their decimal expansions, strange things start to happen and sometimes the division wouldn't come to an end. One can go on forever and ever: another experience with infinite sequences. But these sequences are very special—periodical. Unending calculations come back with the question of the place of irrational numbers on the number line. The square root of 2 turns out to be "squeezed" between two infinite sequences of rational approximations. Calculation of areas of figures even as simple as rectangles give rise to questions leading to infinite sequences. If the sides of the rectangle are commensurable, then the formula: area = length x breadth is easily explained in the frame of the conception of multiplication as repeated addition. But what if the sides are incommensurable? This demands a reconceptualization of the notion of multiplication. Iterating functions, producing sequences of numbers and sequences of geometrical transformations (possibly with the use of computer), approximating solutions of equations, maxima and minima, tangents, areas, velocities are further domains of experience with all kinds of infinite sequences. (Interesting methods of working within these domains in the mathematics classroom can be found in Hauchart 1987.)

This experience can bring about the first act of understanding the notion of CNS which is the identification of a new object worth of study.

2. The predicate: APPROACH

The verb is the most important part of the sentence: where there is no verb, there is no sentence. This is why, after having entered the world of infinite sequences, the most important thing is to distinguish among them those that "approach" something or "tend to" something or "converge" to something. Sequences that "approach" or "tend" or "converge" must, at some moment, become "the figure" in our picture, and the rest of the infinite sequences - the "background". Students who experience this act of identification can be heard exclaiming: "Oh! it comes closer and closer (approaches, tends, converges) but will never become equal (or reach)".

U(lim)-2: IDENTIFICATION OF SEQUENCES THAT APPROACH SOMETHING

This act normally results in the development of a certain sensitivity to convergent sequences. However, this development is not possible without a number of shifts of attention as far as certain aspects of the notions of number and sequence are concerned. In particular, focus on the form of number, or on the stabilization of decimal digits, or on the rule of generating terms of a sequence, or on the set of the terms of the sequence, can all function as obstacles to the identification of sequences that approach something. Below we shall make some comments on these obstacles.

The form we shall use to name an obstacle is analogous to that which Lakoff & Johnson (1980) have used to name metaphors. Metaphors, according to these authors, are symptoms of conceptions, and, as obstacles are also based on conceptions, the use of an analogous form of coding is not misplaced, I hope.

Number: number is an inscription

This obstacle consists in focusing on the form of number and not on its value; the length of the inscription, digits used in it are more important than the place of the number on the number line or in the sequence.

Having to classify (according to a rule of their choice) the following set of series:

A. $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots$

B. $1 + 2 + 3 + 4 + \ldots$

C. $1 - 1 + 1 - 1 + \ldots$

D. $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots$

E. $1 + 1 + 1 + 1 + \ldots$

some students put the series B and E together not because they knew they are both divergent but because in both of them terms are composed of consecutive natural numbers. A and D were put together in another class again because the same digits appeared in consecutive terms; the last class was C and F because the series are both composed of number 1.

The act of overcoming this obstacle amounts to:

U(lim)-3: DISCRIMINATION BETWEEN NUMBER AND FORM OF NUMBER

Another obstacle is:

Convergence: CONVERGENCE IS STABILIZATION OF DECIMAL DIGITS

The above conception develops easily if sequences are introduced with excessive use of calculators or computers and production of decimal approximations of terms. Students may be brought to making unjustified induction jumps and to believe that if they observe stabilization of digits after a hundred terms, then this means that the sequence belongs to the class of convergent sequences. Kuntzmann warned against this obstacle a long time ago (1976) when pocket calculators started to be openly used in schools. In my experiments one student displayed an interesting combination of the above obstacle and the previous one (*number is an inscription*). When observing numerical approximations of terms of a sequence converging to 4 from below (last terms on the screen were: 3.999998, 3.999999) he would say: "here, it tends to <u>nines</u>". Another sequence was "tending to sixes" (0.666666), and still another one, to "zeros" (3.000002, 3.000001). In fact, he focused his attention on the form of the approximations, and not on the values of numbers there were approximations of.

U(lim)-4: DISCRIMINATION BETWEEN CONVERGENCE AND STABILIZATION OF DECIMAL DIGITS

Let us now consider the obstacle:

Sequence: SEQUENCE IS A RULE FOR PRODUCING NUMBERS

A numerical sequence is a function defined on natural numbers and values in real numbers. This function is normally defined somehow; let us call "rule" this definition. A sequence is a synthesis of its arguments, values and rule. If our attention focuses on but one of these elements - there is an obstacle. In the obstacle presently discussed the focus is on the RULE. Sequences would be classified after their rules and not after the mutual relations between the values of their terms. Having to classify a set of sequences, students are able to put identical sequences into two different categories just because they have different rules (Sierpinska 1989).

U(lim)-5: DISCRIMINATION BETWEEN SEQUENCE AND RULE FOR PRODUCING NUMBERS

Another obstacle to overcome is:

Sequence: SEQUENCE IS A SET

Here the focus is on the terms of the sequence (the values of sequence as function). These terms are conceived of as forming a set; their order is not important. It is inessential how the values of the terms change, whether they approach something or not. At the best, the attention may be attracted by the bounds of the set of terms. For example, in the sequence 1, 1.9, 1.99, 1.999, etc., none of the numbers 1 and 2 will be more distinguished than the other (cf. Sierpinska, 1987). This obstacle is, in fact, inscribed in the notion of sequence: the denotation of the concept of sequence (as function) is sometimes described by "situations in which the set of values of the function is more important than the function" (Maurin 1977, p. 17). To overcome this obstacle, one has to make the...

U(lim)-6: IDENTIFICATION OF ORDER OF TERMS AS IMPORTANT FEATURE OF SEQUENCE

3. The subject revisited; identification of *terms of sequence* as the subject of *approach*. Novices often treat convergence as a kind of phenomenon which does not call for naming the responsible agent. They would say "it converges" as "it rains" or "it snows" with an impersonal "it" (cf. Sierpinska 1989b). Such an attitude towards convergent sequences leaves no room for the question <u>of what</u> is converging, in fact.

Convergence: CONVERGENCE IS A NATURAL PHENOMENON

Some students, especially in situations where they are bound to calculate a certain number of terms of the sequence, identify the subject of *approach* with the person who calculates, i.e., with themselves: "*we* are approaching something" as *we* calculate more and more terms. The sequence becomes a sequence of calculations or operations. This raises the question of the infinity of the sequence, puts physical limitations on the arbitrariness of the choice of the epsilon, and further leads to questions concerning the nature of mathematics (is it constructed? discovered?). Of course, one can deliberately choose the constructivist philosophy of mathematics and assume that...

Sequence: SEQUENCE IS A SEQUENCE OF CALCULATIONS

But if this conception is unconscious - it functions as an obstacle.

4. Attribute of SEQUENCE: INFINITE

The focus of attention on the infinite number of terms of the sequence (i.e., of arguments of sequence as function) is another obstacle:

Sequence: SEQUENCE IS A VERY LONG LIST OF NUMBERS

"Very long" may mean many different things in students (Sierpinska, 1987b).

It is exactly this focus on the length of the sequence and not on the values of its terms that is the basis on which the paradox in "Achilles and the tortoise" and "Dychotomy" is built. The stories are told in such a way that the listener's attention is caught by the infinite number of steps to make; the diminishing lengths of the consecutive steps are left in the shade. The number of steps being infinite, it is "obvious" that Achilles will never catch up with the tortoise, and one can never get out of the room one is in.

Conversely, if the number of steps is not perceived as infinite—there is no paradox. This is what happened in an experiment of B. Cała (1989) who inquired into spontaneous explanations of Zeno's paradoxes in 16-year-old students of electronics and their reactions to the usual explanations of these paradoxes in terms of summing up numerical series. The students were interviewed before and after introduction of the notion of limit formally in the mathematics class. Neither before nor after this introduction did the students see any paradox in the stories about Achilles and getting out of the room. Some students said that after sometime the distance between Achilles and the tortoise is so small that it can be neglected. The number of calculations is finite. Other students said that the reasoning is wrong because it neglects the huge difference of velocities between Achilles and the tortoise. The reasoning is obviously wrong, so no wonder why the conclusion is absurd. There was no feeling of paradox.

The feeling of paradox appeared only when the students were shown the explanations in terms of summing up numerical series. In the mathematization of "Achilles" the following numbers were taken:

$V_A = 20$ km/h; $V_T = 0.2$ km/h.

Achilles starts running 9.9h after the tortoise. The time Achilles needs to catch up with the tortoise is then the sum of the series

$$9.9 + 0.099 + 0.00099 + 0.0000099 + \ldots$$

and amounts to 10 h. Now, this was the really paradoxical result for the students. They would say that this sum is 9.9999999... and this is not 10. The two numbers are not equal. And as they got 10 h in solving the problem with the formulas of kinetics, they would say that the above explanation explains nothing. On the contrary, it proves that Achilles will never catch up with the tortoise, since 9.999... just approaches 10 without ever reaching it.

Such an attitude results from, first, the focus of attention on the number of terms of the numerical sequence; second, the conception of sequence as a sequence of calculations; third, the conception of infinity as a certain potentiality to go on further and further with the number sequence.

The two paradoxes of Zeno cannot be solved with the concept of sum of series. This mathematization turns out to be equivalent to the paradoxes. They cannot be solved on the ground of mathematics at all. The existence or non-existence of actual infinity is, after all, not a mathematical but a philosophical question. And philosophy does not give definite answers to such questions. It can only discuss the possible consequences of different answers. The Weierstrassian definition of limit of a sequence in terms of epsilon and N does not solve the problem of whether or not the sequence reaches its limit. Its static and symbolic formulation eliminates this problem from mathematics and makes it senseless to pose it (Sierpinska 1985a).

In order to understand Zeno's paradoxes and be able to appreciate the Weierstrass definition one must become aware of all this and consciously consider one's own and other people's conceptions of infinity, their advantages and limitations.

Obviously, in understanding limits, a particularly dangerous conception linked with infinity is the belief that what is infinite, is necessarily illimited. All convergent sequences, albeit infinite, are bounded. This belief may be linked with the focus on the very large number of terms. It is the shift of attention from arguments onto the values of the sequence that may lead to the discovery of the "bounded infinity" (Sierpinska 1987a, Thomas' story).

Overcoming of obstacles related to infinity seems to be a necessary condition for a conscious synthesis of the concept of sequence. This is why we choose the following order of obstacles and acts of understanding:

Infinity: DIFFERENT CONCEPTIONS OF INFINITY

These conceptions function as obstacles if unconsciously admitted as absolute truths. Below we distinguish one of these conceptions as particularly undesirable from the point of view of understanding limits:

Infinity: WHAT IS INFINITE, IS ILLIMITED

U(lim)-7: IDENTIFICATION OF DIFFERENT CONCEPTIONS OF INFINITY: POTENTIAL INFINITY, ACTUAL INFINITY, BIG UNDETERMINED NUMBER, ARBITRARILY LARGE NUMBER, ...

U(lim)-8: IDENTIFICATION OF INFINITE AND BOUNDED SETS

Convergence: THE PROBLEM OF REACHING THE LIMIT IS A MATHEMATICAL PROBLEM AND THEREFORE THERE EXISTS A MATHEMATICAL SOLUTION TO IT

Philosophy of mathematics: DIFFERENT PHILOSOPHICAL ATTITUDES TOWARDS MATHEMATICS

Again, these attitudes function as obstacles if unconscious and dogmatic.

u(lim)-9: IDENTIFICATION OF DIFFERENT PHILOSOPHICAL ATTITUDES TOWARDS MATHEMATICS

u(lim)-10: IDENTIFICATION OF THE PROBLEM OF REACHING THE LIMIT AS A PHILOSOPHICAL PROBLEM CONCERNED WITH THE NATURES OF MATHEMATICS AND INFINITY

u(lim)-11: SYNTHESIS OF THE CONCEPT OF NUMERICAL SEQUENCE

It is only then that we can speak of a conscious ...

U(lim)-12: IDENTIFICATION OF THE SUBJECT OF "APPROACH"

5. Attribute of SEQUENCE: NUMERICAL

Analysis of the ancient "method of limits" presented in the XIIth Book of Euclid's "Elements" (e.g. Wygotski 1956) brings to our awareness the importance of the concept of real number in understanding the notion of limit. Today, the notion of real number seems so familiar, so omnipresent in mathematics, that we hardly pay any attention, in theorems concerning limits, to assumptions such as "$a_n \epsilon \mathbb{R}$".

Asked, in a problem, to provide the area of a figure, we produce a number; but have we really detached this number from the figure? Do we always care for saying "area of circle" instead of "circle" when we mean the number—the measure—the area?

Perhaps unconsciously, we still make the distinction between numbers and continuous magnitudes which are wholes containing in themselves their qualitative as well as quantitative aspects? Perhaps the convergence of a numerical sequence is something else than the convergence of the sequence of 2^n-gons inscribed in a circle: in the former, the difference a_n - L is a number; in the latter, this difference is a difference of shapes: the higher is n, the less difference is there in shapes of the 2^n-gon and the circle (cf. Sierpinska 1985a, pp. 51-52).

This is why, as next obstacle we admit:

Convergence: THE MEANING OF THE TERM "APPROACH" DEPENDS UPON THE CONTEXT: DIFFERENT IN THE DOMAIN OF NUMBERS AND DIFFERENT IN THE DOMAIN OF GEOMETRICAL OF PHYSICAL MAGNITUDES

U(lim)-13: SYNTHESIS OF THE CONCEPTS OF "APPROACH" AND NUMBER

This act should result in establishing the meaning of the term "approach" in terms of distances and not differences of shapes, positions, etc.

6. Object of *APPROACH*: number called *LIMIT*

Sometimes students do not conceive of "approaching" as "approaching something" but as "approaching one another": as n grows, the terms of the sequence come closer and closer to one another (Sierpinska 1989b).

Convergence: CONVERGENCE CONSISTS IN DECREASE OF DISTANCE BETWEEN THE TERMS OF THE SEQUENCE

This is the intuition of Cauchy rather than convergent sequences. There is no difference between these in the real (in general: complete) domain, but ...

U(lim)-14: DISCRIMINATION BETWEEN THE CAUCHY AND CONVERGENT SEQUENCES

is an important step towards understanding not only the notion of limit but also the notion of real number itself, of the meaning of the Dedekind axiom in particular, and thus of the essentiality of the assumption that $a_n \in \mathbb{R}$ in theorems concerning monotonic and bounded sequences.

U(lim)-15: IDENTIFICATION OF THE *GOAL* OF *APPROACHING* (i.e., of the limit of sequence)

7. Adverbial phrase of APPROACH: ILLIMITEDLY

Students' first conceptions of convergence need not make any difference between approaching of a sequence like 0.8, 0.88, 0.888, ... to number 0.9, and approaching of a sequence like 0.9, 0.99, 0.999, ... to number 1. To make this distinction one has to overcome the obstacle:

Convergence: CONVERGENT SEQUENCES ARE SEQUENCES THAT APPROACH SOMETHING

U(lim)-16: DISCRIMINATION BETWEEN *APPROACHING* AND *APPROACHING ILLIMITEDLY*

(cf. Sierpinska 1989b, the case of Robert).

8. Attribute of TERMS: ALMOST ALL

Another distinction to make is between sequences like:

1, ½, ⅓, ¼, ... and 1, ½, 2, ¼, 3, ⅛, ...

In both sequences infinitely many terms tend towards zero but only in the first one almost all terms tend to zero. In the second sequence an infinite number of terms run away from zero (cf. Sierpinska 1989b, p. 33-34).

Convergence: CONVERGENCE IS WHEN INFINITE NUMBER OF TERMS OF THE SEQUENCE APPROACH SOMETHING

U(lim)-17: DISCRIMINATION BETWEEN *INFINITELY MANY TERMS APPROACH THE LIMIT* AND *ALMOST ALL TERMS APPROACH THE LIMIT*

9. Reference of the sentence as a whole: what is the relation between the subject *terms of infinite numerical sequences* and the object *limit* ?

Let us call this relation "passing to the limit".

In solving equations by approximative methods, calculating areas, tangents, speed of change of variable magnitudes, etc., one can easily come to the conclusion that ...

Passing to the limit: PASSING TO THE LIMIT IS A HEURISTIC METHOD USEFUL IN SOLVING CERTAIN KINDS OF PROBLEMS

On the other hand, formal teaching of the notion of limit based on introducing the formal ϵ-N definition and then proving (by definition) about concrete a_n and L that lim a_n = L may lead the students to developing the following obstacle:

Passing to the limit: PASSING TO THE LIMIT IS A RIGOROUS METHOD OF PROOF OF RELATIONS BETWEEN SEQUENCES AND NUMBERS CALLED THEIR LIMITS

(cf. Sierpinska 1987a).

Hence there is a need to make a ...

U(lim)-18: SYNTHESIS OF PASSING TO THE LIMIT AS A MATHEMATICAL OPERATION DEFINED ON CONVERGENT SEQUENCES AND WITH VALUES IN $\mathbb{R}$

A mathematical operation should be well defined: a sequence should not have more than one limit. This brings forth another condition:

U(lim)-19: SYNTHESIS OF THE NOTION OF PASSING TO THE LIMIT AS A MATHEMATICAL OPERATION AND THE PROPERTY OF MATHEMATICAL OPERATIONS OF BEING WELL DEFINED (AWARENESS OF THE UNIQUENESS OF LIMIT).

Cauchy, who introduced the symbol "lim" did not demand that it denote a single object. However, he proposed to mark sequences having "many limits" with double brackets: e.g., "$\lim\left(\left(\sin\frac{1}{x}\right)\right)$ has infinitely many values contained between -1 and 1" (Cauchy, A. *Cours d'Analyse*).

The above synthesis (19) may lead to some degenerations in the students' conceptions of limit especially if they tend to reduce the new operation to some well known one and to apply the same methods. One such degeneration, very common in students (and having its counterpart in Fermat's method of "omitting certain terms") is:

Limit: LIMIT OF SEQUENCE IS VALUE OF THE SEQUENCE IN INFINITY

Students are led to considering inscriptions like $\frac{1}{\infty}$, ∞, $\frac{1}{0}$ as numbers, and thus to conceiving of numbers like Leibniz and Cauchy rather than like Weierstrass (cf. Sierpinska 1985a, p. 46). In students' exercise books one can find inscriptions like:

$$\lim_{n\to\infty}\frac{1}{2^n}=\frac{1}{2^\infty} \qquad 0.\underbrace{777\ldots}_{n}=\frac{\overbrace{777\ldots7}^{n}}{10^n}$$

In the first of the above inscriptions "∞", and in the second "n" stand rather for "big undetermined number" than for "infinity" and "arbitrary natural number" (cf. Sierpinska 1987b, 1989a).

U(lim)-20: DISCRIMINATION BETWEEN NUMBERS AND CONCEPTS LIKE INFINITELY SMALL QUANTITY AND INFINITELY LARGE QUANTITY

U(lim)-21: DISCRIMINATION BETWEEN THE CONCEPTS OF LIMIT AND VALUE

10. Understanding a formal definition of the limit of a numerical sequence

The sentence "Almost all terms of the infinite numerical sequence approach illimitedly a number called limit of the sequence" can be used as the definiens of an informal definition of a convergent sequence. Let us consider now the conditions of understanding the formalization of such a definition.

In the formalization, all nouns will have to be translated by letters, all predicates—by symbols of mathematical relations. In the traditional algebra, and in physics, letters denote either variable or constant magnitudes. But, it is not this concept of variable we must have in mind when formalizing the definition of LNS. A letter—a variable—will have to be conceived as just a name for an element of a class (or set). The important thing is to know which variables are bounded and which are free in the definition, and not which letters denote "variables", and which "constants". This conception of "variable" is very distant from what is usually meant by it in algebra or physics (cf. Freudenthal, 1983).

Symbolic language: LETTERS STAND FOR VARIABLE OR CONSTANT MAGNITUDES

U(lim)-22: DISCRIMINATION BETWEEN THE USE OF LETTERS IN ALGEBRA AND LOGIC

Suppose we have chosen S for "infinite numerical sequence" (i.e., S is a name for a representative of the class of infinite numerical sequences), t_n for "term of the infinite numerical sequence", and L for "a number called limit of the sequence".

The defining property of a convergent sequence is that almost all t_n illimitedly approach a certain number L. This number is not arbitrary, it depends upon S. L is a function of S, since S may have only one limit. We might mark it in our formalization and write L_S instead of just L. This, however, would make the formalism very heavy, so let us write L.

The first condition is, therefore, that there exists such a number L:

$$\exists\, L \in \mathbb{R}$$

Now the question is: what does it mean, in mathematical terms, that the numbers t_n "approach" L? Approaching is linked with decrease of distance. Can this distance be measured somehow? Modulus is a notion invented just for that purpose: $|t_n - L|$ is the measure of distance between t_n and L. The numbers $|t_n - L|$ are getting smaller and smaller.

Having overcome the obstacles related to the concept of sequence and being aware that the terms of the sequence change not in time but with the growth of n, we can say that: as n grows to infinity, the distances $|t_n - L|$ get smaller and smaller.

Now, in comparing how the numbers 0.8, 0.88, 0.888, ... approach 0.9, with the way in which numbers 0.9, 0.99, 0.999, ... approach 1 we could see that in the first case the distances $|t_n - L|$ could never become smaller than 1/90, whereas in the second case, for any distance (say ε), however small, we can find terms t_n distant from L less than this distance, i.e., such that $|t_n - L| < \varepsilon$

What does it mean that "*we* can find"? This expression has to be de-personalized.

In formalizing the definition of CNS one must become aware that what points to a term in a sequence is its index: the index determines the place of the term in the row of terms and shows the term we are concerned with.

U(lim)-23: IDENTIFICATION OF THE ROLE OF ARGUMENTS IN A SEQUENCE AS INDEXES OF THE VALUES-TERMS

This act of understanding allows for a de-personalization of the notion of approaching, as well as for reducing some conditions on terms to conditions on indexes. This leaves the variable "term of sequence" (t_n) unbounded, because, instead of having, e.g., "$\exists t_n : |t_n - L| < \varepsilon$" we can put: "$\exists n : |t_n - L| < \varepsilon$" This way the experience with sequences that "do not approach really very close to their limits" leads to a tentative formalization in the form:

$$\exists\, L \in \mathbb{R}\ \forall\, \varepsilon > 0\ \exists\, n : |t_n - L| < \varepsilon$$

However, when investigating the logical consequences of this definition we fall on monsters which do not look convergent at all: only some or, at most, an infinite number of terms approach L and not almost all of them. "Almost all" means that all but at most a finite number of terms approach L. Let this number be k. This number depends on how small we have taken the distance ε. The number of terms from the beginning of sequence is visible in the index. So we may say that terms with index greater than k are distant from L by less than ε. Such an index k must exist, whatever the ε. As k depends upon ε and not vice versa, we put:

$$\exists\, L \in \mathbb{R} \quad \forall\, \varepsilon > 0 \quad \exists\, k \in \mathbb{N} \quad \forall\, n > k\ |t_n - L| < \varepsilon$$

This way, all elements of the structure of the definiens have been taken into account. So, finally, we can formulate the whole definition:

Definition: A sequence $S: \mathbb{N} \to \mathbb{R}\ (n \mapsto t_n)$ is called convergent $\leftrightarrow$

$$\exists\, L \in \mathbb{R} \quad \forall\, \varepsilon > 0 \quad \exists\, k \in \mathbb{N} \quad \forall\, n > k\ |t_n - L| < \varepsilon$$

There exists a formalization where the condition $\exists\, k \in \mathbb{N}$ is replaced by the condition $\exists\, k \in \mathbb{R}$. I have seen a teacher parachuting this definition on 17-year-old humanities students. The students were completely lost; they could not see the meaning of this condition. Of course, logically, the two definitions are equivalent, but the meaning of $\exists\, k \in \mathbb{N}$ is different from the meaning of $\exists\, k \in \mathbb{R}$ and therefore, they are not equivalent psychologically. The condition $\exists\, k \in \mathbb{N}$ points to an index and thus to a term of the sequence after which all the terms of the sequence are distant from L by less than ε. This allows to understand the definition even without having completely depersonalized the choice of terms that are in the interval $(L - \varepsilon, L + \varepsilon)$. But if we write $\exists\, k \in \mathbb{R}$ then we must conceive the indexes as numbers, and remember that these numbers are embedded in the field of real numbers. Now, when thinking of sequences we normally do not regard indexes (arguments) as having the same status or as belonging to the same category as the terms (values). Indexes are not numbers; not "nombres" but rather "numéros" (like those you get in a cloakroom). And this point of view has to be overcome if the formalization with $\exists\, k \in \mathbb{R}$ is to be understood and accepted.

Sequences: INDEXES OF TERMS OF A SEQUENCE ARE NOT NUMBERS

U(lim)-24: IDENTIFICATION OF INDEXES OF TERMS OF A SEQUENCE AS NUMBERS

11. Generalization

Further conditions of understanding the notion of LNS would be concerned with generalization and synthesis of this notion.

First, the activity of formalization of the definition of CNS may lead to a more conscious overcoming of the belief that the "problem of reaching the limit" is a mathematical problem. This act of understanding has already the status of a synthesis; this is, in fact, an evaluation of a mathematical result, a perception of its relevance:

U(lim)-25: SYNTHESIS OF THE DISCUSSIONS AROUND THE PROBLEM OF REACHING THE LIMIT IN THE LIGHT OF THE FORMAL DEFINITION OF LIMIT: AWARENESS THAT THE FORMAL DEFINITION AVOIDS PUTTING UP THIS PROBLEM AND IS ACCEPTABLE WITHIN MANY DIFFERENT CONCEPTIONS OF INFINITY

Looking for new domains of application of the concept of limit leads to questions concerning the relevance of the different conditions in the definition of the CNS. What can we change without loosing the general idea of approaching and preserving the passing to the limit as a mathematical operation? It may be tempting to define the concept of limit of a sequence in any topological space. Preserving passing to the limit as an operation demands, however, that we restrict ourselves to Hausdorff spaces. Unawareness of this is an obstacle which has to be overcome.

And so on. There is probably no end to generalizations and syntheses.

12. Recapitulation of sections II.1 - II.11: acts of understanding the concept of convergent numerical sequences.

Acts **U(lim)-1** through **U(lim)-25** of understanding the concept of convergent numerical sequences have been put together in Table 1, p. 53. In the Table, these acts have been divided into the following categories:

1. Acts of understanding (abbr. aU) convergent sequences.
   1.1 aU the particular elements of the structure of the definiens of an informal definition conceived of as a sentence in natural language.
   1.2 aU globally the concept of convergent sequences.
2. aU infinite sequences and infinite numerical sequences.
3. aU the notion of real number.
4. aU symbolic language.
5. aU the natures of infinity and mathematics.

The methodology of "levels of understanding" in epistemological analyses of mathematical concepts (Herscovics & Bergeron 1989a, b) seems to focus on the evaluation of knowledge in students. The methodology of "acts of understanding" is concerned mainly with the process of constructing the meaning of concepts. However, the partial order within the acts

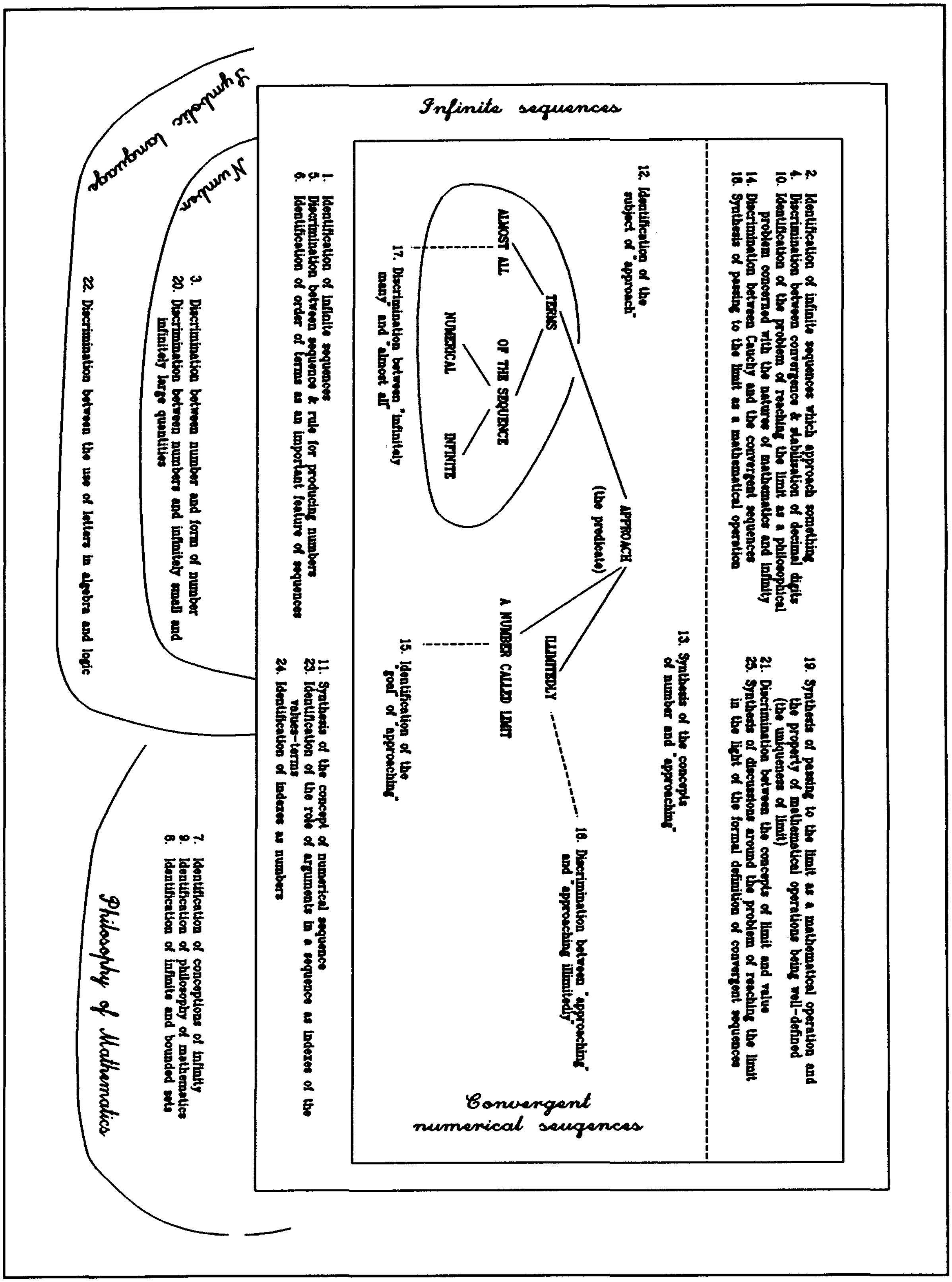


Table 1: Acts of understanding the concept of convergent numerical sequences

of understanding would probably allow for defining something like "depth" of understanding. The depth of understanding might be measured by the number and quality of acts of understanding one has experienced, or: by the number of epistemological obstacles one has overcome. Of course, there is the problem of methods of providing evidence that, in a particular person, such and such act of understanding has taken place. These methods would probably have to be elaborated separately for each act of understanding. There is also the practical educational problem of how to provoke these acts in students within the classroom work and how to check without making detailed interviews that they have occurred.

The methodology of "acts of understanding" is not very precise. Perhaps also, it cannot be made more precise without loss of generality. For example, the choice of sentence(s) the sense and reference of which are analysed is not well defined. In our example of the concept of convergence, this sentence was a part of an informal definition. Has it always to be a definition? This might work with the notion of function (e.g., "changes of magnitude ***Y*** are related to changes of magnitude ***X*** in a well-defined way"), but one can hardly do the same with the concept of number or even such an apparently simple thing as the concept of area of a rectangle (cf. Sierpinska, 1990).

However, in spite of all these difficulties, and whatever the methodology, the usefulness of epistemological analyses of the mathematics taught at different levels seems indisputable, whether for the practice of teaching or writing textbooks or as a reference for all kinds of research in mathematics education.

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Question Period

Does understanding exist only in a mathematical sense or is it possible to understand something mathematical by making a correlation to some other kind of activity and are these comparisons with other activities a basis for understanding? I mean when you find a discriminating and generalizing, you are mainly talking in mathematical field. I am wondering if connecting mathematics to everyday events is understanding also?

I think that in the act of generalizing or synthesizing you might have these things. **Limit** is very mathematical at the point of abstraction, but if you study probabilistic notions you have discriminations with chance and probability which are all involved in real life situations. So there are fields closer to reality than **limit**. We might talk about velocity or things like that which would probably be closer to reality.

I think you were saying, if I understand you correctly, that when you apply something you are showing you understand. The way you use generalization as I understood is not the way many mathematician use it but you define it as an extension of reference. So I would think that with your definition, which coincides with Dienes definition of generalization as opposed to abstraction, means that the more applications you can think of means the more you extend the domain validity and the more you understand. In that sense it is part of understanding so it depends on what you mean by generalization; another poorly defined word.

One of the things that happens when you start to teach fractal geometry is that you are constructing converging series in a geometric sense and the students, if they have not been spoiled by higher education, can grasp this very quickly. They also learn to juggle for instance the Triadic and Quadratic Koch Islands which are very widely reproduced. Both both converge in different ways, but both to finite areas with infinite perimeters and one more infinite than the other. Students sort of geometrically start to run into the semantic trap that we see in algebra. Certainly from my experience looking back to my days in university I did very poorly with algebraic series because they were a list of random telephone numbers that were running across the page. Whereas when I look at the Koch Triadic Island I can ***see*** *the infinite sprouting points and I can* ***see*** *they are confined to an area with a finite area. I think some of the concepts that we have trouble with are troublesome because we put them in algebraic and semantic terms, whereas is not a picture worth a thousand words?*

You took the Dewey example of identifying and growing with the experience of one dog. So "In what sense do students do this with a concept of infinite concept or infinite series" Secondly, in what sense does becoming more formal or more abstract relate to your notion of understanding?

I think that the students meet different obstacles, and all these obstacles are also steps towards understanding. When they start thinking that if the digits in a sequence start to stabilise it is convergent; they have generalised. They have seen one such sequence converge and then they believe all sequences behave this way. If they see something similar to a sequence they have seen before, that will be convergent.

Let me contextualise this for you. I have just been reading Heisenberg's reflections on the notion of abstraction, and he talks about how hard it is for human beings, even for Einstein, to give up reality and go into a different order of reality. He says we must step into a circumstance where we are not talking about the world but ideas about the world, and that *is the pathway to understanding. It seems that people resist doing this, they resist going away from their everyday life. Is this overcoming of cognitive obstacles always happening rather than being a rare thing?*

I have read Heisenberg's book *Der Teil and das Ganze* I loved it very much. I had the impression that this book was on epistemological obstacles, and that is a better book than Bachelard. I thought the book was on epistemological obstacles, but in fact this is a book on understanding. In it he tells the story of how he was asked by a friend "Did you understand relativity theory?" He said "I don't know; the mathematical apparatus doesn't make any difficultly for me. I understand how one thing is derived from the other logically but still this notion of time, relative time, is very confusing to me." To understand is also to find the reference in reality and our concepts of time which we are used to are obstacles to the notions of time and position and momentum that are used in quantum mechanics. To explain what an epistemological obstacle is to a mathematician I take the example of Heisenberg's Uncertainty Principle and everybody has this obstacle; you can't simply understand it.

One thing that puzzled me a bit is that one of your components of understanding was identification of a class of objects that one works with and the example you gave was of infinite sequence. You say that the students have to identify it as an object which is worth studying and it seems to me that one way to realize why sequences are worth studying is that some sequences are convergent and some are not convergent. In other words, how do you actually identify something that is worth studying until you have seen what it is about that thing which makes it worth studying?

You may trust the teacher. That's right because I have ordered it [the example of infinite sequence] in this way but this is only a partial ordering. Sometimes you must go to **u(lim)-6** in order to understand **u(lim)-1**. Of course these things go together, but they are not linear.

Just a comment. I think that sometimes when we generalise and say what children do, as Dr. D'Ambrosio said, we can forget to ask the children. I don't know what I'm revealing about myself but when I was about six years old I was absolutely convinced that cats were female dogs and that they married each other, and I never bothered to check with anybody. I worked it all out by myself and that was that.

I had a colleague that use to teach in Ethiopia and he tells me that amongst the people of Ethiopia they have no generic terms for birds. They classify winged creatures into two categories—eatable and non-eatable. I think here in Canada you will find Eskimos have no word for snow but they have fourteen different words for different kinds of snow. In Sudbury it's "damned white stuff". But Eskimos have a different orientation, they have buildable snow, crushable snow, slippery snow, and they just don't have a category "snow". We have to take great care not to project our culture onto others.

I think in analyzing the notion of "snow" we can go to certain levels of analogies and I think in the culture of Eskimos you must go much deeper to discriminate between many more things.

Discussion of Lecture Two

One thing that I found a little bit difficult to understand myself was the fact that you seemed to be identifying partial understanding with the notion of cognitive obstacle. Whereas I agree that when a learner overcomes a cognitive obstacle there results a new understanding, I am not convinced that the converse is true or rather that this is the only way that the understanding occurs or that partial understanding is always the result of the existence of a cognitive obstacle. To me a cognitive obstacle is existing knowledge which interferes with the construction of new knowledge in this sense my interpretation of this is similar to Bachelard and to Guy Brousseau. I am not sure that we are using the word cognitive obstacle, or epistemological obstacle rather, in the same sense but I believed that partial understanding can be also the result of ignorance and not necessarily because of the existence of an epistemological obstacle. I think both things are quite possible.

I didn't talk about partial understanding and I don't think I have identified partial understanding with cognitive obstacle. I was trying to show that overcoming an obstacle and an act of understanding are complementary images of the unknown reality of an important qualitative change in a person's mind who switches from one way of knowing to another way of knowing. Perhaps at the end I was talking about depth of understanding, because suppose we take the understanding of a concept like *limit* and we have at least twenty or nineteen different acts of understanding to be experienced to understand a little bit of this notion and we can talk about depth of understanding in terms of the number and quality of these acts we have gone through. So of course we can understand something partially; I don't think I've identified some nineteen or twenty acts of understanding the notion of limit but I don't think these are all of them because theories develop; as the notion of limit is embedded in different theories we understand it deeper. We can see it from different points of view and as we see it from another point of view and see it applied to other situations we weren't aware of before, we understand it deeper, so we can talk about partial understanding of the notion, in terms of these conditions we gone have through.

One of the things that puzzles me a bit is that if we look at any content area in mathematics any specific topic any specific concept, we can always do the kind of analysis you have done in terms of what the concept entails, what might be the obstacles in that concept. What message does it give us as teachers in terms of what can we do with that knowledge? I mean there's one sense which says we can get overwhelmed by this because just one little thing "sequence" which is just one part of calculus has already all these things that come into understanding it.

You mean how to turn theory to practice? Well, not with the notion of *limit* but I made a similar analysis of the notion of *function* and I've been trying to apply this, to evaluate a series of text books from the point of view of the way in which the concept of *function* was constructed in them, and I was doing that with my student. We couldn't apply the analysis because I although knew what was behind those acts of understanding and the epistemological obstacles that I mentioned because I knew the history of the notion and I have some experience. But these were my words; they were not operational criteria for evaluating the text book. I gave her the list and she tried to evaluate from this point of view but she couldn't, her results were completely different from mine. So we had to reformulate these acts of understanding in terms of criteria to be applied to text books; for example, the kinds of problems that are used just before the definition. What is the kind of language used, whether the notion of variable is used, are there presentations used, whether the notion of function appears as a model of relationships between variable magnitudes, or whether the notion is introduced in terms of more of less formal definition and then examples illustrating this are given. We had to reformulate these things in terms of items that are included in the text book.

So it is not something that you can *directly* use in teaching, in evaluating depth of understanding, or the depth of understanding that can be provoked by the text book; this is just a basis that you can work upon. I didn't do that yet with the notion of *limit*; I'll have to because another student is working with that. Of course this is just the first thing I have done and I'm trying to apply it, and then perhaps forms of presenting those acts of understanding might change, perhaps. The other notion, of a function, didn't make me change my epistemological analysis of course; I just had to make things very clear so that anybody could understand in the same way to apply it to evaluate a text book.

By the way, in that analysis the book "School Mathematics Project"—two volumes for secondary school mathematics—got the highest score; because we put a high weight on modelling activities and there were very many of them.

I'm not sure if it is appropriate given that your talk addressed the first three questions of seven, but I'm curious about question seven which is "How we measure understanding"?

There is some research measuring understanding in mathematical texts, actually, made by Gagatsis and he is measuring the understanding of a mathematical text with a Cloze test, *teste clozu*, which consists of wiping away every fifth word, for example. They have done that and then they have reflected on what it gives and the relevance and validity of results of such tests and the results are not very enthusiastic, it doesn't give much. So there is research on measuring understanding a text, but I was thinking of measuring understanding by the number and quality of epistemological obstacles you have to overcome. But this is very hard because how can you possibly say that a person has overcome such an obstacle? There is a lot of work to be done to operationalise these criteria. Suppose there is an epistemological obstacle of a certain kind, what kind of question are you going to ask this student to know whether or not he's going to overcome this obstacle? For some it is quite easy but for some other more complicated epistemological obstacles I don't think it is easy. Some people are just aware of acts of sudden illumination and keep talking about it for a day or two and then they forget about it. Sometimes we may ask people to reflect on their own processes of knowing but even two minutes after it has happened you can see it in a different way and then you say perhaps it's not the truth. The real thing that happened, the introspection, is very difficult.

I found interesting yesterday one of the metaphors you used, and it was referred to in the first question, overcoming an epistemological obstacle as being a complementary way of looking at understanding and to help me get a better sense of how you view this interpretation of understanding, I wonder if you might take an example from early school algebra and illustrate a particular cognitive obstacle and how overcoming that is simply another way of looking at the understanding of it.

In early algebra, I think the notion of *variable* and the notion of *unknown*. When solving simple linear equations that are models of situations in word problems and then you switch to linear functions then you have to change your idea of what the x is. In the first it is the unknown and in the function it is the variable, name of an arbitrary object of a class, of a set.

Okay, then the student passes the unknown to you.

Yes, he or she applies it to the functions and then there is a problem of understanding the function, the graph and everything. He has to discriminate between those two.

Would you say then that the use of the letter is a cognitive obstacle?

Yes, x as an unknown, I think.

I would like to go back to the previous question about evaluating understanding. One problem many of us meet is to try to see what individuals, pupils for example, understand of certain things. We organize interviews and the problem is how to analyse the protocols we get in order to say this guy understands more or less, this and that, understands better than and so on. I know there are various methods for the analysing protocols; from your work would you have any favourite methods or would you have suggestions to make about how to analyze such data in order to see what people understand in the sense you mean?

Well I tried this once, it was terrible. The notion of iteration of functions; I followed very closely five persons for a couple of weeks and everything was recorded. Then I tried first of all to identify different conceptions of different parts of the structure of the notion that appeared. I took an informal definition of fixed point and then there were several conditions in the definition and I looked to see whether the conditions were satisfied or not and if it was and how it was understood, for example the notion of *fixed point* as the goal of the iteration. What was *fixed point* for the pupils? There are several conceptions that appeared in the students and I tried to score them, or order them. For some of them *fixed point* was the intersection of the graph with the line $x = y$, because it was in the context of drawing graphical representation of the iteration. But iteration for some of them was "there is a fixed point, an attractive fixed point, if it makes like *this*". This was one conception, just ostensive definition and some other verbalized it more in different terms so I tried to identify those different conceptions, all of them in a certain way, put them on a scale and then follow up the development of those conceptions in each student during those sessions.

I found pupils developing and using an elaborate conception at the development stage and then at the second session dropped because of the context that didn't demand so much of them. This gave me different paths of development of their conceptions. But I missed the data at some points, I was confused; they were saying things and I was not sure whether it is related to *this* conception or to *the other*, how to interpret that was very difficult because they are using very informal language and they are just showing things with their hands. I tried to identify what in the situation proposed to them provoked a positive jump in the conception or a negative jump in the conception. I tried to do all of that.

That's one method but as I said, it is a very difficult one. I don't know if it was my decision to attribute such-and-such speech event of a student to a definite conception, giving it such-and-such a score was really true or not, so I wouldn't propose that to a beginner. There must be, I think, a group of people who look at all the protocols and each does the same thing and then we compare and discuss. I was doing that in isolation and I spent a lot of time on that and it was very difficult. So I don't know, it was one method I tried. It gave some conclusions as to which context, which mathematical contexts, social contexts and which of my own interventions provoked more positive jumps in the conceptions and which did not and which provoked negative jumps and sometimes the conception was worse than before.

I am surprised that in your answer, I think you didn't use the word "understanding" but you used the word "conceptions" all the time. Yesterday too, you carefully defined "understanding" in your way but at some stage you used the word "conception" without definition as if it is quite clear what it means. Could you specify the relation between "conception", and "understanding" the way you defined it yesterday?

A conception is a result of an act of understanding, a conception is a way of knowing. I was trying to make, to define understanding of a concept by acts of understanding; a series of discriminations, identifications and so on have to be made but after that you have a new way of knowing and the conception is a way of knowing that comes after an act of understanding. I was doing the research I mentioned before I ever thought about understanding. I was just thinking in terms of conceptions and epistemological obstacles rather than that, and the epistemological obstacles again are a kind of conception, a way of seeing things.

I will ask a question about the formal approach given to sequence. In your very last utterance yesterday you said that you presented the notion of alternating quantifiers. Now it is clear that such a sequence of alternating quantifiers is a very difficult concept to grasp. So people working in non-standard analysis will say we can replace this sequence of alternating quantifiers by one quantifier. Now this will mean you will have to introduce the concept of infinite numbers and infinitesimals, but then they will say "Oh, when you're studying sequences like that you have to have an intuitive concept of real numbers and it is easy to present them into a concept of hyperreal numbers and to work with this concept. So my question would be if you have any experience or comment about what benefits that could bring in this context.

I have no experience, although children speak as though they were thinking of hyperreal numbers and not the Weierstrass continuum, I don't think they will easily understand the notion of hyperreal number if we introduce it as a series; it is difficult to understand the series in real numbers and if you put the negative indexes there... I have no experience so I can't really say; I wouldn't dare to bring the hyperreal numbers to my students.

You made an analogy with hermeneutics which I found interesting; but it seems to me that that specifically looks at the interpretation and understanding of written text. How does this actually connect with the construction of conceptual schemes in mathematics? I think that we are dealing with two very different kinds of fields. In the analysis of the understanding of text there is a presumption of knowledge, in the construction of new knowledge there is a presumption of a basis on which to build but not of the new knowledge that is going to be obtained. So I think that we are dealing with two different things here.

Yes, of course, I think I mentioned that. I tried to generalize Ricoeur's method of understanding text to understanding mathematical concepts and I ended with the Lakatosian model of development of knowledge. Mathematics is learned in many different contexts, it is not just by reading. I don't think you can really understand mathematics just by reading a text book, you have to talk about it. The formulas are written and you must hear somebody reading them. So you can't really rely just on the text and you learn it in conversation and discussion, dialogues, solving problems. However, I think the model of guess-validation-refinement could be generalized to understanding. When we are in a classroom we make a guess what the teacher is talking about, don't we? First of all it's a guess and then we hear it once again and so on, make a better guess. The first thing we do is to try to understand in a certain way and then we refine it, we try to justify the guess in listening to the next bit of discourse. So I think guess-validation-refinement might be applied to understanding mathematics.

You referred to the notion of "fixed point" and the fact that kids have some different pictures of what this is all about, but isn't that one aspect of understanding? Because very frequently we have a variety of pictures that we look at, at different times depending on what it is we want to see. But what we also learn in order to say we have an understanding is the limitations of these pictures. You know at certain points you can apply this picture at a certain stage and then you shift pictures because this picture is inappropriate at that point and I think that perhaps one of the aspects of developing an understanding is to recommend the limits of these pictures, of how these seem be put together.

Certainly, becoming aware that our conception is just one possible point of view but to become aware of that we must have seen another point of view already. It cannot be inside because inside you don't see anything; you must already have some rival theory to become aware of the limitations of the other.

I just wanted to throw something out in connection with something that is written in the hand out where you refer to a young child; "new concepts must be built up in empty spaces" and then the sentence where I have something perhaps to contribute: "The initial stage of construction of the data bank is necessarily linked with huge amounts of information memorizing mechanically." The only reason this struck me is shortly before I left I picked up the newest issue of American Scientists where there is an article discussing some experimental results with very young children. The suggestion is that Piagetian sensori-motor phase actually is much more short-lived than the current writing suggest; that infants as young as two or three months are already constructing meaning, I guess I was unhappy with this.

It was just a quotation of what psychologists using the metaphor of computer science say about understanding, just to show different theories. The quote is from a book by Lindsay and Norman *Human Information Processing*. That's the metaphor for computer science and there's the question "What's the difference between knowledge and information?" In this theory memory is represented as a data bank, it's not just pieces of information that are retrieved, it's very much inter-connected. A data bank which is structured is perhaps knowledge...

I think this is interesting question because in reading the paper I wasn't quite sure how much weight you were attaching to information processing theory in the construction of knowledge. I had difficulties in trying to see how the notion of an epistemological obstacle would even fit in a theory of information processing. To me it seems to me that you are very much in a Piagetian context of learning when you are talking about the epistemological obstacles because clearly you don't talk about the epistemological obstacles in the acquisition of knowledge at the assimilation level you are talking about the epistemological obstacles in terms of the accommodation province, by which you mean changes in cognitive structures. And therefore, it seemed to me that the references you were making to information processing were not connecting here.

Well, perhaps not but I just wanted to mention them. However, in Lindsay, they say,

> Even if the received information is in obvious contradiction with the previous experience, his or her conceptual structure which constructed such a complicated system of interrelations, stands against any revision.

You see, we have made such a big effort to construct our data bank then some new information comes contradicts something inside us. So we will show resist as much as we can against it so we shall even change the meaning of the new information to fit into our structure because it has cost us so much. I think there is a place for epistemological obstacles even in that theory.

I think Peter Weston has done some studies on hypothesis testing and he is finding that quite frequently people will work out a hypothesis and find contradictory data but still not reject the hypothesis and come back and say hypotheses are dead even after it has been proven wrong in other simple situations.

Did he in fact not show that people looked for those things to support their hypothesis, did not use the scientific approach of looking for those things which would reject the hypothesis. They continuously looked for similar situations to reinforce the hypothesis so that you have a thousand for and one against.

I want to go back to the formal testing. Very often when you teach limits to students the formal approach is an obstacle in itself. How do you see it enter in your concept? Does it come afterwards as the conclusion or here we can express this concept in a formal way that way: is it a conclusion of the whole process or is it a starting point?

No, at a certain moment probably you'll have to understand the formal definition and there is a series of acts of understanding this formal definition and obstacles to be overcome. I didn't speak about that yesterday but it is there, there are quite a lot of things there. These are logical obstacles of course but although I was speaking of a partial order in the acts of understanding these are certainly at the end somewhere and not at the beginning of course.

Working Group A

Chaos and Fractal Geometry for High School Students

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Participants

Rick Blake	William Higginson	Dave Lidstone
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Gila Hanna	Ron Lewis	Rina Zazkis
Peter Harrison		

To most people, fractals are Mandelbrot Sets, those beautiful, complex, infinitely detailed images rendered by computers. And they are. But they are much more than that. Fractals are the shapes of nature, from coastlines, to trees, to galactic clusters, to cheese. Not long ago, if you were known to be teaching fractals you would be asked "What are they anyway?" But awareness of the existence of fractals has grown quickly and now the question has changed to, "Are they really useful for anything?", with a tone suggesting that probably they are not. Ironically, while the art of fractal images can be debated, their value as a tool for analyzing complex systems is now well appreciated by applied scientists in many fields.

Since 1987, Dr. Kaye and I have been presenting lectures, experiments and other activities as enrichment for Sudbury high school students. In the very first session students are engaged in determining perimeter estimates of a lake by doing "compass walks" on a map. As successive estimates are improved with shorter compass steps, we watch them grow to "infinity", or, on log-log graph paper, along a straight line. The slope of this line defines the fractal dimension of the lake. Before long, students are engaged in projects of their own interest where this concrete experience provides not only the tools, but an appreciation of the concept of fractal dimension as it relates to ruggedness or complexity, and some exposure to the concept of self-similarity.

In the first session of the C.M.E.S.G. workshop, participants were given examples of systems where fractals are important in taming complexity. Then we engaged in analyzing the northwest coastline of Vancouver Island using the compass walk technique and pooling data to produce a log-log graph. With appropriate tact, it was pointed out that students although fascinated by the unusual graph paper, are much less intimidated by it than teachers, and so are usually only allotted half as much time for such an exercise. Students have commented that it is concrete experiences such as this that allow them to immediately see the value of fractals, and motivate them to participate in the program.

Next, a well-known fractal, called the Koch Snowflake curve, or Triadic Island, is generated from an equilateral triangle. As the triangle is converted into the fractal island by a recursive procedure, it is demonstrated that while its perimeter is increasing without bound, its area is finite, just like Vancouver Island. Because the Koch island is well-defined, mathematically, not only can its perimeters and areas be presented in terms of geometric series, but its recursive construction can be used to introduce Mandelbrot's formula for fractal dimension. It is found that this island's dimension is close to that of Vancouver Island and to the average value for islands of the world. Thus, the boundary of the Koch Island is a geometrically perfect model for coastlines and other rugged shapes.

While the most obvious difference between this symmetrical and regular figure and the shapes of natural rugged boundaries is the latter's lack of symmetry and regularity they are still statistically similar. This can be understood in terms of having the same data line when estimates of perimeter versus scale of measurement are plotted on log-log graph paper. Their apparent difference is explained in terms of the random arrangements of elements of the rugged natural systems.

Students seem to quickly grasp these concepts, which perhaps are non-intuitive to a teacher familiar with the geometry of Euclid. The hands-on approach to the introduction of fractal geometry has students "doing" fractals before they can verbally define a fractal or demonstrate any appreciation of the formal mathematical foundations.

Before lunch, this working group sorted Smarties (a commercial candy) by colour, and recorded the frequency of each colour. A plot of the organized data on Poisson graph paper produced a straight line, an example of order in random chaos. only if the candies are thoroughly mixed will the data points fit such a line. By way of observation it seemed that soon after the experiment, the frequencies of the candies approached zero, with the few exceptions (people on diets) constituting a further example of Poisson occurrences.

The second session began with a video presentation of an exploration of the Mandelbrot Set, and a short informal discussion of popular books and articles available on fractals. Next, Dr. Kaye presented his essay called "Harmonious Rocks and Infinite Coastlines" regarding advances in methods of characterizing fine particles. Beginning with gross measures such as aspect ratio we have progressed to the sophistication of Fourier analysis of rugged boundary profiles which had been converted into waveforms. The most rugged systems though, such as carbon black profiles, are so complex that only fractal geometry provides satisfactory analysis. However, unlike the Fourier method, the fractal characterization can be performed by students with only compasses and graph paper. This manual method leads the students painlessly to an appreciation of automated image analysis in which a digitized image is characterized by successive perimeter estimates. As well, another technique, mosaic amalgamation, applied to profiles with fuzzy boundaries, can also be performed manually to bring about an understanding of the pixel dilation performed by computerized image analyzers.

The topic then was changed to Diffusion-Limited-Aggregation (DLA) a growth mechanism involving random walks. of the six prize winning science fair projects which our small band of students has produced in six semesters on topics of fractal geometry, two have involved producing fractal trees by electrolysis. This is easily done in a petri dish, and can be demonstrated to a class by placing the entire apparatus on an overhead projector. If the resulting tree is photographed or photocopied, analysis of the image can yield a fractal dimension. Again, the data is obtained with the use of a compass and plotted on log-log graph paper. This time the compass is used to draw concentric circles centred on the origin of the tree. The number of intersections of circle and tree are recorded at each radius and the cumulative frequency versus radius yields a straight line on log-log graph paper. The fractal dimension is deduced from the slope of the data line.

Then it was demonstrated that a Hele-Shaw cell constructed from a petri dish could be used to create viscous fingering patterns resembling DLA when a low-viscosity fluid, such as ink, is injected into a higher viscosity fluid, such as white glue. Some of the many important applications of both electrolytic deposition and viscous fingering were discussed. Finally, a simulation of DLA was demonstrated on the computer. Not only can a printout of computer output be used by students to exercise the techniques for finding fractal dimensions, but students watching the image form, pixel by pixel, can easily visualize the basic notions of random walk, sticking probability, screening, filtering, and density, and possibly even fractal dimension.

In the third day of the workshop we began to hint at a larger framework by which students would come to know fractals in the context of random processes and their links to chaos. Due to the large amount of new material which we hoped to introduce to participants somewhat unfamiliar with most of it, we decided, with their permission, to adopt an expository style at the expense of reducing the number of experiments to only two. First we outlined how 100 computer generated random walks of 100 steps could create enough data to investigate the proposition that, on average, such a random walk will result in a displacement of 10 steps from the origin. The theoretical proof of this involves algebraic expressions and manipulation at only the grade ten level. Next the game of snakes and ladders was discussed. The game, involving several interacting factors to produce a win, has a log-normal distribution with respect to game length. Thousands of games can be played quickly on a computer. The log-normal distribution describes self-avoiding random walks, levy flights, and tracks between occurrences of a given digit in a random number table, as well as the distribution of incomes in a population. With increasing complexity, a log-normal distribution tends to a log-log distribution.

On now to Buffon's Needle, one of the most amazing patterns in chaos, producing an estimate of the value of pi with a process not unrelated to the "chaos game". The chaos game itself was also posed, and later answered by allowing the computer to perform its moves hundreds of times while the audience watched the computer screen and speculated on the ultimate pattern or lack of pattern, which turned out to be a fractal called the Sierpinski Gasket, another basic shape of the new geometry. This led to a discussion of strange attractors and to more computer graphics produced by inexpensive public domain software and shareware.

In the final experiment participants faced off in a simple game of coin toss in which one player earns a point at the other's expense with each toss. The game demonstrates that contrary to common intuition, the lead in a game does not change hands very often, nor does the game usually end in a draw. If a graph of the score in the game is plotted versus time, the horizontal spaces between zeroes have a distribution which is log-log. Thus the concept of intermittency or fractal time can be introduced. Fortunately, a geometric model exists for the pattern of occurrence of these zeroes. The subdivision of a Cantor bar into a Cantor dust defines a fractal set with dimension between 0 and 1. The Cantor set (dust) may be the fundamental fractal to be used to analyze complex strange attractors. So even in the toss of a coin, the very essence of random chaos is tamed. In the end, deterministic chaos is also found to have an underlying fractal structure. And so our grand synthesis brings us full circle (if Euclid will be allowed a word), from fractals, to random chaos and deterministic chaos, and back to fractals.

The Cantor dust has a counterpart in two-dimensions, called the Sierpinski Carpet which corresponds in three-space to the Menger Sponge. The Sierpinski carpet has no area, but is composed of an infinite number of threads. These shapes can all serve as mathematical models for physical systems. For example, the Sierpinski carpet resembles a slice through porous rock in an oilfield, or ore-bearing rock, a Swiss cheese, bacterial colonies in a petri dish, pigment in a paint, pores in artificial bone, a fibrous filter, or a slice of bread. The randomized holes in real physical systems can be explored with a transparent overlay of random search lines, whose intersection with the holes define chords, whose distribution determines a fractal dimension characteristic of the system.

It is the tremendous fascination with fractals and chaos among applied scientists in many fields which had led to a rapidly growing interest in the subject within and without the field of education. What began in 1987 as a pilot project with a handful of gifted and talented high school students in Sudbury has also had other beginnings and is growing and spreading on many fronts. It seems inevitable that mainstream high school students will soon have access to credit courses on fractals and chaos by popular demand, a thought barely fathomable by mathematics teachers whose courses are often compulsory and so sometimes resented by students as a necessary evil. Because of the availability of computers and graphical methods for exploring fractal geometry and chaos, students can now engage in real problems or realistic simulations, Instead of just solvable problems.

It now, however, in retrospect, seems easier to understand the appeal of this kind of mathematics. Aside from the hands-on aspect already mentioned, students also experience some exotic mathematics, not only in the likes of the Mandelbrot Sets, but in encounters with random walks, transcendental pi and e, the square root of minus one, and even infinity, along with fractional dimensions, chaos, strange attractors and "cascades of period-doubling bifurcations". Indeed these are exciting times in mathematics education.

In concluding, I must say that our working group expressed enthusiasm for the course content and for our approach to presenting these topics to high school students, and offered us well-appreciated encouragement for the further development of the programme. Since the C.M.E.S.G conference, the Ontario Ministry of Education has approved the granting of one full high school credit for our course, designated as MOS 4A, Fractals and Chaos, at the grade 12, advanced level, until June 1993.

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Working Group B

New Mathematics Curriculum Visions for Canada

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New Mathematics Curriculum Visions for Canada

A Report of the Standards Working Group

Preface What follows is an outgrowth of the three morning of meetings of working group B. In the group we attempted to familiarize ourselves with the Curriculum and Evaluation Standard for School Mathematics (1989) of the NCTM; to make some value statements about the Standards; and to try to see how they relate to the Canadian scene.

Of course we were limited in the scope of our deliberations, but benefitted from having a whole cross-section of CMESG/GCEDM[1]: mathematicians; university mathematics educators; school mathematics supervisors and practising classroom teachers. In addition, our group had a member of the authoring team of the NCTM Standards and a current member of the authoring team of the NCTM Teaching Standards group.

Our group also found itself very much informed by and continually referencing the opening plenary remarks of d'Ambrosio. There are several explicit and numerous implicit references to his work in what follows.

The Situation of the Mathematics Curriculum

The vision of the mathematics curriculum is structured so that the child or the learner at any level is at its centre. This, of course, is what d'Ambrosio called for in his remarks. It is also congruent with the view that no matter what happens in the learning environment, it is the learner who must make the mathematical sense of it.

Figure 1 below points to the resources in the environment available to the learner. In particular we have highlighted the key roles of the teacher as part of the environment which gives the student possibilities for building up mathematical knowledge. Reflecting the discussions which have taken place in many sessions at CMESG/GCEDM over many years, text materials are considered an element of the learning environment, but not the central one which it is sometimes seen to play.

Putting the children or learners at the centre, we envision them having a variety of experiences out of which they make mathematical sense at a particular time. While it is not identified as one of the resources, we think time is a highly significant in mathematics learning and ways must be sought to maximize the availability of time to the student. It should be noted that this idea takes into account formal scheduled time for mathematics and informal time both in and out of school.

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1 It should be noted that our working group did not have any female members, which was regrettable given our final deliberations.

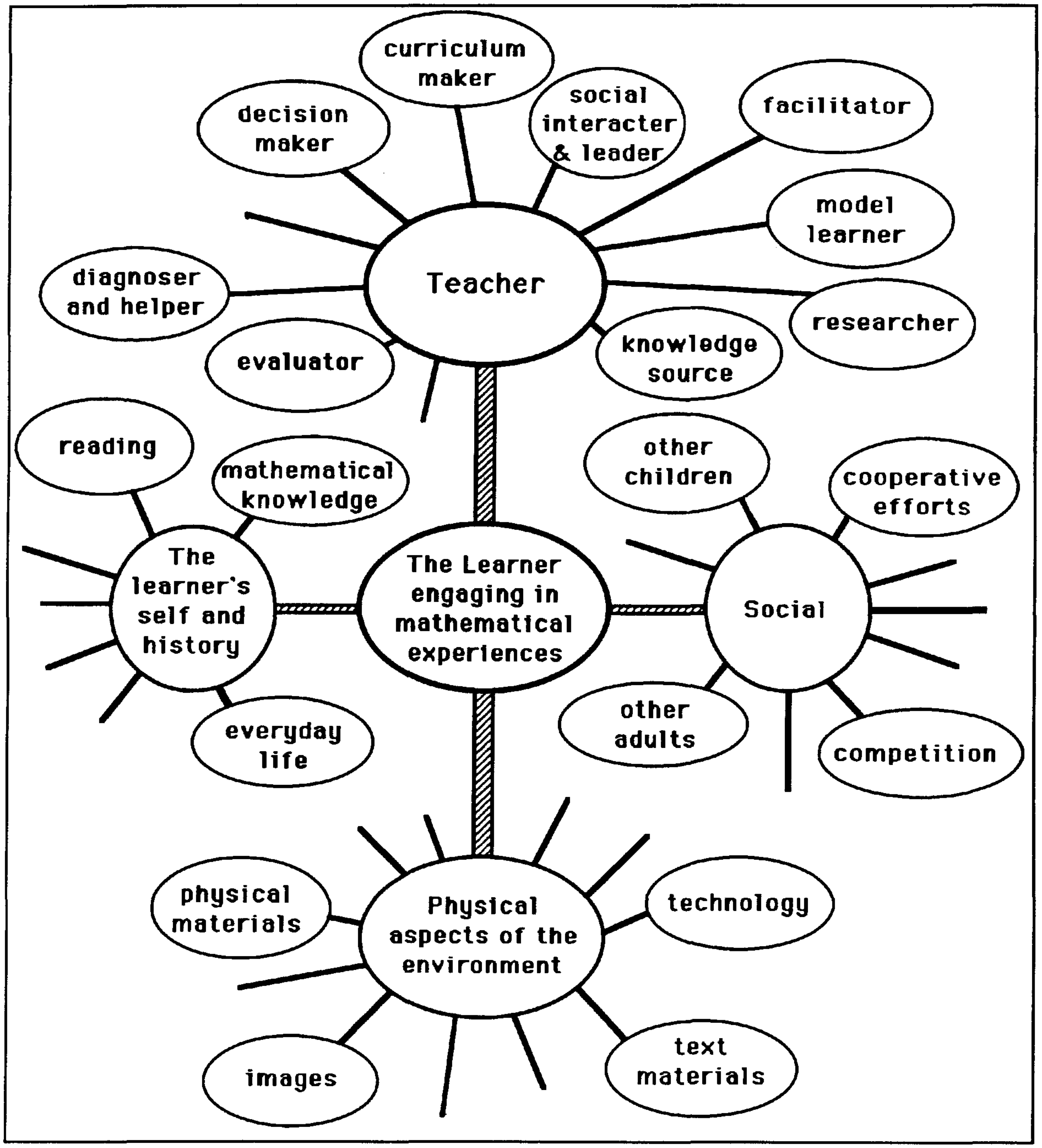

Figure 1 The resources for a learner in a learner-centred situation.

The Activities in a Mathematics Curriculum

The diagram above gives a very general picture of the setting for the curriculum. It could pertain to any subject area and any age level of learner. It is meant to indicate to plethora of possibilities for a mathematics learning environment and the many sided role of the teacher as a key part of that social/curricular environment. Figure 2 below in a sense goes "inside" the central oval in Figure 1.

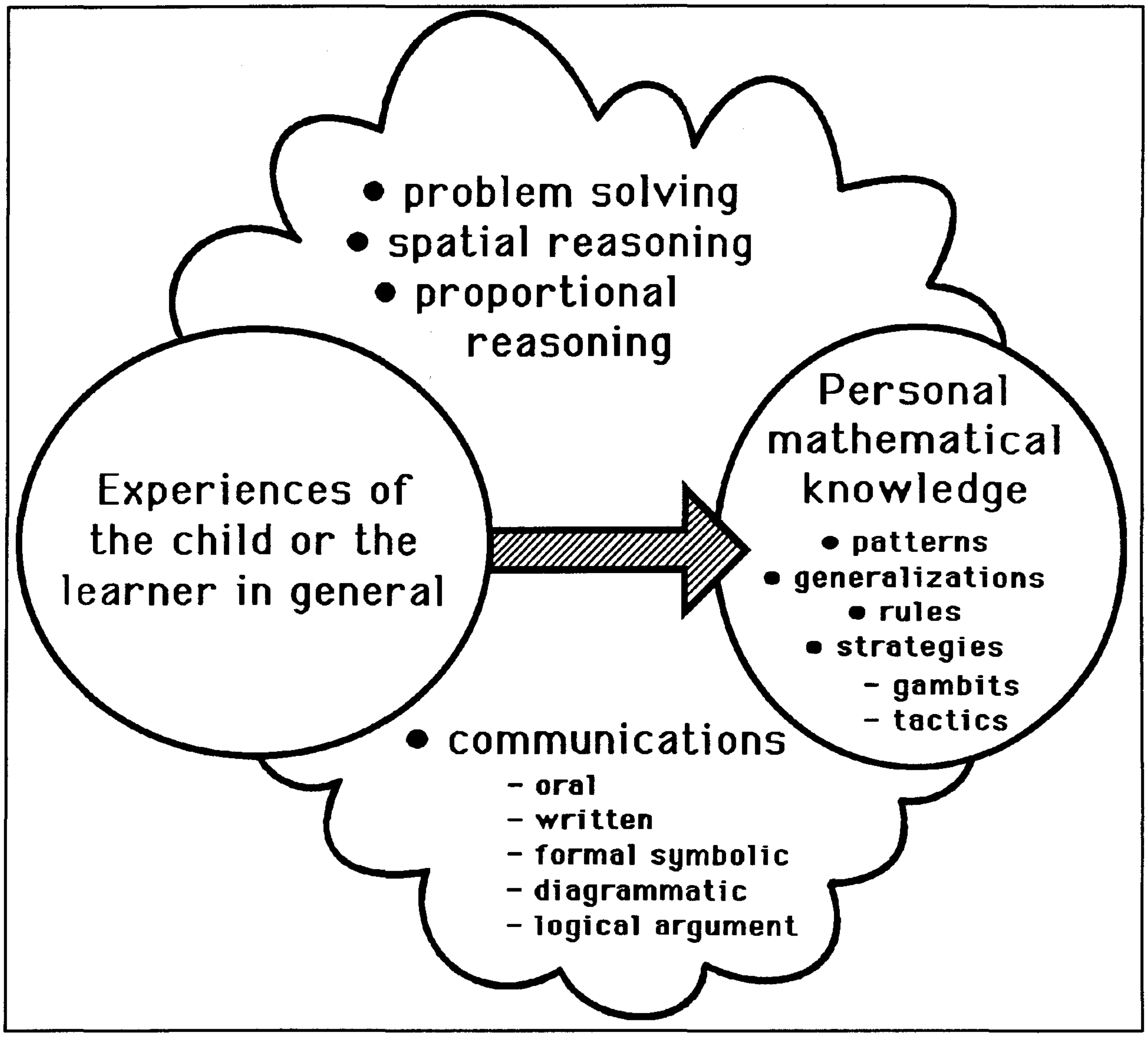

Figure 2 From experience to knowing

The critical feature of this diagram is that personal mathematical knowledge is built from the experiences of the children or learners. Of course, the knowledge is not reducible to the experiences themselves, but comes from the conceptions or ideas which are built from or reflect on the experiences. The "cloud" which forms the link between experience and knowledge contains mathematical thinking and communicating activities which allow the learner to generate, re-present, reflect on, validate, and interact about mathematical knowledge. Just as the curricular setting for mathematics has much in common with that for other kinds of knowledge, so the mechanisms for generating mathematical knowledge have much in common with other kinds of knowing activities. After all the learner is a whole person and not a compartmentalized knowledge box. Still the knowing actions-problem solving, logical reasoning, spatial reasoning... have particular mathematical aspects or interpretations. Any resourceful environment should seek to provoke or provide experiences which stimulate such mathematical processes.

The Mathematics of the Curriculum

The NCTM Standards provide more than 30 different curriculum standards across the elementary and secondary school curriculums. We choose instead to see the mathematical vision of the curriculum as the building up of various mathematical senses. For example, the environment, the experiences and processes in the school noted above, might promote:

- numerical sense; under this rubric one might find such goals as:
  - multiple interpretations of number
  - mental computation
  - estimation
  - use of computational devices
  - sense of large and small numbers
  - sense of direction of numbers

The other mathematical senses which serve as goals, senses of the curriculum might be:

- geometric sense
- measurement sense
- information and data sense (including randomness and probability)
- algebraic sense (including symbolic sense, variable sense and structured sense)
- function or analytic sense.

Of course activities and processes of the mathematics curriculum should be such that these senses are not thought of in isolation. And, in fact, the inter-connection among these developing senses are as critical as the senses themselves. Thus the mathematical vision of the curriculum - for example for the earlier years in school (say K-4) might be seen in Figure 3 (opposite).

The vision for other years of levels might show different emphases but the activities would be aimed at the development of this variety of interconnected mathematical senses.

Concluding remarks

The three sections above represent three completely interdependent aspects of the vision of mathematics for the learner in Canada today. The first section emphasized the social situation and the resources of the curriculum. The second section emphasizes the "mathemae" of mathematics. While the final section considers the actual techniques and senses local to mathematical itself. This vision of the curriculum implies that the processes of mathematics are being used continually to help the learner build inter-connected mathematical sensibilities. Such processes arise out of the experiences of the learner and these experiences occur in the multifaceted environment of a rich social situation in which the teacher has many roles.

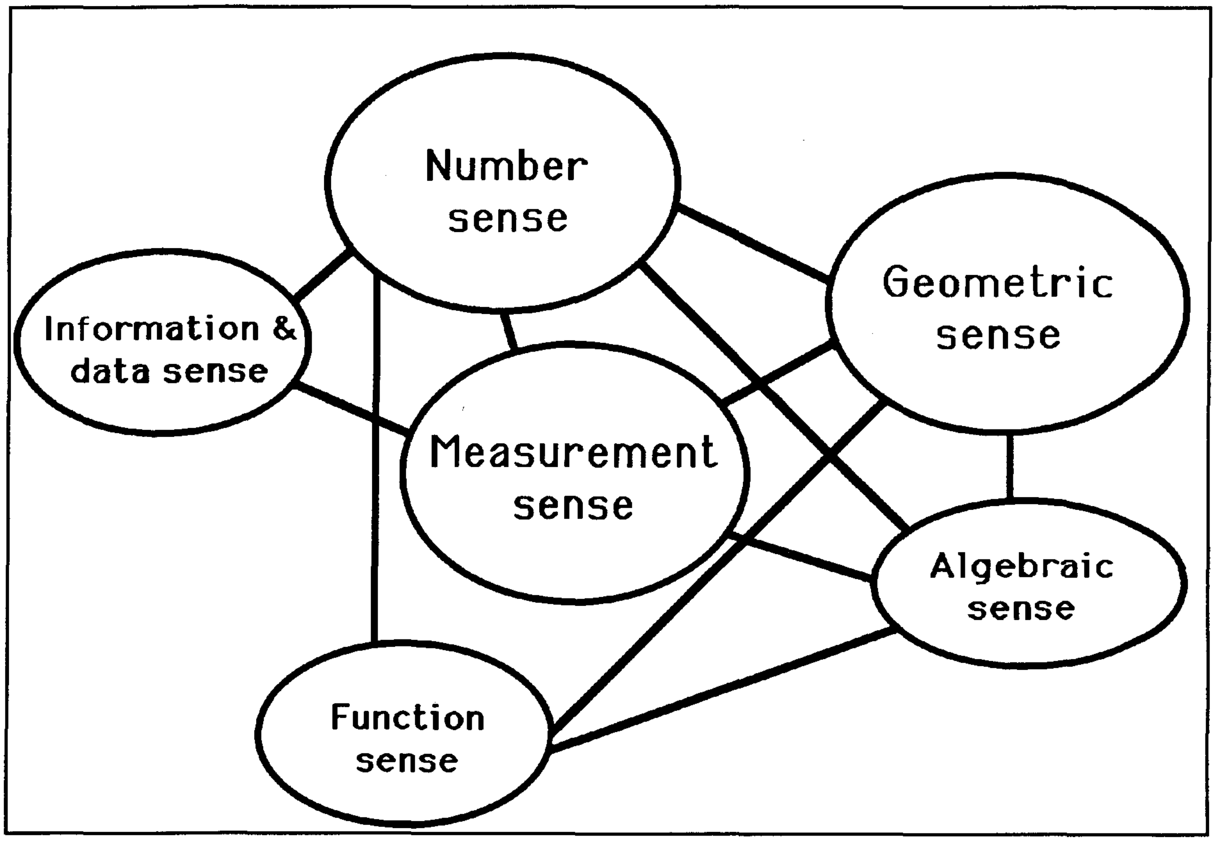

Figure 3 The Senses of the Mathematics Curriculum

Working Group C

Explanatory Models of Children's Mathematics

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A cognitive matrix describing the understanding of early multiplication[1]

A major problem faced by mathematics educators is the teachers' perception of their discipline. On one hand, many teachers do not have a global overview of the topic they teach and, on the other hand, they tend to emphasize the algorithmic aspects of mathematics with little attention paid to the concepts underlying these algorithms. In order to retain the organic character of the fundamental ideas developed in mathematics, we have introduced the notion of a **conceptual scheme**. This was to distinguish it from the notions described in terms of examples and non-examples found in classical concept formation theory. We defined a conceptual scheme as a **network of related knowledge together with all the problem situations in which it can be used** (Bergeron & Herscovics,1990).

In order to provide a frame of reference that might be used as a background to describe the students' construction of conceptual schemes, we developed various **models of understanding**. The model we presented at last year's meeting is particularly useful in guiding us through a conceptual analysis of the fundamental ideas that can be related to the student's physical experience.

Last year we showed how this model enabled us to analyze the notion of early number (Bergeron & Herscovics, 1989) and how we used it in an international study assessing the numerical profile of kindergartners in three different countries (Herscovics & Bergeron, 1989). In order to contrast our approach with the ones found in most textbooks aimed at teachers we will now apply our model to achieve a conceptual analysis of early multiplication. Since two aspects of understanding (procedural understanding and abstraction of pre-multiplication) have been explored in two pilot studies, we are also in a position to describe the tasks and questions we have used in case studies of several young children.

Since our model is relatively complex, it is worth presenting a short description of it. Two specific tiers, the first one pertaining to the understanding of the physical pre-concepts, and a second tier involving the mathematization of these pre-concepts are proposed. Very briefly, we recall the criteria used in identifying each aspect of understanding:

The Understanding of Preliminary Physical Concepts

Intuitive understanding refers to a global perception of the notion at hand; it results from a type of thinking based essentially on visual perception; it provides rough non-numerical approximations.

Logico-physical procedural understanding refers to the acquisition of logico-physical procedures which the learners can relate to their intuitive knowledge and use appropriately.

[1] Research funded by the Quebec Ministry of Education (FCAR, EQ2923)

Logico-physical abstraction refers to the construction of logico-physical invariants (as in the case of the various conservations of plurality and position), or the reversibility and composition of logico-physical transformations (e.g. taking away is viewed as the inverse of adding to; a sequence of increments can be reduced to fewer steps through composition), or as generalization (e.g. perceiving the commutativity of the physical union of any two sets).

The Understanding of the Emerging Mathematical Concepts

Logico-mathematical procedural understanding refers to the acquisition of explicit logico-mathematical procedures which the learner can relate to the underlying preliminary physical concepts and use appropriately.

Logico-mathematical abstraction refers to the construction of logico-mathematical invariants together with the relevant logico-physical invariants (as in the abstraction of cardinal number and ordinal number), or the reversibility and composition of logico-mathematical transformations and operations (e.g. subtraction viewed as the inverse of addition; strings of additions reduced to fewer operations through composition), or as generalization (e.g. commutativity of addition perceived as a property applying to all pairs of natural numbers).

Formalization refers to its usual interpretations, that of axiomatization and formal mathematical proof which, at the elementary level, could be viewed as discovering axioms and finding logical mathematical justifications respectively. But two additional meanings are assigned to formalization, that of enclosing a mathematical notion into a formal definition, and that of using mathematical symbolization for notions for which prior procedural understanding or abstraction already exist to some degree.

The non-linearity of our model is expressed by the various arrows in Figure 1.

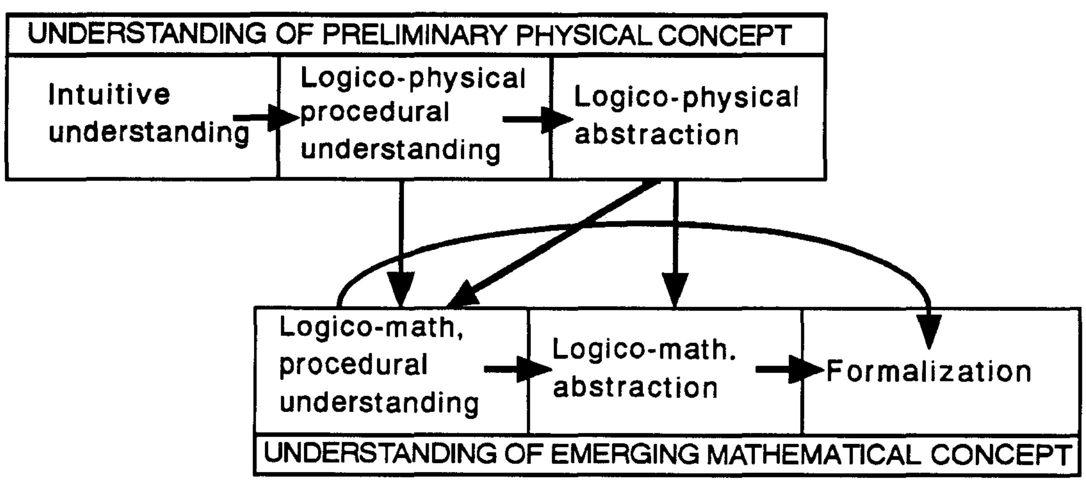


Figure 1 The two-tiered model of understanding

What is multiplication?

Various textbooks use different paradigms to introduce multiplication, usually in the third grade. Some books introduce it in terms of jumps on the number line while others use a combinatorial approach ("How many outfits can be made from 3 different coloured skirts and 4 different coloured blouses?") This variety of presentations shows how widely multiplication can be used. However, they do not necessarily correspond to the easiest initial construction of the operation. Jumps on the number line require a well developed conceptualization of the notion of measure which is far from completed by the majority of third graders (Héraud, 1989), while as reported by Suydam and Weaver (1970), Hervey (1966) has found that the Cartesian product approach was more difficult for second graders than the equal addend paradigm. Thus, it is this latter one that was adopted in our work.

Since the first tier of our model deals with the understanding of the physical pre-concepts we must look for an operation in the physical realm that would correspond to the operation that will constitute arithmetic multiplication, that is the multiplication of numbers. Curiously, when it comes to the other three arithmetic operations, we can easily identify physical operations with each one of them: **addition** refers to the quantification of either a set that is **augmented** by other elements, or the **union** of two sets; **subtraction** usually refers to the quantification of the set after some of its elements have been **taken away**; **division** refers to the quantification of subsets resulting from the **complete partitioning** of an initial set, focusing either on the number of subsets or the number of elements in each subset. However, when it comes to multiplication, most of us tend to define it as "repeated addition" which provides a procedural answer totally defined by a prior operation. Does this mean that there is no fundamental action that may correspond to multiplication? This seems unlikely since division, the inverse of multiplication, does and is linked to equipartitioning.

Piaget and Szeminska (1941/1967) are the ones who first related multiplication to a physical operation when they described it as the iteration of a one-to-one correspondence between several sets: "From a psychological point of view, this simply means that setting up a **one-to-one correspondence** is an implicit multiplication: hence, such a correspondence established between several collections, and not only between two of them, will sooner or later lead the subject to become aware of this multiplication and establish it as an explicit operation" (Piaget and Szeminska, 1967, p.262). In an earlier paper Herscovics, Bergeron and Kieran (1983) have shown that similar results can be obtained by the iteration of a **one-to-many correspondence**. For instance, when young children are asked to make four piles of three cards, they are more likely to achieve this through the iteration of a one-to-many correspondence than through the iteration of a one-to-one correspondence. However, both procedures are possible and hence must be accepted as actions corresponding to the generation of a multiplicative situation. This may look very much like a problem of equipartition. However, it differs from it in the fact that **it need not exhaust**, or attempt to exhaust, the initial set. The children may have a whole deck of 52 cards in hand when they are asked to make four piles of three cards.

That quite early children can generate various multiplicative situations is not too surprising. But can we claim that by iterating a one-to-one or a one-to-many correspondence they are actually aware of the situation as being multiplicative? Of course, this is not necessarily the case. This claim can only be made if there is some evidence that they iterate such correspondences with the conscious intention of making a desired whole. We thus claim that **a situation is being perceived as multiplicative when the whole is viewed as resulting from the repeated iteration of a one-to-one or a one-to-many correspondence** (Nantais and Herscovics, 1989). It should be noted that no quantification of the whole set is involved here. Nevertheless, the perception of a situation as being multiplicative requires an awareness of the number involved in the initial correspondence, the number of iterations, and the resulting fact that all the subsets will necessarily be equipotent.

Using our working definition of the pre-concept of arithmetic multiplication, we have identified criteria related to the understanding involved at the first tier. Some of these criteria have been used to develop a set of tasks and related questions to gather evidence of the different levels of understanding proposed in the first tier.

The Understanding of Preliminary Physical Concepts

Intuitive understanding

A first criterion of intuitive understanding might be the ability to perceive visually the difference between a situation that is multiplicative and a situation that is not. For instance, a set consisting of several equal subsets might be compared to a set consisting of unequal subsets.

Since rectangular arrays are so useful in illustrating multiplicative situations, *a second criterion* might verify if the children will spontaneously perceive the rows, as well as the columns, as equal subsets.

A third criterion might involve the visual comparison of two multiplicative situations in which one of the "factors" is different. For instance, without knowing the total number of objects present, one could compare 4 sets of 5 chips with 4 sets of 6 chips or 4 sets of 5 chips with 3 sets of 5 chips and decide where there are more.

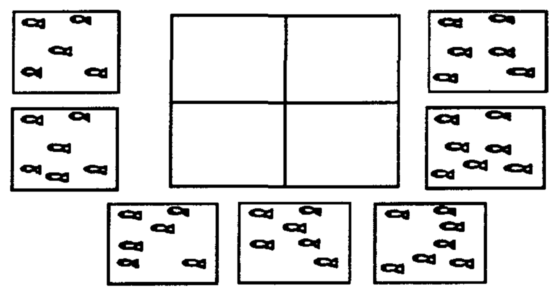
A fourth criterion might involve the comparison of equipotent sets involving various configurations of 9 subsets of 7 objects each. The total number would be large enough to discourage enumeration, but it may bring out the child's awareness of the fact that if the number of subsets and the number of elements in each subset are the same, the whole sets must have the same cardinality. Since no transformation is performed on the sets presented to the pupils, one cannot here infer about any apprehension of the invariance of a set with respect to a spatial transformation. However, such a task might be used at the level of logico-physical abstraction.

Procedural understanding of a logico-physical nature

For the second level of understanding of pre-multiplication, we have identified some criteria and translated these into different tasks and questions. Results of an exploratory study involving 9 case studies have been reported (Beattys, Herscovics, and Nantais, 1990). These children, three from each grade (K-2) attended an urban school in New Brunswick, N.J. In order to illustrate the type of tasks and questions that can be attempted even with younger schoolchildren, we provide here a detailed account of this investigation.

As a *first criterion* for procedural understanding, we used **the ability to construct a whole based on the iteration of a 1:n correspondence**. Two tasks were designed to verify this criterion. The first one consisted in presenting the child with a cardboard and telling him/her that it was to represent a pet shop. The cardboard was divided into four equal parts that were to represent shelves in the pet shop. Along with the cardboard, the interviewer presented a set of 7 aquarium tanks, 4 containing 6 fish, 1 tank containing 5 fish, and 2 containing 7 fish. The children were asked:

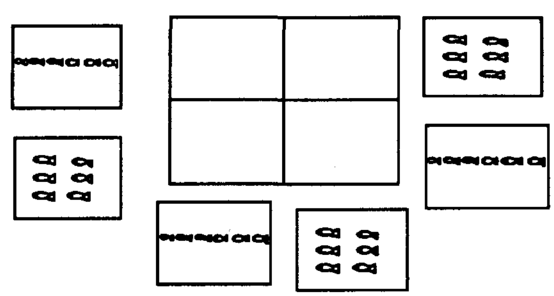
Here is a pet shop. Look at the shelves. We have to fill the shelves with aquariums. But each aquarium must have six fish in it. Can you fill the shelves using some of these aquariums?



The nine children in the three grades responded uniformly. Each one counted the fish in each tank and selected the appropriate ones.

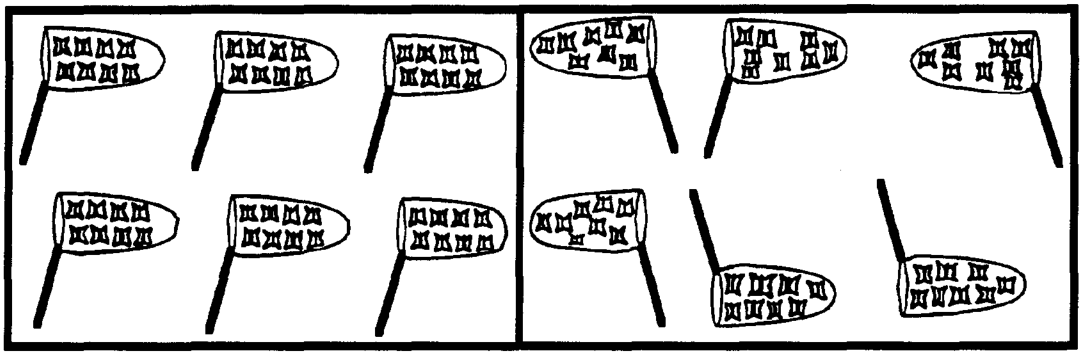
The second task associated with this criterion verified if children would be influenced by different configurations. They were told:

Here are some shelves. Again, we have to fill these with aquariums. But each aquarium must have six fish in it. Can you fill the shelves using some of these aquariums?



As expected, the differences in configuration affected some of the younger children. Two of the three kindergartners selected only the arrays and stated "that's all that are sixes" and "the rest are not the same". The 6 first and second graders all succeeded. But configuration also played a minor role as evidenced by the fact that five subjects chose the arrays first, surely based on the fact that they could be subitized.

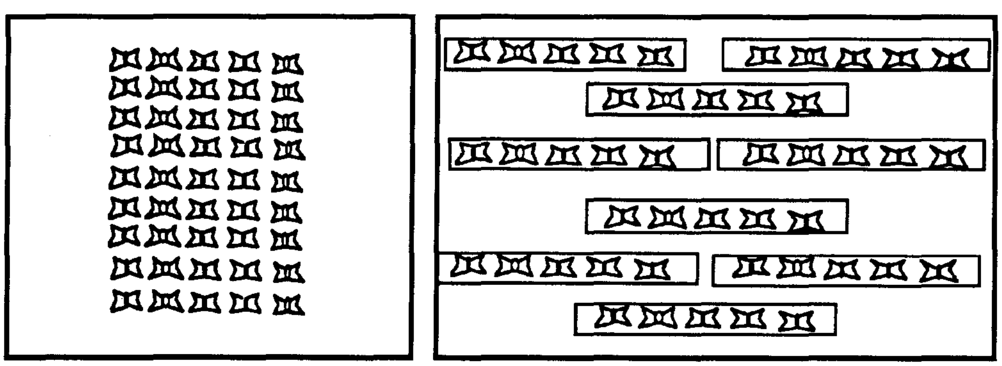
The *second criterion* for procedural understanding was **the ability to perceive the whole in terms of its factors**. The first task verifying this criterion involved two sets of 48 butterflies arranged in 6 nets containing 8 butterflies each. Numbers were chosen large enough to discourage the children from counting the sets One set had the butterflies neatly arranged in the nets in two rows of four, whereas in the other set, the butterflies were randomly spread out in the nets:



Children were told: **Here are some butterflies for you and here are some butterflies for me. Do we have the same number of butterflies?**

Children responded quite differently according to age. None of the kindergartners succeeded in solving this problem. Two of them thought that there were more butterflies in the nets where they were arranged in arrays; one child simply tried a guess based exclusively on an equal number of nets. Among the first graders, one child simply counted all; the second one first counted the number of groups and then counted all; a third child counted the number of butterflies in each net but ignored the number of nets. It is only in second grade that we find evidence of children's perception of a multiplicative structure. One child counted the number of nets in each set and then proceeded to count the number of butterflies in corresponding nets; one little girl provided the clearest evidence by stating that there were "six packs of eight". Clearly, here was a child who could judge the equality of the two sets without knowing their cardinality. The responses of the third child were closer to the ones seen among first graders; he focused only on one of the factors.

A second task associated with this criterion involved two sets of 45 butterflies arranged in 9 sets of 5 (see figure opposite). However, this task was deemed to be somewhat easier since the number of butterflies in each set could be subitized. Again the subjects were asked **Do we have the same number of butterflies?**

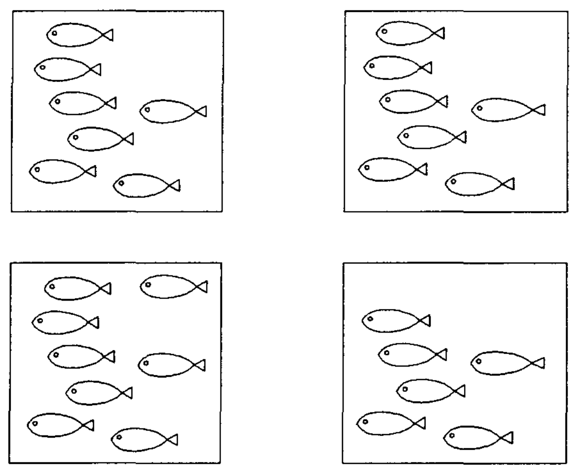


As with the last task, kindergartners made responses based on the configuration of the two sets. Each one identified the array on the left as having more butterflies, alluding to the empty spaces in the set with equal groupings. In contrast, the first graders provided evidence that they could also perceive the multiplicative nature of the situation. One child counted five butterflies in each row and then stated that there were nine rows in each drawing. A second child succeeded by counting 5 in each row and then establishing a 1:1 correspondence between each row in the array and the groupings in the other set. The third child's response was similar to the kindergartners since he believed that the number in the array was larger in view of the spaces in the other set. The three second graders reverted to a counting strategy.

The *third criterion* involved the **transformation of a non-multiplicative situation into a multiplicative one.**

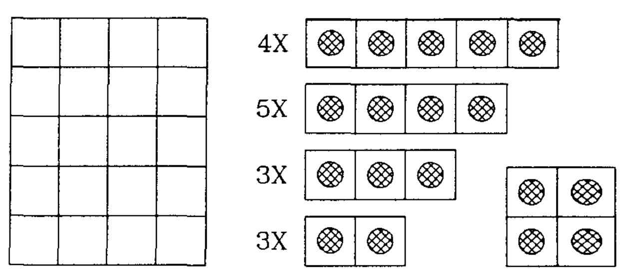
The corresponding task involved a pet shop with four aquariums, two of them containing 7 fish, one containing 6 fish, and the last one containing 8 fish. The child was told:

> **In the pet shop, each tank should have seven fish in it. The fish can be moved from one tank to another. Can you fix up these tanks and put them in the pet shop?**



This task proved to be easy for most children. Two of the kindergartners, two of the first graders, and the three second graders transferred spontaneously one fish from the 8-fish tank to the 6-fish tank.

The *fourth criterion* called on procedures needed to **create a multiplicative situation in two dimensions**. Children were presented with a 5x4 cardboard rectangle and with strips of transparencies of different lengths (five 4-unit strips, four 5-unit strips, three 3-unit strips, three 2-unit strips, one 2x2 strip). We considered that children who could cover the rectangle by choosing strips of equal lengths were in fact creating a multiplicative situation. However, since we were aware of the cognitive problems related to bilinear measure, we struck a somewhat intermediary situation limited to a discrete set. This was done by gluing on the strips stickers of sports figures in each square unit. The children were asked:



> **Can you cover the rectangle with these strips so that there is a sticker in each square? But you can only cover the rectangle with strips that are the same length.**

Since the available strips could correspond to rows or columns, when children had found one solution they were asked if there was another way. All the children found at least one solution. One unforeseen response was that of a second grader who used one 4-strip horizontally and the other 4-strips vertically. The other children focused on either the rows or the columns. Four children (1 k, 1 first grader, 2 second graders) covered the rectangle in two ways with strips of 4 units and again with strips of 5 units. One noticeable difference between the second graders and the younger children is that the latter used trial and error in selecting the appropriate strips while the second graders counted the number of cells in each row or column and then went about finding corresponding strips.

The *fifth criterion* ascertained if the children had **a procedure to verify if a number was "rectangular" or not**, (i.e. if the number of corresponding chips could be arranged in a rectangular array). We chose 15 chips so that the problems might be more challenging than any even number of chips. Children were asked **Can you make a rectangle with these fifteen chips?**

In every case, the child's initial response was to construct a rectangle using the chips only for the perimeter. Hence, an additional requirement needed to be stated: **"Can you give me a rectangle that is filled with chips?"**. None of the kindergartners were able to provide the solution, whereas all first graders did. However, only one of the second graders was successful. All the children who succeeded used a trial and error strategy, most of them starting with two rows.

In evaluating our pilot work on procedural understanding, it is clear that the two tasks related to criterion 1 and the one related to the third criterion (the aquariums tasks) succeeded in investigating our subjects' perception of equal groupings. However, we did not find the right kind of questions that might have conveyed to us that our subjects were perceiving the pet shop as constituting a whole. On the other hand, the questions and tasks associated with the other three criteria enabled us to deal with problems in which the whole was always the topic under consideration.

This exploratory investigation certainly was not presumed to be an exhaustive study. Nevertheless, it brought out some evidence regarding young children's perception of multiplicative structures. The results of the two tasks with the butterflies show that even first graders are able to compare two quantities by considering the two factors.

Logico-physical abstraction.

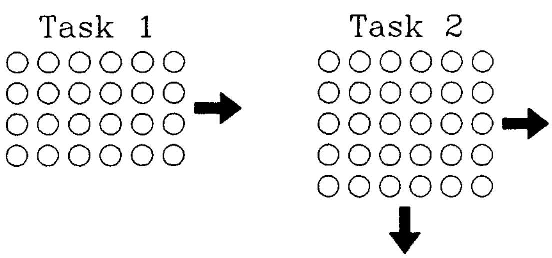
The third level of understanding of pre-multiplication, that of logico-physical abstraction, refers to the construction of invariants with respect to spatial transformations, as well as the reversibility and possible composition of these transformations. This level also includes the generalization of a concept. Applying this definition to pre-multiplication, we have identified specific criteria and translated these into specific tasks and related questions. These tasks and questions were tried out by Nicole Nantais who interviewed

5 first graders and 4 second graders in a school in the Sherbrooke area. Although this pilot study has been reported (Nantais and Herscovics, 1990), for the sake of completeness we provide a detailed description of the tasks, questions and answers.

Pre-test: The first task we suggested to our subjects did not deal with multiplication but verified if they conserved plurality (Piaget's classical 'number' conservation test) and quotity (Greco's comparable test including counting). Two of our first graders and three of the second graders conserved plurality. Regarding quotity, this was conserved by four first graders and three second graders.

Criterion 1: We have used four tasks to assess the children's apprehension of the **invariance of a multiplicative situation with respect to configuration.** Two variables involved in the design of these tasks were the elongation in one or two directions, and the presence or absence of a comparison set.

For the first task, we used 24 chips arranged in a 4 x 6 rectangular array that was stretched in the horizontal direction. In the second task, we used 30 chips arranged in a 5 x 6 array and stretched both horizontally and vertically. For both tasks the questioning was identical:



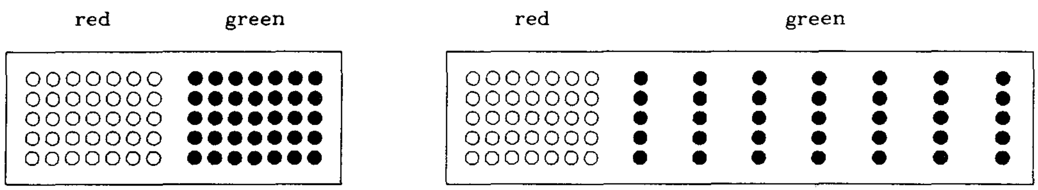
Can you tell me if all the rows are equal? Look at what I'm going to do. (After stretching the array) **Can you tell me if now there are more chips, less chips, or the same number of chips as before I stretched the rows, or do we have to count them in order to really know? Why do you think so?**

The results obtained were quite different in the two grades. All four second graders succeeded on both tasks. Among the five first graders, two succeeded on the first task and three on the second one. Most surprising, among these children three of them had not shown that they conserved plurality.

The next two tasks involved the same transformations but in the presence of a comparison set. Based on prior work dealing with the understanding of plurality, it was thought that this would make the tasks more difficult due to the interference caused by the visually perceptible differences between the comparison sets and the elongated sets. For the third task related to this criterion we used a 5 x 7 array of red chips and another one of green chips. The chips had been cut out of felt and were disposed on a felt board. The array of green chips was then stretched horizontally. The fourth task involved another board with two 5 x 7 arrays of yellow and green felt chips respectively, the green array now being stretched horizontally and vertically. For both tasks, the questioning was similar:

> **Here is a set of red chips and here is another set of green chips. Can you tell me, just by looking at them, if we have the same number of red and green chips? Look at what I'm going to do** (The interviewer then stretched the green array). **Just by looking, can you tell me if there**

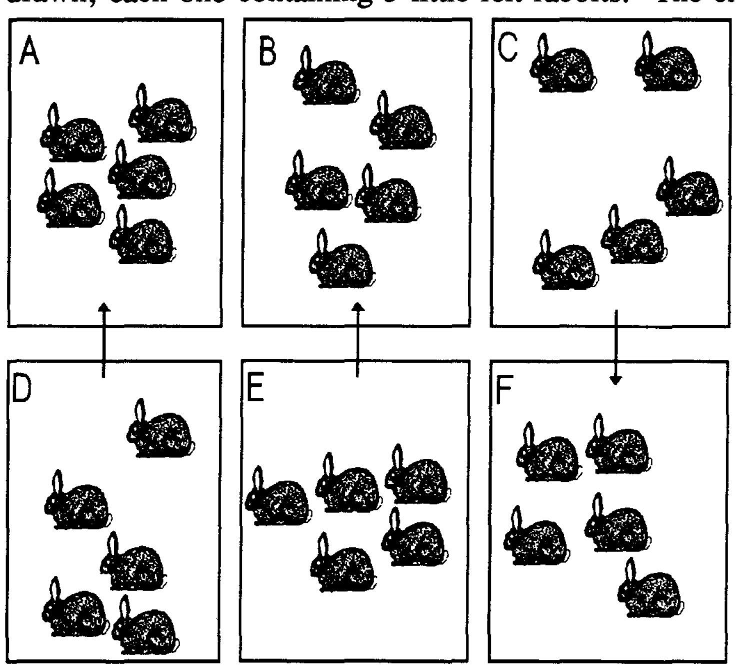
are more, less, or the same number of green chips as red chips? Or must we count them all in order to know?



Task involving a horizontal expansion of the green chips

To the initial query about the two sets having the same number of elements, all children answered affirmatively, most of them basing themselves on the visual similarity. Other children explained it by counting either the rows or the columns but not both. Our subjects had a somewhat better success rate than on the first two tasks. Four of our five first graders succeeded on task 3 and three succeeded on task 4. All the second graders who were given the tasks handled them successfully.

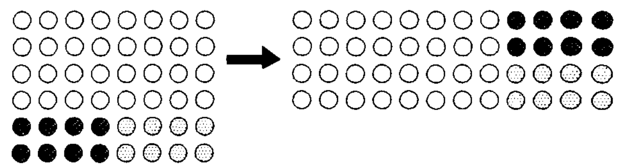
Criterion 2: Regarding the **invariance of a multiplicative situation with respect to the regrouping of the subsets**, four different tasks were used. The two variables involved were the randomness of the elements in the subsets and the presence of a comparison set. For the first task we used a cardboard on which six rectangles were drawn, each one containing 5 little felt rabbits. The children were told:



Here is a farm and this is a barn where rabbits are kept. But the farmer has to repair three cages. Thus he must move some of the rabbits. Look at what I'm going to do: While I repair this cage (D), I will put these rabbits in this cage (A). (Similarly, the rabbits where then transferred from E to B and from C to F). **Do you think that now in the barn there are more, less, or the same number of rabbits as before? Or do we have to count them in order to know? Why do you think so?**

Children were almost evenly divided in their response. In each grade, two children thought that the regrouping of the rabbits into three cages resulted in more rabbits in the barn. Clearly, these subjects were focusing on only one aspect of the multiplicative situation, that of the number of elements in each group. They were not compensating for the smaller number of groups. Those children who thought that the number had not changed justified it in terms of "nothing added, nothing taken away from the barn".

In the second task dealing with this aspect of invariance we used a 6 x 8 array of identical chips and transferred the bottom two rows as follows: the bottom chips on the left side were aligned along the first two rows, the bottom chips on the right hand side were then aligned with rows 3 and 4. The different shadings shown in the diagram are there simply to help visualize this transformation. The questioning then proceeded as in the prior task.



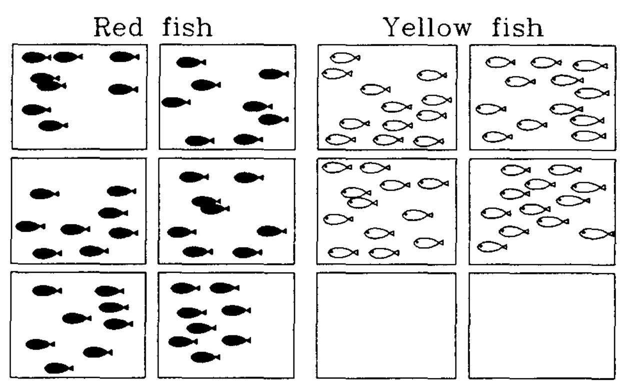
This transformation seemed to have a greater impact on the first graders. Two of them thought the total number had changed whereas none of the second graders did.

The third and fourth tasks verifying this invariance were variations on the first two since the only change was the addition of a comparison set. In task 3, children were presented with two felt cardboard strips representing two pet shops, one selling yellow fish and the other selling red fish. Each "pet shop" contained 6 fish tanks and each aquarium contained 8 little fish cut outs in felt. Two of the tanks with yellow fish were then emptied and the fish redistributed into the other yellow tanks. The questioning proceeded as follows:

> **Here are two pet stores that sell fish to keep in an aquarium. One store only sells red fish and the other one sells only yellow fish. There are eight fish in each aquarium. Do you think that there is the same number of fish in the two stores?**

Following an answer, the subject was told:

> **But in the pet store selling yellow fish, two tanks are leaking and we have to move the fish in the tanks. Look at what I'm going to do.** (The interviewer then removed the yellow fish from the bottom tanks and distributed them one at a time into the other yellow tanks so that the child would be assured that the fish had been distributed

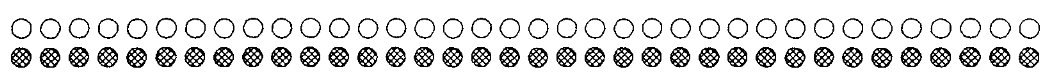


equally). **Now, do you think that there are more, less, or the same number of yellow fish as red fish, or that we have to count them in order to know? Why do you think so?**

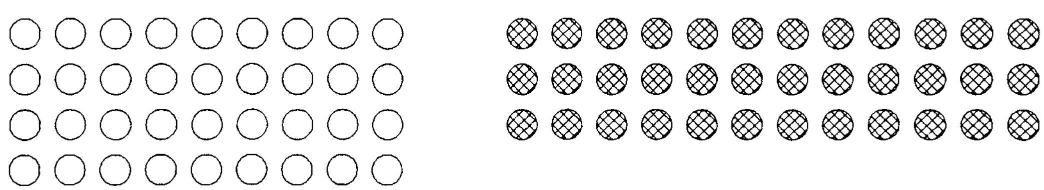
Most of our subjects did not succeed on this task. Among the five first graders, one of them apprehended the invariance of the total number of yellow fish, while three children felt that the numbers were no longer the same; we could not proceed with one little girl since she thought right from the start, before any transformation, that the number of fish in the two pet stores was not the same. Among our four second graders, two children apprehended the invariance whereas two of them did not.

Results on task 4 dealing with this invariance were much better. In this task children were presented with two 4 x 7 arrays of red and green chips respectively. The green array was then transformed into a 2 x 14 array. Three of the five first graders and all the four second graders thought that the total number of chips in the two sets had remained the same.

Criterion 3: Whereas in the previous tasks we had started with multiplicative situations, either sets subdivided into equal subsets or sets displayed in rectangular arrays, the task we envisaged here was to verify if children could perceive the possibility **that the same quantity could be decomposed into two different but equivalent multiplicative situations.** To this end we displayed to the second graders two rows of 36 pink and 36 green chips. We then transformed the pink row into a 4 x 9 array and the green row into a 3 x 12 array. With the first graders we used rows of 24 that were transformed into arrays of 3 x 8 and 2 x 12 respectively. The questions were: **Here are two rows of chips. Do you think that we have the same number of pink chips and green chips**



Look at what I'm going to do. With the pink chips I make four rows of nine and with the green chips I make three rows of twelve. Do you think that the four rows of nine will give me the same number as the three rows of twelve, or that we have to count them in order to know? Why do you think so?



The results indicate differences between the two grades. Two of the five first graders and three of the four second graders thought that the two arrays had to have the same number. The other subjects did not. It is interesting to note here that all the children who conserved plurality on the Piagetian test also conserved it in this task.

Criterion 4: **Pre-commutativity**. In order to verify if children apprehended the commutativity of a multiplicative situation (with respect to the total amount of objects), three different problems were presented. The first one was purely verbal. The children were asked:

> **If I have six bags of marbles and nine marbles in each bag, and you have nine bags and six marbles in each bag, can you tell me if you and I have the same number of marbles, or if we don't have the same number of marbles, or if we would have to count them all in order to know? Why do you think so?**

Children were presented with a sheet of paper on which the information was written in the form of two columns:

Me	You
6 bags	9 bags
9 marbles	6 marbles

In the second problem children were told a similar problem but the objects were put out in front of them (five bags of eight red chips vs eight bags of five red chips) and the third problem involved the comparison of a 5×9 array of circles vs a 9×5 array.

The differences between the children from the two grades were remarkable. None of the five first graders perceived the commutativity of the multiplicative situation. On the other hand, for each one of these problems, three of the four second graders thought that the quantities were the same.

Criterion 5: **Pre-distributivity of multiplication over addition**. In order to verify if children had some inkling about the distributivity of multiplication over addition, our subjects were shown two arrays of white circles (4×5 and 4×6) and a 4×11 array of black circles and asked if the total number was the same.

The responses followed almost the same pattern as for pre-commutativity. Three of the four second graders thought the total number of white circles was the same as that of the black circles, but only one of the five first graders.did The justifications were straightforward: "They're the same. Because these are separated and the others are not. If we put these together..."

In evaluating our study on logico-physical abstraction, we find that the conservation of plurality (assessed through the classical Piagetian test) seems to be a determining factor in the child's readiness for multiplication. This can easily be explained. If anything, the child's failure on the Piagetian task indicates an inability to compensate for the elongation of a row by the corresponding decrease in the density of the row. This inability to account for two complementary variables must also be a critical cognitive factor in the recognition of multiplicative situations, since these always involve two factors: the number of groups and the elements per group.

The general data bear out the importance of the conservation of plurality on the success rate achieved with the multiplication tasks. However, there seems to be one exception to this statement. Four children who did not succeed on plurality should not have succeeded on the tasks dealing with the invariance of the configuration but three of them did. Clearly, some other factors are involved here.

The results on the tasks dealing with regrouping of subsets indicate that for first graders, when randomly disposed subsets are regrouped into larger subsets, the presence of another unchanged comparison set does interfere with the child's reasoning. This is shown by the different results on the rabbit task and on the pet fish task. However, when it comes to regrouping rectangular arrays, the presence of a second set does not seem to alter the results.Nor does such a presence seem to affect the success rate of the second graders.

The task involving the decomposition into two different but equivalent arrays was remarkably related to prior conservation of plurality. The tasks dealing with the axioms of pre-multiplication separated the children in the two grades. It should be noted that although the three tasks on pre-commutativity were expected to be unequally difficult, the results did not bear this out.

This exploratory work shows there is little doubt that second graders are ready to learn arithmetic multiplication. However, this does not mean that one should follow the philosophy of existing programs and stress almost exclusively the development of arithmetic procedures. Of course, these procedures are of prime importance. However, the usual tendency to overemphasize them is practice at the expense of conceptualization. As the two pilot studies show, it is possible to develop many tasks related to the concept of multiplication without necessarily quantifying the total sets. This may yet provide us with a better definition of arithmetic multiplication. Until now, the definition needed to be procedural: "multiplication is repeated addition" However, this only tells us how to answer the question "How many?". Viewing multiplication as the mathematization of multiplicative structures may bring us to consider it as an operation as vital and primitive as addition, subtraction or division.

Understanding of the Emerging Mathematical Concept

A major feature distinguishing the two tiers in our model is the process of quantification. It is only at the second tier that we raise the question "How many?". This necessarily brings about operations involving numbers. Nantais & Herscovics (1989) described the three aspects of understanding the arithmetic operation as follows:

Procedural understanding of a logico-mathematical nature

By procedural understanding we mean the appropriate use of explicit arithmetical procedures. Initially, when young children in grade 2 are asked "How much is three times four?", many will respond by saying that they have not learned it yet. Some will model the problem by making three sets of four and count them starting from 1. While simple enumeration provides an answer, it cannot be considered as a multiplicative procedure since it does not take into account the existence of the subsets. The most primitive procedure that can be considered as being somewhat multiplicative must provide such evidence. This is reflected when the child manages to **skip count** on a number line: 4,8,...12. If no number line is available, the child may remember the first part and produce "4,...,8,9,10,11,12. This **mixed procedure** does indicate an awareness of the two factors involved even if some counting takes place. A more advanced procedure involves **repeated addition**: $4 + 4 = 8$ and $8 + 4 = 12$. Gradually, by grades 4 and 5, children learn to memorize some number facts which provides them with an apprehension of **numerical relationships** that they can use in deriving larger products as for example the product 4×6 which may be obtained by the smaller product $2 \times 6 = 12$ and then the sum $12 + 12 = 24$.

Logico-mathematical abstraction

Gradually, as the child's procedural knowledge evolves, the reversibility of the operations and the apprehension of some mathematical invariants become possible. For instance, the child no longer needs concrete material **to break a number down into its factors.** This inevitably leads to thinking of these factors as also being **divisors**, the operation thus becoming reversible. Knowledge of the multiplication table also enables the child to apprehend the **equivalence of various products** with respect to a given number without having to depend on their different configurations. In terms of **axiomatization** and **generalization**, the **commutativity** of multiplication becomes self-evident and somewhat later, so does the **distributivity** of multiplication over addition.

Formalization

Interpreting formalization in terms of the symbolic representation of the learner's previously acquired knowledge, children first learn the usual **notation** for multiplication and can interpret $\mathbf{4 \times 3}$ as meaning four sets of three objects. They also can recognize an appropriate additive situation as being multiplicative by expressing the sum as a product (e.g. $\mathbf{3 + 3 + 3 + 3 = 4 \times 3}$). On the other hand, when this arithmetic equation is read from right to left, it expresses a form of procedural understanding since it symbolizes repeated addition. Interpreting formalization in terms of axiomatization, the **axioms of commutativity and distributivity** can be crystallized in various notations, a simple one being $\circ \times \square = \square \times \circ$, $\triangle(\square + \circ) = \triangle \times \square + \triangle \times \circ$, and a more difficult one being the use of letters.

By way of conclusion

In order to get an overview of the different criteria used to describe the understanding of early multiplication, we now gather them in tabular form:

The Understanding of Preliminary Physical Concepts

Intuitive understanding	Procedural understanding (logico-physical)	Abstraction (logico-physical)
▪ visual distinction between a multiplicative situation and one that is not; ▪ apprehension of a rectangular array as a multiplicative situation; ▪ comparison of two multiplicative situations on the basis of their "factors" being subitized; ▪ comparison of equipotent sets involving different configurations.	▪ construction of a whole based on the iteration of a 1:n correspondence; ▪ comparison of two multiplicative situations on the basis of their "factors" being counted; ▪ transformation of a non-multiplicative situation into a multiplicative one; ▪ generation of a bilinear multiplicative situation; ▪ verification of a number being rectangular or not.	▪ invariance of a multiplicative situation wrt configuration; ▪ invariance of a multiplicative situation wrt regrouping of the subsets; ▪ invariance of a multiplicative situation wrt its decomposition into different rectangular arrays; ▪ pre-commutativity; ▪ pre-distributivity.

The Understanding of the Emerging Mathematical Concept

Procedural Understanding (logico-mathematical)	**Abstraction (logico-mathematical)**	**Formalization**
▪ skip counting on a number line; ▪ mixed procedure indicating awareness of factors; ▪ repeated addition; ▪ numerical relationships.	▪ numerical decomposition into factors; ▪ apprehension of the equivalence of the different products obtained by the decompositions of a given number; ▪ generalisation of commutativity; ▪ generalisation of distributivity; ▪ reversibility leading to the apprehension of factors as divisors.	▪ introduction of the multiplication sign ▪ replacement of addition string by product; ▪ bi-directionality of product; ▪ generalised symbolization of commutativity; ▪ generalised symbolization of distributivity.

It should be noted that the three levels of understanding included in the first tier are linear. Without prior intuitive understanding, the acquisition of concrete procedures could hardly qualify as understanding. Similarly, one cannot expect the child to achieve any logico-physical abstraction without being able to reflect on the procedures used to generate multiplicative situations. Nevertheless, the model as a whole is not linear. The aspects of understanding identified in the second tier need not await the completion of the physical tier. Well before they achieve logico-physical abstraction, children can start acquiring the various relevant arithmetic procedures by the quantification of problems introduced in the first tier. The formalization of multiplication need not await the completion of logico-mathematical abstraction; the formalization of the arithmetic procedures will occur much earlier than formalization of the axioms.

This work has some interesting pedagogical implications. It suggests an alternative to the age-old tendency of introducing multiplication merely as repeated addition. Instead, it shows that prior to the introduction of this arithmetic operation, one might present children with didactical situations in which they could recognize and generate a great variety of multiplicative problems. Indeed, corresponding to the different criteria used for the different levels of understanding in the first tier, one can develop a broad sequence of activities. The stress on work at the concrete level should not be interpreted as an attempt to diminish the importance of the traditional work on explicit arithmetic procedures. But the prior introduction of multiplicative situations should provide motivation and relevance.

In terms of its relevance to mathematics educators and to teachers, our approach brings out the organic nature of a concept as fundamental as early multiplication. It identifies the operation with a cognitive matrix encompassing the knowledge relevant to its construction and conveys an epistemological perspective. Such a matrix provides the teachers with an overview of the conceptual scheme which in turn enables them to better appreciate the particular contributions of their pedagogical interventions. And just as significantly, while it brings out the importance of the acquisition of various mathematical procedures, it does so in a broad cognitive context that attaches as much importance to the procedures as to the underlying concepts, as well as the emerging abstractions.

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Topic Group A

The Benchmark Programme: Evaluation of Student Achievement Incorporating the NCTM Standards

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ABSTRACT

The Toronto Board of Education's Benchmarks in mathematics are information about student achievement at the end of grades 3, 6, and 8 (approximate ages 8, 11, and 13) to be used by all elementary teachers in evaluating students and reporting to parents. The Benchmarks are in videotape and print form and are packaged in three libraries, one for each grade. The information is the result of interviewing a ten percent representative sample of students at each grade for about five hours each. Students were required to perform a wide range of tasks spanning the curriculum. They formulated and solved problems, worked with manipulative materials, and gave oral explanations. Holistic and analytic scoring were used to rate the performances. In the final holistically scored videotape Benchmarks there are five performance levels and for each level there are the holistic scoring criteria and resulting percentage of students. Each videotape provides sample student performances at the top three holistic levels and there is an unrated performance at the end which the viewer is invited to rate. The video box provides the corresponding print information.

The Benchmarks are not tests. They are descriptions of what our students can do in activities which teachers now use in their classrooms. Teachers are expected to experiment with the use of Benchmarks and gradually integrate effective practices into their ongoing programs.

Topic Group B

First Adventures and Misadventures in using Maple

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First adventures and misadventures in using Maple[1]

The subject of Computer Algebra Systems (CAS) or Symbol Manipulation Systems has come up regularly in the Study Group meetings of the past five years. It was discussed during several Working Groups dealing with the general question of computers and mathematics education. Maple was shown to us at the Waterloo meeting and a subsequent Working Group devoted specifically to Maple took place at the Brock meeting (CMESG, 1989). While several of our colleagues in the Study Group have already accumulated substantial experience in the classroom use of CAS, for us at Concordia this past year was the first 'hands-on' experience with such systems. We would like to report on our initial impressions and reflections stemming from working with Maple and two small groups of students.

Since CAS were first designed as an extensive analytical tool for professional users and not as 'educational' software, we might first try to recapture some of the arguments for the pedagogical uses of CAS.

- The 'existence' argument
  CAS are out there and they are proving to be a useful tool for mathematicians, so students should also become familiar with their use.
- The 'calculating expediency' argument
  CAS allow for increasing the complexity of calculations (be they numerical, algebraic or graphical) at any level of instruction. Their use lets students tackle 'real' problems with realistic data rather than 'text book' problems.
- The 'conceptual shift' argument
  Both instruction and learning can be more conceptual since the use of CAS results in a lessened need to teach routines and in a possibility of students "seeing beyond the calculations".
- The 'mathematisation' argument
  Using CAS is a more active and engaging way of learning mathematics. Their use fosters an environment in which exploring, conjecturing and tinkering are natural activities.
- The 'cognitive support' argument
  CAS, possibly in a modified form, can be used to help overcome known conceptual obstacles.

It is worth noting that *none of the above arguments are specific to CAS*. In fact most of the same arguments have been forwarded to promote the use of calculators, the learning of programming, and the use of numerous software packages (hence the battle lines between opponents and proponents of these arguments have already been well established). There are, however, two things which are different about CAS. First, because they are extensive and extendable, they can embrace all of the above points of view. Secondly, the range and level of mathematical activities which can be effected by their use covers all of undergraduate mathematics. This, in turn, seems to have provoked greater willingness of the mathematical community (read, the 'pure' mathematicians) to integrate computers with instruction.

[1] We would like to acknowledge the generous support given to the project by SSHRC (Grant #410-89-1174) and FCAR (Grant #90-ER-0245).

What often is not articulated explicitly are the assumptions about the kind of students, their conceptual knowledge, the educational goals (individual and institutional) and the changes to the curriculum and to the style of teaching and learning that underlie the different arguments for introducing CAS into instruction. Globally, we might say that these were listed above in the order of importance given to the computer in the learning process (from merely a subsidiary role to a central role), and in the order of emphasis on pedagogy (from strictly content related issues to learning issues).

While these points of view are clearly not incompatible nor mutually exclusive, it might be useful to tease out the different scenarios that each argument carries when it is used as the *principal rationale* for the use of CAS. Taking the standard introductory calculus course as an example, we can see how each point of view entails changes, some necessary and some optional to subject matter, to classroom organization, to basic skills, to learning style, to evaluation, and so forth.

The 'existence' argument is a 'laissez faire' one in which the use of CAS need not call for any specific instructional action or perturbation of the existing curriculum—CAS might simply be picked up by students along the way, just as slide-rules and calculators were, and used when needed. (Indeed, this phenomenon has already been evident with the advent of 'supercalculators'). In the case of the professionally bound mathematics or engineering students, the implicit assumption is that they have the appropriate knowledge of the underlying mathematics so as to be able to use CAS effectively as a tool.

The 'calculation expediency' argument sees in CAS an opportunity to include 'meaty' applications instead of the stereotypical textbook calculus problems. Access to CAS is available to students but it does have to be structured as a lab. Changes to traditional topics are not necessary except for the inclusion of class discussions about CAS, the routines that they employ and the interpretations of their output, i.e. students should acquire 'analogue insight' of CAS (D. Tall's term; Tall & Winkleman 1986).

On the other hand, the 'conceptual shift' argument sees in CAS over and above a computational tool, a vehicle to overhaul traditionally taught subject matter, both in content and order of presentation. It calls for less emphasis on the teaching of those techniques and algorithms that can be handled by CAS (e.g. techniques of integration or of graph sketching) and for a greater emphasis on "teaching for conceptual understanding". A computer lab becomes an integral part of the course and a lot more play is given to 'multiple representations' of solutions by using graphical, tabular, numerical and analytical solutions to problems. By being able to work with a large range of functions, students acquire better understanding of concepts such as limits, differentiability, tangent lines and the relations between a function and its derivative or integral.

The 'mathematisation' argument goes even further than the previous ones in requiring a major change to the traditional teaching and learning *style*. It envisages the students' work on the computer as the central learning activity and it gives importance to a learning style which calls for an active engagement on the part of the student. It supposes that work on the computer precedes the formal presentation in class. Students, for example, may be asked to explore graphically the relations between a function and its derivative and to try to formulate these relations explicitly. On the other hand, traditionally taught calculus topics (including the various routines) are not necessarily altered.

Finally, the 'cognitive support' argument tries to exploit ways in which CAS may allow for different entry points and approaches to the teaching of some concepts. It is a view which takes into account cognitive aspects of students' difficulties and it requires crafting specialized "computer learning environments" out of CAS. Examples are found in D. Tall's suggestions of introducing the derivative via the point graph of the numerical gradient to a function at a point prior to introducing limits, and differentiability of functions via the notion of 'local straightness'.

Our objectives in using CAS

It seems reasonable to look at the various segments of the mathematics student population and to inquire what the appropriate uses of CAS are for such groups. We chose to look at the feasibility of using Maple in our university's 'collegial programme' which comprises pre-university courses (in Quebec's 3-year university system). Students in this programme tend not have the 'standard' student profile; they are enrolled in such a programme either because their studies have been interrupted or because they have changed their field of study and have to pick up mathematics prerequisites. They are usually over 21 years of age, with a gap in their mathematics studies in the 2 years to infinity range. Typically, their previous encounter with mathematics has not been positive and the courses they now have to take are often just accelerated versions of their previous nightmares. It seems reasonable to assume that such students would benefit from a different kind of a learning environment and from a curriculum which is not rigidly structured in terms of prerequisite skills.

Before describing our activities, we ought to tackle an inevitable question: by choosing to use CAS with students at the 'collegial' level (Functions, Introductory Calculus, Introductory Linear Algebra), were we trying to 'kill a fly with a sledgehammer'?. It is clear that work at this level requires very little of the general arsenal available in Maple. How appropriate then is it to use CAS when there are many existing software packages which are more specialized, more modest and often come with a very good user interface? In particular, since functions and graphs play an important role at this level of instruction, why not use some of the better software packages dealing with this topic? (Maple is presently a very poor at graphing, more on this later).

The answer depends on whether one looks at the use of CAS as a 'one shot deal' or as something which can become part and parcel of doing mathematics across different courses and at different levels. Clearly it is the latter viewpoint that justifies the use of such extensive systems. A program such as Maple provides students with a general and extendable tool, with a consistent language and syntax; a tool that they could use throughout their mathematical training.

While either the 'conceptual shift', the 'mathematisation' or the 'cognitive support' could be the main rationale for using CAS with such students, our initial objectives were more modest—we simply wanted to get acquainted with Maple, with its strengths and weaknesses (relative to the student group that we had targeted) and to observe how we and the students interact with Maple. We did not feel ready (nor were we equipped) to handle a whole class, so we worked instead with two groups of students, the first group of five came from an introductory calculus course and the second group of four from a course on functions. Our experimental sessions ran in parallel but not in conjunction with these courses. We were careful in how we presented our enterprise when we solicited volunteers for the study. While we did try to 'sell' Maple as an exciting new mathematical software, we merely said that we would be using Maple for some of the topics covered in the courses and though the approach was to be different, we thought that what the students would learn would prove useful for them in their course work. Because the students participated as volunteers and because they (as it turned out) were having difficulties with their actual courses, we tied ourselves down to the 'official' curriculum more than we would have liked.

The 'Introductory Calculus' Group

The first group of students were already 6 weeks into their calculus course when they started to participate in the sessions. When asked why they came, they alluded to a general interest in computers and they mainly talked about their problems with their course, e.g.

- the instructor does the algebraic manipulations on the blackboard too fast, so they can't follow
- word problems
- not seeing enough examples
- getting bogged down with algebraic calculations

It is interesting to note that Maple exacerbates the first difficulty and can do nothing about the second. (There was some disappointment in the realization that computers 'cannot do word problems'. On the other hand, the students got a kick out finding out that Maple can do the first part of a standard calculus exam—which is of the 'differentiate the following functions' variety. Maple is much better suited to handle the last two of the students' problems.

Because these students had already covered the basic ideas of the differential calculus in class, Maple was used mostly as a way of cementing these ideas and as a verification tool. We tried to show where using Maple may be appropriate to the kind of problems that they brought to the sessions. There was a lot of emphasis on gleaning information about a function (its zeroes, extrema, slope of tangent lines, etc.) initially from its graph and then by analytic means, and the simultaneous plotting of f, f′ (and, sometimes, f″) so as to get a feel for the relations between these functions.

As attractive as an approach which puts a lot of emphasis on graphical representation and interpretation may seem, we had rather a mixed success with it. When things went well (and that often meant that we had chosen beforehand the examples to be worked with Maple), the students were engaged and, at times, excited. However, there were also times in which the students felt that they were not "getting their money's worth". The problems that arose were due to:

- the students' own lack of knowledge
- the way CAS operate in general

- inadequacies specific to Maple.

The students, in general, relied on the default setting for plotting a graph of a function f (domain is the interval [-10, 10] and range is the interval which is just a bit larger than [min(f), max(f)]). When this particular 'window' yielded only partial or hard-to-interpret information about f, the students were at a loss of what might be a better graphical 'window'. They also were not able to judge whether the plot was a faithful representation of f or whether it was an artifact of way Maple handles plots. For example, certain functions were erroneously believed to be identically zero on the interval, others, to have no zeroes at all (see the example of $f(x) = |x - 2|$ discussed below).

Another difficulty in interpreting graphs had to do with scaling. Most CAS 'fit' the graph within a specified rectangle, thus scaling the two axes quite differently. This tended to throw students off balance—not only were the graphs of even familiar functions different than those they have seen in the textbook or in class, but their 'intuitive' notion of steepness and rate of increase (so vital for discussing the properties of derivatives) were being shaken, for example with the default setting for plotting, all linear functions look as if their slope is either 1 or -1.

In other computational situations, particularity when solving analytically for zeroes of a function, the CAS solutions were often incomprehensible; for certain rational functions, solutions occupied several full screens. Here is the output for the roots of $x^3 + 3x + 5$:

```
>
 solve(x^3+3*x+5=0,x);
                      1/2 1/3                     1/2 1/3
       (- 5/2 + 1/2 29   )    + (- 5/2 - 1/2 29   )    ,

                           1/2 1/3                          1/2 1/3
         - 1/2 (- 5/2 + 1/2 29   )    - 1/2 (- 5/2 - 1/2 29   )

                 1/2                  1/2 1/3                     1/2 1/3
           + 1/2 3    ((- 5/2 + 1/2 29   )    - (- 5/2 - 1/2 29   )    ) I,

                           1/2 1/3                          1/2 1/3
         - 1/2 (- 5/2 + 1/2 29   )    - 1/2 (- 5/2 - 1/2 29   )

                 1/2                  1/2 1/3                     1/2 1/3
           - 1/2 3    ((- 5/2 + 1/2 29   )    - (- 5/2 - 1/2 29   )    ) I
```

Maple's own shortcomings as a graphing tool created other problems—it is very slow, particular for non-polynomial functions. It has a useless way of labelling the coordinates on the y-axis and allows no interaction with the graphical window (such as labelling points or graphs, reading coordinates, zooming, etc.)

And, yes, Maple does bomb about once per session—though the students learned to be philosophical about it, nevertheless they were frustrated when they had to restart in the middle of the session, losing all their work.

It was clear to us that the students felt a tension between their classroom work and their work on Maple. In the computer lab, questions had to be posed differently, solutions looked different, new mathematical concepts and ideas had to be addressed, and notation was not the same (though they didn't seem to have any particular difficulty with Maple's rather transparent notation). However, we believe that if the mathematical issues that arose out of work with Maple were to become part of the course, the students would find the (calculus) course more interesting and more meaningful. They would have had a first hand and personal experience working with problems for which these issues are relevant.

We felt uncertain whether the work with Maple helped the students conceptually in their actual calculus course. We were somewhat surprised (and pleased) when they inquired about continuing the Maple sessions with their subsequent course. (Was it Maple or the fact that they had one-on-one help? There was a fair deal of 'illegal' pencil-and-paper tutoring that took place during the sessions.)

The 'functions' group

The four students from this group started their Maple sessions at the same time as their functions course. The emphasis was, again, on graphical representation of functions (linear, quadratics, exponential, logarithmic and trigonometric). We felt, once more, an obligation to stay close to the official curriculum, however we did select more carefully topics in the functions course that we felt were best done with Maple and we structured some activities around each topic. Unlike the students in the previous group who had, on occasion, to work in pairs, these students worked individually.

Of course, some of the same difficulties that were experienced by the calculus group were also problematic for this group. We did, however, manage an end-of-session interview with the students which we include here (reconstructed from notes taken by us—with all the technology around us, we didn't think of a tape recorder!)

End-of-session interview with students

Question: Did you find Maple helpful? Was there an advantage to using a computer over and above working one-on-one with an instructor?

Jill: I saw it as a great calculator. I like to see things visually. So there it's more convenient - instead of wasting time plotting something, I could concentrate on something else. I understood and started deriving formulas for things I'd just memorized.

Denis: It didn't make sense until we did it in class. Maple sessions would have made more sense if we'd done the topics in class first.

Jerry: Because the way things were arranged, you were free to ask questions and do your own things (here he refers to his last two sessions where he was attempting to graph circles). I had to give an allowance for the machine—it took a bit of effort. Sometimes it opened up visually what we were doing in class. Sometimes it would be interesting to toy with it. Other times a relatively simple thing would get pretty involved.

I'm still ambivalent because I felt I could have got more out of it. I'm not computer literate. If I'd had more time...I would like to have seen it be more creative. The scope could have been larger.

Craig: I was baffled when I took the functions course and I had to drop it. I found the interactive help [with the researchers?] more important than the computer. The Maple program is not quite interactive enough. It doesn't prompt. If you type in HELP the computer prints help. In general I was very pleased with the sessions.

Question: If Maple was accessible to you now, would you use it?

Jill: I'm trying to talk my father into getting a modem then I could use it...I would need more exploring with someone.

Craig: The HELP function is quite nice. (He explains how he'd used it.) It's intriguing because you could go in and actually do it. Then modify a few variables.

Jerry: If I'd had time alone I would be able to try out some of the other functions. A lot was too one-tracked. I would have liked a more holistic way of using course content instead of subject by subject.

Question: What was missing? What did you find frustrating?

Jill: If it gave us the text book version first...(A brief discussion follows about how it would be helpful to have the computer display the classic textbook graph of a function such as e^x before the students start plotting the Maple graph with a specific domain and range.)

Craig: Why didn't I start this earlier! Also the lack of feedback from the machine. There was not enough time on it to really get into it. (Here he explains to the others that they can get a Vax account and "practice to your heart's content".)

Jerry: You're covering a topic and you end up with something that doesn't approximate the ideal situation. An hour later you got what you wanted. What did you learn? It's like being a slave to the machine. The machine had a mind of its own. It limited your ability to be more creative, to come up with something new. The machine did what you asked but fell short of your expectations. It's not ideal enough.

Craig: It's just a tool. (Here he explains how a computer is like a hammer or chisel...)

Jerry: It sounds like you're having a devotional relationship to your machine.

Denis: You type in your problem. It say "error". I just had to re-type...there was nothing wrong. I also wanted to see the steps.

Jill: It would be interesting seeing steps as in [solving] a trig identity. If it could be like a tutorial and show you where you were wrong.

Question: We constructed the sessions choosing from the functions course those things that we thought Maple could do best. Should we forget the course and just do some of Maple's exciting things?

Jerry: It might have been more detached from the course. It didn't marry what Maple does best with the course content. When I came to the first session the message

I got was that "It's what you make of it". It fell way short of my expectations. The commands we had were limiting.

Question: Would you have preferred working in pairs?
Craig and Denis: Prefer to do it myself first and group afterwards. (General agreement)

Comments:

These responses to the questions show (not surprisingly) individual differences, reflecting each person's initial expectations, preferences and the way s/he was coping with the actual function course.

Jerry was the most tentative—he was less interested in the actual content of the functions course (and he had a poor mastery of it). Rather, he was looking for a creative medium to work in (to do what exactly?) and consequently found work with Maple too constricting. He might have been more enthusiastic if Maple had a friendly user interface with a more iconic rather than symbolic writing of commands.

Jill, who had a fairly good grasp of the course material was the most explicit in her assessment that work with Maple helped her conceptually (supporting the claim that CAS help the better students and makes things worse for the weaker students). She viewed Maple as a tool to cement ideas that she has already picked up.

Denis, who liked making and testing conjectures, exploited Maple quite effectively for this purpose (see the description in the next section). He also expressed the sentiment that Maple was useful only after a concept was covered in class. It is not clear whether the distinction made between classroom work (which counted) and Maple work (which didn't count) is just an artifact of the way we ran the sessions (after all, the Maple sessions could have been part of the course). It might be that Denis was expressing a preference to see a more formal presentation of material prior to the more experimental work on the computer.

Craig, who had very little post-secondary mathematics but a lot of familiarity with computers, found a 'natural' medium for him to work in. He felt that his experience with Maple has opened a mathematical door for him. He would have functioned best (and more autonomously) in an interactive tutorial-type environment, one in which there are menus, prompts, examples and more self-explanatory error messages.

Some instances of learning with Maple

Each observer describes below an episode or a session where they felt that some significant learning has taken place. Work with the second group (functions) predominates these descriptions simply because it was the most recent.

Pat Lytle:

Jill referred to both the x and y intercepts in the first session while indicating how she would plot a line from an equation. During the second session, Jill described the line

$y = x + 20$ as "for every x, y increases by 20", and that of $y = 5x$ as "a shift to the right or left by a factor of 5", i.e. she made no reference to the graphical representation nor to slope or the y-intercept. I prompted her to look at the simultaneous graphs of the family $y = 5x + b$ (where b was 20, 10, 5, 0, -5, -10, -20) and asked her to comment on the relationship between the algebraic expression and the graph. She stated that the "highest" line was $y = 5x + 20$, etc., but again made no reference to the y-intercept. After several indirect attempts to elicit this information from her, I finally asked her directly about the y-intercept. Jill then immediately realized that she had not, up to this point, made any connection between the algebraic form (the 'b' term), and the point on the y-axis. Seeing all the graphs of these parallel lines made the concept of the y-intercept come together and cemented during the remaining sessions.

Helena Osana:

Craig had plenty of experience with computers as he worked in the Computer Centre at Concordia, but knew very little about mathematics as such. Previous to the sessions, he had attempted a functions course but had not completed it.

At the start of the third session, we looked at the four quadrants and those in which x and/or y are negative. He had a vague notion of where "negative space" existed on the Cartesian plane, but this needed clarification. We then started to vary parameters in the general quadratic equation. For example, we compared $f(x) = x^2$ and $g(x) = 4x^2$. Craig knew instinctively that the "4" would "do" something to the shape of the graph, and we studied what would happen. After we went over the effects of bx^2, $x^2 + c$, and $(x + d)^2$, (b, c, d any real constants), we started combining parameters and effects. I was astonished to see that Craig was understanding the trends and patterns that accompanied these parameter changes. He could correctly predict, for example, what $f(x) = -3x^2 - 5$ would look like in relation to $f(x) = x^2$. The session went very well and Craig himself was pleased with his progress to date.

A highlight occurred several sessions later when I was working with Denis. Craig joined us in the middle of a trigonometry session. We were looking at alterations to the sine curve as a result of phase shifting, and varying amplitude and frequency. Craig had never studied the sine graph mathematically, and had only seen it on an oscilloscope. However, owing to the highly successful session on quadratics from a few weeks prior to this, Craig was immediately able to predict the changes in the sine curve, given the initial plot of $f(x) = \sin(x)$. This pleased me greatly, as Craig was experiencing the "doing" of mathematics for the first time, i.e. the varying, altering, predicting, and comparing of mathematical entities.

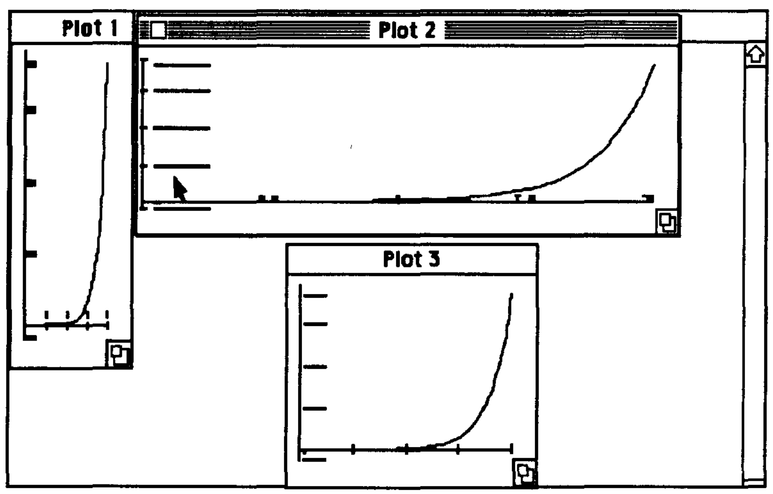
While it was not entirely clear that he actually understood the point-wise effect of varying parameters, it is safe to say that Maple helped him visualize globally how the graph of a particular function changes due to various alterations of parameters.

Joel Hillel:

Maria has not done any mathematics courses for nearly 30 years, though, at the time when she did high school mathematics, she enjoyed it and was reasonably good at it. She had some experience working on a computer terminal in her job as a travel agent.

She was looking at plots of the same function through two different 'windows' and she seemed puzzled. "How can this be [the same function], it looks so steep here and so flat here?" This was not the first time that issues related to scaling of axes and the choice of domain and range have come up. Yet Maria was not entirely convinced that the computer was not doing something wrong in going from one plot to the next.

I was about to bring up the issue of scaling again, but instead I thought of a different approach. I asked her to look at only one of the plots and to play around with 'dragging' the rectangular frame containing the graph. Maria, on her own, ended up boxing the plot within different rectangular frames and seeing how the 'steepness' of the graph was changing accordingly (see, plots 1-3 below).



It was apparent to her that Maple was not computing anything new: "It is just the same graph" she said, and the whole issue of scaling became all at once transparent to her.

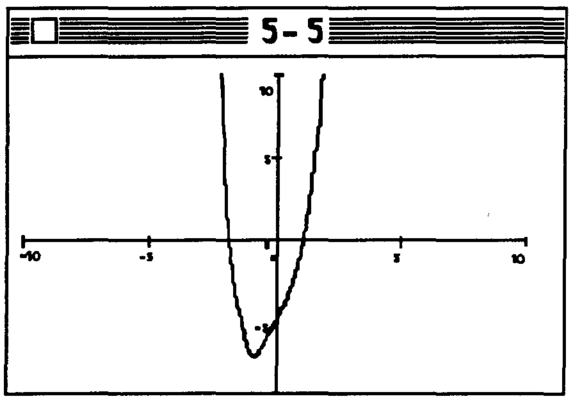
Lesley Lee:

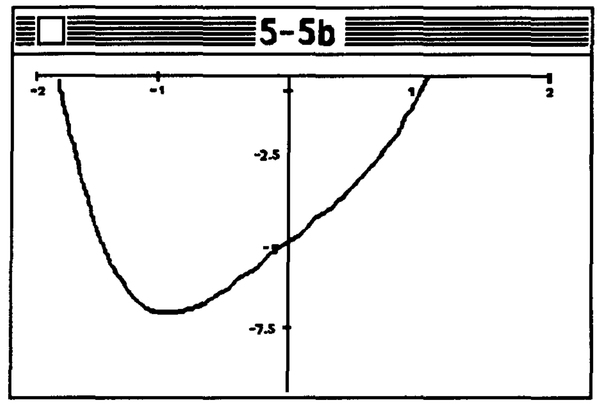
Maple sessions provided students with an opportunity to formulate hypotheses then test and refine them in a way that would be too tedious to do if hand plots had to be drawn. In an hour and a quarter session, for example, one student was able to plot fifteen graphs in order to refine his own theory concerning graphs of polynomials. We trace his progress, from his original hypothesis that all polynomial expressions of power n are parabolas if n is even, to several more refined hypotheses concerning even and odd powered polynomials and the effect of the odd powered terms on their graphs. The session illustrates how Maple can allow students to engage in a genuine mathematical activity.

Denis had established to his satisfaction that the graph of a quadratic expression is a parabola. At this point he formulated a generalization: "As long as it's an even exponent of x it will be a parabola. I know this from class". More formally, this could be expressed as:

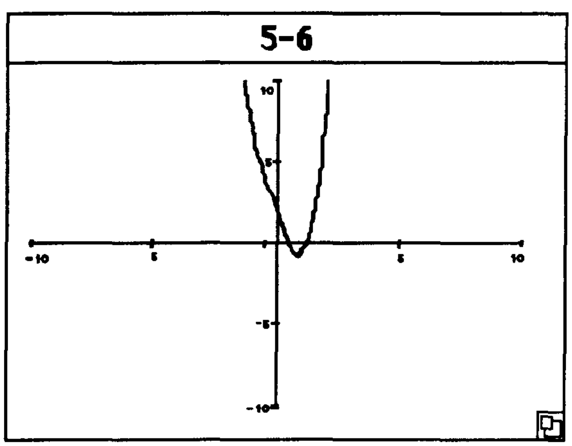
Hypothesis 1: **All polynomials of the form $y = ax^n + bx^{n-1} + \ldots$ are parabolas if n is even.**

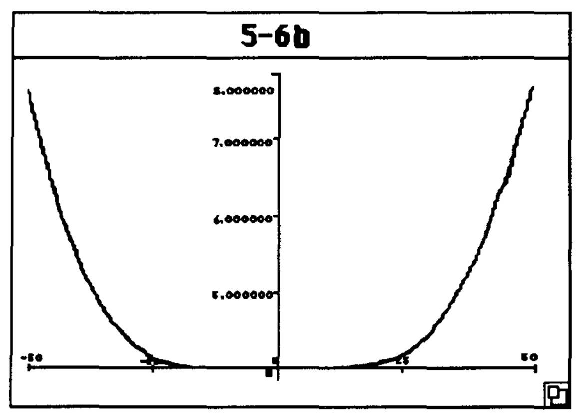
Since Denis had just graphed $x^2 + 3x - 5$, I suggest he change the 2 exponent to 4. He plots $x^4 + 3x - 5$ getting the rather unsymmetrical plot (plot P5-5 below). He then restrains the domain to [-2, 2] and the range to [-10, 0] and the somewhat smoother looking curve (P5-5b) strengthens his belief that he has a parabola.



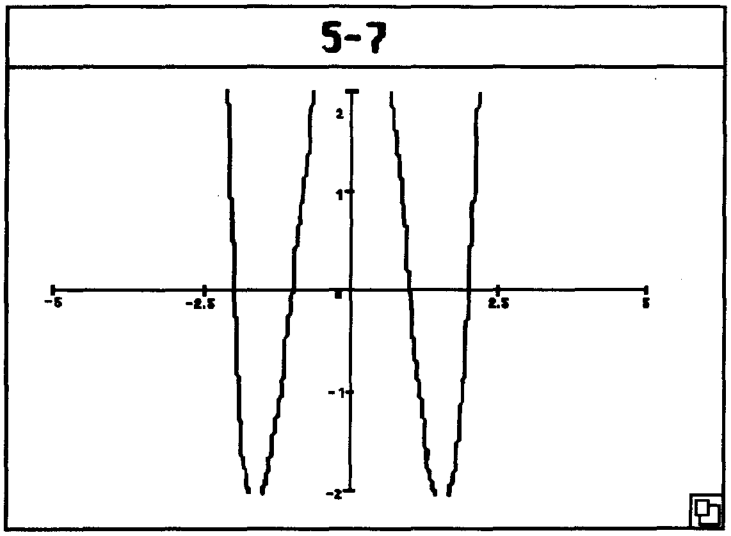


We make up another similar expression, $x^4 - 4x + 2$ (P5-6) which Denis also believes to be a parabola. I suggest we try $x^4 - 5x^2 + 4$ and ask him what he expects the graph to look like. He replies, "A parabola—at this scale it would look like one". And it does (see P5-6b).

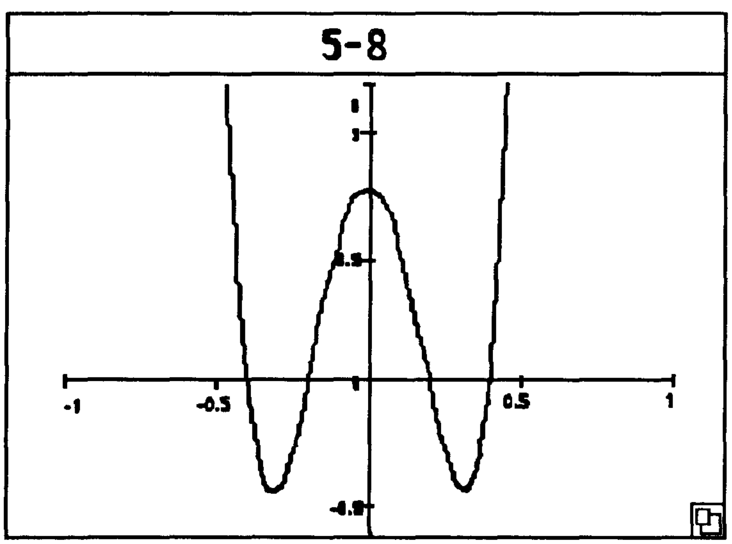




I suggest changing the domain to [-5, 5] and when asked if resulting plot (P5-7) is a parabola, Denis replies "Yes it's two".

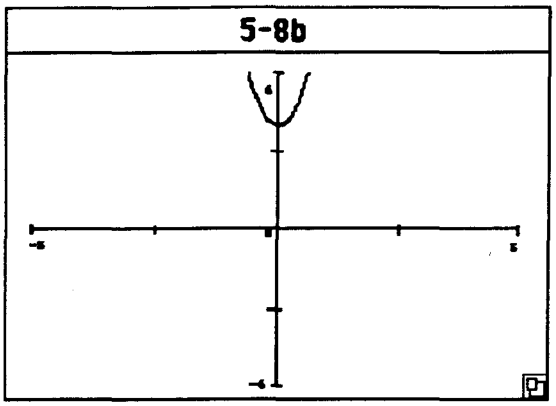


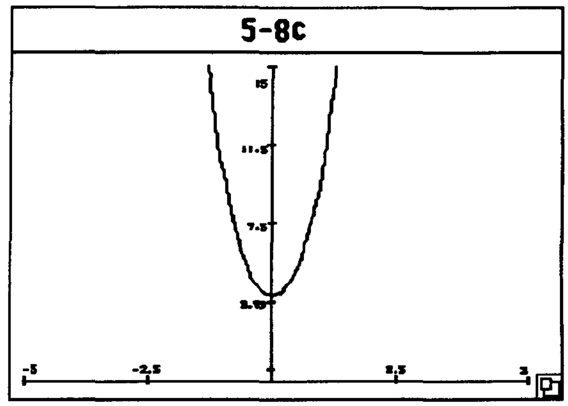
It is only when he tries a range of [-4, 6] that Denis realizes he does not have a parabola (P5-8). Denis focuses on the $-5x^2$ term and says "If we made it a positive it would go down". He may be thinking that this action will cause the centre section of the graph to flip downwards, resulting in a parabola. At this point he refines his theory by imposing the condition that all the terms be of the same sign: "If the power terms are all positive or all negative it's a parabola".



Hypothesis 2: **All polynomials of the form $y = ax^n + bx^{n-1} + \ldots$ are parabolas if n is even and a, b, ... are of the same sign.**

To demonstrate this Denis changes the $-5x^2$ term to $5x^2$ and plots $x^4 + 5x^2 + 4$ restricting the domain to [-5, 5] and the range to [-6, 6] "because it would be off the scale now". Getting P5-8b he decides he "went the wrong way" in his range restriction and tries [0, 15] (see P5-8c).

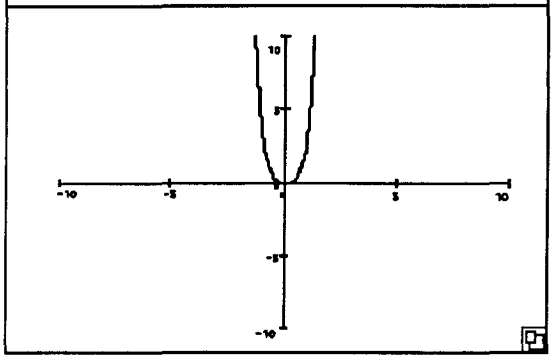




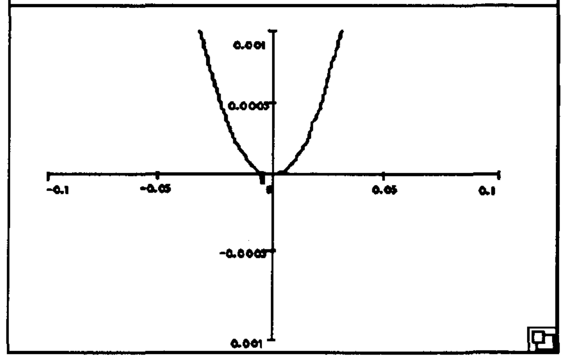
Seemingly satisfied with his latest hypothesis, Denis explores the effect of a negative term by plotting in rapid succession $x^8 - 3x^2$, $x^8 - 3x^2 - 5$ and $x^8 - 5x^4$.

He then tries the plot of $x^8 + x^4 + x^3$ which indicates that he is not sure whether all terms should have even exponents, or simply the highest. Here he develops a strategy which is to plot $x^6 + x^4 + x^2$ (P5-10)—"I want to try this and then introduce an odd power after to see if it messes up my theory. I might have to alter my theory a bit." At this point he repeats the plotting several times, decreasing the domain interval. Plot P5-14 is done with domain [-0.1, 0.1] and range of [-0.001, 0.001].

5-10

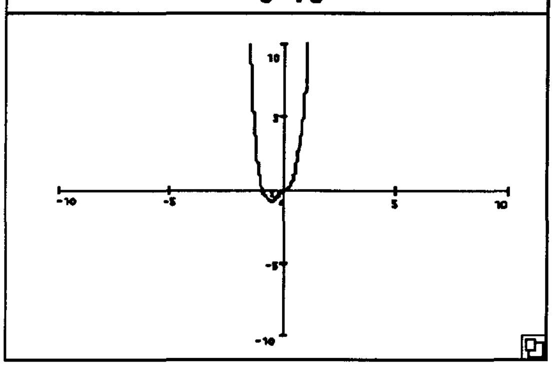


5-14

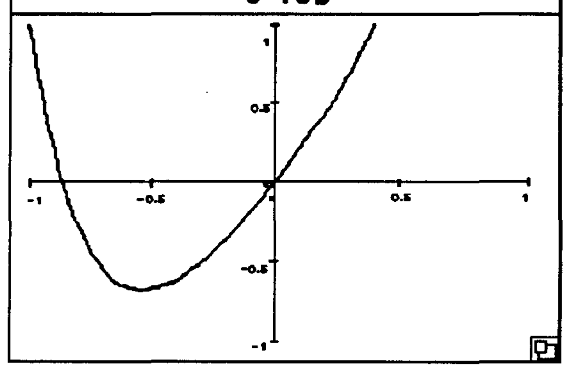


Convinced now that $x^6 + x^4 + x^2$ is indeed a parabola, Denis adds the odd-powered term, $2x$, to test his theory. He realizes that the graph of $x^6 + x^4 + x^2 + 2x$ is not a parabola (P5-15). To be sure he imposes a domain and range of [-1, 1] and concludes that it is definitely not a parabola. (P5-15b). He decides that the "odd powers" give you a "squiggle". He appears to have reformulated his hypothesis to impose evenness on all exponents.

5-15



5-15b



Hypothesis 3: **All polynomials of the form $y = ax^n + bx^{n-2} + \ldots$ are parabolas if n is even and a, b, ... are of the same sign.**

Here I remind him that his original expression $x^2 + 3^x + 2$ was a parabola in spite of the odd-powered term, $3x$. He reformulates his theory by excluding polynomials of degree 2 or less: "Even powered polynomials with terms of the same sign with the highest exponent greater than 2, are parabolas" and goes on to elaborate another theory: "Odd powered polynomials with terms of the same sign with the highest exponent 3 or 5 or more, are squiggly non-parabolas"

Hypothesis 4: **All polynomials of the form $y = ax^n + bx^{n-2} + \ldots$, with $n > 2$, are parabolas if n is even and $a, b, \ldots$ are of the same sign.**

Hypothesis 5: **All polynomials of the form $y = ax^n + bx^{n-2} + \ldots$, with $n \geq 3$, n odd and $a, b, \ldots$ are of the same sign are "squiggly non-parabolas".**

Satisfied with hypothesis 4, Denis now sets to work on testing his 5th hypothesis.

The session concludes here leaving the role of odd-powered terms slightly unsettled. Denis seems to be quite certain of the parabolic nature of even-powered, same-sign polynomials and associates the "squiggle" or distortion of these to the odd-powered terms. It is agreed that we will return to a study of polynomials at the following week's session. In time Denis refined his theory on the graphs of polynomials.

While this session with Denis did not lead to a complete clarification of the graphs of polynomials, it did initiate an important process of questioning beliefs about graphs which had been firmly, and erroneously, acquired in the classroom. Although, at first, Denis appeared to be led on in this investigation, once his hypothesis had been seriously challenged by unexpected graphs (in particular P5-8) he took over the exploration creating his own polynomials and adjusting his hypotheses according to the graphic evidence. The graphing capabilities of Maple allowed him to undertake this genuine mathematical investigation in a way that would have been tedious if not disheartening with the pencil and paper tools available to most students.

So you want to know what's wrong with Maple?

In the spirit of the contemporary Canadian trend of "Canadian Bashing", we take our turn with a few underhanded jabs at Maple.

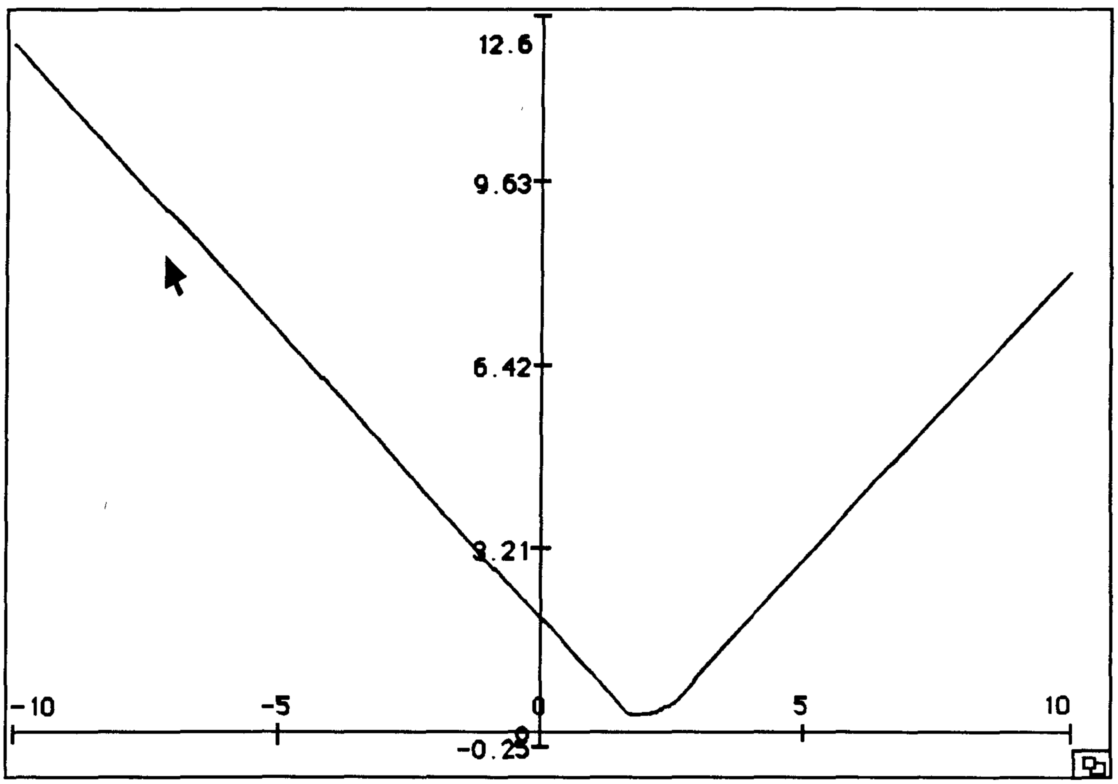
It should be said that each CAS has its Achilles' heel and one can always come up with the type of problems that push the system past its computational limits and produce very unusual responses. So a graph of a quartic polynomial can show a thousand roots (D. Tall's example), limiting processes suddenly start to diverge and solutions are either not found or are nearly incomprehensible as mathematical outputs. These are unavoidable limitations of the systems and one hopes that users will become aware of these and treat all responses of a system with a certain degree of suspicion. What concerns us here are certain things that Maple does poorly and which are avoidable.

Maple, as other CAS, was not designed as a piece of educational software (though, as all other CAS, it alludes to classroom use in its publicity). Moreover, unlike Mathematica, the graphing component (for the Macintosh 4.20 version) seemed to have been tacked on as a last moment afterthought. Unfortunately, it is the part that is most essential for any work in the functions-calculus-analysis sequence. Among the shortcomings (some of which have already been mentioned in the text) we include:

(i) The plot-screen, which is separate from the text/computation screen and unlike the latter, is completely non-interactive. Aside from being able to drag and re-size the plot-screen, one cannot label points, label plots, add to an existing plot, highlight, zoom, pick coordinates, add comments, etc. In other word, all the advantages of using a Macintosh are simply lost. Neither is it simple to integrate a plot into the text-screen.

(ii) Maple is very slow in plotting non-polynomial functions.

(iii) The example of the plot of $f(x) = |x - 2|$ (which came up in one student's work) encapsulates many other problems with Maple plots. It is plotted over the default domain (the interval [-10, 10]).



We note that:

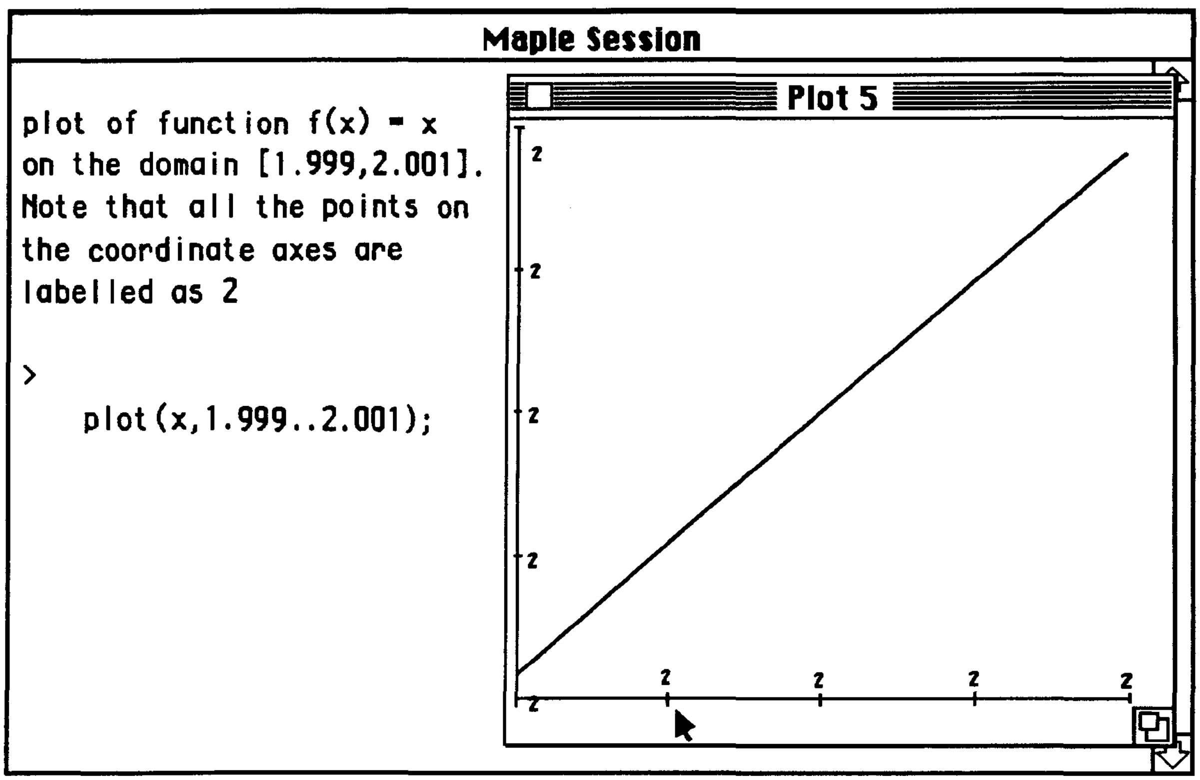
- It is a very poor plot of the function f. It leads one easily to believe that f is strictly positive and differentiable.
- The range chosen by Maple is the interval [-0.25, 12.6].
- The labelling of points on the y-axis is bizarre—the points are neither convenient landmarks nor do they correspond to some special functional values. In fact, the points 3.21, 6.42, and 9.63 correspond, for some reason, to ¼, ½ and ¾ of 12.85 which is the length of the domain interval.

- □ There is a cluster of values around the origin which is very confusing to read. In fact, on a closer look, it is not even certain whether the x-axis is passing through the origin or just below (so maybe the function has a zero after all!)
- □ Even a more local look about the point $x = 2$ (say, the interval [1.7, 2.5]) doesn't show the illusive zero. One either needs to double the number of points plotted (default is 25) or to choose an interval for which 2 is the mid-point (in which case one already knows that $x = 2$ is a special point).

Other problems with Maple plots include:

- □ When plotting a set of functions, Maple doesn't necessarily plot them in the order in which they are listed.
- □ There is no allowance to label points on the x-axis in terms of p when plotting trig functions.

- □ On intervals of the form $[a\text{-}d,\ a+d]$ where d is even as small as 10^{-3}, Maple will label every point on the x-axis as a. For example, as in the plot of $f(x) = x$ on [1.999, 2.001]:



Of course, some of these are just temporary setbacks. The proverbial "new version" is already in the making, which will have a much better graphical package (including 3-D plots) and likely to solve most of the above-mentioned problems. (Expected date for the Macintosh version is Fall

1990.) There are also indications that specialized and friendly 'user interfaces' will become available.

The I-W-S-A question and some concluding remarks

At last year's meeting, Eric Muller raised the following question: When is it Impossible, Wrong, Stupid, Appropriate to use CAS?

This is not as straightforward a question as it seems (and maybe it was not meant as a straightforward question). For one thing, it depends on who the user is and on the kind of knowledge of CAS and mathematics that s/he possesses (here, D. Tall's distinction among 'external', 'analogue' and 'specific' insights is useful). It depends on the intended pedagogical use of the CAS, whether it is to foster calculation, conceptual shift, mathematisation or cognitive support, and on what other instructional supports accompany their use. It depends on whether one is considering present versions of CAS or future ones.

We might say, for example, that studying functions via their CAS generated plots is simply wrong—that there are too many things that could go astray and which lead a naïve user to false conclusions or generalizations, and that the mathematical issues underlying the difficulties are more complex than the concept of functions and graphs. On the other hand, the ability to examine the graphs of many functions and to focus on different graphical 'windows' each of which hides and reveals some features of the its underlying function, may be construed as an appropriate use of CAS. While the issues that come into play are more complex than simple paper-and-pencil graphing, one can argue that they are more accessible to students since they relate to students' realm of experience.

Bernard Hodgson has commented several years ago (Hodgson 1987) that the effectiveness of CAS in instruction cannot be evaluated as long as they remain an extra activity rather than being fully integrated into the curriculum (with all of the changes that are logically implied by such integration). In a sense, our evaluation of Maple suffers precisely from our having used it as an extra activity. While we realised that this would be a problem, we felt that it would have been too big a plunge to introduce major changes in a course without having a first-hand and close look at students working with Maple. We emerged out of this initial experience cautious, uncertain about the I-W-S-A question but willing to venture into a more integrated use (with an eye, of course, to that "improved updated version"). For the student group that we have targeted, the end result of working with CAS may not be better mathematics but more engaging and personal rapport with mathematics. That by itself would already be an achievement.

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Tall D. O. & Winkelman B. (1986). Hidden algorithms in the drawing of discontinuous functions, *Bulletin of the I.M.A.*, 24.

Hodgson B. R. (1987). Symbolic and Numeric Computation; the computer as a tool in mathematics, in Johnson D. C. & Lovis F. (ed) *Informatics and the teaching of mathematics*, North-Holland.

Topic Group C

The SFU joint (Mathematics/Education) Master's Programme

Harvey Gerber

Simon Fraser University

Rationale

The Mathematics and Statistics Department and the Faculty of Education at Simon Fraser University have a long history of close cooperation. For example, Tom O'Shea and I meet frequently to discuss a course, *Mathematics for Elementary School Teachers*, which the Mathematics and Statistics Department offers. This course is required for entrance into the Professional Development Programme of the Faculty of Education. One of our major concerns is in-service programmes for mathematics secondary school teachers.

Our discussions with secondary school mathematics teachers convinced us that there was a need for a different kind of Mathematics/Education Masters Programme for secondary school teachers. The traditional programme consists almost exclusively of education courses. While the traditional programme is suitable for teachers who want to become administrators, or who want to be made more aware of the current research and thinking in pedagogy, it does not satisfy the teacher's desire to learn more about mathematics and to work with like-minded colleagues on more challenging mathematics concepts and problems. The Mathematics/Education Masters programme is designed for teachers who are teaching during the day. Therefore, all courses in the programme meet once a week from 4:30 - 8:30 pm for thirteen weeks. The students remain together throughout the two-year programme.

The Department of Mathematics and the Faculty of Education at Simon Fraser University in order to satisfy the wishes of the teachers, has devised a joint programme stressing the human aspects of both education and mathematics. The programme consists of three mathematics and three education courses. Two of the education courses are *Foundations of Mathematics Education,* an examination of the historical, cultural, and psychological forces shaping the secondary school curriculum, and *Teaching and Learning Mathematics*, the theory and practice of mathematics teaching at the secondary school level with an emphasis on the nature of the learner and the function of the teacher. The third education course is elected from one of the existing graduate courses in the Faculty of Education.

The three mathematics courses are *Geometry*, *Mathematical Modelling*, and *Foundations of Mathematics*. We will discuss this last mathematics course in detail because it gives a flavour of the entire venture.

To repeat, we want the Masters' programme to stress the human aspects of mathematics. Although many of the teachers have taken some graduate courses in mathematics, they do not seem to want, or in fact feel that they would benefit from, advanced or graduate level courses like topology or the theory of rings. Our intent is to show mathematics in the making rather than as a finished product. We are not only interested in the subject, but also in the way it evolved and the reasons for its evolution. The first course we developed, *Foundations of Mathematics*, looks at various areas of the secondary school curriculum (including calculus) from a historical, and sometimes philosophical point of view. The emphasis is on the mathematical problems at a certain moments in history, and how these problems were resolved.

Structure of Foundations of Mathematics

The grade in the Foundations of Mathematics course is based on one paper which the student also present to the seminar. The students are required to submit an outline of their paper, along with a bibliography, by the fourth week. By the end of the eighth week they submit a first draft of their paper, written as if it were a final draft. The comments on the paper are mostly general, with detailed comments for a handful of pages to give the student a concrete sense of what is required. We believe that detailed comments for the entire paper mean that the final version would just be a corrected version based on the comments. The final draft is due on the last day of class. The grade in this course is based on one major paper and one classroom presentation, There are no written examinations.

Because of the primacy of the student paper, we created guidelines to lead student on the right track.

Guide for writing the term paper

1) By the fourth week of class students submit an outline of the paper which should include)a a statement of the problem
   b) the approach in solving the problem, and
   c) a bibliography. The outline should be about one page (two pages maximum).

2) By the eighth week students submit the first draft. This draft should be written as if it were the final draft.

3) The final draft is submitted in the last week. The paper should be about 20 pages. The classroom presentation will on the same topic as the paper.

Term Papers

The following list of titles for term papers in the first offering of this course illustrate the scope of the students' work.

1. Classical Problems and Field Theory (Bisection of the angle and duplicating the square)

2. History of the influences leading to the development of Analytic Geometry

3. The Background on Cantor's work on Set Theory (Work on trigonometric series leading to transfinite numbers)

4. Riemann and the Foundations of Geometry

5. A description of some of the contributions of Cauchy and Weierstrass to Mathematical Analysis

6. The mathematical antecedents of Newton's Calculus

7. Georg Cantor and the Transfinite

8. Negative and Complex Numbers

9. Views on infinity leading to Cantor's theory of transfinite numbers

10. The development of the function concept

11. The development of the normal curve (1600 - 1850) in the history of Statistics

12. The development of algebra as a deductive science

13. The historical development of Non-Euclidean Geometry

14. Ptolemy's Trigonometry.

Conclusion

Out of the first class of fifteen students, fourteen have completed all the course work. It is clear that several areas of the programme are in need of improvement. For example, in *Foundations of Mathematics* we need to introduce material that the teachers can easily present in class and use in their own secondary school classes that captures the theme of the course, One possible textbook for this is *Journey Through Genius - The Great Theorems of Mathematics* by William Dunham, John Wiley & Sons, 1990.

The problems we experience with the programme are relatively minor and, we feel, can be easily resolved as the programme develops over the years.

Ad Hoc Group A

The Development of Student Understanding of Functions

Steve Monk

University of Washington

I am going to talk today about my investigations into the ways in which students understand the concept of function and into the possible paths by which this understanding might develop during high school and the early college years. For reasons that will emerge, I will focus exclusively on functions represented by graphs.

Section 1. The Table/Pictorial Approach.

I will start with an example of a pair of graphs (see Figure 1) I used while interviewing beginning college students in 1975. These students were registered in—or had recently completed—a course in calculus for social science and business majors that colleagues and I had written material for. These graphs appeared on page 1 of that material.

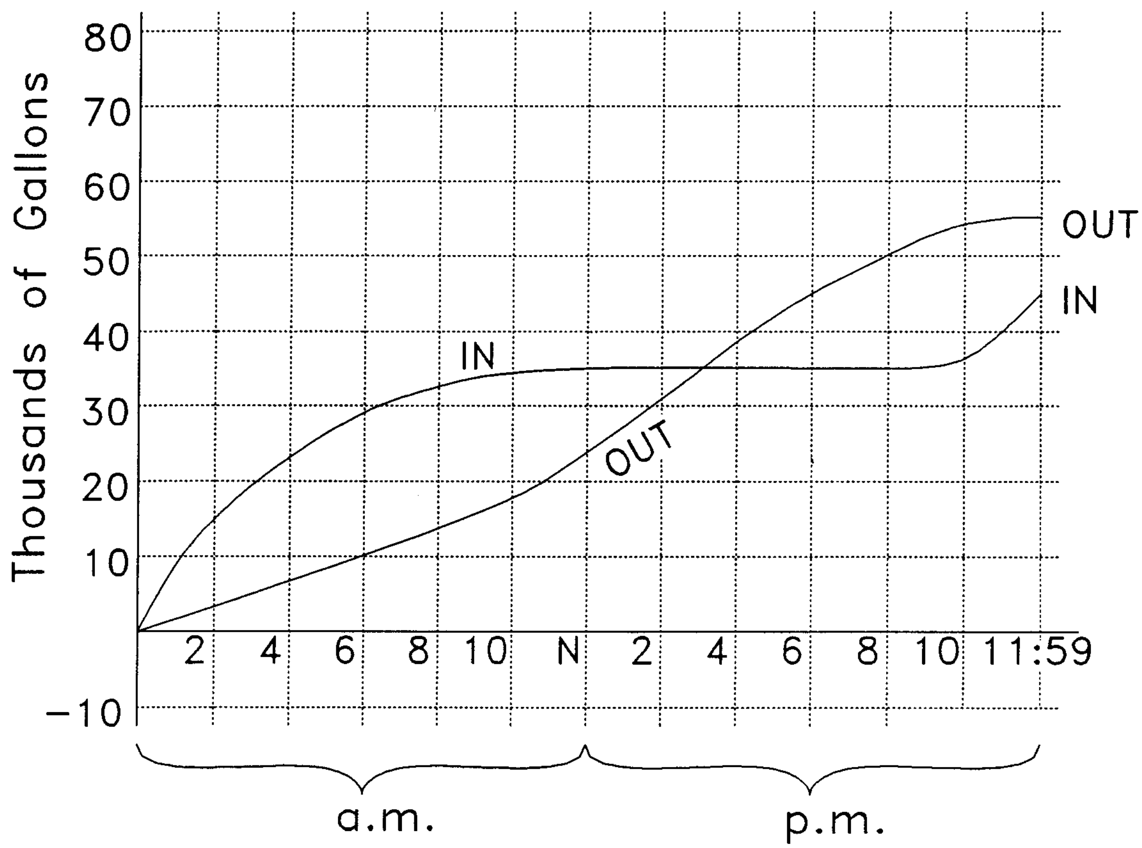


Figure 1

In the class material and in the interview, the student reads the following description of the context which these graphs refer to:

"Gauges are attached to the inflow and outflow pipes of a reservoir, measuring the amount of water that has flowed in and out of the reservoir. The gauges are like turnstiles—they add up how much water has flowed in and out since midnight. They do *not* measure how much water is *in* the reservoir. The graphs below show the amount of water that has flowed in and out of the reservoir over a 24 hour period."

When my colleagues (on this study) and I interviewed these students, we asked the standard questions like:

a. Suppose we assume that there is 10,000 gallons of water in the reservoir at midnight. How much water is in the reservoir at 4 a.m.?

As one might expect, we found that students could generally answer this question, since it requires only that they read the In graph and the Out graph at 4 a.m. and do some arithmetic with the numbers: 10,000 gallons of water was present, 24,000 gallons of water came in and 8,000 gallons went out. However, we were surprised to discover that they had a much more difficult time with such questions as:

b. What is the change in the amount of water in the reservoir from 4 a.m. to 8 a.m.?

For me, as a mathematics teacher, it seemed obvious that in order to answer this more complex question, the students would start to use the vertical distance between the two graphs as an indicator of something like the net change in the amount of water. (Certainly the students who had completed the course would!) But what we saw is a continuation of the same sort of the naive table reading that the students used on the above question a. These students read the In graph at 4 a.m. and 8 a.m. They read the Out graph at 4 a.m. and 8 a.m. They wrote down the four readings they obtained from the graph and combined them by arithmetic to arrive at the answer that an additional 8,000 gallons had come in over this time and an additional 6,000 gallons had gone out. Some of them then concluded that 2,000 gallons more had come than gone out—that (in our terms) the net increase in the amount of water in this time is 2,000 gallons.

We never saw a student spontaneously use the vertical distance between the graphs as any kind of indicator of a quantity of water at the reservoir. Moreover, questions that I would now describe as quite leading, never elicited this notion from these students. In retrospect, I wonder why a student would use this notion here. Perhaps, some do see the vertical distance between the graphs, but even they would probably not use it here, because they don't need it here. (Students are, after all, practical creatures.) But there are situations when the use of this construct (of vertical distance) would seem to be natural and called for. Thus, we asked such questions as:

c. Give a time when the amount of water in the reservoir has increased by 20,000 gallons.

Most experienced graph readers would note that 20,000 gallons corresponds to two vertical units (or "boxes") and they would therefore look for a time at which the vertical distance between the graphs is two units. This would require that they be able to use the vertical distance between the graphs as an indicator and that they be able to translate or relate it to the net change in the amount of water in the reservoir.

However, most of the students did this problem by a kind of "Guess-and-Check" method. They started at 2 a.m., read directly (and recorded) from the graphs that 4,000 gallons had come in and 14,000 gallons had gone out, so that there was a net increase (not their term) of 10,000 gallons. They then went on to 4 a.m., and through the same manual procedure, determined that this time is not the answer because the net increase had only been 18,000 gallons. And, of course, they came upon the answer at the next try.

What we found in these studies is that students strongly tend to use graphs as if they were tables in which one could read the amount of water (In or Out) for each time and do arithmetic on these amounts. There was little or no evidence among these students, of what one might call *Chunking*—of the bits of information given by this "table" into new constructs. They did not form or use what I will call *Concepts in Graphs*. Related to this, of course, is that there was little or no use of the various derived quantities or variables that we might see and use in this situation—such as Net Increase or Net Change in the amount of water over various time intervals.

But, we asked other types of questions about these graphs and these indicated another difference between what you or I might see in this graph and what students see. We also asked such questions as:

> d. Imagine that no water flowed out of the reservoir over the 24-hour period shown, and that the amount of water that has flowed in at any given time is given by the In graph shown. Using the information on the In graph, tell how the water level at the reservoir will change from noon to 6 p.m.

Typically, students first said that 34,000 gallons had come in at noon, and that 34,000 gallons had come in at, say 4 p.m., and the same was true of 6 p.m. They would then begin to conclude that no water had come in over this time. But this conflicted, for many, with their strong belief that, "If no water had come in over this time, then the graph would have to be down on the x-axis, which it is not. They would then conclude that 34,000 gallons was *coming into* the reservoir over this time, so that the level was steadily rising. In effect, they had changed the meaning of the vertical axis from *amount* that has flowed into the reservoir over some time period to *rate* of flow into the reservoir at a given time.

Clearly, there are several likely sources of this error. First, these students do not have anything like the clear, firm distinction we do between amounts and rates, especially in a context they have as little quantitative experience with as water in a reservoir. Moreover, the problem itself has so many different amounts that anyone could easily get confused.

But, the fact is that something prompts these students to begin to act as if the graphs were of rates and not amounts. Perhaps it doesn't take *much* prompting, but something in the graphs causes the switch. When you listen to students describing what graphs like this tell us, you begin to hear them strongly responding to the visual cues in the graphs—but in a rather global, pictorial fashion. They say such things as

> "If nothing is happening at the reservoir, then the graph should be down on the axis."

"If the In-graph is over the Out-graph, then the water level is rising." "When the graphs cross, then things are the same."

Or they refer to gross visual qualities of a graph like:

high	low	peaked	flat
cross	wiggly	going up	going down

Of course, the use of such features is part of anyone's interpretation of graphs, as is the reading of individual numerical values. The point is that students at this level are *restricted* to such features, they are not able to go beyond them. Thus, students approach a graph with two disparate—nearly irreconcilable—views:

> A graph is a *Table* in which one can look up individual values, one at a time, and do arithmetic with them.
>
> A graph is a *Picture* which tells rather directly what is happening in the phenomenon being described.

Not only are these two views inherently limited, but they also make difficult any attempt at synthesizing the numerical and visual aspects of graphs. For this reason, we find students alternating between these two views, using the graph as a table to respond to one question and using the graph as a picture to respond to another question.

In both the material used in that course and the questions put to students in these studies, we proceeded from an underlying assumption that these students were able to use graphs to interpret real phenomena, and were, perhaps, in the process of extending this understanding to functions represented by more abstract representation, such as formulas. These assumptions have repeatedly been shown to be false; most college students are not able to use graphs in any but the most naive fashion. But in my mind, this is not an issue of one representational scheme or another, but rather an issue of the state of their conceptual apparatus for describing complex patterns of change. I continue to use graphs to inquire about and describe students' understanding of functions, but I believe that many of the perspectives gained are more general than this particular representation.

Section 2. Families of Questions.

I have described a rather extreme naive conception of functions represented by graphs—in terms of a graph being a Table and/or a Picture. And I have suggested that, in contrast, those of us who are familiar with graphs have a much richer view, so that we can use graphs in more complex ways. I would like to briefly sketch the approach I take to describing in greater detail the *vast terrain* between these two idealized approaches to functions given by graphs—the naive Table/Picture approach and the approach of an "expert" or "experienced user" (ourselves) who can use graphs in more flexible and complex ways.

To do so, let us consider another pair of graphs (shown in Figure 2) that I have often used, of distance versus time for two cars over a 60-minute time period.

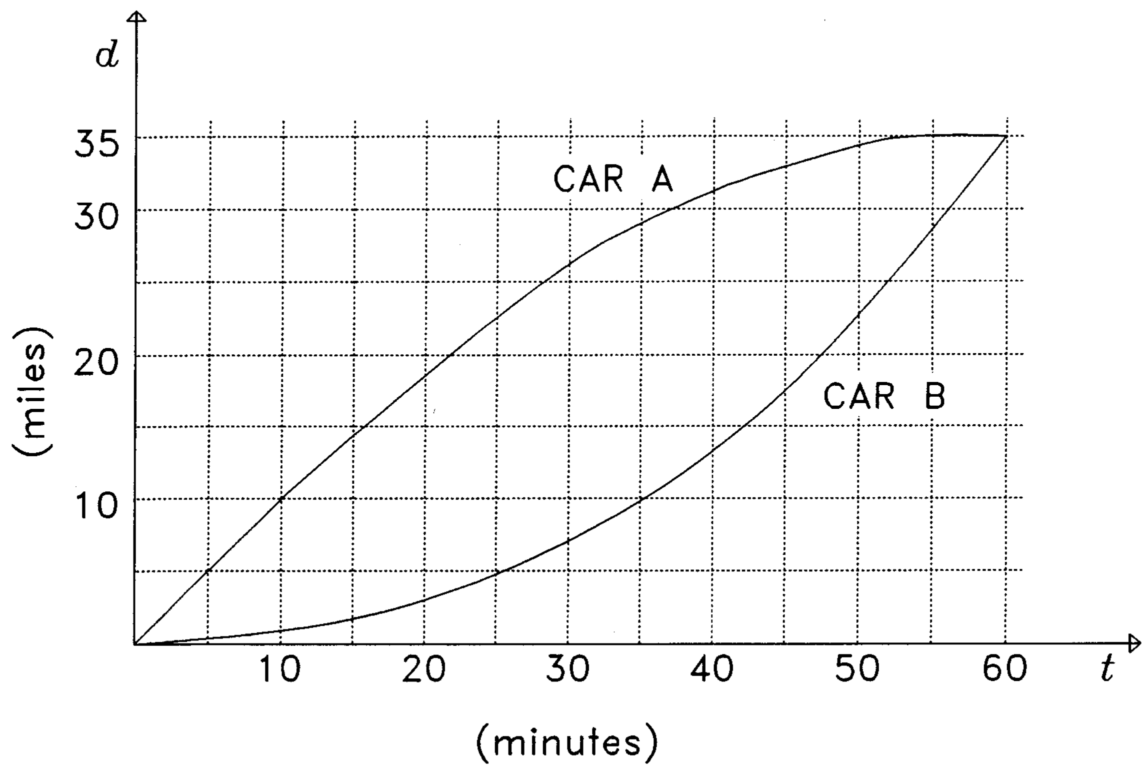


Figure 2

As with the reservoir, there are two basic quantities or variables in this context—the distances travelled by the two cars since the race began. But there are many other quantities or variables (I call them "derived quantities" or "derived variables") that the graph carries information about. And for just about all of these there are natural concepts in the graph that can be used as carriers of information about them. For instance, we can see, for each car:

Distance travelled since some fixed time
Distance remaining at any given time Distance travelled over 5-minute intervals ("displacements")

and, of course, we have

Distance between the cars at any given time

In addition to these distances, there are such rates of change for each car as:

Average speed from the beginning of the trip ("average trip speed")
Average speed over 5-minute intervals ("incremental speed")

In fact, each of these derived variables can be very clearly indicated on the graph by such constructs as:

Vertical distance between graphs Change in height Slope of "Diagonal lines" Slope of "Secant lines"

The way in which I think about the diverse approaches to (or understandings, or conceptions of) functions represented in any manner is in terms of the ease, flexibility, and robustness of an individual's use of the various derived quantities in the context—and this person's use of various concepts in the representation to do this. But, in order to talk about whether or not an individual has a certain capacity, there must be a clearly delineated specific *Task* for that person to perform in which it would be to appropriate to employ that capacity. For me, "understanding" means being able to use concepts in ways that are more or less powerful. But then these uses must be specified. To do this in the case of functions, I have described six general families of *Questions* that can be put to students with reference to the various derived quantities in a context. (These families of questions and their rationale have been described in my paper "A Framework for Describing Student Understanding of Functions", delivered at AERA, March 1989.) The (somewhat descriptive) names of these families of questions are:

FORWARD	COMPARISON	BACKWARD
ACROSS-TIME	ARTICULATION	MULT REPRESENTATION

Thus, for instance, we could focus on the Derived Quantity, "Distance covered by Car A in 5-minute intervals," which is represented by the "Change in the height of the graph over 5-minute intervals." Then we would have:

FORWARD QUESTION: Give the Distance covered by Car A in the 5-minute interval starting at 20 minutes

COMPARISON QUESTION: Which of the following two distances is greater: The Distance covered by Car A in the 5-minute interval starting at 15 minutes OR The Distance covered by Car A in the 5-minute interval starting at 35 minutes?

BACKWARD QUESTION: Give a time when the Distance covered by Car A in the 5-minute interval starting with that time is 9 miles.

ACROSS-TIME QUESTION: Tell whether or not the Distance covered by Car A in 5-minute intervals gets bigger or smaller in the period from 20 to 40 minutes.

ARTICULATION QUESTION: Give a time period over which the Distance covered by Car A in 5-minute intervals decreases while the distance from the start increases.

MULTIPLE REPRESENTATION QUESTION: Which of the following graphs is closest to the graph of Distance covered by Car A in the 5-minute interval for the entire trip shown?

These families originally emerged from a careful study of the operations on functions that are implicitly and explicitly expected of students in college mathematics. But, in fact, I claim that they also apply to the uses functions are put to in high school—even middle school—mathematics courses. In addition, they can be used to describe the uses of functions one finds in subjects outside of mathematics as well as more practical, everyday uses. This is very important to me, since I am interested in how the ability to perform the tasks behind these questions might naturally evolve for anyone, regardless of whether or not this person will study mathematics in college.

One way in which I have used this scheme is for making distinctions between the strengths and weaknesses college calculus students have in using graphs. For instance, I published the results of a study of student responses to questions that were placed on exams in an engineering calculus class. ("Students' Understanding of Functions in Calculus Courses"). In these questions students were shown the graphs shown in Figures 3a, b, c, accompanied by the following questions:

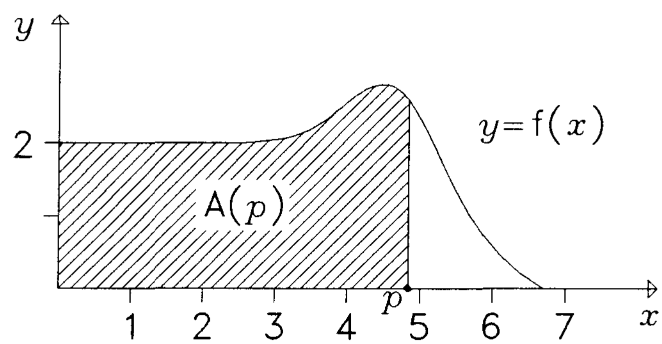


Figure 3a: Area Under the Graph

a. AREA UNDER THE GRAPH. Almost all students could answer the *Forward Question:*

Determine as closely as you can the values of A(l) and A(3).

But only 37 percent gave the correct answer to the *Across-Time Question*

Suppose the value of x increases from $x = 4.8$ to $x = 6.0$. Tell whether the function A(x) increases or decreases.

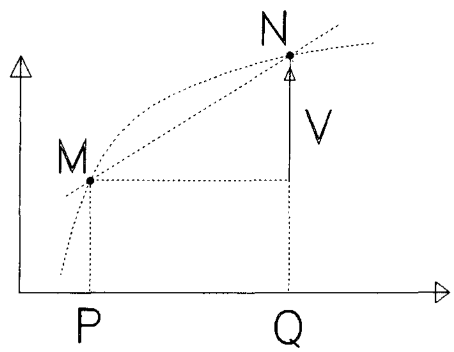


Figure 3b: Sliding Secant

b. SLIDING SECANT. Almost all these students could answer the Forward Question:

> Determine the slope of the line S and the value of the quantity v when M and N have coordinates (1, 6) and (4, 12) respectively.

But only 48 percent could give the correct answer to the *Across-Time Questions*:

> The point Q moves toward P. As this happens, does the slope of the line S (increase, decrease, stay the same)?
>
> As this happens, does the vertical change v indicated in the diagram (increase, decrease, stay the same)?

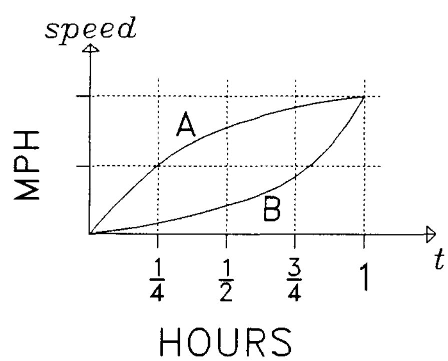


Figure 3c: Two Speed Graphs

c. TWO SPEED GRAPHS. Almost all the students could answer the *Comparison Question*:

> Tell whether or not Car B is going faster than Car A at time t = ¾.

But only 51 percent gave the correct answer to the *Across-Time Question*:

> Tell whether or not the cars are coming closer together in the time period t = ½ to t = 1 hour.

It is worth noting that, in spite of the fact that the students in this course are more advanced and mathematically able than the students in my 1975 study, they exhibit the same weakness in their use of graphs as the students in the earlier study. They can make numerical readings, as if the graph were a table, but when the question is difficult, or invites them to do so, they alternate this view with one in which they see the graph in an overly Pictorial manner.

Section 3. Reading Shape in Graphs.

In this final section I would like to describe some recent studies of mine that involve 9th and 10th grade high school students. The Two Car Graphs I showed before (Figure 2 above) were from a written test that I gave to an Honours 10th grade mathematics class at a suburban high school.

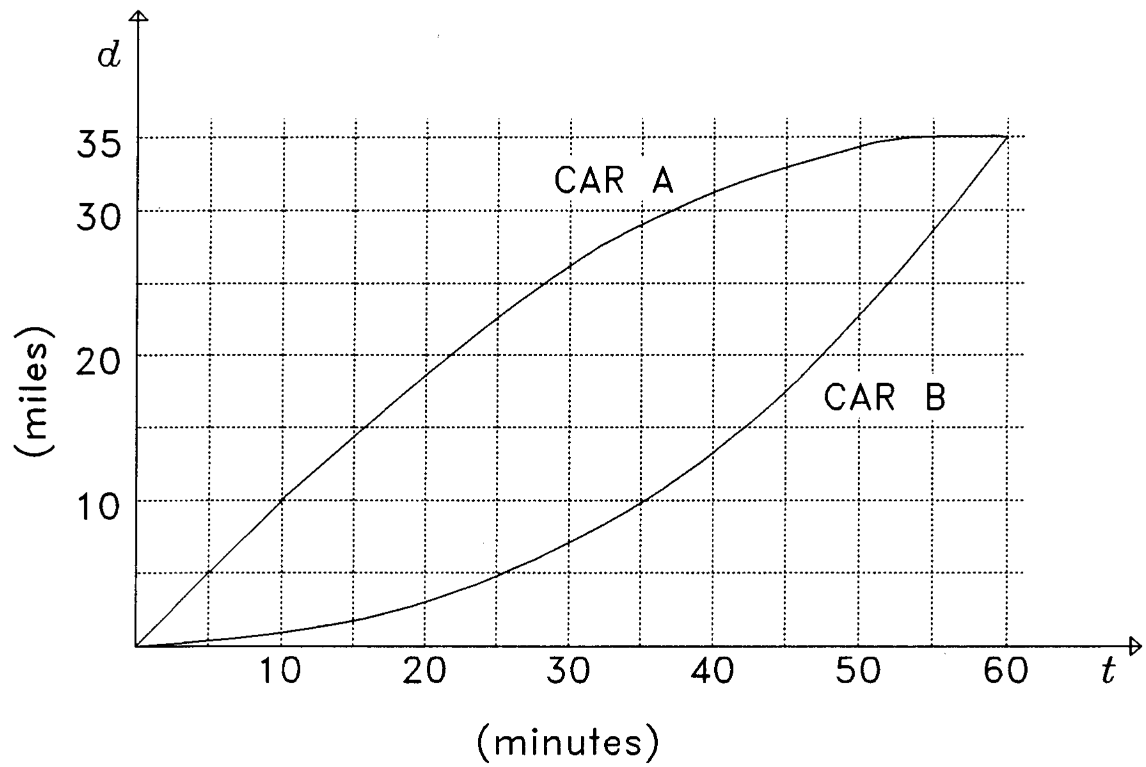


Figure 2

I found that these students could answer the Backward Question: Give a time when the cars are 15 miles apart, but, when asked to tell how they got the answer, many (but not all) of these students, in effect, said that they had done the problem by Guess-and-Check—that they had started at some (randomly chosen) time, read Car A's distance from the start and Car B's distance from the start, subtracted and then checked to see if this was 15 miles. This indicates that these students used the graph as if it were a Table.

I also found that the majority of these students also tended to take a Pictorial view of the graph under certain circumstances. In response to the question: Tell whether or not the following statements are true or false:

> Car A is slowing down over the entire time interval 20 to 35 minutes.
>
> Car B is speeding up over the entire time interval 20 to 35 minutes.

these students said that the first statement is false and the second is true. Thus, they believe that Car A was speeding up, presumably because Car A's graph goes up.

Even though these students gave responses from the predictable Table/Picture approach, the results of this study were not all negative. There were many students who responded to the Backwards Question by using the vertical distance between the graphs as an indicator of the distance between the cars, and there were many who saw that Car A is slowing down. Thus, I have been encouraged that these questions, at these levels of difficulty, are appropriate to study the ways in which younger students do come to understand functions—at least those represented by graphs.

More recently I devised a series of questions to explore how younger students deal with questions of speed on a graph of distance vs. time. It seemed to me that in order to overcome a rigid Pictorial view of graphs, students would have to develop more articulate ways of interpreting the shape of a graph. In the case of a distance vs. time graph, this means coming to have some interpretation of shape in terms of a more or less adequate notion of speed. But most teachers have told me, quite confidently, that students have no idea about how to read speed from a distance vs. time graph. But this could mean several different things: Is it the case that students do not make any speed statements, or is it the case that they make statements that come from conceptions of speed different from ours? With this and other questions in mind, I recently interviewed average 9th graders at this same suburban high school using the graph in Figure 4.

The "story" and questions were as follows:

> Sally drives a racing car along a track for 60 minutes. As the car goes along, a device keeps a record of how far Sally has gone since she started. The graph below gives the distance Sally has gone for each time. For instance, it says that at 25 minutes Sally has travelled 20 miles since she started, and at 45 minutes she has travelled 54 miles since the beginning.

Question 1. How far has Sally gone in the first 15 minutes?

Question 2. Indicate which of the following two distances is bigger?

> The distance Sally travels in the time period 25 minutes to 30 minutes.
> The distance Sally travels in the time period 40 minutes to 45 minutes.

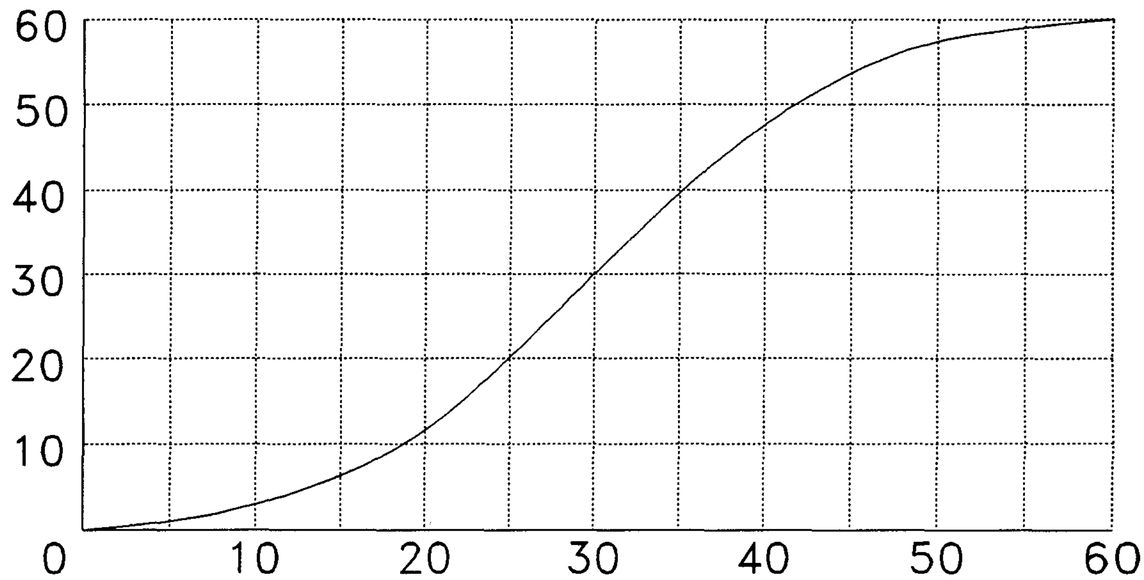


Figure 4

Question 3. a) Give a time interval that is 5 minutes long during which Sally travels exactly 8 miles.

b) Give a another time interval *different* from the one you just gave that is 5 minutes long during which Sally also travels exactly 8 miles.

Question 4. Answer the following question based on the information you are given. If you think there is not enough information to answer this question, please say so.

Is Sally travelling faster at 35 minutes or 45 minutes?

Question 5. As Sally drives along, the distances she covers in the 5-minute intervals vary. Give the best description you can of the pattern of these "distances covered over 5-minute intervals," from the beginning to the end of Sally's trip. Use such terms as:

Gets bigger   Gets smaller   Stays the same

In terms of the scheme I described in Section 2:

Question 1 is a Forward Question on the derived quantity Distance covered. It is asked only to settle the students into the situation and to get them to figure out what the story and the graph tell—before they deal with more demanding questions.

Question 2 is a Comparison Question on Distance Covered over 5-minute intervals. Before I asked more complex questions in this derived quantity, I wanted to find out if they could use it in the most straightforward way.

Question 3 is a Backwards Question on the derived quantity Distance Covered over 5-minute intervals.

Question 4 is a Comparison Question on whatever notion of "speed" these students have.

Question 5 is an Across-Time Question on the derived quantity Distance Covered over 5-minute intervals.

I interviewed six students on this set of questions. All but one student used the graph shown in Figure 4, but, as part of this on-going study, I have also made certain that the information presented on the graph can be displayed in other forms, such as the usual table of times and associated distances, as well as such representations as a bar plot, and a graph with the points at 5-minute intervals clearly marked. In addition, in order to explore the possibility that the things students think about distance/time graphs are primarily artifacts of this one context, I have devised an alternate context in which the variable of distance covered is replaced by the variable of the number of people that have come into an arena over a one-hour period. All of the questions I ask about distance vs time can readily be transformed into this alternate context.

My prediction of how these students would do on these questions, based on what teachers had told me—and on what I see college students able to do—was as follows:

Almost all students would do very well on Questions 1 and 2;

Most students would ultimately answer question 3, but do so by Guess-and-Check;

Few students would attempt question 4 and most would get bogged down in question 5.

I am (mostly) happy to report that I was quite wrong. These students did, overall, quite well at these questions, with significant weaknesses. In a way, one could say that these questions are right at the frontier of what these students can do. I will describe what I saw.

The Backwards Question.

Question 3. a) Give a time interval that is 5 minutes long during which Sally travels exactly 8 miles.

b) Give a another time interval *different* from the one you just gave that is 5 minutes long during which Sally also travels exactly 8 miles.

All the students but one immediately understood what was being asked for in this question. It took this one student a few minutes to realize that he was being asked for the distance covered over a 5-minute time period and not the distance covered since the beginning of Sally's trip. But these students differed in the techniques they used to answer the question. Four of these students quickly chose a strategy for finding 8 miles on the vertical grid as being just under 10 miles, which is a "box" or a "space between the lines." The other two had more cumbersome ways of representing a vertical distance of 8 miles and, not surprisingly, tended toward a Guess-and-Check method. For instance, one seemed to look for a place where the graph crossed a horizontal line and then he would count how many points below this horizontal line the graph crossed the vertical line to the left and how many points above this horizontal line the graph crossed to the right. Then he added these two numbers, checking to see if they added to 8. The other student used several different strategies, at one time he used a variant of Guess-and-Check, going to some (randomly chosen) point on the curve where it crosses a vertical line and reading the height, then going to the next vertical line to the right, and then subtracting. At another time he used the change in the height of the graph.

The Across-Time Question.

Question 5. As Sally drives along, the distances she covers in the 5-minute intervals vary. Give the best description you can of the pattern of these "distances covered over 5-minute intervals," from the beginning to the end of Sally's trip. Use such terms as:

Gets bigger Gets smaller Stays the same

Again, I was surprised that all but one of the students had good comprehension of what was being asked. One student could not understand the question at all, even though she was able to answer the Backward Question on this same derived quantity. She kept hearing that the question was about some other kind of distance not the same as the distance given by the graph reading, but she thought that I was asking about what happens after the time period she was shown—as if I wanted her to make a prediction of what would happen next.

Two students answered the question by making tables of the distances covered over 5-minute time intervals—in spite of my attempts to get them to predict what the pattern of the numbers would be before they completed the table. I am not sure how to interpret such answers. Perhaps they are the result of cautiousness on the part of the student, but perhaps the student does not trust his or her own constructs on the graph sufficiently to give the answer directly. My sense is that one can distinguish between a student who makes a table and reads the numbers to give a pattern and a student who gives a pattern directly from mental constructs made on the graph.

One of these students who made the table described himself as making a table of distance covered over 5-minute intervals as he proceeded to make a table of the distances covered since the beginning of the trip—i.e. to simply copy the information given by the readings of height of the graph. However, when I asked him, after a while, what these numbers represented, he quickly corrected himself and wrote down the distances covered for each 5-minute interval. The other three students all tried to read the answer to the question directly from the graph—by looking at something like the "jumps," i.e. the changes in height of the graph over 5-minute intervals. But all made more or less persistent errors—the result, I think, of their feeling a strong *pull* toward saying that these distances over 5-minute intervals get bigger, which is a response to the graph as a Picture. This is another example of how, even though these students seem beyond an initial Table/Pictorial approach, they relapse back to it, when under pressure.

As an example of this, one student responded by saying "I think that the distances covered in 5 minutes would be bigger and bigger ... (hesitates) ... I can't keep these things straight..." He changed his mind twice about his answer to this question, but kept on puzzling over it, wanting to say that these distances got bigger, but feeling uncomfortable about this. Even after I had thanked him for the interview, he went back to the question and finally gave a resolved answer that these distances get small "at the end," meaning that it happens only at the end, and not over the last 20 minutes. Another student was confident of her answer that the distances covered get smaller, but continued to return to the view that at the very end they get bigger—as if, only at the end does the "upness" of the graph finally assert itself.

An overall observation I would make about these students working on these two problems (Questions 3 and 5) is that they make remarks and comments along the way that indicate that students do look at both the visual and the numerical information on a graph and do expect these to support one another. Sometimes when I have seen students misinterpret shape in a resolved way, I have wondered if they look at the numerical information at all, when the visual information is so compelling. Apparently they do look at both and move back and forth and this is a basis for optimism.

One of the conclusions I arrived at about the weaknesses college calculus students have in using graphs is that they find Across-Time Questions much more difficult than questions that can be answered by looking at one point at a time. I wondered if answering Across-Time Questions is always a difficulty or whether the way in which students respond to these questions depends primarily on the situation. For this reason, it is interesting to note that question 5 is an Across-Time Question. To be sure, this question is more difficult to answer than questions 1, 2, and 3, but these students do seem on the way to mastering it. Thus, understanding functions probably does not consist of some generalized skills like "Making Across-time Readings," or "Answering Backwards Questions." It is more likely the case that each of these capacities is acquired in one context for certain derived quantities, and then in another context for other derived quantities, etc. All of these issues seem to me very contextual: What concept in a graph is being considered; How is it realized in the context; and How well is the context understood?

Comparison Question on "Speed".

Question 4. Answer the following question based on the information you are given. If you think there is not enough information to answer this question, please say so.

Is Sally travelling faster at 35 minutes or 45 minutes?

Their answers to this question are by far the most interesting to me. One student was given a Bar Plot and she alone was not willing to make any speed statements at all—except at the very end. All the others had rich ideas about "speed." All seemed certain that speed and "steepness" were connected. The issue then is, how do they connect them; is this done through a rather primitive intuition that speed and steepness are the same, or do they also relate speed and steepness to other aspects of the graph and context, such as distance covered and change in the height of the graph? It is perhaps significant that when asked questions about the speed at one time or another, all of these students tended to go back to the beginning of Sally's trip and give an overall play-by-play description of Sally's speed, as if speed is the result of an overall pattern and not a "local" property. Probably, this results from their having primarily a qualitative and not quantitative view of these concepts, speed and steepness.

The student who seemed to me to have the most complete answer to this question first gives a play-by-play description of Sally's speed from the beginning of the graph. But when she reaches the middle section, from 25 to 45 minutes, she keys on the fact that the graph is straight there, so that Sally's speed must be "consistent." But then, when the graph begins to "slope downward," she starts to use what I call "distance/time pairs," and says:

> "But then she seems like she's going slower here [at beginning.] Then she keeps it. Then she picks it up. Then she goes slow again. ... It takes her like 5 minutes to go 10 miles. And 5 minutes to go 10 miles—and then it's like—well wait. Then it's 5 minutes to go 8 miles. Then it takes her like 8 minutes to go only about 4 miles. So you see she's dropping off here. She's like speeding up, then slowing down."

I should remark here that none of these students referred directly to rate of change of distance—or any single-number measure of speed. AH used these distance/time pairs, such as "she went 10 miles in 5 minutes," with greater or lesser effectiveness. This tends to confirm my view that, regardless of how we would like students to think about ratio, correspondences among pairs is the way they most readily use this concept.

A second student begins by using distance/time pairs, but he uses an elaborate argument that compares the distance/time pair for the period from 0 minutes to 35 minutes with the distance/time pair from 35 minutes to 45 minutes.

> "...at 35 she's gone about 40—she's going faster—In 10 minutes she doubles
> -adds 5 miles going—so she's going faster at 45. (I ask for explanation.)
> Because at 35 she's gone 40 miles, which is in 35 minutes—whereas you only add 10 minutes, and she's increased 15 miles to that—so she's going faster at 45—because at 35 she's gone 40 miles, but at 45 she's gone 55 miles, and if you look at it, her speed—the distance she goes gets greater—by the time they hit
> 45."

But then, he quite suddenly breaks this off, and looks at the shape of the graph, and, then quickly integrates it with appropriate use of the distance/time pairs.

> "It's a *steep*—Okay—if you look—like—if you go every 10—for 45 from here he goes 15 miles—from 25 to 35 he goes 10 miles—so maybe—lemme see—(computes)—No, the graph's steeper here. I think it would be. I'm gonna change and say it's 35. (I: "What just happened, how did you—?) Because the graph—it looks steeper, it cuts through—two different blocks. This one goes up and kinda slopes off. (I: "So that tells you he's going faster at 35 than 45?") Because it's a constant steady uphill. Here it's kinda downhill."

Another student also tries to compare the distance/time pair for the period 0 minutes to 35 minutes with the distance/time pair for the period 35 to 45 minutes, but he is not able to move beyond this approach. His argument is that Sally is travelling faster at 45 minutes because she travelled 40 miles in 35 minutes, and then travelled an additional 14 miles in the next 10 minutes. This is 4 more than he expects in this time, and so she is going faster.

Finally, there is the student whose interview reveals some of the genuine intellectual work that students must undergo in order to be able to use these distance/time pairs. He is able to read distances and times, but he does not believe it is acceptable to compare the pair associated with one interval with the pair associated with another. For instance, he says:

> "At 35 she's gone 40 miles. At 45 she's gone 54. Let's see—[lots of computations.] I'd have to have like a basis where they are both equal—where I'd have to figure out—some kind of number that is equal for both and go from there. ... Like an equal number."

(In passing, it is worth noting that Medieval scholars had some of the same compunctions.) Then he and the interviewer agree that between 30 and 35 minutes Sally went 10 miles and between 35 and 40 minutes she went 8 miles. But then to the question of comparing her speeds, he responds by saying:

> "Well, see I have to make sure that this number and this number are in some ways similar—so that I can figure out—that they start out even—even though they are different numbers. (I: "So there's some kind of basis of comparison?") Right. So you can figure out which number has gone further—based on an even—thing. You can't just say 30 to 35, you know, 30 started off differently."

His concern is that these numbers can only be compared after one has complete information about the entire trip before the time interval in question.

> "But I'm just wondering on what happened down here [beginning part of graph]. If that could change just a little section...so that, you know, maybe the difference—it could have—you go at different rates—as you go farther—the car could slow down—the car could speed up—whatever—I just feel that you have to start somewhere where you *know* you're getting an even start between the two base numbers"

In the end, rather than being able to arrive at a positive conclusion, he seems to surrender to the messy realities—as if to say, in this world of uncertainty, one must be ready to compromise and make very rough guesses.

> "Well—you figure out you really can't get a good number, because there's no real pattern to it [probably means that it's not constant, but varies]—on this graph, so there's no real starting point for either number—so the best way to do it—an even way to start off is just go from 30 to 40 because if I went from—it's as reduced as you can get—because if I went from 35 to 45 and 25 to 35, then there's more places for error. So I just go with 10 and 8, I guess, because it looks good with the graph."

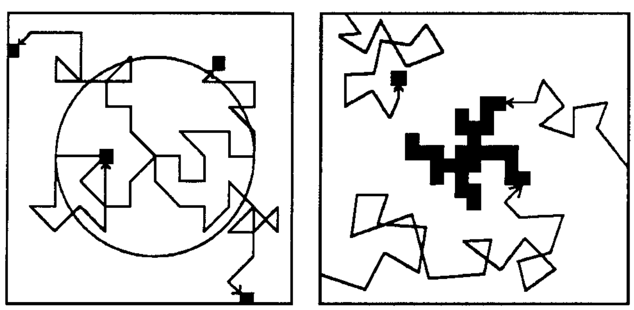
I find myself encouraged by these recent interviews, both as a researcher and a mathematics teacher. After years of cataloguing what students cannot do well and getting ever sharper indications of the dichotomous approach taken by students to graphs (as Tables and Pictures), I believe I now have evidence of things students do surprisingly well, which also provides clues to ways in which students naturally work to integrate or synthesize these two views. These students seem to me to be working at tying together the numerical and the visual information in a graph at the same time that they are working toward a more adequate description of the movement of a car. They don't have available to them an understanding of movement that can be "transferred" to graphs, and they don't have an understanding of graphs that can be applied to a description of movement. Coming to understand graphs is a process, I believe, that goes hand in hand with coming to understand several real contexts that graphs are good carriers of information about. This is a somewhat radical position for a mathematics teacher to assume, because it implies a view that learning mathematics separate from applications is extremely difficult and at the same time, not very valuable. But, at least this view seems to be in accord with the ways in which students' understanding actually develops.

Ad Hoc Group B

Fractalicious Structures and Probable Events

Brian Kaye

Laurentian University



Fractalicious Structures And Probable Events

1. Some Musings On Capricious Events and Clockwork Gods.

Evidence from recorded history and biological information shows that human animals have been equipped with brains identical to those of modern men for thousands and thousands of years. Long before they built their first ancient cities along the fertile crescent of Egypt and Mesopotamia man was capable of inventing Calculus and building spacecraft. However society remained intellectually stagnant for eons before civilization as we know it evolved from the fertile crescent of the middle east. Human brains failed to produce the thought systems we call science until relatively recent times. Depending on which authority you accept as setting the dates, modern science dates from the 15th century. It is approximately only 500 years old. Pioneer workers such as Copernicus (1473 - 1543), Kepler (1571 - 1630), Galileo (1564 - 1642), and Newton (1642 - 1727) developed what we know as the scientific method for studying the Universe. In this method one develops theories as to how the universe works and then devises experiments to test the validity of the theories. If the resultant experimental data is different from the expected pattern of information one modifies the theories and devises further experiments.

Why did the pioneers of the scientific method approach the universe with this "probe and discover" attitude when previous generations of brilliant men had failed to discover the amazing dimensions of the universe and the physical environment?

Scholars differ in their answer to this question but one answer is that the world had to wait for an intellectual monotheism (belief in one God) to develop before what we know as science could flourish. Premonotheistic society believed in a host of manipulative Gods hiding behind nature. These capricious Gods interfered with every day events in an unpredictable way. They were subject to emotions of anger, hate, love, and kindness and were therefore unpredictable. The word capricious comes from a Latin word for goat. Capricious behaviour means to leap about in an unpredictable manner like a goat playing on the hillside. If the behaviour of the universe was a pattern of events created by capricious Gods why bother to look for predictable behaviour. Cause and effect were not part of the "World-thought" system of the Greeks and Romans. In those days to have a happy life required a knowledge of how to make sacrifices to the Gods to keep them happy. Such sacrifices hopefully minimized the mischievous interference of the Gods in everyday life. As the monotheism of Christianity spread over the western world it was initially strongly linked with an authoritarian form of church government which still fostered the idea that God could be influenced by bribes (donations?) and whose modus operandi was set out by the priests of God. Their opinions were not open to discussion. With the coming of the reformation a few brave spirits put aside the theories of the established church and set out to discover for themselves how the universe really worked. Contrary to some popular misconceptions the scientific pioneers were not unreligious atheists. They were often men of deep religious faith who were motivated in their search for knowledge by a desire to know more of the nature of a supreme God who had created the universe and set it in motion. For example when Kepler started to develop his theory of the elliptical orbits of the planets he tells us that he was motivated in his search for a mathematical expression to describe planetary motion by the fact that his observation of the actual movements of the planets showed that there must be another form of orbit other than circular because, "God does not make bad circles."

The pioneers of the scientific method rejected the idea of angels pushing stars around the sky and struggled to understand the mechanics of the force interacting between the moon and the planets. They began to grope toward the grand theories of the conservation of energy and the laws of thermodynamics. As the implications of Newton's Laws of motion began to percolate into the philosophical thinking of the time the ideas of God evolved from that of grand fabricator and ever present organizer of the universe to that of the master clockmaker who had made, wound up and set the universe in motion. To some of those who began to understand the mechanisms of the universe God became a figure that watched its operation from a distance. To the scientist the mechanical unfolding of the universe day by day was the opposite of capricious. All was now foreordained. As our knowledge of the universe advanced man appeared to loose his freewill. From being a play thing of the capricious gods he became a cog in a master clock turning in a preordained pattern marking the passage of time. This mechanical view of the universe reached its peak in the ideas of Laplace (1749 - 1827) who wrote a book on the movement of the earth and the planets. It is said that when Napoleon the Emperor of France looked through Laplace's book he said,

"There is no mention of God."

Laplace is said to have replied,

"I had no need for that hypothesis."

In a few centuries man's search for order using the scientific method had created a mechanical view of the universe from which God, an apparently redundant hypothesis, had vanished.

2. Laplacian Determinism And A Gambling God

The theory that the state of the Universe tomorrow could be predicted from its present state by applying Newton's laws of motion to its mechanical behavioral patterns became known as Laplacian Determinism. After the conversation between Laplace and Napoleon, Laplacian Determinism was the dominant philosophy of science for a hundred or more years. When I was a university student in the early 1950's it was still the assumed philosophy of my teachers even though Heisenberg's uncertainty principle, (put forward in the late 1920's) had caused stochastic flutters in the confidence of the scientific authorities, that all was predictable. (The word stochastic means "fluctuating by chance or in an unpredictable manner.") I was taught that the future behaviour of the universe was as predictable as the movement of colliding balls on a billiards (pool) table. A knowledge of today's configuration of the universe would enable us to predict tomorrow's configuration once we had a big enough computer to carry out the necessary calculations.

I used to think about Laplacian Determinism as I walked along the shore of my native East Yorkshire. I used to look at the swelling waves approaching the shore and think how marvellous it was that the waves were created by the predictable forces of the moon and the sun on the earth creating tides and the drag of the wind created by the rotation of the earth. I used to watch the predictable breaking of the wave creating the foam of surf as the surging wave crashed on the sand. The complexity of the surfs surge and the retreat of the water into the sand of the beach amazed me. I wondered what size of computer was necessary to predict in detail the complex structure of the retreating foam. It must be a very difficult calculation I thought but I still believed in Laplacian Determinism. I was prepared to wait for that ultimate computer program which would predict in fine detail the future behaviour of each filament of chaotic foam. Heisenberg put forward his uncertainty principle in 1927. In this principle it is stated that it is impossible to make an exact and simultaneous determination of both the position and the momentum of any body. The more exact one's determination of one quantity is the less exact is the other. Essentially this means that if one is studying an object one cannot know the velocity and the position without uncertainty. The more precisely one measures the velocity the less certain one is of the position. When it comes to the study of moving atoms constituting the universe this means that our apparently solid bodies are part of a stochastic soup swirling with uncertainty. In the words of Asimov, "Heisenberg's uncertainty principle had the effect of weakening the law of cause and effect and destroyed the purely deterministic philosophy of the universe."

When looking at a scientific principle such as Heisenberg's uncertainty principle one cannot refute it at a philosophical level and yet it seems nonsense to say that because of a swirling soup of uncertainty at the atomic level that we cannot know or predict behaviour at the 'large lump' level of study of the universe. By the 1950's a thoughtful scientist was caught in the middle of seemingly contradictory philosophies of science. On the one hand Laplacian Determinism made the individual feel like a powerless cog; on the other hand Heisenberg's uncertainty principle made one feel like a helpless cork being thrown around in the surf of waves of unpredictable events. My own nagging need to understand the meaning of self consciousness, and my unwillingness to regard my own behaviour as a stochastic pattern in a set of random events, led to a continuous mental struggle to attempt to reconcile the need to accept freewill for my own actions with the view that I was journeying either on a physically predetermined roller coaster of events called the physical universe, or swimming in a sea of uncertainty. In my own personal rejection of the view that Heisenberg's uncertainty principle threatened the entire causative structure of the universe I was greatly encouraged by the fact that the pre-eminent scientist of the century, Einstein rejected this stochastic philosophical view of the universe. When criticizing the gross extension of Heisenberg's uncertainty principle from the atomic level to the interaction of macroscopic (large scale) bodies Einstein is said to have made the comment, "God does not play dice with the universe".

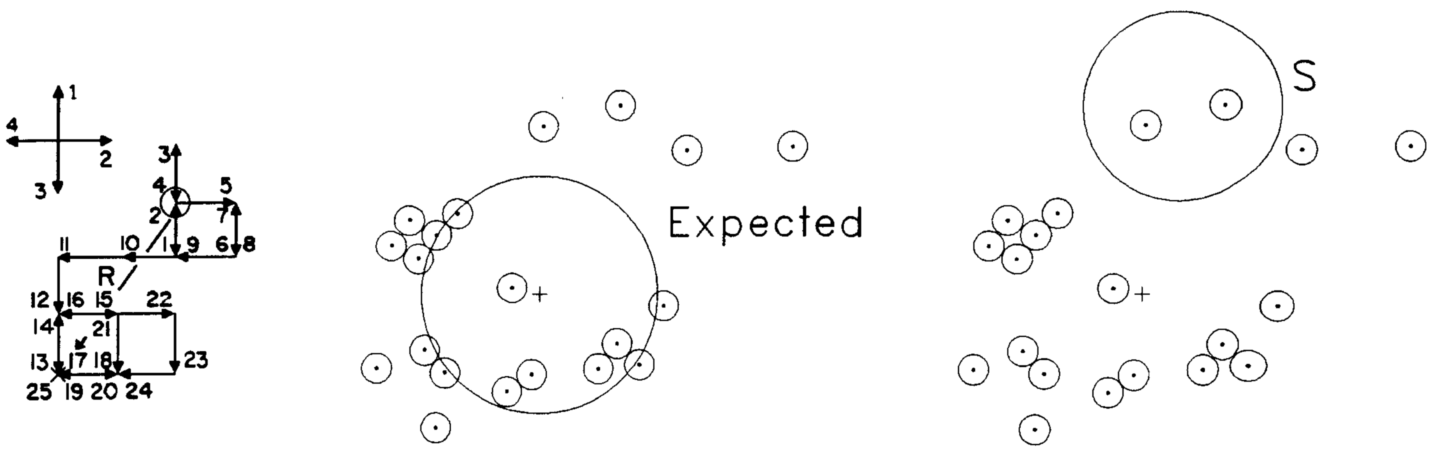
3. The Clash Between Probable Events And Laplacian Determinism

The quote from Einstein's philosophy of the universe given above conjures up the possible image that perhaps God if he exists decides the future of man's destiny by rolling celestial dice. If he really is omnipotent can he predict the outcome of every throw of the dice? This image recalls another historic threat to the theories of determinism posed by the evolution of the subject that we have come to call probability theory. The evolution of probability theory began at the gambling tables of the French aristocracy. The need to be able to determine equitable subdivision of monies left on the table when a game of chance had to be terminated prematurely became a fascinating study involving scientists of the calibre of Pascal, the great french mathematician and other thinkers such as DeMoivre. These and other mathematicians evolved several descriptive scientific relationships such as the Gaussian distribution (more popularly known as the Bell Curve), Log-normal distribution, and the Poisson distribution which have come to dominate much of modern experimental physics. It should be emphasized that these relationships should not properly be called scientific laws but descriptive relationships. One cannot use the relationships to predict the outcome of an individual event only the probable pattern of events for a given system. Furthermore the only real justification for using a descriptive statistical function is the discovery by experiment that a given pattern of events can be described by a given statistical relationship. The statistical relationships set out above are often known collectively as the laws of chance. Some scientists took the pattern of events describable by the laws of chance as further evidence that in our search for meaning in the universe we were treading dangerously on the surface of a stochastic swamp of probable rather than deterministic events. This viewpoint however is a mistaken one arising from loose thinking and a failure to be precise in the use of technical terms. In the late 1970's a new subject arose which has come to be described by the term Chaos theory. This term is somewhat unfortunate and arises from an erosion by popular use of the term 'deterministic chaos'. (When long words cross over from scientific usage to popular vocabulary they quickly erode to short forms as demonstrated by auto for automobile and T.V. for television.) To those who understand the origin of words, the term "deterministic chaos" seems to be a selfcontradictory term. The term chaos comes from the greek word meaning totally unstructured and disorganized. The term Deterministic Chaos arose in science when scientists began to realize that many multi-variate systems in the real world, although essentially deterministic, would remain forever unpredictable because of the extreme sensitivity of the final outcome of a system to the initial conditions. [ ] The fact that an essentially deterministic system can produce a bewildering array of possible outcomes appears to constitute the evolution of the use of the word chaos for summarizing the term deterministic chaos.

From the viewpoint of deterministic chaos the familiar probability relationships are not an abandonment of determinism but rather represent probable outcome of systems that lie within the domain of the subject of deterministic chaos. To illustrate this fact let us consider the problem of a coin being flipped. It is well known that if one is using an unbiased coin the probability of the coin coming to rest with heads or tails uppermost is a fifty-fifty chance. Therefore the study of probable outcome of flipping a coin lies within the domain of probability theory. However any one flip of the coin is a problem in deterministic mechanics with the outcome being very sensitive to many small uncertainties in the quantities used to predict the movement of the coin. If we multiplex this problem and consider the problem of what happens to twenty-five coins tumbled out of a container onto a table, in theory we should be able to predict the position of every coin every time from knowing the mass, position, and tumbling dynamics of the coin. However scientists have learned that from a practical point of view it is not useful to concentrate on the determinism of the individual coin movements and that in a real world we have to be satisfied with observing the probable patterns of events. It can be shown that the probable pattern of heads and tails in a set of coins repeatedly tumbled onto a surface follows the Gaussian distribution function. The observation of the pattern of the coins in this type of problem is a practical adaptation to the real world not a philosophical retreat from the theoretical possibility of determinism. When describing the pattern of coins, one has not abandoned determinism, one has accommodated ones ability to describe the system to reality rather than involving oneself in a hopeless tangle of deterministic calculations.

4. Deterministic Behaviour Of Diffusing Drunks?

To illustrate the relationship between chaotic determinism and probable patterns of events we will discuss a popular example given in the scientific textbooks of the diffusion pattern created by a multitude of drunks staggering away from a central lamppost. The basics of the problem can be appreciated from the systems shown in Figure 1. It is assumed that each drunk staggers away from his lamppost with equal probability of stepping in four directions as illustrated in figure 1.



(a) (b) (c)

Figure 1: Diffusing drunks create a pattern of events describable by probability functions.
(a) A typical 25 step random walk.
(b) Dispersion pattern of 20 drunks each taking 25 steps.
(c) A typical search area, (S) imposed on the dispersion set.

The four directions are allocated the digits 1, 2, 3, 4. The progress of the drunk is then modeled by selecting these four digits in a random sequence using a random number table with each digit determining the direction taken at each step by the drunk in a sequence of steps. We assume that we follow the progress of each drunk for twenty-five random steps. A typical path for a staggering drunk is shown in figure 1(a). One of the surprising facts that can be discovered in such a simulation study is that if we studied the progress of many drunks starting out from the lamppost then on average, provided the steps are of equal magnitude, L, the average distance reached by the drunks after **N** steps is:

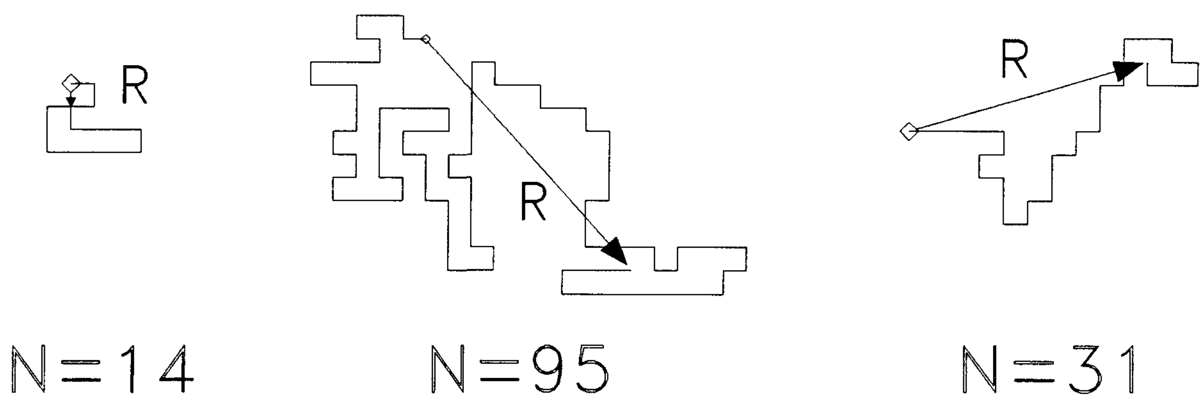
$$\mathbf{L}\sqrt{\mathbf{N}}$$

If we assume that the steps are of unit length, $\mathbf{L}=1$, then for a walk of twenty-five steps, $\mathbf{N}=25$ and the average distance becomes:

$$1\sqrt{25} = 5$$

In figure 1(b) the dispersal pattern of several drunks around the expected value of 5 is shown. Although the drunks are not completely in charge of their progress they most certainly, in a befuddled way, are exercising their free will and determinism as they take each step. Therefore the pattern of progress is the sum total of many causes (each step) interacting at random to produce a pattern of events. In this case it can be shown that, from a first order magnitude perspective, the distribution of distances of the drunks from their anticipated dispersal distance is a Gaussian distribution. In general it can be shown that if one were to put a small search area on the pattern of dispersed drunks, such as the circle "s" shown in Figure 1(c) that the fluctuation in the number per search circle as it is moved around the dispersal pattern is a Poisson distribution.

Another variation of the random walk dispersal of the drunks is the self avoiding random walk. In this type of walk we refuse to let the drunk cross his own path. A self avoiding random walk usually ends up in the drunk being trapped in a position from which there is no legal escape route. This is illustrated by the self avoiding random walks shown in Figure 2 (a), (b), and (c). It can be shown that the distribution of distance from the lamppost for the drunks



(a) (b) (c)

Figure 2: The magnitude distribution of Self-avoiding random walks can be described by the Log-Normal probability function. Shown are three walks that have experienced self-trapping.

undergoing self avoiding random walks, after they reach a position of being self trapped, is a log-Gaussian distribution. These simple examples illustrate how the probability patterns of the statistician are generated by systems properly described as deterministic chaos system with the determinism of individual systems being one level of reality remote from the probable patterns observed experimentally. This brief discussion of the dispersal pattern of drunks also illustrates the general truth that when a pattern of events is generated by the interaction of many causes, then the pattern of events generated by that interaction is describable by one of the several probability distribution functions discovered empirically by statisticians.

5. Deterministic Chaos And Fractal Geometry

Deterministic chaos burst upon the mathematical world in the late 1960's and early 1970's. Another mathematical revolution was created in the late 1970's by Benoit Mandelbrot. In 1977 Mandelbrot published a book entitled, "Fractals, Form, Chance, And Dimension". In this book Mandelbrot created a geometry of rough systems which he developed methods for characterising the rugged structure of natural systems such as river basins, craggy mountains, and fractured rocks. Mandelbrot pointed out that there is no absolute answer to questions such as "What is the length of the coast line of Great Britain?" "How many islands in a lake?", or "What is the surface area of a fractured rock?" Mandelbrot discussed the fact that one can only have an operational answer to questions of this form, such as :

"If I measure the coastline this way then my estimate in this value."

He demonstrated that, as one increased ones resolution of inspection of a coastline the estimate of its magnitude tended to infinity. This fact is illustrated by the data of Figure 3, in which estimates of the length of the coastline of Great Britain are made by striding around the coastline with various step sizes, l, of decreasing magnitude. The estimated perimeters are

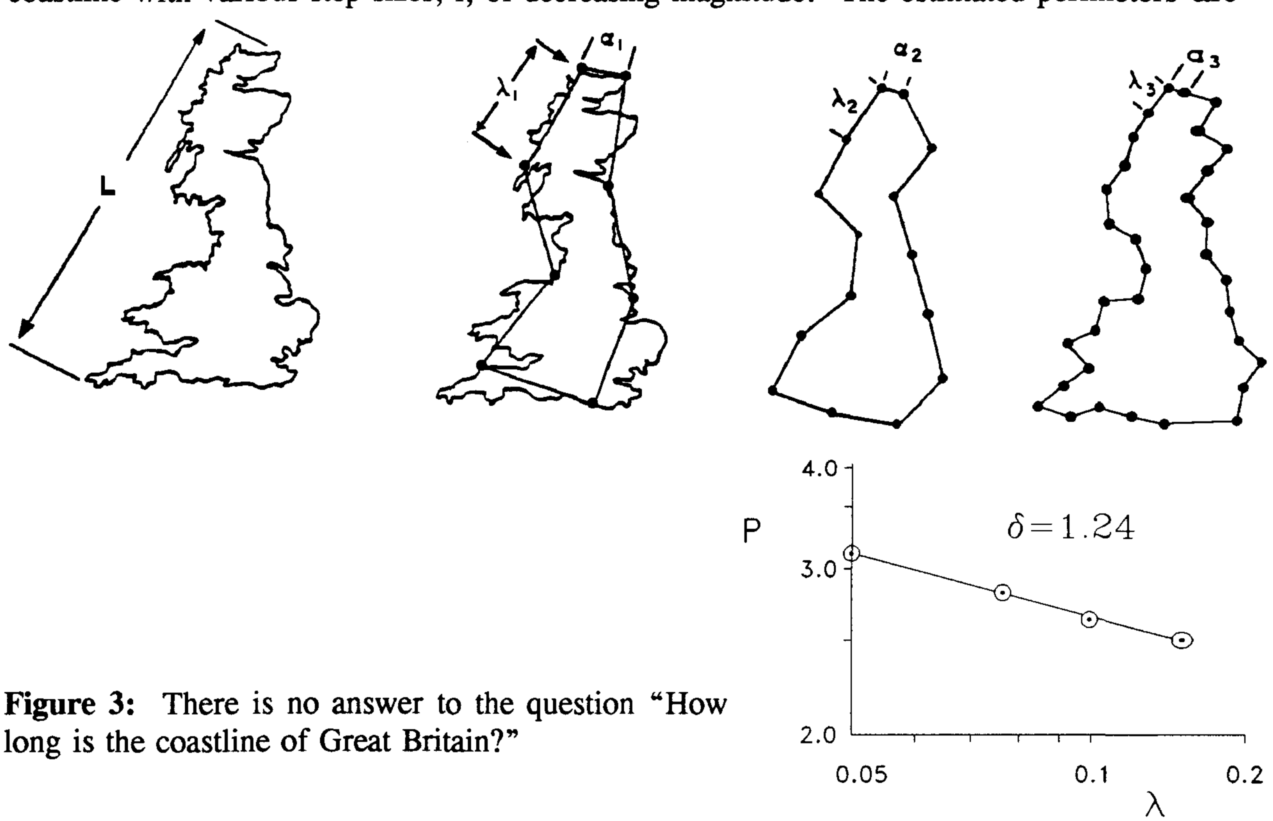


Figure 3: There is no answer to the question "How long is the coastline of Great Britain?"

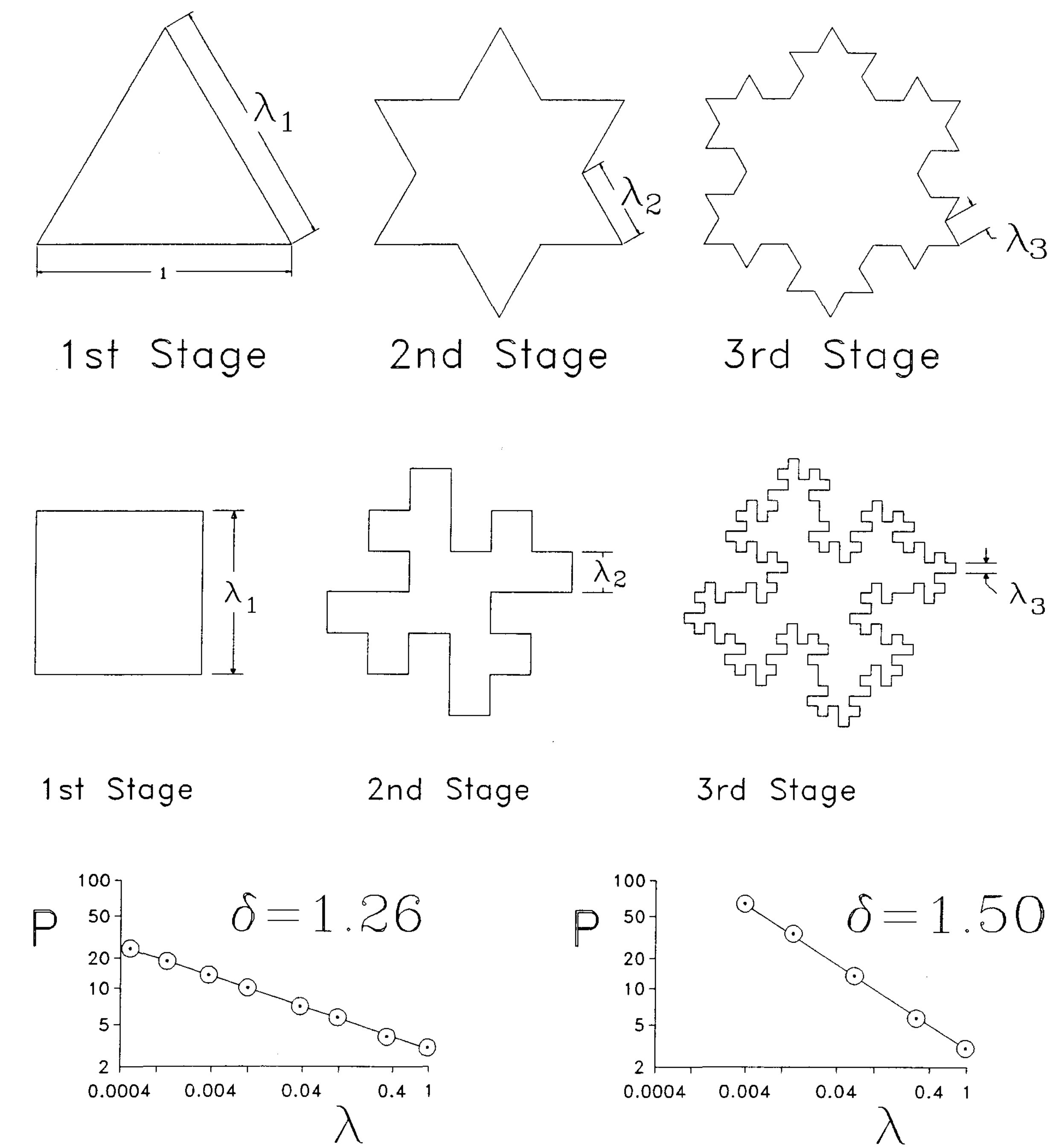
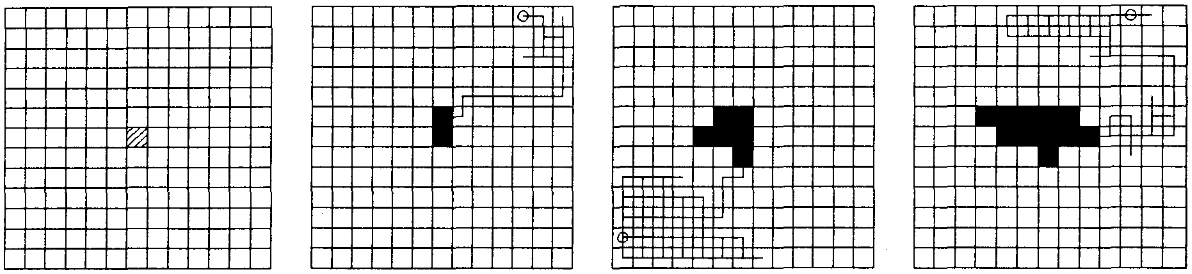


Figure 4: The ruggedness of the Koch Triadic and Koch Quadric islands can be described by their Fractal Dimension.

plotted on a log scale against the values of l, also plotted on a log scale. Richardson had demonstrated earlier that such estimates produced a straight line relationship on log-log graph paper. Richardson thought that his discovery was an ad hoc empirical discovery. Mandelbrot however has shown that Richardson's work was in fact the first discovery of a general relationship that can be used to describe many boundaries. In honour of Richardson's pioneering work the graph of estimated perimeter magnitudes versus inspection resolution data plotted on log-log graph paper is known as a Richardson plot. Mandelbrot showed that one could usefully describe the space filling ability of a boundary by studying the rate at which the estimates of the boundary's magnitude tended to infinity as the resolution of inspection was decreased. In other words the slope of the line on the Richardson plot was definitive of the ruggedness of the coastline. He illustrated this general fact with two theoretical curves known as the Koch Triadic, and The Koch Quadric island. In Figure 4(a) a construction algorithm sequence for both mathematical figures are shown. The increase in perimeter of these profiles with increasing complexity is shown in figure 4(b). Mandelbrot showed that the fractal dimensions of the boundaries can be described in a useful manner by adding the absolute value of the slope of the Richardson plot to the topological dimension of the profile. This combination of topological dimension plus fractal addendum is now generally known as the fractal dimension of the system. Thus the ruggedness of the Triadic island profile is 1.26 and the ruggedness of the Quadric profile is 1.50. The fractional part of these numbers can be deduced from the slope of the Richardson plot but can also be calculated for the Koch islands from theory with the fractal dimension being $\frac{\log 4}{\log 3}$ for the triadic island and $\frac{\log 8}{\log 4}$ for the quadric island. The fractal dimension is proving to be a powerful tool for describing important natural systems. Fractal dimension description of systems can exist in various dimensional spaces. Thus chords drawn on a line space can define a Cantorian set of points of fractal dimension less than one. A flat piece of paper has a fractal dimension of 2.00. As it is crumpled into a ball it acquires a fractal dimension of 2.X where X increases as the volume of the crumpled ball of paper decreases. The study of the fractal dimension of rugged structures is part of a subject known as Fractal Geometry. The structure of a rugged system describable by a fractal dimensions has obviously been produced by the interaction of many causes. Thus for instance a coastline is produced by the combination of wind and wave action along with other factors such as the structure of the rock being eroded, the activities of man near the rock and also geological forces such as lifting of a coastline and volcanic action. Therefore a study of the formation dynamics of fractal structures is obviously a branch of deterministic chaos.

6. Fractal Patterns Of Congregating Drunks

We can generate an interesting fractal structure by reversing the study of the behavioral pattern of dispersing drunks that we considered in the earlier part of this essay. If the diffusional pattern of the drunks was unique to the drinking human population it would appear to be a mathematical novelty however the essential dynamics of the dispersing drunks are to be found in many real systems such as the dispersal of smoke from a chimney stack by diffusion, or the dispersal in a solution of molecules of a given chemical species from a point source. Likewise the pattern of events generated by the drunks giving up their attempt to reach home and reversing their steps to find the central lamppost is a model of electrolytic deposition of material, the growth of fumes in turbulent flames and many other physical systems. To model the patterns generated by the



(a) (b) (c) (d)

Figure 5: The Drunkards Return results in the build up of an agglomerated structure which has a fractal dimension.

(a) The staggering space and nucleating centre.
(b) Path of the first pixel joining the growing agglomerate.
(c) Path of the fifth pixel joining the agglomerate.
(d) Path of the tenth pixel joining.

drunks seeking to return to the security of the lamppost we can use the system illustrated in Figure 5. The checkerboard squares covering the area of activity can be given numerical addresses. The point of entry to the square of the returning drunk is selected at random around the periphery of the activity area. The drunk can be represented by a small square (a pixel in computer jargon) which is allowed to travel at random to the centre of the square by choosing the address of adjacent pixels at random from a random number table. The pixel continues on its staggering track until it either butts onto a pixel which has already joined the lamppost or until it reaches the lamp post. In Figures 5(b), (c), (d), the tracks of the first, fifth, and tenth pixels are shown. It can already be seen from these early patterns that the accumulating pixels begin to branch out from the central lamp post (called the nucleus) to form a multi-branched structure. For historic reasons this type of growth is known as a Whitten and Sander fractal.

In Figure 6(a) a well developed Whitten and Sander aggregate is shown. In Figure 6(b), and (c), modified agglomerates in which various sticking rules have been applied to growth of the system are shown. Thus in figure 6(b) it is assumed that the pixel arriving at the growing cluster still has energy and therefore has a one in ten chance of joining the cluster. In other words it has a nine out of ten chance of moving away again until it again meets the cluster when again it will have a 1 in 10 chance of sticking. It can be seen from these three diagrams that the structure of the system depends upon the way in which the contributory causes are interacting. Further more it can be shown that the three different structures of figure 6 can be described by different mass fractal dimensions. It is well documented that the fractal dimension of a Whitten and Sander aggregate, formed by pixels approaching the growing structure with a one hundred percent probability of sticking for orthogonal encounter with the branches of the growing aggregate, is 1.7.

Our brief exploration of the dispersion of a set of drunks and the study of the reverse pattern formed by congregating drunks helps us to appreciate the significance of the following

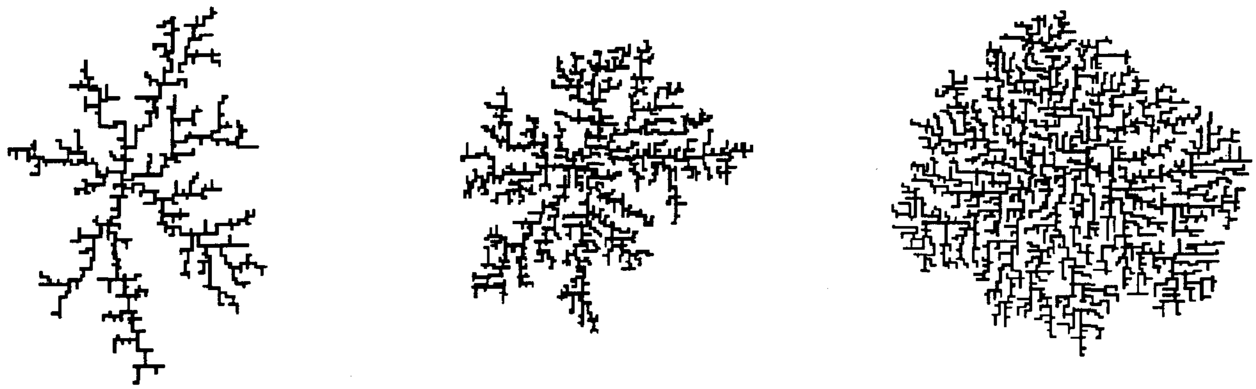


Figure 6: Typical agglomerates grown on a higher resolution grid than that of figure 5 with sticking probabilities 100%, 10%, 1%.

general dictum.

If the random interaction of several causes produces a pattern of events then the structure of that pattern will be describable by the various probability functions discovered by statisticians. If on the other hand the random interaction of the multiplexed causes creates a structure then that structure will have a fractal structure and which will be describable by a fractal dimension.

To further illustrate this dictum consider the picture of the rugged coast being pounded by the seething sea shown in Figure 7. The structure of the coastline was produced by the pounding of the sea and the effects of wind and rain and depends partly on the direction and power of the waves and also on the cracks in the structure of the rocks. At the forefront of the picture there is a leaping pattern of disintegrated foam. If one were to look at the size distribution of the droplets in this foam scientists know that the distribution of sizes of the droplets is probably describable by a stochastic relationship known as the Rosin-Rammler distribution. (An empirically discovered statistical function.) On the other hand the coastline itself is a fractal structure. From a study of the fascinating structures of fractal systems, a new word has been coined for the English language, "Fractal", which means "infinitely intricate" has been combined with Delicious to form the word "Fractalicious". A system which is fractalicious is something which is infinitely fascinating and pleasing to the mathematician. Hence the title of this essay. The coastline of Cornwall shown in figure 7 is a fractalicious structure and the leaping foam droplets are a pattern of probable events caused by deterministic chaos.

7. A Personal Postscript: Freedom Regained With A Limited God?

This essay started in a philosophical mood with the discussion of Laplacian determinism and Heisenberg's uncertainty principle. We discussed the ultimate apparent loss of freewill in a universe which was either a chaotic soup or a pre-determined clock work mechanism. The scientist of today as he contemplates the fractal structure of the coastline can meditate on the

Figure 7: Rugged natural coastlines are fractalicious structures. (Photograph of Land's End, Cornwall, England. Photo by Murray King, Images of Cornwall.)

essential unpredictability of the future pattern of events, unpredictable not because they are indeterministic but because even God may have difficulty in predicting the outcome of chaotic systems infinitely sensitive to small variations in beginning structures. Some scientists feel they can regain their freewill by sharing the universe with a self-limited god. Theologians may have difficulty with the idea that God may have limitations because they let their words govern their thinking. Once they have defined God as omnipotent then they are unwilling to look at whether they are describing God or insisting that God fits their definition. The specialist in deterministic chaos no longer expects even God to be able to predict the outcome of a complex system. Rather the chaos specialist is able to see that in such a universe the unpredictability of tomorrow gives the individual the freedom of choice as to how to react to the patterns of tomorrow. The scientist who chooses to be religious can now regain personal freedom by looking at the fractalicious structures of the universe and awaiting with fascinated anticipation the patterns of tomorrow knowing that tomorrow will bring the individual challenges to allow the individual to exercise personal freewill. Hiesenberg's uncertainty principle is just part of the challenging flux of patterns and Laplacian determinism is seen to be a limited simplistic theory now replaced by the complexities of deterministic chaos. As the subjects of fractal geometry, deterministic chaos and probability theory develop the scientists of tomorrow will wrestle meaningfully with complexity and will no longer believe that to solve all problems all that is needed is a bigger and better computer. Perhaps one of the important results of deterministic chaos will be to once again give importance and prestige to experimental studies of the universe.

Panel Discussion

The Future of Mathematical Curricula in Light of Technological Advances

Moderators

Bernard Hodgson, Université Laval, Québec
Eric Muller, Brock University, Ontario

Panellists

Harold Brochmann, North Vancouver School District, B.C.
Sandy Dawson, Simon Fraser University, Burnaby, B.C.
Gary Flewelling, Wellington Board of Education, Guelph, Ontario
Israel Weinzweig, University of Illinois, Chicago and O.I.S.E., Ontario

(The following provides a summary of the Panel presentation and discussion as seen by the moderators - it is not a verbatim report)

In the time that was available to them the panellists were asked to address the topic by concentrating on the following aspects:

Flewelling - problems of implementation
Brochmann - the present situation in schools
Dawson - teacher education
Weinzweig - exciting future directions.

Flewelling In many schools mathematics teachers were the first to get involved with the computer technology, but in general the population of mathematics teachers as a whole has been the last to embrace it. There are a number of reasons for this.

1. Many teachers focus on the technology itself rather than on its potential for the classroom and the student; for example, some teachers become computer hackers, some attend computer courses rather than mathematics courses, many develop computer literacy courses and these appear to have no impact on the mathematics classroom.

2. The teacher's perception of the technology can be a major obstacle to implementation; for example, some teachers believe that the technology can replace the teacher—half the class is sent to the computer laboratory where no teacher interaction is provided; others cannot perceive how to use a computer when it is available in the classroom. Generally, computers are lost to the teacher when they are located in laboratories and are not also available in the classroom.

3. Mathematics consultants leave the mathematics implementation to the computer consultants.

4. Perhaps the biggest obstacle to implementation is the teacher's perception of what mathematics is all about. If the teachers perceive mathematics as something which you have to master before you use it, they will find it very difficult to use large software packages where a student can explore, generalize, etc... On a more positive note it appears that the most conservative group, the mathematics teachers, are almost ready to start implementing some computer software into the mathematics classroom.

Brochmann In B.C. computers have been introduced as tools, the primary use being word processors to satisfy the demands of English departments. Only recently have graphing packages been introduced and where these are available mathematics teachers are using them in that mode "as tools". The use of spreadsheets is also starting. The fundamental questions have not been addressed—why should everyone receive mathematics instruction? Why should mathematics be taught? How should the curriculum change because of the available technology? In B.C. students in general do not take a significant amount of mathematics beyond grade 10. The main driving force for the mathematics curriculum has been the requirements of post secondary institutions, which is a tragedy and is certainly not in line with the social contract of teachers with society. We cannot expect any significant changes in the curriculum if this central role of the post secondary institution is not changed. One must review the way mathematics is taught and introduce technology into the mathematics classroom starting with the most obvious areas such as statistics, fractals etc...

Dawson Unfortunately, in teacher education, computers have become part of the problem rather than part of the solution. The present mandate is to make every teacher in every classroom a computer user. At the same time there is a major shift in curriculum and methodology starting at the elementary levels to be followed up through the intermediate — to make education child centred rather than subject oriented. Although both these objectives are laudable ones, the pressure of these two simultaneous changes on the teachers may cause a backlash. Teachers find it difficult to see how a computer can be used in a child centred way. The management of the instruction process has to be studied carefully. Teachers must see computers as meeting their own self interest in the classroom otherwise computers will be neglected. The major impact of computers in education has been in the writing process. Beyond this teachers see very little in computers which is in their own self interest. If it does not meet the teacher's need then it does not work. It must become part of their world. One of the reasons why the mandate to make every student teacher at SFU computer literate has not worked is that the mathematics software that is available at the elementary level is drill and practice, which is awful! The entire curriculum at the elementary level has to change to take into account computers. We must also be aware that software comes with a built-in bias, and we must be careful what message is imparted to children. These biases need to be identified and noticed by teachers.

Weinzweig Mathematical concepts arise out of a particular context in response to a recurring problem situation. The context is not always "mathematical". Concrete materials can establish a context and iconic representations provide a critical link between concrete representations and more formal symbolic language. These iconic representations are more easily manipulated than concrete materials but not as easily as symbols. With iconic representations the constraints can be gradually relaxed to reduce the cognitive steps required by the student. The curriculum should focus on higher order cognitive processes and allow skills and more factual knowledge to be acquired through their use within a context. This can best be achieved by investigating interesting, significant challenging problems. Such a situation was proposed which would involve students in a multimedia presentation: the example was a problem faced by an individual employed to coordinate the fighting of a forest fire. Video would present the actual situation, students would then raise questions of personnel, transport facilities, wind speeds, etc from a simulation program and react to the situation developing mathematical concepts within this practical context. In parallel with the development of multimedia one should anticipate networked computer systems to support cooperative and collaborative work. Together these would provide for a networked, real-time shared hypermedia database tool tailored to the particular needs of the classroom.

The panel was allowed a minute to summarize and/or respond to other members of the panel.

Flewelling Weinzweig paints a picture of extreme high tech, systems are now available to implement curriculum change—too little is being asked of the existing software and it is not being used correctly.

Dawson What is needed is inservice on pedagogy and not computer literacy. There are software packages which just reinforce some bad teaching practices.

Brochmann The reason why curriculum change is not happening is because most who have them don't know how to use computers to meet their needs in the classroom. *Weinzweig* The development of computers is so rapid that one must move and look ahead. The computer must be part of the classroom but not the focus. Cooperative learning must be encouraged.

The panel presentation stimulated a lively discussion.

Kieren **The "geometric supposer" is an example of an open ended software. Have teachers used it and rejected it?**

Brochmann It requires too high a level of expertise by the teacher.

Flewelling Use of this type of software should be encouraged.

Harrison **Should students do programming in the mathematics classroom?**

Weinzweig There is nothing wrong with the programming activity but teaching of programming should not replace the teaching of mathematics.

Brochmann Programming is a mathematical activity.

Flewelling Programming is one way of communicating with the computer.

Côté Although there has been good work done with programming, it has very little to do with mathematics education.

Edwards **Supported Weinzweig's view of moving ahead. For example Boxer, the successor to Logo, would soon be available on cheap multimedia systems - these systems should no longer be regarded as the futuristic.**

Gaulin **Raised the question of the role of Computer Algebra Systems for teaching algebra and functions.**

Flewelling CAS unfortunately will be used to do traditional things.

Muller Hopefully these will have a substantial impact on the mathematics curriculum and the teaching of algebra and functions both in schools and universities. These now provide an environment which includes numerical, graphical and algebraic representations. Teachers will be able to concentrate on concepts while students move easily from one representation to the other to facilitate developing a good understanding of the mathematical concept. CAS provide a rich environment for problem solving.

Poland **There is a problem of equity in the provision of microcomputers.**

Brochmann In North Vancouver the situation is close to being equitable for all students.

Flewelling There will always be situations where two students working on a computer at school, one will have five of them at home, while the other will have none.

Weinzweig Prices will continue to go down — if price is a measure of equity.

Kastner **There is at present only one project (Jim Fey, University of Maryland) which is attempting to change the curriculum in light of technology.**

Routledge If a CAS had been available to do the algebra when she struggled through her calculus course, she could have concentrated on the calculus.

Berggren Experiments are demonstrating that symbolic manipulators do work in the classroom.

Hoffman Progress will vary from college to college, some are only now introducing calculator use in statistics, it will take a while before CAS are considered.

Hillel The technology can no longer be neglected. These are forcing issues in mathematics education which must be addressed.

Taylor In Ontario, where the mathematics curriculum of the final school year has been revised, universities are modifying the first year mathematics courses. The whole computer issue is a distraction in this process and is postponed for future consideration.

Muller That is a real pity. Universities should be presenting the cutting edge of the discipline to their students and computers are redefining what cutting edges in mathematics are accessible to undergraduate students.

Previous Proceedings

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