CANADIAN MATHEMATICS EDUCATION STUDY GROUP

GROUPE CANADIAN D'ETUDE EN DIDACTIQUE DES MATHEMATIQUES

PROCEEDINGS

1991 ANNUAL MEETING

University of New Brunswick

Fredericton New Brunswick

May 23-27, 1991

Edited by

Martyn Quigley

Memorial University of Newfoundland

CMESG/GCEDM Annual Meeting
University of New Brunswick
Fredericton, N.B.
May 23 - 27, 1991

Table of Contents

Invited Lectures

Working Groups

Topic Groups

Ad Hoc Group

Appendices

EDITOR'S FORWARD

I should like to thank all the contributors for submitting their manuscripts for inclusion in these proceedings. Without their co-operation it would not have been possible to produce the proceedings.

Special thanks must go from all of us who attended the conference to the organizers, and particularly to Dale Drost, Rick Blake and Charles Verhille who did so much before and during the meeting to make it such an enjoyable and profitable event.

I hope these proceedings will help generate continued discussion on the many interesting issues raised during the conference.

Martyn Quigley

April 10, 1992

Acknowledgements

The Canadian Mathematics Study Group wishes to acknowledge the continued assistance of the Social Sciences and Humanities Research Council for their support. Without this support the 1989 meeting and proceedings would not have been possible.

We would also like to thank the University of New Brunswick for hosting the meeting and providing excellent facilities to conduct the meeting.

Finally, we would like to thank the contributors and participants, who helped make the meeting a valuable educational experience.

In Memoriam: Linda Brandau

With great sorrow we learned that Linda Brandau passed away on April 30, 1991. She was a member of the CMESG/GCEDM Executive at the time. The following memorial text was delivered during the annual general meeting in Fredericton, May 1991.

I first met Linda 11 years ago at the NCTM meeting in Boston. At the time she was heavily involved in her doctoral research under the guidance of Jack Easley at the University of Illinois. When she completed her PhD in 1984, she moved East to teach at a small college in New Hampshire. A few years later Linda moved again, this time to the University of Calgary. She soon heard of the Canadian Mathematics Education Study Group / Groupe canadien d'étude en didactique des mathématiques and decided to attend one of our meetings to see what we were all about. What she found was a group whose thinking and activities supported her own approach to mathematics education.

I would like to take a couple of minutes to communicate to you a little of Linda's perspective in mathematics education by reading two extracts from a paper she wrote last year and presented at the annual conference of the Theory of Mathematics Education group. The title of her paper was "Our Theories, Ourselves."

> The title of my talk implies a wholeness, an inseparability between us as mathematics education researchers, our theories, who we are (both personally and professionally), what we believe, how we behave. "Our theories, ourselves" implies an integrated way of thinking, a view of knowledge for which thought is not separate from feelings, mind is not separate from body, the rational is not separate from what we usually label as intuitive. This view of knowledge differs from the one pervasive in our Western culture, one rooted in a philosophic and scientific tradition based on the work of Plato, Francis Bacon, and Descartes, among others. This Western tradition values qualities we label "thought," "mind," "rational knowledge" and devalues those labeled "feelings," "body," "intuitive knowledge." This language, the one that creates these labels, also creates dichotomies. It is a language that disembodies, that fragments, that separates.
>
> In this separation, we have alienated ourselves from our experience. So-called scientific thought has taught us not to trust our senses or our perceptions, has taught us that the only kind of knowledge that is valid

> exists outside ourselves. In mathematics classrooms, we are creating students who do not trust their own ways of thinking because they are told they must learn procedures in one way--the teacher's or the textbook's, in essence, some external "expert." If that one acceptable way does not happen to coincide with students' ways, their thinking is invalidated. Students then resort to memorization of rules, to viewing mathematics as fragmented bits and pieces of information. Or perhaps they view mathematics as whole but themselves as fragmented?

And the concluding paragraphs of her paper:

> If we complain about the fragmentation of the discipline, then we need to investigate how we are fragmented. If we criticize elementary and high school students for doing very little "critical thinking," then we need to uncover how well we critique our own theories and research. If we do not like the language used by teachers in the classroom, then we need to explore the language we use. If we do not like the values being espoused in education, then we need to examine our values. If we do not like the kind of teaching styles we observe in elementary and high school classrooms, then we need to look at our teaching in university classrooms.
>
> A summary of these ideas was given by Confucius, ". . . wanting good government in their states, they first established order in their own families; wanting order in the home, they first disciplined themselves." Thus, journeying into ourselves is the only way we will begin to alter what we do not like about mathematics education. It is the only way we will begin to articulate our love for mathematics to our students and to ourselves.

Linda, we are all very sad that you cannot be with us today.

Let us observe a minute of silence in tribute to a fine person and a good friend.

On behalf of the CMESG/GCEDM Executive
Carolyn Kieran

Editor's note: The complete text of Linda Brandau's paper, from which the above excerpts were drawn, appears in Appendix A.

Lecture One

Students' Reading and Writing of Mathematics

Colette Laborde

Université Joseph Fourier

Students reading and writing mathematics

Every teaching is based on communication processes and requires various language activities. The teaching of mathematics is concerned with two kinds of problems:

- specific features of the language used in the mathematics classrooms may be an obstacle to the comprehension and the learning of mathematics by the students,
- but one of the aims of the mathematics teaching is to allow students to understand mathematical texts and to use appropriate words and formulations to express mathematical ideas.

Mathematics teachers are confronted with the paradoxical situation to prevent language used in the classrooms from being a source of difficulties but simultaneously to allow students to reach a language proficiency in mathematics. We focused our study of language problems in the teaching and learning of mathematics on the first part of secondary education (11 to 15 year old students) because we think that is a critical time for learning an appropriate use of language in mathematics; the process of acquisition of language lasts over a long period of time and the command of their mother tongue is not completely mastered by students of this age. Mathematics used in secondary education is not so much related to action as in primary school and requires a language more remote from the ordinary one, more formal with specific vocabulary and specific syntactic and pragmatic uses.

The analysis presented here is based on two hypotheses:

- there exist specific problems caused by language activities in the learning of mathematics, problems that are not identical to problems of conceptualizing the mathematical notions
- but these problems are related to the conceptual aspects of knowledge acquisition and we have attempted to search for the dialectical links between the nature of mathematical ideas elaborated by students and the way they express these ideas. We even think that the development of the competence to read and to formulate in mathematics is linked to the development of mathematical knowledge at pupils. And each of these competencies may be used as a hook for promoting an evolution of the other one. This point of view about the development of language has been supported by several psycholinguists (Vygotsky 1962, Oléron 1978, Beaudichon 1982) who stressed the interaction between conceptual, social and linguistic aspects.

Therefore our approach in mathematics teaching and learning considers a language activity as a global activity involving several aspects:

- the content to which the formulations (written or oral) refer i.e. a mathematical content;
- the speaker or the listener as a cognitive and social subject with, in particular, his/her ideas about the mathematical content and his/her knowledge of the language used;

- the situation: moment and place of the language activity, the aim of the activity (giving information, solving a problem for the teacher, convincing somebody...).

Writing as well as understanding a text requires by the speaker or the reader (listener) conceptual operations involving conceptions and representations of the various elements of the discursive situations cited above. If everybody generally views writing as an activity, reading is often perceived as passive. In our approach it is considered as entailing an active construction by the reader and we believe that it is the only way to account for misunderstandings by students when reading problem statements or mathematical texts.

In this paper, three points will be developed:

- the specific features of the mathematical discourse,
- the use of denotations and symbolic expressions by students when formulating in mathematics,
- students' reading mathematical texts.

I - Specific features of mathematical discourse

A widespread opinion among mathematicians claims that as soon as a mathematical text is clear, it must be understood by every other mathematician. This utopian view was (and may be is) shared at least in France by the community of authors of the official texts about the teaching of mathematics. Mathematics is the science of rigour, clarity, precision: all these features appear through the discourse of mathematics that tends to be precise, concise and universal.

To meet these aims, some linguistic uses are privileged in mathematics. A first tool of precision and conciseness consists of a specific writing system made of signs exterior to the natural language such as $+$, $<$, ... together with letters or numerals which can be combined according to specific rules in order to create well-formed expressions like $2x + 7 = 15$. In most cases these symbolic expressions are embedded in sentences written in natural language: "Find the solutions of the equation $2x + 7 = 15$". The economy obtained by this system is very high through both functions of symbolic writing in mathematical discourse.

A first function consists of designating objects: an object can be denoted by only a letter or a symbol (what we call a simple designation) or it can be denoted by a compound expression like "$f(x)$" or "(AB)" (straight line passing through the points A and B in the French school mathematics). The denotation plays the role of a proper noun.

A second function of a compound expression consists of making explicit the relations between the object and other objects: "(AB)" not only denotes a straight line but also gives information about the position of this line in the set of points of the plane. Expressing the relations between objects by means of symbolic expressions is a powerful tool of conciseness and precision and contributes to make mathematical texts very dense.

Because mathematical objects are not isolated but depend on other objects, the precision purpose leads to the use of long complex noun clauses with subtle surface marks of relations between the components as for example in:

"the circle with centre A and radius r",
"the foot of the perpendicular line from A to D",
"the parallel line to D passing through A",
"a tangent line to the circle (C) at M",
"the image of A through a reflection about the straight line D".

When reading such formulations, the student can find a help in his/her knowledge of mathematics to interpret the prepositions or to break down the clause into subparts. We could observe for example difficulties of French students in reading the following problem sentence: "Construis E symétrique de A par rapport à I et F symétrique de B par rapport à J" ("Draw E symmetrical point of A with respect to I and F symmetrical point of B with respect to J").

Some students asked the teacher

"What does that mean: symétrique de A?"

or

"What does that mean: symétrique de A par rapport à I et F?"

The fact that these students did not master well the notion of point symmetry and some of them confused reflection and point symmetry (confusion expressed by the second question where I et F refers to the straight line passing through I and F) prevented them from properly segmenting the clauses. It implies that language, instead of helping students, may be an additional barrier to understanding.

Because mathematical discourse tends to be universal, it objectifies the operations made on objects in deleting any reference to the persons performing these operations and to the time aspects. The mathematical discourse is very often a static discourse with a predominant use of nouns and adjectives and a reduced number of verbs.

All the particularities mentioned above can be found in the mathematical discourse of the classroom even if it differs from this one used for example in the mathematics research papers because the syntactic structures are not so complex. These particularities do not correspond to the linguistic habits of middle school students.

II - Formulating by means of symbolic expressions

From studies carried out on the problems faced by students when formulating mathematics, it came out that they do not spontaneously have recourse to symbolic expressions or even to denoting objects by letters. We refer here in particular to the studies of Lee and Wheeler (1989) about the generalisation and justification power of algebra and by Laborde (1990) about the ways students describe mathematical objects and relations between them. When faced with arithmetic problems whose solution requires the labelling of a number by a letter and the use of algebra, although they were familiar with algebra, students preferred to formulate solutions in natural language. When faced with the task of describing geometrical figures without any labelled elements, students made long and tedious descriptions of geometric relations between the elements of the figure in natural language and did not spontaneously decide to denote the key elements for the description of the figure by letters (Laborde, 1982).

It is interesting to analyze what strategy these students used to refer to objects previously introduced in their description since they did not have the facility of symbolic expressions. They indeed used three kinds of methods:

- repeating exactly the same expression describing the object "the big straight line splitting the rectangle into two parts" for the perpendicular bisector of two parallel sides of a rectangle,
- referring to the moment they drew the object, sometimes with a subtle use of tenses, and
- using distinctive features of the object to be referred; if these features became no longer distinctive in the course of the text, students changed them. For example a big rectangle could become a small rectangle because a bigger rectangle has been drawn.

When students decide to denote some elements by letters, for example points, they had difficulties creating compound symbolic expressions with these letters as for example (AB). They used the expression "the segment going from A to B" instead of "the segment [AB]".

From these observations some conclusions can be drawn and assumptions can made:

- the resistance of having recourse to formulae or algebraic statements may be due to the difficulty the students have in going beyond the context and in deleting reference to time and to action. The change-over from natural language to mathematical symbolism eliminates all these elements which create the actual meaning for the students. Part of the difficulties in using the symbolic expressions originates from the mental representations of the mathematical objects developed by the students,
- the conventions underlying the functioning of the mathematical writing system are not known by the students and not used spontaneously because very often the teaching does not really pay attention to these problems. Even the convention according to which two different objects used in the same problem is not followed by some students in case of mathematical difficulty (cf §III.1).

III - Understanding mathematical texts

The role of language comprehension has been mostly investigated for solving arithmetical problems and several variables that affect the students' representations have been studied:

- how relations between the given and the unknown quantities are expressed and in particular the degree to which they are made explicit (De Corte 1985, Bachor 1987),
- the degree of attraction of some expressions or words like "more" or "less" which may be distracters as well as cues (Nesher & Teubal 1975, Fayol 1990).

Two studies will be presented below that address topics up to now rather unexplored:
- reading geometry problem statements,
- reading long mathematical texts.

III.1 - Reading problem statements in geometry

Reading strategies have been proposed by linguists and psycholinguists to describe reading processes (cf below §III.2). It seems that the following hypothesis is shared by numerous studies: the reader selects among the elements of the sentence a main element of high information, relates to this element the other information items and so constructs a meaning in function of this firstly selected element. If there are too many high information items, the task of the reader becomes complex and a possible behaviour is to select a main information item and other elements but to miss some other subsidiary elements. As mentioned above, a mathematical text has a high density of information; the requirement of conciseness is particularly met by problem statements that attempt to give all needed information for solving a problem in the shortest form. The probability of missing an information item is therefore very high.

In problem statements in geometry contrary to word problems which include superfluous information referring to a real situation, all information items are of equal importance and such an omission changes the problem to be solved. This phenomenon of omission is likely to occur more often with students: they are not acquainted with mathematical discourse, they are mostly confronted with the every day language or with narrative texts that are redundant and in which information items are not of equal importance. What are the variables of the wording and of the structure of the text that may influence the dual processes of selection and omission?

In geometry problem solving, a first step is to draw a figure before even starting to solve the problem. For some problems, the information items can be used sequentially for this drawing activity but for some others they are to be linked and used together. An information item given at the beginning of the problem statement may be used only at the end of the construction. We assume that this kind of organisation provides a greater cognitive burden in the processing of information and requires the memorizing of these items. Thus it may lead to omission of information items by the reader and then in some cases give rise to a conflict. Let us give an example.

Students of grade 8 were given the following problem: "Draw a parallelogram ABCD such that A belongs to the perpendicular bisector of CD". Several students segmented the statement into "draw a parallelogram ABCD" and the remaining part of the sentence. They immediately drew a parallelogram ABCD which of course did not satisfy the further condition. They were then puzzled with the second part of the statement because obviously the perpendicular bisector of CD did not pass through A. Some of them solved the conflict in drawing this bisector and marking A near this straight line designating an undetermined place on the line.

Some observed reading behaviours

Let us present examples taken from an experimental investigation we carried out with students of grade 8[1]. We chose some geometrical exercises in mathematics schoolbooks of grade 8 and for each exercise we made some variations on the wording of the problem, changing the order of the information items.

A first version of a construction exercise taken from a schoolbook was given to students working in pairs:

> "Construire d'un même côté de la droite (AB) deux triangles ABC et ABD tels que AC = BD, AD = BC, et AC > BC" (Construct on the same side of the straight line (AB) two triangles ABC and ABD such that AC = BD, AD = BC, and AC > BC")

Other students working in the same conditions were given a modified version

> "Construire un triangle ABC tel que AC > BC. Construire ensuite un triangle ABD tel que C et D soient du même côté, de la droite (AB) avec AC = BD, AD = BC." (Construct a triangle ABC such that AC > BC. Construct then a triangle ABD such that C and D are on the same side of the straight line (AB) with AC = BD, AD = BC.)

We assumed that the first version could lead to selection of only some elements for solving the problem. The first information item of the version 1 would be omitted by the students because it could not be used immediately in the drawing and because it is not so important as the two triangles ABC and ABD. The three following information items "AC = BD, AD = BC, et AC > BC" are expressed with formulas. We assumed that the first two will be more taken into consideration that the third one because they are equalities and they were given before the inequality. An equality is more constraining and in this sense gives more information.

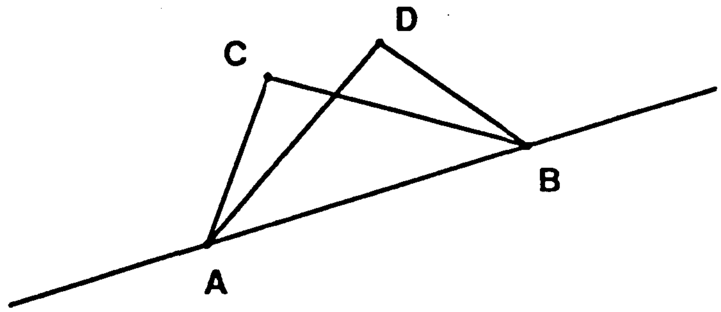
One of the difficulties of the task lies in the fact that all information items have to be used before obtaining simultaneously the points C and D. Another strategy would be to choose randomly a point C but we did not expect this strategy which was not at all suggested by the statement.

The second version was modified in order to avoid the omission of information items by the students: the two information items "AC > BC" and "on the same side of the straight line (AB)" are given at a place in the text where they can be used immediately for the drawing. Another important change facilitating the solving of the problem

---

[1] The same experiment was carried out by H. Emori (University of Tsukuba, Japan) in the frame of a French-Japanese cooperative research programme.

is the splitting of the information item "two triangles ABC and ABD" into two items given separably in the text.



The expected behaviours concerning omission and selection of information items for the first version appeared:

- the information item "on the same side of the straight line (AB)" was omitted by half of the students and even in the verification process of the produced figure when they tended to check wether their figure did really fulfil the required conditions, and
- the inequality AC > BC was also omitted by about half of the students but contrary to the previous item it was taken into account in the verification process.

The first behaviour was confirmed by observations of teacher students in mathematics who systematically did not take into account the item "on the same side of the straight line (AB)" when solving the version 1 of the problem. We assume that the place of this item plays an important role but this omission may also be due to the fact that this item is expressed in natural language and not as a formula. According to an implicit rule, every numeric or symbolic expression given in a problem statement must be used in the solution. The drawing produced with C and D on both sides of (AB) is also visually more balanced.

But in addition, unexpected difficulties and interpretations occurred that were related to the reading of the following symbolic expressions:

- the equalities AC = BD, AD = BC,
- the denotation (AB).

The equalities were not read analytically and several pairs interpreted in a first step the sequence of the two equalities as AC = BD = AD = BC and then as AC = AD and BD = BC and finally as the correct ones. The move from one interpretation to the following one was triggered off by a verification process when the students checked whether the produced figure fulfilled the given conditions.

Example: the sequence of the figures produced by Catherine and Karin

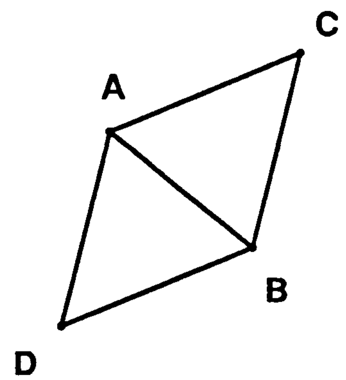


Figure 1: AC = AD = BC = BD

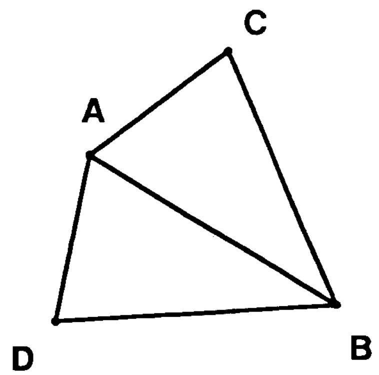


Figure 2: AC = AD and BC = BD

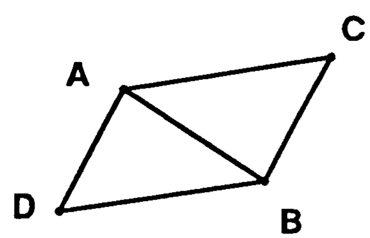


Figure 3: AC = BD and AD = BC

Using these equalities requires a detailed analysis of the expressions and also memorization. The combination of the letters was not usual (the combination AC = AD and BD = BC would have been more usual) and maybe a wrong memorization took place.

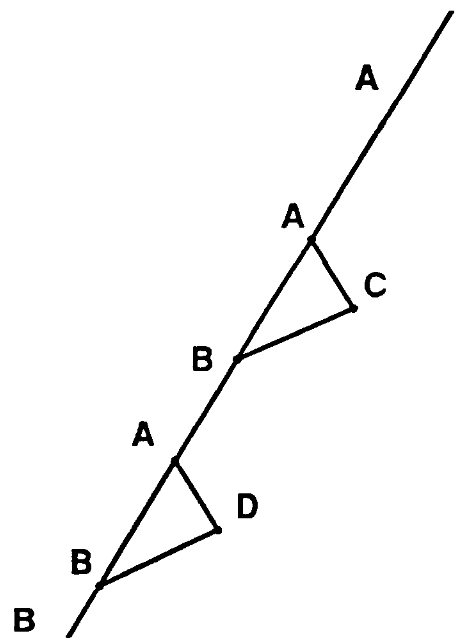
The analytical reading of formulas is considered as taken for granted in the teaching of mathematics; this experiment shows that it is not made spontaneously and must be learned.

The denotation (AB) was interpreted by several pairs as a global name for a straight line. Two behaviours were observed:

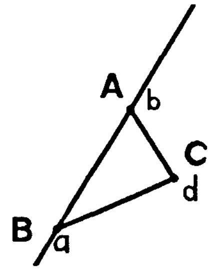
- the students drew a straight line, wrote AB or (AB) near the line and selected two points on the line they called A and B or a and b,

- the students drew a straight line they called AB and chose two points A and B outside the line.

Some students interpreted the sentence "construct then a triangle ABD" as disconnected from the first one and considered that the triangle ABD was not constructed from the first points A and B but from two other points A and B, although they used the relations AC = BD and AD = BC considering in these relations two different pairs of points A and B. So Sophie and Lydie produced the following figure (with the relations AC = AD and BC = BD).



The conventions underlying the usual functioning of the symbolic language are not taken into account by these kinds of answers, leading to an inconsistent figure from a mathematical point of view. It seems that the students are not even aware of rules underlying the functioning of the symbolic language and that they can both use a rule at some places and not follow it at other places in the same exercise, as shown by the previous example. The geometrical difficulty of the problem may even lead them to proposals they never met in their mathematical experience, like Catherine and Karen who could take into account the equalities AC = BD, AD = BC only by the following figure:



III.2 Reading long texts

The texts that pupils are mostly confronted with are the texts of problems to be solved. The reading of these texts can give rise to difficulties as it has been pointed out above. But even if some of the origins of these difficulties are known, and even if it is possible to decrease the linguistic complexity of a text, very little is known about the ways in which pupils read complex mathematical texts and learn from them. Linguistic research on reading assumes that the most widespread reading strategy is what is called the garden path strategy (Frazier & Rayner, 1982). The reader begins a text with a hypothesis about its interpretation (this interpretation results from the features of the text, of the situation and of the knowledge of the reader, cf. our theoretical framework mentioned above) and goes on reading while retaining this hypothesis. If the chosen hypothesis enables the reader to construct both a coherent local and global interpretation, there is no problem, and the reader is not even aware of the reading hypothesis. If the chosen hypothesis is inconsistent with the remaining text, that is if the mobilised knowledge is not compatible with the subsequent information in the text, there is a crisis that makes the reader aware of the hypothesis, leads to a reconsideration of the hypothesis, and to a rereading of the text with another interpretation.

Some reading experiments have been carried out with pupils on longer texts than problem statements; in this case the scope of the coherence between the elements of the text that the pupils have to construct is larger and the garden path strategy is likely to be more visible. In these investigations, the students' reading processes can indeed be described with such a strategy (Guillerault and Laborde, 1986). They also seem to show that pupils have difficulties in rejecting their first hypothesis, and that they prefer to make decisions that are in flagrant contradiction to the text. In this way, information in the text can sometimes be more or less deliberately omitted.

Another important use of reading deals with the activity of extracting information from written texts whose importance progressively increases as the pupil is getting on in years. How students use a written text or their textbook as a learning tool or as a help to solve problems has received very little attention except in some works like in Walther (1981) or Rasalofoniaina (1984).

We carried out a first experiment (Laborde, 1990) aimed at investigating the impact of linguistic features of texts on pupils' comprehension; our aim was to investigate not only local features like the choice of words, the structure internal to each group of words or to each sentence but also global features like the structure of the text and its homogeneity (that is, similarity of structure between parts of the text). We chose four excerpts of French schoolbooks (for grade 9) about the same theme: operations on square roots. These excerpts were one or two pages long and were given to pupils working in pairs. Each pair was asked at first to read carefully these four excerpts and then to write a joint text meant for other pupils who did not attend the lesson about operations on square roots so that the produced text should enable them to learn this theme. 12 pairs of pupils were observed and recorded. These pupils had already been taught square roots, but six months before the experimentation. In this task, the aim of reading is determined by the production activity of a text.

As far as pupils had to express to peers, it was reasonable to think that they would select information items they thought important and that they would transmit them in a form they would judge the most accessible. Possible explanations that they would develop for the receivers of the text gave us additional information about their understanding of the texts. The choice to give four texts to be read makes the task especially difficult. It was intended to allow us to determine the impact of the presentation variables of the same mathematical content on the understanding of pupils. For instance two of the texts had a classical linear way of exposition whereas the two other texts proposed activities to the pupils; one of them should be read in a non linear way because activities introducing properties were given in the margin of the page.

The reading of several texts required a constructive activity from pupils and prevented them from a purely copying activity which could have occurred if they had only one text. Reading the set of four texts required them indeed to construct not only a coherence internal to each text (intracoherence) but also to overcome the differences between the four texts and to construct a coherence between them (intercoherence). The choice of the texts made difficult the construction of the intercoherence, because in particular one of the texts had a coherence remote from the other ones. Three texts presented properties of operations as results of theorems whereas the fourth text presented these properties as simplification rules which were introduced without justification. A conflict should arise between these four texts if the pupils tried to use all of them. To overcome this conflict, pupils had two possibilities: to choose one of the coherences or to construct a new coherence. They could also fail and juxtapose two coherences in their text. It appears from the study of the texts produced by the pupils that they were mainly inspired by the two texts which presented a linear macro-structure corresponding to a classical way of exposition. Is it because these two texts were more understandable for the pupils or because pupils at this grade are already accustomed to a normative presentation?

Reading a long text requires the construction of a global coherence related to the reading hypothesis made by the reader and of local interpretations. These latter are related to the global coherence. It means that a global coherence may affect the local interpretations but that local interpretations may have a feedback on the global coherence. Such an interaction between local interpretation and global interpretation could also have been observed.

Let us give two examples, the first one concerning the intercoherence, the second one concerning the local and global interpretations.

Example 1:

One of the proofs about multiplication of square roots could induce a misinterpretation. It was the following (French version):

> "Soit le produit $(\sqrt{7} \times \sqrt{13})$. Son carré $(\sqrt{7} \times \sqrt{13})^2$s'écrit (puissance d'un produit de réels):
>
> $$(\sqrt{7} \times \sqrt{13})^2 = (\sqrt{7})^2 \times (\sqrt{13})^2 = 7 \times 13$$
>
> D'où: $$\sqrt{7} \times \sqrt{13} = \sqrt{7 \times 13}$$

De même, quels que soient a et b réels positifs, le carré du produit $(\sqrt{a} \times \sqrt{b})$ s'écrit:

$$(\sqrt{a} \times \sqrt{b})^2 = (\sqrt{a})^2 \times (\sqrt{b})^2 = a \times b$$

D'où: $$\sqrt{a} \times \sqrt{b} = \sqrt{a \times b}$$

An important piece of information was implicit: the property that if the squares of two positive numbers are equal, these numbers are equal. The scope of the connecting words "de même" was ambiguous: was it the whole proof or only the writing of the property

$$(\sqrt{a} \times \sqrt{b})^2 = (\sqrt{a})^2 \times (\sqrt{b})^2 = a \times b?$$

The connecting word "d'où" referred to two equalities

$$(\sqrt{a} \times \sqrt{b})^2 = a \times b$$

and the other one

$$(\sqrt{a \times b})^2 = a \times b$$

which was implicit. The text emphasized very much the equality

$$(\sqrt{a} \times \sqrt{b})^2 = (\sqrt{a})^2 \times (\sqrt{b})^2 = a \times b$$

which was introduced by an explanation. And the "d'où" could be understood by the pupils in the reversal sense as indicating the source of a deduction. All these reasons could explain why some pupils were lead to interpret the result of this proof not as

$$\sqrt{a} \times \sqrt{b} = \sqrt{a \times b}$$

but as

$$(\sqrt{a} \times \sqrt{b})^2 = a \times b$$

and to propose in their text two rules for the product of square roots, this latter one and the correct one taken from another excerpt. Their misinterpretation of only one paragraph led them not to recognize a common content between the excerpts and to produce an incoherent text. It was one of the cases of failure in the construction of an intercoherence between the four texts.

Example 2:
One of the texts presented the properties of operations on square roots as simplification rules without justifying them. But before introducing the rule for multiplication and division of square roots, it stated that it was possible to delete the sign "×" in a product and to write only "ab" instead of "a×b". This preliminary remark about a writing convention preceding rules led one pair of pupils to interpret these rules as writing conventions. In this case a small remark misled pupils in the global interpretation of the text because they assumed (reading hypothesis) that every remark of the text had the same conceptual status as the first one, since the text did not mention any change of status.

This experimentation shows the complexity of the processes underlying the construction of a meaning by the reader as far as local and global phenomena strongly interact.

Students seem to experience several difficulties in using language in mathematics as shown by the previous examples. It appears that the use of language in mathematics is not natural for students, not developing as a matter of course. Students are required to imitate the language of the schoolbook or of the teacher but they are seldom taught how to do this. An explicit teaching of language based on activities should be developed in the mathematics classrooms to allow students to improve their language proficiency in mathematics. Some of possible activities based on the theoretical framework presented above have been experimented (cf. Laborde 1990). A collaborative work between teachers of mother tongue and of mathematics should also be organised to convince the students that the quality of the language spoken and written in mathematics is important and that it is a real language.

References

Bachor D. (1987). Towards a taxonomy of word problems, *Proceedings of the Eleventh International Conference for the Psychology of Mathematics Education,* Bergeron J., Herscovics N., Kieran C. (eds), Volume II, pp. 163-169.

Beaudichon J. (1982). *La communication sociale chez l'enfant,* Paris, Presses Universitaires de France.

De Corte E., Verschaffel L. and De Win L. (1985). The influence of rewording verbal problems on childrens' representations and solutions, *Journal of Educational Psychology,* 77, pp. 460-70.

Fayol M. (1990). *L'enfant et le nombre, du comptage à la résolution de problème,* Delachaux et Niestlé, Neuchâtel.

Frazier L. and Raynier K. (1982). Making and correcting errors during sentence comprehension, *Cognitive Psychology* 14, pp. 178-210.

Guillerault M. and Laborde C. (1986). A study of pupils reading geometry, in F. Lowenthal and F. Vandamme (ed.) *Pragmatics and Education*, Plenum Press, pp. 223-38.

Laborde C. (1982). *Langue naturelle et écriture symbolique: deux codes en interaction dans l'enseignement mathématique*, Thèse, Université Joseph Fourier, IMAG, Grenoble.

Laborde C. (1990a). Language and Mathematics, in *Cognition and Mathematics*, Nesher P. & Kilpatrick J. (eds), Cambridge University Press, pp.53-69.

Laborde C. (1990b). Lecture de textes mathématiques par des élèves (14-15 ans): une expérimentation, *Petit x*, n° 28, IREM de Grenoble, pp. 57-90

Lee L. & Wheeler D. (1989). The Arithmetic Connection, *Educational Studies in Mathematics*, Vol. 20, n° 1, pp. 41-54

Nesher P. & Teubal E. (1975). Verbal cues as interfering factors in verbal problem solving, *Educational Studies in Mathematics*, Vol. 6, pp. 41-51.

Oléron P. (1978). *Langage et développement mental*, Bruxelles, Pierre Mardaga Editeurs.

Rasolofoniaina I. (1984). Conditions d'apprentissage mathématique par la lecture, *Recherches en didactique des mathématiques*, Vol. 5, n° 1, pp. 5-42.

Vygotsky L. S. (1962). *Thought and Language*, Cambridge, MIT Press.

Walther G. (1981) Autonomous learning and the reading of mathematical texts, *Journal für Mathematikdidaktik*, Jahrgang 2, Heft 2, pp. 147-77

Lecture Two

Mathematics and Technology: Multiple Visions of Multiple Futures

James Kaput

South Eastern Massachusetts University

Technology, Transliteration and Change: Two Fantasies

Fantasy 1: The Invention of Tape Recording in an Oral Culture

Imagine a society, without writing, but with mathematics. What kind of mathematics would, or could, be possible in this oral culture? One can imagine much reliance on mental computation and estimation, probably involving the use of body parts for indexing the steps in procedures, but it would all be constrained by the limitations of working memory. It would need to be learned via oral means—passed down as were the histories of pre-literate cultures, in oral form, perhaps using poetic devices such as rhyme and rhythm. A mathematics classroom in such a situation would be without pencil and paper or chalkboard. All practice and testing are done orally. Memorization ability would be paramount in determining success. There would perhaps be a mix of group work ("chanting") and individual work, probably with closed eyes. Testing would take the form of oral recitation.

Imagine further that, somehow, a new electronic technology were developed for this oral culture. What might it likely be? Suppose it is tape recording—providing a permanent record of oral events. The first recorders, reel-to-reel, were large, hard to use, and unreliable, but gradually they improved. How would they be used in math classrooms? Only one was available per class and was used to present chanting tasks to the whole class while the teacher worked to test individual students in another (sound-proof) room. Quite a change in classroom life. The big breakthrough occurred when the earphone was invented, allowing for group use of machines without cacophony. Certain wealthy schools were able to set up what would look to us very much like today's language lab with permanently installed tape machines. These assisted practice in learning the facts and procedures of the existing modes of doing mathematics, with their largest contribution taking the form of replacing the teachers as task-providers and performance monitors. Memorization and computation tasks could be given on tape, and students could submit their work likewise.

Certain people with keen interest in the new technology would predict ever smaller, less expensive, more convenient tape recorders—"micros" were what they were called. Students would even have them at home, use them while walking or driving! They predicted dramatic impact of the tape-technology. What could be done in twelve years could probably be done in nine or even fewer years, and learning could occur at any time or place. Teachers were free to individualize instruction. Tape machines had infinite patience, ...

Studies were done comparing rates of learning with and without the new machines, with generally positive results. Scholarly debates ensued about the dangers of intensifying instruction, the social isolation of people listening to individual machines, the quality of the lessons available on tape, the disparity in access to the expensive technology, the economics of producing and distributing tapes, ...

We interrupt our fantasy on the impact of technology on school mathematics to ask what has changed and what has not changed as a result of the new tape-technology? The mathematics or curriculum? Definitely not. Pedagogy? No. The goals? Nope. They were all "transliterated" to the new medium, perhaps made more efficient, but not changed in any *deep sense. the structure of schooling? Not really. The impact of this technology on school mathematics could be described as entirely superficial.*

Let us now enter a new fantasy that involves the deeper changes involved in the application of a fundamentally new medium for doing mathematics, and its impact on school mathematics, something closer to the situation provided by today's, and, ***a fortiori****, tomorrow's computers. While the second fantasy describes fantastic events following the first, it is intended to explore other aspects of the contemporary situation, including:*

(1) the mismatch between established school practice and the new forms of mathematics and learning possible in the new computer medium,

(2) the connection between changing media and changing mathematics,

(3) our egregiously limited expectations and experience regarding the use of computers in schools, and, most importantly,

(4) the stubborn and deep underestimation of the computational medium as means for encoding written mathematics. It is much more than that, but how much more, we can only guess.

Fantasy 2: The Invention of Writing

Writing was invented, on both permanent and erasable media. The world of intellectual work was transformed as people rapidly invented new uses for it in mathematics—first recording oral events so that the mental computations could now be "seen," and information could be stored outside people's heads. This mode of recording and "playback" (known as "writing" and "reading") was, of course, not allowed in schools given its obvious negative impact on the development of memorization skill and, worse yet, motivation to memorize (which had never been overwhelmingly strong).

However, outside schools all sorts of invention were underway: extremely efficient systems for writing and operating on quantities yielding exact answers were developed that were very different from the old halving and doubling techniques, the counting up and down, the finger indexing of steps, the increase-decrease compensation routines. In written math were developed abstract quantities and what was called "algebra," and "analysis." This "new math" was, of course, learned outside of schools—few teachers knew much about it, nor did **their** teachers, many of whom were still debating the proper role of tapes.

Nonetheless, a few daring schools began to teach the new "reading" skill, and at the most advanced level, some "writing" was taught—always in conjunction with tape recorders, which had come to be the primary means by which tasks were presented and student work collected—a few students practised writing what they heard on tapes. Sophisticated arguments showed how this practice (known as "writing-aided instruction") integrated the new technologies into the real process of teaching and learning (memorizing).

Gradually (much here must be left untold—we recount only parts of the story), writing and reading came to infiltrate the schools. Eventually, many mathematics classrooms had a pencil and a pad of paper in the back of each room and a large pad of paper was introduced at the front of every room. This required some physical changes because the tape shelves lined the room and some shelves needed to be moved out, and there were no tables available—they were frequently borrowed from the cafeteria. A bigger problem in older schools was lowering the soundproof barriers (left over from the days before headphones) so that students could see the paper-pad from their bin-seats (each student had a tape-bin under his seat). Some of the new schools even had special "pencil rooms" where each student could visit on a regular basis to do pencil work on a small table, although this proved to be quite disruptive and broke the continuity of the chanting and other tape-work. (Typically, a chanting session was scheduled for 15 minutes, whereas writing sessions seemed to require much longer periods.) Other new schools, copying situations from life outside the school, added a small table to the front side of each student bin-seat—these came to be called "desks."

But this too proved to be disruptive—the paper and pencils always seemed to be in the way of the tapes, and there was the problem that the big soft pencils could only be used to write extremely short messages on the tiny pads that were available to schools. It seemed that there was an almost unlimited amount of paper outside of schools as the economies of scale and new production techniques developed, but it had somehow been decided what (in the words of one wag) "the smaller the kid, the smaller the paper" And the pencils in schools seemed always to be softer and less precise than those newer models used outside of schools. Indeed, some benefactors donated used pencils to schools, where they were gratefully received—and then stored away.

In the early days of pencil instruction, much attention was given to the history of paper and pencils, their design, how to handle them and classify them (for example, the technical hardness differences between a #2 and #3 pencil), such matters as finding well-lit areas to practice reading (classrooms were generally kept quite dark, to help with concentration), finding flat surfaces where one could practice making marks, and storing papers and pencils (there was very little storage space, given all the tapes that needed to be stored).

After a time, as the practical, physical problems were solved, or, rather, were worked-around, other difficulties arose. Among the most serious involved writing the lengthy chants and the mental computation rules on these small papers. The logistics were almost insurmountable, even with careful numbering of papers. And the amount of time spent copying from the large paper-pad and from the special bundles of papers (called "textbooks") detracted from the actual memorization work with tapes (this work was known as "the basics"), which simply seemed intrinsically more efficient

Furthermore, the individual oral recitals that constituted testing could only accommodate reading. For a variety of reasons, tape submissions were not allowed for the important tests. One important reason widely offered was that it was simply impossible to determine by listening to a tape whether the reciter was working from memory or from paper. Lastly teachers were unaccustomed to pencils and paper, which were generally not used in the education of teachers with the exception of training in transcribing tapes onto small pieces of paper and some traditional chanting work based on reading the papers aloud.

Perhaps a word is in order about the "textbook problem," as it came to be known. Almost all textbooks were literal translations of the chants, so many educators wondered why they should be purchased at all given the added costs and complications in using them—what with the disruptions with the tape work mentioned earlier. The publishers, whose major business was in tapes, were likewise not inclined to invest much interest or money in this small and risky market for textbooks. A few textbook companies managed to survive, and an even smaller group produced textbooks containing material other than chants. But their business was at the margins of the large, established and very lucrative tape market, which came to be dominated by a few very large corporations.

But all the above deals with practical matters. We beg the reader to bear with us in more theoretical analyses of the difficulties, because these analyses may eventually help shed light on some deeper difficulties that arose. In education, writing was consistently viewed as the visual encoding of sound, not as means for the expression of new forms of meaning or operating. This view was reflected in practice, in two ways: (1) virtually all writing and reading amounted either to writing the spoken mathematics of the past or (2) attempts by more ambitious educators to teach the new mathematics in the oral, memorization style. Both managed only limited success, and were often in subtle conflict.

A few educators went sofar as to argue that the best way to use the papers and pencils is to **do** mathematics—to write and think using one's own writing as the basis for further thinking, as was the norm outside of schools. But, by and large, students were not expected to **do** mathematics in schools, which, of course, were **places of learning**, that is, memorization. So, given all the difficulty in using paper and pencils—theoretical, practical, political—tapes and chanting continued to dominate school classrooms.

But the story does not end here because, for a variety of reasons, especially the now conspicuous gap between school mathematics learning and the ways mathematics was done outside of schools—pressure increased on schools to teach the new ways of doing mathematics. Influential educators argued persuasively that **students should practice doing the new mathematics in schools**. They argued that teachers and students learn to do the new mathematics and that teacher-educators concentrate on teaching teachers to **do** the new mathematics. But the word "do" was understood to mean that the new curriculum would be based in the earlier grades on speaking arithmetic in the new number system language, and in later grades, speaking algebra. Special care was needed because of the difficulties in speaking complicated statements (younger students had difficulty in saying "parenthesis" and everyone seemed to stumble on differences among "minus," "negative," and "subtract").

Reformers argued convincingly that chanting and memorization be reduced—although, in practice, it seemed that chanting was a good way to learn how to say all that complex new mathematics. Some studies purported to show that performance in recitals of even the new mathematics seemed to be improved as a result of chanting, especially if one used tapes to help with chanting.

Indeed, some educators suggested that recitals could be submitted, or even performed, on paper! Such suggestions met considerable resistance on several grounds. One was the great disparity in types of paper and pencils that might be used (some students had the large paper and pencils that might be used outside of schools—with erasers) which would yield them unfair advantage. A second, nearly decisive objection was rooted in the feeling that the basics were being subverted by paper-recitals. However, literally hundreds of studies were performed that showed memorization skill was not damaged by writing and, in fact, in some instances, it was slightly improved.

Some innovators eliminated virtually **all** chanting. Their students wrote the mathematics instead. Researchers showed that these students "reached basic skill levels" in many fewer repetitions than those who used chanting—as few as 20 repetitions could have the impact of 50 chantings. Remarkably clever writing techniques were developed, one of which deserves special mention because of the controversy it provoked among educational psychologists. This technique (termed an application of the new "cognitive science") involved reorganizing the material to be learned so as to facilitate more efficient writing. Most writing was done sequentially in the order that the material would be spoken, in horizontal rows, as follows:

Left parenthesis, x plus three, right parenthesis, squared \ equals x squared plus six x plus 9
Left parenthesis, x plus three, right parenthesis, squared \ equals x squared plus six x plus 9
Left parenthesis, x plus three, right parenthesis, squared \ equals x squared plus six x plus 9
Left parenthesis, x plus three, right parenthesis, squared \ equals x squared plus six x plus 9
Left parenthesis, x plus three, right parenthesis, squared \ equals x squared plus six x plus 9
...

Younger readers may not recall that "\" was a pedagogical symbol introduced to indicate a pause.)

However, this striking innovation (known as "mastery learning") took the form of listing all repetitions of each word to be "mastered" before moving on to the next word. Applied to the above lesson, it would begin as follows:

Left
Left
Left
Left
Left
...

Despite the uncanny originality of this reorganization of the learning process, the reader likely sees the radical horizontal-to-vertical learning principle. This innovation was shown to decrease the amount of time needed for a given number of repetitions by as much as 30%. After initial enthusiasm, however, the method failed to gain lasting acceptance since, on further study, it was found to actually improve **learning** only for quite short statements, of the type "one times a equals a" (a statement surely worth knowing—that is worth remembering).

But bigger problems loomed as reformers became more insistent that large changes were needed. They declared that teachers and students should learn to speak the new mathematics well, use it to solve problems and reason effectively outside of schools. To this end, it was recommended that forms of reciting be revised, from chanting to extemporaneous spoken mathematics, which was much closer to the kinds of mathematics done outside of schools—although, as became increasingly clear, outside of schools, most mathematics was done in written form. Somehow, the awkward compromise that had evolved was not working, especially as it related to reciting. And, of course, recitations were of paramount importance. While not as profitable as the tape industry, there had developed a rather large recitation industry for creating, administering and judging recitations, with its own systems of rationale for its methods.

We have already noted the subtle conflict between those who thought that using the new writing technology meant writing the old mathematics (which for some was the only mathematics they knew) and others who saw schools as a place where students should learn the new mathematics, that is, where students would memorize the new mathematics—by chanting from tapes, by reading from a paper, or even, by writing the new mathematics—over and over again, in long lists, as described earlier.

Reformers seemed to be saying that even more change was required. They spoke—and wrote—with great urgency and fervour about the need for students to do mathematics, to solve problems, to "construct" mathematical ideas. There seemed to be a deeper difficulty. Did they mean "speaking extemporaneous mathematics," or "writing extemporaneously"—as was done in the new language arts classes? A few people recalled that, historically, before writing, even before tapes, some had suggested creative mathematical activity—the inventing of new techniques, new ways of computing and reasoning. Those suggestions never seemed to be heard except by an earnest, but sparse few. Would the situation change now that writing was available to many students? After all, the situation **had** changed in the world outside of school, where writing predominated.

In the face of this increasingly obvious need for students **to do actual mathematics,** reformers officially insisted that each classroom be equipped with a large paper-pad and pencil and one tiny pencil and pad for each student. In this way, all students would actually get to **see** written mathematics on a regular basis, and, perhaps on a weekly basis at least, write some mathematics themselves on the big pad. Within ten years most schools had at least two large pads per classroom and most students spent at least an hour weekly writing mathematics. Initially, they mostly copied mathematical writing from the paper pad and from the textbooks (this was generally felt to be a more enlightened use of the new technology than copying spoken mathematics from tapes), and gradually they began to solve problems using pencils. And gradually, the pads became larger. Some visionaries began to call for a pencil for every pair of students, and large paper for the teacher that the teacher could even use at home.

Progress was slow until the now famous Odnetnin breakthrough. Odnetnin, a creator and marketer of written games for children, and, later, adults (as its players grew older), made a fortune selling these games—almost every home had them, but no schools ever did. Without warning, Odnetnin bought control of the largest tape seller, applied its sophisticated writing technology to education, and the rest, as they say, is history.

Working Group A

Fractal Geometry in the Curriculum

Peter Harrison

Fractal Geometry in the Curriculum.

Participants: Tasoula Berggren, Don Eastman, Bernard Hodgson, Maurice Garançon, Jacqueline Klasa, Gila Hanna, John McCarthy

Group Leader: Peter Harrison

At the 1990 meeting of the CMESG, Ron Lewis and Brian Kaye led the working group *Fractal Geometry and Chaos for High Schools* (see the 1990 proceedings) and discussed many topics that they had presented to high school students in recent years. It was felt by the program committee that a follow-up working group should be organized in which participants could reflect and comment upon many of the exciting possibilities suggested by Ron and Brian.

Lewis and Kaye have offered material which is very rich in applications. Indeed, professor Kaye is well known for his research and industrial applications of fractals and Ron Lewis has done an admirable job in bringing some of this work into the classroom.

However, Robert Devaney has observed [1991] that

> "... the real importance of [chaos and fractals] is not in the applications that will stem from them. Rather, I feel that these ideas will have their biggest impact in mathematics education. The mathematics of chaos and fractals is at once accessible, alluring and exciting. Fractal geometry offers a wonderful arena for combining computer experimentation and geometric insight."

Devaney is a mathematician and Kaye is a physicist. They at least seem to agree that there is a definite place for chaos and fractals in the school curriculum.

As working group leader, I brought several personal concerns to these sessions. These included:

- some issues related to the value of empirical verses deductive methods in mathematics
- the measurement of student achievement in what is often seen as an "investigative" or "aesthetic" area of study
- the availability of hardware and reasonable software
- teacher acceptance, training, competence and fear
- the possibility that students may become as disdainful of pictures of the Mandelbrot set as they are about work sheets on factoring.

I also brought along a considerable enthusiasm for fractals since I find them interesting and have had some success in sharing this interest with many of my students.

During our first meeting, we spent some time introducing ourselves and describing some of our own background with fractals. There proved to be a considerable range of familiarity and interest.

There was a thoughtful discussion about "how we understand pictures". What do the images of fractals convey to us? Is it possible to gain any appreciation of fractal geometry without a prior familiarity with Euclidean geometry?

There was general agreement that any study of fractals would necessarily involve the use of computers and this would probably require some knowledge of programming and computer technology. Experience seems to bare this out. We discussed the possibility that any boundary that may presently exist between mathematics and computer science would probably crumble with the introduction of fractals into the classroom. Will anybody feel threatened by this?

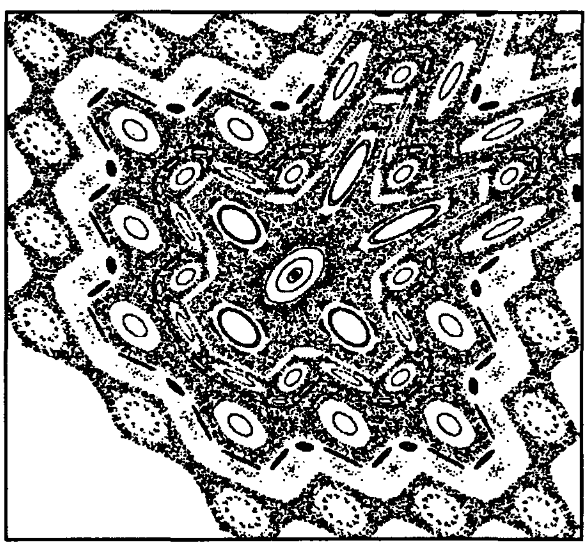
During our second meeting, I presented an overview of a unit on Iterated Function Systems (a "traditional fractal topic") that I presently teach in my Algebra and Geometry OAC (Ontario Academic Credit) course. This unit involves the following "old style traditional" topics:

- transformations
- self-similarity
- random numbers
- determinants
- systems of equations
- probability distributions

I made the following observations based on teaching this material for four years:

- students work very co-operatively and effectively in small groups,
- they develop and acquire an appropriate vocabulary for the topic,
- there is a high level of recognition of internal self-similarities (girls appear to be better at this than boys),
- the students experience an effortless use of computers,
- students enjoy the material.

The group went on to discuss dynamical systems as models of the world. We seemed to agree that students could benefit by witnessing how simple equations can generate complex outcomes. We shared a sense of wonder at the equations for the "Gingerbreadman" or "Rabbit" as investigated by Devaney (1984).



Gingerbread Rabbit: A = 0.99

$$x_{n+1} = 1 - y_n + |Ax_n|$$

$$y_{n+1} = x_n$$

As we sat fascinated by the resulting orbits, we wondered:

- which students would enjoy this?
- are we pursuing legitimate mathematics?
- how can we address the educational (political?) concerns such as evaluation, assessment and accountability?

On day three, we discussed the teaching of "maps". We mentioned "simple" problems such as school boundaries with different geometries. It was pointed out that the *design* of a solution becomes the real assignment. Several examples were discussed:

1. Probability problems:

- Buffon's needle problem,
- find the probability that two randomly chosen numbers ($0 \leq r_{1,2} \leq 1$) will add to less than $\frac{1}{2}$.

2. Difference equations:

- for example:

$$x_{n+1} = px_n(1 - y_n)$$

$$y_{n+1} = x_n$$

  where the map *is* the solution,
- basins of attraction for roots of various equations,
- by studying the maps produced by different root-finding techniques, meaningful observations can be made regarding the different rates of convergence.

This session and working group came to a close with a rather exciting discussion about how the introduction of fractal topics will necessarily challenge most of the dearly held views on what constitutes mathematics education.

We concluded that fractals can expose students to a less familiar side of mathematics by offering:

- open questions
- equations with no algebraic solution
- the interdependence of mathematical ideas
- the construction and investigation of mathematical models
- some of the beauty and fascination of mathematics
- a sense that mathematics is *alive* and *recent.*

I would like to thank the members of our small group for their many insights and observations and friendly encouragement.

References

Devaney, R. L. (1984). A piecewise linear model for the zones of instability of an area preserving map. *Physica D10*, pp 387-393.

Devaney, R. L. (1991). Introduction. *The College Mathematics Journal*, Vol. 22, No. 1, January 1991, p 2.

Working Group B

Socio-cultural Aspects of Mathematics

Compiled by Martin Hoffman

Report of Working Group B: Socio-Cultural Aspects of Mathematics

Opening statement

Working Group B, entitled Socio-Cultural Aspects of Mathematics, endeavored to examine social and cultural issues in mathematics education. Questions such as the following were to be discussed:

- How is mathematics used to understand and/or obscure social issues?
- How can we reconceptualize mathematics to incorporate non-Eurocentric views?
- What are the effects of culture, language, and ideology on the mathematics people develop?
- In addition, D'Ambrosio's notion of ethnomathematics, "the mathematics practiced among identifiable cultural groups", was to be examined, with emphasis on pedagogical implications for the classroom.

The participants were:

Alan Yoshioka
David Wheeler
Pat Rogers
Martyn Quigley
LaDonna MacRae
Lesley Lee
Thomas Kieren
Martin Hoffman (co-leader)
Marilyn Frankenstein
Ubiratan D'Ambrosio (co-leader)
Marcelo Borba

The discussions were wide ranging, covering in some depth topics not specifically listed in the working group description, while only lightly touching upon some topics in the description. Between the second and third sessions of the working group, the participants decided to write a page or so on their definition of ethnomathematics (or some related topic). These brief essays were duplicated and distributed to the participants during the third session. The range, depth, and somewhat unpolished form of these statements provided a basis for the discussion in the third session of the Working Group. It was also felt that these statements, with accompanying commentary and questions, might provide a reasonable picture of the deliberations of the group. The following are the transcriptions, with minor revisions, of the statements that were written by the participants after the second session. These transcriptions were distributed by electronic and ordinary mail to all the participants who then provided the commentary and questions that follow each of the original statements.

We hope that through these devices the reader can share at least a small part of the excitement and intensity that marked these sessions.

Contributions

Tom Kieren

Like any good teacher I have—I think—always tried to be conscious of the knowledge of my students and in any particular circumstance (class, day, particular student, particular incident) would, in some way, effectively use that knowledge. I was likely capable of reflecting or receiving such knowledge and hence had a kind of understanding of "ethnomathematics" in this most local of senses.

It was not until 1984 that I developed an image (a more general understanding) of ethno-mathematics after listening to Ubi (D'Ambrosio) at ICME 5. This image, whose definition I gave yesterday, "the mathematics a person knows just because they live (even without schooling)", became a rather unexamined core of my own thinking about personal knowledge of mathematics. As such I generated and elaborated some aspects of it for myself over the past 7 years.

Yesterday I was stimulated to fold back—to remember and to engage in activities to re-make my own image and understanding of ethnomathematics.

I have been a student—in my own way—of history of mathematics at least for 35 of the 50 years of my life enjoying playing with a number of the ideas of mathematics in terms of their supposed genesis. Our group's work has asked me to re-think this as well.

I now think ethnomathematics—even for a student—must not only be known to a teacher in some way but developed as a core. This, for me, now means that teachers should reflect on the history of maths in their country, and even city. The child can reflect on the "history" of maths in their own family or locale or culture or gender or... If a person is to have intellectual self-esteem and if their mathematics is to grow, then they must, even as a child, be able to name and claim their own history. By this I am not suggesting the formal study of such histories—although for some students at some levels this may be important. But I think the teacher should provide activities which have students investigate mathematical aspects of their own lives and communities and that such activities would have a valued part in any mathematical curriculum.

Ubi said, "Their (i.e., local contributors to mathematics) names do not exist." If ethno-mathematics is to have substance for us and for students such names must continue to come into existence.

Comments

I find this concrete project of looking at the local history of mathematics very appealing; ethnomathematical theory insists that we ground it in experiments like this. I'm just struggling to imagine what it would look like and how we would do it. (Alan Yoshioka)

I think another reason it is important for students to be aware of their ethnomathematics is because self-reflection is a key part of any in-depth learning. Further, class activities involving students teaching their ethnomathematics to other students will increase everyone's self-esteem and understandings. Finally, the search for mathematical ideas in one's general pool of knowledge can lead to a breaking down of the usual split between "practical/concrete" and "theoretical/abstract" knowledge, and the separations among the various academic "disciplines". (Marilyn Frankenstein)

Ubi D'Ambrosio:

I suggest to approach socio-cultural issues in Mathematics Education through the research programme which I call *ethnomathematics*. This is NOT a theory or a concept, or a method, but rather it IS a programme, in the sense of I. Lakatos, on cognition, epistemology, institutionalization and diffusion of knowledge, focused on the evolution of ideas related to what we nowadays call mathematics, in the diverse habitats of the several branches of homo sapiens since the appearance of the species on the planet.

Since the earliest times humans have developed their intellectual capabilities in order to cope with reality and to manage it for their survival and at the same time in order to explain and to understand while struggling for transcendence of their own existence. These intellectual capabilities or mentifacts (complemented by artifacts) are organized as *tics* (from techne = art or technique) of explaining, understanding, managing (for which I use the Greek *mathema*) and of course this developed differently in each different habitat which is emphasized by using the prefix *ethno* (which means essentially culture). Thus...*ethno mathema tics*.

The methodology takes into account cultural dynamics (springing out of the capabilities of communication) as a major force in the build up of knowledge, both horizontally—individual knowledge, i.e., learning—and vertically—organization of experiences and reflections accumulated in a communication through the life of individuals and of the entire community from generation to generation, i.e., history). This methodology calls for a broader way of looking into history. The so-called "official" or documented history gives only the version which justifies the present, it relies on the "winning" ideas, while in dealing with cultural dynamics, the losers and why they lost are as important as the winners and why they won.

This programme has obvious pedagogical implications, mainly by calling on the free flow of cultural dynamics in the classroom. The teacher is seen as managing a process which stimulates socialization and facilitates the flow of knowledge and the constructions of knowledge. Besides, the ethnomathematics programme sees cultural diversity as important for the harmonious intellectual evolution of our species, i.e., for the preservation of civilization, as bio diversity is for the preservation of life on the planet.

Comments

I was alarmed at Ubi's assertion that teacher education programmes must be the means of spreading a sensitivity to ethnomathematical issues among teachers. Just as schools are increasingly, and unrealistically, expected to include prophylactic instruction against all the ills that society thinks itself heir to, so teacher education programmes are equally unrealistically asked to overturn all the (supposedly) undesirable mind-sets of the students. Deborah Lowenberg Ball's paper in a recent issue of FLM is a reminder of the possibilities and impossibilities of methods courses, which is where such reorientation would have to be accomplished. And further, where are the instructional materials that the teachers of these courses would surely need? They are largely conspicuous by their non-existence. OK, I'm not running away in despair but rather saying: find every possible chink of enlightenment among teachers, mathematicians, teacher educators, etc., and start there, not fooling oneself that the time is yet ripe for a global onslaught. (David Wheeler)

"The losers and why they lost are as important as the winners and why they won."(Ubi) "Mathematical history is a rational reconstruction." (paraphrase of Lakatos) "As mathematics develops it reconstructs its own history." (paraphrase of Tahta) Most mathematical history is written from the side of the mathematicians; they are often as militant against errors as fanatics of an organised religion. We cannot expect the Boyers or the Daubens to tell us what we really want to know. Where can we find models of the sort of historical writing that will? (David Wheeler)

This view of history is part of a broader project by intellectuals of the Third World to reorganize dominant cultural practices which glamorize the American Way at the expense of complexly evolving local cultures. As Canadians I think we can identify with their need to dismantle the national "inferiority complex." What seems to be missing from this conceptualization, however, is an account of economic and political domination within this nation. How do local elites use "national culture" to reinforce their power? (Alan Yoshioka)

I liked the question Ubi posed of why mathematics does not obey what he calls the "laws of cultural dynamics," under which the dominant religion, language, etc. of the colonial power becomes hybridized with the colony's prior culture, while - more than any other aspect of Western culture—"European" mathematics has penetrated the rest of the world with hardly any local variation. (Alan Yoshioka)

Ubi observed that in the history of math, the European is the standard against which others are judged, e.g., the Incas were smart because they even knew some trigonometry. I found, in trying to explain how ethnomath might affect a classroom situation, it was very easy to slip into a language of "cultural deficits", e.g., Chinese grammar LACKS an if-then-else construction. I need to find a way of talking about cultural difference which avoids our Anglo-American propensity to rank everything as better or worse. (Alan Yoshioka)

D'Ambrosio brings to mathematics education the more general notion that history has meant "history of the winners". I would like to add that D'Ambrosio's concern also highlights the need to take into account and legitimate the history and culture of the ones who are under attack today and might become the "losers" tomorrow. (Marcelo Borba)

I think the view of ethnomathematics as a "programme" helps focus on the activities involved, connecting the development of mathematical knowledge to the development of human cultures, which in turn are connected to the material conditions of the society. I think it would be fascinating to research a world history of mathematics which highlights how those mathematical ideas which were developed and institutionalized were those ideas needed to justify and extend the hegemony of the "winners". (Marilyn Frankenstein)

David Wheeler

I am certainly sympathetic in general to the acceptance of the socio-cultural influences on mathematics, and to the general ambitions of the ethnomathematical movement. But, skeptical as ever, I utter three (of many) cautions:

1. The dangers of: "mathematics for the gifted few, ethnomathematics or the rest". Thirty years ago, in the U.K. and elsewhere, first language teachers were saying that most students could not appreciate the classics, so perhaps "the less able" students should study popular literary culture instead.
2. Intellectual activity depends on the economy that comes from abstract, symbolic language (hence, perhaps, the tendency of people to use mathematical competence as an indicator of intellectual capacity). We will not be doing students a favor if an emphasis on domestic, local, user-friendly mathematics denies them access to the intellectual power of abstract mathematics.
3. If the reconsideration of what constitutes mathematics goes too far, it will degenerate into "mathematics is all around you" (a catch-phrase of "progressive" education). This both trivializes mathematics by denying it a particular character, and claims too much for it by indiscriminately recognizing its presence in anything).

Comments

I find these cautions worth serious consideration. Marilyn gave the analogy of the debate in the U.S.A. over "Black English," which is as rich and expressive as any other language, and is emphatically not "ungrammatical." But if young black students do not learn standard English, they will lack a crucial tool for engaging the white-dominated power structure, and will be excluded from many jobs. So they should be taught the standard language but in a critical and reflective way which asks why they need to speak in a certain way. (Alan Yoshioka)

Can we be clearer about our goals? Are we primarily concerned about students' access to the "intellectual power of abstract mathematics" or about their access to jobs (or about transnational corporations' need for a skilled labor force)? (Alan Yoshioka)

Could a notion of ethnomathematics that allows for academic math as being one among other mathematics address some of your concerns? (Marcelo Borba)

1 and 3: I don't see "ethnomathematics" as separate from "mathematics". "Ethnomathematics" includes what is currently called "mathematics", in addition to many ideas developed in cultural groups who have not had the power to make their ideas part of our general pool of knowledge. "Mathematics" might have included different concepts if other cultures' mathematical ideas had been taken seriously. (Paulus Gerdes has done this with the networks involved in the Tchokwe

sand drawings from Angola—he has proven new theorems, theorems not in current "mathematics" texts because the questions these theorems answer arise from considering their sand drawings.) I'm not sure of the value of particular labels for the different "disciplines"; knowledge is much messier and interconnected than the labels could ever acknowledge. Indeed, we can't think about every aspect of everything all at once—we do need to focus. But, I'm not sure the most helpful focus is a specific "discipline". It may be more useful and less trivializing to focus on certain "questions". Why not re-organize curricula around knowledge and activities that explore various "interdisciplinary" questions?

2: I agree that all students would learn abstract symbolic language. But, I would not limit a definition of intellectual activity to a dependence on that language. I think one could argue, as I believe David Pimm has in his talk at the April 1990 Political Dimensions of Mathematics Education Conference, that abstract symbolism disconnects mathematical knowledge from the consequences of its use in the world. Further, manipulating abstract symbolic language to solve problems can give a false "objectivity" and "simplicity" to the solution. Has all the mathematical competence out there solved the major intellectual questions of our times—from hunger and hopelessness in the midst of plenty, to a cure for cancer and AIDS? Maybe the kind of intellectual activity that is focused through abstract, symbolic language has impeded these intellectual explorations. (Marilyn Frankenstein)

Lesley Lee

What distinguishes the ethno-math view of history and view of mathematics from any other view? (Referring here to Marilyn's "3 views of ethno-maths".)

Is the pedagogical proposal any different from any progressive student-centered pedagogy? From the pedagogical proposal put forward by constructivists say?

Do women constitute a cultural group and do gender issues fall within ethno math? Is there anything new in ethno-math (view of history and math, pedagogical proposal) that has not already been developed by feminists (gender analysis of the history of math, math itself, and pedagogical practices)?

Is there an ideological stance, political analysis,... behind ethno-math? Can a "chauvinist capitalist pig" be engaged in what could be defined as an ethnomathematical pedagogical practice? (Example: Professor assists "natives" to recover lost mathematical arts in order to produce a video which will then be sold to tourists to earn money to buy a computer for the school.)

Comments

Ethnomathematics as a programme, as a pedagogy or as a view of mathematics is no panacea. It is not ethnomathematics that will make chauvinist pigs become caring men, discover their sexuality, or take into account all the agenda of the feminist movement. Does that make ethnomathematics unimportant then? The stress on social-cultural issues in mathematics has never been addressed with such an emphasis; this might be the new side of ethnomathematics as pedagogy, which, indeed, shares assumptions and principles with other movements. (Marcelo Borba)

I see all the questions raised here as connected to the intersection of "ethnomathematics" and "criticalmathematics" and providing some of the reasons why Arthur Powell, John Volmink and I composed a definition for a "criticalmathematics educator". Yes, not only can there be, but there are people who are involved in ethnomathematics (meaning they describe their work this way and belong to our International Study Group on Ethnomathematics) whose work is concerned only" with (from our point of view) narrow aspects of that discipline, such as recovering "lost mathematical arts" (none that I'm aware who are making money this way—can any of us imagine tourists paying for a video on, say, the quipu?). The Criticalmathematics Education Group intersects the concerns of ethnomathematics and progressive student centered pedagogy (which might not include and ethnomathematical concerns) and economic and political change in all institutions of the society, not just in the educational arena. [Our definition appears in these proceedings in my summary of my talk.] (Marilyn Frankenstein)

Pat Rogers

- I would like to return to the original questions outlined in the abstract. In trying to define ethnomathematics I feel we have lost sight of the socio-political/cultural aspect of our discussions. Our pedagogy must be a political (subversive) act validating the lived experiences of our students and teaching them the rules of "the club" so that they in turn can transform them rather than be excluded or co-opted.

- The development of math is a cultural process itself which in turn has repressed its own process. Example: The decontextualisation that took place in developing the nature of proof took 2,000 years but we have lost that cultural dimension in our practice. This is an illustration of both the positive and negative aspects of the power and authority of mathematics and how they frequently go hand in hand (cf.B.Russell).

- The math which children bring with them to the classroom is itself a cultural and social construction and not "out there". We need a critical approach to this too lest we romanticise it just because it is!

- Half (or more than half) the voices and the names that do not exist in mathematics are female. Gender is a very relevant theme in this discussion contrary to the impression some of us have created. Mathematical discourse as currently conceived is not intended for Her (or for any Other).

- Aren't we really talking about anti-racist and anti-sexist mathematics not ethnomathematics? If not, shouldn't we be talking about it?

Comments

I wonder why I have reservations about the idea of "empowerment"? Maybe because I sense that in some usages it pays only lip-service to the interests of those who are to be empowered. Indeed, I think that just as most of us had to learn, slowly and painfully, that students don't have to be taught to think since they already know how, we may need to learn that we don't have to try to give them powers since they already possess them. What remains important, perhaps, is to help students to an awareness of the powers they own and of the opportunities that exist for using them. The rest may best be described in traditional terms as, say, knowledge, skill and understanding. I want students to have access to those activities they would like to engage in, some of which certainly require mathematical fluency. Mathematical qualifications serve as a membership card for a few, a card they're entitled to have, I think.

I'm trapped now, though, because I don't believe that the "few" can be identified in advance, yet I would not want everyone else to have to master some mathematics that they neither like or need. This is, I suppose, the big question about the educational justification of "mathematics for all". I'm tempted to say that mathematics is a guide to intelligent functioning, but then I can then hardly distinguish this from items of the schoolroom folklore that I have often objected to—"mathematics teaches you to think/reason/solve problems", for example, "it trains the mind", "it makes you logical", and the like. I glimpse the way that ethnomathematics might give me an exit from this dilemma, by providing a quite different sort of justification, but I"m not yet ready to embrace the solution wholeheartedly. (David Wheeler)

Right on! Pat has named the issue of *power* in several ways. One which I'd like to explore is the relation between gender and the process of decontextualization she described. (Alan Yoshioka)

Some of these questions also apply to my comments to Lesley Lee. In addition, it would be interesting to develop a curriculum unit, using the example of proof, about how "the development of math is a cultural process itself which in turn has repressed its own process", especially as the nature of proof has evolved in the recent history of mathematics in such a way as to challenge the notion of mathematics not being an empirical science (e.g., the computer proof of the four-color theorem) and to challenge the notion of deductive certainty (e.g., Rueben Hersch has written about how long proofs are almost certain to contain errors—he claims mathematicians believe the results for other reasons than that they have been deductively proved, reasons such as symmetry with previously believed results, or elegance, or simplicity). (Marilyn Frankenstein)

Marilyn Frankenstein

A few notes from the paper Arthur Powell and I are working on: We feel that teaching from an ethnomathematical perspective involves the underlying assumption that through interacting in myriads of daily-life activities, people know how to think, and, more specifically, they think mathematically. To understand their ways of thinking mathematically we need to consider and re-define conventional notions of mathematical knowledge. We need to learn about how culture—daily practice, language, and ideology—interacts with people's views of mathematics and their ways of thinking mathematically. To enable students to discover that they already think mathematically, and, therefore, can learn "school" or "academic" mathematics, we need to teach in ways that connect their mathematical understandings with an undistorted history of mathematics and with the "academic" mathematics they are studying.

An underlying theme that emerges from our reflections is that the separate categories so commonly made in much academic thought need to be reconsidered. For Freire (1970,1982) this means breaking down the dichotomy between subjectivity and objectivity, between action and reflection, and between teaching and learning. For Lave (1988) it means understanding how "activity-in-setting is seamlessly stretched across persons acting". For Fasheh (1988) and Adams (1983) it means that thought which is labeled "logic" and thought which is labeled "intuition" continuously and dialectically interact with each other. (Further, D'Ambrosio (1987) challenges the static notion that "there is only one underlying logic governing all thought" and Diop (1991) illustrates how the interaction between "logic" and "experience" changes our definition of "logic" over time.) For Diop (1991) it means that the distinctions between "Western", "Eastern", and "African" knowledge distort the human processes of acquiring/creating knowledge from interactions with each other and the world. We argue in our article that underlying all these false dichotomies is the split between practical, everyday knowledge and abstract, theoretical knowledge. Understanding these dialectical interconnections, we believe, leads us to connect mathematics to all other "disciplines" and to view mathematics as one aspect of humans trying to understand and act in the world. We see ethnomathematics as a useful way of conceptualizing these reconnections from a theoretical and a curricular point of view.

References

Adams, H. H. (1983). African Observers of the Universe, the Sirius Question. in I. V. Sertima (Ed.), *Blacks in Science, Ancient and Modern* (pp.27-46). New Brunswick, N. J: Transaction.

D'Ambrosio, U. (1987, September). Reflections on Ethnomathematics. *International Study Group on Ethnomathematics Newsletter*, pp.5-8.

Diop, C. A. (1991). *Civilization or Barbarism: an Authentic Anthropology*. New York: Lawrence Hill Books.y

Fasheh, M. (1988). "Mathematics Education in the Global Village: What can we expect from Ethnomathematics?" Panel discussion, Sixth International Congress on Mathematics Education, Budapest, Hungary, July.

Frankenstein, M. & Powell, A. B. (in press) "Toward Anti-Domination Mathematics: Paulo Freire's Epistemology and Ethnomathematics", in McLaren, P. and Leonard, P., eds. *Paulo Freire: A Critical Encounter*. London: Routledge & Kegan Paul.

Freire, P. (1970). *Pedagogy of the Oppressed.* New York: Seabury.

Freire, P. (1982). "Education for Critical Consciousness". Unpublished. Boston College course notes taken by Frankenstein, M. July 5-15.

Lave, J. (1988). *Cognition in Practice.* Cambridge, England: Cambridge University Press.

Comments

An important thread for me was the business of different logics. Can we go on pulling at the end of that one? It seems to me to have a promising future whether one accepts the whole ethnomathematical story or not (perhaps I should say "any of the "n" variants of the ethnomathematical story") In a 1950's paper, Caleb Gattegno talks of "logics", including a "logic of perception", for instance—reminding me of Bhaskara's ("Behold!") demonstration of Pythagoras' Theorem. Frances Hawkins, wife of David Hawkins, the philosopher of science, called the book based on her experiences working with preschoolers, "The logic of action". Brouwer, we all know, challenged the Aristotelean logic applied to infinitary mathematics. All this is quite sufficient, without even entering the arena of alternative cultural patterns of thought, to explode any simplistic connection between mathematics and logic. Indeed, by taking logic out of the books and locating it in the universal human capabilities of doing, perceiving, and communicating, we might more easily be able to explicate the cultural dimension and show precisely how it affects mathematics. (David Wheeler)

Marcelo Borba

A problem can be seen as a situation which involves an impasse in the flow of life and which is important to (someone's) existence. When a problem results in mathematical treatment, it can lead to the generation of mathematics by the person(s) who was (were) puzzled by this situation. A person is a cognizant being who functions within the language and interpretative code of her or his socio-cultural group. A language is a code understandable only to people who have participated in common past experiences. Each language expresses a way of knowing developed by a group of human beings. One way of knowing is mathematics. Mathematical knowledge expressed in the language code of a given socio-cultural group is called ethnomathematics. Even the mathematics produced by professional mathematicians can be seen as a form of ethnomathematics. Hence, ethnomathematics should not be understood as "vulgar" or second class mathematics, but as different cultural expressions of mathematical ideas.

Does this definition address some of the issues raised on the questions on the program of this Working Group?

Comments

The definition above was not a result of a secluded intellectual exercise. This definition was mainly developed during two weeks of field work in a slum in Brazil. The mathematics of that socio-cultural group was studied and incorporated into a pedagogical proposal for the kids (Freire, P. et al, 1987) in Portuguese. (Marcelo Borba)

The description seems rather bland considering Marcelo's experience of teaching math to kids in a Brazilian slum. What made an ethnomathematical approach important to that community? (Alan Yoshioka)

I like the connections drawn here between mathematical ideas and the language in which they are expressed. It very nicely includes "academic mathematics" in ethnomathematics. However, it also raises questions such as "can one have mathematical knowledge that is not expressed in language?"—I'm thinking here of geometrical or geographical knowledge. Also, which mathematical ideas can be translated in all the languages? Are there mathematical ideas that cannot be translated into other languages? (Marilyn Frankenstein)

Martin Hoffman

I came to the Working Group seeking ways to bring the teachings of ethnomathematics more explicitly and more meaningfully to my classrooms. Although I did not receive many specific answers, I have received much of greater value, including:

1. An appreciation of what one means by ethnomathematics, especially the range of human activities that it encompasses.
2. Several questions and areas for investigation that I will endeavor to attend to during the coming year. Among these questions are:

    1. What systems of validation do my students utilize? (from Tom Kieren). The question will become part of an investigation I will be undertaking (with Arthur Powell) to learn about the culture of the classroom through various students'writing activities.
    2. What is the (proper) role of authority? (from David Wheeler, Martyn Quigley and others). I feel I've looked at this questions before, but so many insights were offered that I feel I need to look again. Powell and Frankenstein (p. 34) make interesting comments on the role of the teacher in this regard "... to be strong influences without being superiors, constraining and controlling the learning environment". Also related to this is the question (of Pat Rogers) on the role of mathematics as a de-humanizing activity.

Comments

"Authority" preoccupies me: our discussion didn't help me get much further with it. The two aspects (out of many) that particularly exercise me are: (1) "the legitimate authority of the teacher". It seems to me that teachers ought to continually face themselves with the question, "By what right do I do (say, tell, ask, demand) these things?" It is, I think, an extremely hard question to answer much of the time, but it needs to be asked because when it isn't, teachers tend to fall back on tradition, customs, self-righteousness or arbitrary power trips. Perhaps it can only be satisfactorily answered in conjunction with an awareness of (2) "the intrinsic authority of mathematical knowledge". Here, I want students to know mathematics "because it is so" and not because they are told what to believe. This is the authority stemming from what I would call truth. Strangely, I don't feel the authority is essentially weakened by acknowledging that the truth involved may be temporary, partial, relative, uncertain, but I do notice a residual difficulty: can I be quite sure that I am working at the level of mathematical reality and am not just seeing what I have been trained to see? (David Wheeler)

Very possibly the systems of validation that students use have been influenced by the competitive, individualistic cultures of world capitalism. Students may not be validating their knowledge because the systems of validation they are using do not consider their knowledge as valid. If you find this, will you and Arthur intervene? What kinds of writing activities are you planning? (Marilyn Frankenstein)

Martyn Quigley

Rather than give a definition of something which, essentially I came to the group to learn about, I want to recount three specific incidents which occurred in my classroom whilst I was a high school teacher. This is to illustrate the 3 dimensions I perceive when I think of the words "social" and "mathematics" together.

1. I was trying to help a 13-year old from rural India with his subtraction. Using counters, I was getting nowhere. Then one kid said, "That's no good sir, where he comes from they count on their fingers." Bingo.

This depicts how all children come to the classroom with "something", they are not the clean slates we teachers often assume they are. For me the trick in teaching is to build upon what is there. This particular student was perfectly capable of adding and subtracting even large numbers, just not the way I was expecting.

2. For 3 years I taught a (grade 7) course which integrated mathematics, geography and science. It was a lot of fun. One day whilst studying human biology we somehow got onto how heart rate increases when the body is under stress. So I had the kids (boys actually, the girls—predominantly Muslim—declined) do the Harvard step test. (In this test, the subject's heart rate is measured and then s/he is made to step up onto a bench and then down again repeatedly for a minute or two. Then the heart rate is measured again.)

   The spontaneous discussion led to the number of seconds in a minute, minutes in an hour, and so forth. Before anyone realized it, we had "done" modular arithmetic. All of it and more besides.

This shows how, for me, it is important for mathematics to come from something which is of direct and immediate relevance. The teacher should not be afraid to ditch whatever was planned and exploit a situation which may simply occur for no apparent reason.

3. I taught one particularly troublesome, but clever, grade 11 boy, Denis. We were doing proof by induction and he was having a hard time. All of the sudden, in the middle of class, he had his "aha!" and he shouted out, "Well, I'll be fucked!" What could I say?

This illustrates how learning should override the artificial nature of the classroom environment. Denis' reaction, when he suddenly grasped what is, after all, quite a subtle principle, would normally be deemed utterly unacceptable language for a classroom.

Comments

The second example raises questions for me about teacher intervention. The fact that all the girls declined to participate in a physical activity would concern me as a teacher. Did you or any of the students comment on this? There are possibilities in this discussion for mathematics learning too—understanding and analyzing overall statistics about "boys" vs. "girls" activities; conducting and analyzing a poll in the school about who would take the step test and why. (Marilyn Frankenstein)

Alan Yoshioka

- The process of taking back, recovering or reclaiming mathematical language from patriarchal and colonial system is a difficult experiment. We are groping for ways to speak the unspeakable and we will stumble along the way. Some of the pedagogical examples I have read or heard seem forced or awkward to me, but I hope we can be patient with each other in our criticisms.
- I have found the discussion tends toward the better explored territory of economic oppression and there has been a lot of resistance to feminist insights that the personal is political. The language that is available to us makes it relatively easy to talk about the amount of work that women perform worldwide without getting paid. It is harder to talk about how women in many cultures are excluded from the power to shape systems of meaning, one of which is mathematics. To do the latter authentically requires a different, perhaps more embodied and emotional, mode of thinking than many of us are used to.
- Let's remember how women's names are removed from history. Let's also emphasize that our conclusions do not simply spring into an existence without history and personal association, but that human actors are creating mathematical knowledge, by naming our own names in the summary document.
- Are we not part of a revolutionary movement? Do we need to say so? For our own sake? For others' sakes?

- I like Henry Giroux's typology [in Marilyn Frankenstein and Arthur Powell's ICME-6 paper] of ideologies underlying pedagogical practices. Instrumental ideology focuses on prediction, efficiency and technical control; this de-contextualized approach may allow advanced mathematical training in order to build more deadly cruise missiles. Interaction ideology, as found in "humanistic" math teaching emphasizes individualized instruction, process over product, alleviating "math anxiety" but omits notions of conflict and power differences along gender, racial or class axes, etc. Critical ideology helps learners reinvent and create the tools to transform an oppressive social reality.

Comments

The point about "do we need to say we are part of a revolutionary movement?" relates back to Lesley's question about how is ethnomathematics different from other progressive pedagogy? Much progressive pedagogy falls under the interaction ideology. Naming our project a revolutionary or as Pat says anti-racist, anti-sexist, etc., I think, helps focus on the goal to change the inequalities and injustices in our society. One of the reasons for trying to spell out a definition for a "criticalmathematics" educator was to describe that label in such a way that the name could not be coopted (the key phrase I think is anti-capitalist-Con Edison may be able to describe its education program for its low wage employees as Freirean, one could even imagine them describing it as "empowering" but would they ever call it "anti-capitalist"?) (Marilyn Frankenstein)

Working Group C

Technology and Understanding Mathematics

Joel Hillel

Concordia University

Franklin Demana

Ohio State University

Technology and Understanding Mathematics

The 20 participants constituted a rather healthy cross-section of the mathematics education community, as they included teachers at the high-school, collegial and university level, as well as teachers' trainers and mathematics school consultants. The participants were, by and large, an already technologically-committed group. Almost everyone has used in his/her teaching either a graphing calculator or software packages such as: Minitab, LogoWriter, Computer Algebra Systems (MAPLE, Mathematica), MasterGrapher, Gauss, Geometric Supposer and the Mathematical Experience Tool Kit. These tools were used in a variety of settings: in the classroom, for demonstration purpose by the teacher, in computer labs held separately from the classroom, and, in the case of hand-held calculators, by both teacher and students as part of the normal classroom activity (see also Appendices 3,4,5,6).

Since most of the experiences of the participants in using technology in the classroom have been positive, the discussions in the working group, rather than asking the question of "should we?", focused on questions of "how to" and "what implications". These questions and the ensuing discussions are presented briefly below. Some of the issues that stemmed out of the group work are elaborated more fully in the appendices.

Curriculum changes: new emphases and dinosaurs

Question: By bringing computational technologies to the classroom, what needs to be changed in terms of

- *curricular goals*
- *textual material*
- *style of teaching*
- *mathematical language*
- *evaluation ?*

This question was raised within the context of the group's "hands-on" work with the Texas Instrument TI-81 graphing calculator. F. Demana demonstrated some of the activities that he and his colleagues have been using in the classroom to develop "function sense". For illustrative purposes, the group looked at graphing rational functions, an activity which highlighted how notions of graphical windows, scaling, local and limiting behaviour, roots and intercepts come into play. This particular activity, as well as more general work on graphing functions, suggested that traditional topics such as factoring quadratics, algebraic manipulations, trigonometric identities and simplification of quadratic forms can be given a much less prominent exposure than at present. As an example, it was suggested that the teaching of factoring can be organized around the Rational Root Theorem. The fact that graphing technologies lead to a change in focus, in problem types, in language and in cognitive demands has also emerged out of the experience of the MAPLE-based pre-calculus course given at Concordia (which was reported on in the previous meeting).

The question of pedagogical changes for an effective use of technology is also brought up in Appendices 1, 2 and 9, and the issue of evaluation, in Appendix 3.

Question: Are we looking at curriculum changes simply because the technology allows us to do certain things that couldn't be done without the technology?

On reflecting on the work with the graphing calculator, one was led to ask what is the role of graphing functions, of rational functions, of asymptotic behaviour of functions, etc. within the overall curriculum goals. What seemed to be needed is to have such concepts embedded in rich "problem situations" in which these concepts become meaningful, e.g. optimization problems or parametric graphing. In some cases, the technology itself might be the raison-d'etre for making curriculum changes, as it creates new opportunities for teaching and learning. Even if this seems to be "putting the cart before the horse", the overall impact on mathematics education may be quite positive. We are probably no longer aware about how much of today's curriculum was driven by paper-and-pencil technology. Our present day emphasis on "linearity" may be a case in point.

Learning issues: blackboxes and the triumph of inductivism

Question: What is it that the students are not learning because of the use of technology?

Among the concerns raised in the group's discussion was that work with calculators and computers over-emphasizes inductive rather than deductive reasoning and practical rather than theoretical mathematics. What, for example, are students' conceptions of irrational numbers if they rely on their calculator for all numerical answers? The consensus was that "it all depends"—it depends on how the technology is integrated in the curriculum, what kind of problems and activities surround it, and what are the overall instructional aims. Demana made the point that the students seem to gain new insights simply because of the kind of questions that using a calculator imposes. One impressive statistic relates to the positive effect of using the calculator-based instruction in the pre-calculus in Ohio State. Students brought up on such approach performed much better on a standard "calculus readiness" test, even though the test was rather "old fashioned" and not at all geared towards the use of technology.

Question: What about the blackbox syndrome?

Here, several approaches were discussed. For example, Brody and Rosenfield, in describing their calculator-based problem solving approach (Appendix 4), mentioned that they give explicit instructions and textual material related to the functioning of the calculator. Others reported that they defer the use of calculators till the students have dealt with the meaning of the actions that they perform. On the other hand, Vanbrugge discussed examples where one deliberately begins with several blackboxes and then builds the instructional sequence around a gradual "unravelling" of the blackboxes (Appendix 6).

Implementing changes: human and material resources

Question: *How to get changes implemented and other colleagues involved?*

Several "local solutions" were offered. For example, in Brock University, all members of the mathematics faculty agreed to at least monitor MAPLE-lab sessions through one complete semester. At Ohio State, Waits and Demana were given a carte-blanche to redesign the pre-calculus curriculum based on calculator use. In a sense, other colleagues had to get involved simply because calculator use was mandated by an official curriculum (see also Appendices 7 and 8).

In terms of material resources, it is obvious that graphing calculators are an appropriate technology to use in the classroom. Despite some of their limitations, they are affordable, transportable and students seem to be quite comfortable using them. However, the gap between calculators and computers is being bridged with the advent of the Palm-Top computers. These very portable computers, though presently still expensive, are quite versatile. For example, the HP-95LX can run a version of the computer algebra system DERIVE. If eventually such computers become as available and cheap as graphing calculators, computer labs may simply vanish.

Appendices:

Appendix 1 (L. Jansson): What change in our teaching knowledge base is required?
Appendix 2 (G. Gadanidis): Keeping up with "technological inflation" in mathematics education.
Appendix 3 (J. Hillel): A Maple-based functions course.
Appendix 4 (J. Brody & S. Rosenfield): A calculator-based computational approach.
Appendix 5 (T. Berggren): Doing mathematics in a computer lab using Maple software: A first experiment.
Appendix 6 (B. Vanbrugghe): Deux exemples.
Appendix 7 (E. Muller): A strategy for teacher involvement.
Appendix 8 (D. Lidstone): Convincing Colleagues.
Appendix 9 (J. Roulet): Technology and curriculum change.

Participants:

Alalouf, Eva
Berggren, Tasoula
Blake, Rick
Boileau, André
Brody, Jozef
Defense, Astrid
Demana, Frank
Flewelling, Gary
Gadanidis, George
Hillel, Joel
Jansson, Lars
Lidstone, Dave
Muller, Eric
Rosenfield, Steven
Roulet, Geoffrey
Vanbrugghe, Bernard
Vickers, Anne
Whitehead, Janet
Wight, Don
Williams, Edgar

Appendix 1:

What change in our teaching knowledge base is required?

Lars Jansson
University of Manitoba

In discussions with several other conference participants and in light of Jim Kaput's remarks I have tried to get a better grip on the notion that our "Pedagogy has to change in the context of technology." It might be helpful to think of this issue in terms of the knowledge base for teaching laid out by Lee Shulman (1987). There is not room here to detail Shulman's work but it suffices to recall that he lays out seven categories in his knowledge base. Four of these knowledge categories seem to me to relate directly to the question of whether or how our pedagogy must change if we are to successfully use technology in the classroom:

- content knowledge
- general pedagogical knowledge
- curriculum knowledge
- pedagogical content knowledge.

The remaining categories—knowledge of learners and their characteristics; knowledge of educational contexts; and knowledge of educational ends, purposes, and values—seem essentially unaltered in the presence of technology in the classroom (except perhaps for the micro-context of certain classroom dynamics).

General pedagogical knowledge, which Shulman suggests refers to as "those broad principles and strategies of classroom management and organization that appear to transcend subject matter," would clearly be altered at the level of technique. Classrooms will be organized and managed differently, but that does not necessarily imply new and different management principles, although principles too may change as our philosophy and basic assumptions about learners and learning change. It does imply that new and different strategies may be used, e.g., more student centred activities with less teacher control are a likely change in classrooms with computers.

The category of content knowledge is likely to undergo change slowly. As teachers begin to use technology there is the possibility that their understanding of mathematics and what it is may shift. For example, the issue (raised within the group discussion) of the inductive/empirical vs. the formal/deductive and their interactions and interrelationships must be dealt with by the reflective teacher attempting to implement the use of technology. This may, in turn, lead to altered notions of mathematics itself, as well as to new understandings of elements such as proof and construction.

Curriculum knowledge is described by Shulman as a grasp of the materials and programs that serve as "the tools of the trade" for teachers. Clearly the implementation of technology in the instructional program requires the development of new materials (including both print and software), and instances of these are widely available. What is less clear to me is how shifts in the content understandings are translated into "programs" implemented in classrooms, although it appears that these things grow out of understandings and knowledge in the other categories discussed here. The NCTM Standards (1989) proposes ways in which curriculum and instruction must change along with a change in our understanding of the nature of mathematics.

Shulman pays particular attention to pedagogical content knowledge. It is "that special amalgam of content and pedagogy that is uniquely the province of teachers, their own special form of professional understanding. It "represents the blending of content and pedagogy into an understanding of how particular topics, problems, or issues are organized, represented, and adapted to the diverse interests and abilities of learners, and presented for instruction. Pedagogical content knowledge is the category most likely to distinguish the understanding of the content specialist from that of the pedagogue." Thus as the understanding of content changes, so must the intersection of content and pedagogy. The ways in which mathematical concepts and understandings are represented seems to be a prime instance of this phenomenon.

To summarize, if we wish to consider analytically the kinds of changes motivated by technology in the classroom, Shulman's categories of teaching knowledge may prove useful. It was clear from the discussions in our Working Group that we are concerned particularly about 1) the nature of the changes in these different areas of knowledge, and 2) ways in which we can seek to change the thinking of others—colleagues, teachers, administrators, consultants, etc.

National Council of Teachers of Mathematics. (1989). *Curriculum and evaluation standards for school mathematics.* Reston, VA: Author.

Shulman, L. S. (1987). Knowledge and teaching: Foundations of the new reform. *Harvard Educational Review,* 57(1), 1-22.

Project Prometheus

In October 1990 the Faculty of Education at the University of Manitoba signed a 3-year contract with IBM Canada to explore the application of computer based technology in the promotion of mathematics and science literacy and to facilitate computer innovation in the mathematics and science classroom. This cooperative project is the first and largest of a number of projects to be initiated by IBM. A second project has been developed with the University of New Brunswick.

The expected outcomes of this project, as described in the original proposal, include 1) scholarly reports and publications and conference presentations; 2) university credit courses for pre- and in-service teachers; 3) professional development programs for in-service teachers; 4) summer institutes for in-service teachers, focusing on computer applications in mathematics and science; 5) a national conference on the use of computer-based technology in enhancing mathematics and science literacy; 6) the development of a computer laboratory and demonstration classrooms at the U of M; and 7) demonstrations in Manitoba schools to assist teachers in integrating computers into the school curricula.

A number of these activities have already begun. In early June 1991 we will announce the approximately 10 Manitoba schools which will have demonstration classrooms of 4-6 computers each, i.e., workstations and a file server linked by a Novell Network all supplied by IBM; or multimedia installations, i.e., videodisc players linked to an IBM computer. We want to investigate computer and videodisc use when the technology is available in the classroom on a daily basis, so that the teacher does not have to book a lab days or weeks in advance. IBM software such as the Mathematics Exploration Toolkit will be available to the schools. Other software may be purchased by the schools or by the Project. (I hope to have the Cabri geometry software also available.) A summer institute will be available for Manitoba teachers of mathematics during the summer of 1992.

Appendix 2

Keeping up with "technological inflation" in mathematics education

George Gadanidis
University of Western Ontario

Keeping up with monetary inflation is an idea we have all accepted and, in fact, become very sensitive to. For example, as the cost of living rises, we expect that our employers will agree to compensatory wage increases, as we have come to see such a corresponding inflation of our wages as our right. After all, it does not increase our "real" income; it merely helps us keep up. The inflation metaphor can be used to understand the reasons for, and effects of, integrating current technology in mathematics education.

Like its monetary counterpart, technological inflation is not new. We have, over the centuries of our distant history and the decades of our more recent past, accepted a wide variety of technological products in our classrooms. We have moved from drawing geometric figures in sand, to using pen and paper, printed text, erasable chalkboards, slide projectors, motion films, overhead projectors, television, and video players. Today, the inflationary pressures come from the increasing use of information technology in our society, and from the availability and affordability of calculators and computers for educational purposes. As professionals, we need to be as sensitive and responsive to this technological inflation as we are to monetary inflation. Just as we expect that our wages keep up with monetary inflation, we should expect that our teaching keeps up with technological inflation. However, keeping up with technological inflation does not in itself lead to a "real" increase in the quality of mathematics education; just as keeping up with monetary inflation does not improve our buying power. The increase in our students' mathematical power that will result as they learn to take advantage of calculators and computers only keeps up with the higher mathematical demands placed on them by a more technological society.

Some educators and technology enthusiasts are arguing that the integration of recent technology will lead to better teaching of mathematics; that is, it will do more than just keep pace with technological inflation. This claim is usually based on the perception that the use of calculators and computers goes hand-in-hand with such approaches as cooperative and experiential learning, and an emphasis on connections between multiple representations of mathematical concepts. The coupling of technology with sound teaching practices is a good "sales pitch", but the link is arbitrary. From our experience we know that given comparable resources and students, different teachers can and do teach very differently. Teaching is a complex human endeavour and cannot be improved through simplistic solutions. The exaggerated claim that more technology will mean better teaching may appear attractive, but it has the negative effect of getting us to continue to ignore the most important agent of change in education: the teacher.

Yes, let's keep up with technological inflation, but let's not forget that it's the teachers themselves who will determine how the use of calculators and computers will affect their teaching style. If we want a "real" improvement in mathematics education let's start by shifting some of our focus from the technological tools available to the teachers that will use them.

Appendix 3

A Maple-based functions course

J. Hillel
Concordia University

After two years of observations of small groups of students working with MAPLE, during the 1991 winter semester we took a section of our pre-calculus functions course and integrated it fully with the use of MAPLE. Our rationale for starting with a functions course rather than a calculus course was based on the idea that one should use a general tool like MAPLE as soon as it makes a good pedagogical sense. Also, our experience has shown us that in order to use MAPLE effectively in the calculus, one needed to be fluent in the particular ways in which graphical representation of functions is handled. Specifically, students needed to know how to manipulate graphical windows and how to coordinate and interpret data from different windows (see last year's proceedings for more details).

In this special section of the functions course, we alternated between classroom and lab work (about 75 minutes a week for each). While we were still constrained in what we could do by the official description of the course and because we had to "toe the line" with the other sections, we did make some substantial changes, notably:

- introduction of new concepts directly related to MAPLE (windows, local and global behaviour, scaling, pertinent numerical and analytical routines such as fsolve, solve, realroots, etc.)
- increase emphasis on the linkage between the different representations of functions
- reduce emphasis on algebraic manipulations, particularly on trig identities
- reshuffling of order of presentation. For example, the behaviour of functions was discussed with some general examples (e.g. abs(cos(x)), before the actual functions were covered in the course. Linear functions were deferred nearly to the end.
- emphasis on active learning—the computer labs were structured on worksheets containing specific tasks and leading questions.

We felt that our approach would make students a lot more "calculus ready" since some of the ideas of what the calculus is about were implicit in most of the activities that we designed. However, we were aware that the payoff of such an approach is dependent on the calculus course which tries to build on the strengths of students' previous MAPLE experience, i.e. the functions, calculus (and linear algebra) courses need to be re-worked in unison and a complementary way.

While final evaluation could have been a combination of paper-and-pencil work and computer work, we opted for only pencil-and-paper work, mostly for practical reasons (insufficient number of computers, not having technical staff on hand in case something goes wrong). Students were told that their exam will have specific questions related to MAPLE. Enclosed are the copies of the two final exams—one for the traditional course and the other for the MAPLE course. The first three questions of the MAPLE final were our "payment of dues" to the standard course. The remaining questions reflected better what we expected students to have learnt from the course.

CONCORDIA UNIVERSITY

Department of Mathematics & Statistics

Course MATHEMATICS **Number** 201/4 **Section(s)** 52, T, TT

Examination	**Date**	**Time**	**# of pages**
Final	April 1991	3 hours	3

Instructor(s)
Harrison, Harrow, Osborne

Materials Allowed:
Calculators permitted.

Special Instructions:
Answer all questions in graph books provided. Show all work.

MARKS

3 1. a) Find the centre and radius of the circle $x^2 - 4x + y^2 - 8y = -4$

2 b) Find the equation of the line passing through $(1,5)$ and $(-2, -4)$.

3 c) Find the equation of the line having the same x-intercept as the line $2x - y = -4$ and perpendicular to the line $-x - 2y = 9$.

6 2. Find the coordinates of the vertex; the equation of the axis of symmetry; the y-intercept, for $f(x) = 2x^2 + 12x + 18$.
Sketch the graph.

3 3 a) In a circle of radius r, an angle of 30^0 subtends an arc of length $\frac{\pi}{6}$ centimetres. What is r?

3 b) $P(-1, \sqrt{24})$ is a point on the terminal side of an angle Θ in standard position. Make a sketch, clearly indicating the angle Θ, and find the six trigonometric functions of Θ.

4 c) Without using a calculator, find the <u>exact</u> values of the six trigonometric functions of 150^0, after first making a sketch with 150^0 drawn in standard position.

4 4 a) If $\log_2 5 = 2.3222$ and $\log_2 3 = 1.5850$, evaluate

i) $\log_2 15$ ii) $\log_2\left(\frac{15}{2}\right)$

3 b) Express $\log_b\sqrt{\frac{xy^3}{z}}$ in terms of the logarithms of x, y, and z.

4 5 a) Simplify:

i) $3^{-2\log_3 4}$ ii) $3\log_2\left(\frac{2}{32}\right)^{\frac{1}{3}}$

4 b) Solve for x: $\log x + \log(x + 9) = 1$.

3 6 a) Use a calculator to determine the minimum number of whole years that the principal P must be invested at the given rate r compounded annually to produce the given compound amount S.

P = \$1000 S = \$3000 r = 10%

3 b) Use a calculator to solve for x: $4^{x+1} = 5^{x-1}$

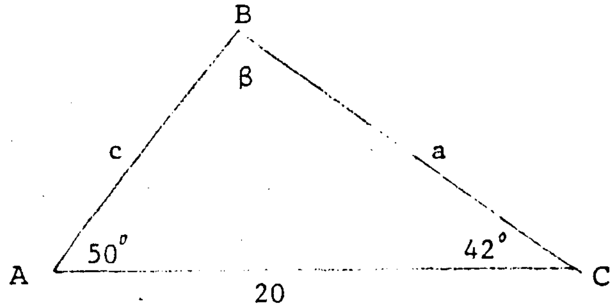
6 7 Assume $\sin u = \frac{2}{3}$, with u in Quadrant **II** and $\tan v = 2$, with v in Quadrant **I**. Find $\sin(u + v)$, $\cos(u + v)$, and $\tan(u + v)$.

5 8 a) Use half-angle formulas to compute the <u>exact</u> value of $\cos \frac{\pi}{8}$

5 b) Find all the solutions for u in the interval $[0, 2\pi)$, given $(2\cos u - 1)(\cos u + 1) = 0$.

5 9. Sketch the graph of $y = 2\sin \frac{1}{2}\pi x$ for $0 \le x \le p$, where p is the fundamental period. What is p and what is the amplitude?

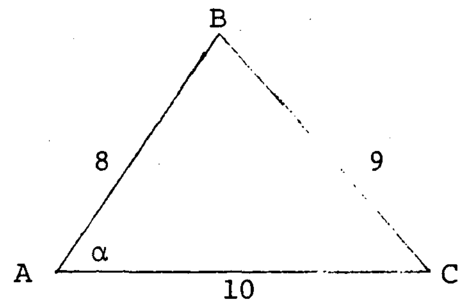
5 10 a)



Find the length c.

5

b)



Find the angle α.

9 11. Prove the following identities:

a) $(\tan u - \sec u)^2 = \dfrac{1 - \sin u}{1 + \sin u}$

b) $\dfrac{\sin 2u}{1 - \cos 2u} = \cot u$

c) $\sqrt{2}\ \cos\left(x - \dfrac{\pi}{4}\right) = \cos(-x) + \sin x$

12 Let $f(x) = \log_2 x$ and $g(x) = 2x - 1$

4 i) Determine $f(g(x))$ and $g(f(x))$

3 ii) Determine the domain of $f(g(x))$

3 iii) Make a sketch of the graph of $f(x) = \log_2 x$, showing the y-coordinate when $x = \frac{1}{8}, \frac{1}{4}, 1, 4$.

3 13 a) If $f(x) = \log_2(x + 3)$, verify that $f^{-1}(x) = 2^x - 3$.

2 b) Write the following expression in terms of a single function:

$$\sin 5t \cos 7t + \sin 7t \cos 5t$$

y

Mathematics 201 Section 02 Instructors: C. Bowers & L. Lee
Final examination: April 26, 1991 Time:19:00-22:00 CC301
Materials allowed: Calculators permitted (Do all work in the exam booklets)

MARKS

1. Solve for x:

8 a) $\log_{15}(5x)=2$ b) $5^x = 9(3^x)$ c) $\log_5(x) + \log_5(x-24) = 2$

2. Solve the following triangles (find all the missing angles and sides):

8 a) A triangle with sides of lengths 18cm., 25cm. and 12cm.

b) A right-angled triangle with a base of 6cm and a height of 7cm.

3. a) Show that $g(x)=5x-2$ is **not** the inverse function of $f(x)=(x/5)+2$.

8 b) Given $f(x)=3x^2$ and $g(x)=0.5x-1$, find $f(g(4))$.

c) Find the inverse function, $f^{-1}(x)$, of $f(x)=-3x-7$.

4. The following table of values has been created from an exponential function, f(x).

x	-2	-1.2	-0.5	1.2	2	3
f(x)	.14	.30	.61	3.32	7.39	20.09

a) Make a table of values for the inverse function.

13 b) Draw the result of Maple command **plot**([f(x),f^{-1}(x)],-10..10,-10..10).

c) Find the base of the exponential function and then find the expression for $f^{-1}(x)$.

d) Is -13.6 a possible result for **fsolve**(f(x)=0, x)? Explain your answer.

e) Draw the graph of f(x)+1 over the interval [-2,3].

MARKS

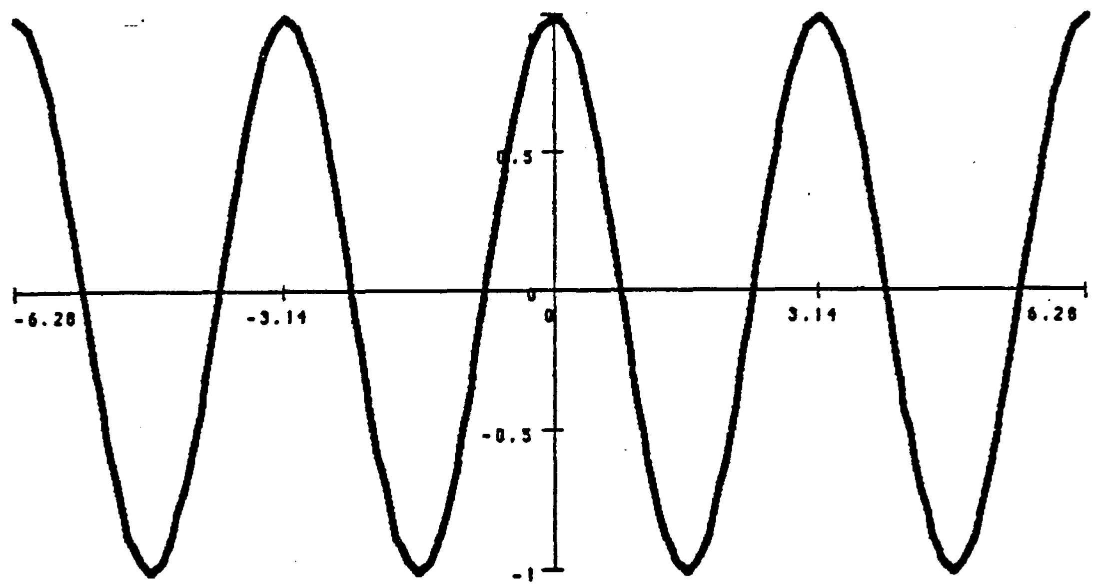
20

5. State all the information you can give about **two** of the three following functions (domain, range, periodicity, intervals where the function is decreasing, increasing, positive and negative, x and y intercepts, coordinates of local maxima and local minima, asymptotes).

$f(x)=(0.3)^x$ $\qquad$ $f(x)=\log_{10}(x)$ $\qquad$ $f(x)=2x^2+4$

20

6. The following plot is the plot of the function y=cos(**b**x).



a) From the graph determine the period.

b) Give the coordinates of two apparent local maxima.

c) Give the coordinates of two local maxima outside the given window, one with a positive x coordinate and the other with a negative x coordinate.

d) Give the coordinates of the sixteenth x-intercept starting from the origin.

e) Copy the above plot in your exam booklet and then add the plot y=cos(x). Give the coordinates of one of the points of intersection of the two graphs.

f) For the given function y=cos(**b**x), estimate the value of **b** and give your reasons for your choice.

MARKS

7. A student looking at the Maple plot of the function $f(x)=x^3+10x^2+11x-70$ on [-10..0] estimated that the graph cuts the x-axis at -7 and -4.

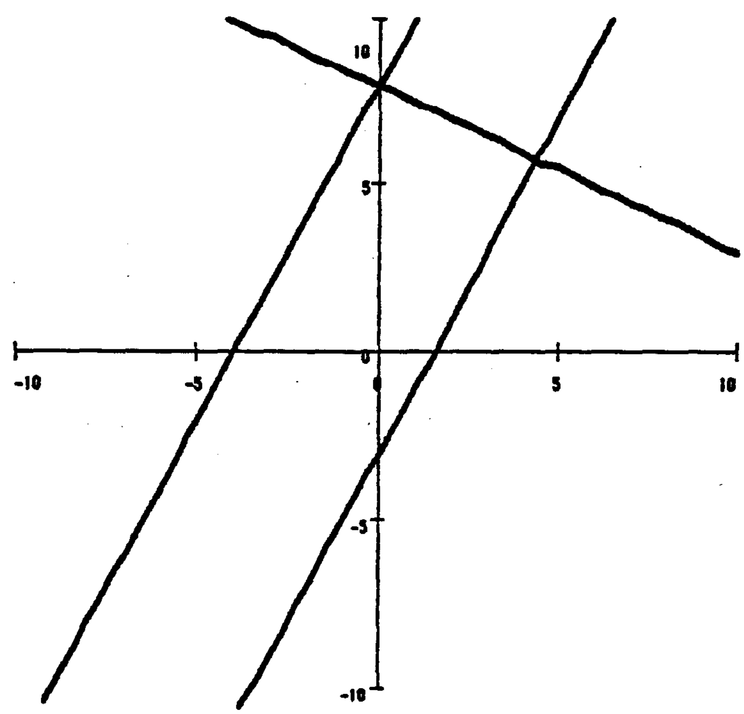
6

a) Check these results to see if they are exact. (Hint: You should be able to do this without plotting a graph.)

b) From the default plot of the function it looks as if the y-intercept is about -65. Check to see if this is correct.

8. In the simultaneous plot of three linear functions below, line*1 represents the function y=2x-3.
Line*2 is parallel to line*1.
Line*3 is perpendicular to line*1 and line*2.
The intersection of line*2 and line*3 is (0,8).

8



a) What is the expression for line*2?

b) What is the expression for line*3?

c) What is the x-intercept of line*3?

MARKS

9. For the function $f(x) = -3x^2 + 43.5x - 50.6$, Maple gives the following result for **fsolve**:

```
• f:=proc(x) -3*x^2+43.5*x-50.6 end;
f := proc (x) -3*x**2+43.5*x-50.6 end

• fsolve(f(x)=0);
                              1.275400990, 13.22459901
```

9

a) What window would you choose to get the x intercepts to appear?

b) On what interval(s) is the function positive?

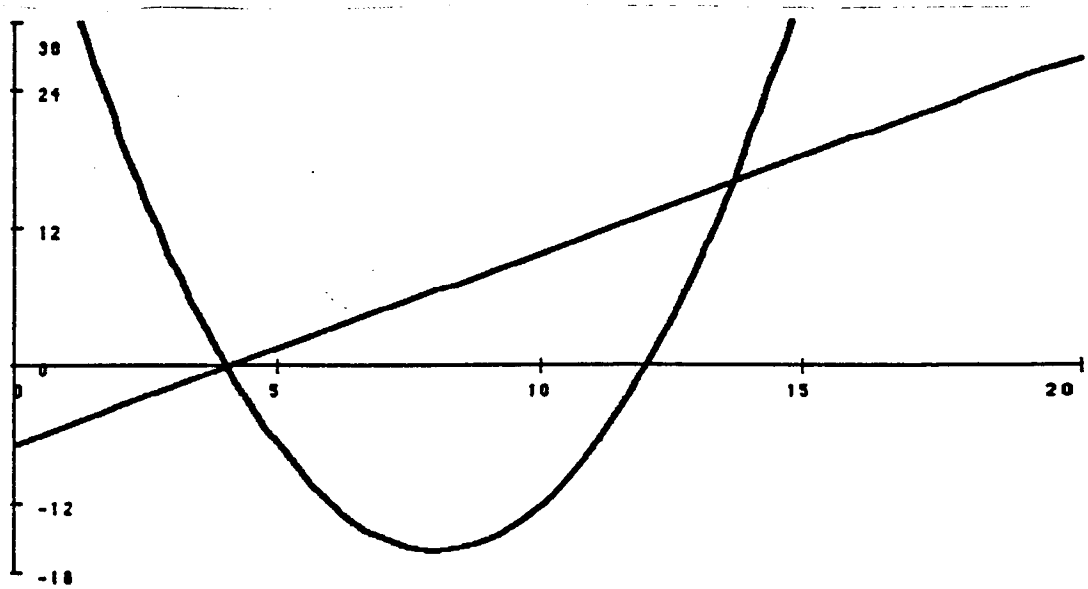
c) Find the coordinates of the local maximum.

d) On what interval(s) is the function decreasing?

BONUS QUESTION

10.

5



a) Give a common x-interval on which **both** functions are positive (simultaneously).

b) Give a common x-interval on which **both** functions are increasing.

c) Give an interval on which one of the functions is positive and the other is negative.

Appendix 4

A Calculator-Based Computational Approach.

Josef Brody and Steven Rosenfield
Concordia University

The presence of modern technology (calculators, graphing calculators, palmtops and computers with either numerically based mathematical software or symbolically-based mathematical software) presents us with a number of issues:

1. How should technology be used *within the current mathematics curriculum?*
   - Should we use the speed of these technologies to allow us to cover current topics more rapidly, i.e. contract/condense the curriculum?
   - Should we maintain the current curriculum while presenting more examples?
   - Should we teach the current curriculum in greater depth?

2. How should technology be used *to change the current mathematics curriculum?*
   - Should we remove from the curriculum topics and concepts which in view of the new technologies may be obsolete? If so, which?
   - Should we introduce new concepts and topics into the mathematics curriculum? If so, which?
   - Should we change the entire pedagogical approach? How? Will this improve student understanding of mathematics? Does this require a complete change in the philosophy of mathematics education?

3. How much should the student know/be taught *about the technology used* in any course?
   - What level of understanding of the hardware is required?
   - What level of understanding of the software is required?
   - What level of understanding is required of the complex interaction between digital devices with their inherent finite representations and the infinite numerical systems of theoretical mathematics that they portray?

In our Computational Approach to Linear Functions and Equations, which we have tried in a Basic Algebra course (MATH 200) with about 200 students, we used a specific calculator, the Radio Shack EC-4021. The EC-4021 was the least expensive ($30.00 CAN.) programmable (40 steps) calculator available. In addition it had the feature of allowing the students to swap the X and Y register values, and this was important in our attempt to help students build a mental model of the calculator. In the course we addressed some of the issues listed above, but not all. Specifically:

a. we did not have freedom to make fundamental changes to the existing curriculum, however we did de-emphasize some topics, such as *techniques of* *factoring*, and introduced alternative approaches, such as use of the *Factor Theorem*;

b. we introduced a new concept, based on computation and *input-output analysis* of available data;

c. we introduced the basic concepts of *difference calculus* together with the concept of functions (linear) and equations;

d. we attempted to stress a philosophy in which *problem posing* is as important as problem solving;

e. we helped the students to develop a simple *understanding* of the *hardware and software* of the chosen calculator.

We started with two observations: students feel comfortable with calculators and rely on them; they understand very little about how calculators work and they use them very poorly (in any complex calculation they write down intermediate steps on paper and re-enter them later). It was our feeling that the student affinity for calculators would be a "hook" into their minds, allowing us to hang some mathematics onto it. Unfortunately, the "hook" was not mounted solidly and so we decided to help them build a mental model of how this particular physical calculator works. The hope was that such a mental model would improve their usage of the calculator, and allow them to speculate mentally about logical or mental calculators and the calculation processes.

To accomplish the task we wrote a manual for the use of the calculator which:

a. introduced the various keys on the calculator and the relationship they had to mathematical topics;

b. explained the role of some registers, in order to help students build their own mental model and understand why the calculator reacts as it does to different combinations of keypresses;

c. prepared the students for the computational approach used in the course.

In the course we used the calculator in two ways:

1. as a *physical instance* of an *input → function → output machine* in order to have a sufficiently complex model for the mathematical concepts of:
   a. a function;
   b. linearity;
   c. inverse linear function.

2. as a *programming machine*, which rapidly supplies necessary data for:
   a. the input-output analysis;
   b. the decomposition of a linear function;
   c. further considerations leading to problem-posing concepts.

At the beginning of the course the students learned about the X-register of the calculator. This allowed us to discuss the concepts of a function and input and output variables using the diagram:

$$\begin{array}{c} x \\ \downarrow \\ \boxed{f(\mathrm{X})} \\ \downarrow \\ y = f(x) \end{array}$$

The above figure indicates that as x, the input value, is placed on the display of the calculator, it is simultaneously placed into the X-register. When the key $\boxed{f(\mathrm{X})}$ is pressed, the content of the X-register is transformed into the output value $y = f(x)$.

In order to illustrate the concept, let us introduce the following example:
A wire is cut into 5 pieces. The second is 2 cm longer than the first. The third is twice as long as the first two together. The fourth is 3 cm shorter than the third, and the fifth is 5 cm shorter than twice the sum of the first, second and fourth.

First we organize the data in the following tabular form:

1st	2nd	3rd	4th	5th	together
a	$b = a+2$	$c = 2(a+b)$	$d = c-3$	$e = 2(a+b+d)-5$	$f = a+b+c+d+e$
0	2	4	1	1	8
1	3	8	5	13	30
2	4	12	9	25	52
3	5	16	13	37	74
⋮	⋮	⋮	⋮	⋮	⋮

We shall use a here as the input variable for the following functions:

$g_b: \ a \to b;$
$g_c: \ a \to c;$
$g_d: \ a \to d;$
$g_e: \ a \to e;$
$g_f: \ a \to f.$

From an input-output analysis it follows that the forward differences, $\Delta_g(x) = g(x + 1) - g(x)$, are constant for all the above functions, therefore the functions are all linear and:

$$g_b(x) = x + 2;$$
$$g_c(x) = 4x + 4;$$
$$g_d(x) = 4x + 1;$$
$$g_e(x) = 12x + 1;$$
$$g_f(x) = 22x + 8.$$

The inverse functions (we use capital letters) are:

$$G_b: b \rightarrow a;$$
$$G_c: c \rightarrow a;$$
$$G_d: d \rightarrow a;$$
$$G_e: e \rightarrow a;$$
$$G_f: f \rightarrow a;$$

and:

$$G_b(x) = x - 2;$$
$$G_c(x) = (x - 4)/4;$$
$$G_d(x) = (x - 1)/4;$$
$$G_e(x) = (x - 1)/12;$$
$$G_f(x) = (x - 8)/22.$$

These functions open doors to many questions that can be posed, e.g.

1. If the 1st piece, a, is 2.3 cm long, how long must the whole wire, f, be?

   The output value here is f and the input value is a. Therefore the function g_f: $a = 2.3 \rightarrow f = 22(2.3) + 8 = 58.6$ cm gives the required answer.

2. When the 3rd piece, c, is 5.3 cm, how long is the 5th piece, e?

   Here we use the diagrams, so that it is easy to see the composition of the functions:

h: $c \rightarrow e$

$$
\begin{array}{lcl}
 & c & = 5.3 \\
 & \downarrow & \\
G_a: & \boxed{\text{(X-4)/4}} & \\
 & \downarrow & a = (5.3 - 4)/4 = 0.325 \\
g_e: & \boxed{\text{12X+1}} & \\
 & \downarrow & \\
 & e = 12(0.325) + 1 & = 4.9
\end{array}
$$

So the 5th piece is 4.9 cm long.

3. What is the minimal length of each piece and of the whole wire?

   Since the initial values of each function are all positive, they are the required answers. That is: $g_b(0) = 2$; $g_c(0) = 4$; $g_d(0) = 1$; $g_e(0) = 1$; $g_f(0) = 8$. In words, the 2nd piece is at least 2 cm long, the 3rd at least 4 cm, the 4th and 5th are at least 1 cm, and the whole wire must be at least 8 cm long.

4. How do the forward differences relate to the problem?

   The forward differences give the multiplicative factors for each function. Therefore they give the speed with which each piece, and the whole wire, change. For example, $\Delta_c = g_c(x+1) - g_c(x) = 4$ shows that with each cm increase of the 1st piece, a, the 3rd piece, c, increases by 4 cm.

5. How long is the 1st piece if the whole wire is:
   a) 125 cm; b) 28.5 m; c) 2.5 cm; d) 13.8 m; e) 23.7 cm.

   Here we have values of the length, f, of the wire. We want to know the corresponding values of the length, a, of the 1st piece. The function G_f: $f \rightarrow a$ and $G_f(x) = (x-8)/22$. We have to evaluate this function for 5 input values, so it makes sense to program it into the calculator. The programming requires the following sequence of keypresses:

   $\boxed{\textbf{2ndF}}\ \boxed{\textbf{LRN}}\ \boxed{-}\ 8\ \boxed{=}\ \boxed{\div}\ 22\ \boxed{=}\ \boxed{\textbf{2ndF}}\ \boxed{\textbf{LRN}}$

   Executing this program (entering an input value and then pressing the $\boxed{\textbf{COM}}$ key) yields:

a) 5.32 cm; b) 129.18 cm; c) −0.25 cm; d) 62.36 cm; e) 0.71 cm. From the above results it follows that in case c) the problem has no solution, i.e. the whole wire cannot be as short as 2.5 cm (a confirmation of a result in 3 above).

Conclusions:

In order for students to explore situations such as the one above, we believe that they must understand the logical functioning of the calculator, such as:

a. the role of the X-register, and consequently also of some other registers;
b. the process of programming the calculator and its relationship to the priority of mathematical operations;
c. the rounding-off mode;
d. some, or all, of the function keys; etc.

The aim of the above instruction/explanation of the calculator is to help students achieve sufficient familiarity with a calculator. This in turn allows concepts in the course to be based on their mental model of the functioning of the calculator. Also, they must develop sufficient ability and confidence in practical use of the calculator to rapidly compute with it so as to explore problems numerically. The diagram notation relates the function concept to the transformation of X-register values by the press of a key. It is also a good tool for teaching composition of functions.

We believe this computational approach to be well suited to the problem-posing philosophy of learning. Polya frequently stressed the composition and decomposition of functions in connection with problem solving and we believe that it is a very important part of learning mathematics. The computational approach also facilitates problem-solving, especially in more complex cases, like the above example, perhaps because students begin by manipulating numbers instead of symbols. Therefore the choice of situations (problems) that can be presented is much wider and less artificial. As a result we believe this approach enables students to learn more because a) they create abstractions from the physical calculator, b) they learn to handle symbols in relation to specific (concrete) situations, and c) they are encouraged to generalize, as in Polya's works, by first simplifying a situation, and eventually posing further questions (generalizing) the original situation. Consequently the approach promotes the development of a deeper understanding of mathematical concepts.

Although we have not as yet had an opportunity to analyze the data collected this past term, our experiences teaching with the computational approach do point to a number of obstacles:

1. Students have a tendency to cling to what they have seen before, perhaps even more so if they have been successful in past. This epistemological obstacle creates a strong resistance to the acceptance of new ideas.

2. Although we have prepared two extensive manuscripts (more than 200 pages together) where the calculator and the computational method are illustrated in examples as well as presented theoretically, there is no comprehensive text book yet.

3. The material in the course where we tried this approach was very condensed (packed with many topics acquired over the years). As a result the students did not have enough time to digest important subtleties. We feel that it is important to plan such a new approach while viewing the entire curriculum, leaving out or diminishing the importance of some current topics, adopting new topics, and integrating other available new tools such as:

    i. Calculators with graphs, like TI-81, CASIO G-8000, HP-28.
    ii. Palmtops or larger computers with:
    iii. Non-specialized numerical software like spreadsheets;
    iv. Specialized numerical software like GAUSS, LINDO, etc.;
    v. Symbolic mathematical software like DERIVE, MAPLE, MATHEMATICA, etc.

Perhaps it is time for a group such as GCEDM/CMESG to initiate a project which would propose a new mathematics curriculum, starting in a particular area such as pre-function/pre-calculus/calculus/numerical analysis, integrating the technologies mentioned above with a new philosophy and approach to mathematical education.

Appendix 5

Doing mathematics in a computer lab using Maple software

A first experiment

Tasoula Berggren
Simon Fraser University

1. Choosing the Software

For the past four years I have been interested in working technology into the teaching of calculus and linear algebra, and I used various textbook software packages and other computer programs. Among those which interested me the most were Maple, Mathematica and Derive. At the 1989 CMESG meetings at Brock University I was exposed to Maple by a lab presentation given by Stan Devitt and by the talk which Eric Muller gave about his work introducing the Maple lab for calculus courses. At that meeting I decided that Maple was the software of my choice. It had a nicer screen picture than the various other software programs, its commands were adequate and its symbolic manipulation was dynamic, although plotting graphs was superior with Mathematica and Derive. Choosing the software was an important step for my future plan which was to undertake as a pilot project for the 1989-90 academic year the introduction of a computer-based calculus lab to a group of first year students.

2. Acquiring Maple and getting the faculty involved

In the summer of 1989 when the Mathematics and Statistics Department of Simon Fraser University was contemplating what software to choose I wrote to the Chair requesting the purchase of Maple and expressed my interest in teaching a group of calculus and linear algebra students with it. I wanted to work with this group of students once a week. The department arranged for a Maple demonstration for its members, and at the end of summer 1989 we purchased Maple together with a site licence. Soon after this I also started working with a group of 25 students, all of whom volunteered for the experience. After a few weeks I invited the Chair of the department and professors who were teaching calculus courses to come to the lab to see the students working with Maple. I also sent a letter to the Chair of the Undergraduate Studies Committee (USC) telling her that I was willing to show her committee how Maple works and to give them a demonstration of what I was doing with the students. I felt it was important to have my colleagues understanding what I was doing and supporting it since I requested the USC to consider the introduction of a computer lab for all the first year mathematics students. I also gave the Chair of USC the computer outputs of Maple work done with the students. Soon after this I left for my sabbatical. Already one of our professors is using Maple for calculus homework at SFU. During the same year also I was in contact with Dr. Steve Kloster who is in charge of the university mathematical software and who supported my work by visiting the class occasionally. Now I understand that the university is going to buy the program to have it available on a broad basis.

3. Driving reasons

The driving reasons which led me to introduce computers to our first year mathematics courses are the following.

i. To use computers as a tool in the teaching and learning of mathematics.

- To help students concentrate their attention on the thinking and the procedure for the various mathematical concepts and give them a chance to do problems which they could not do otherwise due to the long and tedious symbolic manipulations. I also wanted to draw their attention to the algorithmic nature of much of the first and second year work.
- To give first year students the opportunity to use microcomputers early during their university years and to provide them with technological knowledge so they can use computers for their remaining university years, and to do this with an application that is sufficiently powerful so that science students could use it right through graduate school.
- To make students accustomed to mathematical software before they enter their professional careers.

These initiatives led me to start the pilot project. The next stage of this project will be to convince the department that it should require all our calculus students (about 1000 each year) take this computer calculus lab for an additional credit to their regular calculus courses.

4. Pilot Project

Administrative details

- The Pilot Project was announced in the Fall to Differential Calculus Class students and in the Spring to Integral Calculus Class students by the professor teaching the course, by overhead announcements and by posters in the Calculus Linear Algebra Workshop.
- Registration was unofficial, since no university credit was given for the work, but each student had to be serious and stay with the course. It was a kind of a contract between them and the teacher that they will stay until the end.
- The plan of the work was given to the students. There were ten Computer Based Calculus Lab projects, one given each week.

Of the twenty five students who registered eighteen completed the project. Their reasons for participating in this class were diverse:

- Some of them were there for better understanding of their calculus.
- Others wanted to improve their performance.
- Others wanted to learn how to use the programs for their future mathematics and engineering courses, such as differential equations.
- Some students simply liked computers and would take any opportunity to use them.
- Some students were there because they considered this an opportunity to learn how to use computers. In general the students' interests were mainly directed

towards participating in something that would aid them in their studies and in class performance.

5. Grouping the students

Starting the project was hard because of the various interests of the students and the different levels of familiarity with the technical computer knowledge. I valued the students inventiveness and ability to discover ways of doing things, so as a beginning project I suggested they become familiar with the basic menus of the Macintosh by asking me questions or asking those of their fellow students who knew already how to use computers. The seating arrangement was conveniently made so students could help each other. In general students were encouraged to work in pairs but each one had access to an individual computer.

6. Examples of beginning activities

- At first I distributed cards I had made up with the basic Maple commands and explained to students generally how Maple works. The students experimented with such Maple commands as simplifying rational functions, expanding binomial expressions, solving equations, differentiating functions of their choice, finding antiderivatives of these functions and so on.
- In another exercise students were asked to form a cubic polynomial by choosing three real numbers as roots, then expanding the cubic using Maple and finally solving it using the Maple software. This and similar exercises were planned to offer them practice with the commands, to provide them with some feelings about the results and to instill confidence in the correctness of the results they found.
- Practising with the plotting commands was an important part of these beginning activities. They were encouraged to graph trigonometric and other functions, to shift and shrink these graphs as well as change their scales. They graphed several functions on the same coordinate plane, for example $\sin(x)$, $2\sin(x)$, $\sin(2x)$, $\sin(x/2)$, $\sin(1/x)$ and they zoomed in on intervals close to zero trying to identify the various functions from the different slopes of the graphs. They were asked to predict the behaviour of the 1st and 2nd derivatives from the graph of the function or vice versa. (Being able to get the answers quickly after their guess is very educational for students.) Students were encouraged at all the times to interact with each other, not just with the computer. So when they graphed functions, they guessed each other's functions and made observations on the derivatives. For example when they graphed 2^x, e^x, 3^x and their derivatives they observed the shapes and positions of the derivatives relative to the graphs of the original functions, the value of the derivatives at critical points and so on.

7. Problem-solving activities

Each project on problem-solving usually had three parts to it:

- A paper and pencil solution of the proposed problem.
- Expressing the method of solution in a way that Maple understands, feeding it into Maple and using Maple as the tool for all the arithmetic and algebraic computations.

- Expanding the original problem to a general one where the use of Maple is advantageous over the paper and pencil solution.

The students tried to do simple proofs by mathematical induction or the definition of the derivative, which otherwise are left untouched, overlooked or misunderstood. Ideas and problems for this part were taken from the *Maple Calculus Workbook* by Geddes, Marshman, McGee, and Ponzo. Some of them were:

- Show that if the roots of a fourth degree polynomial form an arithmetic progression, then the roots of its derivative are also in arithmetic progression. (From *Math Magazine*, **51**, 5 (1979), page 307.)
- Find the cubic with local maximum at $(-21, 50)$ and local minimum at $(44, -87)$.
- Let A be the region between the graph of the cubic $y = x^3$ and the tangent at a P on the graph. This tangent at P intersects the above graph at another point Q. Now if B is the region between the graph and the tangent at Q (i) show that area B is 16 times as great as area A, and (ii) show that this property is true for every curve of the third degree.

Other problems which we worked out dealt with minimum distance between two graphs, maximum and minimum volumes subject to constraints, Newton's method, Taylor's series, Simpson's rule, area and volume problems, lengths of arcs, centroids and some problems from linear algebra.

8. Homework activities

The students were encouraged to use Maple to check the answers to their homework and try out problems from their examinations but they did not do this as much as I hoped they would. I think this is because of time pressure from their course-load and not because they didn't enjoy using Maple.

9. Goals for the future

My goals for the use of Maple are threefold:

- to emphasize a visual approach to mathematical concepts and results,
- to help the students see the essential simplicity of mathematical ideas and procedures by clearing away the confusion that is often created by tedious algebraic manipulation and
- to use the combination of visual presentation and conceptual insight to improve the students' proficiency in and enjoyment of mathematics.

Since their childhood students have channeled much creativity, though and zeal into mathematics. Working with the students made me more sensitive to their real need to reach an understanding of mathematics and to the benefits they will gain from working out concepts through technological tools. Thus I value my computer work at the Lab with the students and I plan to do it again next year, but I hope this time to introduce it extensively to a whole class, as Eric Muller is doing at Brock University. I will try to encourage the students' creativity and involvement by asking them to formulate problems to work on, in addition to those I give them, something I have been thinking about during my sabbatical year at Harvard.

My goal, of using technology as a tool for teaching and learning mathematics, has begun to be realized with this project. By beginning an interactive computer workshop and to contribute to a more profound comprehension of mathematical concepts and problems I have learned through trial and error how better to encourage the students' self-education. And this, surely, is the basic goal of all education.

Appendix 6

Deux exemples

Bernard Vanbrugghe
Université de Moncton

Il semble que dans l'état actuel des choses, la quasi-totalité des participants du groupe de travail C sur "la technologie et la compréhension des mathématiques" en sont arrivés à la conclusion que la technologie ne peut plus, et ne doit plus, être carte de la salle de classe, compte tenu de son énorme potentiel pour l'enseignement. Le vrai problème est de savoir comment introduire cette technologie de façon efficace dans le curriculum, sans en arriver à faire des mathématiques une science complètement expérimentale ou une science de presse-bouton. S'il est vrai que la technologie permet de faire de l'exploration et de l'expérimentation mathématique, elle devrait aussi faciliter l'enseignement des concepts mathématiques. Il n'est pas facile pour nous, qui sommes habitués/ées travailler de façon traditionnelle avec un papier et un crayon, de trouver des activités pertinentes pour faire de la technologie un outil efficace à l'acquisition de concepts mathématiques. Toute idée que nous pouvons glaner ici et là est un pas de plus dans la bonne direction. Pour cette raison, voici deux exemples simples, utilisables soit dans un cours d'algèbre linéaire, soit dans un cours d'algèbre et programmation linéaire, qui illustreront mon propos.

1 Le concept d'espace vectoriel est un concept mathématique abstrait et, pour bon nombre d'étudiants, un vecteur est avant tout un segment orienté (la flèche est très importante) qui représente presque toujours une force. Qu'une fonction, une matrice, un polynôme puissent être considérés comme des vecteurs, est une affirmation qui leur est souvent difficilement acceptable. Pour surpasser cette barrière, il faut faire en sorte que l'étudiant analyse en profondeur des exemples amusants, curieux, inhabituels, concrets; pour qu'il parvienne à admettre finalement que le mot vecteur n'est pas synonyme de "petite flèche", mais qu'il représente un élément d'un ensemble préalablement défini que l'on appelle espace vectoriel. L'exemple des carrés magiques[1], des pentagones magiques des cubes magiques se sont avérés, selon notre expérience, des exemples extraordinairement intéressants pour faire comprendre aux étudiants les concepts d'espace vectoriel et de vecteur. Après avoir défini ce qu'est un carré magique ainsi que des opérations: d'addition de deux carrés magiques et la multiplication d'un carré magique par un scalaire (opérations identiques à celles définies sur les matrices), on demande dans un premier temps aux étudiants de vérifier qu'il s'agit bien alors d'un espace vectoriel. L'expérience montre que si on s'en tient à ce stade, le but recherché, à savoir la généralisation de la notion de vecteur, n'est atteinte que très superficiellement et est vite oubliée. Par contre si l'on demande à l'étudiant de trouver la dimension de cet espace vectoriel, de trouver une base de cet espace vectoriel, d'écrire un carré magique comme une combinaison linéaire des vecteur de base (carrés magiques de base),

[1] Algèbre linéaire, une approche matricielle. Auteur: Pierre Leroux. Modulo Éditeur.

la compréhension de la notion de vecteur devient profonde. Le fait de choisir des carrés magiques de dimensions relativement grandes, par exemple 6 par 6, met en évidence la nécessité de l'utilisation de la technologie (il s'agit en fait dans toutes ses manipulations de résoudre des systèmes d'équations linaires). En dessinant (manipulant, visualisant) sa combinaison linéaire, l'étudiant comprend la généralisation de la notion de vecteur. Vous pourrez alors lui parler d'un polynôme comme étant un vecteur, le terme passera très bien.

2 L'utilisation d'un logiciel de calcul[2] peut servir de motivation à l'introduction de concepts mathématiques dans la progression d'un cours. Nous avons expérimenté la procédure suivante dans un cours d'introduction l'algèbre et programmation linéaire destiné aux étudiants de la faculté d'administration. Dès le premier cours, l'étudiant est initié l'utilisation d'un logiciel de programmation linéaire. Il le voit complètement comme une boîte noire, où après avoir introduit un énoncé, on appelle la commande "Run ou Go" et dont le résultat ressort quelque part dans un monceau d'informations imprimées sur plusieurs pages. La disponibilité de cet outil permet de s'intéresser dès le début du cours à des problèmes relativement complexes (plus que 2 ou 3 inéquations à deux inconnues) qui peuvent représenter des situations réelles, car on dispose déjà d'un outil (qui pour l'instant est une boîte noire) pour en avoir la solution. La partie mise en équations (modélisation) et solution informatisée, se poursuivra tout au long du cours, parallèlement à l'introduction des concepts de matrice, de pivotage etc. Au fur et à mesure de la progression du cours, le dépouillement de l'information de la sortie de la fameuse boîte est de plus en plus pointu, et permet d'expliquer des termes comme itération, variable de base etc. Finalement, à la fin du cours l'étudiant comprend ce qui se passe dans la machine. Ce n'est plus une boîte noire. Cela ne veut pas dire que l'étudiant ne doive pas faire quelques tableaux simplexes à la main etc., mais l'accent est surtout mis sur la signification profonde des opérations de pivotage, plutôt que sur la manipulation algorithmique de la méthode du simplexe. Ce que nous proposons en fait est une sorte de dissection théorique de la boîte noire, dont on utilise des le départ la puissance pour résoudre des situations plausibles intéressantes. Nous avons appliqué cette stratégie également avec succès dans un cours plus avancé de programmation linéaire devant couvrir l'analyse de sensibilité. L'élaboration des équations matricielles théoriques étaient faite à la main et les calculs pratiques à l'ordinateur. L'utilisation de la technologie comme outil de calcul permet de traiter des exemples ayant une apparence concrète, ce qui suscite l'intérêt des étudiants. Cet intérêt sert alors de fondation à l'assimilation de concept.

---

[2] Nous avons utilisé dans ce cas le logiciel MSLP version étudiante pour compatible IBM.

Appendix 7
A Strategy for Teacher Involvement
Eric Muller
Brock University

An Issue from the Working Group

A question raised by a number of participants in this working group was "How does the mathematics teacher who is using technology in the classroom encourage and support colleagues to do the same?"

An answer to this question is badly needed as the innovative use of technology by a particular teacher in a given course/class has usually ceased when the teacher moves to another course/class.

Unfortunately educational reform is mostly achieved in the political arena; the introduction of technology is likely to demand a very similar agenda if it is to succeed. Each academic institution will require its own approach to solve the problem. Nevertheless it is possible to provide some general pointers based on the administrative structure of the institutions.

In Canada education is a provincial responsibility and in some provinces all schools are under the direct responsibility of a provincial ministry of education while in others area school boards have a certain autonomy. In both cases implementation of technology in the classroom requires leadership from either the board or the provincial authority; otherwise implementation will be subject to the ability of a particular school to pay for the technology and the energy of a group of teachers to integrate it into the mathematics classroom. Only boards or provincial authorities have the power to change the curriculum, purchase the equipment, the software and the support services, and provide the teacher inservice required for continued implementation.

At the university level course content for undergraduate courses may be specified by some departmental committee but the teaching methods, books and other educational materials are normally at the discretion of an individual instructor. This is where a fine line is drawn between content required by the department and the academic freedom of the individual instructor. Thus it is quite possible, in a multiple section course, to have the instructor in one section using technology while the other instructors are not. This short paper describes a strategy which has been successfully used at Brock to implement technology in large enrolment multisection courses. In these courses students are now required to use technology and the course calendar entries have been amended to reflect this requirement. Thus not only is the course content prescribed but the use of technology is mandated.

The Brock Experience

Introducing technology in undergraduate education requires the support of the majority of the faculty in a mathematics department. To achieve this we ran and carefully documented a pilot project thereby introducing other faculty to the use of the technology.

The pilot project was designed to demand very little additional work from the faculty and to provide a non-threatening environment.

Here we consider the introduction of a Computer Algebra System (Maple) into the calculus courses. It should be noted that statistical software and operations research software had previously been introduced successfully in all relevant Brock courses. Nevertheless, the introduction of Computer Algebra Systems was much more controversial as it affected all the faculty which at one time or another rotate through the calculus courses.

In the pilot project one hundred volunteer students from an applied calculus course (600+ students) attended weekly Maple laboratory sessions instead of the regular tutorials. Work performed in the laboratory which was subsequently written up was given some credit—a portion of the credit normally given to assignments. Traditional indicators such as withdrawal rates, failure rates and average grade were carefully monitored. A student attitudinal survey of those participating in the laboratory was taken at the end of the course. Two other interested faculty were involved in the running, but not the preparation of, the laboratory activities.

Verbal reports of these activities were presented regularly at department meetings and a full report providing simple analysis of the data[3] was submitted at the end of the year. The results were sufficiently positive to get the department's support for a one year trial implementation of laboratories and to have three other faculty members volunteer to be involved with a laboratory section. The department also negotiated for a new microcomputer laboratory.

In the second year traditional indicators were again monitored and the student attitudinal survey was continued. The data was analyzed and a second report made to the department which recommended that the laboratory requirement become a permanent component of the course and that all faculty volunteer to experience the use of Maple in calculus by attending a laboratory as advisor. Those faculty who were uncomfortable with computers and/or Maple were assigned a senior student who had experience with the system.

All mathematics faculty have now had experience with Maple in a student calculus laboratory. Some faculty have implemented Maple in other courses; others are not convinced of the value of computer use in mathematics. The important point is that the decision of an individual faculty member is now made in light of their experiences and not in total ignorance of the capabilities of Computer Algebra Systems.

---

[3] Muller, E.R., Maple Laboratory in a Service Calculus Course, *MAA Notes Volume 20 Laboratory Approach to Teaching Calculus*, C. Lienbach (Ed.) 1991

Appendix 8

LIDSTONE'S contribution to the report of the 1991 CMESG working group on technology and understanding mathematics

Dave Lidstone

The issue of how to convince colleagues and administrators of the utility of incorporating technology into mathematics instruction arose at each session. Whereas some institutions will respond to input from within such as the work of Eric Muller at Brock University and Frank Demana at Ohio State, others may need exterior influences. Such was the case at the Langara Campus of Vancouver Community College (VCC, Langara). In response to statistics comparing the status of the mathematics major programmes in British Columbia to those in other parts of the country (see attached) the B. C. Committee on Undergraduate Programmes in Mathematics struck a subcommittee to consider the issue of how to best attract more students to major in mathematics. This subcommittee asked college and university mathematics departments throughout the province to consider a visitation programme similar to that in place at the University of Waterloo. In May of 1991 the department at VCC, Langara responded with the recommendation "that any process of attracting students to mathematics should be preceded by a process making mathematics more attractive to students." In order to determine how to best carry out the latter process the Langara department established a self-directed "Instructional Development Seminar" to study issues of reform, including technology. By July of 1991 the seminar had met twice. The first session addressed an overview of exemplary programmes as documented in literature from the MAA and AMATYC, and the second session addressed the use of technology in the classroom. Although attendance was only about half of the department membership of twenty, reaction seems encouraging. We maintain optimism about the seminar growing.

Appendix 9

Technology and Curriculum Change

Geoffrey Roulet
Queen's University

The tendency of discussions concerning the use of technology in mathematics education to focus on the issues of access to hardware and software and schemes for large scale implementation deflects attention away from more primary issues. While a limited number of sites with continuous access to computers and mathematics support tools are required for research purposes, a call for clusters of computers in every mathematics classroom may be premature. This is not to suggest that the potential applications of such technology are not significant. But introduction of the technology independent of curriculum reform and teacher training may lead to rather trivial applications. If these lower level implementations becomes institutionalized it may be difficult to later move to more significant applications with accompanying new views of curriculum.

It is not all that surprising that graphing software for computers and graphing calculators have been accepted into mathematics classrooms. Such tools can be employed to execute traditional assignments with increased speed and accuracy. It is not a major step for the teacher who has in the past assigned the experiment, "Graph $y = \sin(x)$ and $y = \sin(3x)$ and compare the results", to introduce technological support for the graphing. It is a much more major step for the teacher to question the need for the teaching of algorithms for solving equations and to allow students to employ iterative graphical techniques.

The Mathematics Exploration Toolkit, a computer algebra system marketed by IBM, has strong potential in secondary school mathematics and could significantly motivate curriculum change. Contrary to these possibilities, the developers, in the notes provided with the software, appear to be promoting applications that are mainly electronic versions of traditional exploratory exercises. There is the possibility that the software could become a tool for teaching precisely those skills that such computer algebra systems will render irrelevant.

The 1985 Ontario Curriculum Guideline: *Mathematics: Intermediate and Senior Divisions*, in the introduction, suggests that students should have opportunities to build mathematical models. The scheme provided for problem solving via models is a 6 step process. Five of these steps involve translation between the setting and the model while only one involves manipulation of the model. Yet the advanced level (university bound) curriculum described on the subsequent pages deals almost exclusively with these skills of symbol manipulation.

The availability of computer algebra systems and other software tools should allow the topic of mathematical modelling to become the core of the curriculum. The restriction that applications generate only continuous models using at most second degree functions is removed. Complexity is no longer a major problem with software to execute the algorithms or provided iterative solutions. What is needed is a new curriculum direction to accompany the introduction of the technology.

Extensive teacher in-service may be required. In Ontario those mathematics teachers with the minimum qualifications have had little experience of mathematics as a modelling tool. For many the subject consists of those algorithms that the computer can now execute. Experiences employing the new technology for modelling and examining the curriculum implications are required. The issue is not how to introduce computers and powerful calculators into the schools but what form should the curriculum take as such tools arrive.

Working Group D

Constructivism: Implications for Teacher Education in Mathematics

David Reid

Concordia University

Constructivism:
Some Theoretical Concerns Relating to Teacher Development

Four questions formed the foci of discussions. They were:

- "What is Constructivism?",
- "What philosophies of mathematics are compatible with constructivism?",
- "What does a constructivist perspective imply for mathematics teaching?", and
- "What does a constructivist perspective imply for the development of mathematics teachers?"

In answer to the question "What is constructivism?" a working definition was proposed (by Nic Herscovics): constructivism is a theory of knowledge acquisition which holds that knowledge is constructed by the learner. To this it was added that constructivism holds that knowledge is not only assimilated but also accommodated by the learner. This accommodation consists of the overcoming of cognitive conflicts. It was also noted that constructivism focuses on the learners adaptive processes and mental representation rather than on behaviour, performance, or achievements.

In addressing the question "What philosophies of mathematics are compatible with constructivism?" two characteristics of such philosophies were produced:

1. Mathematics must be seen as a process, not a product.

2. Mathematics must have meanings.

This second characteristic excludes formalism and related philosophies which see mathematics as composed of symbol strings without meaning. The discussion of the status of Platonism revealed some interesting contrasts. There seems to be no contradiction between Platonism and the construction of knowledge. Indeed, the unattainable nature of ideal concepts forces the learner to construct a conceptualization which approximates the ideal. These conceptualizations need not be such that they can be organized into a hierarchy of "correctness" as each might come at the concept from a different direction. In this way each might be seen as equally "wrong". On the other hand replacing the Platonic ideal concept with a socially constructed concept allows each conceptualization to be important to the definition of the concept. This validation of the learner's conceptualization was seen by some as an essential part of constructivism. In practical terms the wrongness or rightness of conceptualizations is not so important as the equal status of conceptualizations. One characteristic which separates constructivism from some other theories of knowledge acquisition is the status afforded to the conceptualizations of the teacher. The teacher's conceptualizations are not "better" and to be emulated, but are merely conceptualizations to be considered along with those of the learner.

The discussion of the question: "What does a constructivist perspective imply for mathematics teaching?" produced a list of things a constructivist teacher does and some comments on the methods suited for doing these things. Some of the things on this list were discussed in more detail than others. What follows is a selection of things a constructivist teacher does:

- Presents problems which the learner can understand, but which require the construction of new knowledge to solve, thus creating cognitive conflicts.
- Provides multiple opportunities for the construction of knowledge; does not expect instant results.
- Allows students independence to construct knowledge; does not intervene except when needed.

In discussing materials it was said that the particular textbooks, software, etc. used are less important than the *way* they are used. Knowledge can be constructed in any situation. Learning from a lecture or from a text is possible. However, it was felt that some methods reflect an awareness of the learner's construction of knowledge more than other methods.

The requirement of not expecting instant results runs contrary to current practice in many education systems. A time structured curriculum conflicts with the additional time demanded by the differing rates at which learners construct knowledge. Some compromises are necessary in current practice. One possible compromise is a constructivist approach to teaching important topics with the learner left to construct the remaining topics based only on a quick outline.

Our last question: "What does a constructivist perspective imply for the development of mathematics teachers?" received a fairly definite answer: there should be absolutely no difference between what we do as mathematics teacher trainers and what we want teachers to do in mathematics classes. Some features which we felt should be included in a constructivist approach to teacher development were:

1. The elements of constructivist teaching outlined above;

2. Experiences in listening to students;

3. Learning to develop models of understanding.

Teachers need also to be faced with situations in which they must construct new knowledge, in the same way that their students must. The problems which are addressed in teacher development ought to be problems which can be recognized as such by teachers. These problems might be drawn from the teachers own experiences and from research. All aspects of the development of teachers needs to be done in a constructivist way. This includes the classes they take in the mathematics department, those taken in the education department, and the experiences they have during practice teaching.

If there was any definite conclusion reached by this working group it was that there are many theoretical issues involved in applying a constructivist model of learning acquisition to mathematics teaching and mathematics teacher education. These issues require continued discussion, and it is hoped that some of the ideas which arose within the working group contribute to this discussion.

Topic Group A

Discussion: What can we say?

Susan Pirie

University of Warwick

Discussion - what can we say?

Analysis of peer discussions, in mathematics classes, and the effects on the pupils' learning

This paper is based on findings from a project entitled "Mathematical discussion — is it an aid to understanding?" Its intention is briefly to give an over-view of this project, look at the rationale for the qualitative methods that were used, outline some of the areas that have been explored and then finally to examine one category of data in greater detail.

The project arose as a result of a British governmental report "Mathematics Counts" (Cockcroft 1982), which advocated, without any quoted research base within which to evaluate its assertion, that discussion should occur in all good mathematics classrooms. The very broad and general title of the project mentioned above obviously required some specialisation within the area of mathematical discussion and the focus has been on pupil—pupil discussion, specifically mentioned in Cockcroft 1982, within the classrooms of teachers who consciously and deliberately use discussion as one of their teaching styles. Detailed descriptions of the initial design and implementation of the research can be found in Pirie and Schwarzenberger (1998a). Here it will suffice to give only a brief outline of these processes. The data were collected by tape-recording small groups of pupils as they worked together in their lessons and these were supplemented with field notes made by an observer in the classroom, who recorded activities on a time-sheet which could subsequently be matched to the tape-recordings. Before any analysis of the data could be undertaken, it was necessary to create an explicit definition of *discussion* by which to identify those episodes that should be examined in detail. The following was proposed and has remained the project's working definition ever since.

- ***Discussion is purposeful talk*** i.e. there are well defined goals even if not every participant is aware of them. These goals may have been set up by the group or by the teacher but they are, implicitly or explicitly, accepted by the group as a whole
- ***on a mathematical subject*** i.e. either the goals themselves, or subsidiary goals which emerge during the course of the talking are expressed in terms of mathematical content or process
- ***in which there are genuine pupil contributions*** i.e. input from at least some of the pupils which assists the talking or thinking to move forwards. We are attempting here to distinguish between the introduction of new elements to the discussion and mere passive responses such as factual answers to teachers' questions
- ***and interaction*** i.e. indication that the movement within the talk has been picked up by other participants. This may be evidenced by changes of attitude within the group, by linguistic clues of mental acknowledgement, or by physical reactions which show that critical listening has taken place, but *not* by mere instrumental reaction to being told what to do by the teacher or another pupil.

It rapidly became clear from the first set of data that was collected, that there would be no simple answer to the research question as posed and indeed that had not really been the expectation. Examples arose both where discussion appeared to advance the understanding of the pupils involved and where it quite definitely confused and misinformed them, and inhibited progress towards the solution of their problems. There existed also examples that might have fitted the definitions if the researchers had been able to understand what the pupils were talking about! These were categorised initially as "Incoherent" and left to one side. The task, therefore, became one of looking for meaningful ways in which one could talk about the data in finer detail and thus produce possible indicators of fruitful or unfruitful discussion.

One of the hypotheses underpinning the research was that the situation in mathematics learning might arguably be different from that in other disciplines, because of the fact that mathematics is the only area in which text is written purely symbolically and therefore does not exactly match the spoken language used. Indeed, the very power of mathematics lies in this ability to represent succinctly, through symbolism, ideas which would be lengthy and cumbersome to express completely in ordinarily written language form. An illustrative example of this problem of the discontinuity between the written symbolism and the verbalisation of a piece of mathematics was forced upon the researchers when starting to transcribe the pupils' discussions. What would be the interpretation by a reader of the writing "¾"? That the pupil had said "three quarters" (a number)? "three fourths" (three out of four pieces, i.e. a quantity)? "three over four" (a way of writing)? "three divided by four" (a process) etc.? *All* this meaning is packed into "¾" but not necessarily all conveyed by any single one of the verbal expressions.

Not wishing to prejudge the outcome of the analysis of the project data by imposing what might be inappropriate categorisations it was necessary to create interpretive categories, grounded within the project's own collected data. The process of theoretical sampling was then employed to inform further data gathering. The data collection was to be "controlled by the emerging theory" (Glaser and Strauss 1968). The intention was to use analytic induction to abstract relevant, essential characteristics of the discussions from concrete cases rather than use statistical sampling and enumerative induction to produce categories based on their generality. A more detailed description of the methodology used can be seen in Pirie 1991.

On the first analysis of the first data-set three features seemed to stand out as being of possible interest. The first of these was what it was that gave the speakers something to talk about. Within this feature the episodes could be classified by whether

(a) the pupils had a task or concrete object as the focus of their talk;
(b) they did not have an understanding of something, but knew this and thus had something to talk about;
(c) they did have some understanding and that gave them something to talk about.

The second feature was the kind of language used, the focus being on the language in which the discussion was predominately conducted and not on the content of the statements made. Again there were three categories which suggested themselves. These were:

(f) the pupils lacked appropriate language; they did not have the correct or useful words;
(g) they used ordinary language;
(h) thy used mathematical language.

It could be suggested that the categorisation of language as 'ordinary' or 'mathematical' would be a somewhat arbitrary activity since 'mathematical' language for young children might be 'ordinary' language for them a few years later. In practice, however, viewing the discussion in the context in which the pupils were working enabled decisions to be made with little difficulty, although some subjectivity was inevitable.

The third feature classified was the kind of statements the pupils were making. It was of course noted that there could be a variety of statements within any one episode. The statements were classified as

(p) incoherent — note that that is to say incoherent to the observers;
(q) operational, or, in other words, about doing specific (frequently numerical) examples of mathematics;
(r) 'abstractive' — statements of generalisations of mathematics.

Episodes could be categorised on each of the three features. Illustrations of the use of these categories occur in Pirie and Schwarzenberger (1988a) and (1988b).

Having thus evolved a tentative working categorisation of the first set of data, the next step was to collect a second set by observing further small groups of pupils. This second batch of data was analysed from a completely different perspective, namely from that of the verbal behaviours of the pupils, looking at both their mathematical behaviour as exhibited by their language and the roles they verbally took up within their group.

At this stage the concern was not with exclusivity nor inclusivity, but merely looking for 'features' which would contribute to later, deeper analysis. Examples of mathematical behaviour classes were "defining", "into algebra", "using materials". Verbal behaviours are illustrated by labels such as "working out loud (together)", "revealing errors", "verbalising for approval (frequently their own)", "pupil as teacher". The strength of the methodology being used was that the first set of data was then viewed through this lens, while the original categories of (a) to (r) were applied to the second set of data.

Thereafter, each new set of data that was collected was analysed and used to confirm all the existing classifications or to suggest new ones. Where new categories were suggested, all previous data were reviewed again in the light of the possible new classification. Two new categorisations that did, in fact, suggested themselves were "investigative problem solving" and "text directed exercise". This collection and analysis process continued until the categories seemed to be stable.

Some of these categories are now in the process of being studied in greater detail. A brief illustration, using a couple of examples of how this is happening are given here.

The verbal exchanges categorised as "pupil as teacher" were defined by the occurrence of characteristics such as 'eliciting', 'directing', 'informing' — characteristics of 'teacher talk' put forward by Sinclair and Coultard (1975). These exchanges have now been sub-categorised by who initiates the interaction; that is whether the role is adopted spontaneously by the "teacher" pupil or whether this pupil is pushed into the role by the pupil who takes on the role of "pupil". Questions that are asked of the data include "How faithfully are the roles adhered to, verbally?", "Does the exchange lead to at least short-term success for the 'pupil'?". See Newman and Pirie (1991) for further details of the analysis of this category.

The interactive effects of "investigative problem solving" and the language the pupils use has been approached in a rather different manner. Whole solution-episodes of up to one-and-a-half hours have been examined for the effects of the often unspoken rules by which the 'game' of pupil group work is prosecuted. The ways in which the problem as originally posed often comes to be subtly and unintentionally altered are being noted. The effects that the pupils' perceived need for verbal, as opposed to paper and pencil, solution methods have on the success of the solution of the problem are being investigated. Pirie (1991b) describes the analysis of one such whole solution-episode in which three pupils are addressing the problem: "74 is closer to 81 than 64, so the square root of 74 is closer to 9 than 8. True? Generalisable? Proof?".

The decision has also been taken to revisit the data that had been put to one side as "Incoherent" and it is this category of data that this paper will expand on here.

One of the dangers inherent in research into spoken language is that much interpretable meaning may be lost between the visual observation of the event and its reduction to a transcript. Communication depends upon more than just the words used: inflection, intonation, silence and gesture all play a part. It was therefore possible that transcripts which had left the discussion looking fragmented and even ambiguous, full of hesitations, repetitions, irrelevancies and non-sequiturs were in fact of episodes in which the participants were confidently communicating with one another by supplementing their utterances and interpretations with body language and shared background understanding. It was particularly this shared background understanding which pupils assumed when communicating with each other, that was thought to possibly hold the key to interpreting some of the category of incoherent exchanges.

On re-analysis those episodes which had been categorised as incoherent were found to be subdividable into those in which it seemed to the observers that no meaning was conveyed from one pupil to another, and those which, although at first sight unintelligible to an outside observer, were, in fact, comprehensible to the children involved, and fitted the definition for discussion, with intonation and subsequent action revealing that mathematical communication had indeed occurred. The following brief extract gives the flavour of this sub-category:

Peter:	Do you get minus 1 to minus 1?
James:	Plus 1 is ... times ...
Peter:	Yeah — 'cos minus 1
James:	times minus 1 is minus 2 ... plus 1 is minus 1.
Peter:	No.
James:	Yes.

The outside observer needs to realise that there is a shared, but never verbalised, understanding that the talk is examining the function $y = x^2 + 1$. A 'translation' of this 'incoherent' passage, with an attempt to verbalise the intonation, might be:

Peter:	Do you get y equals minus 1 when x equals minus 1 ? *(implicit: I do)*
James:	Yes — because substituting minus 1 for x we get minus 1.
Peter:	*(interrupting)* plus 1 is *(thinks: two, no, that is not what I did)* ...No it's times
James:	*(Picking up again from his "minus 1")* times minus 1 *(i.e. minus one squared)* is minus 2. Minus 2 plus 1 is minus 1.

Peter: (*doubtfully*) No.
James: (*with conviction*) Yes!
Peter's face clears and he nods.

An additional feature leading to confusion for the outside observer is the shared, but incorrect, 'understanding' that $(-1)^2$ is -2. It is analysis of this incoherent passage which reveals that these pupils possess this erroneous, but shared, and therefore not disputed, belief. One danger would seem to be that such discussions are indeed coherent to the participants and that shared misunderstandings may become confirmed and consolidated.

Examination of any natural talk, between adults or children will rapidly reveal that few people converse in complete, grammatically constructed sentences, but the problem for discussion of mathematics seems somewhat greater: "The strangeness of the vocabulary, the oddity of the form and the tyranny of the right answer, all combine to make mathematics something that is not ... talked about in any conversational way." (Ensor and Malvern, 1987). Pupils wanting to discuss a mathematical problem frequently do so using everyday language, which may not admit of the precision of mathematical language. Alternatively they can communicate through assumptions of shared understanding of both mathematics content and language which may often result in the episodes that have been grouped as initially appearing incoherent to an observer. There is of course no guarantee that the shared understanding is correct mathematically, as has been shown above, nor is there frequently any attempt to ensure that the assumed understanding is in fact the same for all participants. Consider the following excerpt where the pupils are referring to the list of square numbers 4, 9, 16, 25, ... while working on a problem:

Laura: Try the middle one between them.
Claire: Yes that's what I want to know. Something exactly half way between. So what's it got to be? It's got to be between an odd number and an even number, hasn't it to be half way between?
Laura Well do it in between 9 and 16.
Claire Isn't that half way in between? (points to 6 which was earlier suggested as some number between 4 and 9). 5 and 8, 6 and 7, no. (*Meaning 6 is not exactly half way between*).
Ann 6 and 7 that's true. (*Meaning they are both half way between*)
Laura (*Dismissively*) 6 and 7? 7's between 9 and 16 (*meaning that the difference between 9 and 16 is 7*), which is 3, 3 and 1. So what's 4 and 9?
Ann 13.
Laura 13. 13 is the middle of 9 and 16. (*Turning to Claire*) OK?
Claire I was wondering how you did it.
Laura Because 7 is 3 and 3 on each side and one extra in the middle. Yes, OK? (*Meaning that 9 plus 3 plus 1 will give the middle number 13*)

Here all three pupils have a different meaning for "between". For Claire it is a unique, middle number achieved by pairing off numbers till the middle is reached; for Ann it can be the pair symmetrically in the middle of two numbers on a number line; while for Laura it has the mathematical meaning of "difference between": 16 - 9 = 7. Even the "4 and 9" that Laura refers to are not those with which Claire is working! It is some little time later before the three girls realise that they do not share the same meaning for "between". (See Pirie 1991 for more detail)

An alterative strategy which has been observed is the creation of their own vocabulary, which is rarely openly defined but is certainly taken as understood by all the pupils participating in the discussion. Examples of this include "parallel angles" for the angles created by a transverse to a set of parallel lines and "length of a circle" for the diameter, which is arguably less ambiguous than "length of a rectangle"! "Devising" gives a verb for the process of finding the common denominator - these pupils were heard to use the word "divisor" only in the restricted sense of "common denominator".

The following excerpt shows pupils taking a word, appropriate in one context, and using it with total shared understanding in a seemingly inappropriate context:

Len: There are 5 moves for 4.
Andy: How many number of different moves did you get when you got to 5?
Len: You should have wrote your's down.
Andy: (*Dismissively*) I have! How many number of moves—how many number of moves did you get?
Chas: For 2 it's 1.

In a previous lesson these pupils had worked together on an investigation called 'Frogs' which involved counting the number of times counters were moved to achieve a prescribed rearrangement. In the extract above, however, the pupils are counting how many ways they can arrange a given number of square tiles. How they manoeuvre the tiles to produce the arrangements is irrelevant and not being counted, yet well into the lesson they are still using the language of "moves" to mean "arrangements" without any confusion to themselves. They have implicitly, but nowhere explicitly, created a shared understanding that, say, "in maths lessons we will use 'moves' to mean anything that we have been asked to count." It is interesting to note that several weeks later these same pupils were still using the word 'moves' in relation to number patterns.

In this incident no confusion arose since only the pupils were involved in the exchange, but clearly a participant observer, such as the teacher, overhearing such language might feel constrained to intervene and correct the language. The following excerpt illustrates a situation of teacher intervention where pupils were using a shared, but erroneous, understanding, of the meaning of technical, mathematical language.

Two boys were working together on linear equations, making tables of values and plotting graphs. One of the equations given on the worksheet was $y = 2x + 3$.

Kevin: You fill it. (*referring to their table*)
x - one, x squared - two, x squared plus three - five.
x - two, x squared - four, x squared plus three - seven
x - three, x squared - six, x squared plus three - nine
x - four...Wait, we need some minuses...er...do minus three.

Dave: Easy-peasy. x squared - minus six, - minus three x squared - minus two,...no...minus four, - minus one x squared - minus two, - plus one

At this point the teacher passed by, and, hearing "x squared", stopped.

Teacher: Hang on. Go back a bit. You said "if x is three then x squared is six?"
Kevin: Yeah
Teacher: O.K. Hang on a minute...

He then proceeded to draw a 3×3 square, divide it up into unit squares, and count them.

Teacher: So it's nine, isn't it.
Kevin: What?

The teacher repeated the explanation and demonstration with a 4×4 and a 5×5 square.

Kevin: Oh, yeah. (*doubtfully*)
Teacher: Can you carry on now?
Kevin: Yeah.

The teacher moved away.

Kevin: What's he on about? I know all that stuff about areas and counting things
Dave: Yeah, what we got here's lines.
Kevin: Just ignore him. Your turn. You do x-squared minus three (*writing* $2x$ - 3)

At the time, this piece of discussion neither enhanced nor inhibited the boys' current understanding, although the long term effect cannot be predicted.

Perhaps the most important message to come out of the above examination of the ways in which pupils use shared understandings is the fact that teachers need to be alert to the possibility that they and their pupils may not always share the meanings attached to the language being used. Even use of precise, mathematical terminology, and in particular use of symbolic language, does not carry a guarantee that identical mathematical ideas are being considered and communicated and teachers need to be ever alert to this danger.

The intention at this juncture is not to lay down rules for good discussion, nor even to suggest environments within which pupil-pupil discussion might be advantageous. There is still much more to be learned from the data. It is rather a moment at which to raise some questions and caveats for consideration. How important is it that pupils use mathematical language correctly? In their own thinking? With their peers? When communicating to others? If fluency in mathematical language is considered necessary, how can pupils be helped to achieve correct usage? be helped to gasp the multiplicity of meanings conveyed by symbolic representations? Can pupils acquire by themselves, discussing mathematics with their peers, the sophisticated mathematical notion that different verbalisations of, say, "¾" nonetheless in some sense "mean the same thing", or will this lead to firmly and jointly held restricted understandings? How important is the teacher's presence if correct usage is the goal? Are pupils' learning and problem solving inhibited by a necessity to talk? Should mathematics be considered an activity in which there is no need for 'saying' between 'seeing' and 'doing'? Where all communication is best done using written symbolism? These are not trivial questions and must not be dismissed with the time honoured phrase "discussion helps one clarify one's thinking". Is this indeed true for mathematics?

References

Cockcroft, W. H. (1982). *Mathematics Counts*, Report of Committee of Inquiry Into the Teaching of Mathematics in Schools, HMSO, London.

Ensor, B. and Malvern, D. (1987). *Children Talking Mathematics*, A Report of the Reading University and Berkshire LEA Mathematics/Language Project

Glaser, B. G. & Strauss, A. L. (1968). in *The Discovery Of Grounded Theory*, Weidenfeld and Nicolson, London.

Newman, C. & Pirie, S. E. B. (1991). *The Pupil as Teacher: Analysis of Peer Discussions in Mathematics Classes, Between 12 Year Old Pupils and Effects on Their Learning*, Proceedings of Psychology of Mathematics Education, XV, Assisi.

Pirie, S. E. B. (1991a). Enumerative induction abstracts by generalisation, whereas analytic induction generalises by abstracting, invited paper for the *Conference on Qualitative Research Methodologies*, Montana State University, Montana, U.S.A.

Pirie, S. E. B. (1991b). Peer Discussion in the Context of Mathematical Problem Solving, in *Language and Mathematical Education*, Durkin K. and Shire B. (ed.), Open University Press.

Pirie, S. E. B. & Schwarzenberger, R. L. E. (1988a). Mathematical Discussion and Mathematical Understanding, *Educational Studies in Mathematics*, 19, 459-470.

Pirie, S. E. B. & Schwarzenberger, R. L. E. (1988b). Mathematical Discussion - It's What You Say *and* What You Do, *Mathematics Teaching*, 123, 58-60.

Sinclair, J. McH. & Coulthard, R. M. (1975), *Towards An Analysis Of Discourse*, Oxford University Press, London.

Topic Group B

Partitioning: Shape, Action and Mathematical Thinking in Children

Yvonne Pothier

Mount Saint Vincent University

Partitioning—Shape, Action, and Mathematical Thinking in Children

Key position papers by Kieren (1976, 1980) provided the theoretical context for my initial study of children's partitioning behaviours. In particular, Kieren identified partitioning as a basic process in fraction knowledge acquisition.

Using the method of clinical interaction, children's partitioning behaviours were studied in order to trace the emergence and differentiation of the process of partitioning as revealed in children's attempts to solve different partitioning problems.

Partitioning tasks

The partitioning tasks devised for the initial study varied substantively according to the definition of the unit. Units were of the following form:

1) discrete objects (carton-truck problem);
2) discrete objects with the elements divisible (cookie problem);
3) discrete set with subsets separable (boxed-candy problem);
4) continuous quantity with subsets separable (chocolate-bar problem); and
5) continuous quantity (cake problem, giant cookie problem) (Pothier, 1981).

Insights gained from observations

An indepth analysis of children's attempts to subdivide a continuous whole into equal parts (Cake problem involving rectangular and circular shapes) led to the articulation of a five-level theoretical account of the development of the partitioning process in children. Briefly stated, at a first level, children are capable of partitioning rectangular and circular shapes in half and fourths. At level two, the ability to attain halves and fourths is extended to fractions whose denominators are powers of two. The process is an algorithmic one with no concern for equality of parts. Level three marks a breakthrough in the child's thought in that the algorithm of halving becomes more meaningful and equality is a critical factor in determining fair shares. At this level, fractions with even denominators can be attained. At level four, children search for and eventually discover a "new" first move which enables them to partition a rectangular and a circular shape in thirds and fifths. The dominance of a halving cut as an initial partitioning move is overcome. At this level, theoretically all unit fractions can be attained on a rectangular or circular region. The fifth level, called composition, enables a child to attain composite number denominators in a more efficient manner than the one-by-one procedure of level four. (For a more detailed interpretation of the levels see Pothier, 1981 and Pothier & Sawada, 1983.)

Geometric shapes — a model for fractions

The idea of using partitions of geometric shapes in introductory fraction instruction as an aid to concept development has long been a practice of textbook writers. In particular, it appears that authors assume that children focus on the geometry of the shape and that attaining particular fractional parts is facilitated by shape selection (e.g., selecting a pentagon for fifths, or a triangle for thirds). In working with children, Pothier & Sawada (1984a, 1984b), found that children do not seem to attend to the geometric properties of shapes.

What is it that captures the attention of children when they first engage in partitioning exercises? In analyzing children's partitioning behaviours on different shapes (heart, equilateral triangle, square, parallelogram, regular pentagon, regular hexagon, and a five pointed star), one inference seems overwhelming: rather than focus on shape to decide the number of partitions into which a given shape would be easily partitioned, children have a number of procedures or mechanisms which dominate their partitioning. Four partitioning mechanisms manifested are listed in order of their dominance: halving cut, orthogonal lines, parallel slicing, and truncating vertices.

Children appear to use these mechanisms as "templates" on shapes to attain different numbers of parts. For example, a child will decide to attain four parts on a pentagon, a triangle, or a star shape as well as on a square by constructing a pair of orthogonal lines; or, a child will choose to make three partitions on a triangle, circle, or heart shape as well as on a square or parallelogram by using parallel slicing.

Given the opportunity to decide into how many parts it would be easy to partition a given shape, children choose even numbers on a triangular, pentagonal, or a star shape more frequently than they choose three (on the triangle) or five (on the pentagon or star) (Pothier & Sawada, 1984a).

Why is it that children ignore the geometric constraints of shapes even after considerable practice with introductory fraction textbook exercises? A conceivable answer is that this is a learned disposition.

Learning materials for early fraction experiences focus on diagrammatic representation of geometric shapes such as triangles, squares, rectangles, parallelograms, stars, etc., which are **pre-partitioned** into a given number of congruent parts. Two types of questions commonly posed are: How much of the shape is shaded and can you shade in an indicated fractional part of the shape. Success in such exercises does not demonstrate an understanding of the partitioning process nor does it provide children with fraction models as children frequently attribute fraction names to unequal parts of a whole (Pothier & Sawada, 1984a, 1984b). Contrary to authors' intent, it is possible to complete such exercises while remaining entirely in the realm of whole number thinking. The counting algorithm quickly focuses children on the parts; the geometry of the whole is ostensibly irrelevant. Under these circumstances, is it possible, if not probable, that children learn to ignore the whole?

Benefits of successful partitioning

It is recommended that teachers engage children in partitioning activities. Some introductory activities are described in Pothier & Sawada (1990). As children mature in their partitioning capabilities, the following concepts are learned:

1. An awareness of the geometric properties of shapes including the regularity of a shape, the number of sides and vertices, the diagonals, the midpoints of sides, and the centre point of the shape.

2. Knowledge of such different partitioning techniques as the half cut using different orientations, the parallel slice, the corners truncation, and the radial cut.

3. Knowledge of possible operations on shapes, as for example, dividing sides into any number of equal segments, constructing points in the interior of the shape (e.g., the centre point), and attaining different shaped parts within a given shape.

A coordination of the various capabilities slowly emerges over time and enables children to model fractions on geometric shapes. With at least one concrete model readily accessible, fraction work becomes more meaningful.

References

Kieren, T. (1976). On the mathematical, cognitive, and instructional foundations of rational numbers. In *Number and Measurement*, edited by R. A. Lesh. Columbus, OH: ERIC/SMEAC.

Kieren, T. (1980). The rational number construct — its elements and mechanisms. In *Recent Research on Number Learning*, edited by T. E. Kieren. Columbus, OH: ERIC Clearinghouse for Science, Mathematics, and Environmental Education.

Pothier, Y. (1981). *Partitioning: The development of rational number ideas in young children.* Ph.D. dissertation, University of Alberta.

Pothier, Y. & Sawada, D. (1984a). Children's judgements concerning "easy" partitioning tasks as related to a theory of partitioning. Urbana, Ill: ERIC Clearinghouse on Elementary and Early Childhood Education. (ED 245 836)

Pothier, Y. & Sawada, D. (1990). Partitioning: An approach to fractions. *Arithmetic Teacher* **37** (December): 12-16.

Pothier, Y. & Sawada, D. (1983). Partitioning: The emergence of rational number ideas in young children. *Journal for Research in Mathematics Education* 14, November 1983 307-317.

Pothier, Y. & Sawada, D. (1984b). Some geometrical aspects of early fraction experiences. *Recherches en Didactiques des Mathématiques* **5**, 215-28.

Topic Group C

Using Software to Develop Pupils' Probabilistic and Statistical Thinking

Andee Rubin

TERC
Cambridge, Massachusetts

Using software to develop pupils' probabilistic and statistical thinking

The presentation in this topic group included a description and discussion of three programs: ELASTIC (whose published version is called Statistics Workshop), Sampling Laboratory, and TapeMeasure. A short description of each program follows.

TapeMeasure TapeMeasure (Rubin and Goodman 1991) is a system that allows students to make measurements on a videotape. They can choose particular frames to measure by using a VCR-like interface that advances the tape a frame at a time or by jumping to a particular segment they have previously named. The tool palette contains a ruler, a stopwatch, and a protractor as well as several "mark-up" tools that students use to prepare for measurements.

The investigation we designed involved 7th and 8th grade students exploring the variables that might influence their running speed. Students constructed a list including variables directly measurable in the classroom (e.g., height) and variables measurable only using a videotape (e.g., stride length). They designed an experiment format, including details such as how far to run, and videotaped the race. They then made the relevant measurements from the tape and, using a data analysis program, analyzed the resulting data to determine which variable(s) were most closely correlated with running speed. During the investigation, they made conjectures named them after the person who suggested them, tested their conjectures, and even entertained models beyond the 2-variable linear model as appropriate descriptions of their data.

Figure 1 is the tool palette from TapeMeasure. The first three tools—protractor, ruler, stopwatch—measure angle, length, and time, respectively. The last two allow students to place dots or lines on the screen that remain in place even when the video runs forward or backward. Thus, a student can measure stride length by placing a dot on a runner's foot when it leaves the ground, advancing the video until her foot touches the ground again, placing another dot, and measuring the distance between dots. Measurements made using the tools are added automatically a student's datasheet when the student clicks on the corresponding field.

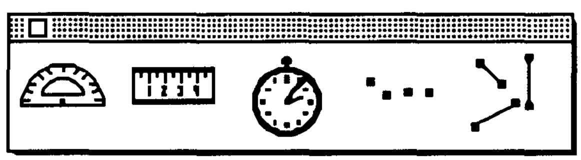


Figure 1 The Tool Palette

Figure 2 is an almost-completed data sheet. Notice that there are three types of fields shown: type-in, click-in, and category. Type-in fields (e.g., "Height") are manually filled in by the student. Click-in fields (e.g., "Stride Length") are automatically filled in by TapeMeasure from the results of a measurement. When these fields are filled, they are marked with a tape icon that can be used to go back to the frame on which the measurement was made. Category fields, such as "Gender" have just a few choices and are implemented, as shown, as buttons.

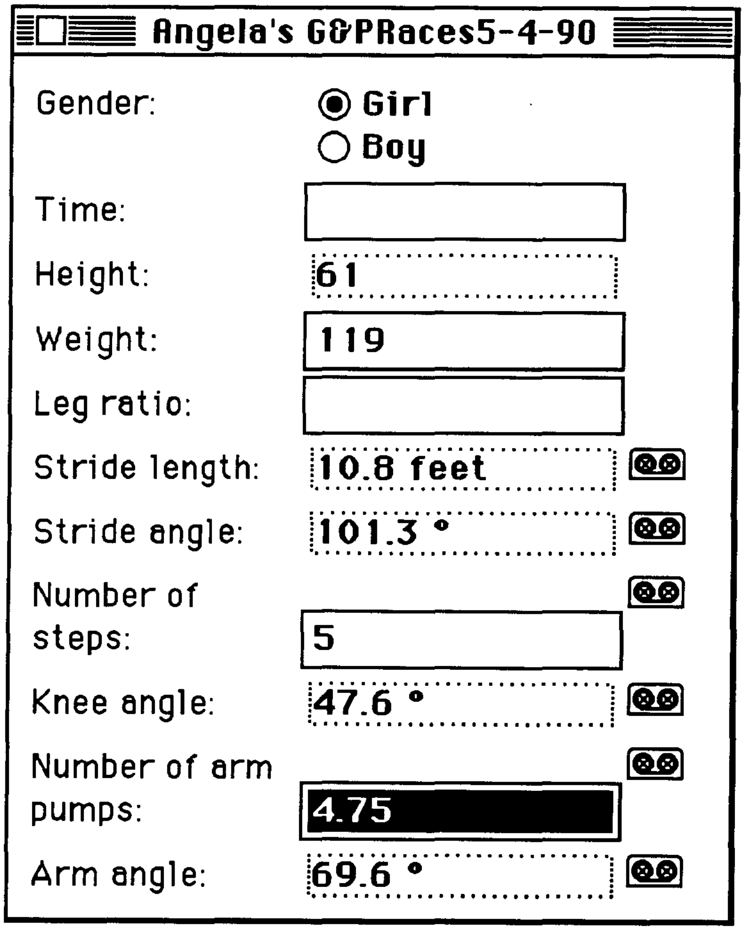


Figure 2 Data Sheet

The investigation was remarkably successful. It was carried on in a seventh and eighth grade classroom in an urban public school over a 6-week period. After designing and carrying out the running experiment and making the necessary measurements, the students analyzed their data. The conjecture that "the higher the weight, the slower the runner" was disproved, much to the relief of a slightly chunky girl. The variables "stride length" and "running speed" were most highly correlated, much to the relief of the researchers who were afraid that error would overwhelm any correlation. One student commented on the high correlation, "All the points are in the suburbs of the line." But by this time the students had decided that one explanatory variable was just not enough. Several expressed a desire to see combinations such as height versus weight versus race time ("If someone's heavy for their height, they may not be able to run fast.") or height versus stride length versus race time. Clearly, they were asking about multiple regression, but the topic was way above their heads. We compromised by showing them three-dimensional graphs of race time, stride length, and height, which satisfied them for the moment.

The success of the investigation was due to two separate factors: the curriculum and the technology. The investigation was well-designed, incorporating a topic of intrinsic interest to middle school students, with enough measurable variables to generate many plausible hypotheses. There were some very open questions (no one really knew which variables would have the most effect), as well as some natural extensions. But the technology itself had its own effect. Using video as data let students return to the source, checking measurements or using a different sampling procedure. Some episodes that were taped incorrectly were discarded, and it was simple to add incrementally to the video data set. Watching the video at standard speed allowed students to see their measurements in context, a reminder of the purpose of the investigation and of the complexity and limitations of taking measurements. Watching at slower or faster speeds gave them markedly different perspectives on the actions and patterns captured on the screen. Here the combination of technology and pedagogical perspective produced an effective learning environment.

Statistics Workshop Statistics Workshop (previously ELASTIC (Rosebery and Rubin, 1990)) includes special facilities for helping high school students learn how to think about numerical distributions and their measures of location (mean and median) and variability (quartiles). *Stretchy Histograms* allows students to manipulate the shape of a hypothetical distribution as represented in a histogram. Using the mouse, they can stretch or shrink the frequencies of the classes in a distribution and watch as graphical representations of mean, median, and quartiles are dynamically updated to reflect the changes. In this way, students can explore the relationships among a distribution's shape, its measures of central tendency, and its measures of variability. They can also use the program to construct histograms that represent their hypotheses about distributions in the real world and compare their graphical predictions with histograms built from actual data.

An incident in a high school statistics class shows how Stretchy Histogram might be used to explore how mean and median provide different measures of location in a distribution. Students were asked to graph the distribution of a sample of 105 people who had been asked how many hours they slept each night. Figure 3 is a copy of the Stretchy Histograms screen for these data. The second problem asked them to add additional data to the right end of the distribution. When they raised the 10-hour bar of the histogram so that 35 people, rather than 10, slept 10 hours (which they could do instantly with the mouse), the median value was, surprisingly to them, still 8 (see Figure 4) Thinking they had done something wrong, the asked their teacher. She suggested they skew the distribution even more and raise the 9-hour bar from 20 to 35 people(see Figure 5).

Imagine their surprise when the median continued to be 8! The teacher was surprised, and confused, as well. It took a long class discussion in which students and teacher participated as equals in understanding how stable a measure of central tendency the median really is. By stretching their histogram even more, they discovered an interesting property of a distribution with many repeats of the same value (e.g., many people who slept 9 hours, in this example). They saw that adding relatively large numbers of new data points to the distribution might not change the value of the median. The explanation of this seeming paradox was that the median, in fact, *did* change, but it changed to another data point with the same value. Thus, the *particular* data point that was the median changed—but its value remained the same.

A different set of activities using Stretchy Histogram is "construction" problems that ask students to create a distribution in which, for example, the median is smaller than the mean or

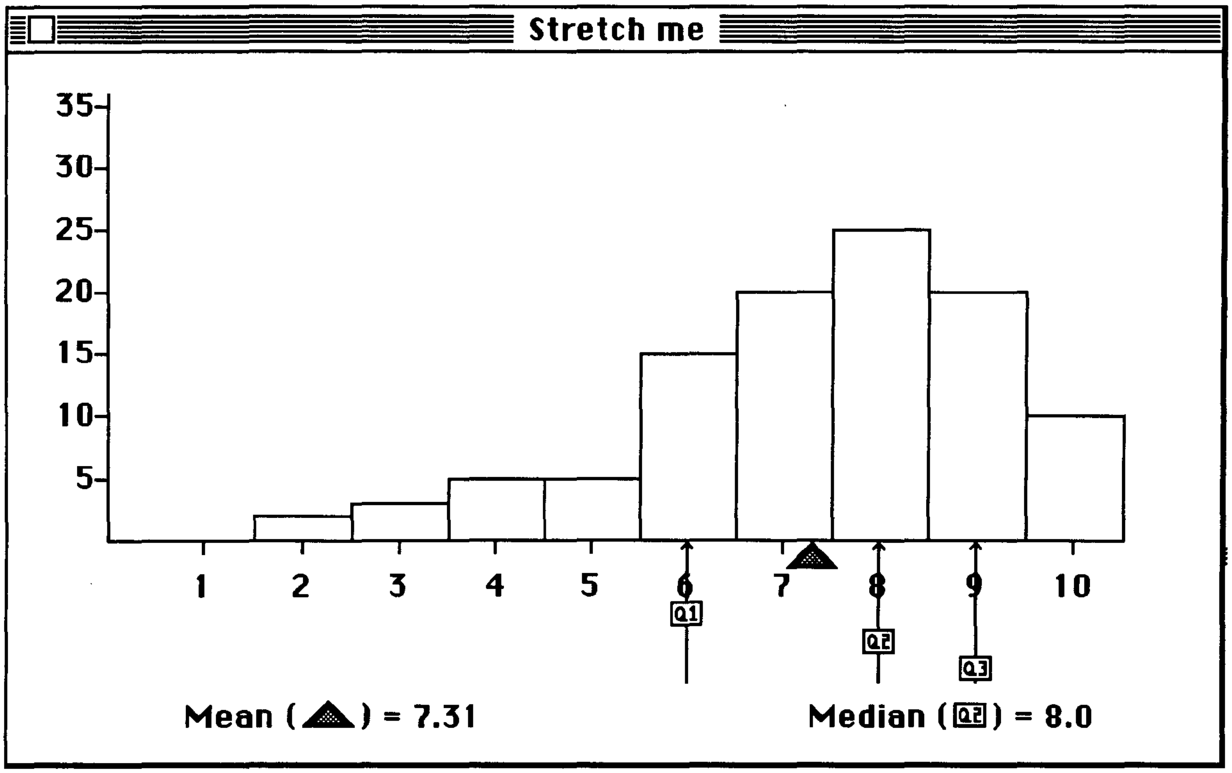


Figure 3 Stretchy Histogram

at least 1 larger than the mean, or in which a quartile is equal to the mean. A comparative construction problem might ask students to create two distributions that differ by only two data points, in one of which the median is greater than the mean and in the other, the median is less than the mean. Such problems require students to go beyond memorizing the definitions and formulas for statistical quantities and to develop a feel for their values based on the shape of the distribution.

Stretchy Histogram's combination of interactivity and dynamic links contribute to its effectiveness as an exploratory learning environment. Its interactivity, the ability to manipulate bars of a histogram graphically, rather than through its numerical representation, makes it easy for students to do many quick experiments with distributions. The dynamic links between the graphical representation and the values of mean, median, and quartiles, provide students with immediate feedback on the effects of their manipulations. In combination, these features make it possible for students to test and retest mathematical hypotheses about distributions and their measures. But "making it possible" is not "making it happen." The nature of the mathematical community in the classroom has the most profound effect on how students take advantage of the characteristics of programs like Stretchy Histograms.

Sampling Laboratory The Sampling Laboratory (Rubin and Bruce, 1991) was designed to be a tool to help students understand this connection between sampling and inference. Like many other sampling programs, Sampling Laboratory allows students to define a population in terms

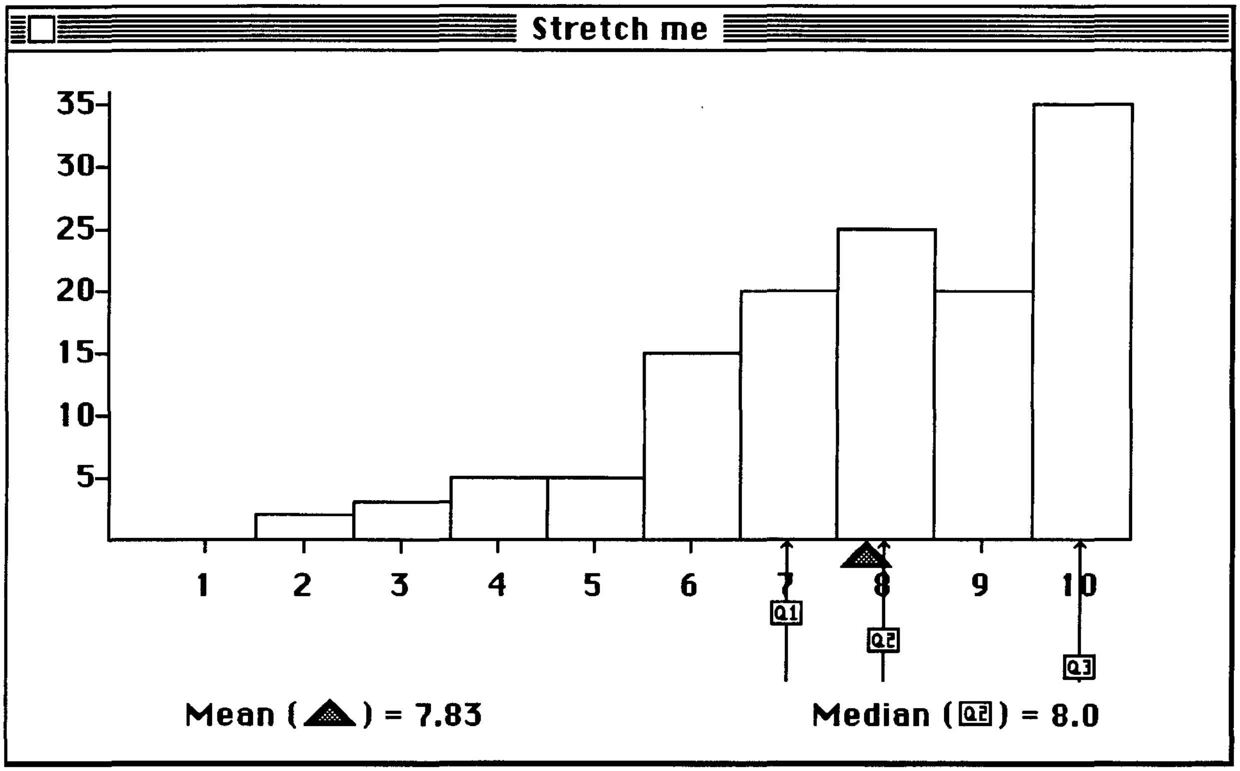


Figure 4 Stretchy Histogram

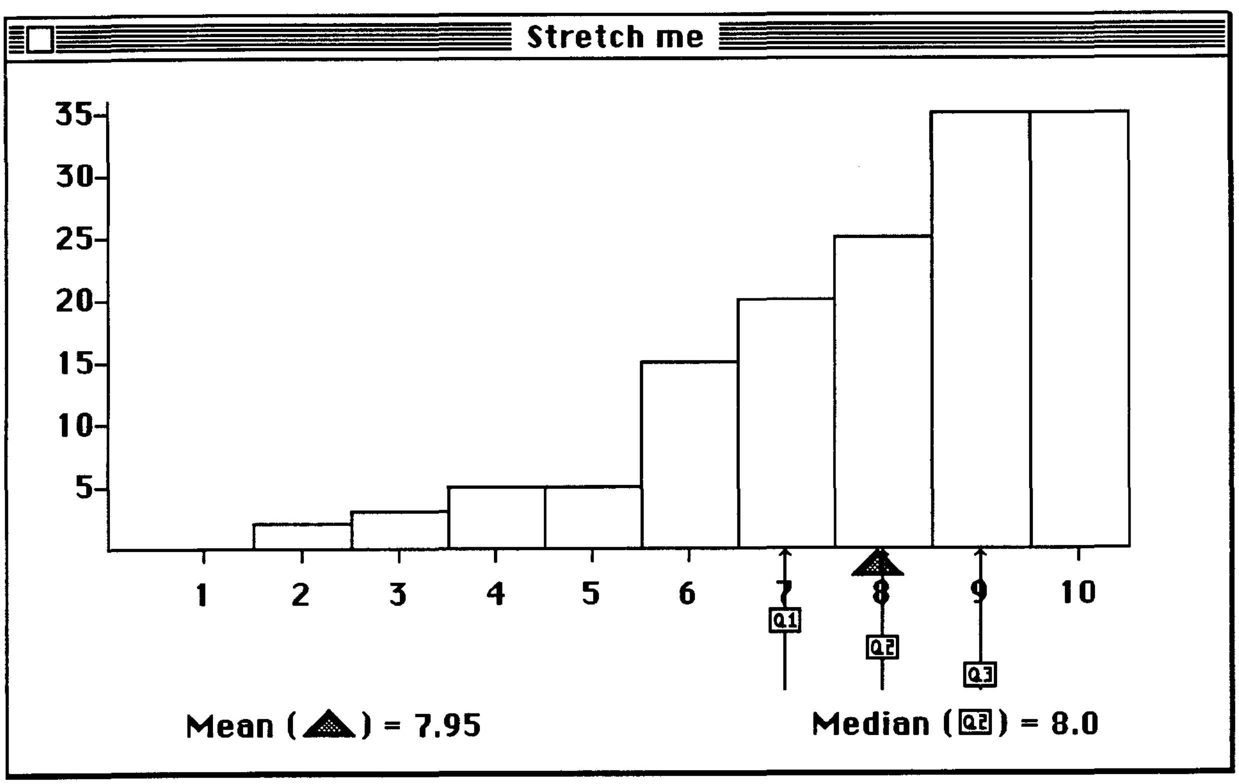


Figure 5 Stretchy Histogram

of proportions and to take multiple samples of a particular size from the population. But it goes beyond this common facility to visually and dynamically link the sampling process to the calculation of confidence intervals. To understand how it supports students' understanding of these connections, consider the following problem that students have worked on in a 10th grade algebra class.

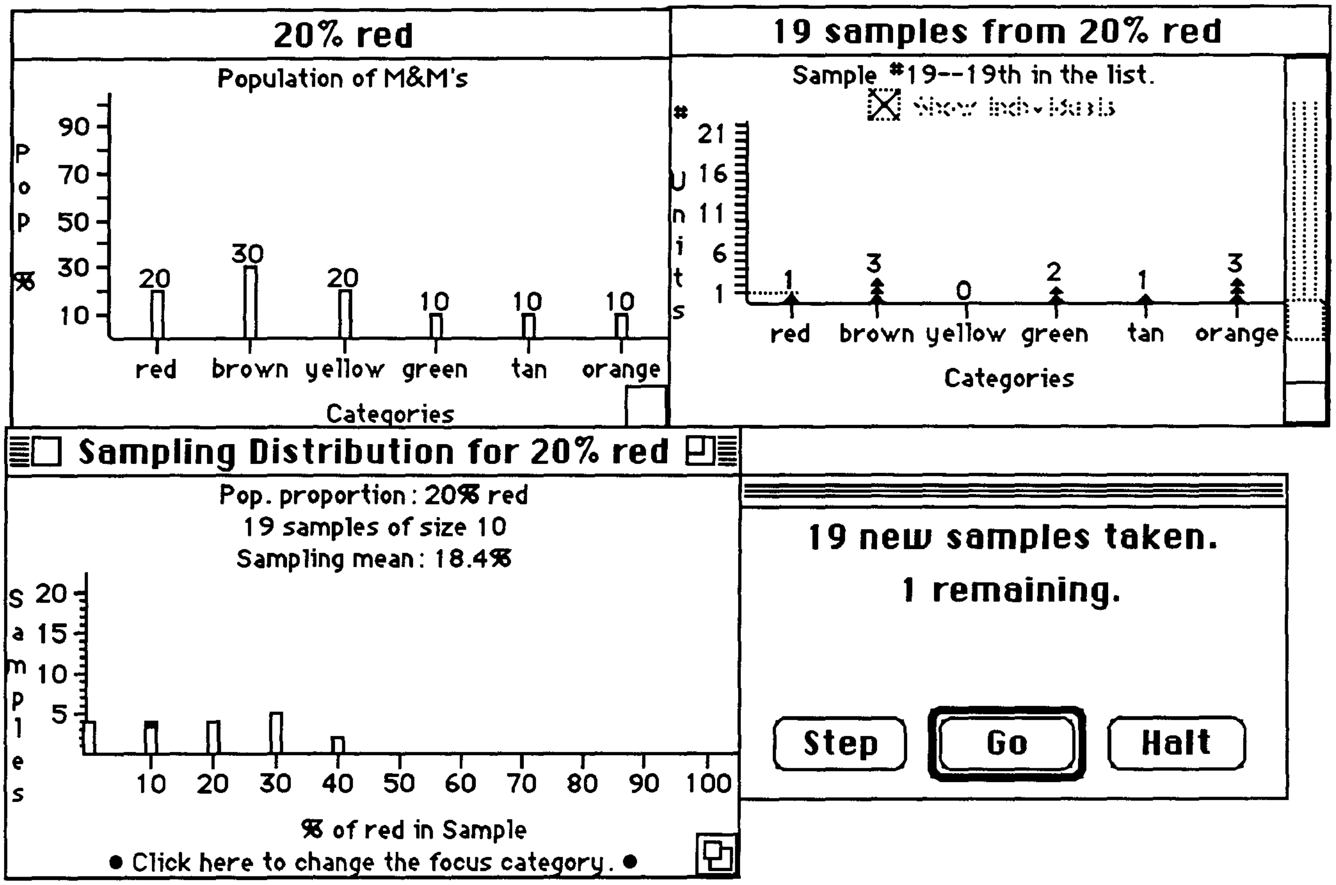


Figure 6 Sampling Distribution

Students explore sampling and confidence intervals in the context of a classic probability topic: M&M colours. They begin by getting a smallish (20-25) sample of M&M's; the question to explore is whether there could be 30% red M&M's in the population. With the Sampling Laboratory, they can define any population by manipulating the weight of various categories. For M&M's, the category values are red, orange, yellow, green, tan, and brown. To explore confidence intervals, students create several populations that differ by the percentage of reds—10%, 20%,..., 50% etc. For each such population, they run a sampling experiment by specifying a sample size and a number of samples to be drawn. Figure 6 shows the production of a sampling distribution for a population of 20% red with 20 samples of size 10.

With Sampling Laboratory, the students can compare sampling distributions for different populations by simultaneously displaying them on the screen. They can also compare sampling

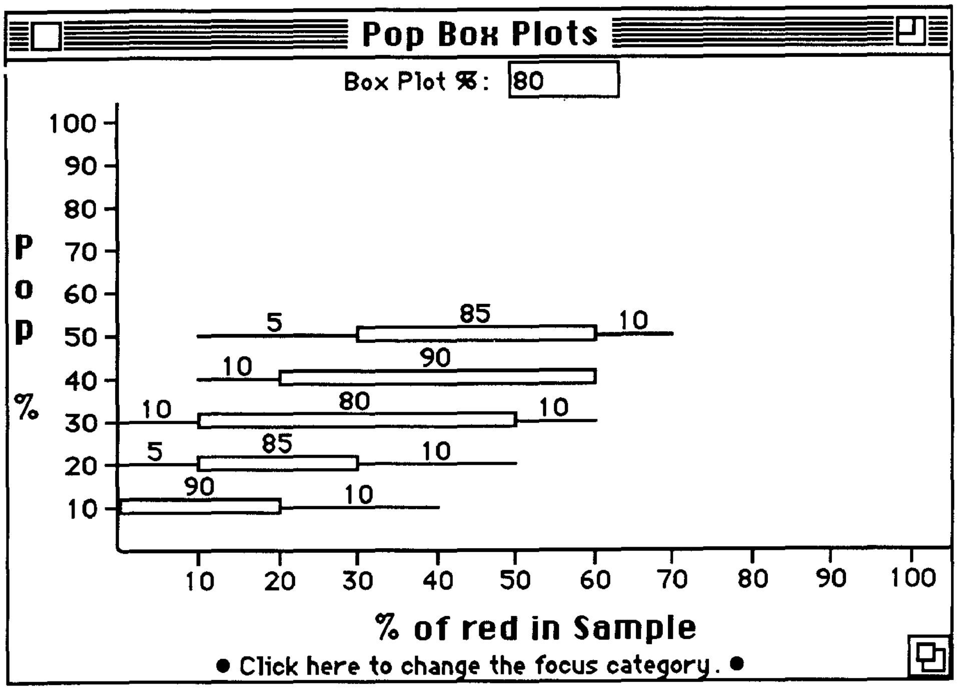


Figure 7 Box Plots

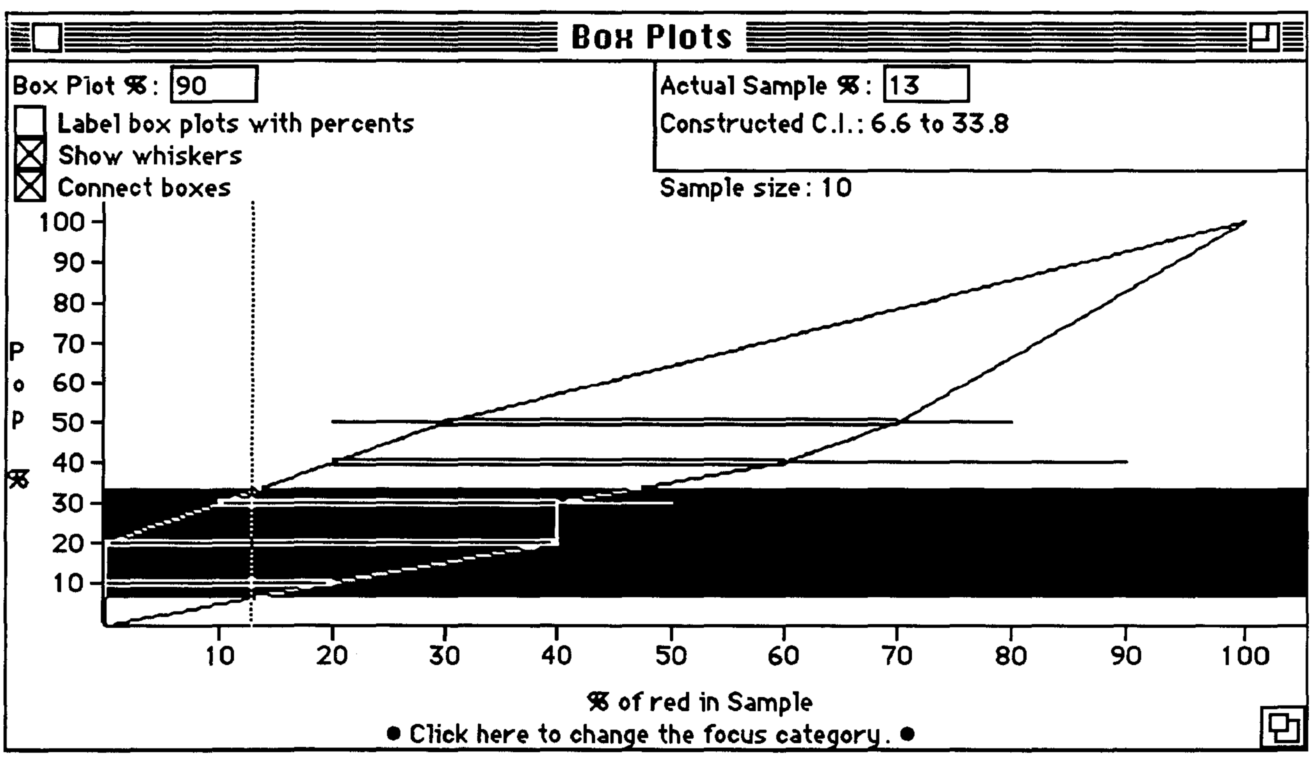


Figure 8 Box Plot

distributions by viewing box plot summaries for each distribution on the same plot. Figure 7 shows 5 box plots for the M&M populations with different percentages of reds = 10%, 20%, 30%, 40%, and 50%. On this graph, students can see the relationship among the box plots, noting how much they overlap. On this same graph, students can also see how a confidence interval is related to these box plots. By clicking on the x-axis at the point corresponding to an actual sample, they can see the confidence interval for that sample (i.e. all the populations with which the sample is consistent) (Figure 8). Moving the line corresponds to changing the sample, and the confidence interval changes accordingly. Students can see how the confidence interval shrinks as the sample gets near the extremes of 0 and 100, and how many populations with different proportions of red are compatible with a particular sample result. They can use this picture to identify the populations compatible with the sample of M&M's they originally received and thus answer the original question: Could there be 30% red M&M's in the population?

Topic Group D

Criticalmathematics Education: Towards a definition

Marilyn Frankenstein

University of Massachusetts - Boston

Criticalmathematics Education

One of the highlights of my professional life has been being discovered by the right-wing. In spite of the fact that the critique in *The New Republic* (Siegel, 1991, p. 35)[1] was based on an inaccurate, trivialized, conception of only one area of my work, and was focused on me as an individual rather than viewing my work as part of an organized international effort to change mathematics education, I was thrilled at that indirect evidence that we are, indeed, having an impact. The author, without any footnotes, incorrectly attributes the creation of the term "ethnomathematics" to me. (The first use of this term that we know of is by Professor Ubiritan D'Ambrosio of the University of Sao Paulo in Campinas, Brazil during a course given through the Organization of American States in 1975.) The author, without any footnotes, inaccurately describes "ethnomathematics"as an area "in which the cultural basis of counting comes to the fore"—Arthur Powell and I (in press) have written an article which includes a summary and categorization of the various definitions of this discipline, which takes up many pages. It involves considering seriously "the mathematics which is practised among identifiable cultural groups "such as children of a certain age bracket, professional classes, people who are expert knitters, etc. (D'Ambrosio, 1985). It involves reconsidering what counts as mathematical knowledge; considering the interactions of culture and mathematical knowledge; and uncovering the hidden and distorted history of the development of mathematical knowledge. The author also seems unaware that there is an International Study Group on Ethnomathematics[2], an official special interest group of The National Council of Teachers of Mathematics, so my work in that area is not that of an isolated individual.

Further, the author is clearly unaware that the major part of my work is in "criticalmathematics" education, also part of an international group of mathematics educators who are trying to transform mathematics education. We formed this group[3] as a consequence of our first conference in October 1990 at Cornell University. As John Volmink, Arthur Powell, and I stated in our first Newsletter (1991)

> We feel it makes sense to constitute ourselves as a group now because the influences of our growing collective body of work and actions will be stronger if it emerges in the context of a defined group with a stated (although not static) philosophy. And we feel the need to raise a strong, alternative voice to the rising conservative influences in the educational arena. (p 1)

We want to complement various other progressive mathematics education groups, such as the *International Study Group on Ethnomathematics* and *The Humanistic Mathematics Network*,

---

1 In addition, my work is criticized in Illiberal Education, Dinesh D'Souza (1991, p.245) and in "Multiculturalism Mobilizes" by Glenn Ricketts (1990, p.57), which appeared in the journal of the right-wing National Association of Scholars.

2 For information, or to subscribe to the newsletter, contact Gloria Gilmer, 9155 North 70th St., Milwaukee, WI 53223.

3 For information, or to subscribe to The Criticalmathematics Educators Group newsletter, contact me at The College of Public and Community Service, University of Massachusetts, Boston, MA 02125.

whose concerns are largely with issues internal to mathematics. Our group adds an emphasis on the political and economic dimensions of mathematics and its education. We also decided to call our organization an educators' rather than education group both to focus on the people who construct the education, and to underline that the ways criticalmathematics educators attempt to achieve our goals—the educational institutions, the curricula, etc.—are more varied, less defined, and more open to debate than our underlying goals. We write "criticalmathematics" as one word in order to symbolize our hope that one day all mathematics education will be critical.

The purpose of my talk at the conference was to present our proposed definition of a criticalmathematics educator and to illustrate it with examples from my teaching practices. We expect that this definition will grow and change as we interact with more and more critical-mathematics educators around the world and as the material conditions of our lives change. The definition encompasses our roles as mathematicians, as teachers, and as concerned and active citizens. The curricular examples I present come from my work teaching criticalmathematics literacy at the College of Public and Community Service (University of Massachusetts/Boston). My students are mainly working class, urban adults in their 30's, 40's, 50's, and older, who have not been "tracked" for college; many of them were labelled as "failures" in secondary school; most have internalized negative self-images about their knowledge and ability in mathematics. Approximately 60% are women; 30% people of colour. Most work (or are looking for work) full-time, have families, and attend school full-time. Most work in various public and community service jobs; many have been involved organizing for social change. Students can work toward their degree using prior learning from work or community organizing, or new learning in classes, or new learning from community service (e.g. students lobbied the legislature and organized for welfare rights forming the Massachusetts Coalition for Basic Human Needs; students, asked by the community, worked with faculty to serve as consultants for the Roxbury Technical Assistance Project to help that community participate in planning its own development). The faculty are activists as well as intellectuals; approximately 50% are women, 30% people of colour; the Dean is African-American, Teachers have less institutional power over students than in most universities, because we don't give grades, and students can choose another faculty member to evaluate their work if they are dissatisfied with the first faculty evaluation. We cannot require attendance or any other work that is not clearly discussed in the competency statement which details the criteria and standards for demonstrating knowledge of the topic which students are studying.

The Definition[4]

As mathematicians, criticalmathematics educators view the discipline:

[4] Our definition is patterned after the "radicalteacher" definition, constructed by Pamela Annas, which appears on the back cover of every issue of Radical Teacher.

- as one way of understanding and learning about the world; they do not view mathematics as a static, neutral, and determined body of knowledge; instead, they view it as knowledge that is constructed by humans;

- as one vehicle to eradicate the alienating, eurocentric model of knowledge, widely taught in the schools, particularly this model's narrow view of what are considered mathematical ideas and who are capable of owning these ideas, and this model's historiography which excluded and distorted, marginalized and trivialized the contributions of women and men from all the world's cultures to what is then considered "academic" mathematics;

- as a human enterprise in which understandings result from actions; in which process and product, theory and practice, description and analysis, and practical and abstract knowledge are "seamlessly" (Lave, 1988) interconnected; and in which mathematics and other disciplines interact, as does knowledge with the contexts of social, economic, political and cultural perspectives.

One focus in my criticalmathematics literacy curriculum, relating to the view of mathematics on our definition, is an attempt to break down the dichotomy between "everyday" mathematics knowledge and "school" mathematics[5]. This split can result in students' feeling that they do not understand any mathematics—many of my students are even surprised to learn that the decimal point is the same as the point whose use they completely understand in writing amounts of money. We discuss Harris' (1987) work showing how this dichotomy intersects with sexism in considering what counts as mathematical knowledge. For example, an engineering problem about preventing the lagging in a right-angled cylindrical pipe from inappropriately bunching up and stretching out, is labelled "mathematics," whereas the identical domestic problem of designing the heel of a sock is called "knitting" and not considered to have mathematical content. We also discuss Joseph's (1987) work showing how this split, combined with racism, results in "academic" mathematics' narrow view of what is considered a "proof." Egyptian and Mesopotamian mathematics are dismissed as merely the "application of certain rules or procedures ... [not] 'proofs' of results which have universal application. Joseph (1987) disputes this definition arguing that:

> ...the word 'proof' has different meanings, depending on its context and the state of development of the subject ... To suggest that because existing documentary evidence does not exhibit the deductive axiomatic logical inference characteristic of much of modern mathematics, these cultures did not have a concept of proof, would be misleading. Generalizations about the area of a circle and the volume of a truncated pyramid are found in Egyptian mathematics... As Gillings (1972, pp. 145-6) has argued, Egyptian 'proofs are rigorous without being symbolic, so that typical values of a variable are used and generalization to any other value is immediate.' (pp. 23-24)

---

[5] The ideas in this section are developed in much greater detail and analysis in Frankenstein & Powell (in press).

Instead, these eurocentric and male-centric views of what counts as a mathematical idea are broadened to include all the different kinds of activities involving counting, ordering, sorting, measuring, inferring, classifying and modelling as they appear in all the world's cultures (Bishop, 1988). These ideas are considered in ways that uncover their connections to the development of human societies, "showing the necessity of any given piece of calculation, measure or pattern for the particular society of which it was a part" (Singh, no date, p. 21). We discuss speculations that, in ancient agricultural societies, the needs for recording numerical information that demarcated the timed to plant, gave rise to the development of calendars such as that found on the Ishango bone, recently dated at around 20,000 BC found at the fishing site of Lake Edwards in Zaire. (Zaslavsky, 1983, pp. 111-112 and 1991) And as African women, for the most part, were the first farmers, they were most probably the first people involved in the struggle to observe and understand nature, and therefore, to contribute to the development of mathematics. (Anderson, 1990, p. 354) Then, as societies evolved, the more complex mathematical calculations that were needed to keep track of trade and commerce gave rise to the development of place-value notation by Babylonians (circa 2000 BC), (Joseph, 1987, p. 27). And this continues to the present day when, for example, military needs and funding drive the development of "artificial intelligence." (Weizenbaum, 1985)

As teachers, criticalmathematics educators:

- listen well (as opposed to telling) and recognize and respect the intellectual activity of students, understanding that "the intellectual activity of those without power is always characterized as nonintellectual" (Freire and Macedo, 1987);

- maintain high expectations and demand a lot from their students insisting that students take their own intellectual work seriously, and that they participate actively as "co-interrogators" (Powell, 1990) in the learning process;

- are not merely "accidental presences" (Freire, 1982) in the classroom, but are active participants in the educational dialogue, participants capable of advancing the theoretical understanding of others as well as themselves, participants who can have a stronger understanding than their students (Youngman, 1986);

- assume that minds do not exist separately from bodies, and that the bodies or material conditions, in which the potential and will to learn reside, are female as well as male and in a range of colours; that thought develops through interactions in the world, and that people come from a variety of ethnic, cultural, and economic backgrounds; that people have made different life choices, based on personal situations and institutional constraints, and that people teach and learn from a corresponding number of perspectives;

- believe that "most cases of learning problems or low achievement in schools can be explained primarily on motivational grounds" (Ginsberg, 1986) and in relationship to social, economic, political, and cultural context, as opposed to in terms of a "lack of aptitude" or "cognitive deficit;

- recognize the reality of mathematics "anxiety", but deal with it in a way that does not blame the victims, and that recognizes both the personal psychological aspects and the broader societal causes;

- recognize that everyone has mathematical ideas; criticalmathematics educators "work hard to understand the logic of other peoples, of other ways of thinking." (Fasheh, 1988)

One focus in my criticalmathematics literacy curriculum, relating to the view of teachers in our definition, is an attempt to break down the dichotomy between teaching and learning.

Students become "co-interrogators" by participating in analyzing what others have said about mathematics learning at the same time that they are learning mathematics. (Frankenstein, 1991) Their own knowledge is extended and deepened by these reflections on the learning process itself. As Freire (1973) reasons

> Knowing, whatever its level, is not the act by which a subject transformed into an object docilely and passively accepts the contents others give or impose on him or her. Knowledge, on the contrary, necessitates the curious presence of subjects confronted with the world. It requires their transforming action on reality. It demands a constant searching. It implies invention and re-invention. It claims from each person a critical reflection on the very act of knowing. It must be a reflection which recognizes the knowing process, and in this recognition becomes aware of the "raison d'etre" behind the knowing and conditioning to which that process is subject. (P. 101)

For example, in my curriculum students discuss the following excerpt from "The Development of Addition in the Contexts of Culture, Social Class, and Race" (Ginsberg, 1982)

> ...in the vast majority of cases, children of different social classes and races demonstrated similar basic competence on the various [mathematical] tasks and used similar strategies for solving them... These results suggest that although culture clearly influences certain aspects of cognition (e.g., linguistic style), other cognitive systems develop in a uniform and robust fashion, despite variation in environment or culture. Children in different social classes, both black and white, develop similar cognitive abilities. The research suggests...that educators must take seriously the notion that upon entrance to school virtually all children possess many intellectual strengths on which education can build ... without the benefits of schooling, young children already understand basic notions of mathematics...If we fail to educate poor children, then it does not help to blame the victim by proposing poor children are cognitively deficient or genetically inferior. We need instead to consider things like motivational factors linked to expectations of limited economic opportunities, inadequate educational practices, and bias on the part of teachers..." (pp. 207-209)

Reflecting on this research that concludes all students come to school with the intellectual development to enable them to do "academic" mathematics, helps students develop confidence in their potential to learn mathematics. They remember instances when they were humiliated by teachers for asking questions, and realize the teacher was wrong; they become motivated to analyze what is blocking their math learning[6] and what situations will help them learn[7].

---

[6] An excerpt from a student journal indicates:

> I think I know part of the reason math is difficult for me. I seem to do math in a disorganized manner ... while doing my homework I became conscious of the amount of scrap paper I used.

As concerned and active citizens, criticalmathematics educators:

- have a relatively coherent set of commitments and assumptions from which they teach, including an awareness of the effects of, and interconnections among, racism, sexism, ageism, heterosexism, monopoly capitalism, imperialism and other alienating totalitarian institutional structures and attitudes;

- believe that good intentions are not enough to define a criticalmathematics educator; criticalmathematics educators are actively anti-racist, anti-sexist, anti-all the other alienating totalitarian institutional structures and attitudes, and they work with themselves, their mathematics classes and their colleagues to uncover, name, and change those conditions;

- view that a major objective of all education is to shatter the myths about how society is structured, to develop the commitment to rebuild alienating structures and attitudes, and the personal and collective empowerment needed to accomplish that task;

- are open to debate about which curricula and which investigations best achieve our goals; there are varieties of criticalmathematics educators; e.g. feminist criticalmathematics educators are not in every respect identical with socialist criticalmathematics educators;

- maintain that "dehumanization, although a concrete historical fact, is *not* a given destiny, but the result of an unjust order" (Freire, 1970);

- are militants in the Freirean sense of the term, committed to justice and liberation: "Militancy forces us to be more disciplined and to try harder to understand the reality that we, together with other militants, are trying to transform and re-create. We stand together alert against threats of all kinds" (Freire, 1978);

- understand that no definition is static or complete, all definitions are unfinished, since language grows and changes as the conditions of our social, economic, political, and cultural reality change;

- also have fun, laugh, and play...

> The figures were all over the paper and at times I couldn't figure out what part of the problem I was working on and what I had figured out before ... I'm going to have to find a method of keeping things organized.

[7] An excerpt from another student journal states:

> I can see now how important that is [to get everyone to participate in the class discussions ... for everyone to help each other] .... The fact that another student can answer the question that one presents is an encouragement ... to put forth a little more effort because if another student can understand enough to answer, then you can understand.

The main focus in my criticalmathematics literacy curriculum relates to the view in our definition of how our roles as concerned and active citizens connect to our role as educators. Criticalmathematics literacy involves the ability to present data to change people's perceptions of those issues and to challenge "taken-for-granted" assumptions about how our society is structured[8] Students learn the meaning of numbers and basic math operations as they are reflecting on real data about economic, political or social issues. The math problems are, within the constraints of covering the entire basic math curriculum, those that arise naturally from the situation as a way of more fully comprehending the issues. Many exercises involve open-ended problem creating, where students consider which ways of expressing the data are more powerful, which ways are deceptive. For example, when students are learning about the meaning of fractions, one problem they work on uses the following passage to answer these questions:

(a) What is Helen Keller's main point?

(b) How do the numbers she uses support her point?

(c) Why does she sometimes use fractions and at other times use whole numbers?

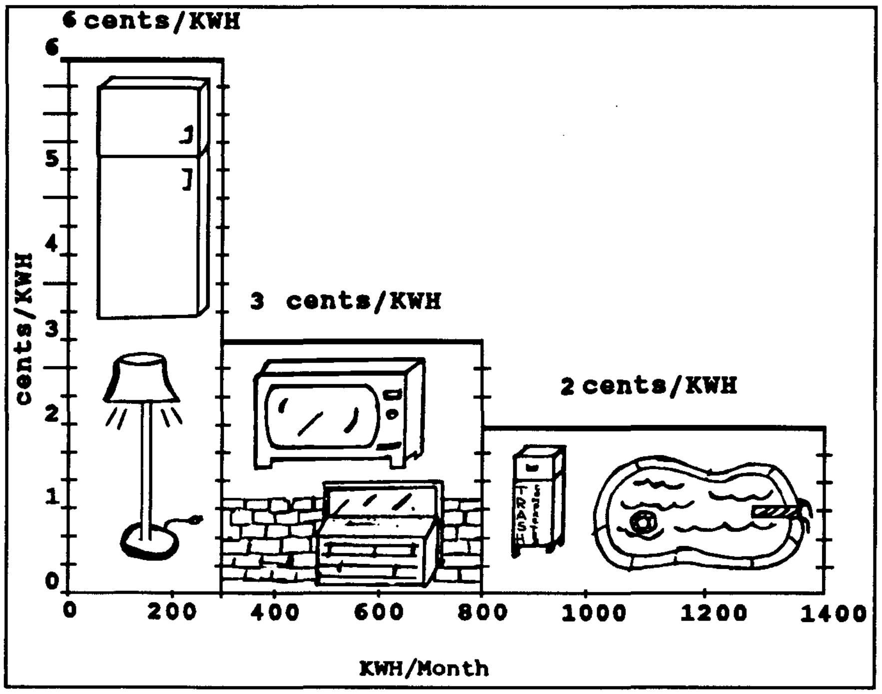
Although Helen Keller was blind and deaf, she fought with her spirit and her pen. When she became an active socialist, a newspaper wrote that "her mistakes spring out of the limits of her development". This newspaper had treated her as a hero before she was openly socialist.

In 1911, Helen Keller wrote to a suffragist in England: 'You ask for votes for women. What good can votes do when ten-elevenths of the land of Great Britain belongs to 200,000 people and only one-eleventh of the land belongs to the other 40,000,000 people? Have your men with their millions of votes freed themselves from this injustice?"' (Zinn, 1980, p. 337)

For another example, we use the following problem to practice multiplication. As we discuss the results, we usually wind up practising division, too, as we calculate the average per unit costs of electricity for various amounts of total usages.

The *Rate Watcher's Guide* (Morgan, 1980) details why low-income citizens who use electricity only for basic necessities pay the highest rates, and large users with luxuries like trash compactors, heated swimming pools or central air-conditioning pay the lowest rates. "A 1972 study conducted in Michigan, for example, found that residents of Detroit's inner city paid 66% more *per unit* of electricity than did wealthy residents of nearby Bloomfield Hills." Researchers concluded that "Approximately $10,000,000 every year leaves the city of Detroit to support the quantity discounts of suburban residents." To understand why this happens use the following graph which illustrates a typical "declining block rate" structure to compute the bill of a family which uses 700 kWh per month and the bill of a family which uses 1400 kWh per month.

[8] The theory and practice of this curriculum are discussed in depth in Frankenstein (1987) and in my text, *Relearning Mathematics* (1989)



The Major Question

Of course there are many questions about my curriculum and about our definition of critical-mathematics education[9]. For me, the major question involves the connections between critical-mathematics education and liberatory economic and political change. My students' evaluations indicate that the course succeeds in convincing them of the importance of mathematics in analyzing issues, and, on occasion, even changes their minds about some of those issues:

> The one big thing I learned: math is not relegated to the classroom but is truly connected with everyday life and the outside world. Learning math truly helps one deal with life better.
>
> I've learned to open my mind to new ideas; I've learned that there are several ways to the same end; I've learned to listen to "all" sides; I've learned to give it a try, even when I'm not sure; I've learned to look deeper at the numbers; I've learned to look behind those numbers.
>
> It made me become more critical and aware of what I read.

---

[9] Starting with our second CMEG Newsletter, we will include a "Definition Dialogue" section which will involve debates about the questions raised by our definition. We view this as an important part of theory building for our group.

I have learned to analyze certain situations and evaluate them with caution because not everything that we read is what it appears to be.

I've learned to question [numbers'] significance as they apply to our society ... I've learned about discrimination, the hidden and misconceptions of learning math and its history.

My experience about our math class has me thinking more about how the "little people" are cheated. I have changed my way of thinking.

But these comments are still a long way from changing society. In another context, concerning the changes in the literacy canon at U.S. universities, Carby, (1990) argues that although on many campuses black women have become subjects on the syllabus, the material conditions of most black people are still ignored. She asks: "Are the politics of difference effective in making visible women of colour while rendering invisible the politics of exploitation?" (p. 85) So if criticalmathematics education leads to more intellectual awareness of the alienating structures of our society, does that mean it will empower people to develop less alienating structures? Using the example of South Africa, Nteta (1988) feels that "revolutionary self-consciousness [is] an objective force within the process of liberation." He argues that the aim of Steve Biko's theories and the Black Consciousness Movement "to demystify power relations so that blacks would come to view their status as neither natural, inevitable, nor part of the eternal social order ...created conditions that have irreversibly transfigured South Africa's political landscape." (p. 61) Fasheh (1982), arguing from his experiences in Palestine, is also optimistic:

> ...teaching math through cultural relevance and personal experiences helps the learners know more about reality, culture, society and themselves. That will, in turn, help them become more aware, more critical, more appreciative, and more self-confident. it will help them build new perspectives and syntheses, and seek new alternatives, and, hopefully will help them transform some existing structures and relations. (p. 8)

References

Anderson, S. E. (1990). Worldmath Curriculum: Fighting Eurocentrism in Mathematics, *Journal of Negro Education*, Vol. 59, No. 3, 348-359.

Annas, P. Radicalteacher Definition, *Radical Teacher*, back cover, P. O. Box 102, Kendall Square, Cambridge, MA 02142.

Bishop, A. (1988). Mathematics Education in its Cultural Context, *Educational Studies in Mathematics*, Vol. 19, 179-191.

Carby, H. V. (1990). The Politics of Difference, *Ms. Magazine*, Sept./Oct., 84-85.

D'Ambrosio, U. (1985). Ethnomathematics and its Place in the History and Pedagogy of Mathematics, *For the Learning of Mathematics*, Vol. 5, No.1, 44-48.

D'Souza, D. (1991). *Illiberal Education*, New York, MacMillan.

Fasheh M. (1982). Mathematics, Culture and Authority, *For the Learning of Mathematics*, Vol. 3, No. 2, 2-8.

Fasheh, M. (1988). Mathematics education in the Global Village: What can we expect from Ethnomathematics? *Sixth International Congress on Mathematics Education*, panel discussion, Budapest, Hungary, July.

Frankenstein, M. (1987). Critical Mathematics Education: An Application of Paulo Freire's Epistemology 180-210, in Shor, I., ed. (1987). *Freire for the Classroom: A Sourcebook for Liberatory Teaching* Portsmouth, NH, Boynton/Cook.

Frankenstein, M. (1989). *Relearning Mathematics: A Different Third R--Radical Maths* London: Free Association Books

Frankenstein, M. (1991). Breaking Down the Dichotomy Between Learning and Teaching Mathematics, unpublished manuscript.

Frankenstein, M. & Powell, A. B. (in press). Towards AntiDomination mathematics: Paulo Freire's Epistemology and Ethnomathematics, in McLaren, P. & Leonard, P., eds. (in press), *Paulo Freire: A Critical Encounter*, London, Routledge & Kegan Paul.

Freire, P. (1970). *Pedagogy of the Oppressed*, New York, Seabury

Freire, P. (1973). *Education for Critical Consciousness*, New York, Seabury.

Freire, P. (1978). *Pedagogy in Process*, New York, Seabury.

Freire, P. (1982). Education for Critical Consciousness, Unpublished Boston College course notes taken by Frankenstein, M., July 5-15.

Freire, P. & Macedo, D. (1987) *Literacy; Reading the Word and the World*, South Hadley, MA, Bergen & Garvey.

Gillings, R. J. (1972) *Mathematics in the time of the Pharaohs*, Cambridge, MA, MIT Press (Dover reprint, 1982).

Ginsberg, H. (1982). The Development of Addition in the Context of Culture, Social Class and Race, 191-220, in Carpenter, T. P., Moser, J.M. & Romberg, T.A., eds (1982) *Addition and Subtraction: A Cognitive Perspective*, Hillsdale, NJ, Erlbaum.

Ginsberg, H. (1986). The Myth of the Deprived Child: New Thoughts on Poor Children, 169-189 in Neisser, V., ed.(1986) *The School Achievement of Minority Children: New Perspectives*, Hillsdale, NJ, Erlbaum.

Harris, M. (1987). An Example of Traditional Women's Work as a Mathematics Resource, *For the Learning of Mathematics*, Vol.7, No.3, 26-28

Joseph, G. G. (1987). *Foundations of Eurocentrism in Mathematics Race & Class*, Vol.28, No.3: 13-28.

Lave, J. (1988). *Cognition in Practice* Cambridge, England, Cambridge University Press.

Morgan, R. E. (1980). *The Rate Watcher's Guide*, Washington, DC, Environmental Action Foundation.

Nteta, C. (1988). Revolutionary Self-Consciousness as an objective force within the process of liberation: Biko and Gramsci, *Radical America,* 21(5), 55-61.

Powell, A. B. (1990). Remarks at *Critical Mathematics Education: Toward a Plan of Action for Cultural Power and Social Change* conference, Oct. 13-14, Cornell University.

Ricketts, G. (1990). Multiculturalism Mobilizes, *Academic Questions.*

Singh, E. (no date). Classroom Practice: Anti-Racist Mathematics, draft manuscript for Open University reader on anti-racist education.

Siegel, F. (1991). The Cult of Multiculturalism, *The New Republic*, Feb 18

Volmink, J., Powell, A.B. & Frankenstein, M. (1991). Criticalmathematics Educators Group Newsletter.

Weizenbaum, J. (1985). Computers in Uniform, A Good Fit? *Science for the People*, Vol. 17, Nos. 1 & 2, 26-29.

Youngman, F. (1986). *Adult Education and Socialist Pedagogy*, London, Croom Helm.

Zaslavsky, C. (1973). *Africa Counts* Boston, Prindle, Weber & Schmidt.

Zaslavsky, C. (1991) correspondence with the author.

Zinn, H. (1980). *A Peoples History of the United States* New York, Harper & Row.

Ad Hoc Group 1

Visions, Practice, Reflection and the Growth of Teachers: A Constructivist Approach to Mathematics Teacher Education

George Gadanidis

University of Western Ontario

Visions, Practice, Reflection, and the Growth of Teachers: A constructivist approach to mathematics teacher education

An earlier draft of this paper was presented at the Division of Curriculum Studies' SEMINAR SERIES, at the Faculty of Education, University of Western Ontario, 5 April, 1991. The author wishes to thank Doug Edge, Barry Onslow and Eric Wood for the careful reading of the earlier draft and for their constructive comments.

Synopsis

This paper outlines and discusses a project carried out with two mathematics methods classes of pre-service secondary school teachers at a faculty of education during the 1990-1991 academic year. The project had two components: (1) it involved pre-service teachers in defining their practice and their visions of mathematics education, and in using reflection as a means of bridging the gap between the two; and (2) it employed a student centred approach by taking advantage of, and building on, the personal experiences, beliefs, and understandings of pre-service teachers. These components constituted an attempt to incorporate into the methods course assignments and activities based on two of the assumptions that provide the foundation for NCTM's standards for the professional development of teachers of mathematics (NCTM, 1989a, p.62; 1991, p.124):

1. *The final success for any teacher is the integration of theory and practice.*

2. *Teachers are influenced by the teaching they see and experience.*

Overall, the following conclusions are drawn. First, the definition and reflective analysis of the practice and visions of pre-service teachers are appropriate and worthwhile activities. Although pre-service teachers are very concerned with the practicalities of "surviving" in the classroom, they can also see the "practicality" of developing an understanding of their practice, their visions, and their path for professional growth. Second, it is important that instructors model the skills and attitudes they encourage their students to learn and employ. This attempt at consistency between visions and practice presents a tremendous learning experience for the instructor involved and parallels the reflective path to professional growth that the pre-service teachers are asked to follow.

Background

A few years ago, while doing graduate work at the Ontario Institute for Studies in Education (OISE), I enrolled in a couple of courses in the department of curriculum. One of these courses was taught by Brent Kilbourn, who asked us to tape-record, transcribe and analyze one of our typical lessons. We also used situational analysis to determine the patterns in our teaching. I found these experiences useful as they provided concrete evidence of the weaknesses and strengths of my teaching and pointed out directions for improvement.

I also took a course from Michael Connelly, who asked us to write stories about ourselves and analyze them with the hope of uncovering themes and getting a better understanding of ourselves as teachers and as people. We also carried out peer interviews in groups of three, where one person was the interviewer, the second the interviewee, and the third the recorder. We also spent a lot of time analyzing John Dewey's Experience and Education as well as sections of a book that Connelly was working on (Connelly & Clandinin, 1988).

Both of these courses helped me realize that my growth as a teacher is: (a) a complex endeavour; (b) my own responsibility; and (c) dependent upon a clear understanding of my visions, my practice, and the gap between the two. When in 1990 I was seconded from my school board to a faculty of education I decided to make it one of my priorities to help my students come to these realizations as well. The question was: "how would I accomplish this?".

Introduction

We are apparently in the midst of a mathematics education revolution, but it's doubtful that many of our students have noticed. Change, it appears, is relatively easy to advocate through visions of what mathematics education should be like but difficult to implement in practice. The current vision, which is thoroughly outlined in the NCTM *Standards* documents, is based on a constructivist approach to the learning and teaching of mathematics. This vision sees students as active learners and teachers as facilitators, which implies that teachers create learning environments that encourage the active involvement of students in constructing and applying mathematical ideas. The current practice, however, is typified by instruction through teacher exposition and extended periods of individual seat-work practising routine tasks (NCTM, 1989, 1991).

The gap between what we believe we should be doing in our classrooms and what we actually do is quite large. Yet there is a glimmer of hope that we can reduce it. This hope is due to the increased realization of the teacher's necessary role as an agent of change, as evidenced by the *Professional Standards for Teaching Mathematics* published by the NCTM (1991). The teacher embodies practice and must be an active participant if the current visions of mathematics education are to be realized. Traditionally we have focused our attention on changing the subject-matter of mathematics, assuming that it will automatically be implemented at the classroom level. But how and what mathematics is taught is inexorably dependent on the teacher. This raises the issue of how to facilitate the growth of teachers so that they take ownership of the construction of their personal visions of mathematics education and of their implementation into practice.

This paper outlines and discusses a project carried out with two mathematics methods classes of pre-service secondary school teachers at a faculty of education during the 1990-1991 academic year. The project had two components: (1) it involved pre-service teachers in defining their practice and their visions of mathematics education, and in using reflection as a means of bridging the gap between the two; and (2) it employed a student centred approach by taking advantage of, and building on, the personal experiences, beliefs, and understandings of pre-service teachers. These components constituted an attempt to incorporate into the methods course assignments and activities based on two of the assumptions that provide the foundation for NCTM's standards for the professional development of teachers of mathematics (NCTM, 1989a, p.62; 1991, p.124):

1. *The final success for any teacher is the integration of theory and practice.*

2. *Teachers are influenced by the teaching they see and experience.*

Rationale

The final success for any teacher is the integration of theory and practice.
The integration of a teacher's practice and visions (or theory) is a goal that is difficult to achieve given the complex nature of mathematics education. However, it is a goal to strive for if we are serious about reform. A qualitative change in mathematics education cannot come about simply by legislating it. Rather, teachers must become involved by taking ownership of the construction of their own visions for reform and by implementing these visions into practice. This requires that teachers become what Schon (1983) calls "reflective practitioners", fostering a dialectic relationship between their visions and their practice.

Teachers are influenced by the teaching they see and experience.
"Teachers' own experiences have a profound impact on their knowledge of, beliefs about, and attitudes toward mathematics, students and teaching (...) these experiences convey messages about what constitutes appropriate teaching and learning" (NCTM, 1991, p.124). For this reason, the instructor felt he should model the constructivist approach to the learning and teaching of mathematics that was the pedagogical emphasis of the methods course. Consequently, the assignments and activities designed for the pre-service teachers who were part of this study had the aim of facilitating the construction of the pre-service teachers' own understanding of what mathematics education entails and what their personal paths for growth may be.

Method

Characteristics of pre-service teachers participating in this study.
The two classes involved in the study were in different programs of teacher education. Students in Class A were part of a co-operative education program in which they had experienced three four-month practice-teaching sessions. They were employed by the school boards in which they were placed and were not supervised by the instructor. Class B had one four-month practice-teaching session which was supervised by the instructor. Class A was relatively homogeneous in terms of age (about 23) and field of specialization (mathematics and computer science). Class B consisted of students who varied in age (between 23 and 45; average 30) and field of

specialization (mathematics with chemistry, physics, general science or physical education). About one-third of the students in Class B had recently changed careers. Class A had 24 students and Class B had 19 students. The instructor had met only four of the students of Class A prior to their coming to the faculty of education. Conversely, Class B was part of a two-week intensive orientation program prior to their practicum and each student was observed teaching twice.

The final success for any teacher is the integration of theory and practice.
Three sets of activities were incorporated into the course to meet the requirements of the first assumption. First, students defined the characteristics of their own practice (in their practicum) after examining lessons they transcribed, stories they wrote, and peer interviews they conducted. The transcript analysis involved tape-recording, transcribing and analyzing a typical lesson. The transcript was analyzed using the Aschner/Gallagher category system (Gallagher & Aschner, 1974), which divides the lesson into thought units which are then categorized according to the level of thinking that they represent. The five categories used are: routine operations, cognitive-memory operations, convergent thinking, divergent thinking, and evaluative thinking. The intent of the transcript analysis assignment was to give pre-service teachers concrete evidence of the general nature of their teaching and to focus their attention on the type of thinking that was encouraged in their classrooms. The transcript analysis assignment was mailed to students near the middle of their practicum. Story writing involved students in writing one story of their choice about their practicum experience and one story which dealt specifically with helping a student understand a mathematical concept. This gave them the opportunity to recreate and reflect upon situations they experienced in their practicum. Peer interviews were conducted in groups of three, with one person acting as the interviewer, the second as the interviewee, and the third as the recorder. The theme for the interview was chosen by the person interviewed, who also analyzed the transcript of the interview. Prior to conducting the interviews, the students shared with one another and discussed the stories they had written. This sharing was done to help them become more familiar with each other so that questions asked and answers given during the interview would be more meaningful to them. It was hoped that the spontaneity of the interview process would reveal new insights from the experiences of the students.

Second, students developed a personal vision of what exemplary teaching entails and outlined this vision in terms of their ideals of the four commonplaces of education (namely: student, teacher, subject-matter, and milieu (Connelly & Clandinin, 1988)) . To facilitate this, the Ontario Ministry of Education Guidelines for Mathematics Education, sections of the NCTM *Standards* documents, and the writings of a variety of prominent authors in education were discussed and analyzed in class, in terms of what we felt they had to say about these commonplaces. The commonplaces breakdown was used to give students a framework for judging the scope and nature of the visions of others, and to encourage them to create a balanced vision for themselves.

Last, students analyzed their visions, their practice, the gap between the two, and proposed future actions which would help to narrow this gap. This process served to connect the activities discussed above and to promote personal reflection and growth.

Figure 1 displays how the reflection activities (shown in italics) were incorporated into the course assignments and evaluation scheme. The transcripts of the lesson and the interview, and the stories written, were read and commented on by the instructor but were not graded. The instructor felt that marks should not be assigned for the low level task of transcribing a lesson and that the grading of personal experiences outlined in the stories and the interview would inhibit expression. The transcript and visions/practice analyses were graded based on the clarity and thoroughness of the discussion of issues raised. It was made clear by the instructor that opinions expressed by the students would not be graded. These activities constituted 20% of the final mark for the course and were assigned in addition to the standard method course assignments, such as lesson and teaching unit preparation, reports on journal articles, and mathematics problem sets. (Class B had one less assignment and one less hour of instruction per week because of the two week orientation.)

Figure 1: Course assignments/evaluation. Bracketed assignments/evaluation apply only to Class A.

creative lesson plan	10%	
problem sets	10%	
journal article report	20%	[15%]
teaching unit	30%	[25%]
[manipulatives lesson plan	10%]	
participation	10%	
lesson transcript	—	
transcript analysis	*10%*	
story writing	—	
peer interview	—	
visions/practice analysis	*10%*	

Teachers are influenced by the teaching they see and experience.
The activities discussed above also help meet the requirements of the second assumption, as they engage students in constructing their own understanding of mathematics education. A constructivist approach was also modelled in the day-to-day class activities designed by the instructor. This was done (a) by focusing the class activities and discussions on the experiences and understandings of the pre-service teachers, (b) by using a variety of cooperative learning, student centred classroom formats, and (c) through the instructor's encouragement of the expression of ideas and avoidance of value judgements.

Results

Once the activities related to this study were completed, graded and/or commented on, and returned, the students were asked to evaluate the approaches used. First, they rated each of the transcript analysis, story writing, peer interview and visions/practice analysis activities based on the degree to which it helped them construct an understanding of themselves as teachers.

Students were also asked to comment on the ratings they assigned. Such feedback would be useful in determining the effectiveness of these activities as perceived by the students and in adapting and improving them in other studies of this nature. Second, students were asked to rate the degree to which the teaching methodology used in the course constituted a constructivist approach. This was an attempt to determine the degree of consistency between the instructor's own visions and practice. Third, they evaluated the nature of the balance between practice and theory in the course. As most beginning teachers are often at a "survival concerns" stage (Fuller, 1969), it would be interesting to see how they perceived a course that included the activities described in the method section, which are usually associated with higher stages of teacher growth. Twenty-three students from Class A and seventeen students from Class B responded to the questionnaire. Figures 2 to 11, shown below, summarize the results (A = one student from Class A; B = one student from Class B; a = mean rating from Class A; b = mean rating from Class B).

Figures 2 to 5 show that Class A consistently assigned lower ratings than Class B. Two factors may have influenced the perspectives of the two classes towards these four assignments. First, the students of Class A had more practicum experience than those of Class B (twelve versus four months). Second, the ages and backgrounds of the students in the two classes were different (see method section). However, the effects of these factors are not clear from this study alone. A factor which influenced the disparity in the rating of the transcript analysis assignment is that the students of Class A were notified of this assignment during their practicum by mail from an instructor whom twenty of the twenty-four students had not met. During a class discussion, several of these students commented that this gave them a poor attitude towards the transcript analysis assignment. The limited communication with the instructor also led to a lack of clarity about the nature of the assignment, resulting in some confusion and frustration. In contrast, the students of Class B had established a strong working relationship with the instructor by the time this assignment was mailed.

The visions/practice analysis assignment was rated the highest of the four. Some students commented in a class discussion that this was the best assignment of the course, tying together most aspects of the course and helping them understand the intricacies of teaching and crucial aspects of their role. This is supported by the nature of the comments made by students in the questionnaire (see Figure 9). The next highest rating was given to the transcript analysis assignment. Figure 6 shows that students found the feedback resulting from this assignment very useful. Many students stated that they were surprised by the minimal evidence of higher order thinking in their mathematics classrooms. They also stated that they gained insights into their strengths and weaknesses, questioning techniques, speech patterns, and classroom interactions and atmosphere. It also became evident in class discussions that students used different criteria for categorizing thought units, as they made different interpretations of similar thought units. This is not surprising given the limited time spent on the difficult task of learning the subtleties of categorization. However, the category averages for the two classes were in the end very similar (approximately 31%, 30%, 31%, 4%, and 4% for the respective categories of the Aschner/Gallagher system). About one-quarter of the students commented that this assignment was time consuming, which is an inherent characteristic of transcript analysis. About 20% of the students of class A commented that this assignment did not result in any new insights. This may be partly due to the longer practicum experience of these students and partly due to the nature of their introduction to this assignment, as discussed above. The story writing and peer interview assignments were rated the lowest. Some students commented that they would have preferred to have written the stories during the practicum, while the experiences they would be writing about were still vivid in their memories. Concerning the peer interviews, some students commented that the transcripts contained misquotes and that the success of the interview varied with the quality of the interviewer. Some students may have felt that these two activities were not as important as the others since no marks were allocated for them. This may have also been a factor in the case of the lesson transcript, which was very time consuming to prepare but was not graded.

Figure 2: Transcript analysis rating.

```
                A           A           B
                A           A           B
                A           A           B
                A           A           B
                A           A           AB            B
                A           A           AB           AB
                AB          A           AB      B    AB
A               AB          AB          AB      B    AB
-----------------------------a-----------b---------------
0               1           2           3             4
no                                            extremely
help                                            helpful
```

Figure 3: Story writing rating.

```
                A
                A
                AB
A               AB          AB          AB
A               AB          AB          AB
A               AB          AB          AB
A               AB          AB          AB
A               AB          AB          AB            B
-------------------------a-----b-------------------------
0               1           2           3             4
no                                            extremely
help                                            helpful
```

Figure 4: Peer interview rating.

```
                        A           A
                        A           AB
                        A           AB
A           B           A           AB
A           AB          AB          AB
AB          AB          AB          AB
AB          AB    AB    AB    A     AB    B
-----------------------ab------------------------
0           1           2           3           4
no                                      extremely
help                                      helpful
```

Figure 5: Visions/practice analysis rating.

```
                                    A
                                    A
                                    AB
                                    AB
                                    AB          B
            A           A           AB          B
            A           AB          AB         AB
            A           AB          AB    A    AB
            AB          AB    A     AB    AB   AB
------------------------------a--------b---------
0           1           2           3           4
no                                      extremely
help                                      helpful
```

Figure 6: Transcript analysis comments.

```
no new insights . . . . . . . . . . . . . . . . A A A A A

time consuming  . . . . . . . . . . . . . . . . A A A A A
                                                B B B B B
good feedback . . . . . . . . . . . . . . . . . A A A A A A A A A A A A A A
                                                B B B B B B B B B B B B B B B B
```

Figure 7: Story writing comments.

```
no new insights . . . . . . . . . . . . . . . . A A A A A A A
                                                B B
fun . . . . . . . . . . . . . . . . . . . . . . A
                                                B B B B
helped to reflect . . . . . . . . . . . . . . . A A A A A A
                                                B B B B B B
```

Figure 8: Peer interview comments.

```
no new insights . . . . . . . . . . . . . . . A A A
                                              B B B B
too brief . . . . . . . . . . . . . . . . . . A
                                              B B B
helped to reflect . . . . . . . . . . . . . . A A A A A A
                                              B B B
sharing of ideas  . . . . . . . . . . . . . . A
                                              B B B B B
```

Figure 9: Visions/practice analysis comments.

```
no new insights . . . . . . . . . . . . . . .
                                              B B
insights into improvement . . . . . . . . . . A A A A A A A A A
                                              B B B B B B B
insights into beliefs . . . . . . . . . . . . A A A A A A A A A A A A A
                                              B B B B B B B
```

The consistency of the teaching methodologies used in the methods course with a constructivist perspective was rated quite high by the students, as shown in Figure 10. This was also reflected in their comments, some of which are reproduced below.

Figure 10: Degree to which students were encouraged to construct own understanding.

```
                                                   A
                                                   A
                                                   A
                                                   AB
                                     A    B        AB
                                     A    B        AB
                                     A    B        AB
                                     A    B        AB
                                     A    B        AB
                       A     A       AB   B        AB
                       A     A       AB   AB       AB
-----------------------------------------a----b-------
0           1          2             3             4
never                                         always
```

Focusing the class activities and discussions on the experiences and understandings of the pre-service teachers.

"(The instructor) always turned to the class to develop a discussion (i.e., from our experiences)."
"We were required to learn from one another and from our practicum experiences."
"(The instructor) waited until our discussion was complete before (he) gave (his) opinion/info so as not to bias our discussion - good idea!"
"(The instructor) rarely gave us the answers but expected us to hash it out on our own."
"When comments were made, or questions asked (the instructor) would not respond himself; this left us having to answer and respond to each other's questions."

Using a variety of cooperative learning, student centred classroom formats.

"Rarely did the class take the form of a lecture."
"Group work was the norm."
"Facilitator is the key word; (the instructor) encouraged us to think."
"Allowed for free and independent flow of ideas."

Encouragement of the expression of ideas and avoidance of the value judgements.

"(The instructor was) not judgemental of our views."
"(The instructor) encouraged us to be independent."
"(The instructor) offered (his) ideas but never claimed that they were better than ours."
"I think (the instructor was) always aware of when (his) preferences showed through and (he) seemed to take steps to encourage us to think for ourselves."

Not enough teacher centred activities.

"I think I would have liked to have a little more knowledge dispensed."

Figure 11 shows that most students felt that there was a fairly good balance between theory and practice in the methods course. This is an indication that students were able to accept the "practicality" of activities which are usually associated with higher stages of teacher growth, even though they may have been in a "survival concerns" stage.

Figure 11: Nature of balance in course content.

```
                                        A
                                        AB
                                        AB
                                        AB
                                        AB
                                        AB
                                        AB
                                        AB        A
                                        AB        A
                                        AB        A
                                        AB        AB
                                        AB   B    AB        A
                              A         AB   B    AB        A         A
------------------------------------------b----a----------------------------------
0         1         2         3         4         3         2         1         0
too                                    good                                   too
much emphasis                          balance                      much emphasis
on practice                                                             on theory
```

Conclusion

Based on the results of this study, a number of suggestions can be made for improving the activities incorporated into the methods course. First, transcription is very time consuming and should be streamlined, perhaps by reducing the length of the lesson section that is transcribed. Alternatively, a lesson can be audio or video recorded, but not transcribed, and analyzed holistically. Second, story writing is best done during the practicum, while the experiences written about are vividly remembered. Third, the success of peer interviews is dependent on the quality of the interviewer. It would be beneficial to discuss interviewing skills prior to this activity. This could be done in the context of skills needed for interviewing students in order to determine their understanding of mathematics concepts or the causes of unacceptable classroom behaviours.

Overall, the following conclusions can be drawn.

- The definition and reflective analysis of the practice and visions of pre-service teachers are appropriate and worthwhile activities. Although pre-service teachers are very concerned with the practicalities of "surviving" in the classroom, they can also see the "practicality" of developing an understanding of their practice, their visions, and their path for professional growth.

- It is important that instructors model the skills and attitudes they encourage their students to learn and employ. This attempt at consistency between visions and practice presents a tremendous learning experience for the instructor involved and parallels the reflective path to professional growth that the pre-service teachers are asked to follow.

References

Arends, R. I. (1988). *Learning to Teach.* New York, NY: Random House.

Cobb, P., Wood, T., & Yackel, E. (1990). Classrooms as Learning Environments. In B. D. Davis, C. A. Maher, & N. Noddings (eds.), *Constructivist Views on the Teaching and Learning of Mathematics* (Journal for Research in Mathematics Education, Monograph Number 4, pp. 125-146). Reston, VA: NCTM.

Connelly, E. M. & Clandinin, D. J. (1988). *Teachers as Curriculum Planners: Narratives of Experience.* Toronto, ON: OISE Press.

Dewey, J. (1938). *Experience and Education.* New York, NY: Collier Books.

Fuller, F. (1969). Concerns of teachers: A developmental conceptualization. *American Educational Research Journal*, 6, 207-226.

Gallagher, J. J. & Aschner, M. J. (1974). A preliminary report on analyses of classroom interaction. In R. T. Hyman (ed.), *Teaching: Vantage Points for Study* (pp. 181-197). New York, N.Y.: J. B. Lippincott Company.

Lovitt, C. et al (1990). Mathematics Teachers Reconceptualizing their Roles. In T.J Cooney & C.R. Hirsch (eds.), *Teaching and Learning Mathematics in the 1990* (pp. 229-236). Reston, VA: NCTM.

Maher, C. A. & Alston, A. (1990). Teacher Development in Mathematics in a Constructivist Framework. In B. D. Davis, C. A. Maher, & N. Noddings (eds.), *Constructivist Views on the Teaching and Learning of Mathematics* (Journal for Research in Mathematics Education, Monograph Number 4, pp. 147-166). Reston, VA: NCTM.

National Council of Teachers of Mathematics (1989). *Curriculum and Evaluation Standards for School Mathematics.* Reston, VA: Author.

National Council of Teachers of Mathematics (1989a). *Professional Standards for Teaching Mathematics: Working Draft.* Reston, VA: Author.

National Council of Teachers of Mathematics (1991). *Professional Standards for Teaching Mathematics.* Reston, VA: Author.

National Research Council (1989). *Everybody Counts: A report to the Nation on the Future of Mathematics Education.* Washington, D.C.: National Academy Press.

National Research Council (1990). *Reshaping School Mathematics: A Philosophy and Framework for Curriculum.* Washington, D.C.: National Academy Press.

Schon, D. A. (1983). *The Reflective Practitioner.* New York, N.Y.: Basic Books.

Steffe, L. P. (1990). On the Knowledge of Mathematics Teachers. In B. D. Davis, C. A. Maher, & N. Noddings (eds.), *Constructivist Views on the Teaching and Learning of Mathematics* (Journal for Research in Mathematics Education, Monograph Number 4, pp. 167-184). Reston, VA: NCTM.

Appendix A

Our Theories, Ourselves

Linda Brandau

University of Calgary

Our Theories, Ourselves[1]

The title of my talk implies a wholeness, an inseparability between us as mathematics education researchers, our theories, who we are (both personally and professionally), what we believe, how we behave. "Our theories, ourselves" implies an integrated way of thinking, a view of knowledge for which thought is not separate from feelings, mind is not separate from body, the rational is not separate from what we usually label as intuitive. This view of knowledge differs from the one pervasive in our Western culture, one rooted in a philosophic and scientific tradition based on the work of Plato, Francis Bacon, and Descartes, among others. This Western tradition values qualities we label "thought", "mind", "rational knowledge" and devalues those labelled "feelings", "body", "intuitive knowledge". This language, the one that creates these labels, also creates dichotomies. It is a language that disembodies, that fragments, that separates.

In this separation, we have alienated ourselves from our experience. So-called scientific thought has taught us not to trust our senses or our perceptions, has taught us that the only kind of knowledge that is valid exists outside ourselves. In mathematics classrooms, we are creating students who do not trust their own ways of thinking because they are told they must learn procedures in one way—the teacher's or the textbook's, in essence, some external "expert." If that one acceptable way does not happen to coincide with students' ways, their thinking is invalidated. Students then resort to memorization of rules, to viewing mathematics as fragmented bits and pieces of information. Or perhaps they view mathematics as a whole but themselves as fragmented?

In master's and doctoral programmes, we are creating graduate students who do not trust what they know, who spend their time researching what is intuitively obvious because they have been told that their knowledge is valid only if externally objectified. Graduate students resort to using only the rules they have been taught, to categorizing rich descriptive data, to viewing research as fragmented bits and pieces.

Fragmentation is the theme of this talk along with positing an integrated way of being in mathematics education research. This integration does not discard the mind, thought, or rational thinking; rather it meshes them with other images, ones expressed in some of the ecological literature, ones concerned with reclaiming parts of ourselves that have been lost. As Susan Griffin states,

> We... have inherited a habit of mind. We are divided against ourselves. We no longer feel ourselves to be a part of this earth... we even learn to disown a part of our own being. We come to believe that we do not know what we know. We grow used to ignoring the evidence of our own experience, what we hear or see, what we feel in our own bodies we have learned well to pretend that what is true is not true... The waters we once swam are forbidden to us now because they are poisoned... But we do not read these perceptions as signs of our own peril... We deny all evidence at hand that this civilization, which has shaped our minds, is also destroying the earth. (1989, p.7)

---

[1] This paper was first delivered at the plenary address given at TME (Theories of Mathematics Education) conference, Mexico, July, 1990.

The ecological movement recognises a need to alter drastically how we think, how we behave. Much of the writing is aimed at awareness, at taking people by the shoulders and shaking them, as in the following by Joanna Macy:

> I believe that we are summoned now, to awaken from a spell. The spell we must shake off is a case of mistaken identity, a millennia-long amnesia as to who we really are. We have imagined that we are separate and competitive beings...
>
> For our own sakes and the sake of all beings, we are called to rediscover our true nature... We are called to break out of the prison we've made for ourselves, turn the key of that isolation cell, and walk out into the world as into our own heart, coming home to the full reach of our being, home to our power, home to our ecological self. (1989, p. 201)

We also need to awaken, need to question what we have been trained to believe is research and theorizing. The current crisis in mathematics education, in North America, is reminiscent of the late 1950's. The Russians put Sputnik into space and the United States rallied to create the new math and science curricula in an attempt to have more and better scientists and engineers. Our current crisis seems to have been precipitated by a steady decline in achievement test scores, in the fear generated by the economic success of the Japanese. The reaction so far has been a series of reports by different groups such as the National Research Council, the National Science Foundation research conferences, the NCTM standards document. While these efforts at responding to the crisis are laudable, as are the changes being implemented in school systems, they need to be viewed not as "the solution", but as the beginnings of a long, hard look into our profession, the beginnings of a journey inward.

A journey of self-discovery, of intense reflection and introspection is the theme of Alice Koller's book *The Stations of Solitude*. Although she is referring to the personal, her ideas can also refer to mathematics education's journey of self-discovery.

> Coming to understand who you are follows no easily delineated procedure. No one sudden illumination comes from you-know-not-where, after which everything falls immediately and perfectly into place. There are, instead, fleeting shafts of light, and it is you who prepares the groundwork for their appearance. They let you go forward a little way, but soon they disintegrate, and you fall back into the dark a greater distance than you had gained. But then you lure out another light, and follow it until it too slips away. So, winning and losing in shreds and chunks, rather than in a single unbruised whole, you find that the work is done. (1990, p.5)

Coming to understand who you are, the journey inward to do so, is the way in which I will explore the theme of fragmentation—how it is manifested in our personal and professional lives, in the way we view knowledge, in how we do research and create theories. I will use an autobiographical perspective and bring the insights gained there to mathematics education.

Such inward explorations have, of course, been discussed by philosophers such as Sartre and Nietzsche and have often been used in literature—Homer's *Odyssey*, Conrad's *Heart of Darkness*, for example. The theme, however, has always related to the famous dictum from Socrates: "The unexamined life is not worth living." At the core of this statement is the belief that the purpose of life is transformation, that this occurs through a process: obtaining self-knowledge, gaining awareness of one's beliefs, values, behaviour patterns, consciously trying new ones; believing that transformation has occurred, discovering old patterns resurfacing, perhaps in different ways, different forms, recognizing that the "old" does not simply disintegrate, but gets integrated into the new.

This process is one that recognizes the inseparability of our personal and professional selves, our present selves from our life history, from other people in our lives, from the world. It demands an awareness of how our theories are meshed with all we think and do, with all we have thought and done.

Journeys inward are often precipitated by crisis, and mine has been no different. In August 1989, I learned that I had breast cancer, needed surgery, and six months of chemotherapy. Needless to say, when one is faced with the possibility of death, serious examination of all aspects of life are undertaken. What has become especially heightened is my sense of time—how I spend it, what I do with it and with whom. I have become especially critical of my professional work. If something does not seem especially "important", it gets shoved aside. I have probed deeply to try to understand what is important to me, and even though that may change daily, I continually question what I am doing and why I am doing it. But my inward journey really began in 1983, another period of crisis, when I was in the midst of midst of my dissertation work, deciding if my marriage was ending, questioning who I was and what my life was going to be. I physically removed myself from my environment by going to Cape Cod in Massachusetts for six weeks. I walked the beaches, read many books, shed many tears.

It was then that I became familiar with Alice Koller's other book *An Unknown Woman*, a work about her journey of self-discovery, one which she undertook by going to Nantucket for several months, a time of solitude. A friend told me about this book because she recognized that I was in great personal crisis, because she knew that I was also headed for physical and emotional solitude, because she knew that books were an entry into my psyche.

I think about what I read, what reading has meant and means to me. I see now, with the usual clarity of 20/20 hindsight, that I used books as an escape, from the World, from my parents, from myself. I always read fiction as a child—Mary Poppins, Dr. Doolittle and Nancy Drew mysteries. I could escape into those books, fantasize, tune the world out, become someone else so I didn't have to think about who I was.

Until perhaps ten years ago, my bookshelves were filled with fiction, with escapism. Today I read more and more non-fiction, books to inspire me, my research, my writing. Is this still an escape? A different one, but changed in form not substance? I don't know. Hopefully books now are an entry into myself, to my journey inward, to understanding my life, who I am. Books are so much a part of me that I couldn't imagine I life without them, I like food, water, the air we breathe. My son often comments, "don't you have enough books already?" I can't understand this view. To me that is like saying, "you ate dinner yesterday, why do you have to eat dinner again today?" Books nourish me, my soul, my learning.

The books I read are important not only because they are part of my intellectual life, hence part of myself, but what I have assimilated from them—the words, concepts, form, style, rhythm—all seep into my professional writing, into my theories.

Milan Kundera's work, especially *The unbearable lightness of being*, has had a profound effect on my thinking and writing. When I first read the book I was impressed not only with its scope, its ideas, its beautiful use of language, but with the form Kundera uses, one which interweaves fictional narrative with philosophical, psychological, and socio-historical discourse. He allows us to see how our individual lives are mirrored in the culture and how the culture is mirrored in our individual lives. Tomas, the main character, exemplifies the fragmentation of cultural and political life in Czechoslovakia in the late '60's. He is a man afraid of commitment, one who has several lovers at the same time, one who continually philosophizes about his state of being, both personally and within a broader existential level, but one who rarely takes action.

What has been so influential about Kundera's writing, both in form and substance, is that it encapsulates what I have been trying to do theoretically. It rejoins, reembodies what I see as having been fragmented—the cognitive and the emotional, the intuitive and the logical, the individual with his or her profession and with the socio-political context.

I enter my past now. It is 1965, my first year of teaching. I am standing in front of the classroom, the blackboard behind me, eighth grade mathematics students in front of me. I am clutching the textbook for dear life. If I let it go, I will lose myself, my security, my authority. I am the textbook. What is contained in it is not only the mathematics I teach, it is what constitutes mathematics for me. It is who I am as a teacher. I talk about the ideas in the book, have students do the problems in it, then use it to check their answer, The book is my curriculum. it keeps me distanced from the students, from learning about them as people.
Students are even seated at a safe distance from me, at desks in traditional rows; separated enough so that no talking will occur. Some of them are bored, yawning, fiddling with their pencils but at least they are not making any noise, or causing trouble. Some students are talking. Some are making spit wads. One flies by me. I yell, "stop it". The words float into the vastness, the sea of faces. I don't know exactly which student threw the spit wad, so my anger is vented at all of them.

My stomach begins to churn. How do I get them to be quiet? to pay attention? I yell some more. Gary, one of the students, stands up and tells his classmates to be quiet, to "give Mrs. Brandau a break". They are quiet for a few minutes but the noise level rises again. The cycle continues over and over and over, day after day after day.

I exist solely from day to day. I live for weekends, for a time when I don't have to face students, when there will be a temporary cease fire in the war between us. I live for vacations, holidays, school assemblies, for any time when I don't have to teach, for when I don't have to fight in this war. This is how the school year progresses.

Then I get called into the principal's office one day in the spring of 1966. I am told that I am not and never will be a good teacher and that I should resign. I am stunned, momentarily shocked, yet underneath relieved. Now the war is oyer, at least the one waged in my classroom.

As I write now, recalling this experience with 25 years of hindsight, I think about who I was in 1965—young, just graduated with my bachelor's degree, newly married, new career, inexperienced, naive, sheltered, fearful, uncommunicative, fragmented, a solid wall built around me so that no one, no pain could permeate. Or so I thought. Life must have been painful, I now realize. But I denied the pain then, buried all feelings so deeply that they, and I, could not be touched. I shared none of my feelings with anyone, not even myself. The word numb is probably most appropriate to describe the being I was then. I lived in a numbness that allowed me to be perfectly functional—to get up in the morning, teach, cook, shop, do laundry, be with friends, and so on. I did not live a life of conscious awareness however. I had no self-knowledge and no desire to have any. Perhaps I was too afraid to uncover and discover who I truly was.

My teaching mirrored who I was then. In my fear, my uncomfortableness with who I was, I slipped into a power struggle with the students. It was easy to do. The students were accustomed to being at war with teachers. And I was trained to participate in such a war, a metaphor for the kind of information provided in my education courses, ones replete with clichés like: Don't smile until Christmas. Be tough at first, you can always loosen up. Don't let the students get the upper hand. There was nothing, either implicit or explicit, that called forth images of honesty or caring. This was a language of power and control, a language that distanced students from teachers.

My educational training (and I use that word deliberately) also focussed on a separation of and fragmentation of mathematics from the people I was to teach. I was trained in mathematics, the education courses I took were not my primary focus and served only to fulfil the necessary requirements for a secondary teaching certificate. Even my memories of methods courses are filled with ideas for teaching the mathematics content not with concern for the students who were to learn these ideas.

Fragmentation marred my personal life as well. The battle for power between myself and my students bled into my private life. (Or perhaps my private battle bled into the public one?) I can now see that I was fighting for independence, a fight that was carried into my marriage. In my struggle, the all I had built around me thickened. The more my husband tried to converse with me, the more I withdrew, the thicker the wall became. Conversation for me then was viewed as a means to winning the war, for someone else to overtake me, to take me over, to have control and power over me and my being.

It is in the early 1970's when I begin work on a master's degree at Illinois State University because, for no other reason, it is in the town where I live. One of my first courses deals with Piaget's theories. I feel overwhelmed, not only by the ideas, but by the terminology used by Piaget, by the professor, and the scholarly articles he has us read.

The scholarly world baffles me. It is an empirical world, one full of statistics, correlations, percents, tables, charts, categories. The reading is so different from what I am used to doing that I come to believe that this is what it means to do and write about research. I take my methodology course, begin to read what is being done in mathematics education, begin to learn the "rules" for being a researcher, a theorist. There are acceptable ways to write—always in the third person, always in a language that seems inaccessible to anyone else but other scholars in your field, always referenced with the work of more respected scholars. There are acceptable ways to think—always logically and rationally. There are acceptable books to read and to reference—always ones by noted authorities, experts in the field, people who have proven themselves, by the quantity of their writings, by the number of research grants (especially multi-million-dollared) they have obtained. There are acceptable forms, not only of doing the research but of analyzing data and reporting it—always a literature review, delineation of methodology, its validity (both internal and external), data analysis, and so on.

I do not critically question any of the research I am reading or the rules for doing it and writing about it, however. Perhaps I feel too much of a novice to express opinions. Or perhaps it is because I do not question much in life anyway.

My research and theories mirror who I am. My master's thesis (done in 1976-77) is an interview study of several junior high school students. I presented them with some problems to solve, ones that were divergent in nature, and asked them to think aloud while solving them. From the transcripts of the audiotapes, I created categories of behaviours, then analyzed the data using a statistical package that did a cluster analysis, created a visual representation.

So although I did a study that involved looking at divergent thinking processes, I did not use divergent methods, either in doing the research or in writing about it. I was interested in creativity but was not being creative myself. I did learn, however, that I was not interested in doing any more quantitative studies. I was not learning enough about how students thought, about the process of how they become memorizers of mathematical procedures.

With encouragement from John Dossey, my master's advisor, I entered the doctoral program at the University of Illinois in Champaign-Urbana. I began speaking with Ken Travers but he immediately directed me to Jack Easley. Thus began a process of transformation—in my thinking, my research, my life.

I share part of my life history not only to evoke resonances in you, the listener, but because our individual lives (both personal and professional) mirror the educational culture and the culture mirrors the individuals comprising it. The patterns we uncover about ourselves can similarly be found in educational systems.

By investigating my own fragmentation, both personally and professionally, a pattern emerges. In general, it is one involving fear, insecurity and a distancing from everyone and everything. These are linked to a clinging to external authority (in terms of textbooks, curriculum, experts in the field). At the core of this general pattern is a rational, logical perspective of knowledge, one that uses a language of power and control.

Specifically, fragmentation is manifested as a lack of awareness of the connectedness between thoughts, feelings, behaviours (both personally and professionally), language. Fragmentation is manifested as a separation between the subject of mathematics, the curriculum used to teach and learn it; the students who learn. This separation involves language, values, and teaching style. When I yelled at students, I both created and sustained the distance between us. Basing my curriculum solely on the textbook shows how I valued external authority, keeping me and the students separate from each other and from mathematics.

Fragmentation is manifested in my research, in the attempt to take conversational data (transcripts with students), reduce it, splinter it into categories. A reflection of my values and what I consider knowledge is shown by the language used, one of distancing—from people, from research that requires writing in the third person, that uses categories to distance the researcher from the researched.

My work with Jack Easley and others at the University of Illinois had profound effects on my thinking, on my view of knowledge. It took at least a couple of years of being immersed in qualitative data (videotapes of Jack working with young children) for me to make the Gestalt switch necessary to stop thinking quantitatively. I don't recall how, or when it happened. I do remember Jack telling me, every time I thought I had "made the switch", to go back and work some more.

The kind of research and theorizing I learned to do in my doctoral programme prepared me well. I learned how to question critically, how to open my mind to new ways of thinking about research, about questions of objectivity/subjectivity, validity and so on. I learned that research does not consist of learning rules, procedures, and methods, but of asking important questions, ones that disturb us, ones that are rooted in the soul.

In 1984, my first job after finishing my Ph.D., I teach at a small college in New Hampshire, struggle with crises on all fronts—personal and professional. On one hand I have struck out on my own, become independent, taken charge of my life. On the other, I have left my son, essentially given my ex-husband custody, gone through a divorce, moved at least 1000 miles away from friends and family. Putting this amount of physical distance between myself and what I view as my past seems necessary. Yet the trauma created by doing so becomes overwhelming.

I have recurring nightmares, wondering if I have made the right decision about my son Mark. I sink into a deep depression, some mornings I am unable to got out of bed. Work is difficult at best. I am team teaching with two people who are on different philosophical planets from me. There is no understanding of me, nor I of them. We work together everyday but do not listen to one another. We converse only with the intent of determining who has power over the other.

Friends tell me to enter psychotherapy. With Carol, my therapist, I begin to discover and uncover who I am. We delve into my relationships with family, husband, friends, colleagues. The impenetrable wall surrounding me begins to crumble a bit, just a bit.

I find myself transferring my work with Carol to my teaching at the college. Conscious awareness of my behaviour begins, as well as alterations in that behaviour. My teaching becomes an experiment in communication between me and my students. I am listening to them so that I can hear not only the words they speak but the messages, often emotional, underlying the words.

No longer is the textbook my curriculum. I create my own, collecting and integrating bits and pieces from many places. My teaching begins to take a philosophic twist. I want the students to explore who they are mathematically, as I am exploring who I am in my therapy sessions.

The present, 1990. What I do now professionally, who I am, is rooted, in part, back in New Hampshire, in those therapy sessions with Carol. My classes at the University of Calgary, teaching elementary mathematics methods, focus on the exploration of beliefs about mathematics and how it is learned. The theme of these courses is the need to listen to children, to their ideas, to their interpretations of mathematical concepts.

My focus on listening, not only to students but to myself and to colleagues, is contrary to the power/control metaphor that was so central years ago when I hear the language I use now, it is one of caring, concern, and closeness, not power, control, and distance.

The closeness I am finding with my students now is not often found in educational institutions. Although it is impossible for me to banish my power completely, because I am, for example, still required to grade students, the kind of power has shifted from my early days of teaching. There is an openness with and between students so that everyone in the class feels his or her ideas are cherished. Ideas may be debated heatedly but there is always a feeling of underlying respect for the person and his or her ideas. The concern with people leads to a curriculum that is not routinized or standardized, but to one that is meaningful to students. It is a curriculum which often allows for interpretation, one for which problems do not always have one right answer and/or one right way to do those problems—usually the teacher's way. It is one that allows for ownership from students, one for which problems and tasks are so involving that students or the teacher cannot stay distanced. It is one that involves all aspects of learning—political, socio-cultural, and so on. Discussion flows like a conversation, taking us in any direction that seems appropriate. Mathematics becomes integrated with people, with emotions, with what happens in our culture, on the earth.

My research and writing has evolved from the quantitative analysis I did for my master's thesis to the ethnographical analysis for my PhD dissertation to what I hesitantly label the literary, philosophic, critical theoretic type of exploration I do now. It is an exploration in form as well as substance, one that involves the integration of the emotional with the cognitive, the rejoining of fragmented parts of knowledge—the logical with the intuitive—the mathematical content with the people who are learning it.

"Coming to understand who you are" is a process, one much more complex than has perhaps been communicated here. It may seem that I have gone from living a mechanical existence to one of conscious awareness, from distancing myself from everyone and everything to one of closeness, from being concerned with power and control to being concerned with caring. This is too simplistic a portrait however. Complexity lies in non-linearity, in the idea that we move back and forth between behaviours and patterns in our lives—new ones, old ones, ones in the process of being integrated into some new form. Disturbing old behaviour patterns often resurface, especially while one works consciously to transform them.

What this has to tell us as educators is that transforming classroom teaching and learning is not only slow, but also non-linear. Even though teachers try new ideas and new teaching styles, they will go back to the old. This may still mean, however that transformation is occurring; it just takes longer to view any significant changes.

What has been, and still is, extremely important for me is recognizing that I cannot do this work alone. At times I have needed therapy, and have always needed supportive friends, ones who would be willing to poke me into conscious awareness of my behaviours, thoughts and language even when I have not wanted this awareness. Teachers cannot transform their teaching alone either. Short-term workshops or in-service days will not work, and never has worked. Teachers need sustained support, especially through discouraging times—when the new ideas learned in workshops are implemented and do not work as well as they could.

This journey inward has also taught me about the wholeness and inseparability of our private and professional lives, of ourselves from others, from the earth on which we live. Much of this learning is rooted in books outside of mathematics education—especially ones that merge traditional medicine with spiritual ideas, ones that are taking the discoveries in quantum physics a step further, such as *The Quantum Self* by Danah Zohar and *The Seat of the Soul* by Gary Zukav. The point is that it is important for us to read widely *and* to bring those ideas back to mathematics education.

At the core of my learnings has been the need to question critically what I am doing and why, to probe what mathematics is, what knowledge is. This would be my answer to the questions "What life does the profession of mathematics education want to lead? What does it want to become?" It is a personal answer. Yet that is all it can be. I can only find my solutions, private solutions. And, in turn, if each of us in mathematics education would do so, we could work together to create public solutions.

To close, I circle back to the themes mentioned at the beginning of my talk—fragmentation, integration, a different kind of knowing, one that is unnameable, because as soon as I label it "intuitive", I am using the language of categorization, of separation. In fact, throughout this talk I have been limited by the language available to me, a language that often has not coincided with what I have been trying to express, a language that has evolved from and has been created by a logical/rational view of knowledge. What is needed is a way of expressing an integrated view, one that does not discard logical categorical concepts but that expands to mesh with what is now unnameable.

To illustrate the direction such integration can take, I share the following bit of stream of consciousness writing. This was done on June 27, 1990 while I was awaiting the results of my latest bone scan.

(1230 PM) I wait, trying to breath naturally, trying to stay calm, aware that if I'm anxious then, I need to "go" with that anxiety, not deny it. This is what living moment-to-moment truly is—being told they need closeups of my elbows, shoulders, left rib cage—wondering if the cancer has spread—trying to detect what my intuitive knowledge says... It helps to write, focus my energy... I stop (12:40), look at the clock, anxiety starts to take over again, nerves, what's taking them so long? Aware of shortness of breath, tightening in my chest, fear—as soon as I write this, it goes—keep writing... difficult to do, keep writing, stay focussed in the moment... What's taking so long? An indication of? Anything? Nothing? Terror. I recall March, my first bone scan. This has been a bit better, only a bit. I watched the screen doing those closeups, when I could, felt more in control...

(12:45) She hands me the x-rays; relief? No more closeups at least.

(1:10) I wait for Peter Geggie. Fear, racing heart, lightness, dizzy... When I looked at the x-rays, they look the same as the March ones did Back to gut feelings, trusting those And they are? Fear at the moment. Part of me says I'm fine. Part of me say, how do I know? We don't learn to trust, or even to discern our gut feelings. Back to living in the head. Objective science. It doesn't make sense that cancer would spread to both elbows or shoulders at the same time!? "Sense" means logic doesn't it? Nervous. Trying to write, monitor feelings while writing... Live in the moment, not anticipate. Yet I don't trust my knowledge, fear is there that

there is a chance it has spread... Living on borrowed time? Meaning "it" will spread eventually, whatever that means ...

(2 pm) Anita comes to get me to look at x-rays with Peter.

What makes this integrated is its back-and-forth nature, between thinking and recording awarenesses of body sensations, what is happening physically and emotionally, in short, what the experience is. When I watch my doctor read x-rays, I also witness this kind of knowledge. He makes statements rooted in his medical, logical background, like "Everything looks fine because if it weren't we'd see very dark spots." At the same time, he also makes ones like: "This shadow here makes me uneasy." Bottom line, what I have learned from him is that I need to "read my body", become in tune with what it is saying to me.

This kind of knowledge, one that goes beyond the logic of the mind, is used not only in the medical profession. Scientists who spend many, many years exploring one theory do the same. They are trusting something inside that is telling them to go in a certain direction. They eventually share this unnameable something through the medium of their work. Gary Zukav speaks to this idea in *The Seat of the Soul:*

> During the years that I was writing *The Dancing Wu Li Masters* after, I was drawn again and again to the writings of William James, Carl Jung, Benjamin Whorf, Niels Bohr, and Albert Einstein ... these fellow humans reached for something greater than they were able to express directly through their work...
> ... I came to understand that what motivated these men was not Earthly prizes or the respect of colleagues, but that they put their souls and minds on something and reached the extraordinary place where the mind could no longer produce data of the type that they wanted... They could not articulate this clearly because they were not equipped to talk about such things, but they felt it and their writings reflected it... (1990, p. 11-12.)

It is this kind of knowledge that not only scientists use, but all of us use daily. When we are jogging, and unconsciously keeping track of our pacing, we are using this kind of "knowing." When we watch our sons and daughters, or, grandsons and granddaughters, count the stairs they are climbing, we are witnessing this kind of knowledge. We are not only observing we are thrilled at some unnameable level just as we are when we are working on a research project or writing an article for a journal, our hunches prove correct, and everything falls into place.

It is this kind of knowledge that we need to cherish and bring forth from our graduate students, the ones who will be the theorists of tomorrow. Yet I do not see much evidence of this happening. At the recent NCTM research pre-conference, I attended a session on the topic of teacher's beliefs. One of the speakers was a graduate student, one who was taking rich descriptive data, forming static categories and was planning to do a factor analysis. No one in the audience seemed to object, including myself, probably because I did not know this person. However, I have spoken up in such a situation, to another student, one who is not in mathematics but who is doing a fascinating study of the image of teaching as it is portrayed in movies. When I asked her why she was categorizing her data, she responded that she did not know what else to do with it.

So even students who are being educated in qualitative research, still fall back on quantitative methods and ways of thinking. They are still searching for rules to do research. A friend who is doing a master's degree in technology would like to do a qualitative study, one that would look at the concept of distance education from the viewpoint of a student taking a course using the technology of the field. His advisor told him that he would not be much help because he did not know how to do this kind of research. As he was relating this story to me, I thought of rule-boundedness. The attitude indicates solely a rational/logical view of what research and theorizing is—that there could be a list of rules and prescriptions for creating knowledge.

I am led back to my title, "Our Theories, Ourselves." We are not isolated in the world of research, we do not create our theories in a vacuum. We exist among institutions, other people—with their rules and conventions. We bring to our research our life history, what we have been taught to value, both personally and professionally. Even when we recognise the inseparability of everything on earth, we still live in a culture that manifests fragmentation at every turn. While the thought of changing this pattern can be overwhelming because of its pervasiveness, we need to try.

If we complain about the fragmentation of the discipline, then we need to investigate how we are fragmented. If we criticize elementary and high school students and teachers for doing very little "critical thinking", then we need to uncover how well we critique our own theories and research. If we do not like the language used by teachers in the classroom, then we need to explore the language we use. If we do not like the values being espoused in education, then we need to examine our values. If we do not like the kind of teaching styles we observe in elementary and high school classrooms, then we need to look at our teaching in university classrooms.

A summary of these ideas was given by Confucius: *...wanting good government in their states, they first established order in their own families; wanting good order in the home, they first disciplined themselves...* Thus, journeying into ourselves is the only way we will begin to alter what we do not like about mathematics education. It is the only way we will begin to articulate our love for mathematics to our students and to ourselves.

References

Griffin, S. (1989). *Split Culture*; In Plant, J. (editor), *Healing the Wounds: the Promise of Ecofeminism* Santa Cruz, CA: New Society Publishers.

Koller, A. (1983). *An Unknown Woman: A Journey in Self-Discovery*, New York: Bantam Books.

Koller, A. (1990). *The Stations of Solitude*, New York: William Morrow & Co.

Kundera, M. (1984). *The Unbearable Lightness of Being*, (M. H. Heim, Trans.) New York: Harper & Row.

Macy, J. (1989). *Awakening to the Ecological Self*, in Plant, J. (editor), *Healing the Wounds: the Promise of Ecofeminism*, Santa Cruz, CA: New Society Publishers.

Zohar, D. (1990). *The Quantum Self: Human Nature and Consciousness Defined by the New Physics*, New York: William Morrow & Co.

Zukav, G. (1989). *The Seat of the Soul*, New York: Fireside Books, Simon & Schuster.

Appendix C

Previous Proceedings

Previous Proceedings

The following is the list of previous proceedings available through ERIC.

Proceedings of the 1980 Annual Meeting	ED 204120
Proceedings of the 1981 Annual Meeting	ED 234988
Proceedings of the 1982 Annual Meeting	ED 234989
Proceedings of the 1983 Annual Meeting	ED 243653
Proceedings of the 1984 Annual Meeting	ED 257640
Proceedings of the 1985 Annual Meeting	ED 277573
Proceedings of the 1986 Annual Meeting	ED 297966
Proceedings of the 1987 Annual Meeting	ED 295842
Proceedings of the 1988 Annual Meeting	ED 306259
Proceedings of the 1989 Annual Meeting	ED 319606