CANADIAN MATHEMATICS EDUCATION STUDY GROUP

GROUPE CANADIAN D'ETUDE EN DIDACTIQUE DES MATHEMATIQUES

PROCEEDINGS

1993 ANNUAL MEETING

York University

Toronto, Ontario

May 28-June 1, 1993

Edited by

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CMESG/GCEDM Annual Meeting
1993 ANNUAL MEETING
York University, May 28-June 1, 1993

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EDITOR'S FORWARD

I should like to thank all the contributors for submitting their manuscripts for inclusion in these proceedings. Without their co-operation it would not have been possible to produce the proceedings.

Special thanks must go from all of us who attended the conference to the organizers, and particularly to Pat Rogers and Olive Fullerton who did so much before and during the meeting to make it such an enjoyable and profitable event.

I hope these proceedings will help generate continued discussion on the many major issues raised during the conference.

Martyn Quigley

April 28, 1994

ACKNOWLEDGEMENTS

The Canadian Mathematics Study Group wishes to acknowledge the continued assistance of the Social Sciences and Humanities Research Council for their support. Without this support the 1993 meeting and proceedings would not have been possible.

We would also like to thank York University for hosting the meeting and providing excellent facilities to conduct the meeting.

Finally, we would like to thank the contributors and participants, who helped make the meeting a valuable educational experience.

Lecture One

What is a Square Root? A Study of Geometrical Representation in Different Mathematical Traditions

George Gheverghese Joseph

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Introduction [1]

A widely accepted view among historians of mathematics is: Mathematics outside the sphere of Greek influence (such as India and China) was algebraic in inclination and empirical in practice which provided a marked contrast to Greek mathematics which was geometric and anti-empirical. A simple illustration of the difference in approach between the Greeks and others was in tackling a problem such as: Solve for x the equation: $x^2 = N$.

The Greeks would seek a geometric solution which involved taking the side of a square of area N while the Indian and Chinese approach would be similar to the algebraic procedure we adopt today of taking the square root of N.

Like a number of such generalisations there is more than a grain of truth, though this is not to argue that all mathematical traditions not influenced by the Greeks were essentially algebraic without any analytical tradition in geometry. The main argument of this paper is that a geometry did exist outside Greek mathematics which had moved beyond mere numerical relations or practical surveying considerations to active search for general proofs. The reasoning in these proofs was mostly based on one basic premise: figures of dissimilar shape can have the same area and that they can be dissected and then reassembled for purposes of proving this fact.

The concept of a square root

A student first introduced to the square root of a number N is told that it is the number which gives N when multiplied by itself [2]. Examples such as:

The square root of 4 is 2 since $2 \times 2 = 4$
The square root of 9 is 3 since $3 \times 3 = 9$

are used to illustrate the point. And a geometrical interpretation shows a square of 9 square units with side (root) of 3 units Or, in a general form, where the square root is the solution to the quadratic $x^2 = N$. This reinforces the view that the square root exists irrespective of the value of N.

The problem really starts when the student has to find the square root of N where $N = 2, 7, 14, \ldots$ The definition of the square root given earlier is not very helpful for it is not possible to find a whole number multiplied by itself which will give either 2 or 7. Neither is a geometrical interpretation sufficiently general to be helpful. No doubt the square root of 2 is the diagonal of a unit square, by applying the so-called Pythagorean theorem. But how would one interpret the square root of, say, 14 geometrically?

Faced with this problem of incommensurability, the reactions of different mathematical traditions are interesting. The Pythagoreans came across lengths which were incommensurable when determining the mean proportion of the two sides of a rectangle needed for "squaring the rectangle". The discovery of the diagonal of a square of side one unit, $\sqrt{2}$, caused such a scandal that a pupil of Pythagoras, Hippasus, who compounded the scandal with public disclosure, was supposed to have perished at sea. The memory of this

[1] This article owes a considerable debt to an unpublished paper by David W. Henderson (University of Cornell) entitled "Geometric Solutions of Quadratic and Cubic Equations". I am grateful to the author for sending me a copy of this paper.

[2] Nor is a definition from "advanced" mathematics particularly illuminating. The square root consists of a certain equivalence class of Cauchy sequences of rational numbers or a certain Dedekind cut.

scandal still remains in the terminology of modern mathematics. Numbers expressible as a ratio of two integers are called *rational* numbers, whereas numbers such as the length of a unit square or the value of π not expressible as a ratio are known as *irrational* ("un-ratio-able"). It is interesting in this context that the etymology of the word "rationalism" comes from the Latin word *ratio*, which is a translation of the Greek word *logo* meaning mathematical ratio, symbolising reason itself [3].

Thus Greek difficulties with incommensurability arose from the attempt to establish a close correspondence between geometric and arithmetic quantities, the result being a heavy emphasis on a geometric interpretation of irrationality of numbers. Because of this geometric bias, the Greeks were not at ease with irrational numbers and consequently operations with numbers were reduced to a narrow geometric realm robbing them of considerable potency in arithmetic.

The stress in other traditions on *operations* with numbers rather than the numbers themselves meant that their mathematics steered clear of any problem with incommensurability. For example in India, surds, known as *karani*, were accepted as "proper" numbers from early times and rules for handling them were developed. Though the rational-irrational classification did not exist in the Indian tradition, the notion of exact-inexact numbers was developed. This is reminiscent of the Babylonian distinction between "regular" and "irregular" numbers. In both traditions, procedures were developed for calculations with these sets of numbers.

The Babylonian square root algorithm

The earliest version of an approximation procedure for evaluating square roots of "irregular" numbers is from Babylonia and dates back about four thousand years. In a clay tablet, held at the University of Yale, the diagram shown in Figure 1(a) appears, with the transliteration of the Babylonian numerals into the Neugebauer notation for sexagesimal (base 60) number system given in Figure 1(b).

The number 30 in Babylonian notation is marked along the length of one side of the square. There are also two other numbers given on and below one of the diagonals. The number on the diagonal converted from the Babylonian notation to ours gives:

$$\begin{aligned} & 1 + 60^{-1}(24) + 60^{-2}(51) + 60^{-3}(10) \\ = {} & 1 + 0.4 + 0.0146667 + 0.0000463 \\ = {} & 1.41421297 \end{aligned}$$

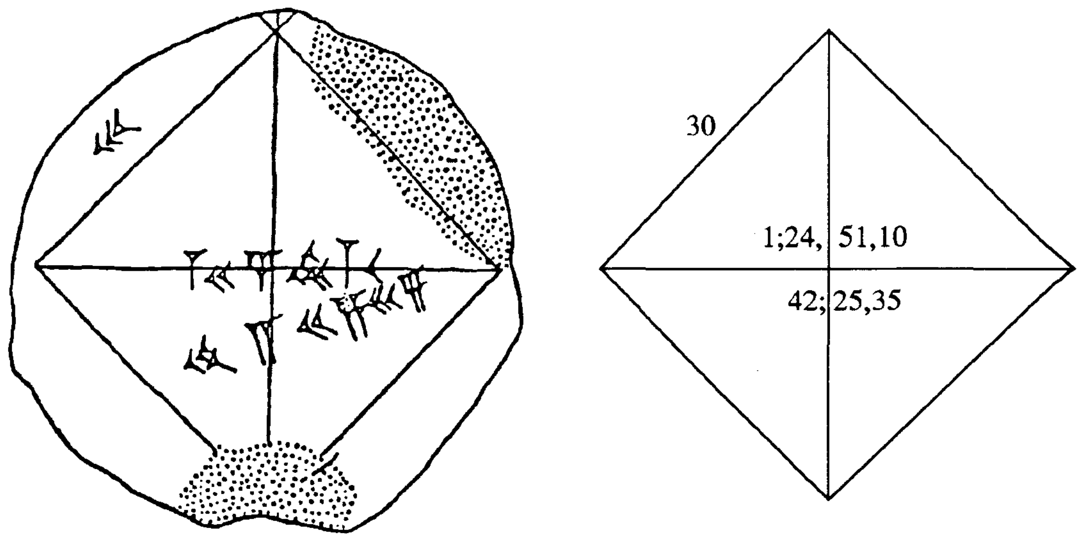
What we have here is a well-known approximation to the square root of 2, with the estimate being correct to five places of decimals! The number *below* the diagonal is the product of 30 (the side of the square) and the estimate of the square root of 2, the number given on the diagonal. This product in decimal notation is 42.426389. What does this quantity represent?

Let $\boldsymbol{d}$ be the diagonal of the square. Applying the Right Angled Triangle theorem,

$$\boldsymbol{d}^2 = 30^2 + 30^2 \Rightarrow \boldsymbol{d} = 30\sqrt{2} = 42.426407$$

which is the number below the diagonal expressed in our notation. Two aspects of Babylonian mathematics are highlighted in this example. First, over a thousand years before Pythagoras, the Babylonians knew and used this result. Second, there is the intriguing question as to how the Babylonians arrived at their

[3] A significant trait of Pythagorean philosophy was its atomistic outlook: Number was perceived as an entity constituted out of an indivisible unit, a monad or primordial atom. The Greeks imagined that the world and its bodies was built "out of numbers." Even Euclid, whose internalisation of the Eleatic tradition (that is, truth is not grasped by means of sense perception but by only reason (*logo*)) freed the subject from the major strait-jacket of incommensurability, still considered number as having a pre-existent reality.



(a) (b)

Figure 1

remarkable estimate of the square root of 2. We may well find the answer in a method known as Heron's procedure, named after an Alexandrian mathematician who lived two thousand years later. Represented in modern symbolic algebra, Heron's procedure is:

Let N be the number whose square root is sought and the positive number a be a "guess-estimate" of the answer. Then

$$N = a^2 + e$$

where the difference (or "error") e can be positive and negative. We try next to find a better approximation for the square root of N which we denote as $(a + c)$.

It is obvious that the smaller the "error", e, the smaller is c relative to e. Thus we impose the following condition on c:

$$N = (a + c)^2 = a^2 + e \rightarrow 2ac + c^2 = e \quad \ldots\ldots \quad (1)$$

If we made a sensible guess for a in the first place, then c^2 will be very small relative to $2ac$ and may therefore be ignored. So (1) becomes:

$$c = \frac{e}{2a} \quad \ldots\ldots \quad (2)$$

Hence, from (1) and (2), an approximation for the square root of x is

$$(a + c) = a + \frac{e}{2a} = a_1$$

Now take

$$a_1 = \left(a + \frac{e}{2a}\right)$$

as the new "guess-estimate" and repeat this process to get a_2, a_3, ..., which are better and better approximations. Implied in this iterative procedure is the assumption (The Completeness Axiom) that the sequence of approximations converges to some real number.

To illustrate this approximation procedure consider the question we started with–how did the Babylonians obtain their estimate for square root of 2 as 1.41421297 ?

Step 1: If $a = 1$, then $c = \dfrac{e}{2a} = 0.5$ and $a_1 = 1.5$

Step 2: If $a_1 = 1.5$, then $c_1 = \dfrac{e_1}{2a_1} = -\dfrac{0.25}{3}$ and $a_2 = 1.41667$

Step 3: If $a_2 = 1.41667$, then $c_2 = \dfrac{e_2}{2a_2} = 0.00246$

and $a_3 = 1.41667 - 0.00246 = 1.41421$.

This is very close to the value for the square root of 2, expressed in decimal notation, shown on the diagonal of Figure 1(b) This procedure for calculating square roots seemed to have been a standard procedure in Hellenistic mathematics, showing the Babylonian influence on the mathematics of that period. A more intriguing question concerns the appearance of another approximation in the earliest extant mathematical writings of the Indians, known as the Sulbasutras (800 BC – 500 BC).

The Indian square root algorithm

An important source of early Indian mathematics derives from a class of ritual literature dealing with the measurement and construction of various sacrificial altars. The Sulbasutras provided such instructions for two types of rituals, one for worship at home and the other for communal worship. Square and circular altars were sufficient for household rituals while more elaborate altars involving combinations of rectangles, triangles and trapeziums were required for public worship. One of the most elaborate of the public altars was shaped like a falcon, or rather like the shadow of a falcon, just about to take flight (Figure 2). It was

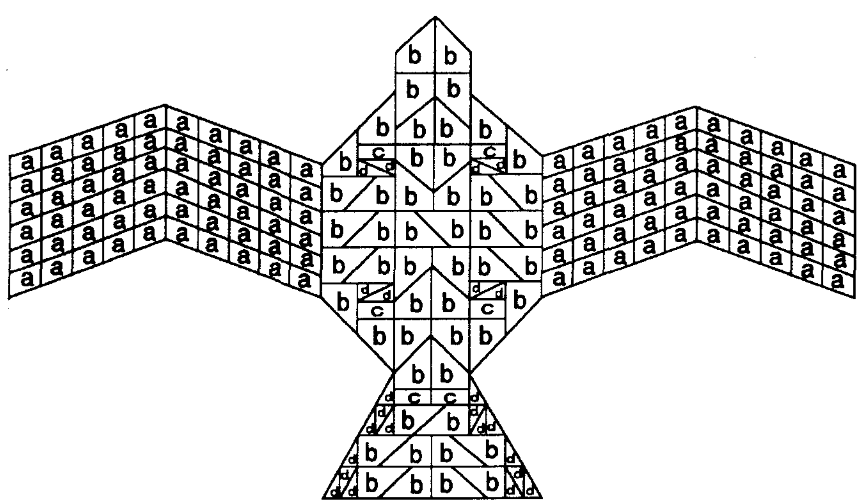


Figure 2: The first layer of a Vakrapaksa-syena altar

believed that offering a sacrifice on such an altar would enable the soul of the supplicant to be conveyed by a falcon straight to heaven.

In Figure 2; the wings are each made from 60 bricks of type a, and the body, tail and head from 50 of type b, 6 type c, and 24 type d bricks. Each subsequent layer was laid out using different patterns of bricks with the total number of bricks equalling 200.

The procedure for evaluating $\sqrt{2}$ arose from an attempt to construct a square altar twice the area of a given square altar, a basic design requirement for a number of constructions. The problem reduces to one of constructing a square twice the area of a given square **A** of side 1 unit. It is clear that for the larger square **C** to have *twice* the area of square **A**, it should have side $\sqrt{2}$ units. Also, we are given a third square **B** of side 1 which needs to be dissected and reassembled so that by fitting cut-up sections of square **C** on square **A**, it is possible to make up a square close to the size of square **C**. Figure 3(a) shows what needs to be done. The instructions in the Sulbasutras may be translated as:

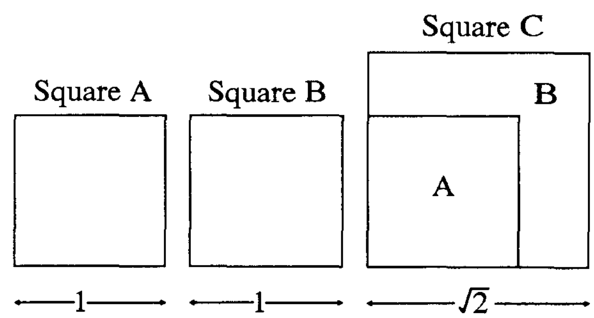


Figure 3a

> *Increase the measure by its third and this third by its own fourth less the thirty-fourth part of that fourth. This is the value with a special quantity in excess.*

If we take 1 unit as the dimension of the side of a square, the above formula gives the approximate length of its diagonal as follows:

$$\sqrt{2} = 1 + \frac{1}{3} + \frac{1}{3 \times 4} - \frac{1}{3 \times 4 \times 34} = 1.4142157$$

A commentator on the Sulbasutras, Rama, who lived in the middle of the fifteenth century gave an improved approximation by adding a fifth and sixth term to the right-hand side of the equation, consisting of:

$$-\frac{1}{3 \times 4 \times 34 \times 33} + \frac{1}{3 \times 4 \times 34 \times 34}$$

which then gave the first seven places of decimals correctly.

It can be shown that the Sulbasutra formula, when used in evaluating $\sqrt{2}$, produces more or less the same result as one obtained from the iterated use of the Babylonian procedure discussed earlier [4]. This has raised the possibility that this method of calculating the value of the square root of 2 may have been derived from the Babylonian procedure. In considering this issue, the following points may be of relevance:

(i) During the Sulbasutra period, there is no evidence from any other source of the knowledge or use of the sexagesimal system of number reckoning.

(ii) There is no evidence that the Babylonians were aware of this precise algorithm and their intellectual heirs, the Greeks, had a number of approximations to the value of the square root of 2 but this precise method did not occur in any of their literature.

(iii) It is quite likely that the basic mode of approach was different in the two cultures: the Babylonian approach being *algebraic* and the Indian being *geometric*.

---

[4] Note that the Indian approximation to square root of 2, expressed in sexagesimal unit, is 1;24,51,10,36,..... compared to the Babylonian value of 1;24,51,10 given earlier.

The Sulbasutras contain no clue as to the manner in which this accurate approximation was arrived at. A number of theories or possible explanations have been proposed. Of these, a plausible one is that of Datta (1932).

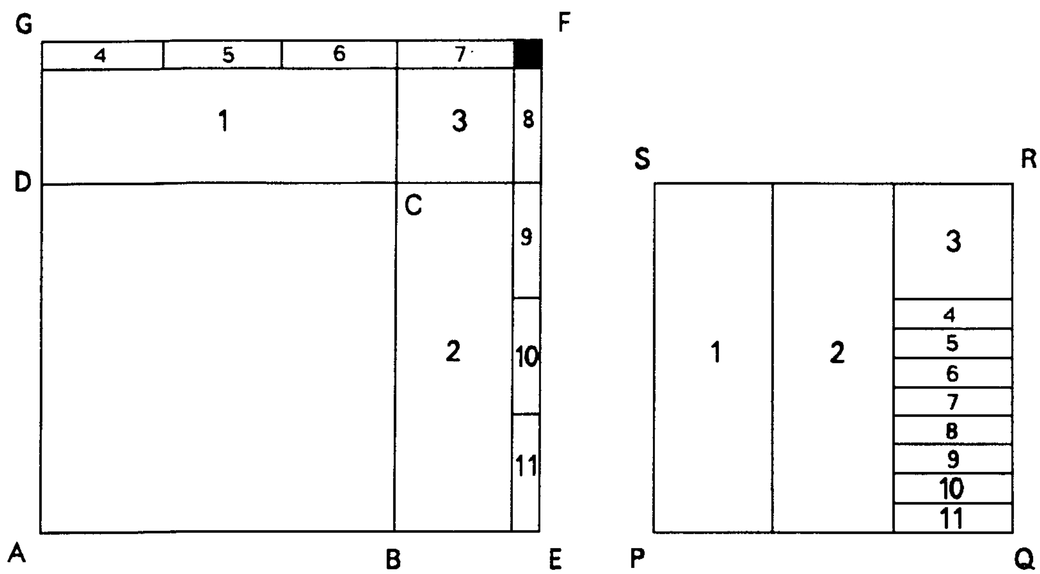


Figure 3b

In Figure 3(b), two squares **ABCD** and **PQRS** of unit sides are taken. **PQRS** is divided into three equal rectangular strips, of which the first two are marked 1 and 2. The third strip is subdivided into three squares of which the first is marked 3. The remaining two squares are each divided into four equal strips marked as 4, 5, 6, 7 and 8, 9, 10, 11. These eleven strips are added to the other square **ABCD** in the manner shown in Figure 3(b) to obtain a large square *less* the small shaded square at the corner. The side of the augmented square **AEFG** equals $1 + \frac{1}{3} + \frac{1}{3 \times 4}$. The area of the shaded square is $\left(\frac{1}{3 \times 4}\right)^2$ so that the area of the augmented square **AEFG** is greater than the sum of the area of the original squares **ABCD** and **PQRS** by $\left(\frac{1}{3 \times 4}\right)^2$.

In order to get the area of the square **AEFG** to be approximately equal to the sum of the areas of squares **ABCD** and **PQRS**, cut off two tiny strips from either side of the square **AEFG** of width x so that:

$$2x\left(1 + \frac{1}{3} + \frac{1}{3 \times 4}\right) - x^2 = \left(\frac{1}{3 \times 4}\right)^2$$

Simplifying the above expression and ignoring x^2, an insignificant quantity, gives: $x = \frac{1}{3 \times 4 \times 34}$

Thus the side of the square whose area equals the sum of the areas of two original squares, **ABCD** and **PQRS** or the diagonal of each of the original square is:

$$\sqrt{2} = 1 + \frac{1}{3} + \frac{1}{3 \times 4} - \frac{1}{3 \times 4 \times 34}$$

What is particularly interesting about this line of argument is its visual mode, a form of argument also found in Chinese geometry [5]. Described in Chinese texts as the "out-in complementarity" principle (or what is more familiarly known as the principle of "dissection and reassembly"), it follows from two common sense assumptions:

(i) The area of a plane or a solid figure remains the same when the figure is rigidly shifted to another place on the plane or in the space

(ii) If a plane or solid figure is cut into several sections, the sum of the areas or volumes of the sections is equal to the area or volume of the original figure.

This mode of demonstration is neither dependent on the existence of well packaged symbolic algebra nor the Euclidean mode of axiomatic deductive inference. A modern variant of this approach is found in a topic in geometry known as Dissection Theory. Let us examine how this theory helps us to understand the concept of a square root.

The Dissection Theory

In Dissection Theory, there is the result that every polygonal region in a plane can be cut up into a finite number of pieces and then rearranged to form a square. Consider the special case of a rectangle.

Every rectangle is equivalent by dissection to a square

The proof of this theorem is well known. In Figure 4, a rectangle **ABCD** (of sides a and b where $a > b$ and square AEFG (of side s) are equivalent with $s = \sqrt{ab}$. The square is placed on the rectangle as shown in the Figure. Draw **GB** to cut **DC** at **H** and **FE** at **K**. Let **DC** cut **FE** at **L**. From similar triangles **KBE** and **GBA**, we have the following result:

$$\frac{KE}{GA} = \frac{EB}{AB}, \quad \frac{KE}{s} = \frac{(a-s)}{a}$$

and since

$$GA = s, \quad EB = a - s, \quad AB = a,$$

then

$$\begin{aligned} KE &= s\frac{(a-s)}{s} \\ &= s - \frac{s^2}{a} \\ &= s - \frac{ab}{a} \\ &= s - b \end{aligned}$$

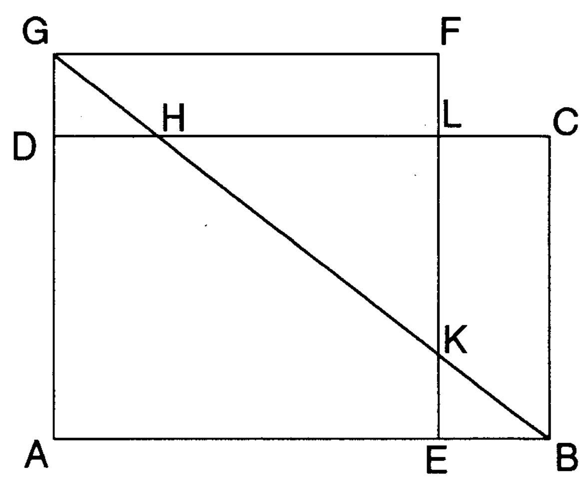


Figure 4

So $\triangle$**GFK** $\equiv$ $\triangle$**HCB** and $\triangle$**GHD** $\equiv$ $\triangle$**KBE**.

[5] This line of argument can be extended to evaluating the square roots of 3, 5, 7.... which could have its pedagogic uses. See Nelson *et al.* (1993, pp. 52-55)

Cut out triangle **HCB** and trapezium **DHBA** from rectangle **ABCD** and assemble square **AEFG** by inserting $\triangle$**GFK** (≡= $\triangle$**HCB**) and trapezium **GKEA** (≡ **DHBA**). This completes the proof by dissection.

The method outlined suffers from two drawbacks. The proof is based on properties of similar triangles and there is often some unease about how satisfactory is the use of such facts in establishing a concrete theory of areas of polygon. The other problem is that there is no attempt at a visual explanation of $\sqrt{ab}$, which is what we are ultimately seeking.

However, the Dissection Theory is perfectly consistent with demonstrations derived from other geometric traditions which have the added advantage of avoiding the assumption that $\sqrt{ab}$ needs to exist uniquely. For purposes of illustration, let us consider two different traditions–the Indian and the Chinese.

The Dissection Theory in the Sulbasutras

In the writings of Baudhyana, the oldest of the Sulbasutras, appears a set of instructions for constructing a square altar whose base has the same area as the base of a rectangular altar:

> *In order to turn an oblong (i.e., rectangle) into a square, take the width of the oblong as the side of the square; divide the rest of the oblong into two parts and by suitable rotation, join these two parts to the two sides of the square. Fill the empty place with an added piece.*

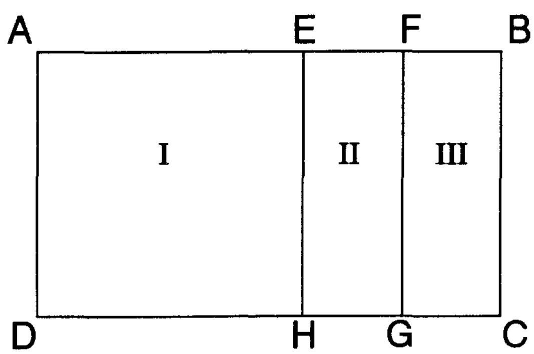


Figure 5a

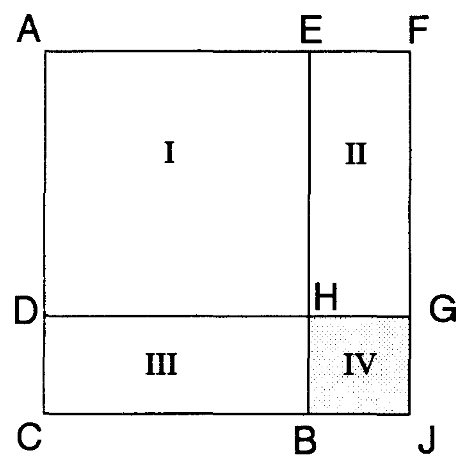


Figure 5b

In Figure 5(a) **ABCD** is the given rectangle in which **AB** = a and **BC** = b. Take points **E** on **AB** and **H** on **DC** such that **AD** = **AE** = **DH**. Join **EH**. Now take points **F** on **EB** and **G** on **HC** such that **EF** = **FB** and **HG** = **GC**. Join **FG**. Label square **AEHD** as **I**, rectangle **EFGH** as **II** and rectangle **FBCG** as **III**. Move **III** with **F** and **G** now located at **H** and **D** respectively. Construct the smaller square IV which is shown as the shaded square in Figure 5(b) to obtain the larger square **AFJC** as the sum of the sections **I**, **II**, **III** and **IV**. So the original rectangle **ABCD** has been transformed into a large square **AFJC** from which the small shaded square needs to be removed. Another result from the Baudhyana's Sulbasutra states:

> *If you wish to remove one square from another, cut off from the larger one an oblong with the side of the smaller one, draw one of the sides of that oblong to the other side; where it touches the other side, that piece should be cut off. By this method the removal is effected.*

Figure 5(c) shows the large square **AFJC** in Figure 5(b) from which the small square (i.e., the shaded square **HGJB** or **IV** in Figure 5(b)) is to be removed. With **B** as the centre and **BE** the radius, construct a circle which cuts **AC** at **K**. Then the square on **KC** is the required square whose area is equal both to the difference in the areas of the larger square **AFJC** and the small shaded square **HGJB** and equal to the area of the original rectangle **ABCD**. The proof follows from the Right Angled Triangle theorem, whose knowledge is evident from a number of references in the Sulbasutras :

$$\begin{aligned} JM^2 &= BM^2 - BJ^2 \\ &= BE^2 - BJ^2 \\ &= \text{Square } AFJC - \text{Square } HGJB \end{aligned}$$

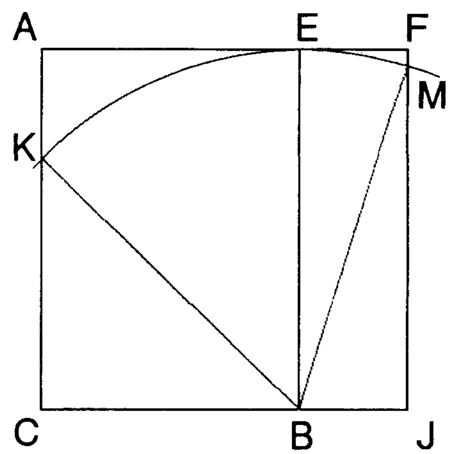


Figure 5c

Figure 6 shows how the side of a square of length $\sqrt{ab}$ can be directly constructed from the original rectangle given Figure 5(a).

Given the length and width of the rectangle **ABCD** is **a** and **b** respectively where b is also the side of the square **AEHD**, then it is easy to deduce that

$$\begin{aligned} MG &= DG = \tfrac{1}{2}(a - b) + b \\ HG &= \tfrac{1}{2}(a - b) \end{aligned}$$

Applying the Right Angled Triangle theorem to triangle MHG, will give

$$\begin{aligned} MH^2 &= MG^2 - HG^2 \\ &= [\tfrac{1}{2}(a - b) + b]^2 - [\tfrac{1}{2}(a - b)]^2 \\ &= ab \\ \text{or} MH &= \sqrt{ab} \end{aligned}$$

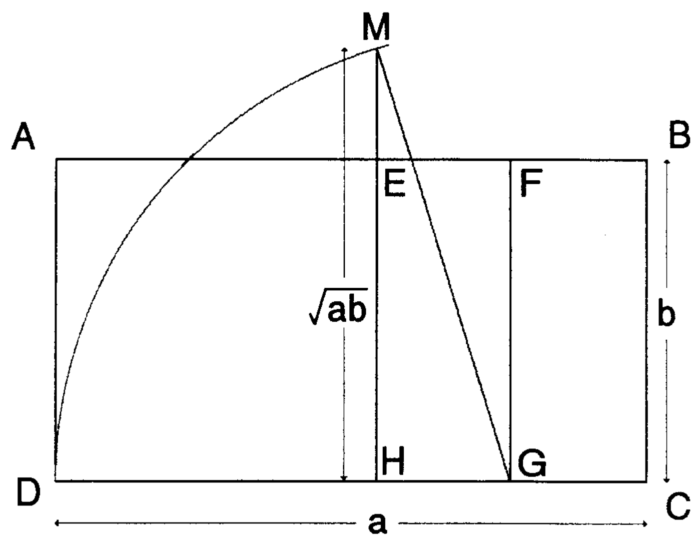


Figure 6

Note that the Sulbasutra approach to square root just outlined has clear conceptual advantages over the Dissection Theory approach discussed earlier. The existence of $\sqrt{ab}$ is established without having to resort to the "Completeness Axiom". No use is made of any facts about similar triangles. There is no need for the area or the sides of the rectangle (or square) to be expressed in numbers. The concept of square root derives directly from the construction of the square and a simple demonstration that its area is the same as the area of the rectangle.

The Dissection Theory in the Chiu Chang Suan Shu

The fourth chapter of the premier Chinese mathematical text, the Chiu Chang Suan Shu (c. 200 BC) contains twenty four problems on land surveying. An important objective was to parcel out land given the area and one of the sides. Consider the following problem from the text:

> *There is a (square) field of area 71824 (square) pu (or paces). What is the side of the square?*
> Answer*: 268 pu*

The algebraic rationale underlying the Chinese approach may be expressed with the following symbolic notation: $\boldsymbol{N}$ is a number whose square root is a three-digit integer. α, β and γ are digits representing "hundreds", "tens", and "units" place value positions respectively. So that if the square of $\boldsymbol{N}$ is a three digit number, ***abc***, then $\alpha = 100\boldsymbol{a}$, $\beta = 10\boldsymbol{b}$ and $\gamma = \boldsymbol{c}$. Therefore,

$$\begin{aligned} \boldsymbol{N} &= (100\boldsymbol{a} + 10\,\boldsymbol{b} + \boldsymbol{c})^2 \\ &= (\alpha + \beta + \gamma)^2 \\ &= \alpha 2 + (2\alpha + \beta)\beta + [2(\alpha + \beta) + \gamma]\gamma \quad \ldots\ldots\ldots\ldots\ldots\ldots (1) \end{aligned}$$

It is simple to extend this formula to include more than three digits by expanding $(\alpha + \beta + \gamma + \delta \ldots)^2$ The Chinese used the above relationship but reversed the procedure and the ensuing calculations that resulted. The procedure is initiated by finding an appropriate value for by "inspection". It is, for example, easily deduced that $\boldsymbol{\alpha} = 200$ (i.e., $100\boldsymbol{a}$ where $\boldsymbol{a} = 2$) if we are seeking r^{th} square root of $\boldsymbol{N} = 71824$. The procedure continues with calculating α^2. This quantity is then subtracted from N. We next estimate the second place of the square root (i.e., $\beta = 10\boldsymbol{b}$), and then form $(2\alpha + \beta)$. We can now work out

$$\boldsymbol{N} - \alpha 2 - (2\alpha + \beta)\beta = \boldsymbol{N} - (\alpha + \beta)^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots (2)$$

The procedure continues along similar lines until the third component on the right hand side of (1) is calculated. If $\boldsymbol{N}$ is a perfect square, the final subtraction of this component from (2) would leave a remainder of 0. The geometric rationale for the algorithmic approach just discussed is found in Figure 7.

We begin by constructing a square ($A = \alpha^2$) with side 200 *pu*. Two rectangular sections, $\mathbf{B} = \alpha\beta$ and $\mathbf{C} = \alpha\beta$, each of dimensions 200 by 60, are added together to give a total area of 24000 square *pu*. To complete the larger square shown in Figure 7, one needs to add a further square section, $\mathbf{D} = \beta^2$, whose side is 60 *pu* and area is 3600 square *pu*. The area of the larger square is
$\mathbf{A} + \mathbf{B} + \mathbf{C} + \mathbf{D} = 40000 + 24000 + 600 = 67600\,pu^2$.

The shortfall that has to be made up is $71824 - 67600 = 4224\ pu^2$. It is seen that this is equal to the area of two rectangular strips of dimensions 260 by 8, $\mathbf{E} = (\alpha + \beta)\gamma$ and $\mathbf{F} = (\alpha + \beta)\gamma$, and a small square $\mathbf{G} = \gamma^2$ of side 8, i.e., $2(260 \times 8) + 8^2 = 4224$ square *pu*. Thus the geometric representation of the procedure for extracting the square root of 71824 is equivalent to finding the length of the side of a square of area 71824 square *pu*. Figure 7 indicates clearly that the side required is $200 + 60 + 8 = 268$ *pu*. It is noteworthy that this method extracting square roots was eventually extended to the solution of quadratic equations. Indeed, the clear connection established between extraction of roots of any degree with solution of the same degree is a feature of Chinese mathematics not present in any other early mathematical tradition.

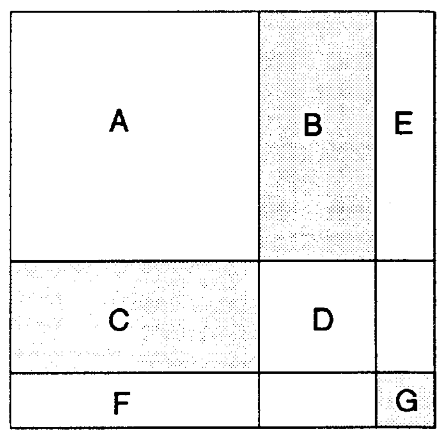


Figure 7

Conclusion

Implicit in the discussion of the square root is the need to pay more attention to the intuitive elements in mathematics [6]. Often in our haste to get to the more powerful analytic tools of mathematics, we ignore the "concrete" meanings and images that are already present. Sometimes it is hard even to recognise that some meaning is missing until a student (or more usually an adult learner) asks in some bewilderment "What does that mean?" or, given a formalist demonstration of something says: "I know it, to a degree I understand it, but I don't *feel* it". Such students can often only make progress, are only satisfied by, a procedure which accepts their psychological state, and works from that to an understanding which fuses, or at least deals with disharmony between, that emotional belief and their intellectual beliefs. Proofs, methods and reasoning should be rather like old fashioned demonstrations, in that they should reflect, not necessarily a chain of deductive reasoning, but rather how the human brain arrived at its current thought. It is clear that in the mathematical traditions we have examined the "geometric" concept of a square root was seen as being important. In spite of a tendency to neglect the geometric mode of argument in favour of analytic ones, based on the relatively recent notions of Cauchy sequences and the Axiom of Completeness, for many students and even some teachers, an intuitive understanding of real numbers and operations with real numbers requires geometry. Compare the geometric imagery of the product of real numbers $\boldsymbol{a}$ and $\boldsymbol{b}$ with the multiplication of two infinite, non-repeating, decimal fractions. Try explaining the product of $\sqrt{2}$ and π to someone with limited mathematical background without the help of geometry. The early traditions that we have examined would not have attempted such a task. Instead they would have concentrated on the visual and intuitive features found in geometrical explanations.

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[6] As an illustration of an intuitively convincing proof, consider the following demonstration of the result that a negative number has no square root, given by Krishna Daivajna (c.AD 1600), a commentator of the mathematical texts of Bhaskaracharya (c. AD 1100).

> *A negative number is not a square. Hence, how can we evaluate its square root ? It may be argued that "why cannot a negative number be a square? Surely it is not a royal command"... Agreed. Let it be stated by you who claim that a negative number is a square as to whose square it is; surely not of a positive number, for the square of a positive number is always positive. Not also a negative number because then also the square will be positive by the same rule. This being the case, we cannot see how the square of a number becomes negative.*

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Lecture Two

Forging a Revised Theory of Intellectual Development

Piaget, Vygotsky and Beyond

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Introduction

During last decade of the twentieth century, we in mathematics education continue to draw most heavily on the work of two scholars, Jean Piaget and Lev Vygotsky, for our theories of intellectual development. These theorists have offered us powerful insights into the human mind and its development, radically transforming our understanding of how children view the world and about how we understand ourselves as individuals within a cultural and historical setting. While these insights inform us about important ways to approach education, they also need revision in light of the current cultural and historical period in North America. A variety of factors have created a critical need to revise such theories, and these factors include changing demographics, a reform climate in education, the creation of new technologies, the press of environment concerns, and issues of power and oppression.

In this paper, I propose to provide brief summaries of radical constructivism, (as one interpretation of Piaget), and socio-cultural perspective, (as one interpretation of Vygotsky). The summaries will include major principles, primary contributions to mathematics education, and potential limitations. In a previous paper (Confrey, in press b), I warned readers of combining these theories too simplistically. In this paper, I introduce a new theoretical perspective which integrates the two theories by means of feminist perspective.

Radical Constructivism

Radical constructivism has one set of roots in the work in the philosophy of science. Starting with Karl Popper (1962), philosophers of science began to challenge the view of science as accretion of information through careful application of "The Scientific Method." Popper argued that it was falsification rather than verification and accretion that directed the development of scientific knowledge, and he drew one's attention to the role of critical experiments in determining progress. Thomas Kuhn (1970) and Stephen Toulmin (1972) followed by presenting differing accounts. They suggested that scientific progress could not be explained adequately by the falsification of empirical results alone. They argued for the analysis of larger structures that would include individual knowledge claims, methodologies, standards, even the forms of proof themselves. Scientific truth began to lose its simple connections to reality. Stability rather than certainty could be achieved based on the robustness of the theoretical or paradigmatic framework within a scientific community. Kuhn proposed a revolutionary view arguing for the incommensurability of paradigms–and for a process of replacement rather than of gradual evolution. Stephen Toulmin proposed an evolutionary view of the development of knowledge, in which it was during periods of change that one's fundamental commitments were revealed.

Philosophy of Science

Lakatos (1970), virtually the only philosopher of this time to apply the arguments from philosophy of science to mathematical knowledge, proposed an alternative framework in which the theoretical hard-core of a research programme remained unassailable directly. It was surrounded by a protective belt of theories and a skin of empirical claims all of which could be abandoned under pressure, if only to ensure the continuation of the hard-core. In his well known book, Proofs and Refutations (1976), Lakatos demonstrated the fruitfulness of applying such a perspective to mathematics, producing a compelling "rational reconstruction" of the Euler conjecture concerning the edges, vertices and faces in a polyhedron. His goal was to challenge the formalists who portrayed mathematics as "authoritative, infallible, irrefutable" (p. 5), and to elaborate the point that "informal, quasi-empirical, mathematics does not grow through a monotonous increase of the number of indubitably established theorems but through the incessant improvement of guesses by speculation and criticism, by the logic of proofs and refutations" (p. 5). Later, particularly with the introduction of computer-implemented and enhanced proofs, philosophers like Tymoczko (1979, 1984), Kalmar (1967) and others begin to document that mathematics too was subject to challenge and was quasi-empirical in nature.

All of these theories coalesced to change the view of science and, to a significant but lesser extent, mathematics, and to make them vulnerable to systematic change, revision, debate and rejection. All struggled to explain the twin processes of stability and change as they admitted relativism into the scientific and mathematical enterprise. And, all of them challenged a simplistic view of objectivity; in each case, the subjective, either as a psychological process or a sociological process was involved inexorably in the process.

In education, particularly in science education, these philosophical debates quickly influenced discussions of student learning. The classic paper by Karmiloff-Smith and Inhelder (1975), "If You Want to Get Ahead, Get a Theory" documented that student approach and observation was also theory laden. Driver and Easley (1978) applied such theory-driven views to suggest that students' misconceptions might be better looked at in light of conceptual frameworks in which there is an interplay between one's mental models and the sensory impressions of a phenomena. From this developed a robust series of research studies documenting student conceptions and how these exhibited persistence, pervasiveness, rationality, originality, and intellectual integrity. Hawkins, Apelman, Colton, and Flexner (1982) and Brousseau (1984) took the position that passage through these critical barriers or epistemological obstacles constituted an important part of learning. In particular, these pioneering researchers led their communities to recognize that much of what was being labelled as student error and misconceptions actually represented legitimate alternative viewpoints. These researchers challenged traditional forms of assessment and learned to listen to students, to propose conditions for conceptual change, and to investigate larger systems of cognitive structure. (See Confrey, 1990 for a review of the conceptions research.)

Piaget

The work of Piaget was exceptionally well-suited to form a conceptual basis to link the philosophy of science to learning theories. First of all, Piaget was a biologist who became a child development specialist, hence he naturally incorporated an evolutionary perspective into his theories. Philosophers striving to make sense of the history of science and developmentalists examining the history of children's ideas formed an intellectual bond.

Piaget was also an excellent candidate to form a bridge for he examined the development in children of fundamental organizing concepts such as space, time, and number and demonstrated the developmental changes that children proceed through given appropriate experience, time and support. The effect of Piagetian work and the philosophy of science was to emphasize the importance of epistemological issues and to challenge the assumption that children's worlds were simply inadequate or incomplete representations of adult worlds. Such contributions were necessary for the formation of a radical constructivist perspective in mathematics education.

In this paper, I distinguish between constructivism and radical constructivism. Constructivists argue for the importance of children's active participation in the building up of concepts. They reject the view that children's minds are blank slates, and they believe that there must be significant discussion and interaction around the variety of strategies that students propose. However, for them, the endpoint of instruction, the character of mathematical knowledge, is seldom questioned. Constructivists generally seek to reproduce in their students the same mathematical ideas that they themselves hold and that dominate modern mathematics. Little investigation is made of the meaning of the mathematical ideas through historical, cross-cultural or cross-disciplinary methods. Generally constructivism is replicative in its goals and only modestly revisionary. The methods of instruction are reformed, and the focus is more psychological than epistemological.

Radical constructivism is a theory whose roots lie in a rejection of illegitimate claims for epistemological certainty. If one accepts the critique that knowledge cannot be shown to represent reality in some iconic way, as a picture of the world, then one is left with a more subjective construction of reality, subjective in the sense that one abandons the effort to factor the human subject out of the process. Although the radical constructivist is relativistic in contrast to the realist, that relativism is tempered by stability which is achieved by the individual in relation to his or her experience. Others exert a significant influence on those experiences. The radical constructivist program assumes that the individual makes sense of experience in order to satisfy an essential need to gain predictability and control over one's environment. Many of the efforts of researchers in this tradition have been devoted to describing how the individual builds up (rather than passively acquires) knowledge of the world.

A Framework for Radical Constructivism

In Confrey (in press a), I argued that the radical constructivist program can be summarized by four "planks":

1. Genetic Epistemology. The construction of knowledge occurs over time; to understand an idea, one needs to examine its construction, ontogenetically and phylogenetically. Piaget explicitly rejected the view that "epistemology is the study of knowledge as it exists at the present moment; it is not the analysis of knowledge for its own sake and within its own framework without regard for development" (p. 1-2). This claim commits an educator to "creating the need" (Confrey, 1993) for an idea, rather than towards informing one of the contents of a knowledge claim. According to Piaget (1970), "The fundamental hypothesis of genetic epistemology is that there is a parallelism between the progress made in the logical and rational organization of knowledge and the corresponding formative psychological processes" (p. 13). This parallelism need not be assumed to argue for a recapitulation argument, that development follows historical routes, but argues only that historical routes are a rich source of diversity that can inform us about alterative developmental routes.

2. Radical Epistemology. "[K]nowledge does not reflect an 'objective' ontological reality, but exclusively an ordering and organization of a world constituted by our experience. The radical constructivist has relinquished 'metaphysical realism' once and for all, and finds himself in full agreement with Piaget who says, 'Intelligence organizes the world by organizing itself.'" (von Glasersfeld, 1984, p. 24). Or, citing Vico, "Veum ipsum factum – the truth is the same as the made." (ibid., p. 27), and "Human knowledge is nothing else but the endeavour to make things correspond to one another in shapely proportion" (ibid., p. 29). By these quotes, we see that radical constructivism has two parts: a) Constructivism rejects a picture theory of knowledge (that we are progressing towards an increasingly accurate view of the "way things really are"), and b) Constructivism entails a requirement that to know something is to act on it, so that knowledge consists of actions and reflection on those actions. "...[A]ll knowledge is necessarily a product of our own acts" (Confrey, 1990, p. 108).

Corollary 1: Recursive fidelity. Constructivism is subject to its own claims about the limits of knowledge. Thus, it is only true to the extent that it is shown to be useful or viable in allowing us to make sense of our experiences and to make predictions.

Corollary 2: Observer's presence. In every epistemological claim, an observer is present. Claims cannot be subjectless; a problem is defined by a proponent. When one accepts an epistemological claim, one is agreeing, or rather agreeing to agree, with the proponents. This means when we seek to speak of cognition, education, problem-solving, mathematics, or learning and teaching, we must take particular care to recognize the role of the observer in the description and analysis of the problem. In the radical constructivist research program, this has meant establishing clear methodological guidelines concerning the importance of "close listening" (Confrey, 1993), the careful conduct of clinical interviews and the articulation of models of student thinking (Cobb and Steffe, 1983). More recently, I have further revised this discussion to include the importance of acknowledging the role of the observer's perspective in the development of student voice and of the importance of using voice to aid us in our understanding of our own epistemological beliefs and commitments (Confrey, in press b).

3. Scheme Theory. The first plank specifies that knowledge can be only understood by examining its genesis. The second plank rejects the view that what is eventually asserted to be knowledge cannot be assured to be "the way the world really is." Both planks lead to the identification of learning, coming to know, as a critical site for investigation. As Piaget (has) said, "Nothing could be more accessible to study than the ontogenesis of these notions. There are children all around us. It is with children that we have the best chance of studying the development of logical knowledge, mathematical knowledge, physical knowledge and so forth" (p. 14). Furthermore, the genetic epistemological position (as stated by Piaget) is that "...knowing an object does not mean copying it – it means acting on it. It means constructing systems of transformations that can be carried out on or with this object. Knowing reality means constructing systems of transformations that correspond, more or less adequately, to reality. These are more or less isomorphic to transformations of reality.[1] The transformational structures of which knowledge consists are not copies of the transformations in reality; they are simply possible isomorphic models among which experience can enable us to choose. Knowledge, then, is a system of transformations that become progressively adequate" (p. 15). How can these transformations be understood?

For Piaget, "the operative aspect of thought deals not with states but with transformations from one state to another" (p. 14). This occurs first at the level of action, goal-directed activity.

When these actions accomplish our goals, we abstract from it. There are two kinds of abstraction: simple abstraction, derived from the object and reflective abstraction, derived from the action on the object. This is reflective in that it moves from action to operation, and in that it involves a "reorganization at the level of thought itself." (p. 18). "Reflective abstraction is not based on individual actions but on the coordination of actions" (p. 18). Operations are the result of reflectively abstracting actions. Operations possess four qualities: 1) they are internalized actions, 2) they are reversible, 3) they suppose some invariant and 4) they exist within a system of operations. Scheme theory is a way to discuss the development of stable and predictable courses of action. For Piaget, a scheme is "whatever is repeated or generalizable in an action." (p.34).

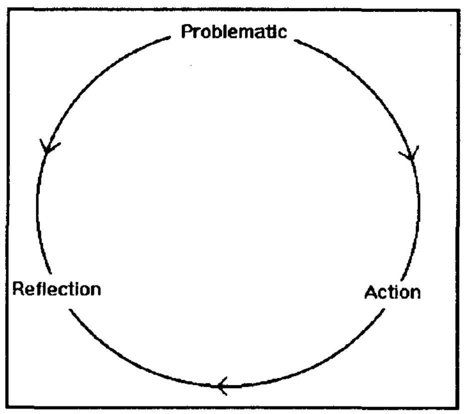

Figure 1

Schemes involve the anticipation and/or recognition of a situation. For the constructivist, a primary role is assigned to "differences." First, there was a difference, a perturbation, which is noticed. For the constructivist, the child must "emerge from embeddedness" (Kegan, 1982, p. 78) in that the newborn is considered to live in an objectless, continuous, timeless "world in which everything sensed is taken to be an extension of the infant" (ibid., p. 78). That is, there is no distinction between what is the infant and what is not the infant; for the infant, there is no boundary. The infant learns to create distinctions which lead to his/her "hatching out" (ibid., p. 80). Without distinction, there is no pattern. A difference creates a perturbation that is a call to action. This perturbation and action to resolve the perturbation are internalized through the process of reflective abstraction. The overall structure that is created, if the sequence of perturbation, action, reflective abstraction is repeated

[1] The phrase "more or less adequately" or "more or less isomorphic" could be interpreted to mean that Piaget intends that the adequacy or isomorphism could be "objectively" checked; a denial of the radical constructivist posifion. It could also be the case that he intends to indicate the distinction that von Glasersfeld has promoted between fit and match. Objective reality may bound our potential solutions in the way that a lock bounds the possible keys that "fit." This does not imply a unique and absolute correspondence in the way that the term "match" does. More or less then, could refer to "fit", an interpretation supported by his next statement that the transformational structures are not copies, but isomorphic models that are selected from experience.

until the action is formed into an operation, is labelled a scheme (Figure 1). For the radical constructivist, the unit of analysis is a scheme and its genesis and modification. A scheme for the constructivist provides what Vygotsky called the "investigable microcosm" (Wertsch, 1985, p. 193). In contrast to Piaget, for Vygotsky the investigable microcosm is the word.

It follows from scheme theory that the child proceeds through stages in development wherein the constructs may not mirror those of adults. Children's views are not miniature adult views; nor are their views missing pieces; nor are they inadequate for the purposes for which they have been built. Children's views are built differently, because the entire task situation may be viewed differently, and because a child's sensory perceptual world for building concepts is different from adults. A child's sensory and perceptual world responds to the world that the child has built, the experiences she or he has had, and the theories she or he has created. The implication of this is that when examining a child's performances, utterances, preferences, or ways of talking, one must not presume that one's own views of knowledge provide sufficient or adequate preparation for understanding that child. It also is suggested by this that the knowledge of the child is epistemologically intriguing, for it provides legitimate (and useful) alternatives to adult knowing.

The importance of scheme theory lies not only in the identification of scheme but in the recursive building potential created by knowing about schemes:

> ... We build this world for the most part unawares, simply because we do not know how we do it. This ignorance is quite unnecessary. Radical constructivism maintains ... that the operations by means of which we assemble our experiential world can be explored and that an awareness of this operating ... can help us do it differently, and, perhaps, better (von Glasersfeld, 1984, p. 18).

4. Model building and the Construction of Others. The discussion of "others" in von Glasersfeld (1982) evolves from a discussion of young children who imbue objects with life, and later give it up, because it does not add prediction and control.[2] He argues for the viability of creating "*models* of Others who ... come to be considered as perceivers, knowers, and intentioned actors, because such an investment does, indeed, make them more predictable" (p. 631). The emphasis on models is essential, for it emphasizes that no privileged access is accorded to our knowledge of others. We remain within our subjective confines, building viable models of the others in our environment through experience. As von Glasersfeld stresses, "... when a subject feels or says that it *understands* an Other, this implies no more than that the cognitive structures which the subject has attributed to its *model* of the Other have so far, or once more, turned out to be viable in the interpretation of the subject's experience of the Other" (p. 632). Researchers in the constructivist tradition have stressed that such models are the only possible product of investigations of children and that they are "the mathematics of children, even though they are not taken to characterize how mathematical knowledge of children really is" (Cobb and Steffe, 1983, p. 13).

I have argued that a constructivist view of instruction recognizes the key role for "reflection, communication/interpretation, and the use of resources" (Confrey, 1985). Rejecting the view that communication is the transfer of information, I stressed the importance of interpretative acts. More recently, constructivist researchers (Cobb, Wood, and Yackel, 1991) have proposed that in a community of learners, there develops a form of knowledge they describe as "taken as shared knowledge," a phrase used to indicate the tentativeness of communication. In our own work, we have preferred to use the term "agree to agree" to emphasize that it is the participants, and not some external observer, who agree to

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[2] This description ignores the viable beliefs of many cultures, including Naive Americans, who derive a cogent view of reality from the assumptions of life forces in many objects which most Anglo-Americans deny. To place such an analysis by von Glasersfeld within a Eurocentric perspective as a socio-historical event reinforces the importance of integrating socio-cultural and radical constructivist theories.

assume successful communication, until such time as those agreements are called into question by either party.

The Contributions of Radical Constructivism

The radical constructivist perspective has contributed significantly to reform in mathematics education. Its primary contribution has been to challenge the stark evaluative climate of the mathematics classroom. Instead of the quick labelling of student answers as right or wrong and the dismissal of differences, the radical constructivist viewpoint has legitimated diversity among individuals as a fundamental part of learning. It has also elevated the discussion of epistemological issues as central and problematic, rather than making the assumption that established knowledge is unassailable. As stated by Susan Jo Russell, constructivism has enabled us to recognize "the complexity of apparently simple ideas" (1993, p. 7). In part, this is because constructivism has demonstrated that understanding what children do requires one to decentre from adult perspective and to imagine how a child's actions and utterances make sense from the child's perspective. And in doing so, mathematics has become an issue of communication and interpretation, and not just the documentation of logical necessity. Finally, constructivists have been significantly effective in challenging the passive mode of learning, putting into practice the use of manipulatives, contextualized problems, the use of small group work, and the coordination of actions, operations and representations.

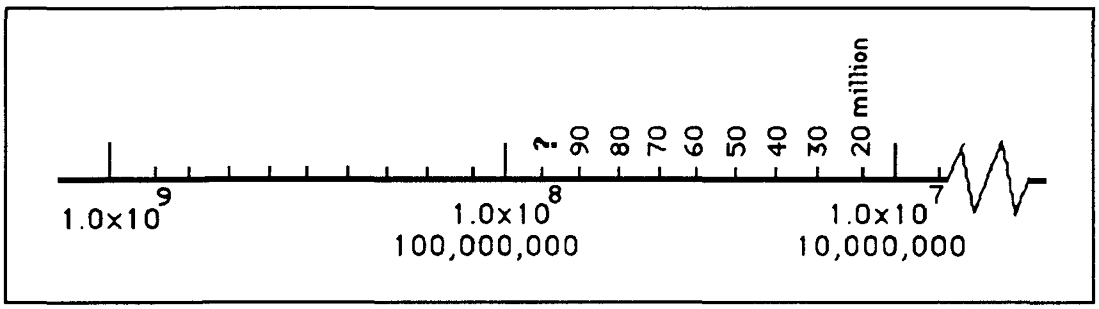

Figure 2

To demonstrate these ideas, consider the following example from earlier work by our research group (Confrey, 1991). We demonstrated how a college freshman named Suzanne built a number line to display a set of historical events from the Big Bang to the present. She did so by building a scale that had two different kinds of units (Figure 2). Her large units were marked in powers of 10 (multiplicatively), and her small units were in additive increments counting from the lower power of ten to the higher one (for example, from 10^7 (10,000,000) to 20,000,000, 30,000,000, 40,000,000 .... 90,000,000 and 100,000,000 (10^8). This gave her nine additive intervals between her powers of ten. At first, we evaluated her display as inadequate as it incorporated a change of units. However, on reflection, we realized that it had a great deal of legitimacy, and that it was we who were responding inflexibly. Our reasons for changing our opinion of the epistemological validity of her approach included: 1) her facility in working with intervals which cut across her multiplicative intervals, 2) her flexibility in moving among the representational forms (from scientific notation to extended decimal notation) and 3) the insights her display provided about scientific notation. Due to her work, we recognized that her visual display paralleled the arrangement of scientific notation which is segmented into the familiar counting numbers between 1 and 10 (like her additive units) and orders of magnitude (like her multiplicative units). Finally, her work set us an appropriate challenge, how to devise a task that would encourage a student to want to create a purely logarithmic scale.

A second example comes from research conducted by Russell and Corwin (1991)[3] and demonstrates the role of communication, interpretation and mathematical reasoning in the constructivist program. In a project entitled "Talking Math," project teachers worked on the construction of pyramids. They cut out and built pyramids with regular bases, triangular, square, hexagonal, octagonal from two dimensional networks. They were asked to formulate conjectures about the relationship between the number of corners and the number of edges and sides. In form, this does not appear to be a very different lesson than what would be advocated by a proponent of discovery learning; however, at this juncture the instructors departed from the discovery learning format. They encountered the following claim by teachers: "The top of a pyramid," claimed some teachers, "should not be counted as a corner. It is a point. Only those at the base count as corners."

The instructors encouraged the teachers to develop their arguments. Some of the arguments which evolved included:

1. A corner should only have three planes coming into it (which is true only of the base corners and the point of the tetrahedron where, as the teachers pointed out, any side could function as the base).

2. The handout defined pyramids as "having a point", thus legitimating their distinction.

3. Corners could be defined as formed by two lines (as in the corner of a sheet of paper) and thus each "corner" of the pyramid is really three (or more) corners.

4. Street corners, corners in a room, all have three planes coming together. However, from the outside, they look like points.

5. Before the pyramids were constructed, they were represented by two-dimensional networks. In each of these networks, the corners at the base appear quite different than did the "point" which appeared multiple times at the top of each triangle which protruded from the base.

Now, the instructors could have simply restricted the debates by introducing the formal term, vertex, and using it to apply to both points and corners. However, by not doing so, one witnessed teachers engaging in spirited debates which were thoroughly mathematical in their character and taught the teachers a great deal about forms of argument. Numerous teachers in the group followed up using the sheet with students and reported similar discussions in those settings.

Limitations of the Constructivist Perspective

It is difficult with a powerful theoretical perspective to differentiate between the qualities of the theory which are limited because of their failure to be successfully implemented and those which are limitations inherent in the perspective. I want to suggest three limitations of the constructivist program:

1. *Many constructivists assume an incremental view of knowledge construction.* As a result, most proponents of constructivism have focused attention on the elementary grades, to the neglect of the secondary and post-secondary instruction. This result is partially inherent to the theoretical perspective wherein a) Piagetian stages imply that there is a movement from concrete operations to formal, abstract operations with developmental level and thus, leave intact the beliefs that at the higher cognitive levels,

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[3] The author of the paper was at one of the teacher sessions at TERC about this problem, hence this report may include details not discussed in the bibliogaphic reference.

contextual influences recede and reliance on formal manipulation of symbols is judged as a more sophisticated way of thinking; and b) a rationalist view of knowledge development is assumed which includes the assumption that there are well defined paths to complex forms of thinking, and that the movement is always from simple to complex. Thus, for instance, in the research on the counting types to additive structures to multiplicative structures (Steffe, in press), we see evidence of a systematic path through counting types to unit types that is of increasing scope but seldom acknowledges competing structures and alternative approaches. I have argued against a singular approach in my own work on "splitting" in which I have posited complementary but independent roots of multiplication/division and ratio to those of counting (Confrey, in press d). In this argument, geometry plays a significant role as a divergent form of thought from number. An alterative view of knowledge development that makes it equally necessary to work at the secondary level would be to assume a less incremental view of knowledge development in which complexity can be lived in and comprehended with increasing depth. Within such a view, context does not simply create the purpose for the goal-directed activity, but creates participation structures that encourage increasing awareness of complexity (Sabelli, in press; Lave and Wenger, 1991; Smith, 1993).

2. *Constructivist approaches can be criticised for positing a universalist or essentialist view of cognition across classifications except age.* Viability typically seems to explain how the character of knowledge changes as a function of age, but less attention is paid to how viability must also lead to differences among children of different cultural background, race or gender. Constructivism has resulted in the documentation of diversity in student method, but little or no discussion exists in the literature to explain systematic differences among classifications of student participants according to culture, race or gender. One possible explanation lies in the tendency for the constructivist program to assert a heavy dependence on the autonomy of the individual. As I have written myself, autonomy is the backbone of constructivism. However, emphasizing autonomy can lead one to devalue or misjudge individuals who resist exhibiting independent judgment, preferring perhaps to gain group consensus, to avoid attention, or to show respect for authority. It is necessary to provide theoretical explanations of group or class-size systematic behaviours or results.

3. *Constructivism may lack an adequate theory of instruction.* In constructivist classrooms, the students are encouraged superbly to articulate their views and to explain their reasoning. However, the teachers, when required to make use of the diversity of ideas, find themselves at a loss. Teachers express a fear of telling–the belief that all constructivism commits them to refusing to inform the discussions with expert opinion or to bring the discussion to premature closure. Constructivism seems to assume that a theory of learning provides an adequate theory of teaching. As result, constructivist classrooms can lack direction and progress, and have been critically described by some as "sharing ignorance." Furthermore, constructivism seems to imply that all imitative behaviours imply rote learning, and are therefore bad. Such a perspective ignores the ways in which imitative behaviours can be transformed into meaningful behaviours, and how the act of imitation is a form of adult-child, expert-novice initiation. A theory of instruction might help to differentiate those occasions on which imitation becomes a charade and in which it is transformed into deeper understanding.

I pose these criticisms as challenges to us as constructivist theorists. And, I propose that elements in the socio-cultural perspective help to form responses to these criticisms.

Socio-cultural Perspective

In the past decade, Vygotsky's work has gained widespread attention and commendation. Indisputably, the work is worthy of careful study. And according to its own fundamental principles, the theory itself is a product of its historical-cultural situation, and in examining it, one needs to first understand it from within that time period and then consider how it might be modified in the light of current cultural and historical times. I will try to demonstrate in this paper that Vygotskian theory can support two opposing interpretations, one which fundamentally supports reform and one which can, in fact, undermine it.

To begin, a summary of Vygotskian theory is offered. Vygotsky proposed to develop a theory of intellectual development that would: 1) recognize a central role for social and cultural influences; 2) build upon the characteristics that separate humans from other animals; 3) create a Marxist psychology (Wertsch, 1985); and 4) contribute to the social program of making literacy accessible to all. It should be kept in mind that Vygotsky's contributions were made at a time when there was a sense of triumphant pride and confidence in the power of science to improve the quality of life, when literacy was seen as positive accomplishment, entailing only gains and no losses, and literacy was assumed to be accomplished in a relatively uniform manner across all peoples (John-Steiner, 1985). My discussion of Vygotskian perspective is organized in six major categories: 1) socio-cultural perspective, 2) Marxist influences of historical analysis and role of labour, 3) semiotics and psychological tools, 4) the dialectic of thought and language, 5) conceptual development and 6) learning and development.

A Framework for Vygotskian Theory

1. *Socio-cultural Perspective.* Vygotsky's central tenet was that socio-cultural factors were essential in the development of mind. In fact, for Vygotsky, the individual emerges from a socio-cultural context. All intellectual development including meaning, memory, attention, thinking, perception and consciousness, evolve from the interpersonal (social) to the intrapersonal (individual). For Vygotsky, "the social dimension of consciousness is primary in time and fact. The individual dimension is derivative and secondary" (Wertsch, 1985, p. 58). "... the very mechanism underlying higher mental functions is a copy from social interaction. All higher mental functions are internalized social relationships." (p. 66) Vygotsky sought to examine "how the collective creates higher cognitive activity in the child." (Vygotsky, 1981, p. 165)

This view entails a paradigmatic shift in the conceptualization of the learning child. The learning child is not viewed as an autopoietic (self-regulating) system who forms connections by modelling others as other self-regulating individuals. His/ Her very identity emerges from social/cultural relations. Vygotsky distinguished two lines of development, the natural and socio-cultural, and for him, it was the socio-cultural that distinguished humans from other animals. Theoretically Vygotsky argued "the two lines of change [the natural and the cultural] interpenetrate one another and essentially form a single line of sociobiological formation of the child's personality" (ibid., p. 41). Methodologically, Vygotsky's emphasis was on the impact of the socio-cultural on the natural. As Wertsch described this, Vygotsky's empirical work concentrated on "...the natural line as providing the 'raw materials' that are then transformed by cultural forces" (ibid., p. 43).

2. *Marxist influences: Genetic Analysis and the Role of Labour.* A second central tenet of Vygotskian theory was Vygotsky's commitment to create a Marxist psychology. Although this quality is being increasingly de-emphasized by Russian psychologists in light of the disintegration of the Soviet Union, its influence on the development of Vygotsky's theory was profound. In this paper, I will stress two components of Marxist theory: dialectical and historical materialism. As described by Cole and Scribner, "A psychologically relevant application of dialectical and historical materialism would be one accurate summary of Vygotsky's sociocultural theory of higher mental processes" (in Vygotsky, 1978, p. 6).

Historical materialism. To examine the development of cognitive thought, one must undertake historical analysis. According to this theory, one examines the conditions and the trajectory that produced a current state of an object, in order to know what that current state is. Vygotsky wrote that a positive picture of child, as opposed to a negative one specifying only what the child is lacking, "becomes possible only if we change our idea of child development in a fundamental way and if we take into consideration the fact that it represents a complex, dialectical process characterized by a multifaceted, periodic timetable, by disproportion in the development of various functions, by metamorphoses or qualitative conversion of one set of forms into others, by complex combinations of the processes of evolution and involution, by complex mixing of external and internal factors, and by the process of adaptation and surmounting difficulties" (ibid., p. 151). This commitment to "genetic analysis," similar to Piaget's "genetic epistemology," suggests that one must examine the genesis of the higher mental functions to understand them.

Dialectical materialism. Marx and Engels emphasized the central role of labour in cultural development. In their work, they argued that it was through the act of production that the truth of an idea was revealed. Davydov took this same position as he wrote, "Productive activity that concerns practical objects–labour–is the basis of all human cognition" (Davydov, 1990, p. 232). Engels himself wrote "The most highly essential and immediate basis for human thought is precisely *man's modification of nature*, rather than nature alone as such, and man's reason has developed according to how man has learned to modify nature" (ibid., p. 232).

Davydov, a follower of Vygotsky and a mathematics educator, stresses that it is through the process of labouring, through the transformations of the objects using tools that an object's incidental conditions are factored out, its invariances can be glimpsed and "their internal, essential properties–the *necessary* forms of their motion," are revealed (ibid., p. 234). The internal or essential characteristics of an object, objectivity in Davydov's meaning, "[i]n contrast to the external, has existence only in a relationship, has a *reflected* rather than an immediate being, a being mediated in itself" (ibid., p. 234).

From these quotes, one gets the sense of the centrality of the activity of labour on cognition for Marx, Engels and subsequently Vygotsky. Also, one learns that the internal character of the object is not a direct perceptual thing but a **mediated relational meaning**. Finally, tools, as the means of transformation of labour, possess a central role as both means of cultural transmission and as intimately associated with the results of labour. Vygotsky wrote, "If one changes the tools of thinking available to a child, his mind will have a radically different structure" (Vygotsky, 1978, p. 126). Built from this central emphasis on tool use in two ways: 1) he argued that linguists who try to understand language development without looking at the use of tools fail to recognize the interplay between the two systems of practical activity (as evidenced by the mastery of tools) and speech; and 2) he proposed that language is itself a type of **psychological tool**. He wrote, "*the most significant moment in the course of intellectual development, which gives birth to the purely human forms of practical and abstract intelligence, occurs when speech and practical activity, two previously completely independent lines of development, converge*" (Vygotsky, 1978, p. 24).

3. *Semiotics and Psychological Tools.* Vygotsky extended the mediational role of tools to psychological tools such as signs systems (language, writing, number systems). Vygotsky saw languages as playing a special role in the development of thought.

> The specifically human capacity for language enables children to provide for auxiliary tools in the solution of difficult tasks, to overcome impulsive action, to plan a solution to a problem prior to its execution, and to master their own behaviour. Signs and words serve children first and foremost as a means of social contact with other people. The cognitive and communicative functions of language then become the basis for a new and superior form of activity in children, distinguishing them from animals (Vygotsky, 1978, p. 28-29).

4. *The Dialectic of Thought and Language.* Briefly, Vygotsky's argument in thought and language developed as follows: Thought and language, argued Vygotsky, have separate roots. Speech, the basis for language, evolved out of gestures and affective responses. It is developed within the context of communication and social interaction. For instance, Vygotsky discussed the development of pointing, explaining it as a movement from grasping, an action, to pointing, a communicative effort to achieve the same end as grasping. "We consider this transitional gesture a most important step from unadulterated affective expression toward objective language" (Vygotsky, 1978, p. 35). For Vygotsky a child's babbling,

crying, even his first words are quite clearly stages of speech development that have less to do with the development of thinking than they are means of social contact.

Thought for Vygotsky, especially the development of logical thought, evolves from the child's activity "the child's experience with physical properties of his own body and the objects around him, the application of this experience to the set of tools, the first exercise of his budding practical intelligence" (Vygotsky, 1962, p. 46). Thus two lines of development, of thought which is non-verbal and speech which is non-intellectual development, then merge around age two. Although current research on infant capabilities suggests much more early development than was posited by Vygotsky, the underlying conceptualization remains robust. At this time, we see the development of verbal thought and inner speech, and this marks the transition whereby, "*The nature of the development changes*, from biological to sociohistorical" (ibid. p. 51).

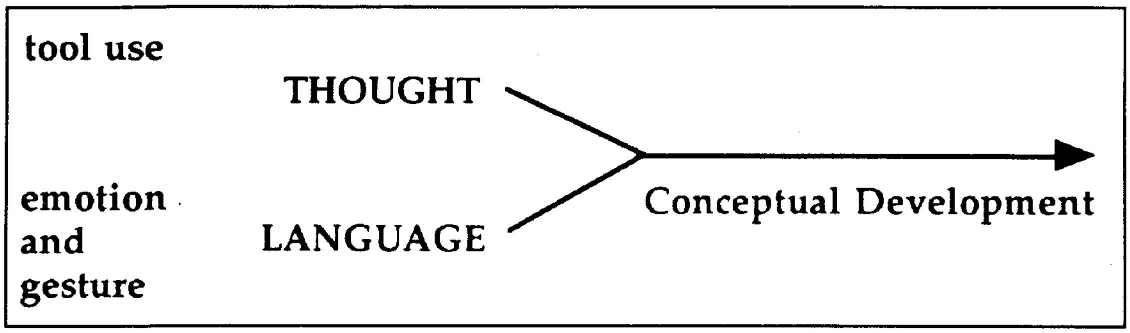

Figure 3

To understand Vygotsky's claim, one needs to recognize his intellectual ties to Hegel. Vygotsky was a dialectician–he believed that by pulling apart the distinct roots of thought and language, one could begin to understand how it is that they mutually transform each other.[4] However, in the end, he describes "the alloy of speech and action" (Vygotsky, 1978, p. 30), suggesting as does Hegel that the dialectic results eventually in a synthesis. Thus, ultimately Vygotsky postulated their unity.

> Although practical intelligence and sign use can operate independently of each other in young children, the dialectical unity of these systems in the human adult is the very essence of complex human behaviour. Our analysis accords symbolic activity a specific *organizing* function that penetrates the process of tool use and produces fundamentally new forms of behaviour (ibid. p. 24).

Thus, in Vygotskian theory, languages, sign systems, possess a central role in the development of higher cognitive thought. In summary, he wrote of a law of development: "A sign is always originally used as a means of influencing others, and only later becomes a means of influencing oneself" (in Wertsch, 1985, p. 92).

5. *Conceptual Development.* Vygotsky's theory of conceptual development is probably the arena of his research programme most in need of modification in light of current perspective. His empirical work was based on classification tasks using what we call attribute blocks (combinations of shapes, colour and size) which were classified into categories and assigned a one-syllable label. The interviewer turned up one block

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[4] A dialectic should not be confused with a dichotomy. A dicotomy implies polarization, conflict in the two positions and the need for choosing one or the other. A dialectic is a talking-back-and forth.

and revealed its label and asked the child to put the other blocks with the same label in a pile without looking at their labels. If there were blocks that did not belong to that category label, the interviewer turned one of these over and asked the child to continue trying. Compared to many similar studies, the interviewer in this study plays a more active role in challenging the child, and words play a key role in the investigation.

From studies such as this, Vygotsky postulated that concepts develop via heaps (unorganized categories) to complexes (family resemblance, collections, chain complexes and pseudo concepts) to true concepts. He warns that a pseudo-concept can appear to be the same as a true concept, however, it does not hold up to intense scrutiny. For Vygotsky, the movement from pseudo-concepts to true concepts must occur with the assistance of an adult. The process of developing a true concept comes for Vygotsky when the child begins to abstract out the concept.

Outside the experimental setting, Vygotsky argued that the way children pick up the adult use of language and use it correctly syntactically before a more complete conceptual development has been achieved is an indication of a pseudo-concept. This process was an essential stage of development, and it is based on imitation. And, of imitation he wrote:

> Children can imitate a variety of actions that go well beyond the limits of their own capacities. Using imitation, children are capable of doing much more in collective activity or under the guidance of adults. This fact, which seems of little significance in itself, is of fundamental importance in that it demands a radical alteration of the entire doctrine concerning the relationship between learning and development in children. (Vygotsky, 1978, p. 88).

6. *Learning and Development.* The changes Vygotsky anticipated in the relationship of learning and development included postulating an interaction between the two. In *Mind and Society*, he criticized soundly methods of "concrete, look-and-do methods" used with retarded students which reinforced their handicaps by "accustoming the children exclusively to concrete thinking and thus suppressing the rudiments of abstract thought" (ibid., p. 89). He criticized any instruction that lags behind development and argued, "The only good kind of instruction is that which marches ahead of development and leads it; it must aim not so much at the ripe as at the ripening functions" (Vygotsky, 1962, p. 104).

Zone of Proximal Development.

For Vygotsky, imitation is not a simple or mechanical process. He claimed, "To imitate, it is necessary to possess the means of stepping from something one knows to something new. With assistance every child can do more than he can by himself" (ibid., p. 103). This view of the central role of imitation led Vygotsky to argue for "a zone of proximal development."

As is well known, the zone of proximal development is the territory between tasks which a student can undertake successfully independently and those which require the assistance of an adult. Built from Montessori's notion of a "sensitive period", Vygotsky stressed the importance of adult-child interactions to propel development along.

Brown and Ferrers (1985) offer a description of the interactions between an adult and child in the zone of proximal development:

> Vygotsky's theory of cognitive development rests heavily on the key concept of internalization. Vygotsky argues that all higher psychological processes are originally social processes, shared between people, particularly between children and adults. The child first experiences active problem-solving activities in the presence of others but gradually comes to perform these functions independently. The process of internalization is gradual; first the adult or knowledgeable peer controls and guides the child's activity,

> but gradually the adult and the child come to share the problem-solving functions, with the child taking the initiative and the adult correcting and guiding when she falters. Finally, the adult cedes control to the child and functions primarily as a supportive and sympathetic audience... Teachers, tutors, and master craftsmen in traditional apprenticeship situations all function ideally as promoters of self-regulation by nurturing the emergence of personal planning as they gradually cede their own direction. (p. 282).

Extensional work by Wood, Bruner and Ross (1974) have argued that "Tutorial interactions are, in short, a crucial feature of infancy and childhood" (p. 89), and have described the concept of a scaffolding process "that enables a child or novice to solve a problem, carry out a task or achieve a goal which would be beyond his unassisted efforts" (p. 90). They stress however, that scaffolding requires that "a learner must be able to recognize a solution to a particular class of problems before he is himself able to produce the steps leading to it without assistance" (p. 90). It is also important to point out that the majority of the tasks in which a scaffolding process has been examined entail such activities as puzzle solving, pattern completion or blocks construction.

Scientific and spontaneous concepts

According to Vygotsky (1962) , schooling is where children are introduced to scientific knowledge. Scientific knowledge for Vygotsky is not so much knowledge of science as it is systematic or taxonomical knowledge. This knowledge is contrasted with spontaneous knowledge which is the knowledge a child develops with "face-to-face" meetings with a concrete situation. Vygotsky writes that "the development of the child's spontaneous concepts proceeds upwards and the development of the scientific concepts proceeds downwards" (p. 108), although he recognizes:

> ...[i]n working its slow way upward, an everyday concept clears a path for the scientific concept and its downward development. It creates a series of structures necessary for the evolution of a concept's more primitive, elementary aspects which give it body and vitality. Scientific concepts in turn supply structures for the upward development of the child's spontaneous concepts towards consciousness and deliberate use. Scientific concepts grow down through spontaneous concepts; and spontaneous concepts grow upward through scientific concepts" (ibid., p. 109).

He suggests that the development of a scientific concept "usually *begins* with its verbal definition and its use in non-spontaneous operations–working with the concept itself." With the spontaneous concept "he ... knows the object to which the concept refers ... but is not conscious of his own act of thought" (ibid., p. 108).

The Contributions of Vygotskian Perspective

Vygotskian theory makes a number of significant contributions to our understanding of intellectual development and mathematics education. In particular, it draws our attention to larger social structures in which educating is embedded. For instance, with mathematics, sociocultural perspective encourages us to consider the role of quantitative thinking in societal organization. For instance, historically, numeration can serve as an instrument of authority as in taxing, census-taking, distribution of financial resources, or the control of reproductive processes. Within the confines of the school, our attention is drawn to the role of mathematics for sorting, standardized evaluation and tracking. This orientation to larger cultural and social structures has been extended by Leont'ev in his work on activity theory. He wrote,

> ... human psychology is concerned with the activity of concrete individuals, which takes place either in a collective–i.e. jointly with other people–or in a situation in which the subject deals directly with the surrounding world of objects–e.g. at the potter's wheel or the writer's desk ... if we removed human activity from the system of social relationships and social life, it would not exist ... the human individual's activity is a system in the system of social relations. It does not exist without these relations. (Leont'ev, 1981, cited in Cole, 1985 p. 151)

Even in social constructivism where interpersonal interaction patterns and classroom routines and norms are carefully examined (Voigt, 1992; Bauersfeld, 1988; in press; Cobb, Wood and Yackel, 1991), there is only a modest discussion of how such interactional patterns reflect a wider societal view of the enterprise of educating in mathematics. Socio-cultural perspective thus sees the classroom as situated in society and the student as a participant in the classroom culture. Radical constructivism focuses on the individual as a self-organizing system, and works its way out towards the conception of a classroom and a society.

Secondly, Vygotskian perspective makes the role of tools in the construction of knowledge a first principle, and hence, the idea that knowledge will change as one's tools change follows as an immediate consequence. This frees one from the assumption that there are enduring concepts that exist apart from the tools of inquiry, and allows one to expect the shifts in conceptual knowledge accompanying different tool use. For example, when two groups of children were given the task of sharing 732 jelly beans among three people, one group chose to use Dienes blocks and another student himself forgetting what value each domino was worth. Hence, he created a more tabular form and listed underneath the unit value of each block. At first glance, the Dienes blocks appears to be a more sophisticated solution, but in fact, it only appeared so due to its easy connections to the decimal numbering system. Upon closer examination, one realized that the student who used the dominoes developed his ability to use whatever units he found most appropriate and hence strengthened an ability to anticipate a choice of units (l00's, 25, 105 and 2's). Notice that these units represent common cultural tools, in particular, coins (quarters, dimes and nickels) except for the use of two's which is part of the most common non-consecutive counting pattern for children, skip counts.)

Hawkins (1974) wrote warning of over standardization of and uniformity in manipulatives. In his view, part of doing mathematics is the overlooking of difference to discern and describe unity within variation. One could argue that the second student exhibited a greater degree of generality in his solution whereas the first child may have revealed a pseudo-concept that was overly dependent on the standardization of the tool. Either way, the diversity in the two solutions seems to be a clear indication of the difference in the tools selected. This attention to the role of tools in mediating knowledge is a clear focus of Vygotskian theory.

The focus on tools by Vygotsky also led him to propose that language is a form of psychological tool–influencing people in a similar way to how a tool influences production. Mathematics is often viewed as possessing the qualities both of a tool and of a language, and the Vygotskian framework can assist us in articulating the comparison and interplay between the two characterizations. For instance, in a classroom of third graders, the two problems were given: "How would you share 696 jelly beans among three children?" and "How would you share 174 jelly beans between two children?" In formal mathematics, the two problems would be considered equivalent, and yet in the school setting where the children were just learning about division, the problems worked out very differently. Some children chose to solve the problem with Dienes blocks and in this case, most did the first problem by splitting up among three: 6 flats, nine longs and six individual blocks. In the second case, they proceeded similarly, representing 174 with one flat, seven longs and four blocks, however, they faced a decision about how to share the flat (10 by 10) among three people. Some did this by segmenting 99 into 33 3's, and adding the one to make 75 blocks which they shared by dividing six longs into groups of two and trading in the final long for 15 blocks shared into piles of 5. A few did the problem by finding 3 50's in 174 (not so easily seen with Dienes blocks) and then sharing 24 among three. Some did the problem by switching the flat to 10 longs and then sharing seventeen longs and four individual blocks among three. In the group discussion, a term *nice parts* emerges as a useful description of how the children chose to segment the whole quantity for easier distribution.

Another group of students tried these problems using short division (no division algorithm had been taught to them in school, but some parents had introduced short division to the students for an earlier problem.) The first problem 3 divided into 696 yielded easily to the method, but the second problem, 3 divided into 174 did not. Many of these students, having abandoned the Dienes blocks on the first problem, were not able to successfully use them then to solve the second problem.

On reflection, it seemed to me that a tool-language distinction and dialectic is useful in interpreting this. The students who used the Dienes blocks were working in ways that were more immediately connected to their use of physical tools. The distance between their solution strategy and the problem interpretation was minimal as evidenced by their easy explanations ("nice parts") and demonstrations of the methods. The students who used the division algorithm were working more with a symbol system whose manipulation and use was more consistent with formal language use. The method was taught to them and they were imitating it. When imitation failed, they had few options to resort to and a return to the materials was not easy for them, because the material methods did not map easily onto the method of short division.

Vygotskian theory helps one to 1) recognize the legitimacy of both methods; 2) draw distinctions between the methods based on their character as a psychological versus a physical tool, and 3) consider how the symbolic (and semiotic) method drew on adult perspective and created a "pseudo-concept" that could be useful in certain situations, but could impede progress in others. However, contrary to constructivist interpretations, socio-cultural perspective asserts a value in having students in the class who could introduce the expert method into the classroom discussion, because it created for all the children a challenge to progress towards long division, a method they had heard about from older siblings and parents. And, it asserts the value of imitation. "Can we figure out a way to do the problem 3 divided into 174 without the blocks?" It seemed that this was perhaps an example of a situation in which Teaming was leading development in a very significant and appropriate way towards long division.

Limitations of Vygotskian Theory

As with constructivism, in a discussion of the limitations of Vygotskian theory, it is difficult to distinguish when the criticisms are of the theory and when they indicate a limited interpretation of the theory. And, especially when considering a theory which by its own principles establishes the cultural and historical precedents, one expects to reconsider a theory's interpretation in light of the current circumstances. In mathematics education, Vygotskian theory has the following limitations:

1. *Vygotskian theory may allow for the neglect or devaluation of concrete activity.* Vygotsky argued for two roots for conceptual development, thought and language. These two strands interweave with each other to lead to mature human development. Neither thought nor language are proposed as superior to each other, except that Vygotsky asserted that it is the introduction of the social-language component that creates the possibility of higher cognitive thought and differentiates humans from other animals. And since Vygotskian theory introduced the role of language in guiding intellectual development, empirically, it focused on language. In fact, Vygotsky (1978) argued for a predominance of speech over activity in the more developed forms:

> Initially speech follows action, is provoked and dominated by activity. At a later stage, however, speech is moved to the starting point of an activity, a new relation between word and action emerges. Now speech guides, determines and dominates the course of action (p. 28).

In developing the dialectic between thought and language, Vygotsky also introduces the idea of language as a psychological tool, to act to influence another. One can interpret Vygotsky as valuing both kinds of tools, but his description can also be argued to communicate that it is symbolic activity as psychological tool which penetrates physical tool use and not vice versa. That this results in a privileging of more abstract sign use over functional practical intelligence can be seen in Vygotsky's original work with Luria where cross cultural studies were investigated. In his categorization scheme, when subjects were given a hammer, a saw, a log, and a hatchet and asked to say which three go together, he and Luria devalued the response of a saw, a log and a hatchet. Wertsch indicated that non-literate populations, tended to resist the experimenters' "suggestions grounded in decontextualized word meanings an hierarchical relationships among them" (Wertsch, 1985, p. 34). Again when given, a glass, a saucepan, spectacles and a bottle, the categorization of the glass, the spectacles and the bottle was valued as made of glass whereas putting the glass, the saucepan and the bottle together on the basis of their practical experiences using these as containers was devalued.[5]

Later, Vygotsky (1978) rejected methods of teaching retarded children which relied exclusively on concrete methods. While rejecting an approach which eliminated abstract approaches in their entirety is consistent with the dialectic, ultimately, he argued for only a weak inclusion of the concrete, "Concreteness is now seen as necessary and unavoidable only as a stepping stone for developing abstract thinking–as a means and not an end in itself" (p. 88).

It appears that within Vygotsky there is an inherent tension between his dialectic Hegelian approach and his Marxist roots. As a Hegelian idealist, Vygotsky was raised as an intellectual, and the preferential valuing of intellectualism emerges intermittently.

2. *Advocates of Vygotskian theory may focus on and privilege language to the detriment of other forms of intellectual interaction and inquiry.* In mathematics, however, and in the sciences, educationally, we see a tendency to give definitions as though definitions were sufficient to guide intellectual development. As Vinner (1983) has demonstrated, such a basis for guiding mathematical development has proven inadequate. In research on 10th and 11th grade students in Jerusalem, 88% of the students could state the definition of function but of those students only 34% acted accordingly.

However, Vygotsky clearly doesn't intend for words to be weakly connected to concepts given his statements such as,

> There is every reason to suppose that the qualitative distinction between sensation and thought is the presence in the latter of a *generalized* reflection of reality, which is also the essence of word meaning and consequently that meaning is an act of thought in its full sense of the term. But at the same time, meaning is an inalienable part of word as such, and thus it belongs in the realm of language as much as in the realm of thought. A word without meaning is an empty sound, no longer a part of human speech. Since word meaning is both thought and speech, we find in it the unit of verbal thought we are looking for (1962, p. 5).

The implications of such a quote are that in interpreting Vygotsky we must insist on the injection of meaning into our discussion of words and recall that the meaning comes from the dialectic between thought and language and not as some have argued from language alone.

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[5] In addition, it should be carefully pointed out that at the time of Vygotsky, when the introduction of literacy cultures in which physical labor predominated, the penetration would have been from language into the use of physical tools. However, I suggest that at the current time when communication devices dominate so much of human interactions and computers are increasingly used to represent physical activities (computer games, simulations, computer design,etc.), there is a need to see how physical uses of tools can penetrate and guide language development. This is particularly the case in mathematics.

Social constructionism, not to be confused with radical constructivism or social constructivism (Bauersfeld, in press) or with constructionism (Harel, 1990) is a theoretical perspective expressed by Gergen (in press) and Shotter (in press) which is closely allied with Vygotskian perspective but which rejects the psychological dimension of Vygotsky. Vygotsky's interest according to Gergen is described as being in "mental processes of abstraction, generalization, comparison, differentiation, volition, consciousness, maturation, association, attention, representation, judgment, sign mediated operations and so on" (p. 10) whereas Gergen is described as being interested in "negotiation, cooperation, conflict, rhetoric, ritual, roles, social scenarios and the like" (p. 10). In social constructionism, language and language games are presumed to make up the whole of knowledge, as witnessed in the following quote from Gergen (in press):

> ... there is nothing about the nature of the world that demands, requires or necessitates any particular linguistic representations. In principle, then, we are free to use whatever configuration of sounds and marking we please on any particular occasion. In principle, this is no more a table before me than it is Gouda cheese or a griffin. In practice, of course, we are not free. By virtue of negotiated agreements widely shared within the culture, we agree to speak of it – dully perhaps – as a desk. Or to put the conclusion more bluntly, all that we take to be the case, – our propositional representations of every thing from physics to psychology, geography to government – gain their legitimacy not by virtue of their capacities to map or picture the world, but through processes of social interchange (p. 9).

If Vygotskian theory is used to draw attention exclusively to the verbal or written language of social interactions, then its influence on mathematical development will be detrimental. Furthermore, even if languaging is used more broadly to include figures and drawings, and symbolic notational systems, and computer software which creates semiotic microworlds, without relating these uses of language to physical activity, it will serve to diminish children's mathematical development. The dialectic must be made truly equal in the contributions of each part, with both acting as mutual guides to each other.

3. *There is a potential to use socio-cultural perspective to reintroduce formalism into mathematics.* This is because Vygotskian theory posits a discontinuity between spontaneous and scientific concepts. This discontinuity is argued to be due to the systematicity of scientific concepts and to their suspension in an orderly, logical set of relations. Bridging that discontinuity to bring students into alignment with scientific perspective is assumed to be the responsibility of the more expert other who does this by appropriating the student's goals until they mirror the goals of the adult. This description is arguably close to mathematics education as it is taught at the university. There a student is presented theorem-proof ad nauseam until the student loses his/her initial perplexity with the value of the enterprise and appropriates the goals of the teacher and becomes a producer of proof by his/herself. Without a more explicit theory of learning, Vygotskian perspective cannot distinguish between teacher-student interactions that will lead to a pseudo-concept and those which will lead to conceptual development.

4. *Vygotskian perspective avoids critical examination of the mathematics itself.* Many of the studies in the Vygotskian tradition focus on how the more capable other assists the novice to become able to independently carry out a task. Examples from mother-child interactions include puzzle-building (Wertsch, 1985) and peek-a-boo and picture book reading (Forman and Cazden, 1985). In these cases, the goals of instruction are typically well-defined and agreed upon. Imitation is viewed as an appropriate means to accomplishing the ends. There is little assumption that the ends themselves might need revision. In mathematics education, we have argued however, for the importance of reconsidering the outcomes of instruction. From close listening to students, we have revised our understanding of mathematics. There is little or no provision for such activity within the Vygotskian framework.

5. *Vygotskian perspective can limit rather than promote or protect diversity in a classroom.* In discussing tutoring from a constructivist perspective, we see authors such as Arcavi and Schoenfeld (1993) warn of the dangers of pursuing student method. In this paper, they document a student's method in finding an equation for a linear function from a table of values that deviates from standard procedure, but which is legitimated by a number of mathematicians. Because the student is initially introduced to equations of the form, $y = x + b$, the student establishes an approach of subtracting the x and y values. However, when faced with an equation of the form, $y = mx$, the student's method is challenged as she finds a linear pattern in the differences, rather than a constant difference. The authors recognize the legitimacy of the student's approach, but they warn of placing undue stress on tutors' knowledge and judgment and of "keeping the student on shaky or superficial ground". As a result, they question whether or not one should pursue a student's method. However, I would argue that their analysis of the student's method was unduly complex, because they relied on algebraic notation to reach the equation. A graphical approach makes it easy to see that whereas $y = x + b$ gives a constant difference between $y = x$ and $y = x + b$, the multiplicative term, $y = mx$ gives an arithmetic progression as a difference. Thus, they question, "running with the student's idea" (p. 11) rather than their own analysis of the mathematics.

I would argue that their analysis is actually more consistent with a Vygotskian than a constructivist analysis, in that the responsibility of the tutor is ultimately defined as "tying knowledge to well-established structures" (p. 11) (though it is unclear whose structures those are). Such an analysis ultimately undermines the radical constructivist approach and endorses a standard treatment for the topic. The concern I am raising is that there is little protection for diversity in the Vygotskian perspective, and no practical way to dissent from the traditional presentation. This seems to be a very serious problem for the theory.

Not all of the Vygotskian perspective leads in the direction of the suppression of diversity. The work on multiculturalism (Vera John-Steiner, 1985, 1991) and literacy has been used for exactly the opposite purpose. However, in mathematics education, the tendency to use Vygotsky to reinforce rather than to challenge the uniform presentation and development in mathematics points to a limitation in the theory. Perhaps it is because so many people assume a universality about the language of mathematics that is quickly challenged from a multicultural perspective.

FORGING A REVISED THEORY

When one considers the theories of Piaget and Vygotsky, one sees places in which their views conflict and when they complement each other. Drawing on their complements while trying to consider their conflicts, I will propose the outlines of a revised theory of intellectual development. As will become evident, I draw upon feminist scholarship in proposing these revisions.

Genetic Epistemology

Both theories share a commitment to the evolution of thought–and the need to look at the process of development to understand the current state of affairs. Both theoreticians possessed a keen interest in philogeny (historical development) as well as ontogeny (lifespan development), and although neither explicitly argues for a recapitulation theory, they both use historical analysis to enlighten their examination of development.

The implications of applying a genetic epistemological approach to mathematics and/or mathematics education are profound. One no longer accepts the formal logical relations as a sufficient warrant for an idea, but instead traces the route of its development over time and over place. Genetic epistemology brings one squarely into the Lakatosian territory of proofs and refutations, where the logic of discovery, the process of acceptance or rejection by the discipline and the logic of justification are all equally explored. Knowledge is not established by examining the immediate facts of the case, but by examining the position of a claim within a theory, by considering its contextual roots and its path of transformation.

For example, an introduction to pi might not only include the calculation of its values and its use in calculating the area and circumference of a circle, but also an exploration of the genesis of pi. In the current introduction to pi, students are taught that pi is a constant and for all circles, the circumference equals $2\pi r$. Quickly the discussion turns to the numerical approximations for pi. Little or no explanation for the development of pi is offered.

An approach through genetic epistemology might begin with the recognition that all circles are similar. Consider the claim that all squares are similar; we find little surprise in an argument asserting that the ratio of the perimeter to a side is 4:1 for all squares. Would not an important step in a genetic epistemological presentation of the ratio of the circumference to the diameter (a ratio we label pi) be the expectation that this ratio should be constant for all circles?

Not all assertions of genetic epistemology are compatible, however, and the distinctions, need examination. For instance, Lakatos' rational reconstruction has inherent in it some remnants of Hegelian idealism. Lakatos (1976) wrote,

> Mathematics activity produces mathematics. Mathematics, this product of human activity, 'alienates itself' from the human activity which has been producing it. It becomes a living, growing organism, that acquires a certain *autonomy* from the activity which has produced it, it develops its own autonomous laws of growth, its own dialectic. The genuine creative mathematician is just a personification, an incarnation of these laws which can only realize themselves in human action. Their incarnation, however, is rarely perfect. The activity of human mathematicians, as it appears in history, is only a fumbling realization of the wonderful dialectic of mathematics ideas. (p. 146).

Lakatos' editors included a footnote to this section which suggested that Lakatos might have modified this statement somewhat had he the opportunity to edit this section himself, however, they also acknowledged that Lakatos did want to admit an "existence" to problems independent of human recognition.

Likewise, in discussions in radical constructivism, there is occasionally an ambiguity about whether what is evolving and subject to adaptation and equilibration is a system of ideas or is a person's conceptual structures. Viability as a quality of endurance, adaptation, comprehension and survival is applied sometimes to the idea itself, as a product of human activity, and at other times, as a component of intelligence as it is evidenced in human action. This is particularly an issue in mathematics where research is often focused on the evolution of a mathematical concept as an isolated entity.

To illustrate this, let's return to the example of pi. Our analysis has led us to explain first why one would believe that pi as a ratio describes all circles. However, it still ignores the question of why one would seek out pi. And, it neglects the fact that for the Greeks, pi was not a number. The history of number and the history of geometry evolved separately, so that pi was originally a ratio that expressed the invariance across proportions such as $C_1 : R_1$ and $C_2 : R_2$ (where C is circumference and R is radius). Finally, our analysis avoids the discussion of why one would wish to find a way to describe that ratio differently than as the ratio of circumference to diameter. The answer lies in two types of cultural activities. The Greeks sought to rectify arc length, that is, to find a straight line equivalent for a given arc, that is, to measure a circumference using a straight line. Not only could one thereby predict circumference given a radius, but one could predict the length of a path of a rolling object. If one creates a mechanical device that hooks up a linkage to a wheel, one needs to be able to move easily between measures (or movements) of curvature and measures (movements) of line segments. One sees in this discussion an even deeper application of genetic epistemology, one that places the investigation squarely within the productive activities of a culture.

The socio-cultural perspective avoids detaching conceptual development from the knower and eradicating the vestiges of Platonic idealism by rejecting the idea of universal concepts. It can be used to legitimate the idea of mathematics differing across cultures and across time. Thus, the genetic epistemology one seeks to include in a revised theory needs to allow for competing, independent strands of development, as well as establishing long term trends that are well integrated into the cultural activities of the time.

In any view of genetic epistemology, one must decide one's position on what constitutes progress. In this revised theory, I would argue for a coherent view of progress locally (both geographically and temporally) but not universally or eternally. To understand conceptual change over longer time periods, one must posit a view of paradigmatic shifts and examine how they come to degenerate, to compete and to replace or be replaced among communities of scholars.

Paradigmatic Metaphors

Margaret Masterman (1970) has argued that one way to capture the sense of a paradigm is to examine its metaphorical commitments. Metaphors have the charming quality of drawing upon elements of tools and physical interactions and those of language and the turn of a phrase. The metaphors underlying Piaget and Vygotsky present an interesting contrast. Piaget, being a biologist, (as well as a developmental psychologist, a philosopher etc.) chooses to rely on the metaphor of evolutionary biology to explain the development of knowledge—and his processes of assimilation, accommodation, equilibration etc. are woven into this metaphor. The human being is viewed as a self-organizing organism who regulates her/his behaviour by acts of problem solving to restore equilibrium. Von Glasersfeld has also committed himself to this metaphor by emphasizing the role of viability in explaining the durability of knowledge. Humans seek to gain prediction and control of their environment and those schemes that produce these outcomes become stable; those that do not fall away.

This is a powerful metaphor for it removes one quickly from the assumption that what is known exists independently from the knower, and that knowledge remains eternal. It allows for a pragmatic view of knowledge in establishing structural and functional relationships, and limits the relativism by placing it larger system of evolution. The metaphor creates powerful ties between one's interaction with one's environment as a system of constraints. Finally, evolutionary biology as a metaphor encourages one to treat development as a metamorphic process of changes and hence to observe carefully how a child at any particular development period behaves in relation to his or her environment.

Using evolutionary biology as a metaphor makes examination of the metaphor itself typically problematic. Our attention is drawn to the constructs in the metaphor while the metaphor itself is taken for granted. This leads us to neglect the fact that evolutionary biology is itself a historical/cultural artifact. It would distract our attention from many issues that feminists have recently demonstrated, for instance how in evolutionary biology has often ignored the role of cooperation in the evolutionary process, stressing the more androcentric trait of competition. And the issues of sociobiology themselves have only entered biology slowly as evolutionary theory has had difficulty in explaining behaviour which is contrary to the survival of the individual but valuable for the group. This concentration on the individual as a microcosm of the species tends to turn one's attention away from the collective behaviour. We can avoid taking the metaphor for granted by placing it in a dialectic with other metaphors.

In contrast to Piagetian metaphor, in Vygotskian theory, we see the Marxist construction of knowledge through labour and production as the primary metaphor. It is through human labour, and the use of tools to create products, that knowledge is constructed. For instance, Vygotsky (1978) refers to Marx, stating, "Marx cites that definition when speaking of working tools, to show that man uses the mechanical, physical and chemical properties of objects so as to make them act as forces that affect other objects in order to fulfil his personal goals" (p.54). Knowledge, according to such a conception, describes the invariances found as a result of the transformation of products through labour. Vygotsky described his commitments to this approach:

> The keystone of our method ... follows directly from the contrast Engels drew between naturalistic and dialectical approaches to the understanding of human history. Naturalism in historical analysis, according to Engels, manifests itself in the assumption that only

> nature affects human beings and only natural conditions determine the historical development. The dialectical approach, while admitting the influence of nature on man, asserts that man, in turn, affects nature and creates through his changes in nature, new natural conditions for his existence. This position is the keystone of our approach to the student and interpretation of man's higher psychological functions and serves as the basis for the new methods of experimentation and analysis that we advocate (ibid., p. 60-61).

A commitment by Vygotsky to historical materialism entails not only an interaction between man and nature, but also a subjugation of nature to man's control and domination. Thus, Engels' concept of human labour and tool use as the means by which human beings change nature and, in doing so, transforms themselves is reflected in Vygotsky. Cole, John-Steiner, Scribner and Souberman (1978) acknowledge this relationship and quote Vygotsky:

> Vygotsky exploits the concept of a tool in a fashion that finds its direct antecedents in Engels: "The specialization of the hand – this implies the *tool* and the tool implies specific human activity, the transforming reaction of man on nature", "the animal merely *uses* external nature, and brings about change in it simply by its presence; man, by his changes makes it serve his ends, *masters it*. This is the final distinction between man and other animals" (in Vygotsky, 1978, p. 7).

The implications of this underlying metaphor are both positive and negative. It is this metaphor that Vygotsky extends to assert that language and other semiotic tools are also subject to historical and cultural forces of change. This inclusion of semiotic tools as psychological tools also allows Vygotsky to suggest that labouring and production create the basis for consciousness.

A labour and production metaphor permits Vygotsky to reexamine development and assert that a classic misconception was to liken development to embryology, and thus to assume the presence of all adult characteristics in embryonic form. His rejection of biological maturation as the basis for development was essential to establish a central role for social interactions. Recall that major goal of Vygotsky is to distinguish humans from other animals – and it is by replacing biological metaphors with labour and production that allows one to escape from the determinism of biology to the social constructivism of activity theory.[6] His choice of focus on language, assumed at the time to be a uniquely human accomplishment (except in parrots who can imitate only) confirms and supports his choice of metaphor. In the Vygotskian metaphor, it is human society and culture that create knowledge, it is the records that we chose to pass on as received knowledge. Our skills in this set us apart from other species. As written by Wood, Bruner and Ross (1976) in their article that introduces scaffolding, "Our species, moreover, appears to be only one in which any 'intentional' tutoring goes on" (p. 89).

These shifts entail powerful accomplishments, however, they also contribute to a view of Vygotsky which accepts and promotes human differentiation from nature. Human beings are assumed to be not quite but almost outside of nature, and their capacity for social organization and for teaching is assumed to far exceed that of any other species. Recent findings in sociobiology challenge such assumptions. And the extensive damage humankind has exerted on the environment has resulted in increased awareness that man cannot consider the environment to be casually subject to his force and transformation, but must learn to view the environment as a resource to be preserved and valued. If one simply contrasts the word "resources" with the Marxist view of "materials", one captures the difference between the spirit of these two metaphors of evolutionary biology and labour/production.

These two theories then express two powerful metaphors for understanding humanity and for modelling and investigating human development. We experience ourselves both as biologically develop-

[6] Note this use of biology seems to be deterministic and naturalisfic rather than developmental or ecological.

mental beings and as productive members of a collective enterprise. Hence, neither can be eliminated from our consideration. It seems obvious that we must consider both to contribute to our view of humanity and its development.

This suggests that both theories are necessary, and that the challenge is to integrate them. I would like to suggest that what has been typically done in the debate in mathematics education is to view Piaget's as an individualistic theory and Vygotsky's theory as a theory about society. Both descriptions seem woefully inadequate and simplistic. Neither description does justice to the strength of the insight of the theorist.

Alternatively, I would propose that both theories lack attention to a fundamental characteristic, of human development, that allows one to avoid placing the individual in tension with the social. This missing component is the role of reproduction in the theories. I claim that neither theory pays adequate attention to the importance of nurture and reproduction in human development.

First of all, the labour and production metaphor focuses on how knowledge evolves from the transformation of objects through the use of tools in the activity of labouring. This seems clearly an insufficient description of parent-child interactions where knowledge is achieved through the guidance of children into adulthood. The metaphor of reproduction that I am suggesting extends far beyond the act of birthing, just as Vygotsky has extended the idea of tool far beyond the physical tool to include psychological tools. Feminist scholars such as Jane Rowan Martin (1985) have argued for the importance of a broader definition of reproduction, writing, "Discussions about marriage, home, family are missing as are discussions about society's *reproductive* processes – a category I define broadly to include not simply conception and birth but the rearing of children to more or less maturity and associated activities such as tending the sick, taking care of family needs, and running a household" (p. 6).

By this argument, education is itself viewed as a nurturing process as much as it is a preparation for work. Caring for the development of the child is more than developing work-related skills; it is nurturing a curious, creative, well-adjusted child, capable of responsible and satisfying interactions with others. It is ironic that to date in education (outside of feminist scholars such as Noddings, 1984; Martin, 1985 Laird, 1988), We have appropriated the language of reproduction, (conception, development, reproduction), but severed their ties to nurture and care. We have coined a term called "social reproduction" and used it to describe the unconscious replication of cultural norms across generations. In my use of reproduction, I seek to disengage it from its implications of unconscious duplication and establish its ties to nurturing and growth.

If one had the opportunity to hear Davydov (1993) speak at the national meeting of the American Educational Research Association in Atlanta, one heard a description of Vygotsky in which nurture was used frequently. This term is absent from the translations of Vygotsky that are currently in English. Davydov attributed his use of nurture to a newly discovered publication of Vygotsky that had been censored. One also heard in this presentation little reference to Marxist roots in Vygotsky, emphasizing far more his connections to Hegel and the dialectic. These changes in presentation, whether due to the release of new work or to the disintegration of the Soviet Union and the rejection of Marxism, may signal changes in the interpretation of Vygotsky in directions compatible to those which I am proposing. Vygotsky, I am suggesting, recognized the importance of adult-child interactions. However, his reliance on the Marxist metaphor limited the scope of those interactions to that of production; I am arguing for its expansion to include also the metaphor of reproduction.

Furthermore, I would argue that despite Piaget's emphasis on evolutionary biology, his treatment of territory signalled by the metaphor of reproductions also inadequate. Firstly, he largely neglects many forms of adult and child interaction in development including imitation. Secondly, he follows Kant in arguing that one's understanding of space, time, number and causality create the fundamental fabric of cognition, and as such he elevates mathematics and physics to the highest plane, and ignoring the importance of human connections. Thirdly, he models mind as the embodiment of abstract mathematical structures. And, finally, as a result, a child's understanding of reality emerges from formal, logical characteristics rather than from simultaneously building conceptions of other human and non-human living beings.

The model that I would put forth would hold evolutionary biology as the umbrella metaphor, and signal humanity's placement in the full ecology of the earth. As a means of knowledge construction and accomplishment, it would be placed within the framework of genetic epistemology. Subsumed within the evolutionary biology metaphor would be the two metaphors of labour/ production and reproduction in a dialectic relation. Each of these sub-metaphors would have a feedback loop to the evolutionary metaphor to signal the need for critical examination of this umbrella metaphor from the perspective of the sub-metaphors. For instance, an examination of whether the mechanisms of selection and variation are sufficient to account for human development would be encouraged by such a feedback loop. See figure 4.

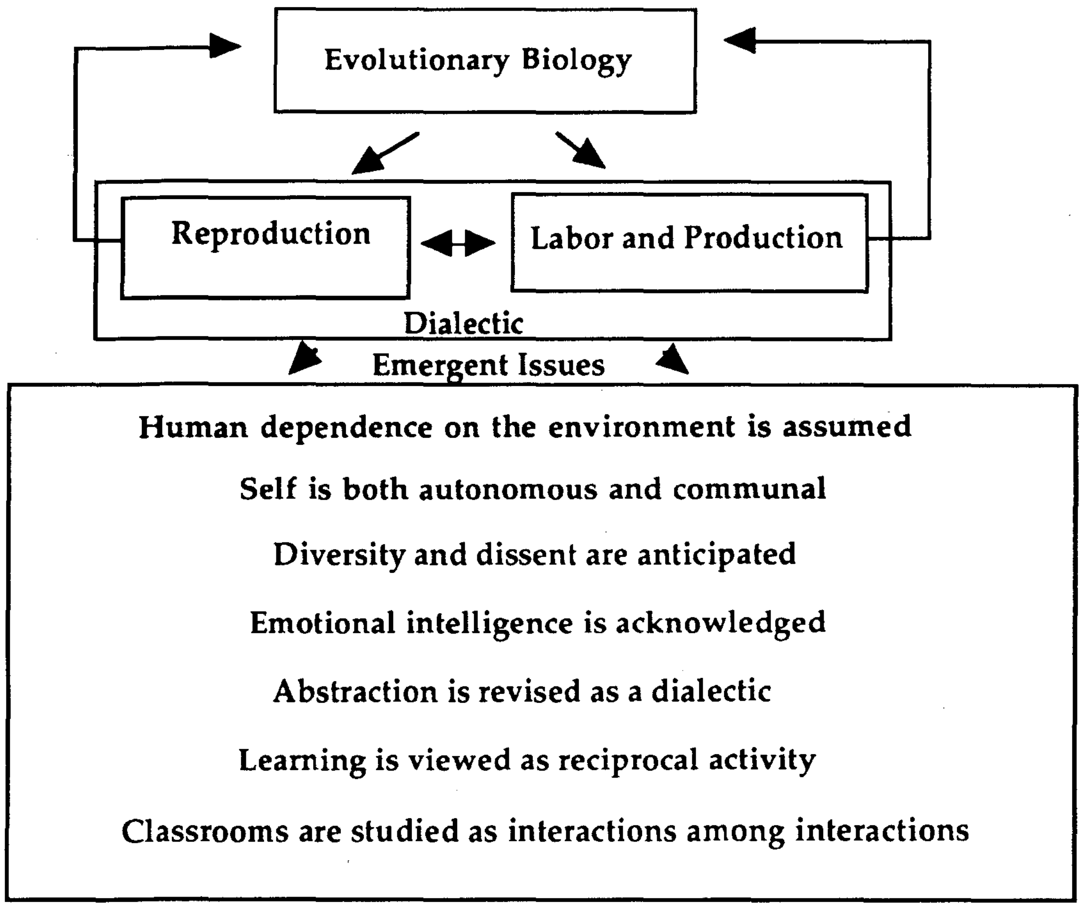

Figure 4

A Framework for a Revised Perspective

From this revised perspective on human intellectual development, a set of issues emerge which have been accorded less than adequate examination within the Piagetian and Vygotskian perspective:

1. Human development depends on the environment.

2. The self is both autonomous and communal.

3. Diversity and dissent are anticipated.

4. Emotional intelligence is acknowledged.

5. Abstraction is reconceptualized and placed in a dialectic.

6. learning is viewed as a reciprocal activity.

7. Classrooms are studied as interactions among interactions.

1. *Human development depends on the environment.* Subsuming the labour/production and reproduction metaphors under the evolutionary biology metaphor makes the statement that in all aspects of educating, one must educate for a global society that includes living and non-living things. It rejects the exclusive perspective in Vygotsky, inherited from Engels' dialectic materialism that views nature as subject to man's dominance and mastery. It suggests that as we consider the materials of labour, we recognize that those materials are also the limited resources of the environment. It obligates us to consider how we change as the environment changes. It reminds us as we seek to mould the environment to meet our needs, we need to respect that other living creatures share our dependence on this environment. Labour and production may be our gauge of progress, of movement towards the accomplishment of human goals, while reproduction brings into our awareness the cycles of human life, and the need to create an enduring and sustainable existence. Just as Stephen J. Gould (1987) has written, history is composed to two dichotomous views of time: "time's arrow views history as an irreversible sequence of unrepeatable events"; while time's cycle sees "apparent motions as parts of repeating cycles" in which "time has no direction" (p. 11). The metaphors of labour/production and reproduction, like time's arrow and time's cycle, place human development into directional and cyclic progressions, only as a dialectic, rather than as a dichotomy.

Vygotsky stresses in his theoretical work the distinction between man and other animals as the basis of higher cognitive development. Placing his work within the setting of evolutionary biology allows one to examine human development not only in terms of our differences from other animals, but in terms of our similarities and common interests. It also encourages the consideration of a more diverse set of beliefs about the capabilities of others (animal, plant and inanimate objects) as regards language, thought, social behaviour and spirit. Fundamentally, embedding human development within evolutionary biology warns us against what Marilyn Frye (1983) has described as "the arrogant eye", the view that "man is invited to subdue the earth and have dominion over every living thing on it. With this view, man sees with arrogant eyes which organize everything seen with reference to themselves and their own interests (p. 66-7). The placement of the reproductive metaphor along side the labour/production metaphor further argues against the use of arrogant eyes to dismiss or diminish female models of development while elevating male ones (Gilligan, 1982; Brown and Gilligan, 1992).

2. *The self is both autonomous and communal.* In Vygotsky, the development of self is described through the act of internalization of social norms. Wertsch (1985) described this as the process by which a higher level process moves from the realm of the interpersonal to the intrapsychological was called internalization. Essentially, Vygotsky argued that "It is necessary that everything internal in higher forms was external, that is, for others it was what it now is for oneself" (in Werisch, p. 62). He saw the act of internalization as transforming: "it goes without saying that internalization transforms the process itself and changes its structure and functions." According to Leont'ev, a student of Vygotsky, "consciousness is a product of society: it is produced. . . Thus the process of internalization is not the *transferal* of an external activity to a pre-existing, internal "plane of consciousness": it is the process in which this internal plane is *formed*" (ibid., p.64). Internalization is " the process of gaining control over external sign forms " (ibid., p. 65).

In the constructivist framework a construction of self is frequently developed in relation to objects. In a careful description of this development of self, von Glasersfeld (1978) distinguishes the construction of self as: 1) "part of one's perceptual experience" and 2) the locus of the perceptual (and other) experiences I am having" (p. 46). The first construction entails creating a differentiation between one's own body and other perceptual items. Watching my son learn to put his hand in his mouth provided a clear illustration that this is learned gradually. At age five months, he began to like to suck on his hand. For days, he worked on bringing his hand to his mouth. At first, he would simply find his hand in his mouth. Then he began to try to bring it there. He would see it in front of him, but he could not seem to bend his elbow intentionally to bring it to his mouth. This was puzzling, because the motion of bending the elbow was not a new one, for he used it regularly in reaching for objects. He solved the challenge in an interesting fashion. If he wanted to bring his right hand to his mouth, he would reach across with his left hand and pull his right hand to his mouth. His visual attention would remain on the right hand to be brought to his mouth. His other hand was not focused on and it would bring the desired object, his hand, to him. After a week more, he could bring the desired hand directly to his mouth without needing the other hand. The difficulty he had in bringing his arm to his mouth can be interpreted as evidence that it was not carrying out the physical action which challenged him, because this movement was not a new one, but doing it intentionally, reflexively controlling his own movements A sense of self is created as the child constructs a permanent entity that coordinates among sensory signals and gains control of one's movements, that is, gains motor control over visual items. It is clear that motor control over the self contributes to a child's sense of self.

The second construction in von Glasersfeld (1978) develops the idea of a self-regulating system, "the concept of an invariant that arises out of mutually or cyclically balancing changes may help us to approach the concept of self" (p. 60). This invariance, he adds, is not a steady resistance but "the invariant the system achieves as in a feedback loop, we find the present act pitted against the immediate past, but itself already on the way of being compensated by the immediate future. The invariant ... consists in one or more relationships, and relationships are not *in* things but between them. If the self ... is a relational entity, it cannot have a locus in the world of experiential objects. ...It manifests itself in the continuity of our sets of differentiating and relating and is the intuitive certainty we have that our experience is truly ours" (p. 60).

The constructivist position presumes that a self develops through one's experiences with the physical world and this self is discussed much more extensively than is the self that develops in relation to others. Kegan (1982), however, proposes a description which is both compatible with von Glasersfeld but also draws on the feminist development of relations. He describes the evolution of the scheme of "object-relations" by first pointing out the etymology of the word, object, as *ject*, to move or throw, and "together with *ob* is the motion or consequence of 'thrown from' or thrown away from" (p. 76). Constructing a scheme of object relations is, in the terminology of Kegan, "a motion, the motion of 'throwing away from' of differentiation, which creates the object, and the motion of integration, which creates the object relation" (p. 81).

Kegan's description of object relations complements and enriches that of von Glasersfeld. In it, he emphasizes that the construction of objects, up to eighteen months, might be better understood as the evolution of the baby and object relation, and that together with this evolution of objects as external from themselves is a loss and anxiety for the child of its own organization, shown up as "separation anxiety". "Emergence from embeddedness involves a kind of repudiation, an evolutionary re-cognition that what before was me is not-me" (p. 82). Kegan's description achieves a fundamental integration of cognition and affect, "because all objects are themselves the elaboration of an activity which is simultaneously cognitive and affective" (p. 83). He defines affect as "essentially phenomenological, the felt experience of a motion (hence,e-motion)" (p. 81). Kegan bridges the emotional aspects of psychoanalysis with the cognitive aspect of Piaget to declare a fundamental principle, basic to constructivism:

> It is the greater coherence of its organization which is the presumed motive (White, 1959), a transorganic motive shared by all living things. A more cognitive-sounding translation of the motive is to say that the organism is moved to make meaning or to resolve discrepancy; but this would not be different than to say it is moved to preserve and enhance its integrity" (ibid., p. 84).

Kegan suggests that the combination of differentiation and integration yields a lifelong theme which David Bakan called "the duality of human experience, the yearning for 'communion' and 'agency'". The desire to preserve independence or autonomy is counter-balanced in turn by "the fear of being completely unseparate, of being swallowed up and taken over, and the fear of being totally separate, of being utterly alone, abandoned, and remote beyond recall" (ibid., p. 107). This balance between the acts of differentiation and integration provide for a theory which:

> recognizes the equal dignity of each yearning, and in this respect offers a corrective to *all* present developmental frameworks which univocally define growth in terms of differentiation, separation, increasing autonomy and lose sight of the fact that adaptation is equally about integration, attachment, inclusion. The net effect of this myopia, as feminist scholars are now pointing out (Gilligan, 1978; Low, 1978), has been that differentiation (the stereotypically male overemphasis in this most human ambivalence) is favoured with the language of growth and development, while integration (the stereotypically female overemphasis) gets spoken of in terms of dependency and immaturity. A model in pursuit of the psychological meaning and experience of *evolution*–intrinsically about differentiation *and* integration–is less easily bent to this prejudice (p. 108-9).

The placement of the metaphors of labour and production with that of reproduction helps to remind us to balance the ideas of autonomy and of connections in our interpretation of models of cognitive development. It allows one to assert that greater coherence can be a goal of intellectual development and not just active control or manipulation.

3. *Diversity and dissent are anticipated.* As discussed earlier, Vygotskian theory seems to offer no way to explain and support invention, creativity and dissent. Without a concept of autonomy, one cannot encourage students to invent new approaches, or to challenge existing ones. Educating for a secure and sound development of one's own potential for acting and reflecting is a basic quality in Piaget but its articulation in Vygotsky is limited. As asserted by Piaget, in the revised theory, the individual is viewed as more than the internalization of society's norms, but as also a product of his/her own experiential path and unique activities of sense-making.

In order to acknowledge the importance of diversity, a theory of intellectual development must assert the value of multiple views. The placement of the theory within a biological evolutionary framework allows for this as a form of biodiversity. Having a broad variety of intellectual perspectives from which to select from is arguably the best assurance of more viable conceptions. The selection procedure is the means of assessing the endurance of the diverse proposals. Hence, selection functions as the framework by which coherence is assured.

I criticized radical constructivist views for their failure to recognize a person's placement in a classification other than as a member of a developmental age-group. In today's multicultural societies, it seems imperative to recognize one's identity as a member of many different sociological groups. Each of these groups has its own identity, values, norms and means of acting. A theory of intellectual development must easily handle these multiple identities. To this end, it seems important to revise the view of autonomy expressed in radical constructivism. Such a revision would start by acknowledging the view that the self is constructed as a viable actor in accomplishing one's purposes. It is physically the most immediate actor, in that we can control our physical actions. However, there is nothing in such a concept of autonomy that denies one the power to create a sense of identity within a dyad, a group or a community. Surely, a mother-child dyad possesses both the physical and the emotional attachments that give it an identity from before birth. A family, a marriage, a partnership, a working group and community can allow one to construct identities in a similar fashion. Identities are not limited to individuals.

That one both forms such relationships anew and one emerges from such relationships commits one to the previously stated view of the self as communal and autonomous. Only according to the revised view, the community is also assumed to have some autonomy in relation to other communities. This revision of the view of autonomy allows for a proposal for a role for dissent. Dissent anticipates the existence of competing views within any diverse population. And, it does not simply assume that these views will be compatible or will equally merit acceptance. Some form of negotiation and resolution of conflict need to be established as an expected part of knowledge development.

To see the purpose of asserting a means of dissent, consider the relationship between the metaphors of labour/production and reproduction. Over history, repeatedly, feminists have documented instances where the society value of reproduction is diminished and subjugated to labour and production. For example, Marilyn Frye (I 983) described the potential connections between the tool/material relation and systems of exploitation and oppression. The ax is used to transform or manipulate the tree to its *telos*. The ax retains its identity while the tree is rearranged to accommodate the ax's purpose. This is arguably the tool-material relationship that underlies the labour/production metaphor. Frye points out that when such manipulation is applied to animate objects by humans we see forms of exploitation. Exploitation may involve killing, but it also can involve the manipulation of other beings through shaping and restrictions such as harnesses, braces, shafts and other paraphernalia, a process referred to as breaking or training. Finally, when the manipulation process is applied to exploit another person or persons, we see the development of systematic networks of forces and barriers that act to reduce, immobilize, mould and shape. Ultimately, Frye points out, enslavement results if the disintegration of the other results in attaching the victim's will, interest and intelligence to that of the exploiter (p. 57-60).

Within the framework of the reproduction and labour/production metaphors, we can see in Frye's description how misuse of the production metaphor as a means to conceive of human relations can result in oppression and enslavement. A means of dissent, as well as a stronger application of the reproduction metaphor, is necessary in order to protect against such imbalances. Thus, dissent becomes a necessary construct to ensure the maintenance of the dialectic.

4. *Emotional intelligence is acknowledged*[7] Bringing the reproductive metaphor into knowledge construction necessitates a reconsideration of the role of emotion in cognitive development. Vygotsky (1962) acknowledges the importance of a relationship between the intellect and affect and wrote "Their separation as subjects of study is a major weakness of traditional psychology since it makes the thought process appear as an autonomous flow of 'thoughts thinking themselves,' segregated from the fullness of life, from the personal needs and interests, the inclinations and impulses, of the thinker" (p. 8). In fact, he argues for locating the roots of language in emotion and gestures: "The preintellectual roots of child development have long been known. The child's babbling, crying, even his first words, are quite clear stages of speech development that have nothing to do with the development of thinking. These manifestations have been generally regarded as predominantly an emotional form of behaviour. Not all of them, however, serve merely the function of release" (p. 42).

Much of the time, Vygotsky (1986) locates emotions in the biological realm and thus views them as having little connection with the processes involved in higher mental thought. He connects them to animal-like behaviour. For example, he wrote: "In the sphere of emotions, where sensation and affect reign, neither understanding nor real communication is possible, but only affective contagion" (p. 8). Before the development of higher level functions, he sees the role of emotions as potentially damaging to higher level thought: "the affective states producing abundant vocal reactions in chimpanzees are unfavourable to the functioning of the intellect. Kohler mentions repeatedly that in chimpanzees, emotional reactions, particularly those of great intensity, rule out a simultaneous intellectual operation" (1962, p. 40).

However, ultimately he does believe that an integration of affect and cognition is possible and desirable. However, a successful integration for Vygotsky depends on having the affect controlled by the intellect rather than having the intellect pushed about by affect. He wrote about:

[7] I am particularly indebted to Elizabeth Rowe for discussions we've had on this topic and for articles she has provided me including her term paper on this topic.

> ...the dialectical law that in the course of development causes and effects *change places*. Once higher mental formations have emerged on the basis of certain dynamic preconditions, these formations themselves influence the processes that spawned them ... Above all the interfunctional connections and relationships among various processes, in particular intellect and affect change (in Wertsch, 1985, p. 190; my italics).

I would suggest that the treatment of emotions as primitive, as requiring release, and as needing to be controlled by the intellect represents an example of how a sole emphasis on a productive model of cognitive development can serve to disenfranchise females' development. In a reproductive model, the expressions of relationship developed through parent-child communication of tone, including approval, pleasure, joy, warning, disapproval, and fear would lead to the development of language, but the emotional character of these exchanges would not fall away nor be cast as secondary to cognitive development. As Vygotsky predicted one would seek the dialectic unity of emotional and cognitive thought and both the intellect and consciousness would be viewed as involving both. Unlike in Vygotsky, an equitable but different contribution would be presumed from each.

Salovey and Mayer (1990) propose to use the term "emotional intelligence" which they define as "the subse ` social intelligence that involves the ability to monitor one's own and others' feelings and emotions, ι.. .iscriminate among them and to use this information to guide one's thinking and actions" (p. 189). They also define emotions as "organized responses, crossing the boundaries of many psychological subsystems, including the physiological, cognitive, motivational and experiential systems. Emotions typically arise in response to an event, either internal or external, that has a positively or negatively valenced meaning for the individual. Emotions can be distinguished from the closely related concept of mood in that emotions are shorter and generally more intense" (p. 186). Finally, Salovery and Mayer create a conceptualization of emotional intelligence which includes: the appraisal and expression of emotion, the regulation of emotion and the utilization of emotion.

As regards mathematics education, the introduction of emotional intelligence into our discussions of mathematics education allows one to assert that both facilitating and debilitating emotions play a significant role in learning, and that the emotional qualities of classroom interactions will exert a significant influence on what is learned. For example, if young women view mathematics classes as prone to embarrassing public exposure, and unwelcome risk and competition, then one can see why young women often fail to persist in mathematics. Incorporating a facilitating view of emotions would allow one to recognize, for instance, that the tendency of young women to seek a deeper level of understanding, because of holding less instrumental views of mathematics, is a positive characteristic.

5. *Abstraction is reconceptualized and placed in a dialectic.* As indicated in the discussion of the limitations of Vygotskian perspective, two interpretations of Vygotsky's theory can be proposed. In one, the dialectic between thought, as evolved from practical intelligence, interplays with language to create a dialectic unity. In such case, both practical activity and facility with signs are necessary to create complex human behaviour and consciousness.

Alternat...ely, one can assume an interplay between practical activity and sign use, but assign to sign use the governing activity to direct practical activity, while neglecting the ways in which practical activity constrains and guides sign use. If this is done, then there is a privileging of the abstraction to the detriment of practical activity. The result may be in fact, a detachment of languaging from practical activity and the development of modes of thought that alienate humanity from everyday activity.

Two versions of this alienation can be witnessed in academic circles. One is the result of over emphasis on social interaction based in verbal exchanges without reference to other kinds of shared or individual activities. The result is that too often the basis for knowledge is assumed to be solely an issue of human negotiation, influence and decision-making without regard for human action.

The second, criticized by feminist scholars, is the disembodiment of the mental from the physical. Cartesian dualism has been followed by a post-modernist tradition that seems to desire the elimination of the body from the discussions of intellectual development altogether. Feminists have made the argument that this distancing from the physical, a form of objectivity, has already made the female body an object of male manipulation. Leaving the physical out of the discussions of knowledge allows oppressive practices to remain as a part of the personal, and not an acknowledged practice of dominant cultures.

An alternative is to point out that in any practical activity there is systematicity once it has been undertaken repeatedly and in multiple forms (Ceci, 1990). Musicians without musical notations surely understand phrase, signature, cadence and rhythm, even though they may not write their compositions down. Gamblers at the track possess deep insight into probability despite their lack of standard notation. Sign use, the development of symbol systems such as algebraic expression, models, musical notation, architectural plans do serve as means of communicating and reflecting about such practical activities, but as with any representation, there is loss as well as gain with their use.

For example, an ellipse can be described as a relation for which $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and in doing so, describes all possible ellipses that are oriented with vertical and horizontal major and minor axes. Alternatively, I can build a variety of tools which trace out ellipses. One which was in use by South African carpenters is to have two narrow perpendicular slotted boards and to place a stick so that a nail at each end of the stick sits in each slot (Millroy, 1991). A hole is drilled in the stick between the two nails and a pencil is placed in the hole to trace a curve. As one drags one nail up and down the vertical slot, consequently dragging the other nail across the horizontal slot, the pencil traces an ellipse. Doing this, one experiences the variety of rates of change involved in the ellipse directly. What one "learns" is not particularly well captured by the equation listed above. Isomorphically, in terms of the loci of points, the same outcome was achieved; epistemologically, the outcomes differ. In carpentry, as in many other trades,mathematics is experienced in action, yet typically when asked, the "experts" deny mathematical competence.

My concern, as I expressed earlier, is that mathematics can easily become detached from its practical activity. And frequently, as writers discuss mathematics, the practical activity is systematically eliminated. For instance, Newton is well known for his contributions to mathematics; his genius is applauded widely. And yet, historically, he owes an unacknowledged debt to Hooke, for Hooke was the curator at the Royal Society for forty years. In this role, he was required to produce each week a physical demonstration of scientific principles. According to V. I. Arnold (1990) many of Newton's insights were formalizations of Hooke's demonstrations, and yet, only to Hooke's law do we currently credit Hooke's achievements. The mathematics was regarded as the formalization of the results in symbolic form. That what we have here is not simply errors of historians but systematic attempts to hide the genesis of the ideas. Newton destroyed the descriptions of Hooke's demonstrations to protect and enhance his own reputation and contributions. Within the framework for cognitive development just proposed such elimination of the genesis of idea would lessen rather than enhance the value and legitimacy of the contribution.

I am proposing that there are two historical roots of abstraction, one which possesses epistemological roots, and the other which is rooted in political oppression and elitism. The history of the term abstraction itself betrays these dual roots as does common usage.[8] *Abstract* comes from the Latin *trahere* which means draw and the prefix *ab* which means away from. It means, withdrawn. It entered into common mathematical discussion in the 17th century in the work of Hobbes, Newton and Leibniz, but its primary use came from the priesthood. It was closely connected with *absolve* and *absolute*, which were originally the past participles of *absolve*, meaning free from sin or imperfection or material consideration. Quoting from the Oxford English Dictionary, a religious tract from 1690, "The more abstract therefore we are from the body... the more fit we shall be both to behold, and to endure the rays of the Divine Light." (J. Norris, 1690, The Beautitudes)

[8] I am particular indebted to David Dennis for discussions we've had on the topic of abstraction and for the historical investigations he conducted as a research assistant.

One path of mathematics thus evolved as highly valued as a form of penance–withdrawn from material considerations. This is the history of pure mathematics where purity was the goal of the penitent. And mathematics is seen to "elevate the mind" and get at the "essence of truth". Mathematics was viewed as the discipline of the mind. And, it is mathematics in this sense, that creates the society of priests, with its implicit ties to monastic training. It is no wonder that mathematics that evolved in this tradition led to the elitism and isolation that characterizes much of the milieu of mathematics departments. And it is ironic that as a number of us challenged "absolutism" in mathematics (Confrey, 1980; Ernest, 1989), yet we did not re-examine its relationship to abstraction.

What is the character of abstraction that does not assume disembodiment and absolution? I would propose three possible solution paths: 1) the recognition of a genuine dialectic between practical activity and sign use, 2) the recognition of the value of multiple forms of representation and 3) the role of action in the act of abstracting.

According to the first path, I suggest that we establish a dialectic of sounded activity and systematic inquiry. Sign use is assumed to make systematic inquiry more accessible.
Conceptual development then becomes the effective synthesis of these two activities, demonstrated by sign use that captures the character of the grounded activity while creating its own systems of manipulation and movement; and grounded activities that predict or anticipate interesting aspects to represent in symbolic forms. By portraying this as a genuine dialectic in which neither part is presumed to require more intelligence, one communicates an important sense of the equity of different forms of human labour. And, one rejects increasing alienation with one's physical being, an essential element of a reproductive perspective.

Secondly, I have suggested the importance of establishing "an epistemology of multiple representations" (Confrey, in press c) in which one recognizes that all demonstrations, whether as grounded activity or as symbolic representation require one to learn to "act" and "speak" in the context, and it is the ability to move among representations that signals intellectual progress. The assumption in this viewpoint is that all representations involve showing and masking, and thus, one no longer seeks to find a single form, the Platonic ideal, but to contrast and compare the different uses of different representations. Accordingly, the history of mathematics becomes not only the establishment of more and more encompassing generalities but the acknowledgment of the value of distinctions within particulars. In this way, contextualization becomes as valuable as decontextualization.

Finally, I would point out that though abstraction may mean withdrawing from the particular objects of a situation; the action that creates the concept is not removed. For instance, symmetry is not abstracted from the activity of folding, but only from the particular of the medium in which the folding takes place. Even such an abstract concepts as cyclic groups, limits, or a derivatives, all have significant ties to actions, and hence, *abstract* may be revised to mean pulling away from the trappings of the construct, while retaining the ties to the action.

Consider the idea of averaging. Students experience multiple competing views of averaging (Mokros and Russell, 1992). One concerns the most likely outcome, another concerns balance and a third concerns equal distribution. Each has simple roots in human activity: based in the ideas of "most", "balance" and "equal sharing". Only the third is easily seen as the sum of the individual values divided by n. Developing the connections to the first two would require rather careful curricular development and activities in probability and in the use of histograms. According to the portrayal of mathematics in this revised theory, conceptual development in averaging would require one to recognize the different forms of action, to be able to represent the idea in multiple ways and to coordinate the grounded activities with the systematic inquiry.

In this revised view, the Piagetian idea of schemes may be useful. Schemes signal a connection to one's goals, forms of actions, means of communication and of reflection. If schemes are then situated within the larger view of activity theory, one creates an understanding of reflective abstraction that makes the concept of pure mathematical abstraction obsolete.

Elimination of the oppressive view of abstraction demands that one disclose the ways in which keeping mathematics mystifying, secretive and unapproachable serves to preserve the status quo of a powerful elite. To a large degree, allowing mathematics to continue to require students to disengage from their personal sources of experience and to learn a system of rituals that make little sense to them but will admit them to the ranks of the elite is one of the most effective forces to maintain this oppression. A critical view of mathematics is essential to change this current state of affairs.

6. *Learning is viewed as a reciprocal activity.* In emphasizing the adult/child relationship in learning, Vygotsky recognized the intergenerational character of teaching and learning. Educating one's young is a powerful part of parenthood, and to ignore the role of such a force in institutions of education is unproductive. Furthermore, Vygotsky emphasized that the learning should lead development, and that a failure to do so will slow down and limit a child's potential.

If, as Vygotsky has posited, learning must lead development, and interactions with adults or a more expert other provide this leadership, then that more expert other must be doing more than "finding out where the child is at." S/He must strive to move the child forward. One way to do so is to postulate a gap between expert and novice knowledge and to allow the child to use the language of the expert, or to perform the routines with the goal that eventually those performances and routines will be transformed from pseudo-concepts to concepts.

The potential issue with Vygotskian approach concerns the nature of the adult-child interaction. Most researchers in Vygotskian traditions consider this a form of apprenticeship, a model closely connected with labour and production. One disadvantage of such a model I argued earlier was that Vygotsky's view suppressed the natural diversity in children. Now, I wish to suggest that bringing in the reproductive metaphor in relation to adult-child and child-child interactions can help to overcome this limitation.

In studies of mother-child versus father-child interactions, researchers have documented that the mothers tend to decentre towards the child's activities and goals and to use the child's goals as a means of educating. Fathers, in contrast, tend to hold firm to their original intentions in interactions and coax the child to strive to accomplish the father's goal. Neither of these can be judged as better, for an extreme version of either will limit a child's growth, leaving the child either without any forward progress or without an possibility of success. Analogously, by asserting both reproductive and productive metaphors in conceptual development, one asserts that both father and mother forms of interactions with children are desirable.

Just as Vygotskian theory fits more closely with the father-oriented view of parent-child interaction, constructivism can be related to the mother-oriented view. Constructivism postulates a gradual but continuous process of growth and transformation from a child's conceptual world to the acquisition of scientific knowledge. If there is one central maxim on which all constructivists agree it is "start where the child is at". And "starting where the child is at" is demonstrably challenging, for it requires significant effort to be put into trying to understand where the child's thinking is, that is, how the child views the problem. Researchers have documented that doing this has a variety of positive instructional outcomes. It increases the child's self-awareness, it allows the teacher to hear how the child views the problem, it focuses on task-based interactions and it encourages the two to find bridges in language. This genuine interest in understanding the student perspective is lacking in Vygotsky's writings, which predated much of Piaget's insights into the diversity in child thought.

And, also missing in the Vygotskian position is the realization that from such interactions, the teacher learns as well. The expert learns how the child sees things, and at the same time, the adult gains new mathematical insights. This view of adult learning, of changes in an adult's content knowledge is missing in both Piaget and in Vygotsky. Only recently have constructivists come to acknowledge that they are not just discovering or articulating student schemes but learning genuine mathematics for themselves (Confrey, 1991). Recently, I have used a distinction between voice and perspective to signal the two kinds of learning that result from a reciprocal interaction between a student and a teacher. Voice is used to refer to the student's conceptions and perspective is used to describe an experienced person's view of the material. I suggested that in clinical or teaching interviews, one seeks to model the student's voice through the perspective of a better informed knower; however, I also pointed out the importance of using the student's voice as a way to reexamine, modify and strengthen one's perspective. Both voice and perspective contribute important epistemological content to the teaching-learning interaction (Confrey, in press b).

A fundamental result of this perspective is the development of a deep respect and support for diversity. If one enters the educational enterprise with arrogance, one's own views of knowledge quickly overpower the insights of the children. However, when the classroom norms are developed in such a way as to promote the exchange of student method and mutual tolerance and respect, the children themselves become increasingly confident of their contributions, and the system becomes self-reinforcing. In both peer relations and in adult-child interactions, the roles as expert, as teacher, as learner and as novice are flexibly drawn. (This does not mean that the teacher ever becomes the same as the student, but only that she or he acknowledges opportunities to take on multiple roles.)

A reproductive view of human development in which cycles of interaction are expected, in which student voice is solicited and valued and in which authority does not come from the dispersal of knowledge but from the creation of a knower is a key quality of empowerment. Jean Baker Miller (1986) in *A Psychology of Women* discusses how domination and subordination of women has limited the development of humanity. She recognizes, however, that there are times in which between two people there is temporary inequality. She describes this relationship as follows:

> The "superior" party presumably has more of some ability or valuable quality which she/he is supposed to impart to the "lesser" person. While these abilities vary with the particular relationship, they include emotional maturity, experience in the world, physical skills, a body of knowledge, or the techniques for acquiring certain kinds of knowledge. The superior person is supposed to engage with the lesser in such a way as to bring the less member up to full parity; that is, the child is helped to become the adult. Such is the overall task of this relationship. The lesser, the child, is to be given to, by the person who presumably has more to give. Although the lesser party often also gives much to the superior, these relationships are based in service to the lesser party. That is their reason d'etre. It is clear, then that the paramount goal is to end the relationship, that is, to end the relationship of inequality.

In this passage, we see Baker Miller recognize the educational purpose for the imbalance, acknowledge that the learning goes both ways and identify the service component of the act of educating.

For example, when asked how a group of fourth grade students identified the prime numbers less than 100, three strategies were revealed. One student went through the process of crossing out every other number, every third number, every fifth number and so on until the primes were left. Another student displayed the odd numbers within groups of ten in such a way that all the odd numbers ending with 5 were in a column and proceeded to look for number patterns. A third student used divisibility rules to eliminate all multiples of 2 and 5, and then when he did not know the multiples of three, he mentally broke the numbers into sums or differences and examined each term for any shared divisibility. For example, to see that 51 was not prime, he viewed it as 21 + 30 and pointed out that both 21 and 30 are multiples of three. Therefore, he argued 51 was divisible by 3. For the same example, another student thought of it as the difference of 60 and 9.

The teacher entered the setting with the instructional goal of having the students learn the rule for divisibility for 2,3,5,9, 11 and to have them explore the patterns in a 100s table. She did not anticipate the solution of the sums and differences. However, using this student proposal she revised her view of the patterns in the 100s table. She found a way to use all four methods successfully in the patterning exercise. One could mark off every nth number (method 1), could find the vertical, and diagonal patterns in the multiples, (method 2), could teach them to test for divisibility (method 3). What the student's proposal of breaking into parts and testing each for divisibility did was to lead the teacher to explore the diagonals in the hundreds table more carefully. It was easy to see that the movement from one multiple of three to the next was down one row and over three spaces (an increase of ten and minus three), but one could also see a pattern of down two rows and over one (an increase of twenty plus one).

This was an example of a teacher learning from students' expressions of diverse method as well as using her expert knowledge to lead students to greater insight in the multiple forms of representations.

7. *Classrooms are studied as interactions among interactions.* One of the most profound influences of Vygotskian theory is his use of dialectic relations, such as the dialectic between thought and language. In a revised theory, I have proposed a dialectic between sounded activity and systematic inquiry. In the revised view, I explicitly claim that both types of activity guide each other.

Classrooms according to this revised view can now be described as places in which children engage in grounded activities and in systematic inquiry. Instead of suggesting however that grounded activities are essentially individual and systematic inquiry is essentially social, I propose an alternative description drawing on the concept of mediation in Vygotsky. Accordingly, one can view grounded activities as actions involving practical activity which are mediated by one's interactions with others. In contrast, systematic inquiry, which involves the communication through the use of signs can be viewed as social activity mediated by one's experience in grounded activity.

This proposal makes it clear that one's physical interactions with materials and tools are influenced by social interactions. Likewise, one's use of signs is influenced by one's personal experiences with grounded activities. Looking at the interactions between these two forms of mediated activity may yield some useful insights into successful educating in mathematics.

Radical constructivism, with its focus on the development of student conceptions and its biological descriptions should admit the reproductive metaphor more easily. Piaget focused on the generative character of human thought. Reproduction plays a central role in evolutionary biology. Bruner and Bornstein (1989) in the introduction to a book entitled *Interaction in Human Development* wrote "Wherever one looked, it seemed to us, there were forms of interaction that were important in their own right, forms of interaction whose nature was somehow not captured by being reduced to the role they played as influences on intra-individual growth factors" (p. 1). And "Again and again, we found, research and theory point to specific interaction experiences and specific times in development affecting specific facets of growth in specific ways" (p. 12). In this introduction, they identify multiple forms of interaction: tutor-tutee, generic and environmental interactions, individual and cultural interactions and so on.

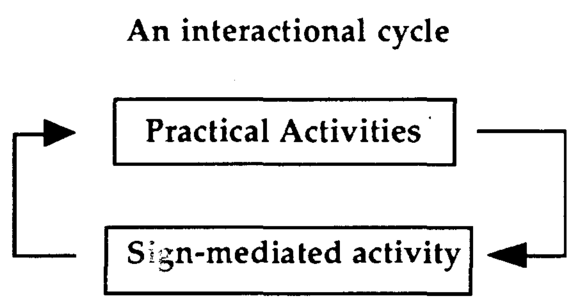

Figure 5

What this does is to draw our attention to the interplay between two factors. When sociocultural perspective leads us to focus so heavily on the verbal interactions among students and teachers (discourse, dialogue, register, etc.), it reminds us to consider at the same time, what children are doing with the elements of the non-human environment. What materials are they working with, what mental operations might they be building? What actions are they taking? What constraints are they experiencing?

For example, when a group of children try to solve the problem: share 162 jelly beans among 3 children, a child who chooses to work with Dienes blocks might approach this by first trying to share the "flat" (10 by 10 square) and that child's choice of materials and strategy will differ considerably from one who chose to make a trade of the flat to produce 16 longs (1 by 10 sticks) and two singles. However, once the language of "trades" is introduced into the classroom, this child's problematic may not be solved directly (how to split the 100 block into 3 parts), but the child will obtain a solution to the problem. How s/he comes to understand this new solution may not mirror how the child understands the method s/he originally proposed. I believe it is an open and question whether both methods will be as meaningful, generative, and enduring. Understanding the interplay between these different forms of interaction is important to understanding the development of knowledge.

One could chose to describe this as an interplay between social and individual; however, I chose not to make social and individual the basis of the interaction in order to keep the dialectic between the two theoretical metaphors at the root of this new theoretical approach. Thus, one's encounters with materials, environmental factors or the natural and physical world reflects the Piagetian radical constructivist view of the world—our biological adaptation to the constraints we encounter; and the interaction with others reflects the way in which our biological, physical, material are shaped by our participation in, or better, our immersion in a cultural perspective. At the same time, we must recognize that the child's likelihood of approaching this problem of looking at sharing one hundred evolves because of our decimal-based number system which has a biological basis in our ten fingers and toes and a cultural trajectory in terms of how that particular system of basis was developed and socially accepted.

CONCLUSIONS

There is no question that all scholars in mathematics education would benefit from the thorough study of both Vygotsky and Piaget. In this paper, I have sought to describe the perspectives of each concisely, hoping that such summaries will encourage further examination of their original work. Examples of how each theory deepens our insight into the classroom were provided as well as outlines of their limitations. In offering these discussions, my hope was to do justice to the contributions of each theory, while creating a tension between them.

In the final section of the paper, I have tried to move beyond the tension in the theories to create a bridge between them. My intent was not to ignore the very real difficulties in doing this. And, rather than create a strictly logical dichotomy between the two theories (such as one is individualistic and the other is social), I chose to select the evolutionary biology metaphor as primary and then to create a dialectic relationship between the subconstructs of labor/production and reproduction. However, the umbrella status of the evolutionary biology metaphor was recognized as tentative and evolving in light of its own historical/cultural character.

The final sections of the paper then articulated the cluster of ideas that emerge as significant in the production- reproduction dialectic. Many were revisions of issues identified as problematic in the two theories. Most revisions were tied closely to the changing circumstances we face in North America at the close of the twentieth century. For instance, I chose to make biological evolution the bridging construct, to recognize the importance of environmental concems. The recognition of how physical tools transform products from Vygotsky was selected as preferable to Piaget's non-contextual treatment of global constructs in part because of its importance in understanding the impact of technological tools at this time. Identifying and stressing the importance of diversity as a fundamental construct is of particular necessity given the multicultural nature of our country and the increasing international influences on all of our lives. Feminist theory was used to argue for a balance in the treatment of connection and autonomy and for a view of adult-child interactions.

Finally, the paper was an attempt to illustrate that in a climate of reform, we need theories to guide us. At the current time, in the United States, the National Standards for Curriculum and Evaluation are serving as a document for reform. Yet, much of that reform threatens to be more a matter of producing the correct slogans than achieving lasting changes in practices. Theories, in that they create systems of thoughts, may be an important vehicle for creating genuine reform. If some of the ideas in this paper can contribute to enduring reform, then it will have accomplished its purpose.

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Working Group A

Research in Undergraduate Teaching and Learning of Mathematics

Joel Hillel
Anna Sierpinska

Concordia University

Research in undergraduate teaching and learning of mathematics

Participants:

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Carr, Donna
Christal, Eleanor
Confrey, Jere
Crowley, Mary
Gillis, Alan
Hodgson, Bernard
Hoffman, Marty
Kieren, Tom
Lidstone, Dave
LoSasso, Anita
Muller Eric
Pimm, David
Pirie, Susan
Rosenthal, Helen
Wolf, Valerie
Yoshioka, Alan
Zazkis, Rina

Guest appearances:

Joseph, George, and Klasa, Jacqueline

The participants in the Working Group were senior high school, collegial, or university teachers as well as teaching assistants. Among their students were included those who are: university-bound, in professional non-university streams, math undergraduates, other undergraduates taking mathematics, teachers (both pre-service and in-service), part-time students and adult learners. The diversity of experiences (both of positive and negative kind), the difference in point of view and in ways of coping with teaching/learning at the undergraduate level resulted in a stimulating and resourceful working group.

The "big question" which acted as an umbrella to the discussions was the following: In what ways is the teaching of mathematics at the undergraduate level different from teaching mathematics at the pre-university levels? Grappling with this question has led us to consider the roles of the various actors in the teaching-learning process, their different expectations, approaches and perceived roles, the meaning of "understanding" at this level and the role of research, relations between mathematicians and mathematics educators, alternate teaching approaches and, the inevitable question of evaluation and assessment.

The report presented here tries to capture some key questions that were raised related to one aspect or another of undergraduate education. It is reconstituted from the various scribbling handed in by the participants right at the end of the Working Group as well as comments written after the meeting was over.

Students and the transition to university mathematics

"Mathematics is just like gardening: the books show you all these beautiful things but never show you people with dirty hands" (Rina Z. quoting a student).

Some of the more salient aspects of the reality of students in a university milieu are:

- they often remain anonymous to their teachers, lost in large undergraduate classes,
- they tend to be a heterogeneous lot with rather different levels of skills and concepts, yet classes are often conducted as if addressing a homogeneous group,
- those in transition from high school to university often have a procedural viewpoint of mathematics and find it hard to conceptualize. They are not prepared for the shifts that take place in university such as from calculus to theory, from proof to demonstration, from applicability to coherence, and a complete lack of reference to tactile experience,
- they remain mute in the classroom.

What do we do to cope with the different preparations and readiness of students to undertake undergraduate mathematics? At some universities, nothing seems to get done and unprepared students simply fail. At other universities there are different attempts made to address these problems such as setting up systems outside the classroom (study halls, crisis centres, learning communities, etc.) or providing specific courses to ease the transition from high school to university.

Some of the implicit assumptions that we, as instructors, make about our students are that they can operate at a formal level and that they are **autonomous** as learners. We have returned to the issue of autonomy of the learner several times during the sessions, asking ourselves

- how autonomous are students really?
- do they know how to take notes in the classroom?
- do they know how to read mathematical texts?
- how can we foster their becoming autonomous learners or should we, on the contrary, help to create a more communal-oriented learning with peer-tutoring, group work and collective assignments, and the likes?
- in what ways can technology help in developing autonomy of students not only as individuals but also as a learning community?

Anna S.: The management of classes at the university level assumes a much higher autonomy on the part of the students than at the previous levels. The students are practically just shown what is the material that they are responsible for. It is assumed that the students are able to take notes, read the notes, read, understand and make use of mathematical texts. There is enough evidence to the effect that many have not acquired these skills. They are passive listeners and readers, they have no idea of what questions to ask oneself when reading a mathematical text, they don't know what they could do to put their "understanding machine" to work (like finding examples, non-examples, examples to show that an assumption in a theorem is essential, specifying variables, making a constant into a variable, etc. etc.).

It is obvious that students need some guidance in these matters, and that the right person that can give them this guidance is their professor, who, for whatever reason, is dubbed as an "instructor" in such things as course outlines. "Instructor" has a kind of military connotation for me: the sergeant gives instructions and the soldier has to execute them without argument. But exactly what we want our mathematics students to do is to "argue"!

One insight for our Working Group was from the doctoral work of S. Frid described by Tom K. Observing students in three very differently taught introductory calculus (including one using an infinitesimal approach), Frid noticed a commonality of student behaviours, independent of teaching styles, which she termed **Collectors, Technicians**, and **Connectors** (corresponding closely, as pointed out by Alan Y. to "received knowledge, procedural knowledge and connected knowledge" in Belenky et al, *Women's Ways of Knowing*).

How much do students' behaviours change over time? What happens when a teacher sees the goal of teaching mathematics as being primarily about ideas and their connections if most students in his or her class are collectors and technicians? What happens to the technicians when they are confronted with Computer Algebra Systems with superior technical capabilities?

It seems likely that the collectors and the technicians have the most difficulty in courses where concepts and proofs play an important part of the mathematics taught while connectors may have difficulties in technique-laden courses.

What happens to connectors when they fail to connect?

Teachers and their Role

The teaching of mathematics at both university and lower levels came under a great deal of discussion. Do we instruct or educate (transmit or communicate)? Do we try or are we at all interested developing intellectual attitudes in students? Do we have a clear vision of what it is that we want to be communicated about our subject? How do we cope with large and heterogeneous classes? What about alternate approaches (computers, encouraging cooperative learning, historically-based approach, etc.) and how can we actually assess their effectiveness?

Anna S.: There is a conviction among university professors that it is enough to understand one's subject well in order to teach (communicate, transmit) it well. There is a tacit assumption here that teaching and/or communication of knowledge is a skill acquired naturally, as a by-product of understanding and not a profession that has to be learned. This belief undermines the very *raison d'être* of mathematics education as a domain of theoretical research and a workshop for elaboration of effective teaching practices.

BUT

At school, a teacher is not an instructor: he or she is the one that is charged to bring up the students, educate them to be both knowledgeable and valuable as members of the society. Is the university professor released from this responsibility of educating?

Ms. Krygowska used to say, in the 70's, that we should not so much teach mathematics as "educate through mathematics". What would this mean today? Krygowska was thinking of "developing more intellectual attitudes towards the world and life" in students. Do we agree with this postulate today, in the post-modern era, which has lost faith in the power of scientific knowledge to solve all our problems? Should "good understanding" of a phenomenon be reduced to its mathematization? What about the so-called "suchness understanding" Maslow speaks about (as opposed to "lawful explanation") which savours the phenomenon, the piece of art, of music, landscape, or mathematics "as such" in all its entirety and fullness?

Participants reflected on their own perception of their roles as educators - what it is they do and how effective it is.

Marty H.: I am intrigued by the computer technique, involving both "manipulation and sight" (D. Pimm) of zooming in and zooming out, as a metaphor for some aspects of my functioning in the role of teacher. When do I zoom in on, say, techniques for multiplying numbers slightly less than a hundred? And what theory (connections?) are made as I zoom out? Is my "screen" able to resolve more than traditional academic math so my teaching might collect and possibly even connect to aspects of the history of math, ethnomath and other areas?

There were quite a number of participants who had considered seriously the role of **computers/calculator** technologies in the teaching and learning process. There was quite a varied experience in using technology - for example, ISETL for teaching abstract algebra (Rina Z.), DERIVE for using the Harvard material for the teaching of calculus (Eva A.), MAPLE for both linear algebra and calculus (Eric M., Alan G., Anna S., and Joel H.), MATHEMATICA for general mathematical work (Marty, H.) and graphing calculators (David, P. and Tom, K.) It was pointed out by Mary C. that from her experience in Dalhousie, students did not particularly like computers as a learning tool. Faculty members also seem to be reticent to use computers in teaching – for some this simply stems from inertia, others are daunted with the practicalities of setting up a functional computer lab which is often frustrating and time-consuming.

Students (particularly collectors and technicians) sometimes find using computers threatening. With less emphasis on learning procedures often go an increased emphasis on conceptual understanding so courses can become, de facto, more difficult for the students.

We also discussed the double role of technology as "amplifier" and "reorganizer" and how one evaluates students' work with computers. Do we distinguish clearly between students who have achieved understanding as opposed to manipulation/fluency skills?

Another alternative to the traditional "transmission model" lecture style is to do with setting up tutorials, workshops, math labs, help/resource centres, and peer tutoring network. Among the questions raised were:

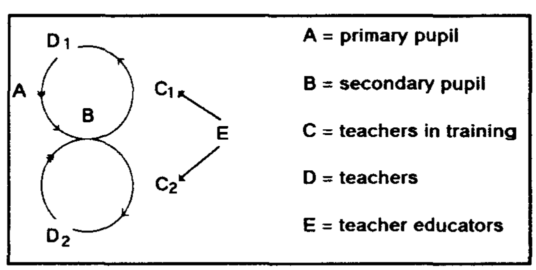
- if we acknowledge the importance of such alternative systems for the teaching of mathematics, how do we actually credit students for participating in such work? How do we credit teachers for setting up and participating such systems? (Jacqueline, K.)
- what role do social interactions play for individual growth?

Responsibilities of mathematicians and math educators

Jere, C. raised the question as to who decides what is legitimate mathematics to teach at the university level. Is it only the decision of the professional mathematics community or by a wider community ? The traditional wisdom is that while school curricula can be left to mathematics educators, university curricula must remain in the hands of mathematicians. This creates a certain tension between mathematicians and mathematics educators which is unique to the teaching of mathematics at the university. Such tension could have positive effects on the teaching contents and methods; then it is the mathematics educators' responsibility to stir up this tension.

We discussed the question of improving the communication between these "two solitudes" of mathematicians and math educators. A particular problem for those engaged in mathematics education is how to gain the respect of their mathematician colleagues – it almost seems that in order to be taken seriously by the mathematicians, mathematics educators must display an understanding of the subject matter that can impress the mathematicians. Some participants, however, reported about positive experiences in their universities and of real collaborative work between the two groups.

Another question was one of the "cycle of responsibilities" throughout the whole education system, particularly the cycle comprising of pupil-teachers-teacher educators. We discussed the known problems related to the attitudes and knowledge of elementary school teachers; that many of the teachers are afraid of mathematics; thus they can only transmit their fears to their students and an unfortunate cycle develops.



Susan P. : My 'lens' for this working group – the way of looking that I am using to interpret what I hear – is that of the gap in the "cycle of responsibility."

Using the language of Tom's and my theory, I am interested in the images that are held for math concepts at stages in the cycle and how they come about and how they are influenced. The focus in our discussions of apparent change between school and university of the expectations of students and teachers/tutors as to the nature of mathematics, the nature of instruction and the nature/responsibility for learning, may hold a key to our understanding the perceived acceptable images that are created for new concepts and which influence expressible/accessible images formed earlier. Does what we do at University destroy or make it harder for students to fold back to earlier images? Does the conviction that school is about *facts* and *doing* while university is about *structure* and *theory* lead to disjoint understandings? Should we explicitly work on this shift of emphasis with new students? Not a course in vigour, but a course in (i) rigour exists (ii) why it is appropriate here, and (iii) how you can learn through such an approach, understanding that is useful when dealing with 'facts and doing'!

None of this was this clear to me before these working group sessions! Certainly a lot of "responsibility" in my cycle lies here!

On understanding

In looking at the question of what is different about university mathematics, we discussed questions such as: What does it mean to understand at this level? In what ways is the subject matter of mathematics different from that at the lower levels? (a shift from calculus to theory, from proof to demonstration, from applicability to coherence).

Kieren, T.: Why might learning university mathematics be different from that which comes before, and what might we do about it?

Susan Pirie and I have been working on a theory of the growth of mathematical understanding in which formalizing as an understanding activity (for example, making concepts which have a general "for all" quality) enfolds and unfolds from several levels or kinds of informal understanding. If instruction starts at a level which "demands" formalizing and if for most students their understanding is not to have a detached quality, then in our language students would have to have available to them already less formal, better understandings (images and properties) to which they could "fold back". In our practice and in our "test" materials - how can experiences for such "folding back" be provided and invoked? Can we do this and start our instruction at a formal definitional level? With each topic?

Our theory suggest other relevant questions which might relate:

- What are the roles of suggested "amplifying" and "reorganizing" tasks in growth of maths understanding of the individual? How can we tell if such tasks act as planned?
- What role does (should) sound interaction (for example, tutorials) play in helping growth of understanding?
- What is the nature of growth in more formal mathematics understanding? We know our questions are meant to provoke such growth. How can we tell if they have such an effect?

Anna S.: At the undergraduate level, students are supposed to be at the "formal operational" (Piaget), or at the "conceptual" (Vygotsky) level with respect to their thinking styles. The contents of the courses and methods of teaching make this assumption. Is it correct to assume that?

Experience shows that even if, potentially, the students are capable of thinking at the level of abstraction and formalization, they are not ready for that with respect to the contents of the courses when they start. They need a certain period of transition from school mathematics, which looks at particular mathematical objects, maybe a few relations between them and focuses on computational skills in the algebraic or differential calculus, to the university mathematics whose main theme are theories of these particular objects and calculi. This is a shift that is by no means easy for even the most gifted students, just as it has not been effortless for the generations of mathematicians in the past. And it is very difficult for the less gifted or less motivated students who end up in mathematics rather than come to study it as a result of deliberate choice.

At school, many things were taken for granted: there were facts, formulas, and the student was only expected to recognize which to use and where to apply it. At the university, professors feel uneasy when they apply a theorem they haven't proved before, and students' assignments often consist in proving some less difficult facts. And this is exactly the mathematics students' biggest headache: they don't know "how to show proofs", they don't know "where to start"; once they have written down an argument, they have no way to check whether it is correct or not, etc.

At the university level, mathematics is looked upon from a systemic point of view: the basic elements are already systems themselves, for example, sets in which some operations are defined, like vector spaces, Euclidean spaces, algebras of linear operators or matrices. We are not dealing with particular vector spaces, we are dealing with classes of vector spaces; not with particular matrices or linear operators, but with linear operators that fulfill certain conditions (singular, non-singular, etc.). The defining conditions are chosen on the basis of their simplicity or logical primitivity, i.e. on the basis of how close they are to the axioms of the theory, and not on the basis of how useful and discriminating the defining property is. For example, in linear algebra, the minimum polynomial of a matrix A is defined as the monic polynomial of least degree of which A is a root. This definition is equivalent to saying that the minimum polynomial of A is a monic polynomial which divides any polynomial of which A is a root. But order in natural numbers (degrees of polynomials) is considered as more primitive that divisibility of polynomials and this is the reason why the definition is as it is. However, in proving that a given polynomial is the minimum polynomial of a given matrix or linear operator, the definition is not useful at all, it is the above mentioned property that comes in handy. For many students, it is this property that "really" defines the minimum polynomial, it best describes it because it is effective, functional. From a practitioner's point of view, the question of "logical primitivity" is irrelevant: let the "foundationists" and logicians bother about it. These people are concerned with systematizing, organizing the field of results that are already well known and well established. Their main problem is not meaning but consistency. But the students don't know the field yet, they only come to explore it, how can they see the purpose of these fussy discriminations? They don't see the point of it and feel discouraged.

Coming back to the assumption that university students are at the "conceptual" level of mathematical thinking, it may be that it is not satisfied in the case of all students. According to Vygotsky, people do not develop scientific concepts spontaneously: a considerable impact on the part of the conscious adults is necessary. However, the interventions of adults (teachers) related to scientific concepts have to interact in a very subtle way with the development of spontaneous concepts of the child. An intervention made in the best of intentions but at the wrong moment will have no effect on the child's thinking. Conceptual thinking develops at the age of adolescence. Therefore, it is quite possible that if the adolescent is not given the opportunity to develop the skills and habits of rational argumentation then the adult will never be able to see the point of mathematical proofs. Teacher interventions at the university may simply come too late!

One question that comes to mind here is: do we close access to university mathematics studies to those students, who, for some reason or another, have not reached the level of more systemic thinking and are incapable of producing a mathematical argument? If we do not, what do we do? Change the curricula? Keep our mathematics courses at a more concrete level? Be satisfied with naive inductive arguments, or arguments based on a single example? Are there other options?

Do we allow such students to become high school mathematics teachers? If we do, we shall contribute to producing further generations of university students to whom the proper teacher intervention has not been done at the right moment of their adolescence. Where should the "circle of responsibility" (Pirie) be broken?

A question for a future working group

One major question that was raised but was really not discussed in the sessions touches on the different practice of mathematicians and mathematic educators relative to the issues of teaching and learning:

> "With respect to the problems of teaching and learning at the undergraduate level we have, on the one hand, systematic research based on surveys, experiments, theoretical reflection, and, on the other, the professorial community's awareness of the educational and epistemological problems and spontaneous innovative changes in teaching practices that are shared via newsletters or bulletins like 'The Teaching Professor', for example. Is there any link between the two streams of reflection? Is there communication between the two groups of people?"

Working Group B

New Ideas in Assessment

David Robitaille
Cynthia Nicol
Heather Kelleher

University of British Columbia

Introduction

This working group, organized by David Robitaille, Cynthia Nicol, and Heather Kelleher, and chaired by David Robitaille and Heather Kelleher, met to discuss new ideas in the assessment of mathematics. Over the course of the three days the discussion focused on three major themes: large-scale assessment, authentic assessment, and implications of new ideas in assessment for teacher education.

The following participants were involved in some or all of the three days of discussion: Gary Flewelling, George Gadanidis, Claude Gaulin, Bill Higginson, Hélène Kayler, Heather Kelleher, Carolyn Kieran, Jacqueline Klasa, Ralph Mason, Tom O'Shea, David Robitaille, and Peter Saarimaki.

This summary of proceedings for Working Group B includes the following:

Part One – Discussion guide provided to participants for the working sessions

Part Two – Overview of discussions for each of the three days.

Part Three – Excerpts from the document *Principles for Fair Student Assessment Practices for Education in Canada* , one of the reference documents distributed to participants.

Part One

New Ideas in Assessment
Theme: Large-Scale Assessment

What is the role of large-scale assessment programs in helping to improve the teaching and learning of mathematics?

Large-scale assessment programs have grown in number and in scope over the past thirty years. The International Association for the Evaluation of Educational Achievement (IEA) is the organization that has the most experience in this domain, having carried out more than a dozen such studies over the past 30 years, including two international studies of mathematics. Development is currently under way for IEA's Third International Mathematics and Science Study (TIMSS), and over 50 countries–including Canada as well as several of the individual provinces–are intending to participate.

The conceptual framework for TIMSS is built around the concepts of the intended curriculum, the implemented curriculum, and the attained curriculum. The overall goals of the study are to describe the teaching and learning of mathematics around the world, and to use data on instructional practices, curriculum, and opportunity to learn to explain the observed differences.

The goal of this session will be to explore the strengths and weakness of large-scale assessments. The following questions might stimulate that discussion.

- What has been learned from previous international studies? Why do we need new studies of this kind, and why is it important for Canada to participate in such studies?
- What are the main threats to the validity of the findings of large-scale studies, be they international, regional, or local in scope? What can be done to improve their validity?
- What sorts of reports from large-scale studies would be of most value to classroom teachers, to curriculum developers, and to researchers?
- Can successful educational practices be exported across national boundaries successfully? Should we even try?

Related Readings

Freudenthal, H. (1975). Pupils' achievement internationally compared. *Educational Studies in Mathematics*, 6, 127-186.

Kilpatrick, J. (1990). Apples and Oranges Again. *Journal for Research in Mathematics Education*, 21(5), 416-424.

Robitaille, D. F. & Garden, R. A. (1989). *The IEA Study of Mathematics II: Contexts and outcomes of school mathematics.*. Oxford: Pergamon Press.

Robitaille, D.F. & Nicole, C.C. (1993). *Research questions for TIMSS (ICC 502).* Vancouver, B.C.: International Coordinating Centre for TIMSS.

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New Ideas in Assessment
Theme: Authentic Assessment

What is "authentic assessment" and what is its role in evaluating students' mathematical learning?

The term "authentic assessment" is currently used to describe a variety of assessment strategies that attempt to align assessment with the teaching/ learning process in a manner consistent with constructivist assumptions. However "authentic assessment" is not the only term used to describe the varied assessment alternatives that address this goal. The following references indicate some of the assessment variations that might fit under the "authentic assessment" label.

In an Educational Researcher (1991) interview, Lorrie Shepard described authentic assessment in this way:

> Use of the term "authentic assessment" is intended to convey that the assessment tasks themselves are real instances of extended criterion performances, rather than proxies or estimators of actual learning goals. Other synonyms are "direct" or "performance" assessments. The intense interest we are seeing in these alternative measures is a response to some of the deadly effects of multiple-choice tests, which are, in turn, the result of the inordinate weight given to traditional standardized tests in the past decade as a key feature of educational reform.
>
> ...The tasks and problems used in authentic assessments are complex, integrated, and challenging instructional tasks. They require children to think to be able to arrive at answers or explanations. Thus performance assessments mirror good instruction, which engages children in thinking from the very beginning. (p. 21)

De Lange (1992) in his description of changes in assessment, proposed that ideas about "authentic" assessment have been influenced by new notions about learning, as well as new goals such as reasoning skills, communication, and the development of a critical attitude. He stated that assessment should reflect current theory of instruction and learning (p 48), and proposed a form of "thinking" assessment that would reflect these new goals. Similarly, Shavelson, Baxter, and Pine (1992) described performance assessment alternatives based on students' performance on concrete, meaningful tasks, as consistent with the emerging constructivist assumptions about learning and teaching.

Wilson (1992) sees a constructivist approach to assessment as consistent with the new and loosely-defined fields of performance assessment and authentic assessment (p.80). Similarly, Clarke and Reed (1992) use the term "constructivist assessment" to describe assessment procedures often labelled as "authentic".

> Constructive assessment practices do more than document a learner's achievement. Constructi ssessment aims to provide the information needed for constructive action on the p all those concerned with facilitating a student's learning. The tasks which typify c tive assessment are characterized by an expressive, frequently open-ended, compor ich provides students with the opportunity to express and display what they have le rather than demonstrate their ability to mimic taught procedures. (p.230)

The California Assessment Program Sampler (Pandey, 1991) uses "authentic" as a major overall descriptor of the assessment program it recommends for implementation in schools (see p. 35). The program promotes the use of four types of assessment: open-ended problems, enhanced multiple-choice questions, investigations, and portfolios. The program's distinction between contrived and authentic assessment provides an example of the current generalized and extended use of the term "authentic".

Our Working Group discussions on the topic of authentic assessment and its role might focus on the following questions:

- What assessment characteristics, purposes, and strategies could be considered "authentic"? What terms other than "authentic assessment" are used to describe these forms of assessment?
- Why is authentic assessment considered to be a more valid form of mathematics assessment, and what are the arguments against its use?
- What efforts are being made to address validity issues in authentic assessment?
- Is authentic assessment appropriate for both large and small scale assessment, for accountability purposes, as well as for instructional and reporting purposes?
- To what extent should authentic assessment include the assessment of mathematical disposition (perseverance, motivation, reflection, etc.)?
- Since assessment practices influence instructional practices, what role might the shift to authentic assessment play in the instructional change process?

Related Readings

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New Ideas in Assessment
Theme: Assessment and Teacher Education

What should teacher education programs aim for insofar as the professional preparation of future teachers of mathematics is concerned in the area of assessment?

Assessment is an essential part of teaching and learning. However, too often changes in curriculum and instruction occur without the necessary changes in assessment methods. The emphasis of reform efforts in mathematics education has been to provide students with opportunities to explore, investigate, reason, and communicate mathematics. Such changes require not only the development of different instructional environments and different roles for teachers, but also the development of different methods of assessing students' understanding of mathematics.

Assessment as an integral part of good, effective, and meaningful instruction has direct implications for the preparation of teachers and the structure of the mathematics education programs in which they are involved. The intent of this session will be to explore possible implications of new ideas of assessment in mathematics education for the professional development of future teachers. General questions and issues around which the discussion may focus are:

- What skills, knowledge, and dispositions do teachers need in order to assess students' mathematical thinking, understanding, achievement, and learning in ways which enhance mathematics instruction and learning? How can teacher education programs be structured and designed to provide opportunities for teachers to develop these skills, knowledge, and dispositions?
- What are prospective teachers' experiences and perceptions about the role, methods, uses, and consequences of the uses of assessment? How might these experiences influence the decisions they make and what implications does this have for teacher education programs?
- To what extent can teacher education programs improve the quality of assessing students' mathematical understanding and achievement?
- To what extent should issues of accountability, meaning, values, and ethics of assessing student learning and of reporting and interpreting assessment results be included in teacher education programs?

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PART TWO

New Ideas in Assessment
Overview of Discussions, May 29-31

Day 1: Large-scale Assessment

After introductions, David Robitaille proposed that the group use the discussion document as a guide for the three sessions and asked for further suggestions for the format of the three days. The group spent some time considering the overall field of assessment and evaluation for the purpose of having a context or frame within which to situate the more specific issues. The following topics were discussed during the morning:

- Peter Saarimaki described a model for identifying the intended audience of a given assessment. The model involves concentric circles expanding to include the student, parent, teacher, school, district/board, province, country, and international audiences.
- Claude Gaulin raised questions concerning the philosophical and methodological differences underlying large international studies, and issues relating to curriculum differences and item quality.
- David Robitaille discussed the differences in using age-based designs as compared with a grade level focus, and the impact of these designs on issues such as those raised by Claude Gaulin.
- Bill Higginson raised concerns about the impact of international studies on teachers and classrooms, and the importance of findings reaching the classroom level rather than having an impact only at a policy level.
- Claude Gaulin described the SAIP use of a pretest to determine which booklet students would complete in the main survey. This was done in an effort to give each student the booklet best suited to his or her ability level.
- Peter Saarimaki asked how Canada was being represented in the TIMSS study. David Robitaille described Canadian participation as being organized at two levels. At the first level, a representative sample of Canadian schools and students would be selected to represent Canada in international comparisons and to be included in the international reports of the study. At the second level, four provinces—British Columbia, Alberta, Ontario (French), Ontario (English), and New Brunswick—had elected to participate at the same level as other countries. This would enable inter-provincial comparisons to be made as well as comparisons between individual provinces and other countries of interest to them. (It seems likely that several states in the United States will participate in this way as well.)
- The issue of reporting large scale assessments was discussed, and how decisions are made about what to factor into the results for meaningful interpretations. David Robitaille described how TIMSS will have an overall report on each of the two parts of the study, but that each country will be provided with data for its own individual report, thus allowing countries to deal with specific issues as they see fit.
- David Robitaille provided background on the IEA's involvement in assessment, and described some of the political issues related to large scale assessments.
- Tom O'Shea suggested questions for the TIMSS teacher questionnaire that could provide valuable information for teachers, such as: "Which mathematical topic is the favourite of your students? of yours? What aspects of your teaching practices (or your teaching situation) would you say contribute most to a successful teaching/learning environment?"

Day 2: Authentic Assessment

Heather Kelleher made an introductory statement summarizing the major points in the discussion document. There ensued considerable discussion of the use of the term "authentic." It was generally agreed that the choice of this label was probably inappropriate, especially if used in juxtaposition with another value-laden term such as "contrived."

Among the points that were raised during the discussion were the following:

- A problem with assessing the work of students done in groups is that some students might be unfairly rewarded or punished if all students were given the same evaluation.
- Peter Saarimaki described the use of a system where each of the five students in a group was given 50 marks to distribute among the group members (including himself or herself).
- A great many reports issued in the recent past call for more variety in evaluation strategies, and call for more formative evaluation. These might include projects, interviews, group work, portfolios, and the like.
- The two volumes published as the report of the ICMI Study Group on assessment in mathematics were recently published by Kluwer. They contain many recommendations about non-traditional approaches to evaluation.
- Authentic evaluation is contextualized evaluation. It implies a process that is coherent with teaching objectives. So, if process objectives are important goals for teaching, they should be prominent parts of our evaluation practices.
- There was general agreement that a major purpose of assessment and evaluation at the classroom level is to inform instruction. George Gadanidis provided an example of portfolio assessment used in his Board, and gave out copies for reference.
- Heather Kelleher described the new portfolio assessment and evaluation program instituted in Kentucky. Teachers are provided with sets of 25 to 30 performance tasks to use with their students, and students build up portfolios of their work on these tasks. Students use a set of guidelines to select between 5 and 7 of their performances for submission to the evaluation process. Those selected should represent a variety of kinds of work as well as different process and content objectives.
- This led to a thorough discussion of the advantages and purposes of portfolios. It was generally agreed that portfolios made it possible to track students' growth over the course of a year, that a wide range of learning goals could be addressed, but that management could be a major difficulty over time.

Day 3: Teacher Education

The focus of the third day was a discussion of the role of assessment and evaluation in teacher education programs. It was recognized that a distinction should be maintained between what prospective teachers should learn about assessment and evaluation as part of their professional education, and what techniques mathematics educators should use in evaluating their teacher education students. Both aspects are important, and both require attention.

The discussion materials prepared by Cynthia Nicole were used to initiate discussion on the topic of what do teachers need to know in order to assess and evaluate the work of their students in a way that will promote and enhance their students' learning. Among the main points that were made in the course of the discussion were the following:

- Tom O'Shea described a new methods course that he was inaugurating at Simon Fraser and that was built around the NCTM Standards documents for curriculum and evaluation rather than around a conventional methods textbook. Peter Saarimaki described a program of in-service education that the Toronto School Board has put in place to introduce in-service teachers to innovations in the areas of evaluation and assessment.
- It was agreed that prospective teachers would need to have views about the nature of mathematics teaching and learning, on the one hand, and assessment and evaluation, on the other hand, that were congruent with one another. That is to say, if they were to emphasize the importance of process over

product in their teaching, the same relative emphasis should be reflected in their approaches to evaluation.

- There was considerable discussion about student resistance to changes in assessment and evaluation where they were concerned, and the inherent conflict of working with students versus evaluating them.
- As an example of a project-based approach to evaluation in methods courses, George Gadanidis described a major project in his methods course in which students work in groups to prepare "chapters in a methods text" based on their readings and assignments during the course.
- Bill Higginson described how he asks students to compile portfolios of their work in his methods course. After about the first six weeks of the course, they submit their portfolios to him for preliminary, formative evaluation. He also discussed the importance of making students aware of the range of possibilities.
- A number of universities have apparently moved away from numerical or letter grades for methods courses and/or practica, in favor of a Pass/Fail system; however, few elementary or secondary schools have done so. It was agreed that instructors of methods courses should, in their own evaluation practices, help prepare students to do the kinds of assessment and evaluation that they would be called upon to do as teachers.
- There was considerable discussion about the importance of establishing and clarifying for students the criteria to be used in assessing and evaluating process-oriented activities. Carolyn Kieran described the criteria she has established in her courses at UQAM for evaluating teaching units submitted by students in her course. The criteria were: the pedagogical creativity of the materials, the internal logic of the unit, the "fit" of the assessment strategies proposed in the unit with the pedagogical proposals espoused, and a number of more commonly used criteria having to do with such things as mathematical correctness, feasibility, and the like.
- Tom O'Shea spoke about the need to have students take on some of the responsibility for evaluating each other's as well as their own work. In one of his courses, students are divided up into a number of "editorial boards" of "journals." Each "board" consists of between three and five students, and they must establish and publish a statement of its mandate and a call for papers. Each student in the course writes two papers and submits them to the editorial boards of his or her choice. The decisions of the boards form part of the overall evaluation of the student's work in the course.
- There ensued some discussion around the question of whether or not it was appropriate to assign students some degree of responsibility for their own evaluation or that of their peers.
- On the topic of whether assessment should lead or follow curriculum, Ralph Mason described a recent situation in Alberta where the January '93 provincial examination in mathematics reflected the spirit of the recently revised curriculum, but teachers had apparently prepared their students for a more traditional examination.
- Tom provided examples from a Connecticut State Department document (Alternative Assessment in Mathematics: The Action in the States and the Reaction From Psychometrics and the Classroom) to illustrate variety points raised in the discussions. Also, for our enjoyment, he provided a description of mathematical cunning and copies of an article he had written concerning that and other Grade 12 math exam heuristics.

PART THREE

Principles for Fair Student Assessment Practices for Education in Canada[1]

Developing and Choosing Methods for Assessment

Assessment methods should be appropriate for and compatible with the purpose and context of the assessment.

- Assessment methods should be developed or chosen so that inferences made about student understandings are valid
- Assessment methods should be related to intended goals of instruction and should be compatible with instructional approaches
- Consideration should be given to the consequences of decisions made based on information obtained
- Multiple assessment methods should be used
- Assessment methods should be suited to the backgrounds and prior experiences of students
- Avoid sensitive, sexist, or offensive content and language
- Translated and imported instruments should have evidence that inferences based on these inferences are valid

Collecting Assessment Information

Students should be provided with a sufficient opportunity to demonstrate the knowledge, skills, attitudes, or behaviors being assessed

- Students should be told why assessment information is being collected and how this information will be used
- Assessment procedures should be used under conditions suitable to purpose and form
- For assessments involving observations, checklist, or rating scales the number of components to be assessed should be small
- Directions to students should be clear, complete, and appropriate
- For assessments involving selection items the directions should be appropriate and consistent
- Unanticipated circumstances that interfere with collecting assessment information should be noted
- Written policy should guide decisions about the use of alternative procedures for assessment information collected from special needs students or ESL students

Judging and Scoring Student Performance

Procedures for judging or scoring student performance should be appropriate for the assessment method used and be consistently applied and monitored.

- Should consider scoring methods before an assessment method is used
- Students should be informed of the scoring procedures before an assessment method is used

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[1] Excerpts from the document: Principles for Fair Student Assessment Practices for Education in Canada. (1993). Edmonton, Alberta: Joint Advisory Committee. (Mailing Address: Joint Advisory Committee, Centre for Research in Applied Measurement and Evaluation, 3-104 Education Building North, University of Alberta, Edmonton, Alberta, T6G 2G5).

- Should ensure results are not influenced by factors that are not relevant to the purpose of the assessment
- Assessment comments should be based on student responses and should be presented in way that students can understand them
- Changes made during scoring should be based upon a problem with the initial scoring procedure
- Students should be informed of an appeal processes

Summarizing and Interpreting Results

> *Procedures for summarizing and interpreting assessment results should yield accurate and informative representations of a student's performance in relation to the goals and objectives of instruction for the reporting period.*

- Procedures for summarizing and interpreting results for a reporting period should be guided by a written policy
- Students and parents should be informed of the comments and grades are formulated and interpreted
- Results used and processes followed in deriving summary comments and grades should be described so the meaning of a grade or comment is clear
- Caution should be given to combining disparate kinds of results into a single summary
- Summary comments and grades should be based on multiple assessment results
- Results should be combined in a way that ensure that each result receives its intended emphasis or weight
- Basis for interpretation should be carefully described and justified
- Interpretation of results should consider the backgrounds and learning experiences of students
- Results combined into summary comments and grades should be stored in a way that ensures their accuracy at the time
- Interpretations of results should consider limitations of assessment methods, problems encountered in collecting, judging, or scoring information, and limitations in basis used for interpretation

Reporting Assessment Findings

> *Assessment reports should be clear, accurate, and of practical value to the audiences for whom they are intended.*

- School reporting systems should be guided by written policy
- Goals that guide instruction should serve as a basis for reporting
- Reports should be complete in their descriptions of strengths and weaknesses of students
- Reporting should provide for conference between teachers, parents, and students
- Appeal processes should be described to students and parents
- Access to assessment information should be governed by written policy
- Transfer of assessment information from school to school should be guided by written policy

Working Group C

Computers in the Classroom: Mathematical and Social Implications

Geoffrey Roulet

Queen's University

Lars Jansson

University of Manitoba

Computers in the Classroom: Mathematical and Social Implications

The effective integration of computers into the teaching and learning of mathematics involves much more than the provision of sufficient hardware and software. Multiple issues and concerns must be addressed if we are to enter the twenty-first century with more than a twentieth century programme delivered with the aid of technology. The eleven members of Working Group C (see Appendix A) began their discussions by reflecting upon the five connected issues presented in the pentagonal model (figure 1) offered by the facilitators L. Jansson and G. Roulet.

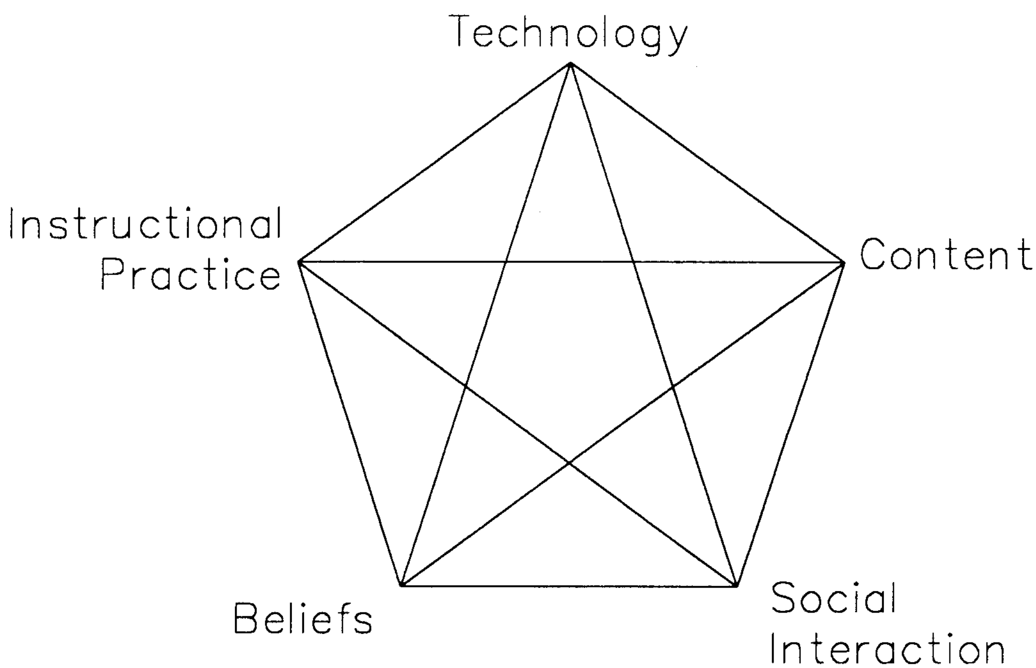

Figure 1: Original Pentagonal Model

Introductory statements indicated general support for this graphic as a framework for discussion but two themes emerged in the early deliberations of the group and generated modifications to the model. Noting the tendency for assessment practices to drive or bound curriculum and instruction it was felt that the issue of evaluation warranted its own vertex. Thus the figure was expanded to became a hexagon. Personal observations by group members together with research results (Hoyles, 1992; Thompson, 1992) suggest the primacy of teachers' beliefs in determining positions and actions concerning all other issues in the model. To indicate this significance for teachers' conceptions, the vertices were relabelled to place beliefs at the apex of the hexagonal model (figure 2).

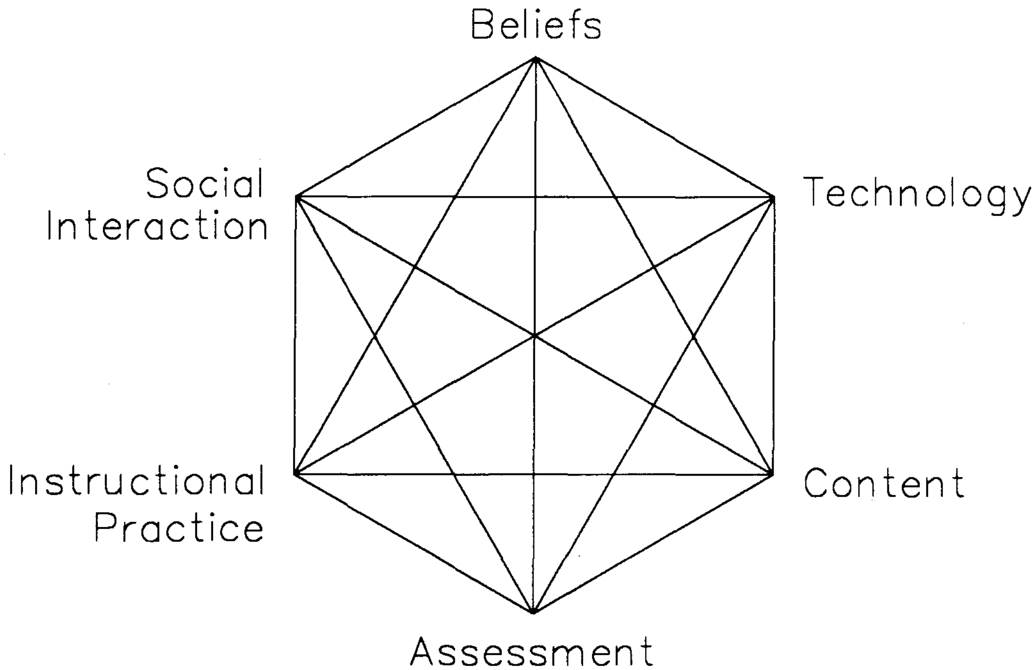

Figure 2: Revised Hexagonal Model

The sides and diagonals of this polygon represent connections between issues and questions that need to be addressed when investigating possible reforms of the teaching and learning of mathematics. But interactions do not just occur in isolation along simple line segments. The model suggests an interplay between all six concerns and various possible subsets, pictured as pentagons, quadrilaterals, and triangles. We noted, for example, that there is much research to suggest that social interactions in the classroom are impacted significantly by the use of technology (see Heid, Sheets & Matras, 1990) and that this in turn affects instructional strategies and perhaps even beliefs about how mathematics is done.

Beliefs

The group attempted to identify a conception of mathematics that could support desired changes at the other vertices of our model. Interaction with teachers suggests that many possess a rather restricted image of mathematics as a collection of algorithms. With this view of the discipline, mathematics tool software is seen as a threat since it effectively makes much course content redundant. What is to take its place? Most teachers have experienced secondary school and undergraduate mathematics that was continually presented as preparation for some future stage. They never get to "do" mathematics. This in turn is the message that teachers present to their own students through their mathematics instruction.

In contrast to the above, the group wished to see teachers who have experienced a mathematics culture and possess a "mathematical disposition" – people who see doing mathematics as exploring, predicting, conjecturing, and explaining. In many ways the conception of the nature of mathematics put forward by group members reflected the social-constructivist view proposed by Ernest (1992) and supported by the NCTM *Standards* (NCTM, 1989). Such an image calls for students to be engaged in collaborative computer supported open-ended investigations (Schwartz, 1989) rather than working with tutorial software that provides instruction toward the development of computational procedures.

Given that many teachers do not at this time hold such a rich "sense-making" view (Schoenfeld, 1992) of the discipline is it possible to effectively integrate technology into the mathematics programme? It was suggested that software that presents linked representations (Zimmermann & Cunningham, 1990) of presently taught algorithms might provide an introduction to computer usage for more traditional teachers. P. Brouwer, L. Pagnucco (using software linking numeric, graphic, and symbolic representations of functions) and S. Rosenfeld (using *Harvard Calculus Project*) reported on software that allows for the dynamic linking of multiple representations (Kaput, 1992) for those with a need to see connections (Hiebert & Carpenter, 1992) for understanding. While it was agreed that technology utilization schemes must, to be successful, address teachers' presently existing views of mathematics, such an approach may be limiting. There must be some programme to push teachers' images of what mathematics teaching and learning could be. The question arises, "Do computer applications have the potential to help alter teachers' beliefs or must changed beliefs precede effective use of technology?"

Assessment

The group noted the conflict generated by the tendency to measure the success of new educational programmes against old goals. If summative evaluation for courses emphasizes the ability to execute paper-and-pencil algorithms there is a disincentive for teachers to employ computer or calculator tools. Research (Heid, 1988; Palmiter, 1991) and personal experience of some group members shows that the fear of a major decrease in mathematical skill may be unjustified. L. Pagnucco reported that graduate level education students at the University of Georgia who take technology supported mathematics courses seem to do better than average in other more traditionally delivered and evaluated courses. Students appear to better construct solutions outside of standard algorithms if they have sufficient opportunity to use technological tools and possess a "sense-making" disposition towards mathematics.

The use of some mathematics tool software tends to encourage an iterative approach to problem solution. G. Roulet reported that when constructing algebraic expressions for given curves, students employing a function graphing tool made initial guesses at the solution and then adjusted the numeric coefficients to obtain progressively better fits. While students appeared to develop underlying concepts they were not successful on paper-and-pencil tests where precise calculation of the coefficient values was required. Others reported similar results and the group emphasized the need for assessment exercises to employ the same technology as that used for instruction.

Students working in a technology supported open-ended investigative environment may construct a "sense" of the concept embodied in the exploration but fail to develop detailed precise algorithms (Perkins, 1991a, 1991b). With the increasing availability of computers and mathematics tool software to perform computations or permit trial-and-error solutions does it matter that students do not learn traditional mathematical procedures? Geometry software such as the *Geometer's Sketchpad* and *Cabri* promote proof by multiple example techniques. Is the reduced emphasis on formal deductive proof a problem?

We need to carefully define the goals of our programmes and make it clear that these may differ from those of traditional courses. Then assessment that is consistent with these goals must be designed along with the selection of content and the development of instructional methods. A restatement of course aims raises the question of universality of goals. Should these be the same for all students? If not, at what age and by what criteria do we differentiate?

Practice

Members of the group shared their experiences working with a wide range (see Appendix B) of mathematics software.

L. Pagnucco. The University of Georgia Mathematics Education Department is currently involved in a number of teaching enhancement projects with elementary, middle, and secondary school teachers of mathematics. L. Pagnucco reported on a high school level activity concerning the factoring of quadratic expressions. Using graphing software, the related quadratic function and linear functions built from the factors were plotted. Two questions arose rather naturally: (a) Where and why will the straight lines intersect the x-axis? and (b) Can the lines be tangent to the parabola? These questions were introduced by consultants on the project to motivate exploration of the connections between quadratic expressions and their linear factors rather than to address any particular content objective. Reaction from teachers was mixed with some having difficulties in generating directions for investigation but with others becoming highly involved and pursuing interesting personal research.

I. Peled reported on a project designed to help elementary school pupils develop schemata for the solution of two-step word problems. The software, developed in Israel, provides a microworld containing different schemata for the solution of various problem types. After receiving a question the student selects a particular schema and is presented with a template into which he or she may insert the problem data. Post-tests conducted without student access to the software showed that previously successful students improved slightly and that weaker ones made significant progress. A significant side effect was expanded teacher mathematical knowledge since many did not originally possess the solution schemata needed for these problems.

P. Harrison. The Ontario OAC-Algebra and Geometry course contains the topics of matrices, linear transformations in the plane, and the solution of systems of linear equations. A computer supported project in fractal geometry has been used to combine these topics. Students, working in collaborative groups, are required to produce the matrices of the transformations that together lead to the generation of a fractal image. The computer serves only to support the mechanics of this project and provides no instruction. Student success and interest is increased by the opportunity to off-load to the computer all the messy and tedious calculations involved in solving systems of equations and generating the fractal images. Evaluation takes two forms: the production of the image via computer and a written test in which students talk about the theory behind the process. Pupils are generally successful at the first task but sometimes have problems with the second. There are major benefits in the affective domain with increased student enthusiasm for mathematics and greater involvement and effort.

P. Brouwer. **Analyzer**, a software package that links symbolic and graphical presentations of functions can help develop connections in Calculus. Teachers who have demonstrated skills in Calculus but lack a strong conceptual base have got numerous "ahas!" when revisiting the subject with the aid of a computer. In particular the connection between the roots of $f'(x)$ and the critical points of $f(x)$ is made significantly more vivid.

Summary and Questions for Research

A number of questions were raised but only briefly discussed during Working Group C's deliberations.

- Social interactions between colleagues are important. How can we build, within mathematics departments, the collaboration needed to support experimentation and change?
- What are the emotional dimensions of computer use? What are the benefits (or liabilities) in the affective domain?
- With the increasing presence of computers in students' homes what is the potential for out of class experiences?
- How is the interaction between user and software different from that between humans? Can the human-computer relationship take on some human-to-human characteristics?

- What is the relationship between software author and user? Software embodies the original purposes and orientations of the developer. What happens when these differ from those of the user?

During our concluding reflections on the past days' discussions it was noted that "seeing how computers are incorporated into practice throws all the issues surrounding beliefs into particularly sharp focus" (Hoyles, 1992, p. 38). Although the planned focus of discussion was to be the "mathematical and social implications" of computer usage, the major theme to emerge from our deliberations was essentially the reverse, "What mathematical and social goals or beliefs encourage the use of technology in the building of a classroom where all pupils are involved in the doing of mathematics?" Computers can be integrated with traditional curriculum and teacher-centred instruction but such a programme is insufficient to meet the needs of a changing society. Technology in the workplace is calling for new sets of skills.

A common theme emerging from reports of successful technology integration appears to be the acceptance of unexpected outcomes. Success can not be solely defined in terms of content objectives met but must also be measured against more long term goals of changed dispositions towards mathematics and the development of problem solving and inquiry skills. This means that those who wish to see increased use of technology must involve themselves in the ongoing debates concerning assessment. With 'standards' presently high on the political agenda, we must help define evaluation processes that reflect our goals and promote the integration of technology.

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Appendix A: Participants

Brouwer, Peter
Cohen, Rina
Harrison, Peter
Jansson, Lars
Kuendigger, Erika
Pagnucco, Lyle
Peled, Irit
Reid, David
Rosenfeld, Steven
Roulet, Geoffrey
Vanbrugghe, Bernard

Appendix B: Mathematics Software

Graphing
A Graphic Approach to the Calculus (Wings/Sunburst)
Analyzer (Addison-Wesley)
Calculus Calculator (Prentice Hall)
Graph Wiz (William K. Bradford Publishing)
Green Globs (Wings/Sunburst)
IMAGE-Calculus (Perfect Systems)
MacFunction (Adison-Wesley)
Mastergrapher (Addison-Wesley)
Math Connections: Algebra I (Wings/Sunburst)
Math Connections: Algebra II (Wings/Sunburst)
Zap-a-Graph (Brain Waves Software)

Computer Algebra/Symbolic Manipulator
Algebra Xpresser (William K. Bradford Publishing)
Derive (Soft Warehouse/Applied AI Systems)
Maple (Waterloo Maple Software)
Mathematica (Wolfram Research)
Mathematics Exploration Toolkit (IBM)
Matlab (The Math Works Inc.)
Theorist (Prescience)
xFunction (freeware)

Geometry
Cabri (Nelson/Brooks-Cole)
Geometry (Broderbund)
Geometer's Sketchpad (Key Curriculum)
Geometric Supposer (Wings/Sunburst)

3-D Imaging
3D Images (William K. Bradford Publishing)

Statistics
Data Analysis (NCTM)
Minitab/Student Edition (Addison-Wesley)
SPSS (SPSS, INC.)
Statview SE + Graphics (Abacus Concepts Inc.)

Other
Gertrude's Secrets (The Learning Company)
Gertrude's Puzzles (The Learning Company)
LOGO Writer (LCSI)
Matrices (NCTM)
Rocky's Boots (The Learning Company)

Working Group D

Gender and Mathematics

Lesley Lee

Université du Québec à Montréal

Gender and Mathematics

Lorraine Gauthier (Women's Studies, York University) gave us a presentation on feminist thought looking particularly at the work of Luce Irigaray (linguistics and psychoanalysis, gendered articulation of the world) and Isabelle Stengers (philosophy of science). A few themes and questions that give some flavour of her presentation and the ensuing discussions are:

- The need to re-examine our conceptualizations of science
- Gendered conceptualizations of the world
- Does mathematics try to exclude itself from feminist critique?
- Do mathematicians and physicists (including women) scare feminists away?
- Chaos theory
- Does sexual morphology influence world views?
- Science as related to psychic mechanisms of man's distancing himself from mother, creating boundaries, objectifying the universe. Women with more fluid ego boundaries.
- Would more women in mathematics change the nature of mathematics? Would they ask different questions?
- What is the link between the feminisation of a profession and its devaluation?
- Is mathematics-in-context more congenial to women, as some authors have suggested? (Some working group participants strongly disagreed.)
- The need for women to learn to laugh at science and not be fascinated by the norms of science
- Science and math communities function like a group of hunters who get to know each other and not the prey. They create the prey's behaviour.
- Ended with some very provocative questions such as: Why must we learn laws of science at the secondary level? As truths? Why not study scientific culture? Can we move off the terrain of reverent objectivity?

Suggestions to CMESG coming out of this discussion:

- That the executive consider inviting someone such as Lorraine Gauthier, Isabelle Stengers or Luce Irigaray as one of the lecturers at a future CMESG.
- That groups such as IOWME and provincial chapters open the door to a feminist critique of mathematics.

Participants in the group (all but one were women) shared their personal reasons for going into math. For many it was an escape from chaos and from the difficulties of social life. Roberta Mura pointed out that this is not supported by data that she collected from a more representative sample of Canadian women mathematicians.

Pat Rogers gave us a brief presentation on gender and math issues from an international perspective based on the papers presented at the IOWME sessions at ICME-7. She took the four stages of curriculum transformation proposed by Peggy McIntosh (1. recognizing a problem, 2. add-on solutions, 3. critique of the discipline, 4. true integration, rethinking the curriculum) and applied the model to the IOWME papers and current work on gender and math.

We recognized that Gender and Math within math education is stuck at stages 1 and 2, that almost no feminist critique of the discipline has taken place from within. Feminist pedagogy, on the other hand, seems to have been better integrated to the mainstream thinking of the math education community.

There was some discussion on whether or not feminist pedagogy had become simply good pedagogy.

Hélène Kayler, who presented the activities of MOIFEM (the Québec chapter of IOWME) advised us that they are planning to look at the question: "What would a feminist pedagogy of mathematics look like?" and hinted that she and possibly Claudie Solar might be willing to lead a working group on this at our next CMESG meeting.

We touched on the issue of concepts and goals of gender and math research being culture-bound and being perhaps more relevant to Western societies than to other societies.

Tasoula Berggren gave us a review of research and action in Canada and the USA and this was supplemented by participants adding what was happening in their areas. An exchange of articles, titles of books, and resource materials was made (or promised) and we closed our discussion with a list of projects/actions/reading for the future.

A bibliography on Women and Math may be obtained by writing to Roberta Mura, Département de didactique, Université Laval, Cité universitaire, QC, G1K 7P4.

The "women" question in mathematics or the "science" question in feminist pedagogy

(Draught version)

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In addressing the question of gender in mathematics, of the relationship of women to mathematics, or mathematics to women, I wish to look at the construction and status of scientific knowledge in general, or rather at the status of physics in particular (so closely aligned to mathematics) and at women's possible relationship to it with very particular reference points – points with which a North American English speaking audience may not be very familiar and which I feel have much to offer: the works of Luce Irigaray and Isabelle Stengers. Irigaray is a feminist psychoanalyst, linguist and philosopher who has developed, over the past twenty years a complex conceptualisation of sexual difference and of women's particular structure of knowledge and use of language. Isabelle Stengers is a philosopher of science who has been working closely with the Nobel prize winner, the physicist Ilya Prigogine, on the importance of the physics of chaos, not only as it pertains to scientific paradigms which it has made redundant but as it pertains to the construct and status of scientific endeavour.

Irigaray and Stengers are useful reference points for this exploration since both concern themselves precisely with the nature and status of physics and ask the very questions which so concern us regarding the nature and status of women's interaction with this "more abstract" field of knowledge. Both address, head on, the oft repeated polemic which argues that a disclaimer of science as a male discourse leaves women on the side of irrationality as traditional discourse would locate us: male/female; rational/irrational; science/poetry; knowledge/intuition. Both, from very different perspectives, look at the very concept of rationality and the claims which science, physics in particular, makes about its epistemology and methodology – claims to rationality and objectivity. They open up new avenues of conceptualisation and therefore new possibilities for redressing the situation of women in science which so many decry.

Irigaray makes explicit reference to Stengers and Prigogine's work as she develops her physico-psychoanalytic and linguistic analysis of the parameters which define the gendered conceptualisations of the world. In her socio-political and historical reading of the construction of science as a discourse and a cultural practice, Stengers shifts the problem from the terrain of how we can get women interested in science or science more adapted to women to the point where we can re-examine our own concept of science and re-situate it in the socio-historic context which defines it and which it defines. For both, context is the key issue to understanding the epistemological constructions of the world which science proffers us. Context is what differentiates male and female "science", what would allow a more "realistic" view of scientific claims to rationality and objectivity and that which would allow us to think differently a science more adapted to humankind – a science more open to women's contextualised epistemology. For both, the laws of probability so "essential" to the physics of chaos, are defined, constrained, effected, negated by the socio-political contexts in which these constructions and reconstructions of the world take place.

As Irigaray, Prigogine and Stengers have demonstrated, the intellectual model or rationality and objectivity, this ideal through which the physical world is analyzed, remains, to varying degrees, according to epoch and paradigms, independent of the physical realities which it purports to study. And it remains, far into the twentieth century, divorced from the psychic constitution of the observer. Irigaray's work echoes much of Prigogine and Stengers work while underscoring sexual specificity by tracing the relationship between the metaphysical concept of "objectivity", of "rationality", of "neutrality" and the sexual morphology of those who produce these concepts. She does not repudiate that there is a relationship between thought and the phenomenal world, that there is a "fit" between the scientific paradigms and the laws they elucidate. Rather, she argues that this relationship is mediated through the sexed body's specific relationship to both – thought and phenomena. Each sex's corporeal experience predisposes it to seeing certain patterns within phenomena rather than others, projects its own material experience of its corporeality onto the phenomenal world. Neither sex can pretend to universalize its mode of thinking nor the limited structures to which this mode happens to correspond in the real world.

Irigaray's conceptualisation of sexual difference as it pertains to epistemology and language centres around two fundamental notions: the relation of each sex to their corporeal origin – the mother and the role which sexual morphology – anatomy – plays in the human interaction with the world. There have so far been three aspects to this dual critique. Of this tripartite binary analysis I will give only a brief overview of the first and third element, concentrating, for our purposes today, on the second – the morphological imposition through analogy of men's anatomical relationship to themselves and to the world. It is here that her reference to Stengers and Prigogine's work lies, and here where the development of a new model of causality, taken from the physics of chaos, is elaborated in relation to her concept of sexual difference.

Briefly the first aspect of her work explores the different psychic structures which develop intrinsically and are socially constituted in relation to our relationship to the lost mother. Men form their sexual identity by differentiating themselves from the maternal/feminine world of their origin. The basis of their masculinity is in fact this distancing from, and in our society this devaluing of, the maternal and hence of all things feminine. But even in a society where the maternal was really revered and empowered, the psychic structure of differentiation would none the less exist. In social terms, within this society, this creates a valued site of subjectivity, a participation in the dominant group. In psychological terms this creates strong ego boundaries between the self and the other, and indeed the objectification of the other – an objectification which will be taken up time and again in men's philosophical and scientific discourses. In psychoanalytic terms it situates the primary interlocutor of intersubjective discourse as a non-interlocutor, as a repository of projected fears, ideals, illusions etc. In linguistic terms, this effaced primal maternal interlocutor is repressed as men privilege male intersubjective discourse and subject/object interactions of philosophical and scientific discourses where the "it" objectified is not always neutral matter but objectified humanity. In her linguistic analyses, Irigaray has shown that men have no problems situating themselves as speaking subjects in the world, in relation to themselves, to the male other (albeit in reverence, in competition, perchance in competitively ordained collaboration and in antagonism), to the nurturing maternal other, to the objectified it.

Women, on the other hand, form their sexual identity by differentiating themselves from and identifying with the maternal/feminine world of their origin. The basis of their femininity lies in this double distancing from and identification with the maternal and hence of all things feminine. In social terms, this means an ambivalent relationship with the revered/devalued maternal and hence with themselves. In psychological terms this creates fluid ego boundaries between the self and the other, and indeed a difficulty, if not reluctance, to the objectification of the other – a propensity for intersubjectivity. Women do not so easily objectify the other as in doing so they objectify themselves. In psychoanalytic terms it situates the primary interlocutor of intersubjective discourse as an ever present yet absent or devalued interlocutor, as a repository of this ambivalence. In linguistic terms, this revered/devalued maternal, this not altogether effaced primal maternal interlocutor emerges in women's discourses as our inability to locate ourselves as subject and the maternal other or her feminine substitutes as primary interlocutor, as the privileged you which she remains in absentia. In her linguistic analyses, Irigaray has shown that women have problems situating themselves as speaking subjects in the world, in relation to themselves, to the male other, to the female other, to the nurturing maternal other, to the objectified it. They manage to do so mostly when they elaborate a negative construct – she did not, I cannot... We will see how this inability or unwillingness to objectify the other, even the object of research, is raised again, in different terms, by Isabelle Stengers' comments on Barbara McClintock's work.

Let me then return to the anatomical morphological constructs with which men inseminate the world of thought as it relates to scientific discourse.

In *Order out of Chaos*, Ilya Prigogine and Isabelle Stengers expose the links between metaphysical philosophy's insistence on one stable universal order and classical science's attempt to universalize the laws of dynamics to cover all natural phenomena. As Prigogine and Stengers demonstrate, recent developments in physics illustrate the specificity of the chance interaction between molecular units and hence the aleatory change which structures undergo. They underscore the fact that structures are valid only within certain contexts, within certain parameters and preconditions, and that the spontaneous changes inherent in most structures lead not to chaos, as western science and philosophy had assumed, but to new structures whose determination lies outside of any preconceived control. (We will see how physicists respond to these new structures which they must perforce acknowledge)

In human terms, what this suggests to Irigaray is the specificity of sex as a precondition which sets the limits, parameters and potential for our interaction with the world. And preconditions, Prigogine and Stengers remind us, are not causally related to effect. They are pure potentialities which may or may not be activated, which are themselves effected by any activation but which nevertheless constitute definite limits beyond which phenomena, contained within them, cannot stray. For Irigaray, the human sexual body is just such a precondition for human existence. Feminist critics who see in Irigaray's corporeal referent a deterministic essentialism remain caught, on an epistemological level, in a linear form of determinism (direct cause and effect) inherited from Newtonian physics. Irigaray attempts to speak in the language of the physics of chaos where deterministic systems exist as boundaries for structures which move randomly in and out of chaos according to certain patterns but not as effects of direct causes. This physics of chaos cannot, therefore, be theorized within paradigms of linear causality. Only context is analyzable: the parameters and conditions which can, at some point, give rise to a series of structured reactions. If the structures are similar they are not identical and if the change of environment can be controlled to elicit change, the move from order to chaos is itself uncontrollable. It is a randomness that has limits and that is often all that can be said. Theory connoted as that which can explain the existence of a phenomena has no meaning here, for the science of chaos can but delineate the parameters of its possible existence.

If nature is only accessible to us through complex mental paradigms, the very nature of those paradigms and their relation to both the "object" under study and the embodied "observer" is, indeed, the issue here. The main point of her thesis is that there is a peculiar fit between what has been excluded from scientific explorations of "nature" and philosophical investigations of the human "subject." The objectified other stemming from the differentiation from the maternal and the sexual morphology of men are the principal parameters within which scientific enquiry has developed.

Like the parameters surrounding the turbulence which is the object of study of this new science of chaos, sex, Irigaray argues, is a defining boundary not a linear causal element. Men's sexual morphology predisposes them to see first the linear dynamic within their experience of the world whereas women's sexual morphology gives rise to more random inter-subjective non-finite patterns within which our experience of the world becomes expressible. But no more than in the physical world are these patterned random occurrences the direct result of a unilateral cause. And just as the phenomena unexplainable by Newton's linear dynamics was not, in itself, unexplainable, was not the chaos which thermodynamics defined it as, women's morphological articulation of the world, uncontained by men's projections, cannot be defined as irrational, intuitive, non-objective.

For Irigaray, what is glaring in its absence in the unisexual and univocal conceptualisation of both metaphysics and classical science is any notion of *difference* other than a quantitative one, of *reciprocity* other than between things sharing the same properties. Reiterating the familiar critiques of post-structuralism, she argues that traditional syntax "is dominated by *identity*, expressing itself through property and quantity by non-contradiction reducing the ambiguity and the ambivalence of plurality by *sets of opposition*: nature/reason, subject/object, matter/energy, inertia/movement. (*Parler n'est jamais neutre* p. 313)

It is the traces of the psychic mechanisms at work in "man's" distancing himself from the natural-maternal world and of his sexual morphology projected onto the world which Irigaray uncovers in her analysis of the scientific and philosophical discourses of the western world. She argues that, whereas the masculine theological concept of the divine and philosophical concept of Being relate more to the desire for parthenogenesis and the ambivalent relationship to the maternal source, the universalizing thrust of the laws of thermodynamics is linked to the discontinuous rhythm of men's sexuality, peaking only to seek homeostasis in a finite, discharge of energy and of matter. Does this not constitute the world, she asks, "in relation to an alternation proper to masculine sexuality: erection-detumescence"? (*Parler n'est jamais neutre,* p. 286-7)

In relation to women, Irigaray articulates the specific morphology of our interaction with the world and the analogous epistemological structures to which it gives rise. I will quote her at length: (my translation)

> I think that the models which Prigogine and Stengers present are more in line with female energy and morphology than the exclusive bulwark of the two principals of thermodynamics applied by Freud to the masculine sexual economy which accommodates itself to the laws of "tension", "discharge" to return to a homeostasis and to the principle of equilibrium. These two laws define the structure of the libido as an accumulation of energy which creates a tension, this tension necessitates a discharge in order to return to homeostasis, to an equilibrium which is, in some way, always the same. These principles, in my opinion, do not account very well for a female sexual economy which is more in line with – which does not mean perfectly – models of becoming through stages of entropy (disorder) and expansion towards a new order, ... The model is therefore not one of tension/discharge and return to homeostasis, but a model of growth without return. If one must give some biological correspondents (or biologico-psychological) it is possible to say:
>
> -women's menstrual cycle cannot be reduced to a model of tension/discharge (except sometimes at certain levels of perception, culture being responsible for this). This cycle never comes back to zero ... Conception and birth also do not correspond to a model of entropy/non-entropy, tension/discharge, and there is never a return to point zero; neither biologically, psychologically nor socially. (*Sens et place des connaissances*, p. 105)

Through these cycles, women are always in a state of movement between equilibrium and disequilibrium, random states whose parameters can be defined but whose actual point and direction of change cannot be defined within the linear causal model of thermodynamics – women are by nature chaotic in the scientific meaning of that word, morphologically, analogously more attuned to the theoretical formulations of post quantum mechanics.

This raises two questions: 1) if the physics of chaos is more attuned to women's morphology why is it that men are still primarily the physicists and how did they discover that to which their morphology leaves them blind? 2) What is the specific form in which sexual difference articulates itself both in discovery and in the language of its articulation?

What follows is a tentative response to these:

1) Briefly, the physicists of chaos continue to attempt to reduce the indeterminacy of phenomena to mathematical formula – theorems of probability – Stephen Hawking's quest. As well, they continue the same response which was characteristic of early reactions to quantum mechanics – chaos is defined in negative catastrophic terms. And here again I will quote Irigaray at length:

> After having imposed truth, the ideal, transcendental unity in the name of a divine and inflexible law, today's experts now preach "randomness" (Reeves), "accident", (Feynman), "ignorance" (Thom), "pluralism" etc., all of this in a context of political fear of new values which appear to be problematic. In any account, the work of today's experts presuppose a "break with the past" (Feynman), patricide (Freud, Lacan), "leaps" (Thom), "explosions" (Reeves), loss of perception (*Le cantique des Quantiques*), and not a becoming, not a memory which learns continuously and changes qualitatively in function of what it learns" (*Sens et place des connaissances*, p. 98-99)

These metaphors are not innocent, they express a discomfort with the loss of the sense of security which a structure analogous to their known experience of the world elicited.

To answer the second question I want to turn to Stengers' work because it is her analysis of Barbara McClintock, the nobel prize winning biologist, which seals the link between her and Irigaray and which provides the beginning of an answer to this second question. It is also her analysis which provides some food for thought for those of us who are interested in the pedagogical question of bringing women to mathematics or mathematics to women.

Isabelle Stengers is as little impressed by the continued quest for a universal mathematical formula to which the world can be reduced as is Irigaray. Stengers' work questions the very claim to rationality based on a specific epistemology and/or methodology which supposedly demarcates science from other intellectual endeavours. For her, the main concern is the unquestioned acceptance "that the sciences have a specific identity, of an epistemological nature, which would define *a priori* the criteria of rationality and objectivity and would allow one to state *a priori* the conditions according to which a problem can be posed in a 'scientific' manner." (Stengers, *Un autre regard*, p. 183). As an example of this let me simply quote Harold Varmus, molecular biologist with the National Cancer Institute in the U.S., who is quoted in Natalie Angier's book *Natural Obsessions: The search for Oncogens*: "You can't do experiments to see what causes cancer. It's not an accessible problem, and it's not the sort of thing scientists can afford to do." (quoted in MS., Vol III no. 6:57) – an obsession with methodology to say the least.

What happens, Stengers asks, if the difference between science and opinion, if the specificity of that enterprise which we call science, is not resolved within the epistemology which science claims for itself? Do we then have a condition similar to that in which Dostoevsky exclaimed: "if God is dead, all is permitted"? If the epistemological and methodological specificity of science does not exist, then there is no science, only opinion. To claim one is not to deny the other. Scientific endeavour does have its own specificity, but it is not that of rationality, neutrality, objectivity etc. and recognizing this would help us move out from under the paralysing grasp of our current idolatry of science and the "scientific" method and allow us perhaps to raise the question of the differing relation of men and women to this endeavour. For Stengers, it is in the historical context of its production and in the socio-political power struggle over ascendancy that the question of science's rationality can best be understood and demystified. It is, she reminds us, "power which hides behind the claim to objectivity and rationality when these are evoked as authorities." (*Un autre regard*, p. 186)

In relation to the questions which we need to ask of science in order to determine the specificity of its endeavour, I will quote Stengers at length:

> Why should this something special be associated to the merits of rationality, of an objectivity which would characterize "science" in general, which would establish a commonality between Newton who was an alchemist and Changeux, who is a reductionist, beyond [and in spite of] their "ideological" differences? And if, instead of trying to define the

> criteria by which we could "judge" Changeux and his neurons, Newton and his planets as scientific, we interested ourselves in the manner in which Changeux became interested in his neurons and interested others in them, including us, in the manner in which Newton became interested in his planets and succeeded in imposing his "irrational" hypothesis of forces acting upon bodies from a distance to the scandalised intellectuals [of his day]? Could we not then, rather than criticize the "objectivity" of the endeavour, ask the person who, "in the name of science" puts rats in boxes or students in statistical format, in what way could his work interest those who do not bow, like him, *a priori*, before the vacuous power of numbers?" (*Un autre regard*, p. 185)

What one finds, she claims, when one does ask these questions, when one studies the practice of science, is neither the pure disinterested rational response to a phenomena which scientists claim nor the self-interested thwarting of phenomena which many critics insist upon – we find no "pure causes" and no "pure arbitrariness". Rather one finds "the establishment of chance associations, unedited, multiple, disparate, between humans and non-humans, links and [associations] which reinvent the practices, the norms and the procedures [of science]. The history of science is not reducible to certain epistemological or methodological criteria. The history of science is a narrative, "neither moral, nor immoral ... rather amoral ... like all history." (Un autre, p. 189) What is recounted in this narrative is "the manner in which certain interests captured and/or were captured, ally themselves and succeed in becoming recognized as legitimate, rational or, to the contrary, are relegated to the sphere of opinion, of subjectivity, of what would be irrational, irresponsible of us to take into account ... It is a question of the political and social relations of force between interests, of the difference between those who have the means to intervene in that history and those whose right to intervene is not recognized" (*Un autre regard*, p. 188). I remind us of the quote cited above from the molecular biologist about what is rational and irrational to ask in relation to cancer research.

So when Irigaray says that traditional scientific thought echoes the metaphysical search for a universal transcendent and analyzes the metaphors of its articulation as an expression a differentiation and objectification, as a refusal of one's embodied humanity, it is not scientific. But when Prigogine, Nobel prize winning physicists, holds, five years later, the same thesis, it is. And so to prove to you that Irigaray is not out to lunch I quote Prigogine, not because he holds the truth, but because he is recognized as holding the truth.

What then is the singularity of this history, this fiction which has the power not so much to discover truth but to create it. According to Stengers, phenomena do not in and of themselves have the power to reveal their own structures, to impose constraints on the intellectual conceptualisations which humans make of the world. It is the role of the scientist to make phenomena speak, to interest themselves in these phenomena in the hope of having them buttress or challenge a certain perception, a certain paradigm, Kuhn would say. If he or she fails, its history remains fiction, if he or she succeeds, it becomes a truth in the scientific sense of that word, at least for a time being and for a certain segment of the phenomenal world. Science, as Irigaray has pointed out, has been obsessed with making each truth fit into a universal concept of the phenomenal world. Having abandoned the universalising question of philosophy in the grand manner of Plato, Aristotle, Kant, Hegel, Marx etc., we displace this question onto scientific endeavours – one universal explanation which accounts, as Kant or Hegel or Marx attempted to do with the social and spiritual world. It is, for Stengers, this "process of truth making, this power struggle, this capacity to convince which is at stake" (*Un autre regard* p. 190)

Scientific debates have little to do with rationality defined in human intersubjective terms. Few scientists attempt to posit their claim to truth by direct rational presentation to others. Rather, and this is what makes science unique, scientists use things, phenomena, and make them make the arguments for them. "They actively seek ways of making of the history of their science, not a purely human history, normalized by human reason, but a strange history where things answer to humans, a history which associated, in ever renewable form, the human arguments and the testimony of things." (*Un autre regard*, 192-3)

The question is how do phenomena, things, come to be recognized as valid and legitimate witnesses to a hypothesis? This, she claims, is not set once and for all even though in some era, it seems that science very rigidly sets out the structures and paradigms within which, and only within which certain phenomena can be considered as possible witnesses and the manner in which they can be so. Classical theory of thermodynamics was such an era and the reverberation of its loss of status can still be felt today in the linguistic metaphors by which post quantum physics is articulated – disaster, rupture, accident, ignorance etc.

It is in the face of those who deem themselves qualified to engage in this endeavour and to disqualify others that Stengers suggests we need to learn to laugh, to not take them so seriously, to not be caught in the trap of their own discourse, for "the fields in which the pretensions which scientists put forth are legitimized in the "name of science, the priority which they propose, the manner by which they define *a priori* what is "rational", what is worthy of their interest and what can be neglected, are open fields, rampant with a multiplicity of disparate interests all equally defined [determined] by the absences of some carefully maintained as by the obsessive presence of others." (*Un autre regard*, p. 194)

This is where the feminist critique comes into play, and, in this instance, Irigaray's specific critique concerning what is ignored, neglected, defined as irrational or non-objective. When science is interpreted as neutral, it ignores entirely this process of selection by which hypothesis and phenomena are included or excluded. Science does not begin in the laboratory, that is but one of its elements and not always the most important.

Stengers cites Koyre, the well known historian of science who argued in his study of Galileo that Galileo was not an empiricist who discovered the laws of motion of celestial bodies through careful accumulation of data, (many experiments he even admits he did not actually do, according to Koyre). Rather, Koyre argues, Galileo was "a convinced platonist, convinced that the natural world was written in mathematical language and therefore, a priori, there were certain questions, proper questions, fruitful questions – certain types of problems which should be left aside because they could not give rise to a mathematical understanding, which he was convinced, a priori, was the only worthy understanding" (Stengers, *L'histoire des sciences*, p. 120). Remember the molecular biologist defining appropriate questions for cancer research. Could we not say this of Einstein, Stephen Hawking, all physicists of chaos who continue to attempt to reduce the phenomenal world to mathematical formulae? It has not been shown that the world in its entirety is so reducible even if sections of it can be codified mathematically. And yet that is what defines physics – everything else is not worthy of the name. What if an entire realm, like the chaos which Newton relegated to the garbage heap, sits there waiting for different questions to be asked of it? What if, as Irigaray suggests, women could ask those questions without always reducing the answers to a mathematical universal?

Stengers is not so sure of the sexual specificity of this "other science," of this other questioning. Is this, she asks, "a perspective with affinity to women's position in our societies? Perhaps, [she answers] in the sense that this position, [of women] to the degree that it has specificity, is a position of a minority – not numerical, for sure, but according to the norms which are meant to define the modes of human existence. And the first challenge to which a minority must respond is to not let itself be fascinated by the norm of the majority." (*Un autre regard*, p. 196-7). I will return to this in my closing remarks.

And yet, when one looks at her commentary on Barbara McClintock, one sees a *rapprochement* to Irigaray's position. She speaks of the manner in which McClintock interrogated the phenomena which was the subject and object of her work – corn – that for so long condemned her work to irrational, unpublishable, un-discussable babble. And this resistance to her manner of proceeding remains active today, claims Stengers, despite Mcclintock's acclaim. What was this manner specific to this women scientist? It is an intersubjective relationship in which both phenomena and researcher participate equally, one in which the understanding which moves comprehension from one level to the next can come in the form of an insight which cannot be demarcated in stages, cannot be articulated as such, cannot be recreated without recreating the entire process of "coming to know". As Stengers puts it: "Discovery, cannot be assimilated to a first phase which could, a posteriori, be replaced by a procedure conforming to an explicit methodology; it is not-eliminable" (Les concepts, p. 180). It is no longer a question of reducing phenomena to the status of witness but of recognizing that "phenomena has meaning which one must decipher" (*Les concepts* p. 181.)

This fits awkwardly with the dominant procedural paradigm of making science. As Stengers so aptly remarks, the current epistemological structure of the sciences are fit "for a hunt in packs not for a solitary stalking" (*Les concepts*, pp. 183-4). A solitary hunter gets to know his or her prey whereas group hunters get to know each other, to identify signals, to work in unison to make the prey do what they want it to do – to frighten, to intimidate, to terrorize it into submission. Just as the group creates the behaviour of the prey which it can then know collectively, scientists "create the object susceptible of being known intersubjectively". But this is a competitive intersubjectivity between men, on men's terms, even in their collaboration. It is not an intersubjectivity between researcher and what is traditionally called the "object" of research, be that object human, animal or inanimate.

Stengers concludes, "The moment where she chose to no longer "use" the kernels of corn, Barbara McClintock made a choice which marked not only the history of women scientists, but also the history of science itself. Barbara McClintock was a woman and that is not insignificant." But here she offers us a sociological analysis: "The art of solitude, the affirmation of a singularity, the acceptance of the marginality which renders so many scientists literally crazy, she had learned them in order to become a woman of science, to conquer that which would have naturally been given had she been a man." For Stengers this attests "not to the discovery of "another" rationality, but the exploration of what reason is capable of when freed from the disciplinary models which normalise it. The exploration of the effective rationality which one can have [experience] when one has a propensity to not feel "at ease" with science. The attempt, not isolated but solid and explicit perhaps, to resist the SOCIAL IRRATIONALITY OF THE SCIENCES" (*Les concepts* p. 186).

And this social irrationality, which is the hallmark of the rationality which men, as the definers of knowledge and culture have imposed on the world, how do we counter it? I return thereby to the challenge which Stengers identified for countercultural work "to not let [ourselves] be fascinated by the norm of the majority." And she has some very practical suggestions to enable us to avoid this paralysing fascination and to ensure that the younger women we seek to lure into science are not trapped by it either and are thus freed to do science on their terms. The escape hatch is sketched for us in the question which Stengers asks of the science curriculum designed for secondary education and which could apply not only to science but to mathematics, not only to women but to all future scientists, not only to secondary education, but in our culture, mostly to university education:

> Why must we, by priority "learn laws" at the secondary level; it is the first thing which is forgotten and it is effectively what we call dogmatic teaching because who speaks of laws speaks of a gigantic simplification, extremely artificial apparatus, reduction of the problem to what the apparatus can treat. There is a kind of contradiction between the law we teach and what it would become if we would put it back into context and if we would make it say exactly what it could say. Must we really study scientific laws as "truths", must we really give laboratory experiments and exercises on those experiments or would secondary education not be the place to teach about scientific culture without concerning ourselves with application, a priori, for interest, for pleasure? As history and not by solving the problems of how we will demonstrate this law in the most simple terms. If we taught the science rather like a culture, like a reservoir of histories of inventions on the part of humans, the problem of research and of the cultural relationship of researchers [both men and women] to science would also be modified by this initial initiation?" (*L'histoire des sciences* p. 145)

And if we spoke of exclusion, of the constant appropriation and renaming of that which is excluded to try to make it fit into the mould, the male mould of universality, of rationality, of objectivity, of neutrality – if we spoke of intersubjectivity, of the import and impact of the maternal interlocutor, of the linguistic effacement of her and of half the human race – if we learn to laugh at the pretensions not only of scientists but of men who hide their power behind the authoritative concepts of rationality and objectivity, if by pleasure and interest we engaged in a particular activity that carries no more and no less importance than others, if we demystified a culture, could we not make more room for another culture, one where women might not only take their proper place but be freed to ask questions that men seem to have such a difficult time asking? To move the struggle off their terrain of reverent objectivity would unmask and free up at the same time – would maintain above all the context of the "object" of study, of the researcher and of the science itself – a truth devoid of its pretentious universality.

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Working Group E

Training Pre-service Teachers for Creating Mathematical Communities in the Classroom

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Training Pre-service Teachers for Creating Mathematical Communities in the Classroom

Three main themes, presented in the conference announcement, guided the discussions of this working group:

(1) the question of values in mathematics teacher education,
(2) dilemmas created in turning theory into practice, and
(3) reflections on current and proposed teacher education programs and practices.

In the first session the question "What qualities do we value in a pre-service mathematics teacher?" was addressed. The participants suggested the following as desired qualities:

Mathematics teachers should:

- understand mathematics and have a good mathematics background
- be able to use mathematical ideas to model problem solving situations
- be intellectually curious and excited about mathematics, and actively engage students in learning mathematics as an intellectual journey
- be concerned for equity in mathematics education
- view mathematics as a way of knowing, not only as a set of content topics
- emphasize mathematical processes as well as mathematics content
- be reflective, responsible, and pro-active
- develop a community of learners in their classrooms
- be confident of their ability to do and teach mathematics, i.e. have no mathematics anxiety or mathematics phobias
- understand that pedagogy and content cannot be separated–that content influences pedagogy but pedagogy also affects content.
- be critical of their pedagogy
- be able to capture "teachable moments" when they occur
- be more concerned with teaching mathematics to all students, i.e. they should make an effort to know their students, to understand their abilities, and to encourage their students to show their thinking
- develop for themselves professional communities to overcome the isolation that so many teachers experience in their professional lives
- take advantage of opportunities to learn from students' errors
- appreciate diversity in learning styles
- become adept at visual representation of mathematics problem situations
- develop qualitative understandings of mathematics
- be aware of alternative algorithms for doing mathematics
- teach mathematics as problem solving and view themselves as learners and problem solvers
- be risk-takers and resist pressures to conform to the status quo
- empathize with and be sensitive to students
- understand that mathematics is not euro-centric and so value other ways of teaching and doing mathematics
- understand why they are teaching mathematics and the importance of learning mathematics
- become aware that they are developing a consistent philosophy of learning
- view the teaching and learning of mathematics as a form of action research.

In addressing the second theme, dilemmas encountered in turning theory into practice, the group tackled the more general issue of problems that arise when attempting to realize our values in teaching pre-service teachers. This discussion began in the first session continued into the second session. Two strategies, writing to learn and learning in cooperative groups, were discussed at some length as ways to help pre-service teachers of mathematics reduce mathematics phobia, increase in confidence in mathematical content knowledge, and come to a greater conceptual understanding in pedagogical content knowledge. Both these strategies have helped pre-service teachers come to terms with their own mathematical histories and their feelings and attitudes towards mathematics. Some highlights of the discussion follow.

Writing to learn can come in many forms including writing for practice, writing as a warm-up, and so on. Students should write with a purpose. In large classes, writing can be a good way for instructors to get to know their students. Writing, for example in a journal, can be even more intimate than communication face-to-face. Maintaining the written record of interactions between students and teachers can also be very valuable.

If group work, social interaction, and cooperative learning are advocated in a pre-service mathematics course, then through the course outline, assignments, and assessment, and the values placed on these areas, the instructor must model these features so that they are perceived by the students to be important. In other words, the entire course must reflect these considerations. Pre-service teachers need to know how to incorporate group work, social interaction, and cooperative learning into their future mathematics classrooms, but they also need to experience it first as learners within the context of a safe exploratory environment–the pre-service mathematics classroom, and they also need to come to value these areas are within the context of their own learning.

Vi Maeers gave an example of such a course presently being offered at the University of Regina. There, all of the methods courses (Language Arts, Social Studies, Science, Mathematics, Health, Aesthetics, and Outdoor Education) are organized around three themes: thematic planning, cooperative learning, and experiential learning. The semester is set up in such a way that all the methods courses concentrate on thematic planning strategies for three weeks, and then the students go into the schools for one week to teach a short thematically planned unit, on which they are assessed. During the second block of three weeks back in the university the concentration is on cooperative learning in all the methods courses. The students not only see the stress and value placed on cooperative learning in mathematics; they see how cooperative learning is valued in all the methods courses they are enrolled in, because they experience cooperative learning assignments in all their coursework, and are assessed on these assignments in a cooperative manner. They discuss it in general terms, and they prepare a week of cooperative learning for the schools, on which they are assessed by the cooperating teachers and faculty advisors. The pre-service teachers see the integrity and cohesive nature of cooperative learning in their own learning and assessment, and because they personally experience it they are more likely to value it and use it in their future practice.

Pre-service teachers need to experience working in groups, but these groups ought to be varied in size, in design, and in purpose. Both group work and group arrangements need to be discussed. What also needs to be addressed in the pre-service mathematics classroom is what to do for students who do not function well in groups or who are not yet ready for group work. A participant asked whether there isn't still a place for individual learning, or does everyone need to be part of a group? How should individual learning styles accommodated? Pre-service mathematics teachers need to experience all aspects of group work and cooperative learning, including assessment issues. They also need to understand the politics of assessment and relationships between group work and assessment. In other words, group work needs to be deconstructed and all aspects of it examined. Others suggested there is a need to question why we do group work and to understand that learning can take place in lecture situations; group work is not a panacea. Eric Wood suggested that pre-service teachers could recount experiences where group work was abused (the group got the mark, but one person did all the work, fairness and justice issues, for example).

The mathematics classroom needs to provide situations for pre-service mathematics teachers to experience mathematics learning of a kind we would hope would be present in school classrooms. It isn't enough to talk about such things as constructivism. The pre-service teachers need to experience learning within a constructivist environment, and need to know the nature and the value of such an environment, so that they to can provide one for their students.

There was a discussion about how to encourage pre-service mathematics teachers to be oriented towards research on their teaching practice. Does it begin with critical reflection on how we are learning mathematics and on the limited exposure we have in the teaching of it to our peers and during short school practica? How, in fact, can we develop a research orientation to our practice (e.g., action research in the classroom)? In this regard, Beatriz D'Ambrosio spoke of her experience in teaching 150 pre-service mathematics teachers who were involved in designing student assessments. Beatriz did not tell the pre-service teachers how to assess their students, but allowed them to generate and test their own assessment procedures. After these teachers designed a task for the students to do, they realized, without any input from Beatriz, the need to talk to the students individually in order to assess their performance. This forced them to examine the research literature on interviewing techniques and student assessment, reading this literature with the goal in mind of constructing their knowledge of teaching. In Beatriz's classroom the pre-service teachers worked in self-selected groups of four or five and reported regularly on their group's progress throughout the project, modeling a work-in-progress approach.

Aldona Kloster and Sandy Dawson of Simon Fraser University described SFU's model of an intensive one year teacher education program in which 28 student teachers, two seconded teachers, and one advisor design the whole experience for the year. These groups have responsibility and power in decision making, including the assessment of candidates' suitability for teaching.

In its third session, the working group reflected on current and proposed teacher education programs and practices. The discussion of this issue can be summarized in point form under the following four headings: Values, Dilemmas, Practices, and Proposals.

Values:

- caring for the students, teaching with integrity, integrating mathematics
- knowing the subject well, feeling confident in learning mathematics and teaching it
- building a sense of community in the classroom
- overcoming mathematics anxiety
- creating a sense of the teacher as a professional, accepted as such by the greater community
- teaching children, not just mathematics
- teacher as decision maker and dilemma manager.

Dilemmas:

- difficulties imposed by paradigm shifts, e.g. from mathematics as content-to-be-mastered to mathematics as problems-to-be-investigated
- evaluation practices which constrain inventive and creative approaches to pedagogy
- the need to overcome teachers' anxieties about systemic constraints. Teachers often move in a context in which the demands of the system make it impossible to attend to their values in teaching.
- working around curriculum constraints. Teachers often do not have the liberty to leave out curricular material when pressed for time because of standardized examinations. Testing practices often determine teaching practices.
- differences between rural and urban practices.
- a need to learn algorithms may contribute to mathematics anxiety
- how do we generate curiosity about mathematics in a mathematics methods class?

Practices:

- Peter Taylor described the use of community or group work, but also a letting go of "covering the curriculum," in a regular course with a specified curriculum. This course develops technical skills in self-paced modules, includes lectures, and has a final examination containing problems.
- The group was urged to aim at successful experiences in doing and learning mathematics, and also at successful experiences at teaching mathematics.
- Vi Maeers suggested considering mathematics learning and mathematics methods learning to be a conversation where the topic is mathematics content and mathematics pedagogy.
- Participants suggested teaching teachers to be greater risk takers and to deal with uncertainty in their classrooms.
- It was suggested that teacher educators should model in the teaching of their methods course the kind of teacher we would like our pre-service teachers to be. Another participant suggested we should all make our own teaching problematic.
- Beatriz D'Ambrosio and Gordon Doctorow argued for embedding group work in the assessment process and using portfolios of student work including critiques by the students of what they have learned and what they think of themselves as teachers.
- Tom Schroeder suggested using "do-it-yourself" vignettes to speak to teachers about their practice. Pre-service teachers can read stories about teaching and reflect upon their own practice or particular things that have happened during particular lessons. The long-term goal is for students to collect their own stories of their teaching, to reflect on their teaching, to look for turning points in their lessons, to identify places in their interviews with students where they could have asked better questions, and so on.
- Richard Allaire proposed having pre-service teachers interview students in their classroom and then, back in the university classroom reviewing the interviews and discussing situations where the pre-service teachers could have asked more or different or better questions of the students in order to help them conduct better interviews in the future.
- Peter Taylor described building a sense of community in his third-year calculus course for mathematics majors by having his students read from books like Whitehead's *Aims of Education* and Milton Mayeroff's *On Caring* and then talk about the readings in class. In this way his students get the message that it is OK to talk about learning and teaching and interdisciplinary studies in a mathematics class.
- Vi Maeers attempts to deal with her pre-service students' mathematics anxiety by asking them to recount their mathematics memories and record their feelings about mathematics on the first day of class, to keep these statements until the end of the term, and then to re-evaluate their feelings about mathematics at the end of their mathematics methods course. Students also record both positive and negative experiences on a personal mathematical time line and are asked to identify anything in the course that may have made them change their minds about mathematics.
- Olive Fullerton described how she has her students write in journals about the most significant thing that they have learned that day – about their impressions, mathematical content or pedagogy.
- Gordon Doctorow discussed having his mathematics students in an alternate high school do community content assignments. For example, they might figure out the mathematics needed for a problem in a bicycle shop, or work in an elementary school as classroom volunteers. Evaluation can be self-evaluation with supplied criteria or the students' own criteria.
- Both Gerry Vervoort and Olive Fullerton suggested engaging the support of the parents in their children's mathematics projects. As an example of building up students' confidence in doing mathematics, Gerry suggested that elementary students be taught number games that they can take home to play with their parents and beat them. A problem of the day could also be given to the students to take home and try to solve with or without the help of their parents.
- Gerry Vervoort also noted the limited amount of time that is available for methods of teaching mathematics.

Proposals:

Teacher educators should:

- investigate ways to make their teaching practices problematic.
- be aware of feminist issues in mathematics teaching and learning.
- require better preparation in mathematics content for elementary school teaching.
- acknowledge that there are constraints in the system and decide what constraints are real.
- enhance their dialogue with their co-workers and become better listeners.
- engage pre-service teachers in the same kinds of activities that they would like them to do with their students.
- encourage subversive activity in students–helping them find the courage to work towards changing the system where necessary.
- support new teachers or any teachers who try to change the system.
- re-evaluate their student assessment practices in the light of changes in pedagogical practices and in curriculum.
- share their outlines for their courses for pre-service teachers.
- develop new types of mathematics courses based on principles of process learning and contextual learning.

CMESG should:

- recognize that it has a responsibility to change the system of teaching and learning mathematics.

The following is a list of persons who took part in the Working Group:

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Topic Group A1

Connected Knowledge in Prospective Secondary Mathematics Teachers

Eric Wood

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Introduction

The idea that a teacher must know something before he or she can teach it to someone else appears self evident. However, the *way* in which teachers know their content also has an impact on their ability to teach and explain various mathematical concepts. In some sense, teaching can be thought of as a way of connecting one's own knowledge with that of the pupils and thereby allowing them to develop a new understanding. Consequently, examining how teachers' mathematical knowledge is connected has important implications for teaching. The research study that is reported here focuses on this aspect of the mathematical knowledge of prospective secondary teachers; namely, the way in which their knowledge of various aspects of the secondary school mathematics curriculum is connected and how they make the connections that are in evidence.

A Conceptual Framework for Connected Knowledge

Using their recent review of some of the relevant literature on teacher knowledge, Fennema and Franke (1992) have tried to develop a cognitive model of teacher knowledge. The integration of Shulman's (1987a) work on pedagogical content knowledge; Peterson's (1988) ideas about teacher cognition; Leinhardt's study of expert teachers (1985, 1986a, 1986b); and, the perspective of Elbaz (1981) with respect to situated knowledge and practical personal understanding, leads them to conclude that teachers' knowledge is a dynamic and multifaceted construct. The diagram in Figure 1 (taken from Fennema and Franke, 1992, p. 162) illustrates the various components of teacher knowledge and how they are interrelated. Although this diagram is useful as a way of illustrating that many different factors need to be considered when developing a model of teacher knowledge, it is overly simplistic because it does not show how each of the various kinds of knowledge can be further subdivided.

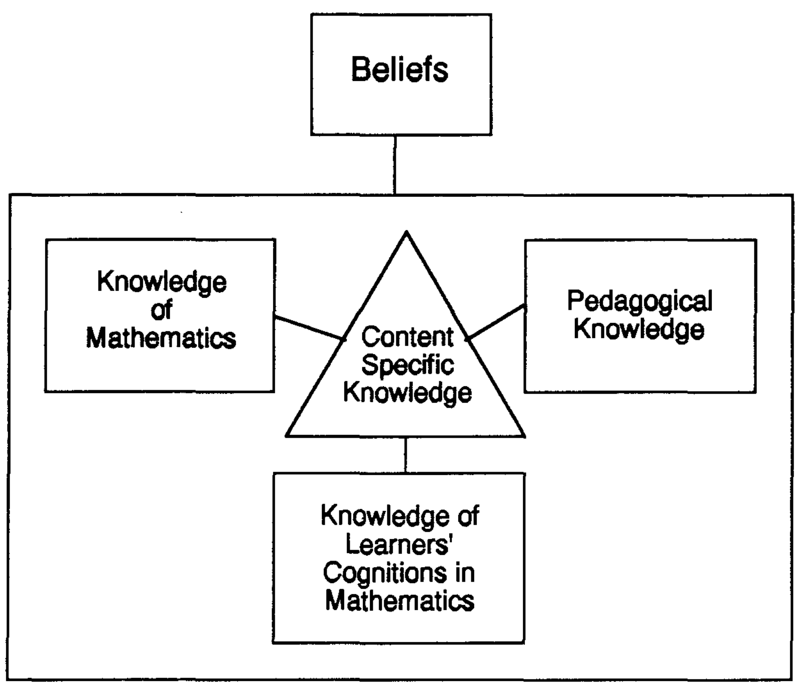

Figure 1 Components of Teacher Knowledge

The category designated as knowledge of mathematics, for example, has a number of different dimensions. Typical categories of knowledge classified under this one title could be, knowledge of the structure of mathematics; knowledge of mathematical procedures and principles; knowledge of the history of mathematics; or, knowledge of how various ideas in mathematics are interconnected. Within each of these subcategories are nested still other organizing structures.

Connections, the subject of this study, can be made in a number of different ways. A teacher could take a particular mathematical idea and connect it to a real world example; to another subject area; to another mathematical topic that the pupil has previously studied; to a pupil's way of thinking; or, to a pedagogical principle.

In order to have a clear idea about what these various connections might look like in practice, the next sections will consider several descriptions of teaching situations that prospective teachers were given to examine in this study; and, how a knowledgeable teacher *might* make various kinds of connections as a way of responding to the difficulties that pupils appear to be having in the vignettes.

Connecting to the Real World

It has been argued that pupils often perceive mathematics as irrelevant because it is rooted in school learning rather than situated in a context that relates to their own lives (Resnick, 1987). It seems reasonable to assume therefore that connecting mathematics to the real world is a useful kind of connection to be able to make. Consider this description of a typical teaching situation:

> The following question is in a homework exercise that you gave your grade 9 pupils:
>
> A woman drives from here to Toronto (a distance of 200 km) at a speed of 100 km/hr. She immediately turns around and drives back but because of the traffic she can only drive at 80 km/hr. What is her average speed for the whole trip?
>
> You notice that two pupils are arguing as they do their work and you go to see what the controversy is about. It turns out that they used different approaches and got two different results:
>
> Pupil #1: Average speed is $(100 + 80)/2 = 90$ km/hr.
> Pupil #2: Time to drive to Toronto is $200/100 = 2$ hrs.
> Time to drive home is $200/80 = 2.5$ hrs.
> Average speed = distance/time = $400/4.5 = 88.89$ km/hr
>
> They want you to judge which solution is better. How would you respond?

Pupils finding this question puzzling might be asked what their final mark would be if they got 5/10 (50%) on a short quiz and 90/90 (100%) on a major examination. The "simple average" $(50\% + 100\%)/2 = 75\%$ is clearly not an accurate representation of the individual's mark and most pupils intuitively realize this fact. They can be pushed further to explain that the 100% mark should have more weight because it is worth more and that their mark ought to be $(50\% \times 0.1) + (100\% \times 0.9) = 5\% + 90\% = 95\%$.

Connecting to other Subjects

Connecting mathematical ideas to other subject areas helps pupils to see where mathematics can be applied and at the same time makes use of intuitive ideas and knowledge that they may have developed in other classes. Using the same example as above, the teacher might ask a pupil who is having difficulty to think about the problem in terms of molecular masses of chemical compounds. For example, if carbon consists of two isotopes, one with an atomic mass of 12 and the other of 14, it does not follow that the mass of a carbon molecule will be 13 because the two isotopes do not occur with the same relative abundance in nature. Obviously it would be of little value to use this kind of example with pupils who did not take chemistry; however, for pupils who do, the discussion makes a nice link with another subject area.

Connections to other Mathematical Topics

Making connections with other mathematical topics can help pupils to see how one piece of the mathematical picture relates to the whole. To illustrate how this kind of connection can be made consider this common teaching situation:

The following conversation takes place between a grade 11 pupil and her teacher:

T: What is $\sqrt{a^2 + b^2}$?
S: I know, that's easy, $a + b$
T: No, that's not right.
S: Yes it is!
T: Could you explain please.
S: Sure. You taught that when there are several operations exponentiation comes first and you also taught us that square root is the same as an exponent of ½ so I did the square root first.
T: Hm...

How would you respond to this pupil?

The difficulty that the pupil is experiencing in this problem relates to his or her misinterpretation of the rules about order of operations and what it means to "do" the square root. From the conversation it appears that the pupil is thinking of this problem as $(a^2 + b^2)^{1/2}$. One way to relate this situation to another area of mathematics would be to ask the pupil to evaluate a similar expression with an exponent but to choose an exponent with which they are more familiar, for example $(a + b)^2$. This question can be done by rewriting the original expression in the form $(a + b)(a + b)$ and then evaluating the answer as $a^2 + ab + ab + b^2 = a^2 + 2ab + b^2$. The answer if done by the pupil's method would be $a^2 + b^2$ and so there is a clear problem.

The connection with another situation involving exponents and brackets which has its own special procedure for evaluation illustrates to the pupil why exponents cannot be distributed and makes use of a previously developed idea. However, the difficulty can also be handled by the teacher's attempt to analyze what conceptual error in the pupil's thinking is leading to the problem. Putting oneself in someone else's mode of thinking is a difficult but useful skill for a teacher; and, it allows the teacher to connect with the pupil's way of thinking.

Connection with a Pupil's Way of Thinking

In the example discussed above the pupil is making a fundamental conceptual error. It is true that multiplication is distributive over addition in the real number system, so $a(x + y) = ax + ay$. However, the belief that an exponent can be distributed in the same way that a number multiplied by a bracket can be distributed, is rooted in the belief that all functions are linear. In fact, this conceptual misunderstanding surfaces throughout mathematics courses in many different guises. When pupils, even in first year university mathematics classes, write that $\sin(A + B) = \sin A + \sin B$ or $\log(x + y) = \log x + \log y$, they are exhibiting the same lack of understanding. Alternatively it may be that in this example the pupil has not made the connection between brackets as they are normally used, for example in $(x + y)^2$, and brackets as *implied* in the expression $\sqrt{a^2+b^2}$.

The teacher must first connect with the pupil's way of thinking to be able to help him or her to solve this problem. If the difficulty is that the pupil does not understand that there are brackets implied in a square root sign then one way of making this connection would be to rewrite the question as $(a^2 + b^2)^{1/2}$ so that the use of brackets becomes explicit. Now the analogy with other similar questions becomes more obvious and the reasons for the error in logic can be realized.

If the response of the pupil is that he or she did realize that there were implied brackets and that they *did* the brackets to get their answer, then the teacher must connect with this way of thinking somewhat differently. It is at this level that arguments need to be made about under what conditions the distributive property holds and whether in this case those conditions have been satisfied. Only after a realization of why the property that they wish to apply is incorrect can pupils then realize the correct method of approach.

This example illustrates that these categories of connections are not mutually exclusive; rather, there are many interrelationships between them. For example, in this case, a teacher who connected to the fact that the student was not interpreting an implied bracket correctly might reason that the pupil needed to see other examples of implied brackets. Questions involving complex fractions where both the numerator and the denominator consist of several fractions that are added or subtracted also have implied brackets. It is for this reason that the typical approach is to evaluate the numerator, then the denominator and then divide these two fractions. In this case, the fraction bar (like the top of the square root sign) is an implied bracket.

This connection could be considered to be a connection with the pupil's way of thinking but it then leads to a connection to another mathematical topic. In some sense these kinds of interrelationships are a product of the desire on the part of the teacher to have the pupil understand the mathematics that they are teaching and they are *all* rooted in pedagogy. Pedagogical connections, therefore, need to be considered in more detail.

Pedagogical Connections

The philosophy that a teacher holds can influence his or her teaching in profound ways (Ernest, 1988). If, for example, the teacher believes that it is important for pupils to understand even routine procedures and why they work, then he or she will teach differently to one who believes that conceptual understanding is unimportant in learning procedural aspects of the subject. Consider the following case in point:

> You have taught the pupils to multiply two binomials by using the distributive property. Although most of the pupils seem to understand what you have taught, a couple just don't seem to be able to understand how to go about it. How would you explain the problem $(x + 7)(x + 3)$ to such a pupil?

A teacher who wanted the pupils to understand this process would likely try to find a way of representing the problem so that the pupils could visualize what was being done in a more concrete way. One simple way to do this is to represent the problem as that of finding the area of a rectangle with sides of length $(x + 7)$ and $(x + 3)$. Once again the clear categorization of this kind of connection as a pedagogical one is not possible. It is a connection that the teacher has made based on pedagogical considerations and yet it does relate the problem to another mathematical idea, that of area. It is the ability of a teacher to represent and transform knowledge that makes student learning possible; so, pedagogical connections are still highly significant. This relationship between and among various components of connected knowledge is represented pictorially in Figure 2.

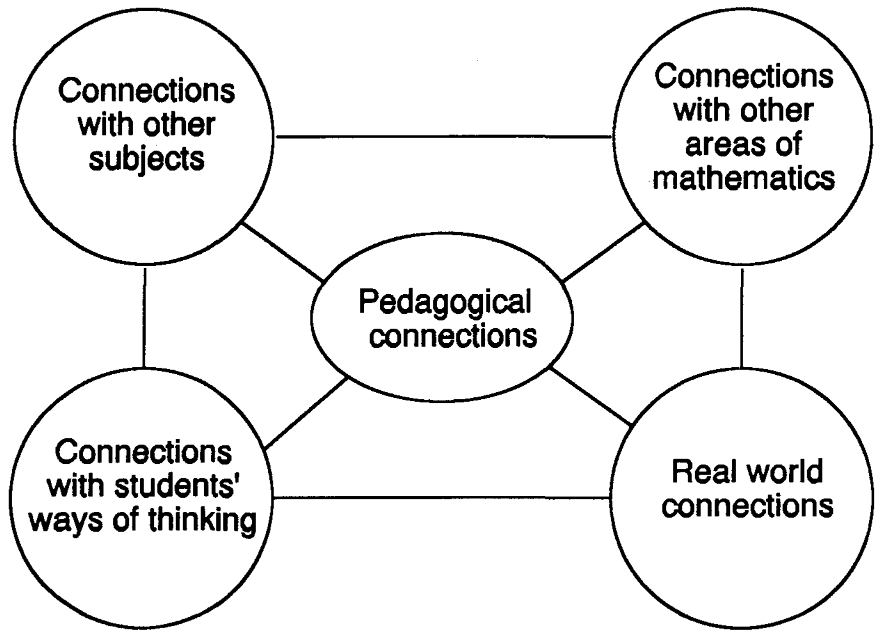

Figure 2 Components of Connected Knowledge

Method

Six one hour interviews with each of eight prospective secondary teachers (with varying mathematical backgrounds) were conducted during February and March of 1992. An earlier survey of 128 prospective teachers using a number of teaching vignettes designed to stimulate them to think about ways of making connections had already been administered and analyzed. This preliminary work revealed some items that were not as appropriate for eliciting information about connections as others. Consequently, 15 items from the original survey pool of 21 were selected to form the basis for the first three interviews.

To provide the respondents with an opportunity to make connections with some items that were slightly more focused on connections, three card sort tasks were also produced. Task one provided students[1] with a series of cards on which were written various mathematical topics, formulae, and procedures. They were asked to take some time to sort them so that items which they believed to be connected were put in the same pile. They were then asked to discuss why they perceived these items to be connected and to explicate the connections that they had identified. This task required all of interview four.

In the second card sort task, students were presented with cards which contained various mathematical definitions and they were asked to sort them into two piles once again. In this case they were trying to decide if the definitions were arbitrary and conventional or whether they were warranted in some way with a reason behind them. Once again a discussion followed the sorting. Task three provided cards which had a variety of common mathematical rules on them. Students were asked to sort these cards into two

[1] The word student is used to refer to an intending teacher while the word pupil is used to describe students in a school setting.

piles: rules they would teach and rules they would not. They were then asked to explain why they had sorted them in this way. These last two card sort tasks were combined in interview five. The resulting interviews were transcribed, coded and entered into an electronic database so that large blocks of textual data could be easily sorted and selected. The process resulted in 951 blocks of text, each of which had one or more codes associated with it. These codes reflected what the block of text was about and allowed selections based on various kinds of connections to be made easily. A total of 425 of these blocks of text had one or more connection codes associated with them.

Developing what themes and patterns were embedded in the data was an evolutionary process. The basic strategy was to work from a relatively atomistic view and then make the searches more and more inclusive until something became apparent. Initial reports that were produced were of records containing individual codes; however, these only rarely illustrated any patterns with respect to the items that were being listed. The next level of search would combine codes that seemed closely related. This second pass would produce a larger number of records in total with larger numbers of items associated with them. As the number of items associated with various codes increased in overall number, differences in the frequency of occurrence of items would become apparent and then some notion of consensus would sometimes emerge.

The Reality of Prospective Teachers' Connected Knowledge

In trying to find what was common about certain text blocks that seemed to be frequently linked to similar types of connections or rarely associated with connections of any kind, broad categories of connections began to emerge. It became clear that some kinds of connections turned out to be much more common than others (for example, numerical justifications) particularly when considered in interactions with content. Conversely, some kinds of connections were rare (for example, conceptual connections) and were only reported for a few items. The patterns reported here focus on commonalities in the data in terms of connections that were often made and those that were rarely evident.

The common connections identified were grouped into two broad (and fuzzy) categories -- strong connections and weak connections. These categories grew out of an examination of the way that connections were commonly made in text blocks that had common codes associated with them. Strong connections were typically similar to those discussed when laying out the conceptual framework. They exhibited a clear link among and between mathematical concepts, other subject areas or pedagogically powerful ways of thinking. Weak connections were often ones that had not appeared obvious to consider at the outset; but, ones that became more obvious as they continued to appear in the data.

Within the strong connection category there were four subcategories identified: mathematical connections; conceptual connections; connections to other subjects; and, pedagogical connections (these included connections to pupils' ways of thinking). Weaker connections were subdivided into four sub-categories also: numerical connections; contextual connections; procedural connections; and pedagogical connections.

Strong Connections

Mathematical Connections

Mathematical connections were connections where a clear link was established between two mathematical principles which at first glance seem to be separate. For example, in one of the card sort tasks, students made a mathematical connection when they saw the pythagorean theorem as a *special case* of the cosine law rather than as a separate topic.

Conceptual Connections

Given the rule oriented way that mathematics is often taught, it is not surprising that some students did not look at the difficulties in the teaching scenarios as conceptual problems. Indeed, conceptual connections were rare, with only a few good examples in the data. One item that did generate a strong conceptual connection from one student was Item 11, which asked students to respond to a pupil's query about the definition of the derivative of a function. In this item a pupil is wrestling with the fact that the derivative is a limit where the denominator of a fraction is approaching zero and he or she knows that division by zero produces an undefined result. Generally, however, there were only a few connections of any kind with this item (9 out of 425 text blocks).

Connections to Other Subjects

Connections to other subjects were perhaps the easiest to categorize because the students made a clear link with another discipline other than mathematics or a "real world" example. This category, however, has few examples with only 17 text blocks out of 425 exhibiting this category of connection. The item that was most commonly discussed in this context was Item five[2], with links being drawn with batting averages, marks and atomic mass.

Pedagogical Connections

The teaching context that underlies most of the instrumentation naturally focused the students' thinking on how to teach things; consequently, it is not surprising that connections with pedagogical ideas and pupils' ways of thinking were more common than the previous categories. Fifty-four text blocks were classified as having to do with a connection with a pupil's way of thinking.

In spite of the fact that many times the students could relate to what the pupils were thinking, there seemed to be a general tendency not to start with the pupil's way of thinking in their explanations. It was more common to find students providing an alternative approach to the problem; re-explaining the situation using essentially the same ideas; or making use of another rule or procedure to try to clear up the difficulty.

Weak Connections

Numerical Connections

It is not surprising that students wishing to make a connection with a pupil's knowledge should make use of numerical arguments as a way of explaining algebraic problems and difficulties; however, although common, these connections were rarely a direct mapping of the numerical situation onto the algebraic one or vice versa. Rather than building on the knowledge of number that pupils possess by trying to generalize this knowledge to algebraic situations, it was much more common for respondents to use numbers as a way of producing a counter-example to show why a particular method of doing something algebraic did not work. Alternatively they used numbers to construct an example to justify that a procedure did produce the correct numerical result.

[2] This was the item that involved finding the average of two speeds.

Contextual Connections

Contextual connections are connections (often correct) that are made on the surface features of a problem rather than on the mathematical principles underlying it. As a subcategory it was quite common with 67 text blocks having this classification as one of their codes. These kinds of connections typically arose in the card sort task which asked for connected mathematics topics to be grouped together. One pair of items on the card sort task referred to exponential growth of bacteria and the compound amount of money invested at a particular interest rate. These two formulae are closely connected mathematically because the exponential growth formula is the limiting case of the compound amount formula when the compounding of interest is instantaneous. Surprising, *no one* was able to draw this link even after considerable questioning and guidance from the interviewer, although most students still believed that they were connected because they both contained exponents. This kind of contextual fixation was common in other pairings as well and often prevented students from seeing much deeper mathematical connections.

Procedural Connections

Another kind of connection which was made on the basis of surface features was sufficiently common to be classified separately and this is the category of procedural connections. In this kind of linkage students related one item to another on the basis of the procedure that was being carried out rather than the idea embedded in the procedures. Many students linked together the cards that contained a binomial multiplication and the multiplication of two numbers using the formal multiplication algorithm; but, only because they were both multiplication problems. Students seemed to look at these as strictly procedural without any mathematical reasoning behind them.

Pedagogical Connections

In some sense weak pedagogical connections can be thought of as missed strong connections. That is, the students' comments are rooted in pedagogical concerns and their remarks may reveal a good insight about pupils' flawed understanding; but, they are relatively weak because they fail to make a clear conceptual connection based on mathematical principles between the pupil's ideas and a way of teaching that might be pedagogically powerful.

Discussion of Results

The fact that students, regardless of mathematical background, are relatively poor at making connections is troubling and counter-intuitive. However, a deeper examination of the various forces and experiences that shape students' thinking before they begin teacher preparation makes the findings less surprising. The high school experience with its rule and procedurally oriented methods gives pupils a false idea of what mathematics is like as a discipline of inquiry. Textbooks, administrators, curriculum guides and provincial testing programs all contribute to this image of the subject. Pupils go on to university with a view of mathematics that does not appear to emphasize connection making and there is no evidence to suggest that this perspective is changed by university study.

University studies are often highly specialized and sometimes poorly taught. Ironically the students who have the strongest preparation in mathematics also know the least about anything else. This specialized study makes them certain about their mathematical knowledge and so they tend to give quick answers in response to pedagogical problems – answers that are often based on procedures or rules. They are not predisposed to think about connections and consequently they do not make them unless pushed to do so.

The interaction between university arts and science faculties and the education schools is often very limited and each group assumes things about the other's role. In the process, the content knowledge of prospective secondary teachers is *assumed* and is not examined in any meaningful way prior to teaching.

Students do not perceive a connection between the world of the scholar that they have just left and the world of the practitioner that they are about to enter. Consequently they do not try to make links between the mathematics that they have studied in university and the content that they are teaching in secondary school. The result of this separation in the students' minds between school mathematics and university study, coupled with the fact that content issues are rarely addressed in any meaningful way with prospective secondary teachers, is that the ability to make connections between and among mathematical topics does not appear to be well developed in this group of prospective teachers.

Implications for Practice

The results of this study imply that there is a gap between the kind of connected knowledge that appears desirable and the level of connected knowledge that prospective mathematics teachers exhibit. Although there does not appear to be a clear difference among students with varying amounts of mathematical preparation it does appear that beliefs, dispositions and attitudes do have a role to play. Furthermore, being required to communicate mathematical ideas seems to have an impact on connected knowledge and understanding. However, despite the fact that this study used two different data gathering strategies and non-traditional techniques for ascertaining prospective teachers' knowledge, finding out what people really know about a subject remains problematic.

Given these implications it seems reasonable to suggest that if teachers are to develop a connected understanding of the subject, some changes in the way that prospective secondary teachers are prepared need to be made. For example, honours courses could be made more flexible to allow for specialists to study some courses which are not narrowly focused on technical detail; but, which consider elementary content from an advanced perspective. This kind of study could be focused on helping students to link their high school mathematics knowledge to the material that they are learning in university. Students also need sufficient flexibility in their programs to allow them to study other subjects that are mathematically related.

Although there are many committed individuals who work hard to provide undergraduate students with the best possible mathematical experience, the reward structure of universities does not make such actions attractive. Professors who are interested in developing innovative courses and teaching methods need to be rewarded for these efforts otherwise they will remain the exception rather than the rule.

Teacher education professors need to realize that there are strengths and weaknesses inherent in the knowledge of *all* the students they teach, regardless of mathematical background. Discussions of pedagogy should be integrated with content issues so that weaknesses can be at least partially addressed. Using cases that generate cognitive dissonance may be one way of having students develop more sophisticated understandings of the naïve concepts that they often hold, even after university study.

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Topic Group A2

Narrative Inquiry in Mathematics Teacher Development

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Narrative Inquiry in Mathematics Teacher Development

Although narratology has existed for a long time, it is only recently that narrative has been adopted as a central focus in conducting educational research, particularly in the study of teaching and in teacher education. Over the past decade, a growing body of literature has been promoting the use of narrative or story not only as object of inquiry, but also as method of inquiry (Butt and Raymond, 1987; Carter, 1993; Connelly and Clandinin, 1988; 1990, 1991; Elbaz, 1991; Solas, 1992). These uses of narrative in the study of teaching are grounded in the growing recognition of story as a fundamental structure of human experience; of teaching as a human activity framed in personal, cultural and historical contexts of the teacher; and of the importance of framing teacher research within a humanistic context.

Although a discussion of narrative inquiry could be useful to researchers interested in mathematics teaching, the goal of this paper is not to consider narrative from a research perspective, but a teacher development perspective. My intent is to advocate the use of the processes of narrative inquiry to facilitate mathematics teacher development. However, to establish the framework for my discussion, I will briefly describe some of the characteristics of narrative and narrative inquiry that make them relevant to teacher research and teacher development.

Polkinghorne (1988) describes narrative as the primary form by which human experience is made meaningful; a scheme by means of which human beings give meaning to their experience of temporality and personal actions; a meaning structure that organizes events and human actions into a whole, thereby attributing significance to individual actions and events according to their effect on the whole. Similar descriptions of narrative and story can be found in the works of Bruner (1990), Carter (1992), Connelly & Clandinin (1988, 1990, 1991) and others. This view of narrative suggests that the narrative one constructs or the story one tells, reflects who one is and thus contains the meaning of one's actions. Thus narrative provides a framework for understanding the past and current events of one's life and for planning future actions.

Narrative inquiry makes use of these characteristics of narratives to understand human experience. To engage in narrative inquiry is to engage in a study of experience, personal experience of the participants of the inquiry. In education, narrative inquiry focuses on personal experience to capture the meaning of specific classroom actions for teachers and students. More generally, narrative inquiry seeks out the deeper meaning of one's actions – the way in which one makes sense of the world – as embodied in one's personal story. Thus it can be viewed as a meaning recovery and a meaning construction process involving the storying and restorying of one's personal experiences or one's self. Connelly and Clandinin (1990) describe narrative inquiry as a process of collaboration and mutual storying and restorying. Thus narrative inquiry shifts the interpretive process from a researcher's interpretation of observed data to a mutual researcher-participant reconstruction of meaning in action. The researcher and participant work together in a collaborative relationship to make sense of the participant's behaviour from the participant's perspective to understand what he or she sees, and how he or she has been making sense of it. Connelly and Clandinin (1990) emphasize the importance of the mutual construction of the research relationship, a relationship in which both practitioner and researcher feel cared for and have a voice with which to tell their stories.

This brief description of narrative and narrative inquiry presents some of the qualities of narrative research that can be adopted in teacher development when the latter is viewed as a process of helping teachers to make sense of their teaching. The remainder of this paper is based on this view of teacher development with particular focus on inservice teachers.

The traditional view of inservice teacher development has been to provide teachers with theories, sample activities and instruction on implementation. This is probably still the predominant mode of working with mathematics inservice teachers. Although this approach has helped teachers gain new insights about teaching and has even helped many to change or enhance their practice, by itself, it seems to be an incomplete approach to have a significant long term effect on teacher thinking. In the past few years, teacher thinking, particularly with respect to how it relates to the teacher's actions, has become a significant consideration in teacher education as the missing link in educating both preservice and inservice teachers. Making teachers aware of their thinking has become a goal of many teacher educators.

With respect to inservice teachers, one of the growing focusses is to help them understand their classroom actions. One perspective taken to accomplish this is not to isolate their professional experience from their life but to view teaching events or teachers' actions as being framed within the context of the teacher's life history or personal story. As Butt and Raymond (1987) point out, "Each teaching action and the thinking associated with it is nested within uniquely personal, situational and contextual determinants and influences. To understand teachers' classroom behaviour, to focus on the behavioural skills alone without reference to the personal context is misguided and liable to prove ineffective." Similarly, Bruner (1990) notes that our actions are not self-contained packages of who we are, but are inextricably linked to our past and future and make no sense in isolation from each other. Our actions are manifestations of how we make sense of the world as individuals. Thus to understand our actions, we do not simply observe and theorize about them, but we look at our history or biography and our intentions to recover the framework that gives meaning to them.

This view of actions and teacher behaviour provides a framework that validates professional development through teachers reflecting on their own narrative and examining their own histories and teaching styles, to understand their teaching and what they bring to the situation. The literature reveals that there could be many benefits of teacher development when it is seen and presented as a process of self-understanding grounded in the teacher's life and work. Connelly and Clandinin (1988), for example, discuss the importance of self-knowledge and self-understanding as a key to professional growth. They argue that self-understanding in the form of reflection on one's personal and practical knowledge of teaching comes before meaningful and substantial changes in behaviour. In mathematics, Celia Hoyles (1992) reported on a set of studies that focused on inservice teacher training where it was argued that it made sense to explore the belief systems of teachers before attempting to introduce changes. Such explorations, however, should include self-explorations and be a part of mathematics teachers' self-development.

At a time when mathematics teachers are being asked to reconceptualize their teaching, self-understanding, particularly with respect to their professional experiences, could help to pave the way for a personally meaningful consideration of changes that are being advocated in mathematics education. Many mathematics teachers who seem to be trapped in a traditional mode of teaching are unlikely to break through this barrier unless they become aware of the nature of it as manifested in their teaching and understand the meaning of it from their perspective. Narrative inquiry provides a means of helping mathematics teachers to know themselves as mathematics teachers. However, narrative inquiry involves processes that are in conflict with the traditional view many teachers (and mathematicians) hold of mathematical thinking as being impersonal and objective. Thus, in my experience, mathematics teachers' initial response to narrative inquiry is resistance because it requires them to look within themselves as opposed to outside themselves. Furthermore, they tend to not see their teaching as something they construct but a state they find themselves in, a state established for them by the nature of mathematics as they perceive it to be. Thus they tend to see reasons and meaning of their actions in the classroom as being external to themselves. In fact, they would rather discuss and analyze their students instead of turning the lens on themselves. But once they are able to penetrate their traditional lenses, they find the experience of knowing themselves to be very liberating. Getting them to this point, however, is less likely achieved if they are simply told to reflect on their teaching. Such reflections tend to result in the recalling and documenting of surface behaviours and surface meanings instead of recovering the deeper meanings that are embodied in the unconscious dispositions that influence their behaviour. Thus they have to be provided with more concrete help in knowing where to look, what to look at and how to look at it to be able to attend to such meanings at a conscious level to bring into focus basic dimensions of themselves and their teaching that are taken for granted. Three ways in which the narrative framework can be used to provide this help are: sharing and resonating in stories of their teaching of mathematics; conducting narrative interviews; constructing autobiographical narratives. A brief account of each of these follows. More details relevant to these processes are discussed in Chapman (1992, 1993, in press).

Sharing and Resonating in Stories

In this process, the teachers share stories of personal experiences in their teaching. The stories must be telling of their classroom behaviours and include as much detail as possible. The stories are not analyzed. Instead, as each is told, the teachers resonate in it. One way in which this can occur is for the teachers to think about themselves as they listen to the story instead of focusing on the teacher (storyteller) in the story. The latter usually leads to a critique of the storyteller, which is undesirable in the narrative framework and is counterproductive in achieving personal sharing since it can silence the participants. Thus it is important for the teachers to focus on themselves and use the shared story to stimulate reflection on their own teaching. They can then share stories that are triggered by the initial story and the process perpetuates itself. Through this narrative sharing the teachers learn from each other and learn about themselves, as behaviours that are taken for granted are brought to the surface and allow them to see themselves in terms of their lived experiences as opposed to their theoretical pronouncements.

Another way of facilitating resonance is having the teachers read articles and/or view videos on mathematics teaching and using them as a means of reflecting on their own teaching to get to different aspects of it that might have otherwise been overlooked. As in the case of the stories, the teachers should not merely analyze the articles or videos in isolation of themselves, but resonate in them by reflecting on experiences in their teaching that reflect or contradict the situations dealt with in the articles or videos. Thus the readings or videos serve to enlighten them about the literature, other teachers' situations and themselves.

Narrative Interview

In the narrative interview, the teachers can take on the role of researcher and participant to investigate each other's situation. Working collaboratively in groups of three or four, the teachers can take turn at being participant and researcher (for example, one participant and two or three researchers). The researchers' goal is to work collaboratively with each other and with the participant to recover some meaning of how the participant makes sense of his/her teaching of mathematics.

The interview should unfold in an experiential framework and focus on lived meanings within the participant's experiences as opposed to merely attributing meanings "out there" to the participant's teaching or theorizing about it. The interview should not be a formal question and answer exchange in which the participant responds to questions in the categorical form required in such formal exchanges. The participant should be encouraged to respond in the narrative form of natural conversation and to share stories of his/her teaching of mathematics or other relevant experiences. In general, the interview should be conducted in a way that requires both the participant and researchers to enter into the "research" situation and partake of it. Thus the interview should be more of a conversation in which the researchers could resonate in the participant's story whenever it triggers memories of their own. The researchers should share of themselves not only to resonate in the participant's situations but to use such sharing, instead of only formal questioning, to get deeper into the participant's personal story. Thus, in coming to know the participant and helping him or her to know himself or herself, the researchers should also gain insight into themselves.

The "data" from the interview and the interviewing process itself form the basis of identifying themes in the participant's behaviour and recovering meanings of his/her teaching. The process can also help everyone involved to better understand how to research himself or herself, that is, how to engage in autobiographical narrative processes to understand his or her own teaching.

Autobiographical Narrative

Autobiographical journals have been receiving growing attention in mathematics teacher education. However, unlike narratives, they generally deal with the teachers' experiences in a fragmented way since each journal tends to focus on an isolated event or behaviour. Autobiographical narratives allow teachers to see their teaching in a holistic way. As Polkinghorne (1988) notes, "Narrative is a meaning structure that organizes events and human actions into a whole, thereby attributing significance to individual actions and events according to their effect on the whole. Thus narratives are to be differentiated from chronicles, which simply list events according to their place on a time line. Narrative provides a symbolized account of actions that includes a temporal dimension."

Three ways (which are not necessarily mutually exclusive) to engage mathematics teachers in the construction of such narratives are:

(1) In conjunction with the two preceding processes (resonating in stories and narrative interviewing), each teacher can construct an autobiographical narrative of his/her teaching, focusing on the specific meanings identified in these processes in terms of how they unfolded in his/her teaching.
(2) The teachers can write journals and stories of specific teaching events and use them to identify themes that can then be used to provide the "plot" of the narrative.
(3) The teachers can trace the development of their teaching using turning points (or lack of them) as focuses of reflection to understand how their teaching has evolved and to establish a basis of restorying their future behaviour.

The narrative should be experiential in that all claims the teachers make about their teaching or the way they make sense of their world should unfold through accounts of personal and professional events or stories. It also should be temporal in that it should portray the teachers' classroom behaviour as it unfolded over the years of their teaching, how the meanings identified evolved over time within their personal lives and some consideration of the future. Finally, to avoid a surface description of all they had learned about themselves and their teaching during the narrative processes, the teachers should focus on only one or two meanings that are most prominent in their behaviour and deal with those in as much depth as possible.

Conclusion

Narrative inquiry provides a framework in which mathematics teachers can use stories of their teaching of mathematics to understand their teaching in terms of the meanings embodied in the stories. My experience in working with mathematics teachers show that these processes can help them to reflect more deeply about their teaching and that this enhanced reflection can result in positive changes in their classroom awareness and their restorying of their teaching. This paper was intended to draw attention to this area of teacher development for further consideration by mathematics teacher educators. However, it has not dealt with the area in great depth. The references can be used to acquire more details to extend what was presented here.

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Topic Group B1

Compétences spatiales géométriques chez de jeunes Québécois du Nord (Inuit) et du Sud

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Geometric Spatial Competencies among Young North Quebecers (Inuit) and South

Abstract

We shall examine the impact of the environment on the devolopment of spatial competency of geometric objects. We shall then present a new typology for classifying spatial competencies. The results obtained from interviews with 10- and 11-year-olds from radically different environments show some interesting contrasts. The findings lead to a questioning of certain aspects of Piaget's theory, and may suggest some adaptations of the current educational model used in the teaching of geometry.

Résumé de l'article

Nous nous intéressons à l'apport de l'environnement physique et culturel dans le développement organisé des habiletés perceptives et représentatives d'objets géométriques situés dans un micro-espace. A cette fin, nous avons élaboré un instrument de mesure des habiletés spatiales. L'objet de l'expérimentation relatée ci-après traite des relations entre ces habiletés spatiales et les types d'espace qui environnent le sujet. L'intérêt de cette démarche est de provoquer éventuellement une diversification des interventions didactiques dans l'enseignement de la géométrie et de toutes autres disciplines touchant à la maîtrise de l'environnement, comme les arts graphiques, qui tiendraient compte des types d'espace qui environnent les sujets.

La perception structurale de l'espace

L'espace peut se caractériser de plusieurs points de vue: physique, social, géométrique, etc. (Alsina et al, 1987) Notre recherche s'est intéressée à la perception d'un espace géométrique. Cette perception peut s'examiner sous un angle formel ou structural. Alors qu'une perception formelle consiste en l'intériorisation quantitative d'un modèle spatial par l'analyse et la synthèse de ses propriétés en termes de rapports, de proportions, de mesures et de coordonnées, la perception structurale considère plutôt l'intériorisation qualitative d'un modèle spatial par l'analyse et la synthèse de ses propriétés topologiques, projectives, affines et métriques (voir Baracs, 1988). Nous privilégions dans notre étude cette dernière approche. "La représentation spatiale est une action intériorisée et non pas simplement l'imagination d'un donné extérieur quelconque." (Piaget et Inhelder, 1948, p. 539)

La matrice du développement de compétences spatiales géométriques

L'instrument que nous avons développé (Baracs et Pallascio, 1981, 1983; Pallascio et al., 1985; Mongeau, 1989), est défini sur la base d'un tableau à double entrée. Une de ces entrées est définie par cinq (5) opérations intellectuelles correspondant à des compétences spatiales ("relation spatiale" et "visualisation spatiale"), alors que la deuxième entrée est définie sur quatre (4) niveaux géométriques.

Les opérations intellectuelles sont respectivement la transposition, la structuration, la détermination, la classification et la génération. La **classification** consiste à grouper des structures spatiales selon un choix de propriétés ou paramètres géométriques communs. La **structuration** consiste à identifier les propriétés et la combinatoire géométriques d'une structure spatiale. La **transposition** consiste à établir les correspondances, les équivalences, et à effectuer le passage entre les différents modes de représentation (physique, linguistique, algébrique et géométrique) et niveaux géométriques. La **détermination** consiste à délimiter les éléments ou les paramètres définis par des contraintes géométriques sur une structure spatiale. Enfin la **génération** consiste à produire ou modifier une structure spatiale de façon à ce que cette structure réponde à certains critères géométriques prédéterminés. Les niveaux géométriques sont les niveaux topologique, projectif, affine et métrique.

Compétences	Opérations	Modes Géométriques			
		Topologique	Projectif	Affine	Métrique
Relation spatiale (Analytique)	Classification Structuration				
	Transposition				
Visualisation spatiale (Opératique)	Détermination Génération				

Figure 1 Matrice du développement de la compétence spatiale géométrique

Les types d'espace

Alors que le **micro-espace** est le lieu de la manipulation de petits objets où il est facile pour le sujet de changer de points de vue par rapport à l'objet, et que le **méso-espace** est l'espace des déplacements du sujet dans un domaine contrôlé par la vue et qui s'obtient par le recollement de micro-espaces connexes, le **macro-espace** est celui qui nécessite une représentation implicite des mouvements relatifs à plusieurs systèmes de références, que l'on pourrait imager par un "recollement de cartes", selon l'expression de Brousseau (1986).

Nous avons cherché à déterminer les relations qu'il pouvait y avoir entre un macro-espace donné et les habiletés perceptives et opératoires appliquées à un micro-espace, comme celui des formes géométriques utilisées dans un test-entrevue élaboré pour valider notre instrument de mesure. Pour ce faire, nous avons choisi et comparé deux groupes de sujets, dont l'environnement macro-spatial est radicalement différent: un groupe d'enfants vivant dans un environnement rural du sud du Québec et un groupe du même âge vivant dans un village Inuit du nord du Québec.

Au niveau du **micro-espace**, les enfants du sud, en milieu rural ou urbain, sont davantage initiés au dessin imaginatif ou figuratif, plutôt qu'au modelage de formes tridimensionnelles, alors que les enfants Inuit sont initiés très jeunes à la sculpture de la pierre à savon, tandis que le papier demeure une denrée plus rare.

Au niveau **méso-spatial**, l'environnement visuel varie sensiblement d'un milieu à l'autre. Alors qu'en milieu rural, les habitations sont des prismes rectangulaires allongés, étendus ou pyramidés (fermes, demeures isolées...) et qu'en milieu urbain les édifices sont essentiellement des prismes rectangulaires, les habitations traditionnelles des Inuit, les Igloo (mot qui signifie "maison" en inuttitut, la langue des Inuit), que les enfants apprennent encore à construire lors de sorties familiales pour la chasse ou la pêche, sont formés de pyramides tronquées, où le parallélisme ne domine pas.

Enfin, au niveau **macro-spatial**, alors que les dénivellations sont variables en milieu rural et fortes en milieu urbain (métro, stationnement souterrain, édifices à plusieurs étages...), c'est plutôt un espace bidimensionnel qui s'ouvre à l'horizon de l'Inuk qui doit compter sur des accidents de terrain épars pour se repérer dans la toundra.

La méthodologie

Le test utilisé, administré par entrevue individuelle, était composé d'une douzaine de tâches ou problèmes à résoudre, couvrant nécessairement une partie seulement de la matrice du développement spatiale, correspondant à l'un ou l'autre des niveaux topologique ou projectif, à l'une ou l'autre des opérations intellectuelles.

Les deux groupes d'élèves comparés étaient composés de 16 enfants, des élèves de 5e année primaire. Un premier groupe (du sud) était formé de 8 garçons et 8 filles, alors que le second groupe (du nord) était formé de 12 garçons et 4 filles, tous et toutes des Inuit, sauf un jeune Amérindien du peuple Cree. Le test, limité à 13 tâches, d'une durée de 40 minutes, a été administré au printemps 1988.

Nous donnerons les résultats des deux analyses statistiques suivantes: un test "t" permettant de comparer les performances des deux groupes de sujets à chacune des 13 tâches qui leur étaient proposées, et une analyse factorielle des correspondances, utilisant la distance du chi-deux pour mesurer les distances entre les sujets.

Les résultats

Nous avons constaté (Pallascio et al., 1990) que les deux groupes s'opposent radicalement au niveau analytique (**relation spatiale**) et au niveau opératoire (**visualisation spatiale**), au niveau des propriétés géométriques, topologiques et projectives, et au niveau des opérations intellectuelles dominantes. Le sexe des sujets n'intervient pas, ni à l'intérieur des groupes, ni globalement.

Les espaces différents qui définissent les environnements des deux groupes de sujets ne sont probablement pas la cause unique des différences observées dans la perception et la représentation des objets géométriques micro-spatiaux. Au niveau méso-spatial, par exemple, certaines constructions coutumières chez les Inuit leur font manipuler des objets aux propriétés davantage projectives qu'affines (p.e.: les blocs de neige servant à la construction d'un igloo sont des pyramides quadrilatérales tronquées et disposées en spirale, et non des parallélipipèdes). Mais les relations et les incidences que nous avons identifiées sont suffisantes pour nous questionner sur la nécessité d'établir des parcours différenciés dans le développement des habiletés spatiales.

Notre objectif à long terme est d'identifier des parcours à l'intérieur de la matrice du développement spatial, qui permettraient de déterminer des cheminements naturels de développement. L'environnement physique et culturel dans lequel se trouve un individu peut influencer son type de parcours. Compte tenu des différents types d'environnement, l'individu pourrait privilégier des parcours plus ou moins abrégés à travers les éléments de la matrice du développement spatiale. Un modèle d'intervention didactique plus "ouvert" ou plus souple pourrait s'avérer très productif pour le développement des habiletés spatiales, chez les jeunes plus particulièrement.

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Topic Group B2

Projet d'Auto-apprentissage Informatisé

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Projet d'Auto-apprentissage Informatisé

Dans cette session, on a présenté un projet de logiciel d'auto-apprentissage des mathématiques. Le logiciel est actuellement en cours délaboration. Certains concepts sont presque terminés, tel le concept de valeur absolue ainsi que tout une partie de la trigonométrie.

Le logiciel s'adresse à des élèves de niveau secondaire deuxième cycle, première année d'études collégiales. Les concepts ou les thèmes traités sont pris à dire en se basant sur des connaissances préalables minimums. Pa example le thème de valeur absolue débute par la notion concrète de distance formelle mathématique que l'apprenant est amené à découvrir. Des manipulations de segments à l'écran permette à de découvrir la signification de la valeur absolue de la différence de deux nombres comme la distance entre ces deux nombres. Utilisant au maximum la représentation géométrique de la valeur absolue comme une distance, l'apprenant est amené à résoudre des équations et inéquations du type:

$$|x - 3| + |x + 2| = 12$$

$$|x - 3| + |x + 2| < 12$$

$$|x - 3| - |x + 2| = 12$$

Les rapports trigonométriques sont introduits ce façon concrète aprés une mise en situation par tes problèmes réèls. Cette mise en situation permet d'introduire les rapports trigonométriques et par la suite de résoudre les problèmes posés.

La particularité du logiciel est de viser à une interaction maximum entre l'apprenant et l'ordinateur. L'apprenant est appellé à écrire à l'écran, à dessiner à l'écran, à déplacer d'appentissage moins monotone.

Le logiciel de base utilisé pour réaliser ce projet est le système auteur *Authorware Professional.* Ce système auteur d'une grande puissance permet de réaliser l'interaction désiré.

Topic Group C

Teaching and Learning University Level Mathematics: Perspectives of a mathematics educator

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Teaching and Learning University Level Mathematics

While preparing for this talk, I ran across *Challenges for College Mathematics: An Agenda for the Next Decade*, a report of a Joint Task Force of the Mathematical Association of America and the Association of American Colleges. In it, 13 areas are identified on which those interested in university mathematics education should focus (see Figure 1). The authors stress that these are issues of context, attitude, and methodology, rather than issues of curriculum. Many of these areas, as you can see, overlap–how can we talk about effective teaching, for example, without thinking about learning?

I'm not going to talk about all of these "challenges" tonight but I think they represent an excellent list of the issues facing mathematics education, particularly at the university level. Before moving on, however, there is one area which I think has been overlooked and which I would like to add to the list: assessment. As the other challenges are addressed, we are going to find that we need to identify and apply new assessment strategies. These should be aligned with instruction and should elicit a breadth of information about a student's abilities and progress. Assessment results can also provided information about instruction (what is successful, what needs more attention, and what needs to be dropped) and about the program. Quizzes, tests, and examinations, at least as they have traditionally been designed, will not sufficiently meet these needs.

1. Learning
2. Teaching
3. Technology
4. Foundation
5. Connections
6. Variety
7. Self-esteem
8. Access
9. Communication
10. Transition
11. Research
12. Context
13. Social Support

Figure 1: Challenges for the 1990's

What I *am* going to focus on tonight are some of the observations, experiences, and conclusions which have emerged from working for the last several years with an "experimental" calculus class at Dalhousie. These will touch on at least 7 of the areas identified above – learning, teaching, technology, self-esteem, communication, social supports, and assessment. I'll start by giving you the background for the experimental course, then discuss what we've done in the teaching and learning areas. If we have time, I want to engage you in an activity similar in structure to one we use with our students. Finally, I want to discuss some of the general factors which I think might contribute towards the revitalization of undergraduate mathematics.

Background History

The Mathematics Department at Dalhousie has traditionally taught first year calculus in a lock-step approach. It's a fairly safe bet that on any given day, all instructors are lecturing on the same topic. Students are told that they can attend any tutorial, regardless of which class they are registered in. All sections have common weekly assignments, term tests, and final exams. The instructors I work with in the experimental sections found this format – inflexible in scope and in sequence, rigid in timing for tests, etc. – unacceptable and set out to do something different.

It's hard to identify just when the idea for the "experimental" course started. As I recall, one of the big factors in its birth was serendipity. While attending a symposium sponsored by the President of our university on improving university teaching, I happened to be sitting behind a man I knew from the

Mathematics, Statistics and Computing Science Department. After it was over, we had a casual talk about what we thought might be learned from the session. Several months later, he called to tell me that he was thinking about introducing computers into the calculus course, that the Department had given him the go ahead, and that he wanted to continue the conversation we had started at the symposium. The rest, as they say, is history.

The first year, we set out to introduce technology into the calculus curriculum. Using a very simple, user-friendly, symbolic manipulation package, TRU BASIC, we were able to have students engage in problem solving, rather than merely doing mechanical exercises. Our students wrestled with problems which came out of a real context, rather than the contrived atmosphere of the textbook. And, with the time freed from routine calculations, we found that we could stress concepts rather than algorithms.

My (volunteer) role in this program was both central and peripheral. I was not the teacher. I suppose I might call myself the pedagogical advisor. The first year, I went to every class, took notes, made suggestions about possible activities, critiqued lessons and lesson plans, encouraged the instructor when he was down, conducted interviews with students, and tried to distil from all of this what was REALLY happening in the class. The second year, when a second instructor became involved, I continued in the same role, although I couldn't go to every class. I also facilitated discussions between them. They were not used to sharing ideas about teaching and learning and found that I was a productive and non-threatening catalyst.

The presence of the computer in the calculus curriculum was motivational to the students. On a simplistic level, they liked interacting with the machines. It made them feel that they were up-to-date. On an instructional level, once students grew comfortable with this break from what they thought a mathematics class *should* be, they liked the types of problems they could solve – those which emphasized interpreting results, used graphics, were examples of real-world modelling, supported their intuition, and encouraged hypothesizing.

As you might expect, we also ran into some difficulties as we introduced the technology. For example,

1. There were no computer-based materials available to us at that time so we had to develop our own. This took time; we sometimes went down the wrong path.

2. Our own classroom was not well set-up for using the machines. When we wanted to demonstrate something using the technology, we rolled a computer into the classroom on a cart. This worked against being spontaneous (unless we brought the computer every day).

3. Students had some problems getting on the machines when they needed to use them. The computers were frequently booked by others. This is changing now. There are more machines available; hours are better; more and more students have their own machines.

4. Not all students were comfortable around technology. We had to help them overcome this anxiety.

5. We were not quite sure how to adjust our evaluation procedures to recognize (or incorporate) the use of technology or, indeed, the other teaching strategies we tried.

As we explored integrating the technology into the calculus curriculum, other aspects of our teaching also changed. Group work, for example, seemed to be a natural outcome of the students working on the machines. They wanted to discuss with each other what they were doing and what they were finding. These "other" changes will be discussed a little later.

Others in the Department didn't show much interest in what we were doing. They might have been mildly curious that we used the computer, but it was obvious that we were spending a lot of time doing what we were doing – reconceptualizing what mathematics we wanted to emphasize, developing instructional materials for the students, exploring "alternative" assessment strategies, marking, re-thinking our own roles as teachers, etc. Not many wanted to spend their time in the same way. Whenever discussion did arise however, the question we were asked – and it was a fair one – was "Are these students doing better mathematically?" These teachers were wary. If *they* were going to consider doing any extra work, or making changes, they wanted to know it would pay off.

We, of course, were also interested in this issue. We know from anecdotal accounts how positively the students reacted to the course. Students informally let us know how much they liked coming to class and how much they thought they learned. I was also told this in the formal exit interviews which I conducted with students at the end of each term. We also know that, after the first year, the two experimental sections filled up first when students enrolled in courses. The word-of-mouth reports back to the high schools were positive – and I don't think the reports were that the course was easy.

I tried one year to measure students' attitudes about mathematics and about themselves as students of mathematics. I had control groups and our groups. However, the invigilator for the post-test forgot to have the students put their student numbers on the tests, thus we couldn't compare pre- and post-test results for individuals.

When we were permitted by the Department to develop our own tests, we discussed the possibility of asking some of our colleagues to rate the difficulty level of the problems we used. The two instructors for "our" course felt that these colleagues could judge what would be conceptual problems, problem-solving problems, mechanical problems, etc. relative to the expectations of the standard calculus class. We felt that we were routinely giving challenging problems, ones which were conceptual in nature, requiring more than merely the application of algorithms. The students felt this too. We wanted someone with some distance from the course to confirm or challenge this. Unfortunately, we don't have those appraisals yet.

This year, we were plunked back into the mainstream. We were "allowed" to teach anyway we wanted but our students had to complete the same sets of assignments, tests and examinations as all the other Calculus students on campus. In large part, because of these restrictions, we did not use the computer as part of the class. The curriculum we were locked into and the questions we were forced to use for evaluation did not lend themselves to using the technology well.

I wish that I could report that our students outshone the others – that they did outstandingly on the common tests, but I can't. Indeed, if anything, our students did a bit worse. We tried to massage the data. We looked to see if our students entered the course with lower academic backgrounds than the other students – they didn't. We tried to see if we had a significantly different number of students from other provinces than the comparison sections – we didn't. We even looked at gender to see if there were any anomalies there. There were not.

Despite this frustrating experience, we were gratified when the colleague who took over the section for the second term reported back to us how talkative our students were. She indicated that they were asking excellent questions, often conceptual and probing. She wondered what we had "done" to them.

What we learned this year from merging traditional assessment techniques with alternative teaching methods, or rather, what we had reinforced, is the importance of aligning assessment with instruction. We knew this, of course, – when we were in control of our own testing we did this. We used group tests, we evaluated writing assignments, we let students have access to computers during testing, we let students bring in support materials, we gave "untimed" tests, in that they were take-home test, etc. Comparable techniques should be used for both instruction and evaluation!

The exit interviews I mentioned earlier revealed something which surprised us. You will recall that I said that we started the course with the idea of integrating technology into the curriculum. We found that this was NOT what the students were finding the most enjoyable or profitable about the class. What they were excited about was their increased sense of self-esteem and the sense of community which had developed amongst the class. This arose in large part through the revised roles of both the students and the teacher. Students were actively engaged in mathematics through the types of problems we were exploring and actively engaged with each other through extensive use of group work. The teacher was a facilitator, not just a lecturer. The way we were developing the course was changing the culture of the classroom, their mathematical experiences, and ours.

So what was this classroom like? At the start of the experimental program we had lots of discussions about what our goals were – what we valued. We acknowledged, quite sensibly, that we couldn't teach them all the mathematics we might want them to know. Thus, what we wanted to do was help the students learn how to learn math, what questions to ask, how to know when they have solved a problem, etc. We wanted them to learn how to read and write mathematics. We wanted to emphasize concepts rather than techniques

We also acknowledged that most students come to university expecting learning to be passive and boring; chalk-and-talk, bite-sized problems to be solved by techniques provided by the textbook section in which the problems appears, feelings of anonymity, etc. When students start classes, they often find their expectations are confirmed. They are lectured to in LARGE classes. There is no opportunity for interaction between instructor and students. Timed paper-and-pencil tests are routine.

We have tried to break this practice. We are trying to get students to become active learners and, in particular, we are trying to get them to think about their mathematical thinking. To do this, we have used many of the ideas currently fashionable in the literature about secondary school learning. Figure 2 presents a (partial) list of the mathematical "actions" we promoted. For each of the actions there is also a mention of at least one of the "means" we used to accomplish this.

Not all of these worked as well as we might have liked. Some, like the JIGSAW, we want to reconsider to see if the time and energy spent were the best way to get the results we wanted. BUT, for the most part, we and the students are satisfied that we are on the right track.

Had time allowed, I wanted to have you participate tonight in one of our most productive cooperative learning techniques, CLUE CARDS. Students, working in groups of four or five, share information which will lead to solving a problem. They then work collaboratively to find the answer (if it exists). It is an easy technique for the students to learn; it is an easy technique for the instructor to develop materials for. Those interested in knowing more about this cooperative learning structure might consult the article by Crowley and Dunn which is listed in the reference section.

Introducing Change

One of the questions my colleagues and I have been considering is "What are the factors which influence teachers to change their teaching practices?" Obviously, there is not an easy answer to this. The following ideas have, however, emerged.

1. I don't think changes will occur unless individual faculty members/departments of mathematics begin to feel UNEASY about what they are currently doing. *If it ain't broke why fix it?* One of the factors currently upsetting to many faculty is the high failure rate of first year calculus students. While some faculty may feel that separating the wheat from the chaff is one of the roles of calculus, many feel that too many capable students have difficulty with the subject. They are beginning to examine why.

   A related area which hits close to home, and which might promote a renewed look at teaching, is declining enrolments. In recent years, the number of mathematics majors, and graduate students has declined dramatically causing institutions to examine why and look for ways to make programs attractive.

2. The COMPUTER might also promote change. Indeed, there is evidence of this with the calculus reform movement. Used well, the introduction of the computer into the mathematics curriculum will change the emphasis in the content covered and the nature of the problems explored.

   (This coming year, we have a commitment to using computers in all of the Dalhousie first year calculus courses. It is seen as an appropriate and up-to-date modification in the program. I am concerned, however, because the faculty's agreement to "using" the

Activity	Means for Promoting Activities
▸ talk about mathematics	▸ cooperative learning activities such as CLUE CARDS
▸ write about mathematics	▸ using prompts such as write a letter to a sick friend about–, or try to explain to someone whom you are talking to on the phone–,
▸ students reflect on –mathematical ideas –their mathematical growth –how they feel about studying mathematics –how they feel about class	▸ Think-Write-Pair-Share activity ▸ collect a mathematics portfolio ▸ keep a weekly journal ▸ write a "one-minute" paper at the end of class on what you found confusing in today's class
▸ identify what is important in a unit	▸ devise a crib sheet for a test
▸ think about how to teach mathematics to others and, therefore, about what is mathematically important and about how mathematics is learned.	▸ the cooperative learning strategy, JIGSAW
▸ foster perseverance	▸ a long term project, open ended questions
▸ acquaint students with the tools of mathematics	▸ computer, calculator, readings
▸ see the instructors wrestling with problems	▸ tackle untried problems in front of students

Figure 2: Promoting Active Learning

technology was elicited on the promise that *nothing* else in the course would change. I can't see this happening but...)

3. Widespread implementation of the ideas presented in the *Curriculum and Evaluation Standards for School Mathematics*, published by the National Council of Teachers of Mathematics might also precipitate changes at the university level. If students come to university, not bored and passive, as I described earlier, expecting mathematics to be relevant, using technologies, familiar with a range of teaching and evaluation strategies, etc., this might force some changes in how university level mathematics is taught.

4. Another factor which might contribute to classroom change is that many teachers are bored. The professors involved in our project talk about the satisfaction they receive from teaching these courses in this different way. The changes are revitalizing how they feel about teaching. They were excited by the new course in a way they had not been previously when delivering the standard material in the standard way. They also found that they enjoyed working with others and sharing their teaching insights and questions.

5. Perhaps related to this last is DEMOGRAPHY. I read somewhere that it is a somewhat common pattern for "mid-career" professors to become less interested in doing research as they age, and more interested in teaching and students. Given the demographics of our university faculty, if this scenario is the case, we might see some interest in teaching. Although, I also wonder if we don't just get more set in our ways as we grow older.

6. The institution needs to place a higher VALUE on teaching than is currently the case. In our university, we say that teaching is an important criteria for consideration for tenure and promotion. In practice, however, it appears that this is not always the case.

7. RESEARCH about what happens in university mathematics programs, what are effective teaching and learning strategies, etc. is beginning to take place. Many of these studies show that programs which provide a supportive, active environment, using, for example, techniques like cooperative learning, build mathematical self-esteem and self-reliance. (Davidson, 1990; Sheets and Heid, 1990) Perhaps, if this research is introduced to the mathematicians, and if it is readable and relevant, they will become interested in examining their own programs and classes. As it is, most mathematicians seem to teach as they were taught. The method worked for them; why won't it work for their students? What gets overlooked with this mindset, however, is that not all of their students are going to become research mathematicians. We need to consider how to reach those other students too.

8. Last but not least, MATERIALS which support changes is needed. This is certainly beginning to happen with the calculus, through the calculus reform movement. Materials which integrate computer and mathematics, group work and mathematics, writing and mathematics are emerging. As an example of the latter, one of my colleague and I are in the initial stages of producing a book of readings – aimed at students – which can accompany the introductory calculus course. Writing will be a key element. We are also developing a bank of cooperative learning activities. Things like these need to be shared.

Once people are interested in exploring change, what do they do next? An excellent starting point is the Action Plan presented in the booklet *Moving Beyond Myths: Revitalizing Undergraduate Mathematics* published by the National Research Council in 1991. I like it because it directs attention to a variety of "shareholders" in the education process. Among these are faculty, departments, institutions, professional organizations, and government. Suggestions on ways to begin changing are presented. For example, faculty are urged to learn about learning; departments are told to develop teams of individuals to explore educational possibilities; institutions are urged to provide resources for such innovations. Figure 3 presents some highlights from that plan.

The Role of the Mathematics Educator

Given who we are, that is mathematics educators, and not necessarily mathematics instructors, what can be our role in the changes which are going to occur in university mathematics teaching? We can assist mathematicians in their teaching, as I have done. We can conduct, or help them conduct, research into how students learn college level mathematics. We can explore the effectiveness of "alternative" teaching and assessment strategies at the university level. We can share with the mathematics instructors the (readable) literature which does exist. (This last has a been one of the more effective methods that I have found to gain an entre into that community. Using short articles, offered in a collegial spirit, is a non-threatening way to begin dialogue on the subject of teaching.) No doubt, you can think of other activities appropriate for your situation.

The collaboration which I have had with the mathematics department has also revitalized my teaching. I am able to put into practice many of the ideas I talk about with my education students, and, because of my circumstances, have not had a chance to try out with high school students. As well, working with the instructors has brought me closer to the Mathematics Department. Indeed, because we are a small School of Education at Dalhousie, and because our Bachelor of Education program is limited to the secondary school level, I am the only person in mathematics education. It's lonely. Working with the mathematics department has given me a larger community. From their accounts, they have found the liaison profitable too.

Teaching and learning at the university level is an area ripe for development. As an example, I was told that a few years ago this organization held a working session on this subject and very few members attended. This year, when there was again a working group on university level teaching and learning, over 20 members participated. The interest is there in the larger community too–mathematicians, mathematics educators, administrators, etc. Research into the field is being conducted. Relevant literature is beginning to appear. As mathematics educators, we are well positioned to promote the growth and improvements which can occur. We need to seize the moment.

Faculty:

- learn about learning
- think about how to teach as well as what to teach
- involve students actively
- emphasize practices known to be effective
- model good teaching for future teachers
- use technology
- teach the students you have, not the ones you wish you had

Departments:

- build a team of faculty to carry out experiments, and expose all faculty to the results
- start a departmental seminar on issues of teaching and learning
- support teaching assistants in their initial classroom contacts

Institutions:

- provide resources
- emphasize effective teaching

Professional Societies:

- investigate and publicize successful instructional models
- stimulate networks

Figure 3: Selected actions from the Action Plan *Moving Beyond Myths*

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Ad Hoc Group 1

Windows and Mirrors: Metaphors for Computer Screens

David Pimm

The Open University

Windows and mirrors: metaphors for computer screens

"The computer screen is a window" is a metaphor, not a description of fact. I have written elsewhere (Pimm, 1987) about certain aspects of metaphor in mathematics. As with other metaphorically-perceived situations, where the metaphor is offered as an initial means of gaining experience of a new phenomenon, it is important that it be taken literally early on if it is to have its full effect. However, it is equally important later on that its metaphoric status also come to light – in this case, so that a separation between mathematics and machine can take place.

The most pressing problem I see is that of novices being able to see beyond the screen. Similar problems arose in working with Logo where the teacher wanted attention to the code that was generating the screen effects, whereas the pupil's attention was frequently taken up with the screen itself, and the status of the generative language was reduced to a mere epiphenomenon. Simone Weil (1952, p. 128) writes of the transference of consciousness into an object other than the body itself as being characteristic of increasing skill and apprenticeship. Here, the invitation is apparently the reverse.

If the metaphor of computer screen as window has some problems, how else might we see it? One possibility is as a mirror, one that allows our own mental functioning to be perceived, both by us and others. Yet with either of these images, the source object is missing. I do not generate the images in the computer seen as mirror, even if I do interpret what they seem to me. They are images and not pictures, even if photographs are subsequently made of them (such as beautiful ones available of fractals).

David Lodge (1984, p. 295), in his novel *Small World*, reports some delightfully creative translations into Japanese of Shakespearian play titles: the best when retranslated being 'the flower in the mirror and the moon on the water' for *The Comedy of Errors*. In the book, a Japanese interpreter explains:

> It is a set phrase – it means, that which can be seen but cannot be grasped.

"That which can be seen but cannot be grasped" is a perfect description of computer screen objects fabricated from light. To the extent that the images stimulate and inspire us, they can act as sources of imagery for human mathematical activity of immense power. To the extent that they take over the imaging and become the actual objects, rather than merely 'transitional objects', they can detract from our working with them for mathematical ends, instead becoming passive watchers.

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Ad Hoc Group 2

Pre-formal, Formal, and Formulaic Proving

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Pre-formal, Formal, and Formulaic Proving

In classifying mathematical proofs Lakatos (1978, p. 61) distinguishes between formal proofs, and pre-formal proofs. Formal proofs are those which might appear in a respectable journal of mathematics. They follow a certain form and rules of presentation. Pre-formal proofs are informal, and often occur in the notes and discussions of working mathematicians. Blum & Kirsch (1991) also describe pre-formal proof. They note that it makes reference to "sense experience – direct appeal to first principles." (p. 184) and describe it as a chain of correct but not formally represented conclusions which refer to valid non-formal premises. These premises might be real objects, geometric-intuitive facts, or intuitively evident statements (p. 187).

I have adopted the terms *formal* and *pre-formal* to refer not only to proofs, but also to the process of proving. *Formal proving* I use to refer to the careful reasoning and use of formal language which leads to a formal proof. During the formal proving process the person proving is aware of the underlying meanings of the formal language used. This awareness guides the proving process. *Pre-formal proving* lacks the formal language and explicit rules of inference that characterize formal proving. The reasoning employed does follow logical principles but the person proving is not aware of them.

In education one encounters a third type of proving: *formulaic proving*. Formulaic proving produces formal proofs, but by a process different from formal proving. The distinction concerns the meaning attached to the elements of the proof by the person proving. While a person engaged in formal proving is capable of "decoding" the proof and unpacking meaning from it[1], a person engaged in formulaic proving operates mechanically on strings of symbols, without attaching any meaning to those symbols. The proof produced might well be mathematically valid, but from an educational standpoint it is a failure. In the following I will present examples of pre-formal proving and evidence of formulaic proving by mathematics students as part of an ongoing exploration of the role of proof in mathematics education.

The ability to prove pre-formally is a necessary, but not sufficient, precondition to the ability to prove formally. A person able to prove pre-formally, but not yet able to prove formally, proves un-self-consciously. That is, they are not aware of their proving process, only of its result: a sense of conviction and explanation.

Consider this example of pre-formal proving by a university humanities student with a weak mathematics background. She is trying to establish a conjecture she has made, that Fibonacci numbers whose indices are divisible by three are always even. (In all the transcripts which follow, ellipses "..." indicate deletions to improve readability and dashes "–" indicate pauses. "R" is the author.)

Beth: ... The multiples of three work out to be even because the other two, when you add the Fibonacci numbers the other two are odd and then so it would come out to be even.

R: *How do you know the other two are going to be odd?*

Beth: I don't – that again is looking at the little charts and they seem to work out that way

R: *So you've made a conjecture that, the two Fibonacci numbers before one that is a multiple of three will both be odd.*

Beth: Because, no, because you, if each Fibonacci number is the first one plus the second one equals the third one, – the first, it starts out, well, then you would be adding two odd numbers together and get an even number, and then you add, oh, that's the same thing, I see, you'd say, then the next one then is odd, so you'd add that to the even and then you'd come out to another odd, but then I don't necessarily know that the, that the next number after an even number would be odd so –

R: *Can you think of any reason why the next one after an even number should be odd?*

Beth: – Because the one before the even number was odd.

R: *How does that make the one after the even number odd?*

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[1] In discussions after this paper was presented Joel Hillel pointed out that mathematicians are not always able to unpack meaning from the formal proofs they produce. The characterization of formal proving is difficult, and I now believe that formal and formulaic proving are not two distinct activities, but rather two ends of a continuum.

R: *How does that make the one after the even number odd?*
Beth: Because if you add an even number to an odd number then it comes out as odd.

Beth provides a pre-formal proof by mathematical induction of her conjecture, but she would be unable to produce a formal proof, as she is unaware of the reasoning process she uses. With suitable instruction it might be possible to make Beth aware of her pre-formal proving methods, and help her to make them formal. It might also be possible to teach her formulaic proving involving mathematical induction. In the case of another student, Greg, that has happened.

At the time of the interview from which this transcript is taken, Greg was a second year mathematics major. He had taken a discrete mathematics course in which proof by mathematical induction was taught, and other courses in which mathematical induction was used by the instructors in their lectures. Among university mathematics students his mathematical abilities were average. He is working with Eileen, who has a strong mathematical background. In the following transcript Greg and Eileen are working on a problem posed to them by Dr. Sierpinska, ("S" in the transcript). The problem is to show that $2^{\underline{n}}$ is an upper bound to the number of partitions of the plane produced by *n* lines. Greg and Eileen have discussed whether the maximum number of partitions results when all lines share a common point. Greg had claimed that if all the lines intersect the number of partitions would be $2^{\underline{n}}$, however Eileen presented the possibility of three lines forming 7 partitions. After some investigations they concluded that the maximum number of partitions was not formed by concurrent lines. Dr. Sierpinska directed them to look back to the statement they were to prove.

Greg: It's kind of evident but it's kind of hard to show. It's hard to show that either that line will intersect all those other lines ... which will double it and yet it still won't exceed that value, but it will be equal to it.
S: How many more parts will you get when you add another line?
Greg: when you add another line?
S: yes
Greg: maximum, multiply the number you had by 2, maximum
S: yes that's it, because it can intersect in at most two parts
Eileen: Umhmm
Greg: two parts
S: each of those can intersect –
Greg For instance you have let's say 6 lines
Eileen: if we, probably, if we have ... *n* lines it gives us, ... double the parts
S: yes
Eileen: then if we add one line it can not divide ... more than we had by half
S: so at the most you double ... so it gives you the proof
Greg: I think that's proof enough. I can't, a could see –
Eileen: by induction, you know
Greg: assuming that your maximum is n and adding an extra line that extra line you add cut through and intersects each and every one of the others, at a particular point then the maximum you can have is 2 times that value, yeah
S: yes
Greg: that's obvious to me

At this point they have been through the pre-formal proof. Greg has no objection to the proof. In fact, it is "obvious" to him. Eileen and Dr. Sierpinska then wrote out the induction step algebraically and Greg expressed his distrust of mathematical induction.

Greg: you can show me a lot of points, but what, what just because you assume something, why are you assuming it to be true in every case?

Eileen: But you made it without any problem here
S: were you convinced by those, those –
Greg: I'm convinced through word not through induction, and if this is indeed the case when you showed it here, in this particular case, I can see it.
S: Umhmm

Greg's comment: "if this is indeed the case when you showed it here" is a telling one. He does not see what they have done as being the same as formal mathematical induction, but he does see that they are telling him it is. In this case then, he has accepted a proof by mathematical induction, but because of its pre-formal nature to him, he cannot see it as such. When Greg was taught mathematical induction it was in the context of proving formally such identities as:

$$\sum_{i=1}^{k} i^2 = \frac{k(2k+1)(k+1)}{6}$$

It is in the context of such proving that he has doubts. Given that he does not connect the pre-formal proof they did earlier with the proving by mathematical induction he did in his discrete mathematics course, it is reasonable to suggest that he did not ascribe meaning to those formal proofs. The proving he did was formulaic, not formal.

The important thing about formulaic proving is the form of the proof, and students who engage in formulaic proving are often quite aware of the forms of their proofs. One characteristic which I have observed in the proving done by high school and university students, is a preference for a form I describe as the *identity* form. In their proofs, the statement to be proven takes the form of a tentative equation, often written with a question mark above the equals sign. This equation is then manipulated until a blatant identity (such as $x = x$) is produced. Beth and Ann, another humanities student, make this point:

Beth: well see that works, when you have $2N$
Ann: It equals $2N$
Beth: $2N$ I like that those sort of proof
Ann: Apple equals an apple. No hesitation, no doubt.

Greg's dislike of mathematical induction may be related to the difference in the form of proofs by mathematical induction from the identity form:

Greg: It doesn't make much sense
S: *Mathematics doesn't make sense?*
Greg: No, mathematic makes sense. mathematics is very logical,
S: *Except this, except this*
Greg: Exactly, I don't like this at all ... I can move forwards, I can move backwards, but I have to move somewhere to show my conclusion, now it's either I can prove it wrong or I can, Like I said we can imply things from right to left or from left to right but ... we're going to come to a final agreement, but this doesn't, You have ... no concrete way of knowing that this is indeed proof enough. That's what I don't understand.

In a proof, Greg says, "we're going to come to a final agreement, but this doesn't."

Greg, and students like him, seem to have taken to heart the Formalist model of mathematics as "nothing but a meaningless game played with meaningless marks according to certain formal rules agreed upon beforehand" (Boyer, 1985, p. 661). This model may have been passed on to him by teachers who accepted the Formalist model of mathematics, or he may have developed it in response to attempts to teach him to prove formally which did not ensure that the connection with pre-formal proving was made. Ways in which teachers can encourage students to make such connections seem worthy of investigation. Blum & Kirsch (1991) have argued that greater emphasis on pre-formal proofs in schools would allow such connections to be made. This approach is distinct from that explored in some research on proof, which focuses on ways of developing formal proving from other formal activities, such as computer programs (e.g. Dubinsky, 1986, and Sfard, 1988).

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Appendix B

Previous Proceedings

The following is the list of previous proceedings available through ERIC.

Proceedings of the 1980 Annual Meeting	ED 204120
Proceedings of the 1981 Annual Meeting	ED 234988
Proceedings of the 1982 Annual Meeting	ED 234989
Proceedings of the 1983 Annual Meeting	ED 243653
Proceedings of the 1984 Annual Meeting	ED 257640
Proceedings of the 1985 Annual Meeting	ED 277573
Proceedings of the 1986 Annual Meeting	ED 297966
Proceedings of the 1987 Annual Meeting	ED 295842
Proceedings of the 1988 Annual Meeting	ED 306259
Proceedings of the 1989 Annual Meeting	ED 319606
Proceedings of the 1990 Annual Meeting	ED 344746
Proceedings of the 1991 Annual Meeting	ED 350161

There was no Annual Meeting in 1992 because Canada hosted the Seventh International Conference on Mathematical Education that year.