CANADIAN MATHEMATICS EDUCATION STUDY GROUP

GROUPE CANADIEN D'ÉTUDE EN DIDACTIQUE DES MATHÉMATIQUES

PROCEEDINGS
1995 ANNUAL MEETING

University of Western Ontario
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University of Western Ontario, May 26-May 30, 1995

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EDITOR'S FORWARD

I wish to thank all those who contributed reports for inclusion in these Proceedings. The care they took in preparing both a disk file and a hard copy of the report, together with camera ready figures made my work as editor a pleasant task. The value of these Proceedings is entirely the credit of the report authors.

These Proceedings will serve to jog the memories of those who participated in the meeting and hopefully will help generate continued discussion on the varied issues raised during the meeting.

Yvonne M. Pothier
Mount Saint Vincent University
August, 1995

ACKNOWLEDGEMENTS

The Canadian Mathematics Education Study Group/Groupe Canadien d'Étude en Didactique des Mathématiques wishes to acknowledge the continued financial support of the Social Sciences and Humanities Research Council. Without this assistance, the 1995 meeting and these Proceedings would not have been possible.

We would also like to thank the University of Western Ontario for hosting the meeting and providing excellent facilities. Special thanks are due to Doug Edge, Education Department and Mike Dawes, Mathematics Department, for their time and work prior to and during the meeting to make the experience pleasant and enjoyable for all participants.

Finally, we would like to thank the guest lecturers, Working Group leaders, Topic Group and Ad Hoc presenters, and all participants. You are the ones who made the meeting an intellectually challenging and all round worthwhile experience.

SCHEDULE

Friday May 26	Saturday May 27	Sunday May 28	Monday May 29	Tuesday May 30
	0900-1200 Working Groups A, B, C, D	0900-1200 Working Groups A, B, C, D ---------- 1200-1300 CMESG AGM	0900-1200 Working Groups A, B, C, D	0900-1030 Panel on "Masters Programs" Tom Kieren (mod.) Anne Sierpinska Rina Zazkis ------------ 1100-1200 Working Groups Reports Closing Plenary
--------- 1630-1730 Opening Plenary Working Groups Intro-duction	1315-1340 Ad Hoc Session ------------ 1345-1500 Discussion Plenary I ------------ 1530-1730 Topic Sessions A & B	1400 Optional Excursion I The Sights of London ---------- 1715-1900 Dinner in Art Gallery	1315-1340 Ad Hoc Session ------------ 1340-1500 Discussion Plenary II ------------ 1530-1730 Topic Sessions C & D	1400 Post Conference Optional Excursion II Stratford
1900-2030 Plenary Speaker I Michèle Artigue	1900-2030 "Saturday night at the movies" Discussion of Selected Film & Video Clips Vi Maeers	1900-2030 Plenary Speaker II Kenneth Millett	1900-2030 Panel on "Teaching Growth and Effective-ness". Sandy Dawson Kathy Heinrich (mod.) Kenneth Millett	

INTRODUCTION

These Proceedings cannot capture the spirit of what takes place at our annual meeting. They do, however, provide a record of the results of our work and discussions. The keynote addresses are included, and in some instances, the Proceedings is the only place these appear. It is clearly the case that the only printed record of the deliberations of the Working Groups and Topic Groups is contained in the Proceedings. It is important to have such records, both as a marker of what the various groups have accomplished in their study in any particular year, but also as a bridge for other groups in future years to build upon the work already done. For those reasons alone, the Proceedings are valuable.

But alas, the Proceedings cannot tell the whole story, so this introduction is designed to give those new to the organization, or those who have not had the opportunity to attend an annual meeting, a taste of what the meeting is 'really' all about. During one of our early meetings at Queen's University, where four of the first six annual meetings were held, Izzie Weinzweig joyously sampled the many varieties of ice cream offered at the Queen's residence hall. Many of us were amazed at the prodigious amounts he could consume, but while he ate his ice cream discussion was lively, funny, critical, and searching about important happenings in the arena of mathematics education. Izzie's ice cream escapade is notorious in CMESG/GCEDM circles, as is the mid-night pizza run. Events such as these are also part of the annual meeting. This introduction will try to capture aspects of that spirit, items which are not recorded in the printed reports of working and discussion groups.

It was in the fall of 1977 that a couple of young mathematics education scholars, Claude Gaulin and Tom Kieren, gave two of the three keynote addresses at the inaugural meeting of what was to become the Canadian Mathematics Education Study Group—Groupe Canadien d'Étude en Didactique des Mathématiques (CMESG/GCEDM). Both went on to become presidents of the group. Though he wasn't a keynote speaker at that meeting, the group's first president, David Wheeler, was one of its organizers. As much as any set of mathematics educators in Canada, these three epitomize the spirit and diversity of the Group. And what is that spirit they so ably display, and how did they and their colleagues in the Group encourage and foster that diversity for almost twenty years now?

The spirit is embodied in the intellectual playfulness of the Group. This manifests itself in a host of ways. It is there in the serious though not somber ways in which current issues and topics in mathematics education are 'studied', that is, where time is given during the three full working days of the annual meeting for an opportunity to 'go deep', in the vernacular of the day, to listen carefully, attentively, and without pre-judgement to the interests, ideas and experiences of one's colleagues. These study periods, called Working Groups (WGs), meet three hours each day, and provide opportunities for elongated discussions, not just the oftimes brief, superficial conversations one experiences at conferences where there are presentations hourly. Though the leaders of the WGs do extensive advanced planning, they really are the 'provocateurs' of the study once the group is assembled. It is the working group members themselves who determine the particular pathway the group takes during its deliberations. In the words of David Wheeler, the philosophy behind this structure was that "...people [could] work collaboratively at a conference on a common theme and generate something fresh out of the knowledge and experience that each participant brings to it."[1] And members are diligent about guarding against a WG becoming the

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[1] Wheeler, D. (1992). The Origins and Activities of CMESG/GCEDM. In Kieran, C. & Dawson, A. J. (eds.). *Current Research on the Teaching and Learning of Mathematics in Canada: Les Recherches en Cours Sur l'Apprentissage et l'Enseignement des Mathématiques au Canada*, p. 6. Montréal, QC: CMESG/GCEDM.

platform for a particular point of view, or being dominated by the leaders. In fact, though leaders make detailed preparations and plans, provide extensive and elaborate occasions for group and group member thinking, it is usually not long into the deliberations that their plans fade into the background, and the discussion goes in directions not anticipated by the leaders. This is not to say that the groups operate in a random or disorganized manner. Rather, the 'orders' of the working group are what Bohm would call implicate rather than explicit, and arise from the discussions occasioned by the interactions among group members.

There are other times as well when members are given opportunities for extended periods of study. Topic Groups (TGs) are sessions where individual members present work-in-progress, and invite and solicit feedback from their colleagues. These sessions are not meant to be one way informational sessions—indeed, efforts to have such a format would be frowned upon by the group—but rather are opportunities to present 'three-quarter baked ideas' and have them critiqued in a supportive and caring environment.[2] Ad Hoc groups serve a similar function, but these are events which are so current that it was not possible to include them in the program prior to its printing and circulation. Nonetheless, it is important to note that the organization of the annual meeting provides the time and space for Ad Hoc groups to occur, and they invariably do. The in-depth study of questions and issues in a conference setting does seem to be the prime characteristic of CMESG/GCEDM, what Wheeler has called 'its study-in-cooperative-action,' and the heart of the intellectual playfulness of the Group.[3] The scope of the topics discussed can be seen in the listing of the focus of WGs over the years which are listed in the Appendix. What is noticeable about that list is the central concern the Group has had "...with teacher education and mathematics education research, with subsidiary interests in the teaching of mathematics at the undergraduate level, and in which might be called the psycho-philosophical facets of mathematics education (mathematization, imagery, the connection between mathematics and language, for instance)."[4]

The Group's playfulness also has a social aspect. Over the years the 'excursions' have developed a reputation for inventiveness and surprise and wonderment. We have toured the plains of Saskatchewan and Manitoba, sipped wine under the waterfalls on the Sea-To-Sky highway of BC's west coast, enjoyed Shakespearean and Shaw festivals in Ontario, toured the Plains of Abraham in Québec, sampled galleries and markets in New Brunswick, and hiked up and around St. John's. Marty Hoffman makes the rather dubious claim to fame of being the one who initiated midnight 'pizza runs', a tradition which has grown in frequency, size and inclusivity as the years pass. One night in Fredericton saw almost all conference registrants crowded into one very small, and overwhelmed, pizza parlour. In Regina, there were so many people prepared to wander the town in search of the 'perfect pizza' that the run took place over two nights.

The diversity which the Group achieves is accomplished in a number of ways. First, and perhaps most importantly, the Group has always sought to attract mathematicians as well as mathematics educators to its gatherings. The Group has been relatively successful in this venture with roughly a third of the Group's membership being drawn from the ranks of professional mathematicians. Moreover, recent years has witnessed a greater involvement by college and CEGEP mathematics instructors, a move widely applauded within the organization. Concerted efforts have also been made to have school people involved

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[2] To use a phrase Uri Leron is fond of and has written about. Uri was a most welcome non-Canadian visitor to the Group's Annual meeting in Regina in 1994. He argues that many ideas are more than just 'half-baked'. Some are better than that and are at least 'three-quarters baked'.

[3] Op. cit., p. 7.

[4] Op. cit., p. 5.

with the annual meeting, but typically this involves only teachers and provincial association representatives for the region where the conference is being held. It is a sad truism that not many teachers can obtain travels funds to attend conferences. Unfortunately, university, college and CEGEP instructors may soon be facing the same funding difficulty, if they aren't already. Nonetheless, the Group attracts a broad spectrum of the mathematics and mathematics education community across the country, something no other organization in Canada accomplishes.

The shifting location of the annual meeting is also a source of diversity. Since education is a provincial responsibility in Canada, it is difficult to get a 'fix' on what is occurring in all parts of the country with respect to mathematics education. Moreover, it is difficult to comprehend and understand the diversity which exists across the country, dictated by local settings without actually visiting and living in, however briefly, parti-cular regions of the country. We have been fortunate to have been hosted by universities all across the land, at incredibly reasonable costs, in ways which allowed us to experience the richness and diversity of Canada as few others in the general population ever have the opportunity of doing.

In their own way, the four presidents have brought their experiences of the west coast, the prairies, French speaking Québec, and English speaking Quebec, to bear on the focus and direction of the organization, and thereby fostered an understanding of the diversity of our country. It will not escape note, however, that all the presidents have been male. Over the past decade, however, the Executive itself as well as the cast of plenary speakers, working and topic group leaders at the annual meetings have been gender balanced. The increased participation of women in the Group has also led us to make changes in the programme components, such as the small group discussion format after the plenary talks, aimed at making our deliberations richer and more inclusive

Keynote speakers also contribute to the diverse points of view to be examined by the Group. While the Group is Canadian, with only a small handful of members coming from outside the country, it was always foremost in the minds of those planning the conferences that the organization should not become parochial in its viewpoint. Efforts were made, therefore, to ensure that the keynote speakers were (1) foremost authorities in their areas of interest, those at the so-called 'cutting edge' of thinking in their field, and (2) brought a non-Canadian viewpoint to the Group. A quick perusal of the list of past speakers included in the Appendix will be sufficient to convince even the most skeptical that the Group has been successful in attracting leading mathematicians and mathematics educators to attend its meetings. These speakers don't just come, deliver their lecture, then leave, but they stay with us for the entire conference, participating in the WGs, the TGs, and the social events which embellish the 'headier' aspects of the meeting. They are active participants working right alongside our members. Moreover, one keynote speaker typically represents the mathematics education field, and the other the views of professional mathematicians. And for financial reasons, one is typically from a location 'close' to the site of the annual meeting, and one from some distance removed from that site. In all of this, the attempt is to invite individuals who will stretch our thinking, who will challenge our home-grown ideas, who will broaden our educational horizons. Sometimes these efforts are successful, sometimes not, but it still seems worth the effort.

Though rich in tradition and perhaps wedded to a particular format and way of working, the Group nonetheless continues to evolve.The face of the organization is gradually changing as individuals new to the field make their presence felt.They have begun to lead working and topic groups bringing with them perspectives and experiences new to the field. Discussion formats which are more inclusive for both new and long term members are being tested, adapted based on experience, and then adopted. Recognition, and a place on the program, is being given to those who have recently completed doctoral studies. The format of topic groups is being modified to give them greater exposure and opportunity for fuller discussion. As with most of life, some things about the Group change, while others stay the same.

But "study–étude" remains the central focus of the Group and perhaps its greatest strength and defining characteristic. This is as it should be if the Group is to be true to its origins. And if the ice cream is being kept cold in case Izzie comes along, and if the local pizza parlours are stocking extra supplies, then you will know you are in a location where the annual meeting of CMESG/GCEDM is being held.

A. J. (Sandy) Dawson
President, 1993-1997

PLENARY LECTURES

Plenary Lecture 1

THE ROLE OF EPISTEMOLOGY IN THE ANALYSIS OF TEACHING/LEARNING RELATIONSHIPS IN MATHEMATICS EDUCATION

Michèle Artigue
University Paris, Paris

I. INTRODUCTION

More and more, mathematics education is considered as an autonomous scientific field, capable of defining its own problems and methodologies. Nonetheless the question of developing or maintaining relationships with related scientific fields remains an essential one. In this text, we focus on the relationships with epistemology which, in my opinion, are fundamental.

First we clarify our use of the word "epistemology" by underlining what we feel is of crucial interest in epistemological work from a didactical point of view, namely, reflection on the nature of mathematical concepts, on the processes and conditions for their development, on the characteristics of present as well as past mathematical activity, on what constitutes the specific nature of one mathematical domain or another.

Such a reflection is necessary in mathematics education, for several reasons, the following being of particular importance :

- our work in mathematics education is governed, implicitly if not explicitly, by our epistemological representations and we have to be as clear as possible about them,
- a strong and privileged contact with mathematics via the educational world tends to distort epistemological representations, to shape them in order to make them compatible with the way mathematics is living in this educational world, and to reduce our mathematics to the taught mathematics.

So epistemological work is necessary to be able to look at this educational world from the outside, to make its epistemological choices apparent and questionable.

By making its epistemological choices apparent and open to discussion, epistemological work helps us to gain an extrinsic view on education. This epistemological work has taken different forms in the didactics of mathematics. In the first part of this text, without pretending to be exhaustive, we will give some idea of its variety and richness. In the second part, we will evoke its limits.

II. THE ROLE OF THE THEORY OF EPISTEMOLOGICAL OBSTACLES IN THE DIDACTICS OF MATHEMATICS

For a French didactician, the word "epistemology" immediately evokes the theory of "epistemological obstacles" initially developed by the philosopher G. Bachelard (Bachelard, 1938) and transported into the didactics of mathematics by G. Brousseau, 20 years ago (lecture given at the CIEAEM Conference, in Louvain la Neuve, in 1976). As stressed by A. Sierpinska in her book "Understanding in mathematics", the idea of "discontinuity" inherent to this theory can be found in many philosophers before and after Bachelard from Bacon and Husserl to Lakatos and Kuhn, not to mention others. Perhaps more than others, this theory dramatically put to the fore discontinuity, by considering that new knowledge is always founded, in some part, on a rejection process.

Bachelard (1938) wrote in "La formation de l'esprit scientifique."

> Reflecting on a past of errors, the truth is found in a real intellectual repentance. In fact, one knows always against some previous knowledge, by destroying ill built knowledge, by overcoming that which in the mind itself is an obstacle to spiritualization (p.13).

This dramatic position is specially questioning for mathematics educators as it radically disqualifies the still dominant illusion that mathematics learning can be organised along a smooth path, where knowledge increases gently step by step, with some necessary reorganisations, of course--everyone has heard about Piaget's theories of assimilation and accommodation--but fundamentally in a process capable of avoiding major ruptures and the disturbing paradoxes of the didactic contract they induce (Brousseau, 1986).

At the opposite end, the theory of epistemological obstacles is based on the fact that ruptures are the normality, that we cannot directly learn definitive forms of knowledge, that progress necessarily requires some kind of rejection of what has been for a time, often a long time, a motor of progress.

Initially, Brousseau exploited this notion to analyze the persistent errors of pupils in the extension of numbers from whole numbers to rationals and decimals and to question the dominant status of errors in the educational world (Brousseau, 1983). Later, the field of mathematical analysis and especially the notion of limit became a field of interest for the development of this theory within the didactics of mathematics, through the works of B. Cornu first (Cornu, 1983) and then A. Sierpinska (Sierpinska, 1985). After an in-depth historical study, Sierpinska produced a structured list of obstacles[1] which marked out the historical evolution of the concept of limit and she proved that such a list could be used in order to interpret persistent difficulties encountered by present students. Beyond that, Cornu's and Sierpinska's research clearly showed the existence of different kinds of epistemological obstacles :

- some could be linked to common and social knowledge about limits,
- some could be traced to the under-development of crucial notions such as that of function,
- some were linked to the over-generalization of properties of familiar finite processes to this infinite process according, for instance, to the "continuity principle" stated by Leibnitz,

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[1] The list was structured around four categories: "horror infiniti" grouping both obstacles linked to the rejection of the status of mathematical operation for the limit process and obstacles linked to the automatic transfer of methods and results of finite processes to infinite ones; obstacles linked to the concept of function; obstacles linked to an over-exclusively geometrical conception of limit; obstacles of a logic nature.

- some, not of minor importance, could be linked to more philosophical principles and beliefs about the nature of mathematical objects and mathematical activity, for instance about the status of infinity.

At a theoretical level Sierpinska, with reference to Wilder and Hall, (Sierpinska, 1988 and 1994) integrated this diversity by considering "mathematics as a developing system of culture and a sub-culture of the overall culture in which it develops." This cultural conception of mathematics leads to the identification of three levels in mathematical culture, in on-going interaction:

- the formal level, that of unquestioned principles and beliefs,

- the informal level which is the level of "tacit knowledge, of unspoken ways of approaching and solving problems," "of canons of rigour and implicit conventions,"

- the technical level which is the domain of rationally justified, explicit knowledge.

According to Sierpinska, epistemological obstacles are situated at the first two levels and this location has some important consequences for the strategies we have to develop in order to overcome them.

This approach in terms of epistemological obstacles is often associated with a search in the history of mathematics, for significant and fundamental problems which permit an organization of the teaching process that would be epistemologically more adequate than the usual ones. Research developed in Louvain la Neuve under the direction of N. Rouche and, specially M. Schneider's thesis entitled "Des objets mentaux aires et volumes au calcul des primitives" is typical of that direction (Schneider, 1989). It is based on a close analysis of students' behaviour when faced with a field of problems, mainly adapted from historical ones. This analysis tends to prove that the perception of surfaces (respectively volumes) as the piling up of segments (respectively surfaces), similar to that developed by Cavalieri and others in the seventeenth century, although not explicitly taught, is present in the mental representations and informal mathematical culture of today's students. Schneider has shown that this fact can explain some frequent and persistent errors in the calculation of areas and volumes, as well as some difficulties in understanding the modern process of integration.

From there, a teaching strategy is designed where problems are first chosen to highlight the productive character of this perception of geometrical objects and make explicit corresponding informal reasonings. For instance, as shown in Figure 1, students have to explain why the area of the parallelogram is the same as the area of the rectangle, or why the volume of the solid delimited by the cylinder and the half-sphere is the same as that of the cone.

In the next step, problems which fired controversies in the seventeenth century were used to attack the epistemological obstacle (called by the author "the obstacle of heterogeneity of dimensions") derived from this productive perception: It results from the simultaneous, uncontrolled and often unconscious use of geometrical objects of different dimensions in the calculation of areas and volumes.

For instance, students have to explain why the lateral area of the cone is different from its base area even though there is a one-to-one correspondence between the circles which compose each of them. Or they have to explain why the formula obtained by summing up the lateral areas of the cylinders which compose the solid of revolution, in Figure 2, does not give a correct value for the volume of this solid.

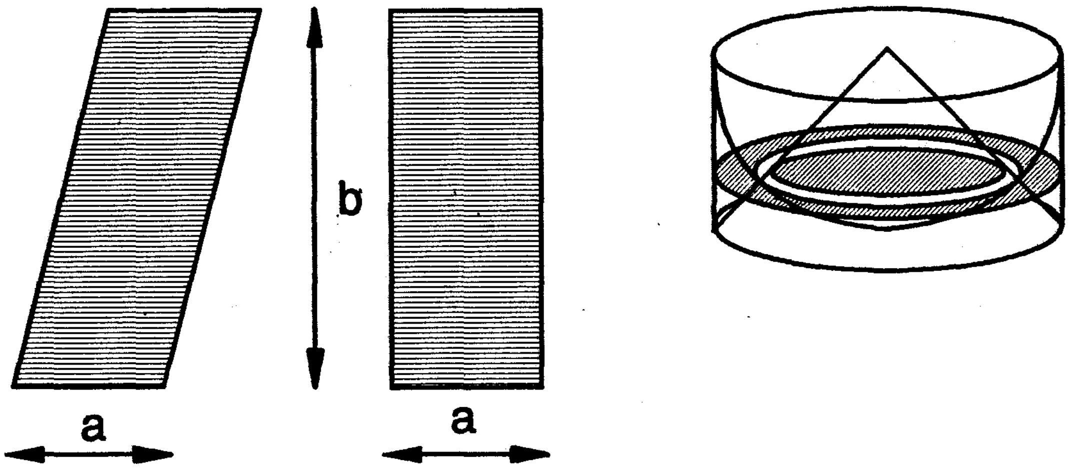
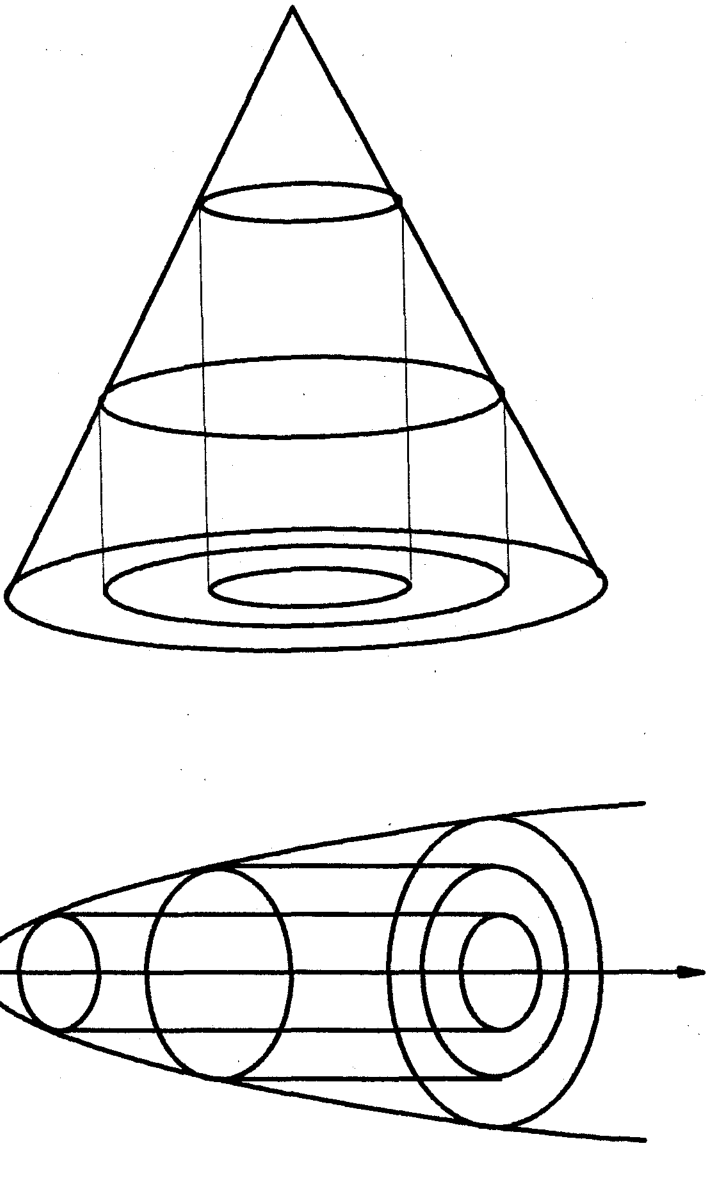


Figure 1

The productive character of an intuition in terms of indivisible

Before leaving this point, let us stress that such a focus on strong and necessary ruptures in the growth of knowledge, is not necessarily subordinated to historical analysis in didactical research. For instance, M. Legrand (1993) recently developed a strong epistemological analysis in order to understand the relationships between algebra and analysis, referring only to the actual prevalent vision of the field of analysis, as the field of "approximation, majoration, minoration", as expressed by Dieudonné. He showed that students must construct their knowledge in mathematical analysis both on and against their previous algebraic knowledge. For instance, in the same way these students have to radically modify their relationship to equality when passing from numerical thinking developed in our elementary schools to the algebraic thinking developed in junior high schools, they have once more to radically modify this relationship when passing from algebraic thinking to analytic thinking. For example, in order to prove that two objects a and b are equal, most often in analysis, we do not use equivalence processes directly as in algebra—such a strategy is often out of reach and, even if it can be used, generally it is not the most effective. Instead, we try to prove that for an adequate norm or distance, the norm of the difference between a and b or the distance between a and b is less than any positive real number.

Entering analytic thinking means understanding that this kind of detour will generally be worthwhile. At a more advanced level, students will have to understand that, in analysis, in order to prove that a family F1 of objects possess a given property, a similar detour will often be used: prove the property for a family of simpler objects F2, then prove that each element of F1 can be considered as the limit for an adequate topology of elements of F2 and that the property at stake is conserved through the limit process. In the same way, students have to change their habits with the treatment of inequalities and many other objects, familiar but in another world—the algebraic world.



$$\int_0^4 2\pi\sqrt{x\,(4-x)}\,dx$$

Figure 2

The epistemological obstacle of heterogeneity of dimensions

Legrand points out how traditional teaching, shaped by the illusion of continuity, is insensitive to these problems and tends to leave it up to the students to deal with these crucial ruptures and reconstructions.

III - EPISTEMOLOGICAL ROOTS OF GLOBAL THEORIES IN DIDACTICS

Beyond attracting attention to ruptures and discontinuities in knowledge, epistemology also plays an essential role in the more global theoretical frameworks we develop. Many examples can illustrate this point. I would like to deal with this aspect through one example: the framework in terms of "tool-object dialectic and setting games" developed by R. Douady in the last decade (Douady, 1984) and widely used in French didactics.

III.1 - The Tool Object Dialectic

The "tool-object" dialectic is based on the following epistemological distinction: a mathematical concept can be attributed two different status:

- a "tool" status when it is thought of and used as a tool in order to solve specific problems,
- an "object" status when it is considered in its cultural dimension, as a piece of socially recognised scientific knowledge, and studied for its own sake.

The study of the history of mathematical concepts suggests that most of the time (but not always) mathematical concepts come into being first as tools and the tool is the basis for the construction of the object.

This distinction appears as a strong characteristic of mathematical concepts and mathematical activity. One could also evoke the distinction between the structural and operational dimensions of mathematical concepts introduced by A. Sfard (Sfard, 1991). Here, the term "structural" refers to a treatment of mathematical concepts "as if they referred to some abstract objects" and the term "operational" to a description in terms of "processes, algorithms and actions." The distinctions introduced by Sfard and Douady do not match exactly. In other words, one cannot equate the "tool" and "operational" dimensions nor equate the "object" and "structural" dimension.

Sfard's analysis is essentially set up around the cognitive processes linked to the transition between processes and objects. Douady"s analysis refers more to an analysis in terms of mathematical problems. Thus, when speaking of the tool dimension of the concept of function for instance, one refers to the use of this concept in order to solve problems internal or external to mathematics; when speaking of the process or operational dimension of this concept, one refers to a procedural view of the concept in terms of input-output system as opposed to the static and structural vision developed within set theory. Nevertheless both approaches stress the complementarity and duality of the dimensions they identify in mathematical activity; both point out the historical anteriority of one of these dimensions (the tool dimension for Douady, the operational dimension for Sfard) and the necessity to pay some attention to these characteristics in didactic transposition processes.

This is again a challenging distinction for mathematics educators as the traditional teaching tends to reverse this natural order and introduce objects which only later have to be used as tools in different contexts.

In Douady's research, the tool-object distinction does not appear as a consequence of an historical epistemological study. It is essentially induced by her analysis of contemporary mathematical activity.

Nevertheless, history of mathematics shows the relevance of this distinction. Let us take the example of the history of complex numbers. Indeed, what appeared on the mathematical scene during the sixteenth century, in the work of Italian algebraists such as Ferrari, Cardan, and Bombelli was first a tool and even an implicit tool, via the audacious extension of a technique for the solving of equations of the third degree, known as Cardano's technique[2]. This technique was extended to an a priori illegal case (negative square), which led to the introduction of new operational signs such as the "piu di meno" and "meno di meno" notations introduced by Bombelli in his "Algebra" (1572) to deal with the addition and substraction of square roots of negative quantities.

These new expressions, of course, did not immediately take on the status of number. They remained for a long time, without any meaning, pure syntactical objects whose manipulation was governed by the principle of permanence stated later by Leibnitz. Thus, during the seventeenth century, they took on the status of convenient intermediaries in calculations, but only calculations which started with ordinary numbers and led to ordinary numbers.

Slowly, however, this status evolved. Tools, explicitly identified, and named (a specific article is devoted to them in the Encyclopedia by d'Alembert), the imaginary quantities went beyond the single context of equations to enter other domains such as trigonometry with Moivre (1722). They also began to be treated as autonomous variables in functional expressions, especially developments in series, which were so important at the time. Moreover, imaginary quantities became necessary instruments in the formulation of such important results of the eighteenth century such as:

- the fundamental theorem of algebra,
- the unification of spherical and hyperbolical trigonometries, via Euler's formulae.

The tool thus started to present undeniable characteristics of a mathematical object, but it remained a purely symbolic one, defined but not constructed, not susceptible to real interpretation. It was only during the nineteenth century that imaginary quantities acquired their present status of fully legitimate objects. This was achieved in two distinct stages: first via the geometrical interpretation proposed independently by Wessel, Argand, Gauss and others, then via Cauchy's and Hamilton's constructions which finally founded them algebraically.

The kind of analysis this brief presentation attempts to summarize has, in our opinion, an important role to play. It helps the mathematics educator to become more aware of the denaturations didactic theories often suffer when, from a tool for understanding the functioning of teaching/learning relationships, they become tools for acting on educational systems and are more or less consciously shaped in order to become compatible with them. For instance, the above historical analysis reminds us that the relationships between the tool and object dimensions of mathematical concepts are complex and dialectic, much more complex and dialectic than they ordinarily appear in the very simplified versions of the tool-object dialectic proposed in most educational papers. The complex number object is not

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[2] In order to solve the equation $x^3 = a + bx$ by searching

$$x = \sqrt[3]{u} + \sqrt[3]{v}$$

as Italian algebraists did at the time, one is led to find u such that the square of u-a/2 is equal to (a/2)2-(b/3)3. When the initial equation has three real roots, this quantity is negative.

of the tool-object dialectic proposed in most educational papers. The complex number object is not created suddenly by some miraculous institutionalisation process on the basis of activities of problem solving in which it only had a tool status. Before being fully legitimised, it already produces generality and is engaged in more complex processes.

The dialectic between the tool and some pre-object is established very early and plays an essential role in the evolution of the two dimensions. A restrictive interpretation of the theory such as: first comes the tool, then comes the object and they both develop dialectically, appears rather inadequate. Moreover, the proof of effectiveness in problem solving is not enough to guarantee the status of object: the mechanisms of acquiring an institutional legitimacy are much more subtle.

Beyond this function of epistemological vigilance, such an historical work help us to question our theoretical categories necessarily built in a limited context. For example, Douady formulates the hypothesis that most concepts obey the tool-object dialectic process. What does "most" means exactly? Is it necessary to introduce other categories and what could be the consequences of other distinctions on our didactics theories ?

The research we carried out on quaternions (Artigue and Deledicq, 1992), clearly shows that these new numbers entered the mathematical scene in the middle of the nineteenth century, directly as objects. This was achieved in a generalisation process whose aim was to extend to geometrical three dimensional spaces the possibility of an algebraic calculus opened up by the use of complex numbers in plane geometry.

Also the recent work on the history of linear algebra (Dorier, 1990) shows that fundamental concepts of modern linear algebra such as that of vector space in axiomatic form, dimension and rank result more from an unifying process than from a tool-object process. This unifying process was aimed at connecting different problems already solved by mathematicians and was to give the means for elaborating this connection formally. This unifying character is not without influence on their development. Dorier points out for instance that it took a very long time for mathematicians, working in this area and used to solving problems with non intrinsic methods even in infinite dimension, to really appropriate these concepts and understand their importance. He also stresses that the development of functional analysis and theory of Banach spaces played an essential role.

No doubt other epistemological distinctions could be made about the status of concepts that we teach, beyond the tool-object distinction. For example the notion of "proof generated concept" introduced by Lakatos (1976), could be of some inspiration to analyze the problem mentioned above and reflect on adequate ways for introducing concepts such as the fundamental concepts of linear algebra. For instance, Robert and Robinet (1993) formulate the hypothesis that teaching processes for what they call "generalizing and unifying concepts," have to obey specific strategies and that metamathematical dimension can play an important role.

III.2 - Setting Games

Another facet of Douady's thesis is the notion of "setting games." Beyond the tool-object dialectic, she identifies another characteristic of mathematical activity which seems to play an essential role in the growth of knowledge: the fact that mathematical concepts function in different settings (complex numbers, for example, are algebraic objects which can function both in algebraic and geometrical settings). What clearly appears when we observe the research work of mathematicians is that they often play with these different settings, in order to progress in the problems they have to solve (cf. Douady and Douady, 1994, for such an analysis).

Roughly speaking, we can schematize the process as follows:

The initial problem stems in one setting (say setting A) and the work inside this setting allows to attain some state, say state 1A where it seems to be stopped. The translation in another setting: say setting B (necessarily imperfect) allows to transform the problem or some sub-problem considered at state 1A into a new one and thus pass from state 1A to state 1B. Then the work in setting B allows to progress until state 2B and a translation back to setting A allows to get a state 2A which one could not get directly.

This analysis leads Douady to theoretically organize the didactic transposition of the tool-object dialectic around situations which can be worked in several settings and to consider the changes in settings, carefully managed by the teacher, as an essential means for solving the eternal problem of the filiation between old and knew knowledge. Her thesis illustrates the potentials of this theory in a long term engineering work at elementary school.

Before leaving this point, let us stress that the description given above of "setting-games" is a very schematic description. Most often, mathematicians do not switch from one setting to another and the interplay between settings is better described in terms of changes in dominant roles: one setting is devoted to technical and formal mathematical work, others are used at a more heuristic level, in order to plan , help some choices, control mathematical activity... This dialectic interplay looks evident for instance if we come back to the construction of quaternions evoked above and analyze the long research process--it took 13 years--described by Hamilton in the preface of his book "Lectures on quaternions" (1853).

Even if Hamilton's problematics had evident geometrical roots at the beginning of the process, the dominant setting was an algebraic one. Hamilton wanted to generalize to triplets the construction he had previously made with couples of reals in order to algebraically define complex numbers. During this phase, the geometrical setting seems to appear punctually, with a heuristic role:

- it guides the choice of undetermined products for the six different unities at play as Hamilton simultaneously introduces two products; an internal and an external one:

  > There still remained five arbitrary coefficients [...] which it seemed to be permitted to choose at pleasure: but the decomposition of a certain cubic function combined with geometrical considerations, led me, for the sake of securing the reality and rectangularity of a certain system of lines and planes, to assume the three following relations between those coefficients (Preface, p.25).

- it helps to understand why the different trials made always give a product which is not regular:

  > The foregoing reasonings respecting triplets systems were quite independent of any sort of geometrical interpretation. Yet it was natural to interpret the results and I did so, by conceiving the three sets of coefficients [...] which belonged to the three triplets in the multiplication, to be coordinate projections, on three rectangular axes, of three right lines drawn from a common origin (Preface, p.22).

After transposing the algebraic product in terms of line products, Hamilton was able to interpret nul products in terms of orthogonality of systems of planes and lines. Not finding the way of solving this problem of non-regularity, Hamilton changed his mind and gave the dominant role to the geometrical setting, trying to generalize to three dimensional space the geometrical interpretation of complex number product: angles, rotations, and lengths were then the main tools. But once more, algebraic setting remained present with a controlling role. Several different generalizations were for instance rejected as they did not obey the distributive principle of algebraic product. And, when, once more blocked, Hamilton came back to algebra, the geometrical setting remained present in order to guide the definition of the product of units and then to find a geometrical global interpretation of the product, first defined algebraically.

Obviously, when we decide to capture such essential features of mathematical activity within our theoretical frameworks, we have to be epistemologically vigilant to the risk we encounter of loosing their essence by over-simplifying them for our educational purposes.

IV - Limits of epistemological work from a didactic point of view

Up to this point, we have stressed the importance of epistemological work. This epistemological work has evident limits.

The educational genesis of concepts cannot match their historical genesis. Obviously, the cognitive functioning of present students can hardly be identified to the cognitive functioning of present or past mathematicians. Historical problems which led to the construction of one or other mathematical concept most often cannot easily be transposed to current teaching. In order not to distort epistemological values and adapt to our present students, a difficult transposition work turns out to be necessary.

Changes in mathematical culture are also evident. Thus, epistemological obstacles identified in history are only candidates for obstacles in the present day learning processes and, conversely, non historical formal and informal forms of knowledge can act as obstacles for our students.

As far as complex numbers are concerned, for instance, the situation of today's students for whom complex numbers are directly introduced as legitimate objects which are endowed from the start with punctual and vectorial geometrical representations, cannot be compared with the situation of Italian algebraists of the sixteenth century and even with that of their successors. What obstacles are resistant to these differences? In fact, what we can immediately transpose from the theory of epistemological obstacles is the fundamental question: why and how do our students have to change their conception of numbers, their numerical and algebraic informal habits in order to cope efficiently with complex numbers?

The answer to such questions cannot avoid taking into account, beyond the cognitive and epistemological dimension, the social and cultural aspects of present mathematical education. We will illustrate this point with two examples.

IV.1 - Teaching Differential Equations at University Level

The first example refers to personal research on differential equations. In 1986, I had been involved for several years in mathematical research in differential equations and I was struck by the epistemological inadequacy of teaching in this area for students in their first two years at university. Teaching focused solely on the methods of algebraic resolution typical of the functioning of the field in the eighteenth century. It appeared impervious to the epistemological evolution of the field towards geometrical and numerical approaches. What constraints could explain such an obsolete stability of teaching? Was it possible to find another equilibrium which would be epistemologically more adequate? Was this possible at a price acceptable by the didactic system, and, if so, how?

Research began by an epistemological analysis. It showed that, historically, the field of differential equations had developed in at least three settings:

- the "algebraic" setting where the fundamental problem is mainly that of finding exact solutions (in finite or infinite terms) or discussing the possibility or such solutions,

- the "numerical" setting where the fundamental problem is to find approximate solutions and control these approximations,

- the "geometrical" setting where the fundamental problem is the geometrical and topological study of flows associated with equations or families of equations.

Teaching for French beginners was focused on the first setting and gave students an erroneous image of the field. They were convinced that every equation could be exactly solved and that researchers in this area were only looking for the missing recipes.

Epistemological analysis clearly showed that epistemological constraints contributed to explain the characteristics of present teaching, mainly:

- the long domination of the algebraic setting in the historical development of the field,

- the more recent appearance at the end of the nineteenth century in Poincaré's work of the geometrical approach,

- the difficulty of problems associated with the geometrical approach such as structural stability problems as well as of problems of exact integrability which marked the development of the algebraic setting past the eighteenth century, and, last but not least,

- a development of the three settings without strong interactions which contributed to their relative isolation.

But soon it appeared that these epistemological constraints were reinforced by cognitive constraints (Artigue, 1992b), as, for instance:

- the flexibility between the graphical register of representation with drawings of flows and associated curves such as isoclines, and the algebraic and symbolic register of representation with the differential equation, associated equations, inequations and functions required by the qualitative resolution of differential equations; indeed qualitative resolution requires a permanent interplay between these two registers and obliges students to coordinate various levels of flexibility, by taking into account not only functions but also their first or second order derivatives,

- the delicate use of elementary tools of analysis required by qualitative proofs when they were presented in their academic form.

What appeared also, was the fact that both types of constraints were strengthened by didactic constraints among them:

- the ordinary tendency of didactic systems to avoid cognitive difficulties in mathematical analysis by favouring algebraic and algorithmic processes; usual algebraic resolution of differential equations, which is essentially algorithmic at this level of teaching, is in accordance with this tendency; on the contrary, qualitative resolution cannot be reduced to algorithmic processes

- the devalued status of the graphic register of representation in the French educational system which created a didactic obstacle to the necessary acceptance of graphical reasonings, at this level of teaching.

In order to answer the questions at stake, it was necessary to identify and understand the real strength of all these constraints as well as their interrelations. We tried to do this and then exploited the analysis in order to build a realistic teaching strategy which better respected the current field's epistemology. The engineering product was then experimented and progressively adapted with undeniable success (Artigue, 1993).

Undoubtedly, epistemological reflection was an essential part of this engineering work, in some way its starting point, but we had to go far beyond it in order to reflect on the possible ways for acting effectively on the current educational system and succeed.

IV.2 - TANGENT CONCEPTIONS AND THEIR EVOLUTION AT SECONDARY LEVEL

The second example we will use is a research on the notion of tangent by C. Castela (Castela, 1995). Its aim was to clarify the development of secondary school pupils' conceptions of tangent and to explore the effects following this development in teaching.

At the beginning of secondary school, pupils first encounter the tangent to a circle. This object is a geometrical object endowed with specific properties:

- it does not cut through the circle,
- it touches it at only one point,
- it is perpendicular to the radius at the contact point.

All these properties are global and do not bring into play the idea of common direction. Moreover, in order to help the pupils to become aware of the abstract status of the figure, teachers often insist on the fact that although to the eye, circle and tangent seem to merge locally, they have exactly one common point. In the same way, this tangent is linked to secants but secants of a given direction which, when moved, help to visualize the change in the number of intersecting points.

In high schools, the teaching of analysis introduces another point of view on the tangent:

- it is a local object with which the curve tends to merge locally,
- it is also the line whose slope is given by the derivative.

Obviously there is no direct relationship between these two objects and it is legitimate to wonder how students manage the transition, if it does work, from "circle" conceptions to "analytic" conceptions. We can wonder also whether, in this transition process, the tangent to the circle is itself a posteriori reconstructed, by integrating the characteristics of more general tangents and becoming the prototype of the tangent to close convex curves, or if it remains s it was before, isolated from the analytic tangent.

Research was carried out through the analysis of text-books, students' and teachers' questionnaires, the students" questionnaire. We will focus here on the students' questionnaire in which various curves and lines were proposed to the students and they were asked to judge for each of them whether: "the line is tangent to the curve at point A" and then to justify their answers.

About 400 students completed the questionnaire. They were at different levels of schooling and from more or less scientific orientations. The answers obtained highlight local adaptation processes which are set up in the long term. Indeed, as could be expected, in high school, before learning analysis, the large majority of students demonstrated coherent conceptions linked to what we have called above the "circle" conception. Differences occurred as some of them blended all the properties of the circle tangent while others seemed to focus on one of them. After the teaching of derivatives, landscape became more chaotic, even though all but one of the items (a curve locally merged with its tangent) obtained high rates of success. This exception scored only a 50% success in "terminale", the final year of secondary school, even though the derivative had been introduced from the notion of linear approximation.

It appears as though, progressively, while remaining an anchor point, the circle conception gradually gave way through various processes:

- by admitting prototypical exceptions such as inflexion points,
- by rejecting prototypical cases such as angular points,
- by integrating in a more or less coherent way some elements from the analytic conception.

Justifications such as the following, for instance, attest it:

There is only one point of intersection and it is a maximum.

There is one common point and the curve approaches the line tangentially.

All these processes continued until a swing towards an analytic conception of the tangent could occur. This analytic tangent in turn became the dominant object in relation to which the cognitive network was reorganised. Note that, however, even in scientific "terminales", only 25% of the students taking the questionnaire presented reasonings sufficiently homogeneous to allow one to suppose that such a swing had occurred. Perfectly correct answers were still accompanied by justifications globally incoherent, from one item to another.

The description given above is expressed in cognitive terms. Historical identified conceptions can be used in order to interpret, at least partially, the answers obtained and it was effectively done.

I am not convinced that this is the more pertinent approach. Both epistemological and cognitive approaches encourage us to set the behaviour of our students in a cognitive rationality. Thus they tend to underline the fact that adaptation processes at play in schools are as much adaptation processes to the educational institution as cognitive mathematical adaptation processes.

What conditions the observed adaptations and their limits? The cognitive characteristics of students? The kind of situations they have encountered about tangents? The way they perceive the demands of their teacher? The status of the notion of tangent in current secondary education?

Data collected provide us with elements for answering these questions. On one hand, analysis of textbooks shows that the question of the relationship between circle conceptions and analytic ones is completely absent from teaching: either the tangent to the circle is not mentioned when introducing derivatives or it is considered as a transparent example. Answers to teachers" questionnaire show that they are no more sensitive to this problem. On the other hand, one of the "terminale" classes, a non scientific one, appeared to be non typical. It was clearly better than all the other classes in its results: this was a class in which the cognitive reorganisation was not left entirely up to the individual student.

Finally, what this research mainly shows is not the way in which students might construct the concept of tangent and the conceptual difficulties linked to this learning process. More essentially it shows the game the students play with this notion in school and the way they optimize this game.

The adaptations carried out, although chaotic and globally incoherent, are quite adequate to allow these students to play their role of students correctly. Indeed, for the marginal object the tangent is in current secondary teaching, this role simply consists in:

- knowing how to recognize simple cases of non-derivability : vertical tangents, angular points,
- knowing how to determine the equation of a tangent, eventually with given constraints,
- knowing how to draw particular tangents on a graphical representation of function, especially tangents corresponding to extreme and inflexion points, vertical tangents.

It is clear that epistemological and cognitive approaches have to be articulated with approaches which include much more detailed analysis of the didactic situations proposed to students, of the way mathematical adaptations combine in these situations with more contractual and institutional adaptations, and of their possible effects on the learning process.

We also need more macroscopic approaches allowing analysis at an institutional level, to understand how the didactic transposition processes we observe function and how one can effectively act on them, to understand how transition processes are managed by institutions and why, to understand how the personal relationships to knowledge our students articulate with the institutional relationships which define the norm.

In French didactics, the theory of didactic situations, first developed by G. Brousseau (Brousseau, 1986) and the anthropological approach developed more recently by Y. Chevallard (Chevallard, 1992) have these aims.

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Plenary Lecture II

TEACHING AND MAKING IT COUNT

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Some mathematicians and mathematics educators say that, "Talking is not teaching and listening in not learning." Few, however, are able to follow through by successfully implementing teaching strategies derived from this principle. Fewer still are those whose work is organized to provide concrete evidence of the results. In these remarks, I will discuss my own thoughts and efforts to document the purpose, nature, and results of some aspects of my educational work experience.

INTRODUCTION

Colleges and universities are looking at how to better evaluate and reward the contributions of individual faculty members to the teaching mission of the institution. Professional and national organizations are similarly looking into this question and issuing reports considering changes from current practice. For example, the Mathematical Sciences Education Board convened a task force, of which I was a member, to "synthesize their knowledge about (1) critical issues in understanding what constitutes faculty growth and effectiveness in undergraduate mathematics education; (2) the kinds of mechanisms for promoting faculty involvement in educational activities that seem to be effective and how they might differ from institution to institution; and (3) existing models for documenting a faculty member's growth or a department's effectiveness, including those from other disciplines that can be adapted for use by the mathematics community." A preliminary draft of an issues paper, "Teaching Growth and Effectiveness," was written. This draft and my own personal experience at the University of California at Santa Barbara are the stimuli for thoughts expressed in these remarks.

How might we, as mathematics professors, work to create a process that could lead to increased recognition of teaching excellence? I will describe some of the key features of such a process, describe some examples of ways in which I have been working toward these goals and, share my understanding of the visible results of this work. The principal questions, I believe, are:

- What are the sorts of things that should be considered as an important and evaluated aspect of our professional activity in mathematics education?
- What can we do to capture evidence of the quality of this work?
- How can we successfuly describe this work to those who need to know about it?

In responding to these questions, a clear understanding of the mission of the college or university and the department within which we work is required. What are the current appointment and promotion policies? How do they provide for the inclusion and evaluation of information concerning educational work? Who will be reviewing the evidence and how well prepared are they to evaluate the importance of your work? This work must be understood as contributing to the success of the institution. The results must be sufficiently conspicuous if we and our colleagues are to appreciate their importance.

Although individual situations do not generalize to those of our colleagues at other institutions, I propose to use my experience at the University of California at Santa Barbara as an informal case study setting for my remarks on these questions. At the very least, this will provide a matrix to provide a structure for the key elements of my thoughts. First I will discuss some general characteristics of the UCSB. Next, I will present some examples of mathematics education work at the UCSB. The two areas are "outreach to underrepresented students" and "renewal and reform of calculus." I will discuss the goals that we have been trying to achieve and the ways in which we have been trying to measure progress towards these goals. In an elaboration of the calculus reform effort, I will present an example of how I have been trying to communicate the nature of this work. Finally, I will discuss my own assessment of these efforts and my thoughts about future work.

A RESEARCH UNIVERSITY CONTEXT: THE UNIVERSITY OF CALIFORNIA AT SANTA BARBARA (UCSB)

With the arrival of Chancellor Barbara Uehling about a decade ago, the University of California at Santa Barbara undertook a series of long range planning efforts to inform the development of the campus. Chancellor Uehling convened several large meetings that brought administrators, faculty, staff, and students together to imagine the academic future of the campus. The result was the eventual adoption, in a process parallel to ones on sister campuses, of the Academic Planning Statement: University of California, Santa Barbara. This statement has been reviewed and slightly revised in the intervening period but remains, even with the recent arrival of Chancellor Henry Yang, the authoritative expression of the goals of the campus. It provides the principal components for the evaluation of proposals for new initiatives or the performance of campus programs such as academic departments. As one might expect, this document has a great deal to say about research and rather less to say about education and public service. These are the three components of professional work by which a faculty member's performance is evaluated.

> **Mission:** As a campus of the University of California, UCSB shares the University's overall mission of teaching, research, and service. In its teaching function, UC preserves knowledge and passes it on to succeeding generations of learners. In its research function, UC develops, interprets, applies, and disseminates new knowledge. And in its service functions, the University applies the intellectual and creative abilities of its faculty, staff, and students toward solving the problems and enriching the lives of the people in California, America, and the world.
>
> The mission has been interpreted in terms of five goals.
>
> **UCSB Goals for 1992-1995**
>
> 1. Ensure a diverse faculty of exceptional quality as the foundation for academic excellence
> 2. Increase stature of research and creative activity.
> 3. Ensure excellence in both undergraduate and graduate instruction.
> 4. Continue the commitment to diversity and quality of the student body at UCSB.
> 5. Enhance the quality of life on the campus and in the surrounding communities.

Educational work with undergraduate students is mentioned once under the elaboration of goal 2:

2(b) Increase undergraduate and graduate student participation in and production of original research and creative activity.

Goal 3 is central to our consideration of educational work and I am including a fuller discussion of it:

3. Ensure excellence in both undergraduate and graduate instruction.

UCSB strives to integrate research and creative activities into all levels of instruction -- to promote an exciting intellectual climate and provide instruction that is both timely and visionary. Curricula for undergraduates aim to instill an appreciation for the humanities and fine arts and an understanding of the natural and social sciences. Graduate programs are characterized by efforts to involve students in research early in their careers, and to provide both formal and informal socialization to the profession for Ph.D. students before they advance to candidacy.

<u>Objective of Goal 3:</u>

(a) Develop and maintain high quality undergraduate, graduate, and professional schools and programs that anticipate and respond to the changing needs of society as well as take advantage of the University's strengths and opportunities.

(b) Increase opportunities for student interaction with ladder faculty, particularly at the lower division level.

(c) Continue to provide an appropriate number of courses and section offerings that will enable students to make timely progress towards degrees.

(d) Continue to improve the instructional environment by seeking optimal student-faculty ratios in courses, and through development of state-of-the-art facilities, instructional delivery and support systems, TA training, and new materials.

(e) Continue the enrichment of the curriculum through interdepartmental course offerings.

(f) Establish and maintain appropriate procedures for measuring and rewarding effective teaching, both undergraduate instruction and graduate student mentoring.

(g) Maintain high levels of student performance in commonly accepted outcome assessment measures.

(h) Expand the opportunities for undergraduate internships.

Colleges and universities are increasingly expected to articulate more clearly what educational outcomes they envision for degree recipients. What does a degree form UCSB mean in terms of competencies and knowledge? What skills will students need in the 21st century? Does exposure to courses in a given sequence lead to the desired outcomes? Answering such questions may require the institution to define more carefully its curricular goals before attempting to measure student performance.

> Due to budgetary constraints, it may be difficult to increase opportunities for interaction between ladder faculty and lower division students on a large scale through smaller class sizes or freshman seminars. Nonetheless, a variety of informal contacts can still be encouraged. Examples include: (1) Involvement of undergraduates in faculty research projects; (2) Inclusion of faculty at residence hall events; (3) Informal departmental receptions or events for lower division students; and (4) Encouragement or insistence that students attend at least one faculty office hour. Such activities help younger students understand that they can approach faculty without hesitation, and aid the mentoring process.
>
> Much has been written recently on the importance of mentoring graduate students as part of their training for professional life. The size and focused specialties of many of UCSB's graduate programs encourage close mentoring relationships. Superior graduate student advising and mentoring should be rewarded the same recognition as superior teaching of undergraduates.
>
> Rewards for exceptional teachers could include research assistance or lectureships comparable to those for research.

Operationally, the University of California, Santa Barbara appears to continue to subscribe to these goals and objectives even though, as in many large institutions, there may be some differences of opinion within the administration, staff, faculty, and student communities concerning the extent to which these statements should inform decisions or influence the establishment of policies, strategies, actions plans, the implementation of specific efforts, or the evaluation of units or individuals.

For the purpose of this discussion, the critical arena is the evaluation of faculty performance for appointment and advancement. In the University of California, reviews occur at many levels. An individual faculty member's case for promotion or advancement originates within the academic department and is reviewed by concerned administrators and, in some cases, by our Academic Senate Committee on Academic Personal. At the most critical junctures, a confidential ad hoc faculty committee is formed to provide an in-depth faculty analysis of the case. Depending on the level of appointment or advancement, the final decision is delegated from the University's Regents to the President of the University, the Chancellor of the Campus, the Academic Vice Chancellor, the Provost, or the Dean of the proposing college.

The research component of any case for appointment or advancement continues to be the most critical dimension and dominates both in quality and length of presentation. In certain cases, if there is acceptable scholarly productivity, evidence of exemplary teaching or public service can favorably influence the decision. Favorable recommendations principally based on teaching or service do occur but are unlikely at those steps considered to be career decisions. The crucial factor is often the attitudes about the importance of educational work or service held by the individual faculty members involved in the consideration of the candidate. They often seem to be less willing to place the weight of their recommendation on evidence of outstanding teaching or public service. While this does not always determine the ultimate outcome of the review, the recommendation of a faculty committee can significantly delay the appointment or advancement. At the critical career points, these opinions appear to be given even greater weight and they, therefore, can effectively prevent appointment or advancement.

Another key aspect of the evaluation system is great variability in the willingness and ability of a department to assemble credible evidence in the areas of teaching or public service. The candidate may also be unwilling or unable to provide the necessary information. There is a perception that some senior faculty members consider a candidate's investment of effort in areas of teaching or public service as evidence of misplaced priorities or a faltering or failing research program. A candidate may decide that submission of evidence of such work could harm their chances for advancement and decide to divert attention away from their achievements in these areas by failing to report them or by diminishing their significance in comparison to their research accomplishments.

Even in the most supportive environments, faculty members are likely to be unprepared to assemble the necessary credible information on which to base a credible assessment of educational accomplishments. At UCSB, while there are certain guidelines that must be followed, there is effectively a great latitude in the preparation of the case for appointment or advancement to allow interested candidates to submit a wide range of evidence of educational and other professional work. The following information from UCSB's procedures establishes the content of the department's submission to support advancement:

3. The basis for the departmental recommendation, including:

   (a) Teaching: General characterization of performance.

   An itemized, chronological (by quarter) list of workload since the last successful review. This list should include: quarter and academic year; course number; course title; course format; unit value; enrollment; share of teaching assignments; evaluations available. Identify any new courses taught or substantial reorganization of old courses.

   Statement of the normal teaching workload for the department overall. A brief explanation of any deviations from this workload.

   Summary of teaching evaluations for each course. (If Senate evaluations are used, summarize those numerical ratings which most reveal instructor and course effectiveness. Raw data are not to be submitted with the departmental recommendation, although they should be kept on file in the departmental office for a minimum of two years in the event that reviewing agencies would want to examine them in a particular instance.)

   Number of graduate committees: How many committees chaired? General membership on how many committees? Degrees completed during the review period?

   Undergraduate and graduate student letters of evaluation (recommended for promotion, desirable for merit increases), or the brief student comments which often appear on evaluation sheets.

   Evaluation of teaching effectiveness by departmental colleagues, other faculty members, or the candidate's self assessment (if he/she chooses).

   Succinct analysis with emphasis on the significance of the candidate's performance.

   (b) Research . . .

While this describes the content of the departmental submission with regard to teaching, the Academic Personal Manual provides more specific information concerning the evaluation process and the criteria themselves:

The criteria set forth below are intended to serve as guides for minimum standards in judging the candidate, not to set boundaries to exclude other elements of performance that may be considered.

(1) Teaching—Clearly demonstrated evidence of high quality in teaching is an essential criterion for appointment, advancement, or promotion. Under no circumstances will a tenure commitment be made unless there is clear documentation of ability and diligence in the teaching role. In judging the effectiveness of a candidate's teaching, the committee should consider such points as the following: the candidate's command of the subject's continuous growth in the subject field; ability to organize material and to present it with force and logic; capacity to awaken in students an awareness of the relationship of the subject to other fields of knowledge; fostering of student independence and capacity to reason; spirit and enthusiasm which vitalize the candidate's learning and teaching; ability to arouse curiosity in beginning students, to encourage high standards, and to stimulate advanced students to creative work; personal attributes as they affect teaching and students; extent and skill of the candidate's participation in general guidance, mentoring, and advising of students; effectiveness in creating an academic environment that is open and encouraging of all students. The committee should pay due attention to the variety of demands placed on instructors by the types of teaching called for in various disciplines and at various levels, and should judge the total performance of the candidate with proper reference to assigned teaching responsibilities. The committee should clearly indicate the sources of evidence on which its appraisal of teaching competence has been based. In those exceptional cases when no such evidence is available, the candidate's potentialities as a teacher may be indicated in closely analogous activities. In preparing its recommendation, the review committee should keep in mind that a redacted copy of its report may be an important means of informing the candidate of the evaluation of his or her teaching and of the basis for that evaluation.

It is the responsibility of the department chair to submit meaningful statements, accompanied by evidence, of the candidate's teaching effectiveness at lower-division, upper-division, and graduate levels of instruction. More than one kind of evidence shall accompany each review file. Among significant types of evidence of teaching effectiveness are the following: (a) opinions of other faculty members knowledgeable in the candidate's field, particularly if based on class visitations, on attendance at public lectures or lectures before professional societies given by the candidate, or on the performance of students in courses taught by the candidate that are prerequisite to those of the informant; (b) opinions of students; (c) opinions of graduates who have achieved notable professional success since leaving the University; (d) number and caliber of students guided in research by the candidate and of those attracted to the campus by the candidate's repute as a teacher; and (e) development of new and effective techniques of instruction.

All cases for advancement and promotion normally will include: (a) evaluations and comments solicited from students for most, if not all, courses taught since the candidate's last review; (b) a quarter-by-quarter or semester-by-semester enumeration of the number and types of courses taught since the candidate's last review; (c) their level; (d) their enrollments; (e) the percentage of students represented by student evaluations for each course; (f) brief explanations for abnormal course loads; (g) identification of any new courses taught or of old courses when there was substantial reorganization or approach or content; (h) notice of any awards or formal mentions for distinguished teaching; (i) when the faculty member under review wishes, a self-evaluation of his or her teaching; and (j) evaluation by other faculty members of teaching effectiveness. When any of the information specified in this paragraph is not provided, the department chair will include an explanation for that omission in the candidate's dossier. If such information is not included with the letter of recommendation and its absence is not adequately accounted for, it is the review committee chair's responsibility to request it through the Chancellor.

(2) Research and Creative Work . . .

How are these policies actually manifested in the evaluation and advancement process at a research university? How might they provide opportunities to place greater value on educational work in mathematics that is of a larger grain than the teacher—student interaction. Unless the work is of a carefully and traditionally oriented mathematics research character (allowing it to be evaluated under the research rubric), I believe that, in practice, there is little opportunity to have it really considered under the present guidelines. Despite this, I have tried to work within the "teaching" framework. This approach, at least, provides an opportunity to present information concerning the educational work. Unfortunately, at a research university, few faculty members feel that they have sufficient competence to evaluate the quality of teaching and these larger educational efforts. As a consequence, we naturally avoid such questions. We also have similar problems in evaluating the quality and importance of administrative and other professional work, for example service as an officer of a national professional organization or a rotator at the National Science Foundation. In extreme cases, this work may be dismissed as so less important than research as to not merit serious evaluation.

Thus, having spent very little time ourselves in formally studying research on teaching strategies, curricula, or assessment methods or in looking into the best ways to prepare graduates for future mathematics teaching careers we often feel ill prepared to evaluate the work of others in these areas. Indeed, until recently, there have been very few opportunities in research universities to discuss questions of teaching mathematics. The traditional complaints of lack of student preparation or ability are still present. The question of how to establish standards to prevent their enrollment in courses or, failing that, how to insure adequate standards are used to separate the most able students from the others is also discussed. While, for some faculty members, a 40 - 60% failure rate in an introductory course is a "badge of honor" indicating the preservation of "standards," it is now the case that others interpret this as an indication that a large percentage of students are failing to learn. This failure represents as much a failure of the institution and its faculty as it represents a failure on the part of the students. I believe that such failed courses are often "instructor" centered rather than "student" centered. Discussions about the distinction stimulate discussions about many wider educational issues within our departments today. In addition, the ongoing "calculus reform efforts" provide many more opportunities to discuss educational issues.

How might one recognize educational success? Is it our job to maximize the learning and success of students? Is it to prevent under prepared students from enrolling in courses through the use of placement exams? Is it only the work done in a formal examination setting that constitutes meaningful assessment of a student's ability to "do mathematics?" Or, do such things as graded homework assignments and group projects contribute to the assessment of the student's ability to do the mathematics of the course? What is the appropriate role for graphing calculators in a mathematics course, in the learning of mathematics at the collegiate level? What about computers? These are a few among the large collection of questions that are being discussed.

Despite the increased frequency of conversation about mathematics education, we still have too little experience and a very limited range of ways in which to recognnize and evaluate educational work. We seldom visit each other's lectures, rarely look at syllabi or course materials such as notes, examinations, class projects or student work. Student evaluations are used to gain some understanding of the relationship between the student and the instructor but, for some, these have only limited credibility. Too often, these are the only sources of information about a colleague's education work and seem to define the evaluation process. For larger scale educational efforts, it seems that only work that is funded by governmental agencies is held to be good and important, the more funding, the better and more important the work.

In the MSEB draft issues paper, we identified a couple of approaches to presenting evidence. Individual faculty members can develop professional briefs analogous to the material assembled for an evaluation of one's research contributions. They can assemble a teaching portfolio in a manner similar to the practice in other disciplines. Departments can create "portraits" to better report on its accomplishments and help identify the specific contributions of individual members of the department to making progress toward achieving the goals of the institution.

Each of these, however, requires individual faculty members to collect and analyze information. This fact challenges us to develop ways to gather, as a regular part of our educational work, information that makes its goals explicit, how this work is accomplished, how we determine if we have been successful, what we have learned from the work and, finally, how it will influence our future efforts. This information must be collected in a manner that is credible to our colleagues. It must be presented in a manner that is accessible and acceptable to them. There seem to be very few models for this that are easily (without a radical change in our professional culture) transferable to the situation of the typical mathematics professor. In saying this, I am not so much concerned with the situation of those mathematicians who are nationally and internationally visible for their leadership in mathematics education. While some of these individuals have encountered serious problems of professional advancement they have, typically, been involved in projects that have lead to the publication of books, reports or, other educational materials. The barriers that face them are, I believe, of a different character from those that I wish to address in these remarks.

I am most concerned here with those persons whose circumstances include a mix of teaching, professional service and, research or other scholarly activities. Often their careers have, at one time, been principally devoted to research. The teaching and educational work has been mostly restricted to classroom teaching, direction of graduate student research and, the mentoring of postdoctoral students. Some aspects of their current work now include a greater focus on educational issues: work of a character that is not of the typical "departmental committee" level or intensity. It may extend beyond the boundaries of the department or institution. One example might be the creation of a training program for graduate teaching assistants that goes beyond the "how to teach mathematics" to include papers and discussions on how people learn, on varieties of teaching strategies and their strengths and weaknesses or, on how to help students move from the memorization vision of mathematics toward a more thinking/creative experience. It might be the introduction of graphing calculators into the undergraduate curriculum and an evaluation of their impact on student learning. It might be the creation of an undergraduate computer laboratory and the evaluation of what software best promotes the kind of learning sought in the course or program. It might consist of working with mathematics teachers in an elementary school district to review of prospective curricula and texts or in planning and implementing a mathematics "in service program" to prepare for the adoption of new materials.

At UCSB some of the key elements required to assess and reward a wider range of mathematics education work seem to be in place: there is sympathy and support from the administration, at least so long as there is no direct conflict for research resources; there is an alignment between mission and goals of the university with respect to educational work; the formal review procedures appear to allow a component that values teaching and; they allow the inclusion of a wide range of material evidence of education work. This consistency of the public, political and, fiscal policy with respect to the educational performance of the university may eventually rebalance the process of advancement and reward in favor of greater recognition of educational work.

THE ENTERPRISE OF EDUCATION: SOME EXAMPLES

Two colleagues at UCSB, Bill Jacob and Julian Weissglass, and I have dedicated significant energy to work of a wider educational spectrum than teaching courses and serving on departmental committees. Bill and I maintain a substantial mathematics research effort, supervise master's and doctoral students as well as postdoctoral students, but Julian's focus has been mainly on mathematics education for many years. In my case, a typical workweek consists of about sixty hours or more; about 20 hours are devoted to research and the remainder to various educational efforts. While this configuration has remained essentially constant for about five years, it is radically different from earlier periods in which the proportions were reversed.

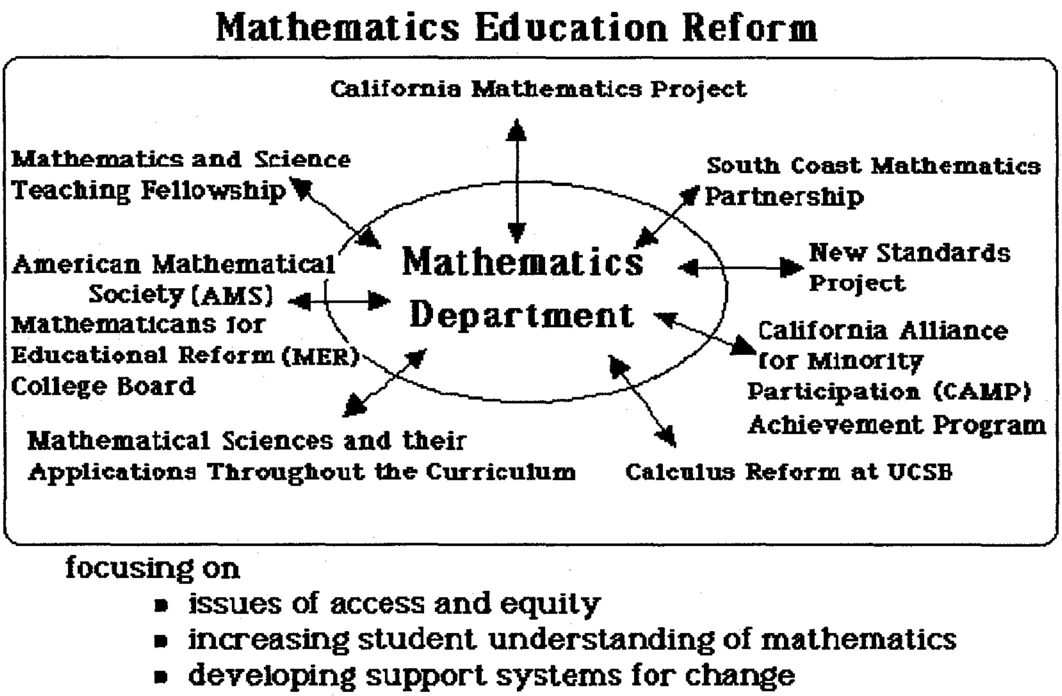
What kinds of things have we been doing at UCSB? One significant dimension is work with K-12 teachers, both preservice preparation and ongoing professional development. One aspect of this work is supported by grants from the National Science Foundation, California Eisenhower funds, private foundations, and the University of California in addition to departmental resources. Except for some limited awareness of the size and source of some of these grants, most of our colleagues have little understanding of the range of these activities and their purpose. At the request of our department chair, who wished to better understand this work, we prepared an inventory. The central theme that emerged from our effort was the commitment to access and equity. Much of our work has been directed to encouraging increased participation and success of women and underrepresented groups in mathematics and science careers. This includes working with administrators in districts and schools and teachers and their students. This includes working with undergraduate students interested in mathematics and science teaching careers. And this includes work on statewide committees and with national organizations sharing these concerns. Figure 1 shows a modified version of an inventory prepared for our chair. I have changed it to reflect only my activities and to be current.

Each of these efforts involves different levels of participation and responsibility and, awareness, understanding, encouragement and support from colleagues, department, campus and university. And controversy! Within the department and campus there have been faculty members taking public and private positions in opposition to work directed at increasing the participation of women and minority students. With the recent promulgations from the Board of Regents of the University of California ending affirmative action, there is even greater opposition to this work and greater confusion about the commitment of the University of California to "outreach" work that is sensitive to gender or ethnicity.

For the purposes of these remarks, I will focus on work directed specifically to underrepresented students (the Mathematics Achievement Program (MAP), the California Alliance for Minority Participation (CAMP), and the Mathematics and Science Teaching Fellowship Programs) and the renewal and reform of the department's calculus program.

Access and Equity

About ten years ago, Uri Triesman, Julian Weissglass, David Sprecher (Dean of the College of Letters and Science and a professor of mathematics) and I met and discussed the possible creation of the Mathematics Achievement Program (MAP). MAP was dedicated to increasing the success of underrepresented students in those calculus courses that were a barrier to progress in mathematics based degree programs. With funding from Dean Sprecher provided to the Department of Mathematics to support graduate assistants, academic workshops were immediately created. When Julian took a leave



An Inventory of Ken Millett's Education Related Activities

- Undergraduate Committee, member
- Mathematics 3ABC Coordinator (First Year Calculus)
- Mathematics Achievement Program (MAP), Co-chair
- California Alliance for Minority Participation (CAMP), UCSB Regional Director
- South Coast Mathematics Partnership (SCMP)
- Mathematics and Science Teaching Fellowship (MSTF), Director
- NSF Mathematical Sciences and Their Applications Throughout the Curriculum Initiative, Co-principal Investigator
- College Board (Western Regional Council chair elect & UC Academic Affairs Representative)
- New Standards Project (Mathematics Advisory Committee)
- American Mathematical Society (AMS) Subcommittee on Undergraduate Education
- California Mathematics Project (CMP), Chair of Advisory Committee and Executive Committee
- Mathematicians for Educational Reform (MER) Forum, Departmental Network Advisory Committee
- UCSB Academic Senate: Committee on Educational Policy and Academic Planning, Affirmative Action Committee

Figure 1

Mathematics Education Reform

of absence to work for the Mathematical Sciences Education Board, Bill Jacob and I became co-directors of MAP, a partnership that continues today. Under our leadership, the program underwent significant changes in response to our evaluation of its successes and failures. We learned that new approaches were required to bring students into the MAP workshops. When the campus received a Howard Hughes Medical Institute grant, Bill was the "mathematics person," the MAP work expanded to the precalculus course and the calculus courses taken by life science students. We explored a variety of ways in which to make the MAP work more immediately valuable to the participants. To initiate new students to the MAP, we began a collaboration with the campus' Summer Transition Enrichment Program (STEP) by taking responsibility for the mathematics program. During the academic year, the extension of the MAP work to additional calculus classes was followed by an extension to the first year physics classes.

In 1994, I became the Regional Director of the CAMP program. This brought substantial amounts of new funding to UCSB. These funds supported the extension of the MAP model to courses in chemistry, biological sciences, probability and statistics and, additional courses in mathematics and physics. In addition, this funding supported a summer undergraduate research initiation program and advanced undergraduate research during the academic year as well as the summer. This enlarged program is now called the Achievement Program (AP). It now includes support of undergraduate interest groups in mathematics and the physical and life sciences. A weekly undergraduate research lunch meeting was initiated to encourage students to explore research opportunities.

Julian Weissglass' time is now entirely devoted to several large mathematics education projects funded by the National Science Foundation. Bill Jacob has accepted significantly increased responsibility for teacher preparation programs in the Mathematics Department and the administration of a large Eisenhower grant funding the summer internship and teacher professional development program of the South Coast Mathematics Partnership. With recent changes in the Eisenhower program, he has had to take on the responsibility of developing a wider funding base for this program. As a result, I now have expanded responsibility for the work of the Achievement Programs spanning several departments and three colleges.

With base funding from the University of California's Community Teaching Fellowship Program and supplementary funding from the National Science Foundation, the Mathematics (and Science) Teaching Fellowship Program was expanded to twenty fellows. This expansion brought the Teacher Preparation Program in the Graduate School of Education more extensively into the collaboration. The Mathematics and Science Fellows, cooperating Teachers, Senior Fellows, and invited UCSB Faculty Members and the Project Coordinators are participating in a series of monthly mathematics or science education seminars. The seminars focus on critical educational issues related to providing equitable, challenging mathematics and science experiences for all students. Participants study the most current reform efforts underway within mathematics and science education at the local, state and national level. In addition to working with middle school/junior high and high school students within secondary science classrooms in the Santa Barbara District, the Science Fellows are documenting their experiences by keeping a weekly teaching diary. At the end of their internship, each Fellow will write a summative reflective paper about what they have learned, what concerns or issues they have, and what new understandings they have acquired through their involvement in the project.

What strategies have we tried to make this work more visible? First, because so many people seem to measure the importance of such projects by the number and amount of grants that support it, I have asked each funding agency to send a copy of the award letter to our chancellor and the appropriate department chair. The result has been significantly increased awareness of the work within the campus administration. I regularly receive notes from the chancellor and various deans expressing their appreciation for these efforts and pledging their support. In addition, the chair of the department has commented at department meetings about the value of various elements of the work.

Second, we regularly invite administrators and faculty members to meet with student participants in our programs. This helps make this work more understandable to them. For example, at the end of the summer undergraduate research program we organize a research colloquium at which students present a report on their results or research progress. Just extending these invitations seems to have been quite helpful and they are frequently accepted. Faculty members appear to enjoy and benefit from opportunities to have academically centered interactions with undergraduate students outside the classroom/course arena.

Third, we send copies of announcements of undergraduate opportunities and the names of successful students to key administrators and faculty members as well as making a public announcement. For example, the announcement of the twenty Mathematics and Science Teaching Fellows as well as the names of the teacher with whom they are working was sent to campus administrators, to leaders in the school districts, to chairs of the departments of the students as well as, the local press. The Achievement program publishes a quarterly glossy magazine in which we have been able to have some of our first year's work featured. The most recent edition included a three page description of several of our new programs and the successes of the participants.

Fourth, we have created a homepage for the Achievement Program with links to affiliated projects and departments. The purpose is to make the goals and range of activities of the Program known to interested students as well as to members of the wider campus community. I am hoping that this will increase the level of support among all constituencies. The homepage will provide links to student research abstracts and news of student achievements as well as announcements of new programs and application procedures. It will also provide individuals seeking more concrete information concerning this educational work with an up to date source.

What is the result of these efforts? Because of the success of the Achievement Program, the campus has provided funding for new initiatives. One example is the three week summer bridge Mathematics and Science Institute (MSI) that I proposed for creation during the 1995 summer. The response of the Dean and Provost was immediate and enthusiastic. They provided the support required to invite 25 students from underrepresented groups having the highest levels of preparation for study in mathematics and science to participate in the program. Funding has been pledged for 1996 and support from the UCSB Foundation is anticipated.

At the departmental level, the situation is challenging. First, efforts directed to encourage women and underrepresented students are controversial for some faculty members. Second, there is a significant reduction of funding for traditional departmental courses. Additional resources are required to renew and reform the undergraduate program in calculus. As a consequence, the levels of support historically provided by the administration for the mathematics work of the Achievement Program are now allocated to be used at the discretion of the chair. The dilemma of whether to support outreach or to spread these resources across the "traditional" program has been an ongoing source of stress almost since the birth of the MAP. Thus, while there has been measurable progress in awareness, skepticism about the legitimacy, effectiveness and, importance of this work at the faculty level continues to be fashionable. Even success provokes opposition. For example, success in assembling financial resources and support has been attacked because of a belief that the money was made available for educational programs to the detriment of traditional research program support. Even if educational work is accepted as useful, critics attack the choice of targets. "Why don't you get funding for ... instead of...?"

Fortunately, there is strong support for this work among several key administration and department leaders. We need to develop a more stable mechanism for support of this work. Keeping the necessity and value of a wide range of mathematics education work in view remains a critical and continuing component of our work. In order to make a lasting change in our educational system, all stakeholders must subscribe to significant portions of a coherent set of goals and strategies that structure their work.

RENEWAL AND REFORM OF CALCULUS

In many ways the UCSB efforts to renew and reform calculus instruction began with the realization that the collaborative work and the worksheets and projects prepared for students in the MAP workshops did not receive the level of student effort that was required to promote the intellectual growth to which we aspired. One way to change this, we proposed, was to insure that this work was necessary for success in the course rather than being an optional component. This required a new text and a new approach to instruction and assessment of student accomplishment. After some review of candidates, the draft materials from the Calculus Consortium based at Harvard (CCH) was selected and permission to use them in several pilot sections of the course was secured from the Department's Undergraduate Committee. During these two years, we kept track of student performance in these pilot sections (for students participating in MAP but open to others on a space available basis) as well as those in the "traditional curriculum sections." Students soon understood that the pilot course required more work and some decided to enroll in the easier traditional course. We found that those who remained actually did better in subsequent courses than the traditional course student. But, on average, they had a lower self image of their mathematical skills. Because of the choice of the text, we began using graphing calculators in the pilot course. We also increased the intensity of effort to prepare the graduate students working in these sections.

Following the two year pilot period, we held a series of departmental meetings that included the participation of graduate teaching assistants. The purpose was to discuss future directions in the calculus course for the physical sciences and engineering. After much discussion and a very close vote, the Department decided to adopt the CCH curriculum for all sections. Bill Jacob led the effort to organize workshops held prior to the start of the Fall 1994 quarter. These workshops included participation of undergraduate tutors, staff from programs concerned with students enrolled in the calculus sequence as well as graduate assistants and instructors. The variability of experience between sections in the 1994/5 courses led to the conclusion that even more preparation was required to achieve the level of coherence needed for student success over the year long sequence of courses.

This first effort was the catalyst for increased discussion of the curricula and teaching strategies among faculty members and graduate students. It has required a continuing revision of the Department's TA training program. It has stimulated the creation of seminars concerned with mathematics education (both elementary, and secondary as well as, undergraduate). And it has rekindled the discussion of student preparation for college level work. While one hears the time honored complaints about university students incapable of thinking mathematically, of learning mathematics, of doing simple algebra, of getting a degree in this or that major, ... there are also discussions about how to improve teaching or to better evaluate student understanding. Conversations about the course expectations (we don't have common exams) are rather frequent. We seek greater comparability of grading standards across course sections. While there were no longer classes with 60% failure rates, as had occurred before, the current range is between 15% and 40%, depending on the instructor.

For the 1995/6 year, effort to provide information to graduate students, instructors and, others increased significantly. With the decision to use another "reform" text for the life sciences calculus course, there were no longer any "traditional" calculus courses being taught at UCSB. All instructors and graduate assistants were asked to participate (but it seems that few instructors actually did so) in a much more intense workshop to prepare for the fall quarter. The TA training program became even richer with regular meetings scheduled for even the veteran TA's. Using ideas and materials from our own work and that at the University of Michigan as a foundation, I prepared a draft Instructor's Guide that included course goals, rough syllabi, and discussion of key issues. These were used during the workshop and provided to all instructors. Workshops for students and faculty on the use of graphing calculators were held prior to the start of classes and during the first two weeks of instruction. Overall, much more was done to prepare for the fall quarter's calculus classes than in previous years.

Once again, however, there were serious problems. Many of our instructors did not attend any of the workshops and some ignored the Department's syllabus, goals, and suggestions in the Instructor's Guide. Several graduate students had a difficulty making the transition from the undergraduate experiences to the level and approach of the courses to which they were assigned at UCSB. And once again, there were questions concerning the appropriateness of the graphing calculator use in calculus courses. Some instructors forbade their use despite the Department goal asking for the appropriate use of technology such as calculators in calculus courses. Others skipped key sections and, even, chapters despite the departmental syllabus provided in the Instructor's Guide.

As a consequence, we continue to have energetic discussions of strategies to improve the undergraduate mathematics instruction. One result of the reform effort is a new proposal to build collaboration across the engineering, mathematical, physical and life science disciplines. This has been developed with support of the chancellor and with the concerned deans serving as principal investigators. In contrast, however, one engineering department has proposed to take over the teaching of the second year calculus. They propose to insure that engineering students (about two-thirds of our second year enrollment) receive an adequate preparation for their engineering classes. Developed without discussion or consultation with mathematics department faculty members, the proposal indicates that an understanding of the calculus reform efforts and a commitment to collaboration are not widely shared across the College of Engineering. Because of this, the Academic Senate Committee on Undergraduate Courses is considering the request and representatives of the two departments are looking into next steps that may be taken in collaboration.

A CONCRETE EXAMPLE: AN EXCURSION INTO CALCULUS REFORM

A critical element of any successful renewal and reform effort is the sharing of what has been learned with colleagues in a manner that informs them and, ideally, enlists their participation, support or, at least, acceptance of continuing work. As I have argued earlier, one must provide examples and information in a form that is accessible and useful to these colleagues. One example of this was my work to put together an instructor's guide for the first year course. This guide included goals for the courses, rough syllabi, discussions of the critical aspects of the teaching that seemed to promote student success, discussions of questions frequently asked by both instructors and students, and examples of problems and student work with an elaborated mathematical commentary on the evaluation of this work and its mathematical content.

In order to make much of this information available to instructors, graduate assistants and, most importantly, students in the first year calculus sequence for mathematics, physical science and, engineering, I am creating an official departmental homepage for one of the multi section/multi instructor calculus courses. I believe that this will encourage greater coherence between the various lectures of the course by making public the goals of the course (for example the extent to which graphing calculators should be part of the student experience) as well as the level of achievement that is generally expected of successful students.

The creation of homepages for courses is quite easy to accomplish and is accelerating, especially in large courses. Often these include tests taken and graded via the computer. Some times there are problem assignments and investigations that use the computer to present information and support student work with this information. In biological sciences, students may look at three dimensional models of molecules that they are able to move in space in order to help them develop greater three dimensional understanding of these conformations. Similar examples exist in physics, chemistry, engineering, and mathematics. Class news groups and virtual discussions are supported, sometimes with access to "virtual visiting experts" whose work may be provided on the computer and to whom questions can be posed. More and more frequently, departments support homepages with links to those of individual faculty members (where their work is presented) and to courses. This vehicle, I believe, is an opportunity to present examples of educational work in a very public manner that can help inform our colleagues and others. It can provide remote colleagues with access to this work and could help enrich and expand a professional community concerned with educational reform. Recently a colleague approached me with the suggestion of creating an electronic journal concerned with undergraduate education reform. Preliminary discussions suggest to me that this might be a project whose time has come.

Among the things that I believe are good candidates for the calculus course homepage are examples of "group investigations." These are longer and larger mathematical projects designed to stimulate student exploration of fundamental aspects of calculus, to provoke greater student collaboration and conversation about the substance of calculus and, to provide an opportunity to work on and write about "open ended" problem situations. During the recent few years I have collected and indexed examples of such investigations, syllabi, exams and, any other course related materials. These are available to temporary instructors and graduate students many of whom are teaching such a course for the first time. We have also used these materials as resources for our workshops and seminars on teaching the reformed curricula. By making these available to the departmental community has provided one method to inform others about the goals and results of the aspect of our mathematics education work.

What is the context within which we are trying to teach a "reformed calculus" course? In addition to large class meetings, I usually have from 100-150 students in each class, we ask students to meet with graduate students in a workshop setting of about 25 students each (recently reduced from 35-40 students). I describe these as class workshops rather than discussion meetings. In the latter, students typically sit and copy graduate student demonstrations of worked out solutions to problems while, in the former, students work on course material during that time, often in collaborative groups, and to report on the results of their efforts to the other students. In addition to the "homework problems" I assign "group investigations" that are meant to be done over a period varying from a few days to several weeks. These are designed to stimulate work with basic issues of calculus and their manifestation in contexts where they are useful. In some cases, new concepts that will be more fully developed in class and elsewhere are first introduced here. There are many reasons for doing this but an important one is to create purposeful situations in which students must discuss mathematics with other students. I like to think of this as part of an optimization problem: maximizing the amount of time each student spends talking about mathematics in contrast to listening to instructors talk about it.

For the occasion of this meeting, I took the first steps toward the creation of such an investigation to be shared with the participants as a means to promote discussion of both the nature of our work at Santa Barbara and as a means of making the critical dimensions of this work more understandable to colleagues. The following version is the one I plan to use in my second quarter calculus course during the Winter Quarter, 1996 at UCSB.

The Investigation: "Just In Time" Management Strategies

A calculus investigation created especially for the 1995 Annual Meeting of the Canadian Mathematics Education Study Group (CMESG) in London, Ontario from May 26 through 30, 1995.

Goal: The purpose of this investigation is to analyze discrete data by using fundamental concepts of differential and integral calculus. A secondary purpose of the investigation is to increase team problem solving capacity, ability to identify key issues, to use calculus concepts imaginatively and, to produce a concise mathematical report.

Assessment Dimensions: Completeness, mathematical correctness, and clarity of written report.

Problem Statement: The CMESG Card Company is a large producer of mathematical birthday cards. In order to keep overhead costs low, the Company proposes to maintain as small an inventory of raw materials and employ as small a permanent work force as possible and still be able to meet the production and shipment schedule to deliver the birthday cards on time. To meet any "peak season" needs, it would receive raw materials and hire temporary employees "just in time" to meet the peak season production requirements. The Chief Executive Officer has asked your group to create a production schedule that would most efficiently manage the flow of raw materials, plant staffing, and shipment of the mathematical birthday cards.

- The production and shipping of these high-tech mathematical cards takes approximately one half month.

- The cards must be in the retail locations about two weeks prior to the intended recipient's birthday.

The CEO has prepared questions whose answers she wishes to have available for the Board of Directors at their next meeting.

Write a two page report for the Board of Governors, including supporting data, graphs, analysis, supporting the cautious management alternative.

(1) Should the Company base its planning on four month average birthday data (this is the average number of birthdays in that and next three months) or on the monthly data? How do these differ? Which would lead to a more conservative management plan?

(2) Based on either the four month average data or on the monthly data, when might the Company need to accommodate peak quantities of raw materials and employ temporary workers? (This is done when the demand is more than 10% above average.)

Extra Credit What behavior of the birthday function will lead to a local maximum of the four month average function?

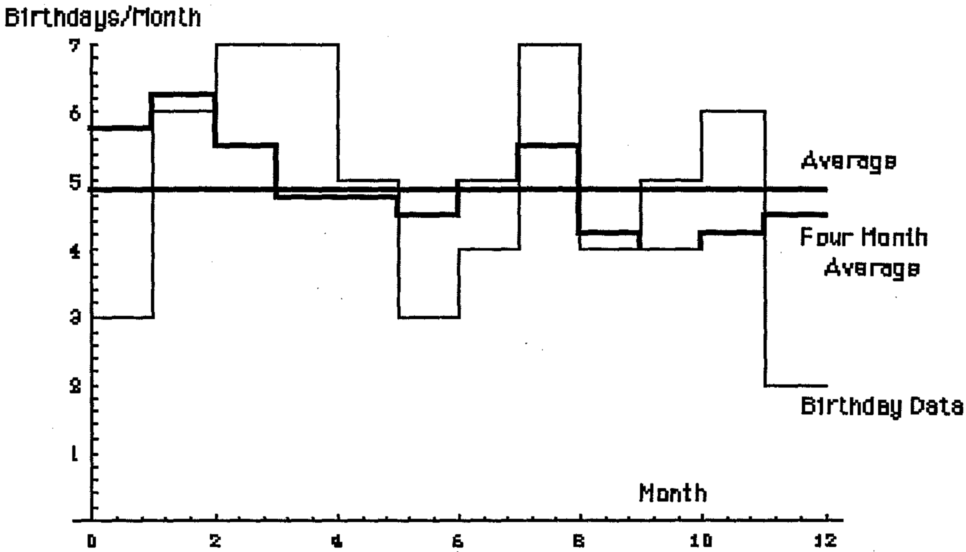
Extensions Does your analysis suggest any extensions or alternative ways to approach this problem?

Discussion of Implementation:

In my class of about 140 students, birthday data is collected during one of the lectures preceding the assignment of the investigation. Here I will use the data collected from participants during the CMESG meeting as a measure of the distribution of birthday months among mathematically inclined persons. We assume that the market for cards will be proportional to the number of mathematically inclined persons having birthdays during that month.

A Model Strong Response:

The four month average data, A[T], is one-quarter the integral of the birthday data, B[T], from T to T + 4. Graphs of A[T], B[T] and, the average number of birthdays per month are presented on the same axes. The average number is 4.9 birthdays per month. Therefore 10% threshold is 5.4. The four month forward average data has smaller fluctuations than the monthly data (it is "smoother") with its behavior anticipating the monthly data. While the monthly data has three peaks, in March/April, August and, November, the four month data has two peaks, in February and in August. With a lead time of one month required, the monthly data suggests hiring temporary staff and increasing materials during January through March, again during July and, finally during October. The four month data, on the other hand, recommends hiring supplementary staff during December through February and during July. These planning methods differ in that the four month data suggests materials and hiring earlier during the winter period and not at all in October for the Fall peak demand. The four month data is more conservative as it anticipates demand and seems to suggest a smaller expenditure of resources to meet demand as it smooths out the peaks and valleys.



Extra Credit. The four month average data will have a local maximum A[t] has a critical point. A' [T] = (B[T + 4] - B[T])/4 = 0 is the condition for a critical point. It is a local maximum when A '' [T] < 0. This means B ' [T + 4] < B ' [T].

An Extension: Looking at the graph reminds us that the one year integrals of the three functions are equal. While we might be tempted to hire a constant number of employees over the year and produce cards in advance, this strategy requires a larger facility for storage of materials and finished cards. By looking at the differences between the monthly data and the four month average, it might be possible to determine if that peaks and valleys occur in such a way that the valleys provide sufficient production time to accommodate the peak demands represented by the peak months. By computing areas, it appears that this is the case as December/January are five below the average while February, March and, April total about five above the average. The other two peaks are preceded by comparable valleys. This suggests that, with a small amount of storage space, the company might be able to avoid hiring temporary employees. This is advantageous from the perspective of maintaining a high level of quality through an employee training program and avoiding any short term employees with less training.

ASSESSMENT RUBRIC AND HOLISTIC GRADING

I have been using a "holistic" grading scheme to assess some elements of student work in my classes for several years. I would contrast this approach with the traditional "atomic" grading, in which students receive large amounts of "credit" for small fragments of work related to the problem, with an approach that focuses on the global level of success of the student in addressing the stated problem. Small fragments need not accumulate into a significant amount of credit unless they clearly tie together in a coherent and successful approach to the stated problem. Furthermore, the clarity of presentation of the work is an important element of the evaluation. As most students are not familiar with this approach, I regularly remind them of the standards by giving them a copy of the following generic rubric. It is also important to work with the graduate assistants and any undergraduate grader regularly. These interactions are, perhaps, the most powerful occasions during which to look at questions about teaching and learning.

Generic Evaluation Rubric: The goal of this investigation is to apply fundamental concepts of calculus to the analysis of birth month data. Participants produce a report describing and supporting the conclusions reached in the investigation. The assessment is accomplished according to the following rubric:

***5 points*:** Fully correct and complete work including explanations and possible extensions.
***4 points*:** Essentially correct and complete work, including explanations. No significant mathematical errors, although there may be some superficial ones or small gaps in supporting discussion.
***3 points*:** Mostly correct and complete work, including explanations. No important mathematical errors (there may be a few minor ones or some small gaps in explanations but most of the key elements are present).
***2 points*:** Partially correct work but key elements are missing, are not correct and/or essential aspects of the explanation are missing.
1 point: Some indication of understanding but essential aspects not addressed or explained.
0 points: No work submitted on the problem.

LOOKING TO THE FUTURE

Although I feel we have had some important success in increasing awareness and support of mathematics education efforts within our institution, there is clearly much more that must be accomplished. Faced with significant demands for attention to a wide range of crises, university, college and departmental administrations must receive more and better information in order to accord this work the value it deserves. The competition between research and educational work must be better balanced and valued. Many of my colleagues across the United States have been reacting strongly to the proposal at the University of Rochester to abandon the Ph.D. program in mathematics, to reduce the size of the mathematics faculty and, to support the teaching of undergraduate mathematics classes by a mixture of regular and temporary faculty (at a significantly reduced cost to the university). While there are many reasons to which one may point for justification of this proposal, critical ones appear to be the recent ranking of the department quality in a U.S. opinion survey, the insulation of the research program of the Rochester Mathematics Department from wider university community, the perceived lack of interest of Mathematics faculty members in insuring an adequate (if not exceptional) undergraduate mathematics program and, the sense that the quality of work and commitment of faculty members to undergraduate students was very low. The views expressed by their colleagues and by professional organizations, such as the American Mathematical Society that even sent a visiting team to the campus, seem to have had little influence.

The result in many Mathematics Departments in the United States appears to be increased levels of anxiety and discussion about the state of affairs within their own institutions. This certainly has been the case at Santa Barbara. I hope that these discussions will provide an opportunity for a more balanced approach to the various aspects of the department's mission. Once again we have been reminded that we must do a much better job at understanding and explaining what we do. Both our research and our educational efforts must be understood as being important to the institution and effective in achieving its goals. In particular, if mathematics education work is to be sustainable it must be understood and valued by our colleagues, by our institutions, by our governmental agencies and, by the general public.

This continues to very difficult to achieve. Despite the fact that more than half of our undergraduate students and almost all of our graduate students will pursue careers in which teaching and mathematics education play the dominant role, there is only a very limited level of resources and attention provided to support their preparation for this work. At the graduate level, we have turned down on several occasions requests to create advanced mathematical education courses for our students. Recently, it was decided that information concerning opportunities to pursue a graduate program in mathematics education would not appear in departmental publications despite the interest of several faculty members.

If we are to be more successful in balancing research and education, better information must be more widely available. Our colleagues must continue to be provided with a survey of activities and credible explanations of how these support the mission as well as how they might be able to participate or support this work. For example, newly created courses for future mathematics teachers must be better described and the results (in terms of student accomplishment) provided to the faculty members. Reports of mathematics education meetings and workshops, mathematics education materials (at all levels), announcements of seminars and meeting and, reports of successful efforts to develop extra mural funding for mathematics education work must be distributed within the institution and the mathematics education professional community. The more widely information is shared and discussed the better able to respond to requests for an assessment of the importance of this work. Just as we share research efforts and accomplishments with colleagues around the world, so must we also develop a professional community around the mathematics education work that includes members for whom this is only one facet of their professional work.

In an important sense, our education work includes educating other faculty members and enlisting their assistance and support in improving the educational effectiveness of our institutions. Each of us must have a "marketing plan" that includes an assessment and characterization of the potential market for supporting this work, a setting of specific concrete goals so that we will be able to determine if we have been successful or not, the identification of the strategies or action plans that we shall implement in our effort to achieve these goals and, an ongoing evaluation that helps us make accommodations or adjustments to successes and failures. We must be significantly more determined in our effort to promote the importance of mathematics education work within our academic institutions and our professional organizations. Surely this is different from the way that many of us understand our professional responsibilities. The MSEB issues paper, Teaching Growth and Effectiveness, is only one small step in the middle of a long journey to create and implement a truly robust professional base of material and knowledge. These will be required to adequately evaluate and recognize work in mathematics education that is not of a traditional research nature. By sharing these thoughts and experiences, I hope to encourage others to take up this work. The experience of working with other concerned mathematics educators at meetings such as the one for which this report was prepared has been invaluable to me. I hope that we will all continue to recruit, encourage, nurture, sustain concerned colleagues and, find new opportunities for collaboration in the future.

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WORKING GROUPS

Working Group A

AUTONOMY AND AUTHORITY IN THE DESIGN AND CONDUCT OF LEARNING ACTIVITY

Gary Flewelling, Arthur, Ontario
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Participants:

Judy Barnes	Marina Padamadan
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Douglas Franks	Geoffrey Roulet
Lynn Gordon-Calvert	Joan Routledge
Anita Losasso	Elaine Simmt
Mhairi (Vi) Maeers	Susan Stuart

INTRODUCTION

This working group, facilitated by Gary Flewelling and Vi Maeers, met to explore the concepts of autonomy and authority within the design and conduct of a classroom learning activity, and to share the experiences and understandings of all the participants in relation to these concepts. Realizing that many different understandings of the concepts of autonomy and authority would most likely be present in the working group, we decided to begin, not by defining our terms, which could have been interpreted as an act of authority, but rather by orchestrating an actual classroom learning activity, and from the participants' experience associated with the activity, developing understandings of the concepts to be explored. The 12 participants were thus given a slab of clay and some 'tools' with which to create decorative designs on the clay. The direction was given to "find the math in the activity."

The three days of exciting interaction around the concepts of autonomy and authority can be considered as three acts of a play, the title of the play being "The Potter's Microworld." The following is an outline of The Potter's Microworld.

Day 1, Scene 1: Entry to the Potter's World

Brief Description:

This was the day we got our hands dirty as we individually messed around physically with the clay, creating our designs with the tools given to us, and with other tools we found around us. This was a physically active, hands-on, concrete-experience day. It could be said that we were building an image, using physical tools. This image would become a mental tool or a mental construct which could then be used to further explore the topic of the working group. The intent of this experience was for all of us to have a common shared foundation from which we could establish and foster discussion. The first half-hour of this day was therefore spent in individual construction and in observation of and interaction with others regarding their constructions.

Day 2, Scene 2: Expanding the Potter's World

Brief Description:

On this day we expanded the conversation from day 1 by 'playing' with the 'mental image' of the previous day's experience and 'transferring' the mathematical attributes of the clay to other mathematical microworlds (e.g., calculus, geometry). One of our members had stated at the outset that her interest in being in this working group was to explore how to "create space for mathematical conversation." What we explored on day 2 was what this (interactive) 'space' might look like.

Day 3, Scene 3: Meanings, Metaphors and Models

Brief Description:

On day three we explored definitions and interpretations of autonomy and authority, beginning with a dictionary definition and some quotes from articles. This was a day of ambiguity as we examined the meanings of terms. The meanings we gave the terms became fuzzy and we found it difficult to distinguish the interpretations. We began to use them interchangeably, through the use of metaphor. The metaphorical lenses through which we worked with the concepts of autonomy and authority enabled us to see the significance of clarifying and contextualizing what we mean when we use the terms. We also proposed some models to help us think more clearly about the concepts under study.

INTRODUCTION TO THE CONVERSATION

Autonomy is a frequently used word in educational policy, curriculum, and reform documents. It appears to be in fashion to cater to the needs of the autonomous learner—at least on paper. Exactly what does the species autonomous learner look like? What is an autonomous learning space? When we think of a teacher being in authority what does this term denote? It is the experience of both Gary and Vi that the terms autonomy, the autonomous learner, and the idea of authority have been 'dropped' into 'conversation' to appeal to and to appease the current agenda for school reform, without any careful description or contextual interpretation of what these terms mean. Therefore, in pseudo-constructivist fashion, one can determine a personal 'truth' without adhering to the common or general interpretation, or to the elusive 'deep meaning' of the concepts. Without adequate definition, example, clarification, and contextualization, this word can be used to generate discussion on positive and negative trends in education. In a narrowly-defined sense autonomy is considered 'all good' and to be sought after and authority 'all bad' and to be avoided. There are many different contexts within which these terms exist and, consequently, many different interpretations of these terms. Clearly, the meaning and experiences of autonomy and authority, held by those in positions to influence what is to be taught and how it is to be taught, need to be addressed. The intention of this working group was to explore the different conceptions of autonomy and authority held by the participants, as these conceptions emerged from a common activity, and as they were influenced by and expanded on within the interactive community of the working group. The intention was not to present 'finished' definitions and polished interpretations. The facilitators and the members of the working group were all participating in this evolving conversation.

The Conversation—the story told by the participants, organized and embellished by the facilitators. The part inside the [ ] and anything within double quotation marks denotes quotes from the working group participants.

DAY 1, SCENE 1

At the start of day 1, after being given the clay and asked to find the mathematics in the activity, it was evident that some people wanted more structure, wanted a clearer idea of what to do, of our expectations for them ["I would begin with a clear statement of the problem"]. For some, the clay provided a rich curriculum space ["math starts coming out as you're doing it" (i.e., playing with the clay)], while for others it was constraining, restrictive ["The medium of the clay poses constraints, as do the students' mathematical histories (experiences), as do the teachers' directions"]. Some participants could not see, initially, the relevance of the clay microworld for their particular focus in mathematics ["I can make mathematical connections, but is this the best medium to do it with? Given the available resources, what is the best medium? I wouldn't use clay to do that. I'd use computer software. What does clay add to the activity? It's important to have effectiveness and efficiency. For the kid, the math has to have efficiency. I don't have the luxury to do whatever I want."]

As the activity progressed and the conversation evolved, it became evident that everyone had created a unique and very interesting 'structure.' Tools and methods for creating 'structures' were shared and products were displayed. In this making and interacting phase, *The Potter's Microworld* was conceived. (We did not plan to have a potter's microworld—this idea emerged from the group, which had now become a community of potters).

The initial directions for this activity prompted some participants to be concerned about the lack of initial structure and the responsibility of the teacher in a pedagogical relationship ["The child has the right to ask questions and the teacher has the responsibility to respond." "We must not forget the relationship with the children."] One of our members was concerned about betrayal of trust. She felt that it was all right to be open and flexible at the beginning and thus allow and enable the child to create any structure, but if we had an ulterior motive and really did have a specific structure in mind then she considered that a betrayal of trust. ["Some activities begin as open and then become closed. Betrayal. The child thinks that you really didn't want him or her to do or know anything. If there is a direction, then lay it out on the table. Do you really value my stuff?"] This valuing of one's contributions, we felt, is essential in an open-ended assignment.

We discussed the initial lack of structure with the question "What structure could be used to increase the probability of certain things happening?" Clearly, a teacher has an intention in every pedagogical decision. If the intention in the above activity was for the children to discover the properties of symmetry, then that would be a closed and somewhat orchestrated and directed assignment. If, however, the intention was to "behave like potters, do what potters do, explore the tools that potters use, create the kinds of designs that potters create, and share how and what we've created within the community of potters" then this kind of open-ended, genuine, trustworthy intention can be offered to the children as a rich open curriculum space for them to explore clay, but within that exploration they will create designs and be able to share some mathematical aspects of these designs with others. The children would thus "be given the scaffold without being given the building already built" and the teacher could "initiate the conversation (from what the children have made)—it should arise from the activity in a reasonably natural fashion." As one member said, ["Math becomes meaningful as we use the tools and materials we are given."]

The medium of the clay poses a constraint—it can be considered an authority in that it initiates certain actions and precludes others. You cannot just do anything with a lump of clay. The authority on the part of the learner is that the learner dominates the clay and the tools used on the clay, manipulates and creates from the clay an autonomous, self-directed design. As mentioned above, the child's own previous experience with clay, or experience in getting his/her hands dirty, or experience in admiring finished clay products all come into play at the point of interaction with this medium. The teacher's (albeit somewhat limited) instructions also pose a constraint and could be considered authoritarian in the sense that choices are being limited by the instructions. But as choices are being limited in one sense, they are being opened in another, because now, within this limited structure, the child is truly free to make all sorts of decisions and choices regarding what materials to use, what design to create, and what kind of finished product to arrive at. Even at first if there is discomfort, "it's not necessarily bad to be uncomfortable. There's autonomy in discomfort. Out of discomfort comes learning." The child when placed in a space where he is not being told what to do, where personal decisions have to be made, may be temporarily in a state of disequilibrium, but will soon accommodate to this state.

At the end of day 1 Gary and Vi felt that the group had indeed entered into the conversation around the terms autonomy and authority, had interacted physically with an interesting and stimulating medium, had begun to make connections to their specific area of mathematical interest, and had seriously considered the issue of structure, both initially, in the form of directions, and within the activity. How much structure is necessary or efficient to give the children "to increase the probability of certain things happening?" We had explored autonomy at the start of the activity and within it and as one member said, "The students didn't have autonomy about the setting, but they did have autonomy within the setting." We all agreed that some structure on the part of the teacher was essential (structure is being used here synonymously with authority). The teacher makes a sound pedagogical decision to involve the children in a particular activity. The children may not, initially at least, have much choice over that activity, but, providing the activity allows space within it for choices to be made by the children, then we can say that an open-ended autonomous curriculum space has been provided. At the end of day 1 we considered the limitations of the actual activity of the clay and began to think of this activity more as an idea which can be connected to other areas of mathematics and outside the mathematics curriculum. The concept of the possibilities evoked by this microworld can easily be transferred to other microworlds. ["The microworld space is transferable—we don't have to be locked into the clay."] With the mental image of day 1 activities, and of autonomy and authority within the context of this activity, we entered day 2.

DAY 2, SCENE 2

At the start of day 2 we reviewed some notable comments from day 1 and thus summarized some of the big ideas generated from the first day. Some people had expressed concern regarding the limited possibilities of this microworld to effectively teach certain concepts and skills. For instance, "Within this microworld, where is the place for drill and practice?" or "How can we ensure that within this microworld certain skills are definitely embraced?"—e.g., creating or recognizing symmetric designs. Others viewed the microworld, which we offered, to be too big—["it could go anywhere"] and felt that certain boundaries or barriers needed to be erected to "navigate our way through the microworld." To partly address these concerns we developed an outline series of 'lessons' around the micro (mathematics) world of the potter. The lesson topics were as follows:

The Micro (Mathematics) World of the Potter

Lesson 1:
The Potter's Pots (the physical making of the pots—the work of day 1)

Lesson 2:
The Potter's Convention. (The sharing of the finished 'products' and the sharing of the tools and methods used to make the pots. This we also did (very quickly) on day 1.)

Lesson 3:
A Potter Family and its Tools. (A group of children could form a 'family' and create a special design significant of the family, using a set of 4 or 5 'tools.' These tools could be selected from a tool bank or they could be created by the family. The family needs to choose its tool selection wisely and be

able to justify this selection to others. This family thus becomes expert in this specialized set of tools and design and can 'apprentice' new craftspeople and/or go 'on tour' with their new craft.)

Lesson 4:

A Potter's Family Manual. (The 'family' group can now prepare a set of written instructions for the design and use of their tools and the making of the actual design. The design may illustrate a 'family's' life story—this could also be written into the manual.)

Lesson 5:

An Archaeological Dig. (In this dig all the tools and/or designs could be buried in sand and when retrieved could be connected to the families who designed them--whoever retrieves the find must be able to say why that tool or that design best belongs to a specific family.)

The entire series of 'lessons' suggested above could occur within a two-week space of mathematics time. In addition to the more general, open-ended activities outlined above, other more prescriptive and closed activities can be worked on. For instance, the following prescriptive side-trips would connect with the overall microworld:

Side-trips

Side-trip A: Shape of the Clay. This influences the design (investigation day 1).

Side-trip B: Creating Special Designs. Potters need to practice their craft and one of the things they like to create is symmetry—a day could be spent specifically creating symmetric designs. Likewise, a day could be spent on angles, on texture, on repeating patterns, or on ancient number writing—e.g., Egyptian Hieroglyphics. If there is something very specific that a teacher would want the children to learn about in connection with the clay, then that specific concept can be introduced in the form of a connected side-trip. This is still part of behaving like potters and playing with the tools of potters. Now, in addition to the physical tools that are available, the children can use mental tools, in the form of remembered patterns, textures, etc. The teacher can be said to be "in authority" in that she or he has provided the microworld space—the physical material and the directions, but even in something very prescriptive as creating symmetrical shapes, the children still have room to make choices in the creation of their designs.

Side-trip C: Using Auxiliary Devices

Side-trip D: The Pottery Designs of Indigenous Peoples

Side-trip E: The Commission. (One would create a custom-made design to satisfy the needs of some customer.)

Assessment

In all the above scenes and side-trips assessment can be built into the activity. Again, to be honest to the child, to not "betray" him/her (i.e., to not ask for one thing, but assess another) we must ensure that appropriate assessment accompanies each activity. For instance, in lesson 1, when everyone is creating designs on clay, all we may need to assess is appropriate use of the clay, appropriate behavior with the tools, and appropriate interchange of information--in other words, the conversation should be focused on the activity of making designs on clay, and the tools and methods being used. In the archaeological dig, which occurs at the end of a sequence of activities, assessment can be more focused on processes such as convincing, judging, connecting, and justifying. The children should know at the outset, when they participate in this dig, that we will be looking for instances of certain kinds of behavior. In all the activities described above there is structure 'imposed' by the teacher as a probability enhancer for desired student action. This structure increases the purposefulness of the activity, helps to focus the activity, provides opportunity for assessment, and provides the occasion to enable learning to occur. We believe that the structure of the activity frees the children to make decisions and bring forth behaviors that enable learning. Without such structure, learning opportunities are limited and erratic.

Microworlds

With the above ideas before them, the conversation continues. Again, as in day 1, the conversation is documented in narrative, with participant quotes interspersed. We opened the conversation with the following comments: "The microworld becomes a probability enhancer for what we want the children to learn." (When a community of learners 'lives' for a while within the microworld, this microworld becomes a rich curriculum space for many curriculum goals to be met. Too often a microworld is used for narrowly defined goals, thus ignoring the potential of this space). A microworld could provide a text or script to follow, thus giving specific teacher and student directions in sequence so that both teacher and students 'get it just right.' In so doing, however, the 'life' of the microworld would change and children may not come to view the experience of living within this microworld as being authentic. The 'tools' and the 'structure' can be provided 'up front' by an 'external expert' or they can be created from within the microworld by the teacher and the students. The activities outlined above have emerged as possibilities for us now because of our actions within the microworld—they were not pre-determined independent of action before the activity began.

Some concern was expressed regarding the wide extent of possible behavioral outcomes from the microworld experiences and the possible resulting 'watering down' of mathematics. ["This microworld is rich in integrative opportunities, but we have to be careful that the math doesn't disappear. Any subject can get lost in the integrative experience"]. Clearly, the microworld is not a space for anyone to do anything in. There needs to be structure built into the initial intention, and built into the ongoing 'navigation' through the world. This structure will first be designed by the master crafter (or potter)—the teacher—and thereafter by both the master and apprentice potters, as they make decisions together about the route to take. The ongoing design thus becomes more collaborative.

Teacher Training

Connected to the concern regarding the many possible outcomes and pathways to follow is the concern for the wide range of knowledge necessary for the teacher to have to be able to appropriately interact with the children. Some people felt that the teacher would need to know all the mathematics related to all the different pathways that children might take. To do this would place an inordinate burden on the teacher. The comparison was made with teachers of language arts--do they have the knowledge of and know **all** the answers to **all** the children's questions? Is it a problem that they don't have **all** these answers? ["It's interesting how we can be this kind of teacher in language arts and not in math."] Some participants felt that children's learning would be restricted by the lack of mathematical understanding of their teachers—again referring to the need for teachers to know more mathematics.

We discussed the perceived problem that seems to exist across the country that many pre-service elementary teachers do not have the necessary mathematical understanding to teach mathematics for conceptual understanding. ["Pre-service teachers don't have a thorough understanding of math. They can't plan a (math) program and they can't be clear as to where this will go;" "Some pre-service (elementary) teachers don't know any of the answers. They are algorithmic bound. They don't know what questions to ask."]

We discussed the fact that the children's learning would not be as rich and meaningful "if the teacher doesn't understand the depth of the math--lots of mathematical opportunities will be missed." If we can accept the premise that many pre-service elementary teachers, and perhaps many currently practicing teachers, do not have a deep thorough conceptual understanding of the mathematics they are teaching, we are in a position to do one of three things: (1) not expect that teacher to teach mathematics in any manner except textbook driven, that is, "follow the script and you can't go wrong;" (2) attempt to change that teacher's view of mathematics and view of the learner. This can be done in part by immersing that teacher in rich mathematical learning experiences where she or he experiences success as a learner and begins to see the possibilities of teaching mathematics the same way. Placed in this kind of situation, teachers "learn to search for answers and get enthusiastic about math;" (3) provide materials, some activities, and inservice support to encourage the teacher to re-orient beliefs and teaching strategies. We pondered the idea that "perhaps there should be minimum requirements for 'living' in this microworld," but came to the conclusion that if we waited for most teachers to acquire these pre-requisites we may have a long wait. Perhaps we could begin the change process by discussing beliefs and pedagogy, but we also need to "put people into experiences, where the 'life' of the experience shapes the belief."

Also connected to this part of the conversation of "knowing all the math" is the criticism given of some teachers who do profess to know all the mathematics and try to foolproof mathematics for the children. According to one member, "Kids have already failed in the present system. If we prepare teachers to know all the math, to foolproof math for kids, how is that going to be a genuine experience for the kids? Kids know things in action. Kids and teachers need to be part of the conversation." The conclusion here is that teachers do not need to know all the mathematics, but do need to have a particular orientation to mathematics and to the learner, be able to engage appropriately in a pedagogical relationship, and be able to initiate and nurture a mathematical conversation.

Drill and Practice

We discussed at length the notion of drill and practice or rote learning. Some people could not rationalize the use of such practice within the microworld we had created. However, as the conversation evolved, it became apparent that "drill within a microworld can be purposeful. Potters need to practice their craft." One participant told the story of her young child practicing putting on and taking off her shoes, refusing any help in the process. This child was exercising autonomy in that she was initiating the activity and working in a self-directed, independent manner. On subsequent occasions, however, this same child, for exactly the same activity, required assistance or support. This would therefore lead us to believe that an autonomous activity is only autonomous if it is perceived as such by the agent within it. With all the best intents and purposes, a teacher may intend to create an autonomous space, but it is the child who determines whether or not that space is autonomous for him or her at that point in time. Autonomy is thus a state determined by the agent with respect to time and space, considering social, emotional, physical, and intellectual factors—that individual's total history comes into play with each perceived autonomous situation.

We closed day 2 by becoming concerned about the terms autonomy and authority ["Are autonomy and authority useful ideas within the pedagogical relationship. I'm worried in my province. What does autonomy mean?"] and by becoming confused by the boundaries of meaning of the terms. Many of us were using the terms interchangeably. We also spoke of the "dialectic between the ideas--the authority on the part of the teacher in initiating an activity provides the occasion for the child to contemplate autonomous activity." The environment or context (e.g., the potter's microworld) can provide the occasion for autonomy or authority. "We can write ourselves into the conversation." "The microworld (and the mathematical conversation) is seen as living, growing, dynamic."

DAY 3, SCENE 3

We reviewed day 2 by reading some notable comments generated by the participants. In response to the question "What does autonomy mean?" and to the noticed confusion around the meaning of the terms we began today with an attempt to "define the terms," something we had systematically avoided doing until now because we wanted the terms to have meaning on a personal level based on a common experience. The initial experience and resulting conversation have both helped to expand and give context to the terms autonomy and authority.

Defining the terms

Autonomy: Barnhart (1988, pp. 66-67) (and with a lot of help from our Greek expert Tasoula Berggren, a working group participant) states that autonomy means independence or freedom (as of will, one's actions, etc.). This word has had an interesting etymological history. It appears to have come to us by way of earlier French (autonomie, 1596) and from Greek (autonomia, from autonomos, living under one's own laws or being independent. Auto—self, and nomos—custom or law; thus, it literally means a law unto oneself.) Brown (1993, p. 153) outlines autonomy as meaning freedom of will, independent, freedom from external control or influence; personal liberty. In common parlance the word autonomy has come to mean the right of the individual or group to be self-governing and has thus become associated with a political right.

Piagetian autonomy, according to Kamii (1994) refers to the ability of a group to be self-governing—in the moral as well as the intellectual realm. "Autonomy is the ability to think for oneself and to decide between right and wrong in the moral realm and between truth and untruth in the intellectual realm, by taking all relevant factors into account, independently of rewards or punishments. The opposite of autonomy, in the Piagetian sense is heteronomy. Heteronomous people are governed by someone else because they are unable to think for themselves" (Kamii, 1994, p. 673).

Within autonomy can be considered the right of the individual to have the opportunity to judge mathematical activity as being authentic to them (Pirie & Kieren, 1992). Also within autonomy can be considered the right of the individual to have opportunity to make decisions regarding personal well-being, behavior and values, and academic achievement. People (students) should be able to make choices about academic issues and social and behavioral issues (Kohn, 1993).

How can an individual exert autonomy within a learning community or how can an appropriate space be provided for individuals, in relation to other individuals, to develop autonomy within the community?

Authority: Barnhart (1988, p. 66) states that authority comes from author which means literally one who causes to grow, hence the founder, author, backer, from augere meaning to increase, the 'th' coming from the Greek. The word author first appeared in English in the early 1500s.

Brown (1993, p. 151) also refers to authority as originating in the word author which means (1) a person who originates, invents, gives rise to, or causes something (not only an immaterial thing, but a condition or an event; (2) a father, an ancestor; (3) the writer of a book, essay, article, etc., a person who writes books, the editor of a journal; (4) a person on whose authority a statement is made, an informant. Authority has come to mean (Bernhart, 1988, p. 66) (1) the power or right to enforce obedience, moral or legal supremacy; the right to command or give a final decision; (2) derived or delegated power or authorization; (3) those in power or control; the governing body; a body exercising power in a particular sphere. Brown (1993, p. 151) also states that authority means the power to influence action, opinion, belief; power to inspire belief; power over the opinion of others, authoritative opinion, intellectual influence; the power to influence the conduct and actions of others; personal and practical influence, commanding manner, etc. A person whose opinion or testimony is to be accepted; an expert in any subject.

We had some of the above definitions on an overhead projector for the purpose of initiating discussion around them. Our concern was more focused on 'whose autonomy is it?' (i.e., who decides whether or not the action is autonomous) than 'what is autonomy?' If the teacher designs and initiates an activity, the teacher can be said to be making an autonomous decision as that teacher has the perfect right to make decisions regarding the well-being of others. Within this design should be the element of autonomous intention on the part of the student. In other words, the teacher is striving to design an activity, the purpose of which is to provide an occasion for autonomous activity on the part of the student. The question is how will this teacher's autonomous act be perceived and interpreted by the student. A teacher could also be acting in an autonomous manner by designing an activity, the purpose of which is not to provide an occasion for autonomous action on the part of the student. This kind of activity, as we have already discussed, is considered prescriptive and closed and possibly of little educational value. The student could hardly be described as exhibiting autonomous behavior if his or her choices are mostly determined by the constraints of the activity and directions of the teacher.

We proposed a rubric of autonomy and authority, as follows:

Design	Conduct	Outcomes	
*	*	*	Teacher Intentions
*	*	*	Learner Perceptions

The top left triangle of each box represents autonomy and the bottom right triangle represents authority. As the above chart demonstrates, a teacher can intend the activity design to provide an occasion for autonomous action on the part of the student, and can also create an interactive climate during the conduct of the activity, again with the intention of occasioning autonomous behavior, but the student may not perceive the situation to be autonomous. This student may not be able to engage in such autonomous behavior at this time or in this space—there may be social, emotional, physical, or intellectual inhibitors to autonomous action. The student and not the teacher thus determines action within the 'space' provided.

The conversation on day 3 began with animated debate around the meaning of authority, focusing initially on different types of authority. Some quotes from our members follow:

- "A teacher may have to exert authority in the behavioral/managerial sense. In an intellectual sense there might be a standard truth and kids may not be getting it. We may intervene intellectually in a way that will lead kids astray (and curtail the mathematical activity). Maybe we should stay out of it. The materials have the power to re-focus/re-direct."
- "There are three areas where authority can be manifested--three areas of decision-making. Authority can be seen in the design and provision of the learning activity--how much authority is exerted from the teacher and from the child? It can be seen in the doing phase, in the use of space, time, and

material--the balance may differ. It can also be seen in the learning outcomes. We can be very controlling and very authoritative and fully direct each area or we can differentiate that authority among the design, the conduct, or the outcomes."

- "The degree of authority in the design, conduct, and product can vary within any activity, can follow a developmental path of lessening authority, or can fluctuate at any time, with any child, during any activity, at any time of the school year."

- "Authority and autonomy are not constants that behave in any ordered predictable manner."

From the few quotes above it has become clear to many of us that what some of us consider an autonomous act (e.g., the choice of or design of an activity) may be considered by others to be an act of authority. We agreed that if the intention was for the teacher to create an opportunity for the student to act in an autonomous manner, then the action on the part of the teacher could be considered autonomous, regardless of how that action was perceived and interpreted by the child. The 'curriculum,' planned by the teacher, may be quite different from that lived by the child or as one member said, "In directing their own learning and defining their own learning, the students' curriculum comes to the surface. If we want autonomy and authority to be located in the learner then we need to respect their autonomous acts." As children engage in an activity and make decisions about their learning the 'curriculum' as lived by the students becomes observable and we can thus interpret whether they have demonstrated autonomous activity.

The conversation took a diversion into assessment/evaluation and the notion of children working in groups and getting assessed for this group work. In such a situation, a student "may give up autonomy and lose out to the group." Other children may dominate the group with their autonomous actions, thus being perceived by the group as authoritative. If a group can function as a learning community, where the roles of community life are lived, then the group can develop a group sense of autonomy.

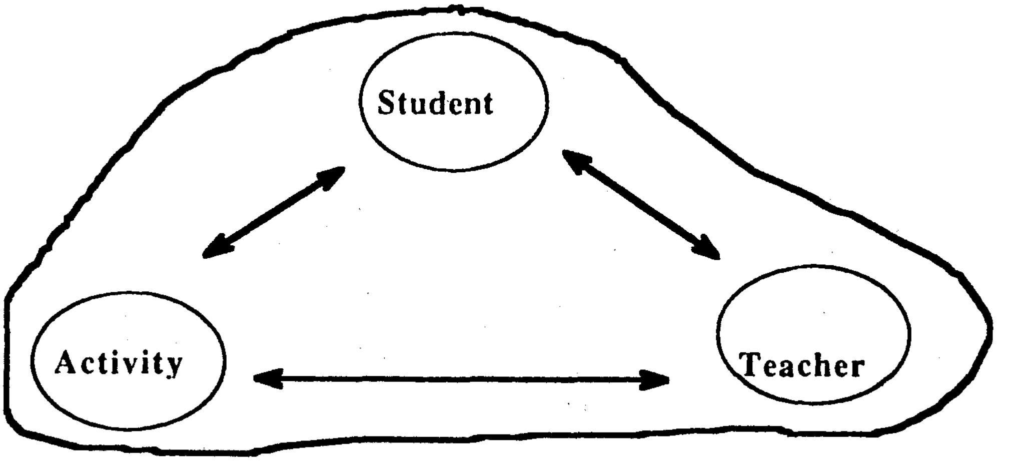
Our conversation then took an interesting twist into the world of metaphor. One member offered the following metaphor (slightly paraphrased): "If authority was an island, what would the ocean be? An island is bounded, framed, cut off from, isolated, unconnected, constrained, and the ocean is free, flowing, dynamic, connected and connecting." This metaphor was challenged by someone else: "I see the island as dynamic as the vegetation changes with the weather, the tides flow in and out, leaving different parts of the island exposed. The ocean can become more closed as coral reefs that support life die, thus eliminating life in them, which destroys other life dependent on the reef for its existence."

The above metaphorical lenses through which to view autonomy and authority can be said to be in conflict. How can autonomy be both the island and the ocean? A closer investigation of the examples of the island and the ocean reveal some interesting characteristics as each relates to autonomy or authority. It would appear that the island or the ocean can be associated with both autonomy and authority as autonomy can be bounded or structured and constrained while simultaneously it can be freeing, dynamic, and connected—it's all in the way you see it and live it. The important element in both the island and the ocean is that each requires an order, a structure in which to exist or as one member says, "the interpretation of the metaphor oscillates and boundaries become fuzzy and somewhat indistinguishable. If we take away the structure of the microworld, whether it be the rain that gives life to the vegetation, or the reef that gives life to plants and animals, then we take away the opportunity to live—or learn. We need structure."

Within the microworld, which we as a group created, or within another mathematical microworld, what does it mean to be a functioning member? What does it mean to contribute adequately to the environment of that microworld? If the action of the member is observed as appropriate then "adequate conduct" (Maturana & Varela, 1992) can be said to have taken place. If the rain stops and all vegetation dies on the island, then the island may cease to exist and will be considered to not be engaged in appropriate action or adequate conduct. The island is a living, dynamic system, dependent on many factors for its sustainability. Take away a crucial element in the island's ecology and that island may cease to function adequately. The microworld of the potter (or any mathematical microworld) is also a living, dynamic system, dependent on many factors for its sustainability. Inadequate conduct within the mathematical microworld may occasion it to engage in inappropriate action. What teachers and children need to negotiate is the terms of that conduct which would constitute adequacy. Once this is established and agreed upon, then appropriate action (= learning) would most likely occur.

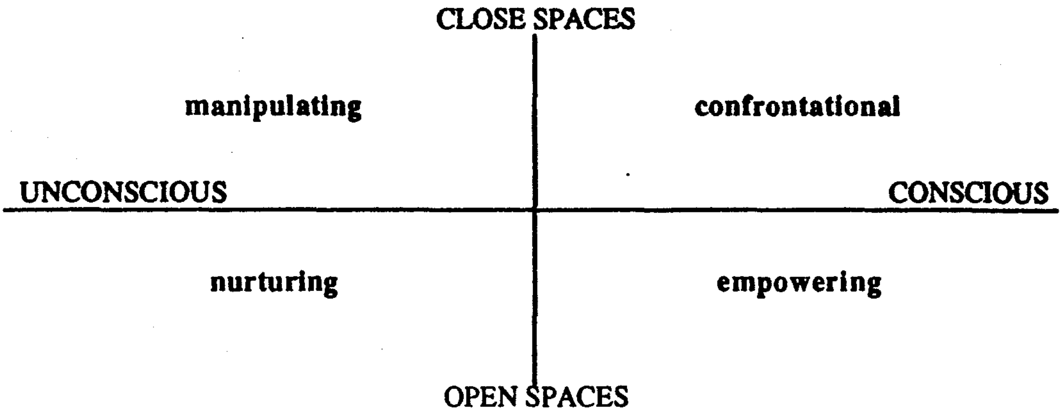
Models of Autonomy and Authority

The first model we proposed was a triad arrangement illustrating the student, teacher, and activity in interaction, within a medium (offered by the structure of the activity). It was depicted as follows:

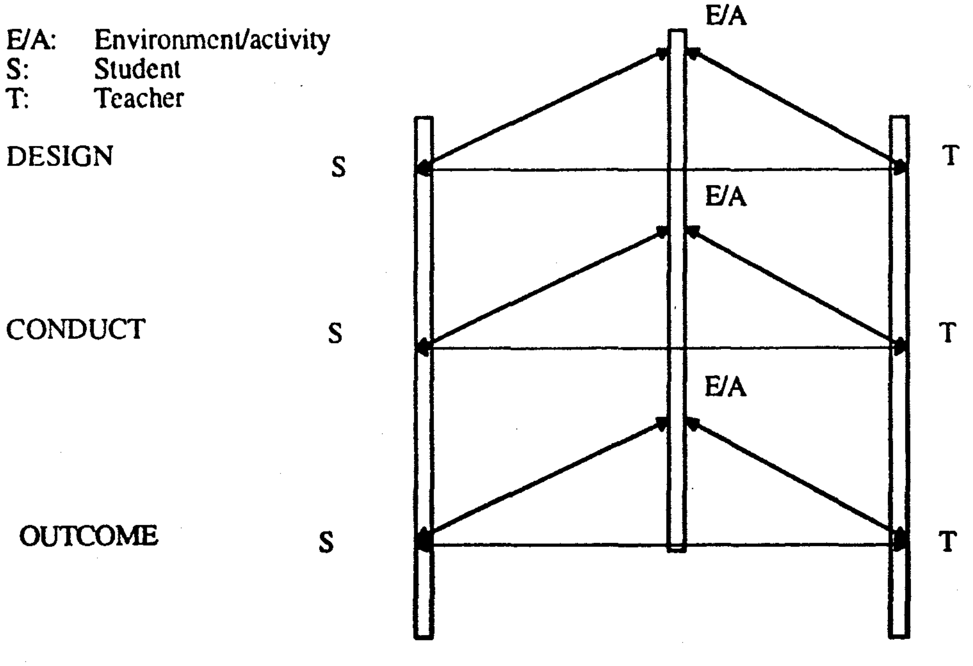


The structure of the above model is seen as fluid, with boundaries, but not necessarily fixed and unchangeable ones. The arrows are different lengths to denote that the relationship at any time between any two members of the triad is also fluid. The idea to be focused on is the constant state of being or living in a relationship to something or someone else. As one member put it, "We at best can only open or close spaces for children to 'live' in. If we open spaces in a conscious way then we are empowering children. If we can do that then perhaps the children can become autonomous learners. The question is, 'how do we best help children 'live' in that space?" This same person then proposed another model which helped all of us better understand the open or closed nature of learning. The following model is based on the work of Carl Tomm from his clinical practice in Calgary.

The most important message to be taken from this model is that if teachers **consciously** plan to create an **open** space to facilitate autonomous action, then an **empowering** situation for the child may result. Many lesson plans are written in a conscious, but confrontational manner, allowing very little choice, thus closing the opportunity to learn, and creating a prescriptive situation where the child follows explicit directions or tries to guess what's in the teacher's head.



The third model we proposed was the original triad set in a 3-dimensional manner, as follows:



The above model can be used at the different stages of design, conduct, and outcome to examine the teaching/learning situation and the degree of autonomy and authority demonstrated. The teacher is the major agent in the creation of an open or closed space, but the activity and environment also play a large part. The student, in interaction with the environment and/or the teacher, needs to determine, if for him that space is open or closed. "How the student reacts to the task and becomes engaged in it will direct the degree to which autonomy and authority are fostered."

The above metaphorical lenses for describing autonomy and authority and the above models illustrating the dynamics of the relationships which evoke autonomy and authority are valid in their use. The lenses and the models become our tools for examining and interpreting the concepts of autonomy and authority.

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Working Group B:

EXPANDING THE CONVERSATION: TRYING TO TALK ABOUT WHAT OUR THEORIES DON'T TALK ABOUT

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Our intentions in this working group were to engage with a range of recent commentaries and criticisms of mathematics education in ways that might influence our perceptions of our own teaching and research practices. In particular we were concerned with those approaches to academic discourse that are stretching our senses of what discourse in the field of mathematics education should be.

We were delighted to find that our colleagues shared our desire that our discussions would be founded on attitudes of openness and attentiveness, that they be about each of us and how the written commentaries and criticisms might speak to each of us, rather than attempting to defend our current beliefs and practices from them. Similarly, in providing opportunities to notice and generate a diversity of opinions we were not aiming to achieve a reconciliation of those differences, but rather to embrace the postmodern recognition "that the texts [we] produce as well as those [we] consume will always contain contradictions and ambiguities and violate 'correct speech' in some way" (Cherryholmes, 1994, p. 201). In brief, we were questing for an awareness of our own weak links or pressure points, the blindingly obvious but somehow still hidden elements of our current understandings. Our goal was meaningful participation within a postmodern discourse, rather than analysis of postmodern discourse from a modern academic stance.

Throughout the meetings of the working group, the ambiguity inherent in the task announced in the title asserted itself. We could not speak about the unspoken any more than we could step outside our own prejudices to take an objective look back upon our stances. The title was only a starting point for the goal the group developed, to create a space where we could listen to one another carefully, each endeavoring to catch glimpses of the unformulated biases which are always inherent in what we say and do. This is the invitation in the ambiguity of our title: although we cannot say what we have not yet seen, as we express our formulated understandings, we can perceive aspects of our conceptions that may be founded on uninterrogated perceptions. In our saying what we already see is the possibility of seeing the edges of the hidden.

As group leaders, we employed two main strategies in our efforts to structure the conversations without predetermining their particular nature. First, we identified three key topics (one for each day of our meetings) and, second, we gathered and presented a series of "common texts" as starting places for discussion. As regards the first point, our three issues were *the nature of mathematical knowledge*, *the place of mathematics in formal education*, and *mathematics pedagogy*. Our common texts took various forms, ranging from assembled lists of quotations addressing the topic at hand to such cultural artifacts as cartoons and movie clips to an assigned homework task to various accounts of particular teaching experiences. These texts proved to be generative of meaningful interaction each day.

We both feel that the often-uncomfortable but crucial ambiguity of the interactions within the working group will not submit to direct re-presentation in spoken or written text. Thus we chose to frame our oral report during the last session of the conference within the format of a final exam. In the spirit of writing from a postmodern stance, we hope that the limits of the format themselves are generative of meaning for the reader. In this case, we hope that the appearance of multiple choice, true-false, and fill-in-the-blank exercises makes apparent the limitations of the one-zero (either-or, right-wrong) logic normally underlying these modes of presentation. Thus this test is, hopefully, both an honest reflection of the group's conversations and an invitation to other conference members to think about the same ill-defined topics by accepting traditional well-defined boundaries for concepts (and forms for presenting concepts) only as a starting point.

1. Mathematics ...
   a) = (cultural weapon) + (intellectual haven)
   b) = (human activity) + (absolute knowledge)
   c) = conversation(universal + contextual)
   d) $\subset$ school mathematics
   e) R us.

Our first day was developed around the question of the nature of mathematical knowledge, prompted by the leaders' shared (or, for the radical constructivists among us, *taken-as-shared*) conviction that our sense of what it means to know mathematics (or, for the Deweyans, Dienesians, and Brunerians among us *to do mathematics*; or, for the enactivists among us *to be mathematical* (we promise not to do this any more if you keep reading)) is inextricably intertwined with our teaching of mathematics.

The first of our common texts in this project was a series of brief quotes, drawn from diverse sources and intended to remind us of the diversity of opinion on the issue. Included among them, for example, were the following:

> Whereas a scientific generalization is readily admitted to be fallible, the truths of mathematics and logic appear to everyone to be necessary and certain. (Ayer, 1946, p. 72)
>
> The social constructivist thesis is that objective knowledge of mathematics exists in and through the social world of human actions, interactions, and rules. (Ernest, 1991, p. 83)
>
> To *do* mathematics is to do research, which means to create mathematics as a poet creates poems. (King, 1992, p. 4, original emphasis)
>
> Mathematics is not a team sport, and the idea of group problem solving is fallacious. (Saxon, 1995, p. 289)
>
> [An] intriguing aspect of mathematics that seems to distinguish it from the arts ... is the extent to which mathematicians ... collaborate on their work. (Barrow, 1992, p. 267)
>
> [We must place] what many now call "western mathematics" in its rightful position ... namely, as one of the most powerful weapons in the imposition of western culture. (Bishop, 1990, p. 51).

You might choose to quickly compare one or more of these quotes we used as common text to the choices in test question #1, above. As you come to see how easy or hard or impossible it is to slot each of the texts into a particular answer, you might have a sense of how this selection of ideas played a profound role in shaping the ensuing discussion. The diversity of opinions offered by group members exceeded even that portrayed within the common text. Josef Brody captured our members' sense that a right answer isn't the goal: "We don't know what mathematics is or what it's about, and that's a good thing. If we ever find out, we're finished."

At first, the most prominent feature of this discussion was the tendency to employ "extremes" or "absolutes" (as represented by choices *a, b,* and *c* to test question #1). Correspondingly, the most empowering element of the discussion as it continued were the values we derived by attempting to pair each extreme with its opposite or complement. To phrase it in mathematical terms, each time a conjecture *A* was suggested, *not-A* came forward in its own terms. However, the discussion did not leave these dyads in contradiction. Nor did we pursue for long a singular position on a continuum with the two conjectures polarized. Rather, we found ourselves talking around the haziness of the boundaries of mathematics, attentive to the simultaneously illuminating and obfuscating natures of each duality. For instance, one point made was that the "absolute" status of mathematical knowledge is the least absolute thing about the discipline. Conversely, to suggest that mathematics is strictly a human activity not about a physical world seemed most meaningful if we acknowledged that the subject matter has arisen only through the conversation of humanity with the biosphere which situates all human activity.

One important theme that arose in the discussion, and thus helped shape it, was the role of mathematics (as the subject of our perceptions) in shaping our perceptions (and the actions that arise from those perceptions). Choice *e* in test question #1 was sponsored by this notion: that, whatever its character, mathematics can only be considered along with considerations of who we are. This took form in the conversation's particular attentiveness to the moral status and ethical implications of our conventional privileging of mathematical knowledge. The group's participants sympathetically acknowledged, "the hegemony of mainstream epistemologies" (Code, 1991, p. 12), and that we are directly implicated in the ways mathematics affects those who have relationships with and within the discipline.

One idea seldom arose directly into a central place during group conversation, but in retrospect we both feel it might represent our group's quest to continue to problematize what math is for us. We tend to think of mathematics (or, more generally, knowledge itself) as if it were a thing (an object or set of objects). Following Artigue (this volume), the "knowledge as object" metaphor may have been a great cognitive obstacle (and, conversely, could now be a great cognitive opportunity) for our group to incorporate into our discussions of mathematics and education. When mathematics becomes a thing to us, it seems we are destined to ask where it is located, in elements of the world or in each knower or in each community of knowers. Is this the foundation of the dichotomy of knower/known, including its alternative forms of mind/body, thought/action, self/other, which are so often a central target of varied critiques of mathematics education? Perhaps this suggests a direction for extending the tendency we established in conversation to cut away action and ideas founded on belief in unambiguous definition as a hallmark of mathematics.

As mentioned, the group accepted that our lack of consensus or sense of completion in our conversations was actually a positive outcome. This continued in our discussions of a second common text on the first day, a brief clip from the movie *Stand and Deliver.* In the scene selected, we were presented with brief glimpses of the pedagogic styles and philosophies (à la Hollywood) of Jaime Escalante's teaching. In the ensuing discussion, we came to focus on the question of whether or not what was happening in his classroom can be thought of as mathematics, and opinion was split on the matter. This issue served to bridge our discussions between the first and second day of the meetings.

2. School mathematics ...
   a) = (cultural weapon) + (intellectual haven)

b) = (human activity) + (absolute knowledge)
c) = conversation(universal + contextual)
d) $\supset$ mathematics
e) R us.

There was considerable discussion over the first two days around the issue of the status of school mathematics relative to the mathematics of mathematicians. Some of our group contended that school math, as enacted in such classrooms as Jaime Escalante's, has nothing to do with mathematics. Others argued that, given the tremendous public support for his manner of teaching and the uncritical acceptance belief in the importance of mathematics in achieving one's goals in life, we might be committing a grave error to dissociate what was occurring in his classroom (and in thousands of other classrooms) from mathematics proper. To do so would be to endorse the belief that mathematics is a well-bounded, pristine discipline, characterizable as socially, racially, culturally, and gender neutral. And so, as the argument went, like it or not, school mathematics is a part of mathematics, and often the only part of mathematics some people see.

Again, we never reached any sort of consensus on this issue. At times the tension of juxtaposing research mathematics and school mathematics had the dynamics of pressing two magnets together, matching ends inward. Is it their sameness that kept us from getting them to touch? Or is the polarity about us? As we attempted to characterize school mathematics as having the complexity we had already attributed to mathematics the previous day, the same descriptors, the same tensions, the same uncertainties arose.

This was a dynamic element for many of us in the group: when we recognized our eagerness to distance ourselves from school mathematics as we see it practiced, we recognized that our subsequent discussions needed to help us re-implicate ourselves with mathematics as and where it is learned.

3. We teach mathematics because ...	ANSWERS
a) mathematicians need students.	________
b) of its place in our cultural heritage.	________
c) we always have.	________
d) it is a discipline our children need.	________
e) it is one way to make sense of the world.	________
f) it's fun.	________
g) it is the great equalizer.	________
h) we are symbol-manipulating, pattern-noticing beings.	________
i) it is a better social filter than gender, class, race, etc.	________

If we had a cynical moment, it was during our exploration of the question, Why teach mathematics? Like our discussion of the nature of mathematical knowledge, this one was framed by a series of brief quotations intended to represent the diversity of opinion from persons inside and outside the field. Much of that diversity is reflected in the list of rationales included in question #3, all of which are drawn from our discussions.

An accident of design in the second day's common text prompted two related but distinct discussions. Again the quotes provided a diverse set of claims, but due to their length we ended up with two separate overhead transparencies. The group found in the first transparency a common concern for issues of what should comprise a formal mathematics curriculum, while the second transparency triggered conversation that was generally more concerned with expressing and challenging rationales for including mathematics (and/or assigning it a privileged place) in schools today.

Here are four shortened samples from the first transparency. Do you perceive the same common concern?

> We teach number sense instead of drilling. ... We don't require students to memorize facts or learn to use algorithms. (Mercer, 1992, p. 415)
>
> [Students who have] automated the skills required to use the concepts can become, with just a little, encouragement, effective problem solvers. (Saxon, 1995, p. 288)
>
> "Real-life" examples make concepts easier to grasp—from showing a child how to make change at the grocery store to plotting the arc of a football for a college calculus student. (Berger, 1993, p. 62)
>
> Counting change, measuring carpet, or balancing one's checkbook requires only the slimmest knowledge of mathematics. From early on I wondered why such pedestrian activity required so much schooling. (King, 1992, p. 278)

Perhaps it is predictable, given the significance of 'meaning' and 'understanding' to many of us interested in mathematics education, that members of our group aligned themselves readily with the first and fourth of these quotes, and had more difficulty accepting the suggestions for their practice which are implicit in the other two quotes. As our conversation addressed particular aspects of who should be learning what, and how, our shared uncertainties turned us to the deeper question of why we teach mathematics. Here, the quotes on the second overhead helped the group find and maintain a more critical edge. Here are four examples, again abridged.

> [The] use of mathematics in our society is almost always linked to an attempt by one group or individual to secure control of property. (Tate, 1994, 483)
>
> The most troubling part of my concern is my worry that the mathematics that is emerging as "mathematics for all" may be as useless in the next century as the worst of school mathematics is in this one. (Cuoco, 1995, p. 186).
>
> Because most positions for mathematicians exist in universities, mathematicians must teach. Therefore students of mathematics are required. (Noddings, 1994, p. 92, on the work of Herbert Mehrtens)
>
> [Our] assumption of one overarching purpose, namely progress, has blocked any serious discussion of alternative ends to education. The ends, or purposes, of education come, without serious reflection, from external units of the society: The state, churches, military, industry, or "the market place." (Denton, 1991, p. 25)

As the group turned to pursue ideas triggered by this collection of text, both of us perceived an aspect of the conversation that confirmed our prior hopes: Overwhelmingly, speakers interacted with those rationales that related to the place of mathematics in shaping the speaker's own identity, both in community and individual terms. In other words, the text of others was serving well to help us weave a text that was about us. Choices *b*, *e*, and *h* from test question #3 triggered only general consensus, while we explored much more thoroughly the choices that sponsored the diversity of perspective among us.

Should we structure mathematics education by considering how the discipline might define our perceptions, or be more centrally concerned with issues of utility, of empowerment, and of sorting? In grappling with this tension in global and theoretical terms we found our words were too seldom about ourselves, creating a comfort that comes of disengagement, of saying what we are not questioning. In response, we moved to a more specific discussion, inquiring about what we have done and what we might do in our own practice. The next test question's format implies disengagement from action, but the choices reflect the sense of I-will commitment the conversation suggested.

4. You have 100 energy units to devote to the improvement of your mathematics teaching over the next year. Plan their allocation.

________ a) Homework will extend the lesson's conversation.
________ b) Students will only do homework if they see its relevance.
________ c) I will give no homework.
________ d) I will not <u>state</u> what is to be learned.
________ e) Each student will help to determine what we do.
________ f) Students (and I) will understand that learning about themselves (myself) is not different from learning about mathematics.
________ g) I will listen.
________ h) Questions for which the answers are already known will not be asked.
________ i) I will value and encourage errors.
________ j) I will address conditions that affect my classroom from outside its walls.

The second day closed with an attempt we both hoped would link the claims of the day about what mathematics should be in education to the challenge of making our beliefs into teaching practice. We assigned a homework question! Do you perceive any relevance to the conversation as we have been describing it, for a question on common-sense misconceptions of limits? Neither did most members of the group, although a few enjoyed the novelty and perhaps the interlude of well-defined inquiry that the question offered. Offered with at least partial honesty, in that neither of us knew the answer and both wanted to hear some, it was nevertheless viewed by most members of the group as a parody of the irrelevant and de-contextualized story questions that pass in most math textbooks as problems to solve. Thus, as much because of than in spite of, its mismatch with the intentions of any of the group members, it served as a powerful trigger for starting the conversation on the third day. (See Peter, 1961, p. 252 if you feel any curiosity regarding the seed whose fertility was clearly due to the soil into which it was cast. Ralph is still hopeful of an explanation that even he can understand, and Brent is willing to give scores out of 10, including part marks for neatness and defining your variables, for those who so wish.)

Thus our discussions began the second day with a critical examination of the role of homework in our conceptions of school math. Some claimed that homework should not be assigned, ever, while others felt more comfortable with accepting the centrality of assigned work and marks based on productivity, and teaching well without challenging that aspect of the didactic contract (Brousseau, in Hoyles, 1988; Laborde, 1989) by improving the relevance of what is assigned to learners.

Considerations of relevance continued, as our conversation changed its form significantly. To foster a practical rather than theoretical tone for the group members' closing considerations of re-visioning their (and our) own practice, we chose to sponsor as text particular examples of practice which two of our members could offer. Tom Kieren went first, providing a vignette of grade three children participating in a "missing fraction mystery," responding to the following prompt: A fractional amount is missing. We know that it is more than one fourth and it is less than three fourths. Can you find an amount that it might be?[1] Tom spoke to the diversity of student response, emphasizing his belief that any attempt to explain their answers and strategies in terms of a causal relationship between his actions and theirs is not very

[1] For those interested, a fuller account of some of the details of this curriculur event is available in Davis, Sumara & Kieren (in press).

helpful. He also suggested that the notion of mathematics as something "out there" (i.e., independent to learners) is seriously challenged by these students' actions, since they did not appear to be converging on any sort of pre-determined understandings despite the relative narrowness of the initiating question.

Our discussion of this common text explored the teacher's place in learners' understandings. In particular, considerable discussion revolved around Tom being personally implicated in these students' understandings, just as the students were personally implicated in their own and each others' understandings. This relates to the transferrability of this occasion as curriculum available to other teachers and students, in other contexts, as well as to the popular conception of mathematics itself as being context- and personality-free. Even with the recent emphasis on embodied cognition in mathematics education, have we made an adequate account of the role of interactions between particular bodies? We might do well to adjust our focus to a broader phenomenal level: away from the classroom as a collection of individual learners and toward a conception of the classroom as unity with an integrity of its own—a body consisting of the interacting bodies, held together by an emergent common knowledge.

Lesley Lee provided our final common text with her description of an assignment she uses in undergraduate mathematics education classes. In brief, the assignment for each student begins with the selection of a non-routine problem on which the student thinks aloud while working independently and continuously for approximately 30-45 minutes. The student's words are audio-recorded and they keep reasonably well organized records of their paperwork. At the end of the time, regardless of whether the problem is solved, students select the "best" five minutes of the recording (by their own criteria for "best"). They accurately transcribe it, and append this to a report about why they chose that section, what they learned about themselves and about problem solving and learning over the process of doing the assignment.

Lesley reported that this task has proven invaluable as an occasion for students to recognize such issues as the sorts of spaces you must create for yourself for doing mathematical work, the challenges of orally narrating one's thinking to submit it to a teacher for judging, and the "presence" of the teacher throughout the activity even though the student was alone for its duration. Most importantly, Lesley suggested, learners learn about their own relationships with mathematics, often revising their sense of themselves as mathematizing beings.

The group's conversation followed from Lesley's observations in pursuing the relational dynamics of this and other tasks. In particular, the conversation credited the possibilities of each student in such an assignment addressing any of a number of issues that tend to remain uninterrogated for preservice and practicing teachers: the making of mathematics by individual learners, the relationship of the learner to the mathematical activity, the relationship of the mathematical activity to the broader goals of the educational endeavor, the relationship of the educational endeavor to the knowledge embodied and sanctified as curriculum. Thus, in arousing our thoughts to the creation of mathematical thought within a formal educational structure where the role of the teacher is significant but only determined by the actions and choices of the learner, we found ourselves closing our working group with conversation which wove together the themes of all three days, without prematurely closing any of them. We hope this report does likewise. Perhaps one last test question can help.

5. (Examiner's note: In addition to re-viewing what was (un)covered, a test should also point forward. That is the purpose of this *Take Home* component.)

   Sawada (1992) suggests that, with regard to the 5 W's (+ H) of decision making for mathematics teaching, we tend to consider things in the following order:
   WHAT is taught;
   WHEN it's taught;

HOW it's taught;
WHY it's taught;
WHO is taught.
WHERE it's taught;

PROPOSE A NEW ORDER.

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Working Group C

FACTORS AFFECTING THE TRANSITION FROM HIGH SCHOOL TO UNIVERSITY

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INTRTODUCTION

The primary focus of our discussions was the student mathematical culture at the post-secondary level: the mathematical background of students arriving from high school; their experiences in first year mathematics courses; the behaviours students display within mathematics courses; and some features of the ideal mathematics culture we desire for students.

Our secondary focus was faculty culture: the current culture as it reinforces and/or responds to aspects of the student culture; some features of an alternative faculty culture we would like to encourage; and ways of developing continuity across the transition from high school to university.

Our conclusions can be summarised in three categories:

- *Goal:* to achieve a fundamental change in the students' mathematics culture across the transition.
- *Strategy:* to work on making a fundamental change in the teaching culture of post-secondary mathematics faculty.
- *Approach:* teacher development for post secondary mathematics teachers, preservice and inservice; dialogue within institutions to support changes in pedagogy; dialogue between institutions to develop continuity and support common directions for change.

This structure will also organise this report, beginning with the student culture to refine and define this goal, then moving to the faculty culture, and then the approaches and activities needed to support the desired changes.

CURRENT STUDENT CULTURE

Any effective faculty response must begin with the students we have and not the students we wish we had. Therefore, we start with a description of the current student culture, including how institutions and the current faculty culture reinforce the current situation.

The Culture Students Bring With Them

Evidently, there is a discontinuity between what happens to students in high school and what they experience at university in terms of teachers' approaches to knowledge, teaching practices, and forms of evaluation. While these experiences differ widely and the following list is probably not exhaustive, these are attitudes and behaviours we all agreed were fairly common features of the first-year student body entering our own institutions. Students:

- wish to understand the mysteries of the universe;
- want to collect credits, maintain grades;
- believe that hard work leads to high marks;
- expect to be given the algorithms and taught the tricks;
- believe that mathematics is a collection of skills and ancient facts; not creative but imitative;
- are biding their time, waiting to get into their "real" programme;
- regard mathematics as an obstacle on their route to something else more desirable;
- have high expectations that this experience (and them) will be different;
- have a fragmented knowledge of mathematics;
- believe they will be anonymous and the professors won't care;
- expect to be more independent;
- see studies as just one more thing to do;
- seek a balance in lifestyle in the face of diversity of experience;
- are away from home for the first time;
- fraught with money worries.

The Culture of First-year Mathematics Classes as Experienced by Students

The following list is based on our own experience and on the results of several informal studies group members have conducted at their own institutions. It describes students' perceptions of their experience in first year mathematics classes:

- classes are huge and impersonal;
- knowledge continues to be fragmented;
- university is a ritual, devoid of any real meaning, but producing a grade;
- a feeling of being a nobody, alone and unconnected in a competitive environment;
- dreams are shattered ("many of you don't belong here");
- grading and evaluation are hard to predict, so they play "guess my game";
- grades are dropping;
- many suffer a battering of self-esteem;
- collect credits and maintain grades;
- "all I want is an A but you want me to understand";

- everyone is expected to learn at the same pace;
- scrambling to keep up with the pace of lectures and the coverage of material;
- more freedom and a need to assume more responsibility for their own learning;
- faculty do care but initiative for contact lies with the student;
- living in a more adult world, with accompanying high demands and expectations.

Faculty Perception of Post-secondary Student Culture

- lack of skills (independent learning, reasoning, connecting, ...);
- high expectations leading to lack of self esteem and shattered dreams;
- the tyranny of content;
- an "imitation" learning paradigm;
- students in service courses are uninterested and resentful;
- low tolerance for ambiguity and a clash between intuition and formal reasoning;
- all they want is an A but I want them to understand;
- lack of clarity as to where the responsibility for student learning lies;
- students don't try to make contact with the professor;
- tension between the intentions of students and faculty.

Characteristics of Current Post-secondary Faculty Culture

- little dialogue within institutions, even within the same course over multiple sections or multiple semesters;
- little dialogue between institutions and levels;
- no support for innovation in teaching;
- little familiarity with the use of technologies in teaching;
- the tyranny of content;
- the "imitation" mode of teacher education at the post-secondary level (we teach the way we were taught);
- other disciplines using mathematics as a filtering device;
- the culture of research/grants/papers/presentations where issues such as these are seen as intrusions on research time.

THE DIRECTION OF CHANGE

The culture we imagine developing for first-year university students is one where the classroom atmosphere invites and stimulates questions and students' active participation. In this classroom of our dreams, learning is a positive experience where students are offered a rich variety of approaches to learning through independent and collaborative group work in order to develop their visual, algorithmic, abstract, oral, and written abilities. Continuity from high school to university in such areas as approaches to teaching abstract reasoning and problem solving might best be achieved by designing a mathematics curriculum covering the full span of the years from grade 9 to the second year of university. The focus of such a curriculum, in addition to the usual content goals, would be on investigating open-ended problems and giving students a more authentic mathematical experience.

CHANGING THE FACULTY ROLE

The key to improving the student culture is to change the way faculty approach their teaching role. First, faculty must recognise that students bring a culture with them to the classroom. They need to meet students where they are and encourage and support them in their learning by means of alternative, more student-centred, pedagogical strategies (such as group work, writing) and a variety of methods of assessment (such as group exams, shorter exams, no exams, projects, portfolios).

Currently, faculty are so consumed with 'covering' content that there is often no room in a course for students to do any mathematics. The curriculum must be restructured by reducing the amount of content to be covered, and increasing the process components. There should be a renewed concentration on ideas, more work on problems, fewer pre-requisites for courses, and greater cohesiveness between courses and sections of the same course.

The mathematics we teach should be meaningful to students: this might mean adding labs and 'applications' in the broadest sense, including capturing students' imagination with novel 'applications' such as the 4-dimensional geometry of hyperspheres and hypercubes. However, it is important that this be done in an integrated, rather than an 'add-on,' manner.

Although students must be responsible for their own learning at the university level, they also need and deserve assistance from time to time. We need to build mechanisms for helping students to form study groups, to work collaboratively, and to read mathematics on their own. At the same time, there is a need to develop more classroom interaction between students and faculty, and ways of assisting students to deal with difficulties and the tragedy of shattered dreams.

Our best teachers should be teaching first year courses. Instructors of first year courses need to work together to achieve more coherence in first-year offerings: agreement needs to be reached on issues such as the aims of a course, the essential content, and the assumed background preparation of students. Faculty values need to be re-examined: for example, final examinations should reflect what is valued and deemed important in a course.

SOURCES OF CONFLICT FOR FACULTY

Many university faculty live in a culture of research/grants/ papers/presentations where issues such as these are seen as intrusions on their research time. The sabbatical is 'release from teaching' to concentrate on research. If such faculty get co-opted into meetings with students and parents, their attitudes may soften, of course, but will this necessarily improve first year teaching?

On the other hand, the 'popular teachers' can be the ones that perpetuate an erroneous notion of the nature of mathematical activity by prescribing it too much and making it too safe for students. 'Student evaluations' and 'academic integrity' can become rallying cries that do not directly address the student culture that now develops in the classrooms or the culture we desire.

We will need open debate within departments to address theses issues and clarify the need for and directions for change. Clearly, it will be critical to support individual faculty members considering making educational changes. Evidence indicates that the first experiments by faculty produce reduced evaluations and unease in the classroom, though long-term persistence can generate dramatic changes in the classroom culture and improved evaluations. In this, and in all similar projects, the active support and involvement of the departmental chair is crucial. (There is an interesting analogy here: a student's change in culture may also have its tough spots in transition, possibly including lower marks. The teacher's active support is crucial!)

FACULTY DEVELOPMENT FOR POST-SECONDARY TEACHERS

Post-secondary faculty interested in changing their teaching practice require support, training, and progressive instructional development. Several interesting and promising directions for change and support were discussed.

- Dorothy Buerk described the First Year Seminar project at Ithaca College which resulted in response to concerns about the transition. Preliminary research indicates that students are overwhelmingly positive about the experience which provides information on academic and personal resources, as well as preparation in learning and writing skills. Faculty who lead the seminars are assigned extra credit hours because of the intensive faculty-student workload.

- Faculty can be supported to attend conferences on issues like calculus reform or collaborative learning and then report back to colleagues in their department. Direct financial support for pedagogical development helps give it a similar importance to 'research'.

- One way to ensure a change in teaching at the post-secondary level is to provide graduate students with experience and training in teaching. Examples of existing programmes were described by Ken Millett, Pat Rogers and Walter Whiteley. Anecdotal evidence of the impact of such programmes is encouraging: in many departments, students with documented experience in teaching are the ones getting the jobs. Examples were given of graduate students with excellent academic records but no teaching experience who have found themselves unprepared for a job market where hiring committees increasingly ask more probing questions around teaching issues (for example, "What is your teaching philosophy?"). Potential candidates are now being asked to present a short teaching seminar on a typical undergraduate topic as part of the interview process.

- Given the considerable anecdotal evidence that future teachers imitate their former teachers, it is essential that faculty present an approach to teaching which is worth imitating. Cross-appointments between the mathematics department and the mathematics component of the faculty of education are also helpful and bring expertise in teacher development into the department.

DIALOGUES FOR FACULTY CHANGE

To assist students in adapting to the university classroom culture in the meantime, we should develop ways to bring teachers from the different levels together to discuss curricular issues and to enable teachers in one milieu to experience the student situation in another. Several examples that work were described:

- Harvey Gerber (Simon Fraser University) has organised evening meetings, held once a semester, with local high school mathematics heads to discuss matters of mutual concern. He is assisted by the university liaison office who also send a representative to the discussions. The agenda for each meeting is developed collaboratively. Rather than a 'university telling the schools what to do' approach, the idea is that we can all serve our students better if we discuss and share our difficulties. For example, a tour of university classrooms and laboratory facilities gave teachers a better understanding of the obstacles we face in the university setting.

- Ken Millett discussed a joint project he has developed over the past fours years with Bill Jacobs at Santa Barbara. In cooperation with local schools, community colleges, and other universities, the project pairs undergraduate students who are interested in a career in mathematics teaching with high school teachers in the summer for an 8 week classroom experience. The project receives $400,000 in NSF funding and both students and teachers receive payment for participating. The project has had the effect of bringing the partners together, improving communication and easing the transition. It has sustained because there is work being done that is important to both institutions.

- There was a brief discussion of the meetings organised by Steve Halperin (Toronto) between university and high school teachers aimed at developing a common approach to Ontario high school mathematics. Unfortunately, this initiative disappeared without explanation, when the Ministry showed little interest and key organisers moved on to other activities.

- Ideas developed collaboratively have a better chance of being viewed as useful and practical by teachers. This was contrasted with the general ineffectiveness of school visits by university professors. Recent moves by the CMS to become more involved in mathematics education issues and to involve all partners in education were viewed as welcome and encouraging.

- Students are critical agents in transmitting the culture from the university to the high school, and they could be mobilised to visit high schools telling students what to expect and providing peer-support to incoming students at the university.

- Other ideas discussed were: organising a "Day of Reflection" for high-school teachers, faculty and students to learn from each other about specific issues like calculus reform or lessons learned from the transition process from grade eight to grade nine; the role of math contests in providing another opportunity to bring people together.

BEYOND DIALOGUE

George Gadanides urged that, while invitations to university faculty to attend regular high school department heads' meetings in the local area are easy to generate, these discussions must move beyond providing a forum for discussion to action for implementing change. Once we have defined the issues, if we value certain changes we have to put energy into implementing them, expressing our ownership of this issue and our responsibility for action. For example, such meetings could examine student misconceptions, and hence treat them early, especially those that have proved to be persistent teaching issues. Teachers could visit university classrooms and vice versa. Perhaps the CMS could explore what department chairs might be willing to do to help facilitate change. Basically, it comes down to the need for structured forums for discussion and implementation at the high school/ university interface. A variety of approaches are likely to work best, from top-down through the CMS or CMESG, to bottom-up approaches like individual postings on electronic bulletin boards and local initiatives like Harvey Gerber's described above.

Since the study group met, Kathy Heinrich has written to all chairs of departments across Canada asking them about their interactions with high schools. She received only three replies:

- Queen's University (problem sessions and a one week summer enrichment programme for students, speakers at teachers' professional development conferences);
- University of New Brunswick (math contests, participation on Provincial Department of Education committees concerned with province-wide testing, speakers at teachers' professional development conferences);
- University of Alberta (week long conference on calculus which is also attracting teachers).

Several additional proposals for following through on the recommendations in this report were offered:

- Short articles, based on this report, should be written for the CMS Notes and other provincial journals.
- Workshops in 'transition' issues should be offered at provincial and national teachers' meetings.
- Models should be collected and disseminated of successful first-year programme developments, thereby creating a 'culture of possibilities'.

BEING EFFECTIVE

In conclusion, Ken Millett emphasized the importance of taking lots of small steps that work and can carry the process to a larger result, rather than attempting it head on. It is important in all our efforts, if we truly want change, to assess where are the opportunities, the places of leverage, the sources of influence, money and control, and to move the system indirectly. It is also important to consider the long term: develop strategies that can be sustained, where people will support one another in the medium term and will involve additional people over time.

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Working Group D

GEOMETRIC PROOFS AND KNOWLEDGE WITHOUT AXIOMS

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PRE-CURSOR

We had already started work before we met, by declaring a focus via the following pre-announced questions:

1. What is geometric intuition? How does it affect and effect our understanding?

2. What role does our imagination have in increasing our knowledge and understanding?

3. Can we imagine infinity? Does it matter?

4. How can we be rigorous without axioms?

5. Are precise definitions always desirable in geometry? Are they always possible?

6. How do we geometrically understand the real numbers?

7. Why is spherical geometry (the geometry of our planet) missing from almost all textbooks? Why is two-dimensional geometry apparently easier and three-dimensional harder, when we live in three dimensions?

8. What use are geometric axioms? What power do they give us?

9. For what are geometric models needed? What about pictures and diagrams?

10. Is knowledge gained without axioms inferior to knowledge from formal axiom systems?

11. How can we have proofs in geometry without axioms? Do we want to?

12. When are we satisfied with our understanding? When are we certain?

We never addressed any of these questions directly in the group, but by the end of our nine hours together, we found we had said something about almost all of them.

INTRODUCTION

Blessed Cecilia, appear in visions
To all musicians, appear and inspire:
Translated Daughter, come down and startle
Composing mortals with immortal fire.
(*Hymn to St Cecilia* by W. H. Auden, 1942)

Saint Cecilia is the patron saint of musicians (do we not need one for mathematicians?), and is being invoked to 'appear in visions.' The Greek word *theorema*, whence the word 'theorem,' also means 'vision.' So theorems in mathematics, we are etymologically invited to think, result from seeings—literally, visions. And the sense of sight is one of the core metaphors in English for 'understanding' (e.g., I *see* what you mean)—the other being touch (e.g., *grasping* what someone means) from which the powerful metaphor of mathematical *manipulation* derives (for more on this, see Pimm, 1995).

But there are other key words in this short verse. The next is 'translated' which evokes translation across languages as well as (parallel) mathematical shifts, which also links to Dick Tahta's reformulation of a Taoist saying, "The geometry that can be told is not the true geometry, but algebra." And Caleb Gattegno often observed how geometry is inherently unstable: it keeps getting turned into algebra.

Although 'composing' too, has particular resonance in a mathematical group such as this, it is finally 'immortal fire' that we wish to draw attention to. Firstly, this image speaks of the passion and the warmth— the life—rather than the disembodied cold—the devoid of life—that *inspiration* (breathing in) can bring.

> For Lacan, mathematics is not disembodied knowledge. It is constantly in touch with its roots in the unconscious. [...] But theories [of psychology] that use mathematical formulation are seen as 'cold,' 'impersonal.' Definitionally, something that is cold leaves out the warmth of the body. (Turkle, 1981, p. 247)

Secondly, it points to the investments of permanence, of immortality, of *not* changing, in those 'truths that never change' (also Auden, *Hymn to St Cecilia*) that some people place/find in mathematics, and in geometry in particular.

PART A: CIRCLES

> Do not just pay attention to the words; instead pay attention to meanings behind the words. But, do not just pay attention to meanings behind the words; instead pay attention to your deep experience of those meanings. (Tenzin Gyatso, The Fourteenth *Dalai Lama*)

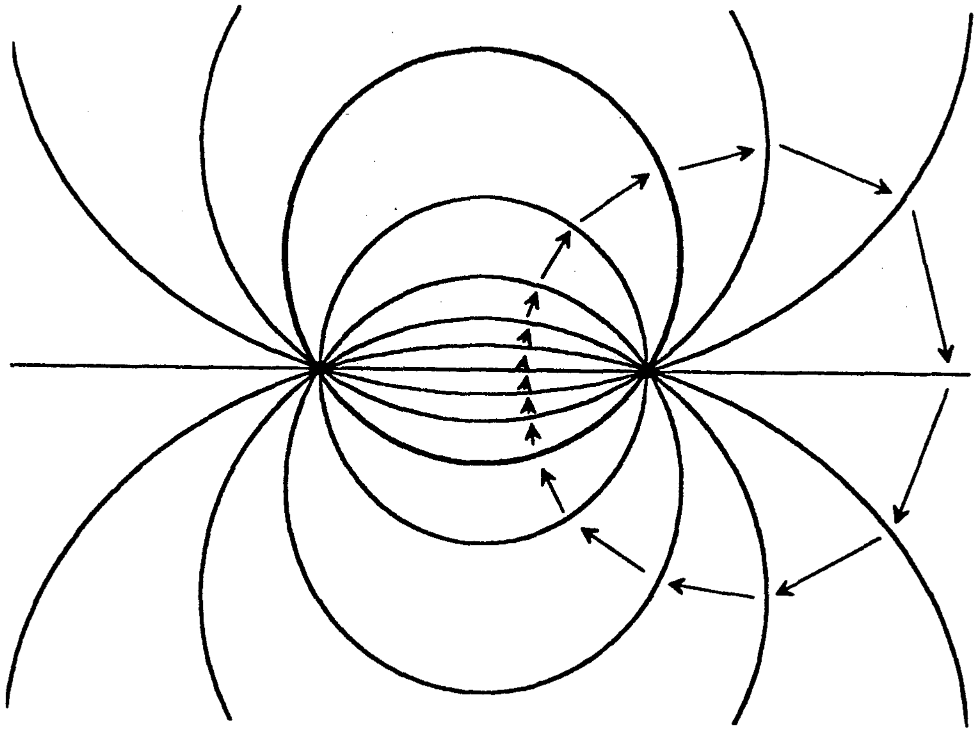
A short (3 min.), silent Nicolet-Gattegno film was shown with the intention of providing an introductory focal experience for encountering obliquely some of the tensions in these pre-announced questions rather than attempting to tackle them head on. The film and our varied responses to it were discussed at length.

It will be hard to give a feeling for this discussion unless the reader has some images in mind. The means available in print to do this are limited, but provide a useful illustration of precisely some of the issues that were under discussion. As you read, pay attention to any images that form, how the words interact with them, and how you work on them.

We cannot provide you with direct experience of this film. We can only offer you words—which you might use to generate your own images—or a sequence of drawn 'stills' (perhaps of perceived key transitions) that suggest the motion, the chronological sequence of frames. Words are quite different from images.

Here, then, is a first 'telling' of the early part of the film.

Imagine a stationary point and a moving circle (which also changes its size). Imagine, at first, that the circle floats freely in the viewing window and then it attaches to the point and rotates around the point while increasing and decreasing in size. Next, a second stationary point appears and the circle attaches itself to both points and then (staying attached to both points) the circle grows larger and larger until it escapes the confines of the viewing window and then appears in the viewing window as the straight line determined by the two stationary points. Imagine that motion of the circle continues, until the circle re-enters the viewing window from the other side. Then imagine other circles joining in and playing with each other.

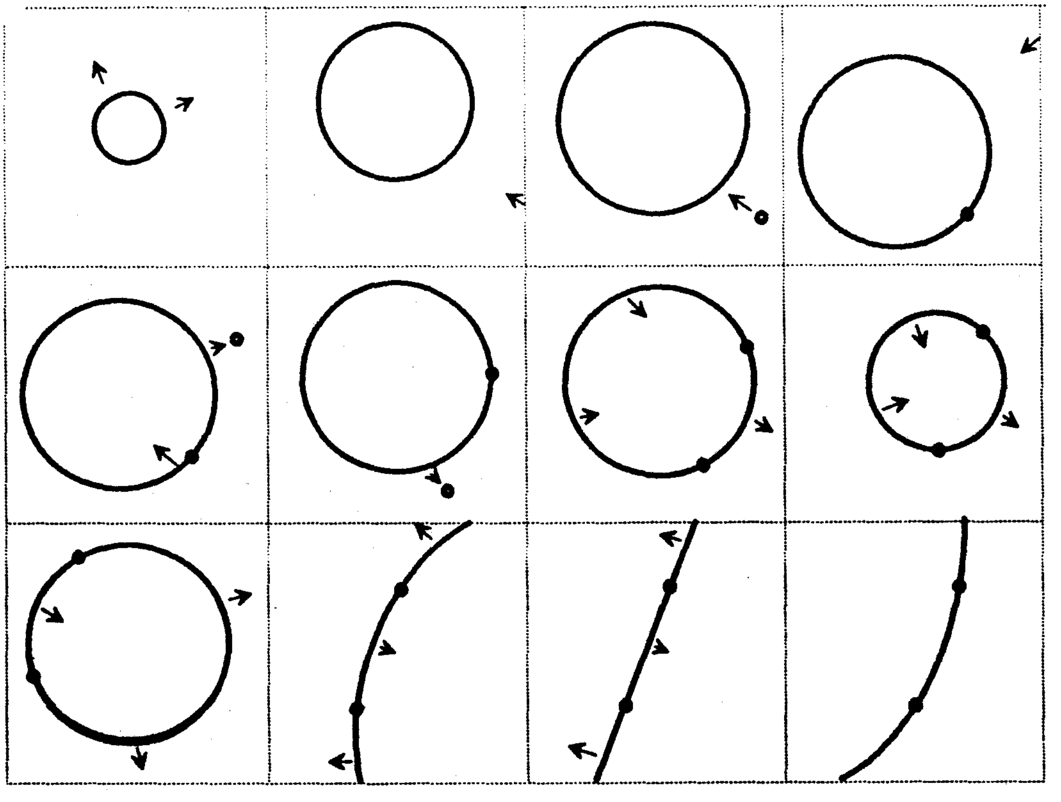


Notice that this image did not appear in the film: it is a composite, attempting to indicate the motion through superimposition, resulting in a false image (in that it is not any particular image from the sequence) that nonetheless conveys some sense of the whole.

But also bear in mind the Taoist saying referred to above: in telling one story of part of the film, we may have turned the geometry inherent in the images into algebra.

Here is a second partial 'telling'.

> A red circle appears and moves around the screen, growing and shrinking in a smooth and continuous way. A point appears and eventually the circle passes through the point and is 'captured' by it; and from then on, it passes through that point. A second point appears and 'captures' the circle from the first point, then both points capture the circle, and finally the circle gets larger and larger.



The above stories are only that: a familiar mixture of description and interpretation. We could instead have given a structuring overview: for instance, this film is about circles and constraints, first none, then one, then two.

Of course, other accounts are possible in response to the same short sequence of images. For instance, is it the 'camera' that is moving and 'actually' the circle that is remaining the same? Is it the circle that captures the point, or *vice versa*? More subtly, was it one circle changing size and position, or was it a whole sequence of different circles being illuminated in turn, one per frame, which brings us back to another key feature of geometry, namely continuity.

In a group discussion of this film at a different conference, someone remarked, "I never actually *saw* a family of circles superimposed at once on the screen. It was only over time—things stayed in my head." Someone else asked, "Why did we read this film as mathematical?". A third commented, "Moving images have multiplicity. With words, it is hard to attend to what is *not* being said, whereas the unsaid, the undemonstrated, can be present and functioning with images."

With a circle that is moving continuously around the screen shrinking and growing, for example, there are at least two different ways of seeing what is going on: one circle moving and changing size and position, or the picking out of a (large) selection from an infinite set of all possible circles. Random (rather than continuous) illumination of possibilities can help to suggest the latter perception. The above accounts are very brief, yet they can be used to generate complex sequences of images—a word here being worth a thousand images.

The time sequence allows the film to be read as a story, and the continuity of image (the close proximity of circle size and position in adjacent frames) invites a 'reading' (seeing?) of a single circle (one *object*) changing, rather than a succession of distinct circles being attended to in turn. For that to be initially the more salient, a sense of discontinuity (jerky, flashing movement) may be required. The plane can be viewed as being empty until a circle is placed there, or it can be viewed as made up of all possible circles permanently present, from which one can be singled out and brought to our attention. Curiously, words can invoke continuity directly, whereas the discrete nature of film images can only invite it.

In the working group, points discussed and expressions used (which reflected a shift in how the film's images were being construed) included:

'I believe I saw ...;'

'I thought I knew ...;'

'... seeing things that aren't there.'

'... the appearance of movement;'

'... infinity is off the screen, but not that far away;'

The stirring discussion ranged over birth, rebirth ('get rid of all form to get back inside') and reincarnation ('circles re-embracing on the other side') as well as how a film could evoke a sense of mystery. The film offered a sense of inside becoming outside ('circles like arms flung wide in an embrace'), a switch of field and ground, negative space, and the mystery of how to transform something into its opposite. There was the implied continuity and presumed regularity of images once off the screen: We think we know what is happening off-screen, even though the film could be a report from a regular part of a highly non-uniform surface.

The playful, rhythmic nature of the film, very smooth and continuous, gentle even if fast, was much commented on. The fact that there is no parametrisation of the circle, it is a single seamless object, with no points on it (until placed there, as is necessary with software such as *Cabri-Géomètre* and *Geometer's Sketchpad*) excited interest.

The final ideas explored concerned freedom and constraint, movement and change: The eternal could be seen as unchanging, fixed, or changing in a regular, ever-repeating cyclic way. It was possible to identify *with* the circle (as with a character in a play or novel) as well as seeing it 'merely' from the camera's apparent point of view 'outside' it. But the more determinedly this is done, the more pedagogically desirable it might be to insist that the film portrays circles, not one but an infinite class. As was commented on above, the linear flow of time invites an (algebraic?) story narrative: yet the film provides both melody and harmony (the richness of what was happening at any given moment and how that balance changed and developed over time).

The film offered, among other things, an experience of infinity.

PART B: STRAIGHTNESS

> A straight line is a line which lies evenly with the points on itself. (Euclid, *Elements*, Definition 4)

> Wisdom will save you from the ways of wicked men, from men whose words are perverse, who leave the straight paths to walk in the dark ways, [...] whose paths are crooked and who are devious in their ways. (The Holy Bible, *Proverbs* 2:12-15)

> Verily, this is My Way,
> Leading straight, follow it:
> Follow not other paths:
> They will scatter you about
> From His great path:
> (Holy Qur-An, *Sura VI*, verse 153)

The geometric line is an invisible thing. It is the track made by the moving point: that is, its product. It is created by movement -- specifically through the destruction of the intense, self-contained repose of the point. Here the leap out of the static into the dynamic occurs. [...] When a force coming from without moves the point in any direction [...], the initial direction remains unchanged and the line has the tendency to run in a straight course to infinity. This is the *straight line* whose tension *"represents the most concise form of the potentiality for endless movement"* (Kandinsky, 1979, p. 57).

Take a piece of paper and draw a straight line on it. Now bend the paper (without stretching it), so that the line is no longer straight in three-space. We express this situation by saying that the line is extrinsically not straight. But is the line intrinsically straight on the sheet of paper? That is, is it straight if you consider the paper to be the universe? Is the line then straight in that universe? Or, consider a two-dimensional bug which crawls on the surface of the paper such that the bug has no awareness of any space off the surface of the paper and is not influenced by gravity (because with gravity operating, vertical is very different from horizontal). Will the bug experience the line on the paper as straight?

> Argue that distances (as measured along the surface of the bent paper) and angles have not changed and thus that the bent paper surface will intrinsically have the same geometric properties as a flat piece of paper.

> Argue that the bug would experience the line on the paper as straight.

> Argue that the bug would experience the line as having the same (local and intrinsic) symmetries as straight lines on (in) the plane.

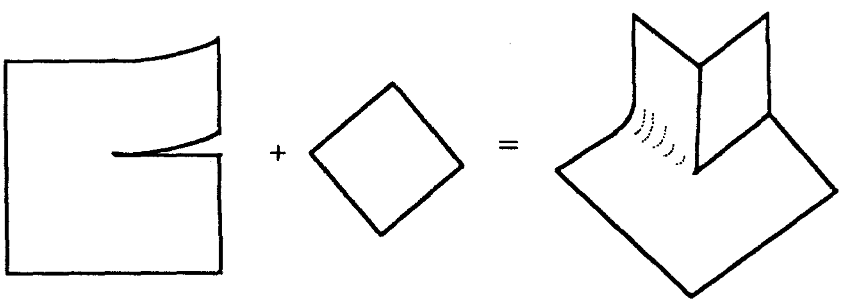
The important thing to remember here is to think in terms of the surface itself, not the surface *in* three-space. Always try to imagine how things would look from the bug's point of view. A good example of how this type of thinking works is to look at an insect called a water strider. The water strider walks on the surface of a pond and has a very two-dimensional perception of the world around it.

To the water strider, there is no up or down; its whole world consists of the plane of the water. The water strider is very sensitive to motion and vibration on the water's surface, but it can be approached from above or below without its knowledge. If you find a pond with water striders, by moving slowly (so as not to disturb the surface of the water with air currents), you can actually touch the water strider with your finger. Hungry birds and fish can also take advantage of its two-dimensional perception.

This is the type of thinking needed to visualise the intrinsic geometric properties of any surface adequately. To an outside observer looking at the bent paper in three-space, the line drawn on the paper is curved—that is, it exhibits *extrinsic* curvature. But relative to the surface (intrinsically), the line has no *intrinsic* curvature and thus is 'straight.'

Be sure to understand this difference. Our experience of our physical universe is intrinsic (for instance, we know how to avoid furniture when walking across the room)—we have no extrinsic experience to draw on, other than through metaphoric extension of how we think about (and in) two dimensions. For more on such an approach to geometry, see Henderson (1996).

Now, take your sheet of paper and make a cut from one side to the centre of the sheet and tape onto the two sides of this cut the 90° corner of another sheet of paper. You could call the resultant surface a 450° cone, because of the number of degrees around the centre (cone) point. Think of the plane as a 360° cone.



What would a bug experience as straight on this surface?

Is 'not-turning' the same as 'shortest'?

Can every pair of points be joined:
- by a straight path?
- by a shortest path?
- by a not-turning path?

Discussion From the Group

1. Straightness involves being indifferent to left or right. It has also to do with a sense of being blinkered and only being able to look ahead (following the 'straight and narrow', in common parlance). Jocasta, in *Oedipus Rex*, urges Oedipus to forget: "From this day on, I wouldn't look right or left."

   In part, it is the more general notion of symmetry that connects straightness to indifference. One way of seeing symmetry is in terms of indistinguishability or indifference. To move in a straight line is to move preserving this symmetric even-handedness. Straight lines have/possess/exhibit particular symmetries.

   Straight lines have reflection symmetries—fold a sheet of paper along the line; fold perpendicular to the line folding one half of the line back onto the other. Reverse direction (180° turn) part-way along and go back in a straight line—straight lines have half-turn symmetry at *every* point (an S-shaped curve has this property at its centre, but not at every point). Straight lines can be rigidly moved (translated) along themselves, such as with trombones and furniture drawers (if the slide is not straight, the drawer won't work). Circles also can be rigidly moved (rotated) along themselves (constant curvature). Screw motion or helical motion is a combination of straight line and circular motions.

2. Imagine a machine for hearing sound; it is not easy to judge equality directly, but use it as a means for reducing difference between two ears, a self-correcting system. The acoustic world is different from the visual world. You are always at the centre of a sphere of sound. Pascal claimed the acoustic world is a sphere with centre everywhere and edge nowhere.

3. At the Chelsea Flower Show (in London, England) this year, there was a Japanese garden which drew attention to a characteristic ethos of such gardens which is the absence of straight lines in order not to encourage any hurrying. Straight is efficient, economic both of time and energy; straight is frugal, offering strictly the minimum, miserly.

4. Is straightness a local or a global property? A part of a line can be straight even though the whole is not. Are we concerned with infinite extent (or extendibility) or non-retracing of steps?

5. There is an issue of verification: have I gone in a straight line (to be decided after the event) versus how to be sure I can generate one (to be decided ahead of time).

   Need some reference to judge, perhaps a point towards which I am moving straightly. Do I know how to start where I am and go straight ahead (for this you need to know ahead of time where you are going)?

   Lying evenly between its points, a line which does not turn (minimal effort), arising from placing one foot in front of the other. Is this Kandinsky's 'tension'? How can I check if I am doing this? What if I am blindfolded? How would a bug know?

6. Does straightness inherently involve sight? William Ivins (1946), once curator of prints at the Metropolitan Museum of Art in New York, has written a book on art and geometry, and draws attention to the differing balance and predominance of the tactile-muscular and visual sources of space intuitions present both in Greek art and mathematics and subsequently.

   > In any continuous pattern the hand needs simple and static forms and it likes repeated ones. It knows objects separately, one after another, and unlike the eye it has no way of getting a practically simultaneous view or

> acquaintance with a group of objects as a single awareness. Unlike the eye, the unaided hand is unable to discover whether three or more objects are on a line (p. 4).

So there are different perceptions according to whether the human sense of touch, sight or sound is employed.

> Line of sight is actually very inaccurate: a cabinet maker planes *two* boards together at the same time and then flips one up onto the edge of another to compare and check for light between them. This 'folding' is an extrinsic test.

The notions of 'not-turning' and 'shortest' coalescing on the concept of 'straight' is embedded in our language and experience from very early on. Shortest and not turning coincide in Euclidean plane geometry: not so on 'closed-in' surfaces. When looking at the 450° cone, we can experience this coalescing as separating or breaking apart.

Announcement: Theorem 1 "There exist pairs of points on the 450° cone through which there are no straight lines (that is, lines that do not turn)."

Response: Prove it, show me where.

Announcement: I can see that it is true, but I don't yet know how to say *why* it is true. I have had my vision, but I cannot yet share it with you. We cannot know what people perceive, only what they say. ('Whereof we *cannot* speak, thereof *must* we remain silent'—the final words of Wittgenstein's *Tractatus*, with our emphasis in the English. The original German is *Wovon man nicht sprechen kann, darüber muss man schweigen*; another less 'biblical' rendering is "what we cannot speak about we must pass over in silence.")

When am I satisfied with a geometric understanding? When I can give an explanation for *why* it is the case. Even after I was certain, the proof still does something for me.

Proofs are not about certainty—they are about communication and offering whys? Certainty implies stability, but not growth. The stability of such knowledge connects to Michèle Artigue's observation (in her CMESG plenary lecture) that the curriculum is only stable when it has become obsolete. And to Marshall McLuhan's observation (see e.g., *Laws of Media* by E. and M. McLuhan) that when one technology becomes superseded and surrounded by another, the former technology becomes an art form (e.g., 'penmanship').

McLuhan provided many other examples of this process: the Earth once surrounded by Sputnik becoming an art object; likewise, film 'surrounded' by television becoming an art object. Part of McLuhan's argument is that the obsolete technology becomes the 'content' of the new one, while the new one remains unseeable. (Danish Maths educator Ole Skovsmose has talked about the way mathematics 'formats' society, a computer disc metaphor, a point which connects strongly here with this sense of invisibility, and hence a loss of the sense that it could be otherwise.)

'Euclidean Geometry was stable for me and now it isn't.'

'Differential Geometry is not dead; it is not completely made.'

That shortest and straight do not coincide is connected to the fact that on the 450° cone not all right angles are equal to one another. (Euclid's mysterious Fourth Postulate requires that for *his* geometry 'all right angles be equal to one another'.) Euclid defined a right angle as the angle formed when two lines intersect in such a way as to result in four congruent angles. So, at the 450° cone point, right angles have 112.5° each; elsewhere, they each have 90°. We are so used to deciding on geometric properties in terms of properties of numbers, their measures that so frequently substitute for them, but which takes us out of the realm of the geometric. To us in such an arithmetic/algebraic frame, Euclid's fourth postulate seems hard to comprehend; surely right angles are *defined* as 90° angles.

What was it that Euclid (or a possible inserter of Postulate 4 into a later 'version' of the *Elements*) had seen? The plane is the only cone for which all right angles are equal to one another, so this postulate definitely helps to specify where Euclidean geometry takes place.

There is a theorem from differential geometry, a paraphrase of which reads: If all right angles are equal to one another on a surface, and all geodesics can be continued indefinitely (which is effectively Postulate 3 from Euclid's *Elements*), then (a) each pair of points can be joined by a shortest path, (b) every shortest path is non-turning, and (c) every non-turning path is locally shortest.

How are we to decide what is worth knowing?

Issues About Definition

There is a story (sometimes associated with social constructivism) that in mathematics we construct many possible definitions and then we 'need' socially to agree on one definition. This was not the experience of the working group. We came up with three definitions of 'straight', and rather then settling on one we glorified in the diversity and richness that the three notions gave us. It seemed natural to hold and explore the complexity and interrelations. We were comfortable with the collection of definitions and our sense of connections among them. It is not so much that the differing definitions contradict each other, but rather that they enrich and supplement each other and point out differing points of view and differing aspects of our deep experiences of 'straightness'. For more discussion of definition, see Pimm (1993) and Henderson (1996).

Issues About Consistency

There is a common notion that is important in mathematics to be consistent at the level of notation or definitions. However, it is an empirically observable fact that neither mathematicians nor mathematical texts are consistent with one another. For example, nine different definitions of angle were found in plane geometry texts in the Cornell University mathematics library, and calculus texts disagree as to whether the function f(x) = 1/x is continuous or discontinuous. Even within one single text (see, for example, the notion of derivative in most any calculus text), consistent notation is not in generally possible, because different notations have differing experiential power in different contexts. Recall Ralph Waldo Emerson's observation: "A foolish consistency is the hobgoblin of little minds." Shackling ourselves with this sort of consistency would be foolish and limiting.

But: What does the consistency of mathematics consist of? Is it all arbitrary?

Issues About Arbitrariness

'Arbitrariness' is a dangerous notion. There may be more than one possible starting point, but that does not mean that the starting points are arbitrary. Differing contexts and differing points of view bring with them a demand for differing starting points. It is not so much that we *couldn't* choose another, but rather that we *wouldn't* want to do so. Why stop doing what is real?

If the choice is arbitrary, then it does away with the need for any discussion. In many discussions about mathematics there is an easy slide from "if it is not absolute" to "then everything is completely relative/ arbitrary"—from Perry position 1 to Perry position 5 in the blink of an eye (see, e.g., Copes, 1993). In many of these discussions, there apparently is no middle ground that might allow choice among alternatives, or allow for the decision not to choose.

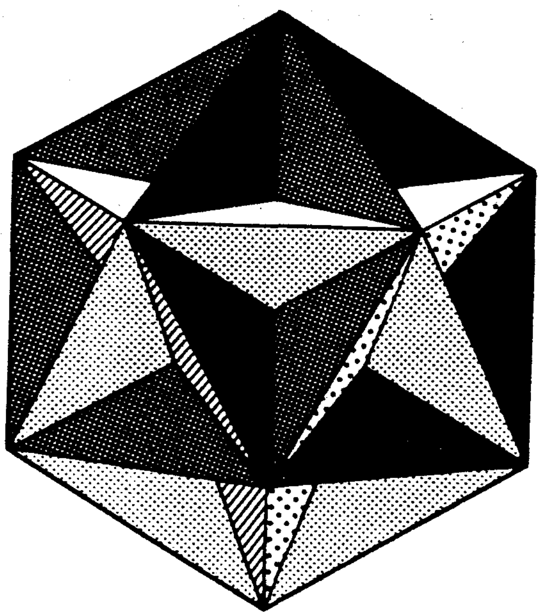
In the working group, we chose not to choose, but rather to hold onto the complexity of the multiple definitions. A metaphor for this is the notion of wave/particle complementarity in physics. Both together is far better than either separately. As an elementary mathematical example, consider the question of whether a square is a rectangle or not. Or whether 1 is a prime or not. In most contexts, it seems like it is less undecidable than not worth deciding, and also there will be a loss by doing so. In a particular context where choosing is important, then it is often clear from the context which choice is appropriate.

Issues About Axioms

Once you have access to a mathematical area through questions, constructions and tasks to explore, we did not want axioms (and axioms seldom offer such access by themselves). We note that in the working group people did not want to talk about axioms.

PART C: THE GREAT STELLATED DODECAHEDRON POSTER

A static, coloured image on a poster was offered for consideration in conjunction with the Ivins quotation given below. The version below substitutes differential shading for different colours.



The invited focus concerned the move from seeing to saying, in the light of:

- first, the Tahta-revised Taoist motto;
- second, the previous day's experience of attempting to express visions as theorem-statements;
- third, the possibility of remaining with the whole through gazing, rather than needing to identify elements.

Simone Weil writes:

> Method for understanding images, symbols, etc. Not to try to interpret them, but to look at them until the light suddenly dawns. Generally speaking, a method for the exercise of the intelligence, which consists of looking (1952, p. 109).

Ivins (1969) writes of the role of symbolism and syntax in the creation of hand-made images historically, contrasting what he calls 'visual statements' with collections of word symbols.

> Thus while there is very definitely a syntax in the putting together, the making, of visual images, once they are put together there is no syntax for the reading of their meaning. With rare exceptions, we see a picture first as a whole and only after having seen it as a whole do we analyse it into its component parts. [...] This leads me to wonder whether the constantly recurring philosophical discussion as to which comes first, the parts or the whole, is not merely a derivative of the different syntactical situations exemplified on the one hand by visual statements and on the other by the necessary arrangement of word symbols in a time order. Thus it may be that the points and lines of geometry are not things at all but merely syntactical dodges (pp. 61-2).

Any process which holds images must have a syntax: for instance, the grid of the geoboard. Colette Laborde (1995) makes clear that computer Cabri-drawings have the sequential-temporal syntax of verbal statements, the product of the process of 'explicit description' which necessarily precedes their generation. Cabri-drawings have memories—their generational history forms part of the figure and the machine insists that they be 'read' this way. Thus, Cabri-drawings provide a new and mathematically important instance of Ivins' 'rare exception.'

With text you have to put it back into time as you read. With an image you can just gaze. The configuration holds everything there together for you, so you can put your own time ordering onto it, in your own time. It does not matter where you start. You may not be able to see the whole thing at once, but you can have an image of the whole thing.

'*L'esprit géométrique est l'esprit fidèle*'—there is a need for the geometer to keep faith with the integrity of the image.

'What is it to be a geometer?' is a different order of question from 'What is geometry?.' What is it to think/act like a geometer?

IN SUMMARY

> In mathematics, as in any scientific research, we find two tendencies present. On the one hand, the tendency toward *abstraction* seeks to crystallize the *logical* relations inherent in the maze of material that is being studied, and to correlate the material in a systematic and

orderly manner. On the other hand, the tendency toward *intuitive understanding* fosters a more immediate grasp of the objects one studies, a live *rapport* with them, so to speak, which stresses the concrete meaning of their relations (David Hilbert, 1952, p. iii).

Deep is different from formal. Our understanding of straightness is very fragile.

'Straightness' has split into three different notions:

(a) no turn, indifference to left or right;

(b) geodesic;

(c) homing beacon, no discernible difference in either ear, balance.

A theorem is a vision that has been socially accepted.

Proof precipitates an experience and answers the question 'Why?'.

Seeing (experiencing) is the destination; saying is the path.

The film offered an experience of infinity and a sense of geometry evolving in time. Not only could we experience infinity, in some important sense we could not avoid experiencing it.

From the poster, the difference arose between an analytic perspective (which decomposes the image into elements) and an active, holistic gaze.

Also, it focused our attention on the issue of the existence, support and valuing of multiple seeings for mathematics. The geometric culture is apparently not interested in ambiguity, and multiplicity is something to be got rid of; the artistic community actively promotes it. Yet, as Chevallard (1990) observes: "Mathematics is a perfect example on which a *celebration of ambiguity* could be founded."

There is a tension between a fluid, open (not completely made) mode and a formal, restricted mode in mathematics. There is room to live, to breathe: some can relish the fact that there are differential equations that cannot be solved. In mathematics, you do not have to enter a completely made world.

One Reflection (David Wheeler)

The last part of the Nicolet-Gattegno film, and the stellated dodecahedron poster, seem to disturb us. We don't know what we are seeing—or, more accurately, we don't know "what to make of" what we are seeing. One wonders why we are not content with the experience, as we may be with the geometrical configurations in our environment that we "don't see", that we don't attend to. Perhaps what is at stake here is whether we are content to "have" a visual experience, or insist on "holding it" too. Without "making something of it" ("making sense of it?"), we can't hold on to it. Indeed, perhaps the word 'experience' should be reserved for those impacts that *have* left a mark on us, and not be used for those things we never noticed although they were there. (The maxim: "experience is the best teacher" only works if the experience referred to is "held" and still available to us.)

Two techniques for aiding geometrical thinking were offered to us by David Henderson. One, to take a familiar geometrical concept ("straightness") and attempt to analogise it in a new sort of space.

As we found, this required us to break apart component subconcepts ("shortest distance" and "not turning") that seem fused in the Euclidean concept. The second technique was "imagine a bug." This technique evokes our own conscious awareness of aspects of our physical behaviour through our imaginative identification with the actions of the bug. Both techniques emphasize the magnitude of our geometrical resources:

- our own repertoire of physical behaviours;
- awareness of how we monitor our physical behaviours;
- imaginative identification with other "possible" or "impossible" behaviours.

None of this touches the realm of two-dimensional configurations and "representations" where geometry is often supposed to begin. Indeed, the two-dimensional realm cannot be "made sense of," without connecting it in various ways to the world of our own physical behaviour.

What are the characteristics of the geometer, past or present? I'm aware how little I've thought about this question relative to the attention I've given to, say, "What is geometry?". As an entirely revisable beginning, I'll suggest:

- the geometer trains and uses his (her) intuition to maintain the presence of the 'whole' of a geometrical situation through all the focusing, analytic, scanning activities of his (her) geometric eyes;
- the geometer knows when to suspend judgement, when to "gaze";
- the geometer refuses to delegate any of his (her) thinking to algorithms;
- the geometer knows that a configuration (situation) has no beginning and no end: no part is more primitive, or basic, than any other.

ACKNOWLEDGEMENTS

We are grateful for comments by Susan Gerofsky, particularly on the work of Marshall McLuhan.

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TOPIC SESSIONS

Topic Session A1

BUILDING COMMUNITY: EXPLORING THE POTENTIAL IN A LOCAL MATHEMATICS CONTEXT

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INTRODUCTION

The notion of "community": What do we think of when someone mentions that term? When the reference is particularly to mathematics education communities, what meanings come to mind? Do we think of particular groups of mathematics educators? Does our understanding of "community" change with the group of mathematics educators we are thinking of—e.g., elementary or secondary mathematics teachers, college or university mathematics teachers, mathematics teacher educators?

It seems we in mathematics and mathematics education have a large number of structures available to serve us, and through which we may serve our colleagues. There are international and national, provincial and regional mathematics education organizations to which we may belong, and which act on behalf of all mathematics educators. We have a shared interest in the mathematics education of all citizens, and a shared interest in the field of mathematics. We have increasingly extensive and varied means by which we may communicate quickly and easily with each other. Form and function seem to be necessary elements in any claim by a group to identify itself as a community. While it seems reasonable to speak of a mathematics education community, it may be more appropriate to speak of a "community of communities."

My present concern is with the possibility of "community" in mathematics and mathematics education at the local level. In this paper I will describe the activities of a small group of mathematics educators in a small city, and an exploration of "community" as it may apply to the group. This group of five secondary mathematics teachers and a mathematics teacher educator has been together for almost a year and a half. We share an interest in the use of non-routine writing in the secondary mathematics classroom. We are a group with a shared interest; what would it mean to say we are a small, but genuine, community of mathematics educators?

The first part of this paper presents some descriptions and meanings of "community." In the interests of space and the objectives of the paper, I will keep these descriptions brief. The list itself is not intended to be comprehensive; it is intended only to give a sense of the possibilities. The second part describes the experiences of the local group. Part three closes the paper with some comments and questions about the notion of and potential for a local community of the "kind" envisioned here.

PART ONE: SOME MEANINGS OF "COMMUNITY"

A: Sandel's Three Conceptions of Community (In Mason, 1993)

In his article, "Liberalism and the Value of Community," Mason (1993) provides a brief overview of Michael Sandel's (1982) three conceptions of community. These are:

- The "instrumental" community, wherein "'individuals regard social arrangements as a necessary burden and cooperate only for the sake of pursuing their private ends'" (p. 222);
- The "sentimental" community, in which "participants share final ends, regard cooperation as a good in itself, and experience ties of sentiment as a result" (p. 222); and
- The "constitutive" community "according to which persons 'conceive their identity—the subject not just the object of their feelings and aspirations —as defined to some extent by the community of which they are a part'" (p. 222).

B: Dewey's Concept of Community (In Schultz, 1971)

According to Schultz (1971), Dewey viewed "community" as emergent, evolutionary and process oriented, and having a strong pedagogical function which was to be interpreted broadly, not limited to formal educational contexts.

For Dewey,

> There must be <u>common interests derived</u> from <u>shared perceptions</u> of the common life, needs and goals of the community. These must be taught by the members of the community and continuously learned and relearned and modified by them as a community operates by a pedagogical enterprise for the achievement of its foreseen and freely agreed upon ends [emphasis in original] (Schultz, 1971, p. 323).

C: Community as a Moral Notion (Mason, 1993)

Mason (1993) notes that community "often functions simply to pick out a group of people each of whom have something in common: members of the same linguistic community share a common language; members of the same cultural community share a culture; members of the business community share the same occupation" (p. 217). Some, such as philosopher Alasdair MacIntyre, believe that the family, the neighbourhood, the city, and the nation are exemplary models of community which serve as a ground for identity definition. These groups may have shared values which help unite them, Mason claims, but they are not communities in the moral sense. These latter communities are built upon "relationships which possess particular moral qualities: mutual concern and the absence of systematic exploitation" (p. 217). Furthermore, this moral concern is more than sentiment or feelings for each other, as in the Sandel "sentimental" conception of community.

D: Four Stages to the Development of Genuine Community (Peck, 1988)

Peck (1988), in his book *The Different Drum*, describes four stages that groups of people typically progress through in their development as a genuine community. Without this passage, they are unlikely to achieve this ultimate status. These stages are:

- Pseudocommunity, in which members are nice and polite to each other, and there is a desire to be agreeable, to keep feelings about oneself and others to oneself. It is a conforming environment in which individual differences are minimized or ignored.

- Chaos, during which group members disagree and challenge each other, with the desire to convert or heal. Individual differences are clear and in the open. Members' efforts are intended not to ignore these differences but to obliterate them.

- Emptiness, a painful step during which group members need to empty themselves of such barriers to communication as personal expectations, preconceptions, prejudices, ideology, and solutions, and the need to fix, solve or control.

- Community, marked by the transformation from a collection of individuals into genuine community. This transformation requires "little deaths" from individuals, and also "group death" and "rebirth." In genuine community, members have a deeper understanding of each other and accept individual differences; members share specific task(s), and a strong mutual respect.

A similarity with Mason's (1993) moral notion of community is evident here. Peck's (1988) view of community also suggests a face-to-face group, local, and relatively small.

E. Professional Community (Field or Discipline) as a "Community of Understanding" (e.g., Science Community, Mathematics Community)

Within the notion of "professional community," knowledge is relative to the given community. Scientists and mathematicians who are steeped (i.e., educated/trained) in their respective fields "are members of a community of understanding ... who participate in a certain framework of interpretation applied to all those subjective experiences which fall within a certain category" (Hatcher, 1990, p. 105). These can be very small, select communities, for example, a "handful" of people on a research team. Furthermore, a commitment to the community implies a strong and shared regard for standards of behaviour, knowledge, and performance (Greene, 1989; Peters, 1975).

Dewey's notion of the pedagogical function of the community clearly applies to "professional community." It is also evident that we may speak of nested professional communities.

F. Learning Community: The Classroom as Learning Community

The notion of the "learning community" is currently a major educational theme (e.g., Ball, 1989; Jacobs, Manley, & Ware, 1988; Talbert & McLaughlin, 1992; many others). The goal is to establish the classroom of learners as a learning community. The development of an environment of trust, shared goals, and mutual aspirations is fundamental to reaching this goal.

Jacobs et al. (1988) describe an attempt to establish an entire university's general education program as a learning community because, they note, such programs typically consist of discrete courses and thus lack "intellectual coherence," an incoherence which is reflected in a pluralistic society where commonality and community appear difficult to achieve.

Summary, Part One

The definitions and descriptions of "community" vary, some have more to say to our understanding of "mathematics education community" than others, but all have something to say to the notion.

I envision major aspects of community, when considered in a deep sense, to be:

- shared interests and goals,

- mutual understanding and freely agreed upon goals,
- cooperation—and more: mutual concern for members and an absence of systematic exploitation,
- consensus building, but accepting differences,
- normative with regard to levels and forms of knowledge, types of behaviour, and a "framework of interpretation",
- committed to recognized standards,
- fundamentally pedagogical in its attentiveness to ongoing learning and relearning.

The focus for me is the issue of the status or the vibrancy of the mathematics education community at the local level.

PART TWO: THE LOCAL SECONDARY MATHEMATICS WRITING PROJECT—THE EXPERIENCES OF OUR GROUUP (FORMING A COMMUNITY?)

In this section I will describe the local context and the formation of the writing group, its human composition, the group's (and my) initial goals and its focus once it got underway, group processes, the things we have done and not done—all issues related to the development of community.

The Local Context

There are two school boards, a separate (R.C.) and a public. Between them they operate six English high schools in the city of 55 000, and the surrounding area. A District Mathematics Council was established in 1992 to promote communication and professional growth among area elementary, secondary, college and university mathematics teachers. At a fall, 1993, meeting of the Mathematics Council I proposed the writing project to the secondary mathematics teachers in attendance. This was followed with visits to mathematics departments of city high schools which did not have representatives at the Council meeting.

My Personal Goals

My personal goals for the project were that it would provide:

- an opportunity for me to become better acquainted with area secondary mathematics teachers (I was relatively new to the area);
- professional development opportunities for all project participants by exploring the potential in a non-traditional secondary mathematics instructional form;
- an inter-school opportunity for a group of secondary mathematics and post-secondary mathematics education teachers who shared certain common goals and interests to develop as a "genuine" professional community of mathematics educators;
- research opportunities for the study of (a) the use of non-routine writing in the secondary mathematics class, and (b) the development of a professional community of local mathematics educators.

The formation of the group was more important to me than the particular focus of writing in the secondary mathematics class. This could be modified if there was a need to do so.

The Proposal

In the proposal I suggested as a project focus that we attempt to develop as

A Community of Mathematics Educators Developing and Implementing Writing Strategies in the Mathematics Classroom
I suggested as a procedure that project participants:

- Meet to decide upon what form(s) of mathematics writing activities the group would initiate in classrooms,
- Choose an approach to implementation:
  - Decide upon which grades, course levels;
  - Develop instructional strategies;
  - Develop student assessment criteria and strategies;
  - Develop means of assessing the value of the project (formal/informal).
- Implement the chosen writing activity(ies) in selected classes;
- Monitor the progress of students and activities.
- Meet regularly as participating mathematics educators to discuss our current status:
  - Problems, successes, insights, ideas for change, improvement as necessary, etc.;
  - And attempt at least informally to assess the value of the project on an ongoing basis, and at end of the year.

It was my *desire* that group members would choose a single writing activity (for example, journal writing), or a very small number of related writing activities in order to facilitate development, implementation, and evaluation, and especially, group discussion. The more we had in common in terms of our respective classroom writing experiences, the richer and more meaningful our discussions would be, and the closer and stronger we would grow as a local community of mathematics educators, I reasoned.

Getting Started

The writing project actively got underway in January, 1994. At our first group meeting, five teachers attended. Three were from the separate board, and two from the public board. The degree of teaching experience varied greatly, from three years to well over twenty. The courses taught by these teachers ranged from grade 9 to grade 12 and Ontario Academic Courses, and included both general and advanced (academic) levels.

We began by sharing why we were there; I reviewed some of the literature on writing in mathematics, and the proposal. The teachers commented on ideas in writing communication they had tried, ideas they thought they would like to try, and particular areas of student development they were interested in such as, increase effective use of mathematical language, improve general and technical writing skills, and enhance student understanding of mathematical skills and knowledge. There was only limited interest in finding out students' "feelings" or views about mathematics, which they felt was the focus of journal writing. Writing activities which tended to be more formal and structured, and which could be graded, were of particular interest, although the particular writing activities of interest did vary among group members.

It needs pointing out that none of us regarded ourselves as having particular expertise in this area. I had access to more literature on the subject, but I had not field-researched this topic before, nor had I any experience with mathematics teachers (especially secondary teachers) who regularly had students undertake non-routine writing activities in their classes.

Consequently, this project was established as very much an exploratory venture. We would, if possible, gradually work toward an action research environment. The approach the group adopted was largely non-prescriptive:

- The approach of focusing on a single writing approach, or even a few selected writing strategies, was not chosen.
- Teachers chose those forms with which they felt most comfortable, which they felt appropriate for their classes, and which they felt could be managed given perceived constraints on their time.
- These ranged from more informal but still structured writing forms, such as logs, to formal activities in which grammar and structure were important, as for example, research projects.

> The over-riding concern of group members was students' ability to communicate mathematically orally and in writing.

The Meeting Process

The group members meet every four to six weeks. Often someone is unable to attend. At our meetings, which are tape recorded, we discuss:

- What each person is trying, or is going to try;
- The perceived results of activities which have been tried;
- Group activities (we sometimes divert our attention from the classroom to pursue other ideas, such as spearheading the Math Council's development of a local clearinghouse for

mathematics-related books and literature, or the publication of a local journal with articles by area mathematics educators—volume 1 of f(un)ction was published earlier in 1995);

- General concerns, issues, and reflections of members.

The diversity of personal interests, secondary mathematics classes taught and types of student writing attempted make for both an interesting and a challenging environment in our meetings. The focus at our meetings often shifts with the speaker but the interests we do share appear to be strong enough to sustain us.

We, collectively, have attempted many writing activities in our mathematics classes since we began. We have presented our work in progress at both national and provincial conferences. We do plan to continue into 1995-96. There is potential for networking provincially with secondary mathematics teachers with similar interests. Nevertheless, there clearly are issues we need to engage in, if we are to move beyond the comfortable plateau we appear to have reached, and more fully become a community of local mathematics educators.

PART THREE: BUILDING COMMUNITY

The brief examination of "community," and our work and experiences as a group (potential community) over the past year and a half, have led me to consider a number of points of reflection and questions (which, in the time remaining in this presentation, I would like us to take up).

Some Comments and Questions

How do we support, build and sustain a meaningful local mathematics community? How does the "writing group," for example, grow from being a collection of mathematics educators with a shared interest in the field and a shared interest in certain pedagogical perspectives, to something approaching a genuine community? An increased sense of shared interests, manifested, I would suggest, in group activity(ies) in which all members of the group have a deep and well-defined role and commitment, should raise the potential for heightened community.

This project is more than, and different from a loosely defined research project that needs tightening up. But it does lead to questions regarding the place of research in groups such as this, and the implicitly/explicitly defined "roles" of the members. I have interviewed (and audiotaped) most of the teachers individually, and some of the students in small groups. Although I have not pursued the research aspect of our writing and group development to date as much as I had originally proposed, I believe our activities are still principally seen as my "project," which, while not unexpected, does, I think, hinder somewhat the potential for community. Defining my role, which is different from the classroom teacher's, is an ongoing task for all members.

When we speak of a community of learners we tend to think of the classroom. We can extend this notion to include groups such as the "writing group" in which the learners are the teachers, the educators. Thus the classroom becomes much more broadly defined to include all those spaces in which we are engaged by some aspect of our project. But, does this mean a loss of intensity, of focus?

We each live in many so-called "communities;" the institutions in which we work (schools, universities, etc.) significantly shape our professional lives. Thus, the saying, "school as community." We may have much more in common with our colleague down the hall who teaches another subject, than with the mathematics teacher in another school in the same city. What possibility is there for a genuine mathematics education community in this environment?

I felt a need at this time to make an individual presentation on this topic; I look forward to future papers which will, I hope, almost always be collaborative productions.

More Questions About Community

If I have not posed enough questions on this subject here are a few more to close with.

1) Do all teachers together constitute a community of educators?

2) Do all secondary mathematics teachers necessarily constitute a community?

3) Do all teachers of mathematics in a local area necessarily constitute a community?

4) Do all mathematics teachers in a single institution necessarily constitute a community?

5) How does collaboration become community?

6) Is it reasonable to think that a "single" shared interest (such as writing) can lead to community? (Does it matter?)

7) Does the overlaying of research support or hinder the development of community (in the short term, long term)?

8) Do group members as action researchers: support or hinder the development of community? (Not every member wants to commit to research).

9) What degrees of local mathematics education community are reasonable or feasible, given our diverse life and professional interests?

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Topic Session A2

CRITICALMATHEMATICS: OBSERVATIONS ON ITS ORIGINS AND PEDAGOGICAL PURPOSES

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ON THE GENESES OF CRITICALMATHEMATICS EDUCATION

The notion of criticalmathematics education is linked to how the Criticalmathematics Educators Group (CmEG) developed. However, just as it is difficult to specify the exact beginnings of any organization, idea, or movement, it is difficult to do so for the CmEG. Nevertheless, the Criticalmathematics Educators Group, the particular organization to which I, as well as other members of CMESG, belong began as an action project, stemming from a conference at Cornell University in October 1990.[1] This conference, "Critical Mathematics Education: Toward a Plan for Cultural Power and Social Change," was organized by John D. Volmink, then an Assistant Professor of mathematics education and now Associate Vice Chancellor of the University of Natal in Durban, South Africa. The organization grew out of conversations between John Volmink, Marilyn Frankenstein, and me concerning how to further the goals of the conference. Our organization, the Criticalmathematics Educators Group, which has a newsletter that circulates to over 500 subscribers around the world, has other organizational and intellectual roots.

These organizational and intellectual roots extend back several decades and across several continents as well as involve concerns for the social, cultural, political, and economic contexts of mathematics education. Among national and international organizations concerned with mathematics education, three groupings of mathematicians and mathematics educators have influenced CmEG's concerns towards social and cultural issues: the Interamerican Congress on Mathematics Education (CIAEM), the International Congress on Mathematical Education (ICME), and the African Mathematical Union (AMU). Of these organizations, according to D'Ambrosio (1990), discussion in both CIAEM and ICME experienced a qualitative shift in and around the middle to late nineteen seventies. At first, concerns of those organizations focused largely on the structure and content of different curricular innovations and preoccupation with conditions of program implementations. D'Ambrosio describes that in CIAEM by 1975 in Caracas, Venezuela "[e]ven though considerable space continued to be dedicated to discussion of programs, the more crowded sessions, with more discussions and wider presence and repercussion, were those dedicated to discussions of a social and, even, of a political nature...such as 'mathematics and development..." (p. 11, my translation). Similarly, in ICME3 by 1976, in Karlsrube, Germany, coincidence with the presence of the Third World, profound inquiry began into the "position of mathematics in education systems" and the "negative that can result from a mathematics education poorly adapted to the distinct socio-cultural conditions, be it in Third World countries, or in countries with large

[1] Among those who attended the Cornell conference in 1990 and participate in CMESG are Dorothy Buerk, Kelly Gaddis, David Henderson, Marty Hoffman, and me.

industrial development" (D'Ambrosio, 1990, p. 11, my translation). These discussions continue to influence proceedings of CIAEM and ICME.

In Africa since the late 1950's, leaders in political, scientific, and educational spheres expressed concern for and began developing solutions to the devastating impact of colonial, and latter neo-colonial, educational structures and textbooks along with the colonial disruption of Africa's scientific development, which was set in motion at the beginning of the European trade in African slaves. As contemporary initiative to search for solutions, the African Mathematical Union through its conferences and its Commission on the History of Mathematics in Africa (AMUCHMA)[2] disseminates information on current research into the history of mathematics in Africa, in part, so that up-to-date historical information, much of it refuting ideas of the non-existence of mathematics in Africa before the Greeks, might be included in schoolbooks. Moreover, the AMUCHMA publishes information on research into ethnomathematics and its pedagogical uses.

This interest of mathematicians and mathematics educators in Africa and Europe as well as in South and North America for the intersection of issues concerning culture and society with mathematics and mathematical education as expressed in various international forums eventually lead to insisting that these issues occupy more substantive space at ICME conferences. At the sixth meeting of ICME, in Budapest from 27 July to 3 August 1988, Christine Keital, Peter Damerow, Alan Bishop, and Paulus Gerdes organized what became known as the Fifth Day Special Programme on "Mathematics, Education, and Society." During this special conference-within-a-conference, presentations and discussion from individuals representing all continents focused not only on sociological but also cultural influences on mathematics education as well as such influences of mathematics education on societies.[3]

As significant and successful as the Fifth Day Special Programme was, it was not immune to criticism. Many people felt that to have had a separate day instead of the presentations and discussions being an integral part of the ICME-6 implied a marginalization or ghettoization of individuals and issues. Others expressed a dissatisfaction with the somewhat elitist structure of the sessions themselves. Furthermore and importantly, many agreed with Noss's evaluation that "while discussion of social and cultural issues was an important break with the dominant psychological paradigm of mathematics education, there was a clear role for a more explicit *political* focus" (Noss, et al., p. vii). Consequently, in reaction to the Fifth Day Special Programme, Richard Noss, Andrew Brown, Paul Dowling, Pat Drake, Mary Harris, Celia Holyes, and Stieg Mellin-Olsen organized a conference in 1990, at the Institute of Education, London University, titled "Political Dimensions of Mathematics Education: Action and Critique" (which later became known as PDME-1). It is important to note that, at that time, mathematics educators in England were in the throes of an unparalleled program of government intervention in the school curriculum. In mathematics it meant an imposition of an elitist and skill-based view of mathematics learning along with national testing, destroying three decades of progress. Naturally, these concerns

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[2] The Commission on the History of Mathematics in Africa publishes a newsletter, *The AMUCHMA Newsletter*, in Arabic, English, French, and Portuguese versions, that is available free of charge. To request inclusion on their mailing list or back issues (fifteen have been published), write to its chair, indicating the language version you prefer: Paulus Gerdes, C. P. 915, Maputo, Mozambique (email: pgerdes@up.uem.mz).

[3] Papers and discussions from the Fifth Day Special Programme are collected in Keital, Damerow, Bishop, & Gerdes (1989).

dominated a significant portion of discussions at the conference since progressive mathematics educators in England had organized to challenge the government's curricular intrusions. [4]

PDME-1 was an invitational conference and invitees were required to submit a paper in advance of the conference. Beside being a reaction to the Fifth Day Special Programme of ICME-6, the work of Stieg Mellin-Olsen provided a theoretical starting point for PDME-1. His book, *The politics of mathematics education*, published in 1987, was one of the first to question explicitly "how personalised knowledge" (a psychological notion) can be discussed without using notions of power "such as politicisation, conflict, and oppression, when the potential learner may be in the midst of bitter struggles for civil rights" (Mellin-Olsen, 1987, p. xiii). Addressing the opening of PDME-1, Mellin-Olsen suggested that the conference be a forum for those "doing research on the political nature of education;" that it "strengthen the political aspects of our work"; and that it "foster collectivism..." (Mellin-Olsen, 1991, p. 1). He went further to insist that the conference be

> a support system for political action and critique directed against governments. In particular those governments which in extreme ways exercise an educational politics which prevents its citizens access to knowledge, in particular mathematical knowledge.
> Not only majority groups of a population, but also other groups, which because of colour of skin, religion or other stigmas, are prevented a democratic access to mathematics knowledge (Mellin-Olsen, 1991, p. 2).

The conference was lively and largely successful. Participants elaborated ideas of power, conflict, and oppression in mathematics education and political critique of traditional educational structures. An important debate that occupied participants concerned the differential political implications of multicultural and anti-racist perspectives in the teaching and learning of school mathematics.

It's fair to say the most people felt nourished by finding an international community of like-minds. Nonetheless, aspects of the conference drew criticism. Some asked, "Where's the mathematics? Others felt that certain aspects of the conference's organizational structure were elitist. Indeed, some participants criticized the conference for its emphasis on critique over action: We organized no fire brigades! Also, some participants expressed disappointment and annoyance that certain individuals were consciously not invited.

Stemming from PDME-1, the problems of community and action were utmost on the minds of the organizers (J. Volmink, M. Frankenstein, and A. B. Powell) of the Cornell University conference "Critical Mathematics Education: Towards a Plan of Action for Cultural Power and Social Change." As such, this conference coming just six months (October 1990) after the first PDME, can be viewed as a reaction to and an extension of PDME-1. However, two aspects of the PDME conference were not followed: (1) academic papers were not the currency of the conference, which is not to say that the conference was anti-intellectual, and (2) invitations went out to several individuals who had been excluded from the first PDME.

To understand the role of the three organizers of the Cornell conference, we must first understand another of the intellectual roots of criticalmathematics. This root is associated with the work of Marilyn

[4] At the same time in the United States, some mathematics educators were explicitly challenging skill-based curricula for working-class students. One such effort in adult education is represent by the publication of Marilyn Frankenstein's book, *Relearning Mathematics*. See Frankenstein (1989).

Frankenstein. Seven years before the 1990 conferences in London, England and Ithaca, New York, Frankenstein (1983) published an article in the *Journal of Education* titled, "Critical mathematics education: An application of Paulo Freire's epistemology". This theoretical article was later re-printed, with classroom examples appended, in Shore (1987), *Freire for the classroom: A sourcebook for liberatory teaching*. Frankenstein's work is important from several perspectives, however, I want to draw attention to how it influenced the agenda of the Cornell conference and the later formation of the criticalmathematics educators group. In Frankenstein (1987), we have the following statement:

> Knowledge of basic mathematics and statistics is an important part of gaining real *popular, democratic control over the economic, political, and social structures of our society*. Liberatory *social change* requires an understanding of the technical knowledge that is too often used to obscure economic and social realities. When we develop specific strategies for an emancipatory education, it is vital that we include such mathematical literacy. Statistics is usually abandoned to "experts" because it is thought too difficult for most people to understand. Since this knowledge is also considered value-free, it is rarely questioned. In attempting to create an approach to mathematics education that can lead both to greater control over knowledge and to *critical consciousness*, it is important to have an adequate pedagogical theory that can guide and illuminate specific classroom practices (p. 180, emphasis added).

The title of the Cornell conference, "Critical Mathematics Education: Towards a Plan of Action for Cultural Power and Social Change," points directly to Frankenstein's work. The notion of 'critical' is borrowed from Freire's emancipatory pedagogy in literacy and is rooted in a Marxian or dialectical materialist interpretation of critique. That is, critique must attempt to clarify the economic and, thereby, political conditions for the development of ideas. Critique has two orientations: critique of a theory (a set of ideas) and critique of the very object of the theory (the real situation that gave rise to the set of ideas). For example, consider as a real situation, the state of affairs that many college students, if given the choice, would not register for a mathematics course, coupled with the idea that normal people are either good in the sciences or the humanities but not both and more often in the latter. In addition, this double orientation of critique, the purpose for Marx of being 'critical', is to challenge *and* change taken-for-granted ideas and conditions of life so as to improve social life.[5]

As I mentioned earlier, the Cornell conference was in reaction to PDME-1. One important reaction concerned action. At the Cornell conference, a real attempt was made to develop action plans that would have a life beyond the conference itself. One such action was the development of an international organization, with a newsletter.[6] In each issue of the *Newsletter* of the CmEG, we publish our definition (see Endnote 1)—one that is evolving and subject to debate.

A commitment to action defines and distinguishes the CmEG. We have held receptions to introduce others to CmEG at meetings of the Mathematical Association of America (USA and Canada), the National

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[5] It's worth noting that this interpretation of the notion of 'critical' differs from the use of concept in Critical Theory of the Frankfort School. There the notion has to do with interdisciplinary study including philosophy, sociology, economy, history, psychology to interpret social life for the purpose of making radical improvements. This idea of 'critical' forms part of the theoretical foundations of Skovsmose (1994).

[6] To date, we have published 4 issues of the *Newsletter* of the CmEG. To subscribe, contact Marilyn Frankenstein, Applied Language and Mathematics, College of Public and Community Service, University of Massachusetts, Boston, MA 02125-3393.

Council of Teachers of Mathematics (USA and Canada), at Socialist Scholars Conferences (USA), at conferences on the PDME-2 (South Africa, 1993) and PDME-3 (Norway, 1995), at meetings of the Mathematics Commission of the National Education Coordinating Committee and AMESA (South Africa), at various meetings of mathematics educators in Brazil and in Southern Africa, and at the ICME-7. Besides informational receptions, at some of these meetings we have organized panel discussions on topics such as "Critiques of Traditional Paradigms of Mathematics & Science Education," "Socialist Pedagogy & Criticalmathematics Education," and "Access, Equity, & Change: Mathematics & Science Programs of Inclusion." We have taken CmEG into the streets by organizing mathematicians to participate in the largest US demonstration against the United States war in the Gulf, carrying placards declaring "Use Math Against Poverty & Disease Not for War," "War Doesn't Add Up," and "Mathematicians Against the War." We have also engaged in providing material support to mathematics education at the Universidade Pedagógica (formerly, the Instituto Superior Pedagógico) in Mozambique. Furthermore, we have been successful in challenging and stopping this distribution of offensive educational material by an organization of the mathematical establishment in the United States (see Baldassarre, Broccoli, Jusinski, & Powell, 1993 or Powell, 1994).

SOME PEDAGOGICAL PURPOSES OF CRITICALMATHEMATICS EDUCATION

My involvement in issues associated with criticalmathematics education emanate from three sets of interests. Briefly, as an African American, I have been keenly interested in examining and struggling against those forces that impede the full participation of oppressed groups in the United States, particularly African Americans, from exercising their right to create and enjoy mathematics. Second, as a mathematics educator, whose epistemological stance is influenced by Gattegno, I am challenged to engage students of mathematics and student teachers in using their awareness, in the Gattegno sense (see Gattegno, 1987, 1988; Powell, 1995) to produce mathematics. Third, as eone interested in civic action, I am interested in the use of mathematics education to enable students to acquire a critical stance on their learning and the teaching of mathematics as well as on the social, political, and economic structure of society. In what follows, I borrow from Powell (1993) to indicate how criticalmathematics education causes one to raise questions about the personal and political (ideological) purposes of mathematics education among left-wing or progressive educators.

At the university, my primary responsibility is teaching students who enter the university with underdeveloped abilities in mathematics. Most of the students I teach enter the university through a special program designed to give access to students whose entrance profile in mathematics and English fall below minimum levels for regular admission. The majority of these students are African American and Latino. Typically, the kinds of courses given to such students reflects the notion that to build their basic skills, one needs to provide them with lots of practice in mechanical operations. One of my students, Rohan, as a condition of his acceptance into the university, attended the six-week college 'readiness' program in the summer. He was required to do so since the university considers him an "at-risk" student because of his low academic profile, his intended participation in the university's athletic program, and his weekend job as a disc jockey.

Using a graphing calculator, students in Rohan's class had to explore a mathematical rather than a contextual situation to explore, which they later related to a contextual situation. They investigated several families of curves for a set of given single-variable functions. At first, to know which unknown to vary in a given function, for example, $l(x) = e^{-kx}$, they had to distinguish among the independent variable, constants, and parameters. Specifically, the functions they investigated included these:

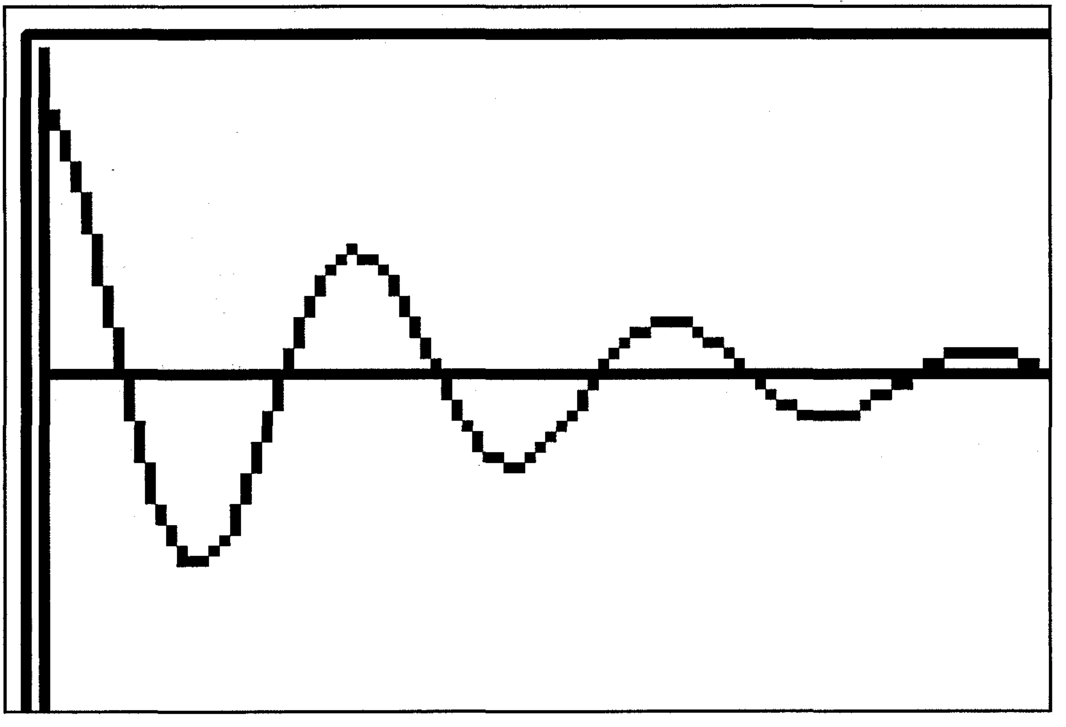
a) $f(x) = ax^n + b$,

b) $g(x) = a/x^{1/3} + b$,

c) $h(x) = a \sin x$, and

d) $j(x) = \cos(wx + d)$.

By combining these functions and adding parameters, students crafted new functions. Having done this, they constructed functions to correspond to given curves. For example, students attempted to determine the algebraic description of a function whose graph corresponds to the shape of the curve in below in Figure 1.



A curve that represents damped harmonic oscillation.

Figure 1

Afterward, they related their function to the contextual situation of a damped harmonic oscillator such as a mass at the end of a spring (a vibrating spring) or a mass at the end of a cord (a pendulum). Finally, students explored questions such as the ones below aimed to increase their awareness of relationships between the curve and the movement, affected by friction, of a pendulum:

1 For the function, where $f(t) = ae^{-kt}cos(bt + c)$ is the displacement (in inches) of a mass from its rest position, t is the elapsed time (in seconds), and a, k, b, and c are parameters, determine parameters values that give a good fit to the following data:

Time	Displacement
0	10
4.5	9.2
9.4	8.9
13.9	8.3

2. At what time does the pendulum first pass its resting position?

3. At that time, what is the speed of the pendulum?

At the end of the investigation, students wrote laboratory reports. Endnote 2 contains Rohan's revised laboratory report. In this laboratory report, Rohan first states a conclusion of the laboratory: that the curve in Figure 1 "represents the distance a pendulum travels over time." He then states that his goal is "to find a function whose graph resembles" the curve in Figure 1 and begins his exposition by reviewing his knowledge of the functions he believes are involved in the curve of Figure 1, namely some form of $cos\ x$ and e^x, and in turn provides illustrations of their graphs.

Separately, Rohan discusses each of these functions. Concerning the cosine function, he shows the effect of multiplying the independent variable by a constant, $cos\ bx$. After illustrating how different values of the parameter b affect the period of the cosine curve, he explores the effects of multiplying the dependent variable, $cos\ bx$, by a constant, a, and explains that the value of the parameter a affects the amplitude of the cosine curve. Reflecting on the curve in Figure 1, he notices that it is a decreasing function of x and suggests that the graph results when the cosine function is multiplied by some decreasing function.

The decreasing function of x that Rohan chooses is e^{-kx}. He notes that e^{-kx} decreases as x increases and that as the value of k increases the graph of e^{-kx} decreases proportionally slower and approaches the x-axis. He then speculates that "[s]ince e^{-kx} is a decreasing function whose graph decreases slowly along the x-axis as the value for 'k' gets smaller, and the graph of the distance a pendulum travels over time is a decreasing cosine graph along the x-axis, I modify the function $f(x) = acosbx$ to $f(x) = ae^{-kx}cosbx$." In earlier sections of his report, he demonstrated his awareness of the effects that the parameters a, b, and k exert on the behavior of his modified $f(x)$. With a suitable choice of parameter values and with a restriction of the domain of $f(x)$ to $0 \leq x \leq 15$,[7] he states his discovery that the graph of the function $f(x) = 5e^{-1/32x}cos4x$ resembles the one given in Figure 1.

[7] Although Rohan writes the interval as $0 \leq x \geq 15$, in an interview, he demonstrated that he knows well how to use inequality notion to indicate intervals on the real line.

Among other things, this example illustrates the power of a graphing calculator to allow the study of functions and physical situations that educators traditionally consider beyond the mathematical reach of pre-precalculus students. Typically, students in advanced calculus study this function as the solution to a problem in physics that, applying Hooke's Law and Newton's Second Law, leads to a second-order linear differential equation that models the movement of a vibrating spring or a swinging pendulum affected by friction due to the resistance of air.

Just as important, this laboratory report exemplifies how a graphing calculator and writing are tools for forcing awareness. By experimenting and focusing his attention, Rohan forced his awareness of the effects of varying the values of parameters of $f(x) = ae^{-kx}\cos bx$ and the corresponding graphical changes. Aided by a graphing calculator, he constructed a function whose graph resembles the one in Figure 1. Writing the laboratory report forced him to articulate connections between information embodied in algebraic and graphical representations of functions. The connections he articulated are displays of mathematical awareness that he forced. Moreover, to inform and persuade his audience, transactional writing obligated him to arrange his awarenesses in logical order that, because of his analysis, possibly lead him to other insights.

In the investigation, he worked as a mathematician in generating knowledge that was new to him. Not only this but also his description of his discovery resembles the writing of mathematicians. That is, his description follows a rather linear, logical order. First, he carefully states his conclusion, then discusses what he knows, and illustrates his knowledge appropriately with both algebraic and corresponding graphical examples. Characteristic of writings in textbooks and professional journals, he reveals neither the affective states he experienced during his struggle nor the messy, perhaps even non-methodical, scratch-pad work in which he engaged. Those affective and messy experiences remain private.

This raises an important instructional point. It is valuable to point to such students as well as to an entire class the contrast between the messy affective and cognitive experiences of their work and the emotionless, logical exposition of their laboratory reports. This may serve to demystify mathematics texts and to make students cognizant that the struggle of discovery is likewise usually missing from the narratives they read in mathematics texts.

The pedagogy that informed this work indicates an ideological position: through educating their awareness in contextual *and* mathematical situations, all students, including working-class African American and Latino students can generate mathematical knowledge. This pedagogy responds to instructional practices about which we can ask: What can be the evaluation of students that puts lecture and rote practice as the central mode of instruction and that holds students hostage to the minutia of textbook discourse?

Questioning and interrogating dominant pedagogical practices remains a critical task of mathematics educators. In the United States as in South Africa and elsewhere, progressive mathematics educators struggle alongside individuals from underrepresented groups in mathematics to open its doors. Many mathematics educators argue that students mostly identify with mathematics when it arises out of concrete or "real world" situations. Consequently, progressives interested in liberatory pedagogy have exulted the primacy of the applications of mathematics for schools as a way of attracting more students to the discipline. This position stems from a concern for the overemphasis of mechanical drill-and-practice exercises and the scant attention or trivializing the utility of mathematics to solving practical problems of everyday life. Furthermore, this position is motivated by concerns for the pervasive racism, sexism, and classism that closed the door of mathematics in the face of the oppressed.

Progressive mathematics educators argue for the primacy of applications in school mathematics most strenuously in discourses about involving underrepresented groups in mathematics. Though this position merits serious attention, nonetheless, it errors in the opposite direction. As Slammert (1991) noted, "[t]o this day, mathematics remains unpopular and inaccessible, creating serious misconceptions about its inherent nature, and about who can and who cannot do mathematics" (p. 206). Indeed, the inherent nature and the landscape of mathematics contain more than modelling and applications; to do mathematics implies more than to perform statistical investigations into wage distributions as a measure of economic injustices. For sure, at present, applications and critical analyses of the political economy of society are crucial and largely neglected in school mathematics curricula. However, claims of the primacy of modelling and applications in mathematics may support an ideology that posits the oppressed as incapable of being motivated by the abstract nature mathematics.

As the above example of crafting a function that approximates a curve illustrates, students from oppressed groups and whose mathematics have been underdeveloped due to negative forces in their society can, nevertheless, generate mathematics knowledge at various levels of abstraction. Moreover, abstract mathematics can be the starting point for involving people in generating mathematics. The project of interrogating practice ought to extend even to ones that claim progressive political dimensions since pedagogy ethnographically reveals ideology.

NOTES

Endnote 1

criticalmathematics[8] educator *n*, first used in the early 1980's as various mathematics educators began a dialogue about how to connect their concerns in mathematics education with their critique of society and their commitments to social, economic, political, and cultural empowerment. By 1990, small groups of criticalmatheamtics educators were collaborating on projects, including writing and holding conferences. By the end of 1990, "criticalmathematics educator" was written in its present form as two words, and was synonymous with (the archaic) "mathematics educator." (See the related term "radicalteacher," New Words: A Postrevolutionary Dictionary, ed. Pamela Annas, 1998, back cover *Radical Teacher*, present issue.

1. **as mathematicians, criticalmathematics educators view the discipline** (a) as one way of understanding and learning about the world;they do not view mathematics as a static, neutral, and determined body of knowledge; instead, they view it as knowledge that is constructed by humans; (b) as one vehicle to eradicate the alienating, eurocentric model of knowledge, widely taught in the schools before the 1990's, particularly this model's narrow view of what were considered mathematical ideas and who were capable of owning these ideas, and this model's historiography which excluded and distorted, marginalized and trivialized the contributions of women and men from all the world's cultures to what was then considered "academic" mathematics (and is now incorporated as part of criticalmathematics); (c) as a human enterprise in which understandings result from actions; in which process and product, theory and practice, description and analysis, and practical and abstract knowledge are "seamlessly" (Lave, 1988)

[8] The following definition is from the *Criticalmathematics Educators Group Newsletter* and is published in each issue.

interconnected; and in which mathematics and other disciplines interact, as does knowledge with the contexts of social, economic, political and cultural perspectives.

2. **as teachers, criticalmathematics educators** (a) listen well (as opposed to telling) and recognize and respect the intellectual activity of students, understanding that "the intellectual activity of those without power is always characterized as non-intellectual" (Freire and Macedo, 1987); (b) maintain high expectations and demand a lot from their students, insisting that students take their own intellectual work seriously, and that they participate actively as "co-interrogators" (Powell, 1989) in the learning process; (c) are not merely "accidental presences" (Freire, 1982) in the classroom, but are active participants in the educational dialogue, participants capable of advancing the theoretical understanding of others as well as themselves, participants who can have a stronger understanding than their students (Youngman, 1986); (d) assume that minds do not exist separate from bodies, and that the bodies or material conditions, in which the potential and will to learn reside, are female as well as make and in a range of colors; that thought develops through interactions in the world, and that people come from a variety of ethnic, cultural, and economic backgrounds; that people have made different life choices, based on personal situations and institutional constraints, and that people teach and learn from a corresponding number of perspectives; (e) believe that "most cases of learning problems or low achievement in schools can be explained primarily on motivational grounds" (Ginsburg, 1986) and in relationship to social, economic,political, and cultural context, as opposed to in terms of "lack of aptitude" or "cognitive deficit;" (f) recognize the reality of mathematics "anxiety," but deal with it in a way that does not blame the victims, and that recognizes both the personal psychological aspects and the broader societal causes; (g) recognize that everyone has mathematical ideas; criticalmathematics educators "work hard to understand the logic of other peoples, of other ways of thinking" (Fasheh, 1988).

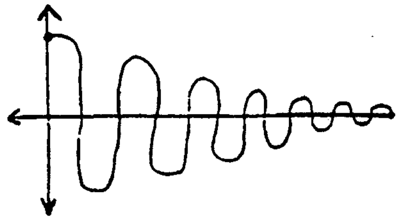
3. **as concerned and active citizens, criticalmathematics educators** (a) have a relatively coherent set of commitments and assumptions from which they teach, including an awareness of the effects of, and interconnections among racism, sexism, ageism, heterosexism, monopoly capitalism, imperialism and other alienating totalitarian institutional structures and attitudes; (b) believe that good intentions are not enough to define a criticalmathematics educator; criticalmathematiccs educators are actively anti-racist, anti-sexist, anti-all the other dehumanizing totalitarian institutional structures and attitudes, and they work with themselves, their mathematics classes and their colleagues to uncover, name, and change those conditions; (c) view that a major objective of all education is to shatter the myths about how society is structured, to develop the commitment to rebuild alienating structures and attitudes, and the personal and collective empowerment needed to accomplish that task; (d) are open to debate about which curricula and which investigations best achieve our goals; there are varieties of criticalmathematics educators; e.g., feminist criticalmathematics educators; (e) maintain that "dehumanization, although a concrete historical fact, is <u>not</u> a given destiny, but the result of an unjust order" (Freire, 1970); (f) are militants in the Freirean sense of the term, committed to justice and liberation; "Militancy forces us to be more disciplined and to try harder to understand the reality that we, together with other militants, are trying to transform and re-create. We stand together alert against threats of all kinds" (Freire, 1978); (g) understand that no definition is static or complete, all definitions are unfinished, since language grows and changes as the conditions of our social, economic, political, and cultural reality change; (h) also have fun, laugh, and play...

New Words in the Sciences: A Postrevolutionary Dictionary. Marilyn Frankenstein, Arthur B. Powell, and John Volmink, co-editors, 2001.[9]

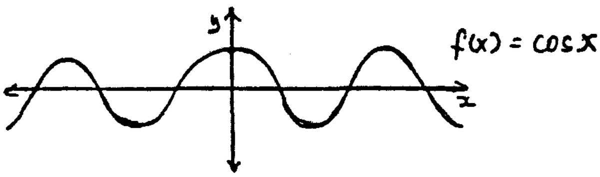
Endnote 2

[With minor changes and improved legibility, this student rewrote his report.]

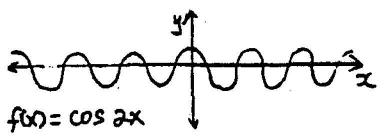
Rohan's Pendulum Write Up Dec. 9th, 1992

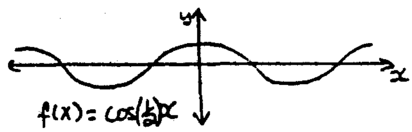


The above paragraph represents the distance a pendulum travels over time. I am trying to find a function whose graph resembles the above. From my knowledge of functions and their graphs, I know that the cosine function *f(x) = cosx* looks like graph below.

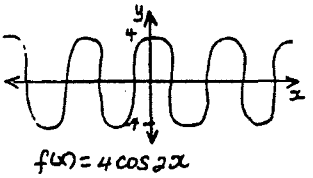


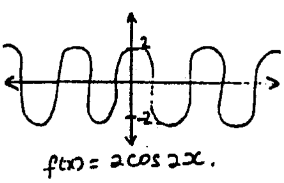
Looking at the graph of the pendulum and the graph of the function *f(x) = cosx*, I know that *cosx* is involved in my equation. Using the graphing calculator T1-85, I modify the equation, rather the function, *f(x) = cosx* by putting in a constant "*b*" in front of the variable *x*, namely *(f(x) = cosbx*. For example, *f(x) = cos 2x* and *f(x) = cos(1/2)x*.





Looking at the two graphs *f(x) = cos 2x* and *f(x) = cos (1/2)x*, I notice that as the value of "*b* increases the speed of the graph also increases. Exploring more with the graphing calculator, I notice that when I put a constant "*a*" on front of *cos bx*, namely *f(x) = acosbx*, the amplitude or height of the graph increases as "*a* increases. For example, *f(x) = 4cos2x* and *f(x) = 2cos2x.*





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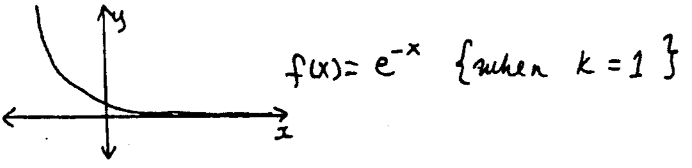
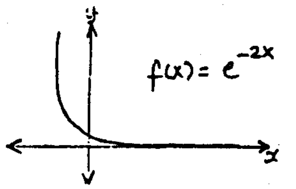
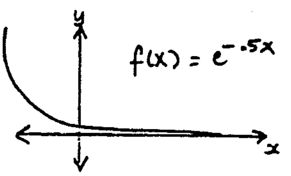
[9]This "reference" represents the wishful thinking of the editors of the *Criticalmathematics Educators Group Newsletter.* We hope and struggle for a substantially more just and peaceful social, political, and economic life on planet Earth in the year 2001.

From all these observations, I know that the constant "*b*" determines the speed or rate of the graph and "*a*" determines the amplitude or height of the graph. Since the graph of the pendulum is decreasing as *x* gets larger, I know that some decreasing function times *acosbx* will result in a decreasing cosine graph. From my knowledge of decreasing functions and their graphs, I know that the decreasing function $f(x) = e^{-kx}$, where "*k*" is a constant, when graph looks like the graph below if "*k*" is 1.

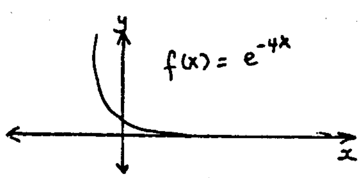


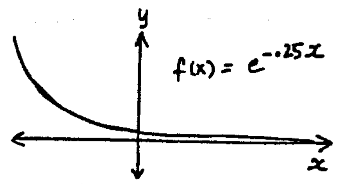
Modifying the function $f(x) = e^{-kx}$, using the graphing calculator, I notice as *x* increases the function e^{-kx} gets larger. The graphs below illustrate for each case respectively.



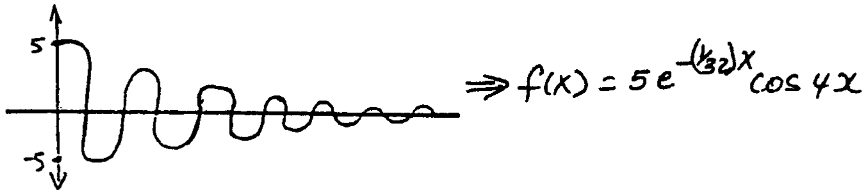


Also, I notice that as "*k*" increases (meaning as the value for "*k*" increases) the graph decreases rapidly along the *x-axis* and as the value for *k* decreases the graph decreases slower along the *x-axis*. The graph below illustrates each case.





Since e^{-kx} is a decreasing function whose graph decreases slowly along the *x-axis* as the value for "*k* gets smaller, and the graph of the distance a pendulum travels over time is a decreasing cosine graph along the *x-axis*. I modify the function $f(x) = acosbx$ to $f(x) = ae^{-kx}cosbx$. Since I want the graph of the function $f(x) = ae^{-kx}cosbx$ to look like the opening graph, I use the graphing calculator to modify the parameters of the function $f(x) = ae^{-kx}cosbx$ and also change the domain to $0 \leq x \leq 15$. After modifying the range and the parameters of the function $f(x) = ae^{-kx}cosbx$, I came up with the function $f(x) = 5e^{-(1/32)x}cos4x$ whose graph is illustrated below.



Part Two

Question: From the function $f(x) = ae^{-kx}cosbx$, find the following data points.

Time	Displacement
0	10
4.5	9.2
9.4	8.9
13.9	8.3

Since when *time = 0* the *distance = 10* we know that "*a*" in the function $f(x) = ae^{-kx}cosbx$ is going to be 10 which is the amplitude of the graph, therefore $f(x) = 1\text{-}e^{-kx}cosbx$. Using the first function formulated $f(x) = ae^{-(1/32)x}cos4x$ and the graphing calculator, I should be able to find the function of the given data points. Using the graphing calculator, I plot the given points first. After plotting the data points, I graph the function $f(x) = 10e^{-(1/32)x}cos4x$. I notice that the graph didn't match the data points plotted. Also, the speed of the graph was too fast. Since "*k*" determines how the graph travels along the *x-axis* and "*b*" determines the speed of the graph, I modify "*k*" making it -1/64. When I graph the new function $f(x) = 10e^{-(1/64)x}cos4x$, I notice that the curve were hitting the points but was going too fast. Using the graphing calculator, I modify "*b*" making it 1.35. when I graph this it was the closest approximation of the data points. Therefore the function for the data points is $f(x) = 10e^{-(1/64)x}cos1.35x$.

Question: At what point does the pendulum first pass its resting point?

Since the *y-axis* represents the distance and the *x-axis* represents the time, using the graphing calculator, I found out the first time $d = 0$ is at $t = 1.4444$. Therefore, the pendulum first passes its resting point when $t = 1.4444$.

Question: What is the speed of the pendulum at $t = 1.4444$?

Since at $t = 1.4444$ the distance from its resting point is 0 and the initial point where the pendulum starts is (0,10). Therefore we know that the distance the pendulum has travelled is 10 units (10-0). Therefore using the formula *rate = distance/time* we can calculate the speed of the pendulum at $t= 1.4444$. Therefore $r = 10/1.4444 = 6.9232$.

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Topic Session B1

WOMEN AND CHANGE: EXAMINING THE VOICES OF RELUCTANT MATHEMATICS LEARNERS

Dorothy Buerk
Ithaca College, New York

Note to the reader: The material presented here draws heavily on two published papers:

"Enhancing Women's Mathematical Competence: A Student-Centered Analysis" by Janet Kalinowski and Dorothy Buerk (1995). *National Women's Studies Association Journal* 7(2), 1-17.

"Getting Beneath the Mask, Moving Out of Silence"
by Dorothy Buerk and Jackie Szablewski (1993).
In Alvin M. White (Ed.) *Essays in Humanistic Mathematics*. Mathematical Association of America Notes #32, 151-164.

The Kalinowski and Buerk paper is the source of the description of Women's Ways of Knowing and the quotes and analysis of the writings of Ginger and Ann. The Buerk and Szablewski paper is the source of the description of the epistemological process students seem to move through when learning mathematics and is the source of the quotes and analysis about Jackie. Should you want to use any of this material in your own work I would prefer that you cited or quoted from the original sources rather than this compilation of those sources. Please contact me for reprints if they are not readily available to you.[1]

With that introduction, let me begin sharing with you my thoughts on

WOMEN AND CHANGE: EXAMINING THE VOICES OF RELUCTANT MATHEMATICS LEARNERS

Mathematics has a public image of an elegant, polished, finished product which obscures its human roots. It has a private life of human joy, challenge, reflection, puzzlement, intuition, struggle, and excitement. Mathematics is a humanistic discipline, but the humanistic dimension is often limited to its private world. Mathematics students see the elegant mask, but rarely see this private world, though they may have a notion that it exists. Their exposure to mathematics, limited to its public image, often leaves them silent and insecure.

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[1] I can be reached at: Dorothy Buerk, Mathematics and Computer Science Department, 1212 Williams Hall, Ithaca college, Ithaca NY 14850-7284. Telephone: 607-274-3587; e-mail: buerk@ithaca.edu

Mathematics is a combination of both its public image and its private world. In this talk I will argue that our students need to experience both of these mathematical domains, and most particularly our female students need to know the private world of mathematics.

I will introduce you to three students, Jackie, Ginger, and Ann, all first-year college students. They all took a course from me where journal writing was an important component and their journals are the primary source of their words presented here. I will ask you to imagine them in your mathematics classroom and listen to their voices and to their growth in confidence in their abilities to think mathematically.

First, I want to introduce you to Jackie. Jackie and I wrote a paper together and so I have an extensive introduction to her ideas in her own words. I think she will set the stage nicely for the ideas that I want to share with you today.

Jackie writes:

> I was one exposed only to the public image of mathematics. To me, there seemed no room for interaction with the content, no possibility of connection with the ideas.

Picture her as she describes her experience with mathematics as

> ... the role of the tourist who merely looks out at the sights that surround [her] as they travel past in a blurred rush. There was no chance of stopping, of touching those mathematical concepts that lay out there.

Think about one of your southern neighbors, like myself, deciding to see and know Canada by taking a one week tour— one day in Halifax, one day in Quebec City, one day in Montreal, one day in Toronto, one day in London, one day in Calgary, and one day in Vancouver. Jackie would NOT say that she knew Canada. She would more likely say that she wanted to stay longer in each place to see art and culture, to sense the Anglophone and the Francophone worlds, to meet some people and get to know them, to gain a sense of history. She wants to know each place on her own terms and in her own way.

Several Quotes from Jackie:

> It was true, I had taken Calculus in high school and had even done well. One would think I had to be connecting with math. But I viewed my mathematical experiences much like building a house that was supposed to be for me, according to someone else's design.

> I never really liked what I was building and never felt comfortable with the tools I was using. Meaning I never felt sure of my mathematical background. If I was using defective tools or materials who was to say that the house would not fall in at any moment. Thus though I "built the house," it was for someone else. I was but the [contractor] attempting to follow the blue prints accurately enough to please the owner, 'cause after all, I certainly never felt I "owned" any mathematical ideas, or understood them well enough to call them my own.

> I did not feel that I could interact with math on a personal level and therefore did not feel as though I owned or could own any of the mathematical ideas and concepts with which I was presented. I found math simply an exercise of going through the motions and "believing for" someone else, rather than for myself. I was frustrated. [As] I began to understand some of that frustration, I wrote [to a mythical math teacher]:

I realize that in order to help us realize all that already exists in the world, in order to guide us through all the worlds of mathematics, you must keep to a strict itinerary. If you didn't, we would not be exposed to all we must be exposed to in order to reach the destination of "mathematician," "chemist," "well-rounded person." But don't you see that in your well intended efforts to show us all the "landmarks" of those worlds, you are not allowing us to touch? How can we come to say that we believe in a thing, a concept, an idea, if we ourselves do not know it is real?

I wanted to touch, to believe for myself. But I knew that interaction required process and what kind of process could there possibly be with an already finished and perfected product. It didn't matter that I had taken Calculus, or any other course for that matter. I still felt estranged from math, aggravated by it, afraid of it. I was math phobic.

As in the cases of other phobias, though you would rather be afraid of heights than face the challenge of a roller coaster, you might find the roller coaster appealing after gaining the courage to try it. In the same way, one may find the concepts of math exciting and exhilarating after building up enough courage to at least attempt to open the mind.

The question remains, how do we go about "building up enough courage"? What does it take to "at least attempt" to try? The first step to conquering fear is naming it. Yet how is it possible to name a thing that renders you voiceless? Unlike English class, math was not a place for ideas in process. You could not say or share something you were thinking about. You could only share with the class completed, perfected thoughts and I simply had no such thoughts concerning math. [I remember in high school] we would just get out of our English class where I never shut up and then we would come right to Math and there I would sit dumbfounded, lost, unable to speak. I was intimidated by those of my classmates who could rattle off formulas and understand them by the time our teacher finished writing them on the board.

Because of my inability to connect and interact with math in the way it was presented to me, I was left voiceless and because of this I could say "I hate math," to anyone 200 times over. I could say, "I am no good at math," but I saw no possibility of saying, "I am afraid of math," and then talking about why. For one, no one had ever asked how I felt about math. Why should they? Either you can do it or you can't; either you like it or you hate it; like all else in math, it's black and white. You don't talk about it. You don't write about it. You just do it. And even if I could talk about my feelings concerning math, that did not automatically provide me with the environment in which to do so. I, like most people, need a supportive, understanding and encouraging milieu where I feel at least somewhat comfortable before exposing vulnerabilities like math phobia.

Jackie sees mathematics as a discipline where she can have no ideas of her own, where there is no room for her own thinking. She has rejected a simple focus on right and wrong. She is estranged from mathematics and afraid of it, but she is cautiously aware that courage and encouragement might help her. She needs mathematical experiences that are not constrained by a right/wrong emphasis. She wants to touch, to feel, to have her own experiences, to own her mathematical ideas. She also wants process; she wants to share ideas in progress rather than just finished, polished products. She has intellectual sophistication and complex thinking patterns which allow her to function in these ways in other disciplines, but not in mathematics.

The epistemological model described by Belenky, Clinchy, Goldberger, and Tarule in *Women's Ways of Knowing* (1986) helps me to understand Jackie and the other two students I will introduce you to. Based on an in-depth consideration of women's voices, this model illuminates important similarities and differences in the ways that women make meaning and I can apply it to their meaning making in mathematics. The theory posits five different ways of knowing, perspectives from which women view reality and draw conclusions about truth, knowledge, and authority. Three of those ways of knowing are relevant to our conversation today: received knowledge, subjective knowledge, and procedural knowledge.

Received knowers conceive of themselves as capable of receiving ideas but not creating them. They learn by listening to the voice of authority. The received knowing perspective is not uncommon in mathematics classrooms. Judith Jacobs (1992) tells the story of a student who insists that -2 plus -7 is +9 despite an intuitive understanding that the answer is negative 9. When asked "Why?" she answers with the rule, "Two negatives make a positive" (p. 47). In received knowing, as in this example, the rule takes precedence over thinking. The received knower sees all problems as closed-ended; the only way students can possibly arrive at the right answer is to slavishly apply the rules that they have been taught. Received knowers strive to replicate their teachers' thinking while discounting their own.

Subjective knowers focus on their own thoughts and attend to their inner voices. They "think that there are as many right answers as there are people listening to their own inner voices" (Belenky et al. 1985, p. 18). Subjective knowers are more difficult to identify in mathematics and science classes than they are in literature classes, for example, since instruction in science and mathematics usually leaves little time for subjective, intuitive explanations to be aired. The subjective stance, when it is heard in the mathematics classroom, presents a major challenge to teachers. Since subjective knowers need only turn inward to know what is right, they fail to value teachers' comments. Academic discourse with subjective knowers is limited; arguments based on logic and reason carry little weight in these students' eyes.

A procedural perspective requires careful observation and systematic analysis. Procedural knowers "work hard to cultivate the voice of reason" (Belenky et al. 1985, p. 20). Belenky, Clinchy, Goldberger, and Tarule have identified two distinct forms of procedural knowledge: separate and connected knowing. Both separate and connected knowers attend to the process as well as the content of knowing, but separate knowers use detached, objective methods of evaluation while connected knowers come to know by participating in caring relationships with what is known--both people and ideas. Connected knowing builds on personal experiences. It is a more introspective and reflective mode than separate knowing. Although Clinchy's research (1989) makes it clear that these two modes are not gender exclusive, her work suggests that more women than men show a propensity for connected knowing. I believe that mathematicians employ the connected mode when considering mathematical questions in private, but use the separate mode, the public mode, to describe their work to their colleagues and students through both their professional writing and their textbook writing (Buerk 1985). The objective stance of separate procedural knowing has until recently been the standard orientation in the traditional mathematics classroom. Jacobs (1992) has argued that the connected approach has much to offer "if what we really want from people is an understanding and ability to use mathematics" (p. 49).

Many students, like Jackie, enter our classes believing that mathematics is something that they can have no ideas about; that every question has an answer; that every problem has a solution. Teachers know these answers and teach methods to reach them. Students learn mechanically, following the rules precisely, without questioning or reflecting. These are the views of received knowers in mathematics.

These received knowers seem to go through a developmental process in changing their view of mathematical knowledge and this process is one I believe we can facilitate. A first step is to listen to their own ideas and to acknowledge them and own them. This requires a supportive environment where students feel safe enough to try out their untested ideas. It requires a group of classmates who will be supportive. Some classroom work in groups, some open-ended mathematical situations, some written explanations that are responded to without being graded—or only graded on the second reading all help to set this environment. In this setting students can begin to speak and can begin to own the ideas they have spoken. In addition, in mathematics this move away from received knowing seems to involve acknowledging that mathematics was and is made by people. This happens before students become comfortable with the theory, the algorithms, or the procedures that are so necessary in mathematics.

I try to create an environment in my classes that helps the students become aware of both diversity and uncertainty in mathematics and gain a more personal, subjective view of mathematics, by working in its private world. The students begin to realize that their own experiences are worth considering. They still believe, however, that knowledge is the product rather than the process of the inquiry. They can now learn from the experience of peers, for peers as well as authorities (teachers, textbooks) have ideas worth hearing. They begin to mold ideas, but often without being willing or able to support them. According to Belenky et al. (1986) the subjective knower is still dualistic in the conviction that the right answer exists, but now believes that truth resides within the person rather than in the external world.

Subjective knowing is particularly important since it does offer the student the opportunity to look inward, to trust her own inner voice and to begin to acquaint herself with her own needs and desires. The internal voice present in subjective knowing marks the emergent sense of self and sense of agency and control, so important to learning.

Students next become aware that intuitions may deceive; that some "truths" are truer than others; that theories can be shared and expertise respected. They become aware that it is the process of constructing knowledge that is important. They move away from subjectivism, learn to respect established procedures and their own insights, listen carefully to other points of view while evaluating their own. We need to help our students move through this process in their view of mathematics. Reluctant students become empowered and then become less fearful of the procedures and techniques needed to do mathematics.

The authors of *Women's Ways of Knowing* note: "We found that women repeatedly used the metaphor of voice to depict their intellectual and ethical development; and that development of a sense of voice, mind, and self were intricately intertwined" (p. 18). Jackie, uses the voice metaphor often and our paper documents Jackie's struggle to gain her mathematical voice.

Jackie struggled because she saw mathematics in a received knowing mode, but rejected that right/wrong orientation. Her intellectual sophistication and connected procedural knowing perspective in areas other than mathematics allowed her, in a supportive environment, to meet various mathematical challenges profoundly. By the end of a semester Jackie, who finally "closed her eyes, took a deep breath, and dove in," could claim her mathematical experience and present it clearly, powerfully, and personally. Jackie had come to my course able to "do" the mathematics asked of her in her high school courses, but considering herself "math phobic," feeling disconnected from the mathematical ideas she studied. Although an enthusiastic, reflective, outgoing person in most settings, she found herself voiceless in a mathematics classroom.

She became involved in mathematical activities and learned to listen to her own intuitions, to make conjectures, and to share her incomplete thoughts with her classmates and with me. She began to develop her private mathematical voice in creative ways, to reflect on her own thoughts, and to listen to my questions, comments, and suggestions, as well as those of her classmates. She experienced the private world of mathematics, became excited about the mathematical ideas that she studied, and gained confidence by developing those ideas and discussing them in the private world of her journal.

With great difficulty, she kept her voice and spoke in the public domain. She made meaning of the mathematics she studied for herself and learned to share that experience finally through our joint paper with all of us.

I hope that you now see Jackie clearly and see her in some of your own students.

Now let us look at two other women who have been in my classes.

Imagine Ginger, a woman with a vivid imagination and a great sense of humor. She had a brick wall in her head that was constantly under construction or demolition depending on how she was faring in her mathematics course. That brick wall separated her from the concepts and ideas of mathematics. Can you picture Ginger cowering behind the brick wall?

Ginger writes:

> At the beginning of this course I had a mental block against math. I picture a brick wall in my head, with little construction men laying more and more bricks on my block to make it stronger.... There have been times (my most frustrating) when this mental block has been at a towering height, and times when it's been only three rows of bricks compared to ten million rows.

When this brick wall/mental block was large, Ginger's mind was blank. She could not make any progress toward the solution of a problem, and she had nothing to add to class discussions about mathematics; she felt helpless. You have Gingers in your classes I am sure.

Mental blocks are easily understood when seen in the light of received knowledge. If truth and knowledge are conceived in rigid categories of right and wrong, attaining the right answer becomes extremely important. The students with mental blocks worry about doing what the teacher wants them to do rather than thinking about the problem at hand. They strive for good grades and are devastated by poor or average marks. Ginger wrote, "I am a person who likes to have things exact because I'm terrified of getting something wrong." Fear of error contributes to the construction of mental blocks and blank minds.

Ginger writes about one experience in my class.

> When we were first asked to think about the [assignment] in class, I really panicked. I am not a "mathematically thinking" person whatsoever and my mind went totally blank.
>
> I was feeling very frustrated with [the problem]. I mean, it was impossible! ... I was giving myself headaches galore.... I was even hoping I would never have to deal with the ... problem. Then Dorothy told us it would be on the midterm. Oh well, it was inevitable. So ... I sat down ... and proceeded [to solve the problem]. Of course, being my old doubting self, I thought I most likely did it wrong, I was way off track.
>
> So, today, I went to see Dorothy for help. She looked at my work and said to go ahead and make a journal entry of it, because I'd done it right.... The way I felt was "ELATED"! I got the problem. And I understand it. And I did it all by myself. That is definitely a confidence booster and a mental-block-wrecking ball. I feel so much better now. I mean if this problem was so hard and I figured it out, I am definitely ready to try and solve any problem I am given. And will I save money not buying Excedrin (HA HA). Math, here I come!

Ginger's panic and blank mind in response to a mathematical problem made it very difficult for her to take any steps toward its solution. Her exuberant use of capital letters in describing her elation emphasized the extent of her joy in tasting mathematical success. She began to develop an awareness that mathematics is something she can do. Curiously, despite her claim that she solved the problem all by herself, this description of events documents her extreme reliance on the authority of the teacher to declare the solution right. This dependence on the voice of authority was consistent with the received knowledge perspective that was pervasive in Ginger's writings.

Many of my students' feelings about mathematics range from occasional complaints about frustration and difficulty to often repeated sentiments of hatred, dread, and avoidance. Students' negative feelings may be expressed through mental blocks, which Ginger describes, or an active rejection of mathematics which Jackie felt.

Many of my students see mathematics as rigid and rule-bound. Even the more able students who did not suffer from mental blocks frequently believe that their own thoughts, approaches, and processes were of no value in solving problems if they did not get the teacher's answer in the end. Mathematics is then often rejected as a course of study because of these perceived characteristics. Mathematics rejection can occur if mathematics is a last holdout of received knowledge when students are moving beyond that perspective and developing more active orientations in other subjects. Students move out of received knowledge by questioning and rejecting authorities at the same time that they evolve an appreciation of their own ideas and thought processes. When students who were developing complex epistemological perspectives in other disciplines and in other areas of their lives conceived of no other way to understand mathematics but to see it through the lens of received knowledge, they found mathematics very unfulfilling and avoided it. Jackie had this experience. While she was a procedural knower in most areas of knowledge, she came to my course believing that mathematics could only be seen through the lens of received knowledge and she felt voiceless in that situation.

Next, let me share some excerpts from the writings of Ann, a sociable student who participated readily in informal conversations but who said very little in class discussions of mathematics.

Imagine Ann as she writes:

> On the first day ... I was LOST. I was so confused, so I decided to try and "pass through" this course without saying or doing anything. Granted, I don't talk much anyway, but I just assumed I could remain silent, without thought all semester.

In my class her plan was to passively withdraw, but I expected active learning, often in dyads or small groups. Midway through the term I assigned a "change paper." The assignment was to describe a specific moment when the student realized a change in her thinking, in her mode of acting, or in her feelings about mathematics during a classroom mathematical episode.

Ann writes:

> My "moment of change" took place during the tile experiment. Before this time, I felt totally incapable of doing the work in this class.
>
> In a group, especially one-on-one, I feel more compelled to work. This is only because I feel bad for the poor soul who got paired with me. I feel like I owe my partner something: I can't just leave her on her own, even if my help would only be minimal. Well, we tried the problem and we talked about life in general. We'd socialize for a minute and work for a minute.

> Eventually, we really got a lot down on paper.... Suddenly, in came the dawn, I saw a sequence in the total number....
>
> At the very moment I first saw a connection in the problem, I lost my feeling of helplessness. With Laura's help I had found a solution to a problem I originally thought was impossible. At that moment, I felt like I had unlimited capabilities to do anything I wanted to do. It sounds extreme, but it's true. I realized that when I set my mind to something, all I have to do is think and look inside myself for an answer. Somehow, I really believe that all the answers to all my questions are deep inside me. The problem isn't how to "find" a solution, it's how to "get it out" from within myself.

Like Ginger, Ann also experienced initial difficulty in thinking about the mathematical problems presented in the course. Forced (her word) from her consciously adopted strategy of silence by a dyadic exercise, she experienced problem-solving success in a flash of mathematical insight. She went from feeling totally incapable, to the idea that she had "unlimited capabilities." With her realization that all the answers were somehow inside her, mathematical knowledge became internally experienced as opposed to externally given. Her words suggest a subjective knowing stance.

Here we have three women. We have Jackie who acts like a procedural knower in many areas, but remains a received knower with respect to mathematics. We have Ginger who seems to be a received knower in most areas including mathematics. We have Ann who seems to be a subjective knower in general, but still acts as a received knower with respect to mathematics. With this in mind let us look at the wants and needs of these three women in a mathematics classroom.

Jackie sees mathematics as a discipline where she can have no ideas of her own, where there is no room for her own thinking. She has rejected a simple focus on right and wrong. She is estranged from mathematics and afraid of it, but she is cautiously aware that courage and encouragement might help her.

She needs mathematical experiences that are not constrained by a right/wrong emphasis. She needs acceptance of her ideas in process. She wants to touch, to feel, to have her own experiences, to own her mathematical ideas. She wants process; she wants to share ideas in progress rather than just finished, polished products.

Jackie has much to say in private, that is, in her journal, and gradually finds more to say in the public domain. She is bright, honest, and reflective and a joy to work with. I was, however, not always prepared for the depth of her struggle as she worked to gain her mathematical voice. See Buerk and Szablewski (1993) for Jackie's story.

Ginger and Ann were much more challenging in the classroom. Their fears ran deep. Their desires to avoid mathematics and to remain invisible, and their tendencies to succumb to mental blocks, seemed ever present. As these students talked about the source of their fear and insecurity with mathematics it became clear that the demand for quick, correct, and public answers can cause anxiety in young received knowers. Both Ginger and Ann describe such experiences in elementary school when asked when their mental blocks in mathematics began.

The need for encouragement and validation is another concern. Some procedural knowers write that it is important for teachers to encourage a student's ideas no matter how bizarre they may seem. They find that this encouragement helps then grow, develop, and become more confident in their mathematical skills.

Compare these ideas to Ginger's statement:

> In the beginning, I was a scared mouse. I wouldn't have raised my hand if someone paid me. Now I know I may not always have the right answer, but I have the confidence to try. Also, if I do ask a question in class, and if everyone happened to laugh, I'd be like "Hey, so I'm clueless in math! Give me a little patience and help me understand!"

Ginger made great strides in confidence, but there is a big difference between having ideas judged bizarre and having them judged wrong. At worst the procedural knower may anticipate being considered a creative thinker. Ginger, a received knower, by contrast, imagined others laughing at her ignorance, in a manner reminiscent of her traumatic experiences in elementary school. She feared being wrong. Received knowers like Ginger take a bigger chance than procedural knowers do every time they speak in class. Received knowers' efforts to speak, to ask, and to try need more support than those of procedural knowers. Students like Ginger need to hear that each of their attempts to solve problems is important and a valuable step in their thinking process.

Ginger has a mental block that grows and shrinks. For her each new mathematical experience is a new crisis. Her confidence slowly grows, but she remains vulnerable. She needs coaxing, encouragement. She needs immediate validation that what she is doing is useful, valid, good and she should do more of it. What we validate in the beginning may be very primitive. She wants her teacher to hold her hand and help her.

Ann, too, was a student who took big risks when she spoke in class. Like Ginger, Ann also feared failure. As a subjective knower Ann had thoughts of her own creation, thoughts she knew were right because they felt right to her. Since that was her only criterion for judging what was right, all external evaluative processes seemed arbitrary. In thinking and speaking their thoughts, students like Ann run the risk that they will be judged wrong and will fail according to some rule system which they do not understand or which seems unfairly applied. Their fears about offering their ideas run deep, for to them, the rejection of an idea is like a rejection of oneself. Ann and students like her also need special support for sharing ideas and speaking in class. They challenge teachers to articulate and follow their own evaluative criteria while simultaneously appreciating students' individual, intuitive ways of understanding.

Ann was forced from her plan to be invisible in the classroom by a problem solving activity done in pairs. For Ann as for Ginger each new mathematical experience was a new crisis. She needs coaxing, encouragement. She needs validation, but that validation may be for something quite idiosyncratic. She needs safety and structure to push her on. She wants to be heard in a safe environment. She seems to be saying force me to talk, but give me safety.

These students' words come from journal writing. Journal writing with extensive response is quite powerful for many students, but these journals are labor intensive both for the teacher and for the student. We all have students in our introductory mathematics courses, in our liberal arts mathematics courses, in our service courses that have the views, concerns, fears, and epistemological stances of Jackie, Ginger, or Ann. These students need the same kinds of encouragement to develop their own mathematical ideas, to begin to value those ideas, and then to evaluate those ideas. They need to be challenged (forced) in a supportive environment. Jackie was encouraged to share her undeveloped ideas with me for my private response. As she came to see that environment as safe, she blossomed, approaching mathematical situations with creativity, depth and the support of mathematical theory. Ginger and Ann would wait for the answer to be given, but it was not. Ginger was "forced" to try because the question would be on the midterm exam. Ann was "forced" to participate because she had a partner. The challenge allowed them each to have one sweet experience of success, a first step toward active learning in mathematics.

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Topic Session B2

RESHUFFLING THE BAGGAGE

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ABSTRACT

Mathematics is commonly perceived to be a discipline dominated by clarity. The technical model of applying prescribed techniques to produce so-called right answers fosters this perception. Typically many short questions requiring minimal time represent the core of the mathematical learning experience in classrooms. Beneath this guise lurks the prerequisite dilemma. Considerable amounts of instruction and time are devoted to demonstrating sufficient mastery of a knowledge base so that one will be ready to start the next course. The development of thinking skills tends to be lost in this product-based pedagogy. Many results of this model are frightening. Consider the prospective elementary teacher who knows of no other mathematical ways or the university student who wonders why math was never difficult before that calculus course. These people may very well represent the next generation of mathematics educators. How about the student who doesn't realize that division can make quantities larger?

We need to acknowledge that our students bring along plenty of mathematical baggage. The way in which it has been packed often does not bode well for the development of mathematical learning. As educators, we are challenged to upset the neatly packed suitcases and promote learning through investigation, problem solving, and discourse. This topic group is intended as a forum for addressing this issue. One may wish to consider some of the following questions as starting points for discussion: How can we constructively integrate this prior experience into the development of an enhanced appreciation of mathematics? How might we move towards changing the learning environment in mathematics classrooms? How do such things as confusion and errors fit into the learning of mathematics? What sort of evaluative models would genuinely support a move towards a process-based curriculum? We have some ideas and look forward to hearing what you have to say.

DISCUSSION

The abstract provides a context for the discussion that follows. More precisely it represents an overview of someof the ideas that guided the discussion in London. The process of developing an abstract provided a challenge that took shape over time. Peter Taylor and myself were invited by Eric Muller to lead a topic group discussion. Upon considering the possibilities, a direction emerged. Perhaps this direction is most effectively captured in a message from Peter which read:

> I'm having trouble with this one. I feel that there's something quite wrong in the way we mathematicians teach—somehow our students remain strangers to our land, somehow they never feel at home, somehow when they come up to explain something they remain awkward and uneasy. I want much less material—somehow I am convinced that the scientific edifice will not come tumbling down if we do less and play more.

This concern is validated through much of my own experience. Many students would not realize that mathematics educators may spend hours on problems that they may not actually solve. Nor do they appreciate the reality that many mathematical problems are unsolved. This lack of awareness is not surprising when one reflects upon the most familiar forms of mathematics education. Typically the teacher orchestrates a relatively polished performance in which glitches are minimized and risks are compromised.

Compare this experience to a theatrical performance being presented to a public audience. The viewers understand that hours of rehearsals have preceded the actual performance before them. The assumption is that the rough edges have been cleaned prior to the showcase event. It is my own feeling that mathematics classes may come across this way to students. Whether this is good depends upon one's personal perspective. Considering the goals of the "performance", my sense is that it is not a model to which we should aspire.

Perhaps it is our own weakness that does not encourage us to entertain the concept of struggle in our own classes. It seems fitting here to offer some thoughts from Steve Brown, a teacher who has challenged me to learn more about my mathematical beliefs and biases. Brown (1993, p. 115) writes:

> While we as teachers may create a state of angst by hiding our legitimate mathematical and pedagogical confusions, we run a greater risk of misrepresenting the lived experience of inquiry when we discourage students from honouring their own doubts, ambivalences, disharmonies. No field of inquiry grows in the absence of perceived anomalies; neither does the individual develop in a non-problematic environment.

A critical point leaps out at me when I reflect upon the theatrical performance. The audience develops a minimal appreciation for the intricacies of the acting profession. What does it take to be an actor? Few people would return to the same performance over and over without wanting something more than the plot or the content.

How does this relate to a mathematics class? An effort to appreciate mathematical learning is diminished by only seeing the final performance. We need to challenge ourselves to bring the students behind the stage. It is there that they can really do mathematics. The remainder of this paper outlines some interesting possibilities for placing a different face on mathematical learning. They tend to take one of two forms: philosophical questions to consider or actual models that are familiar to me.

Peter Taylor uses a problem-based approach to teach first year calculus at Queen's University. The classroom experience focuses attention on the process of problem solving. A basic conceptual framework is presented in his book, *Calculus: The Analysis of Functions*. Taylor (1992) outlines how his approach differs from traditional calculus courses in that it places greater emphasis on mathematical modelling, qualitative analysis of functions, and process. Problems are selected that are intended to engage students in the mathematical process through group work, projects, writing, reading, and problem solving. It is particularly important to point out another aspect of the approach. "Covering" content is not the focus. Students play an active role in shaping how the course may unfold.

Problem selection is critical in any course which centres on problems. The role of the problems must be well understood. If their role is to stimulate investigation and discourse, then it follows that they should not lend themselves to simple answers. Many problems with unique exact solutions may prove to be inappropriate, though misconceptions or counterintuitive results may provide us with some striking exceptions. Surprises tend to motivate dialogue on their own. Avital and Barbeau (1991) identify lack of analysis, unbalanced perception, improper analogy, improper generalization and false symmetrical reasoning as possible sources of misguided intuition. They proceed to provide numerous examples that are well suited for discussion in the undergraduate mathematics classroom.

An alternative approach employs the students as problem posers. The idea of problem posing is elaborated upon by Brown and Walter (1990). One problem posing strategy known as "What-If-Not?" is quite practical to employ. The starting point may be very familiar. One example used by the authors is the Fibonacci Sequence. Characteristics are listed: For example, the first two terms are ones; successive terms are found by adding the preceding two terms. The what-if-not aspect invites us to explore what may happen if some of these characteristics are modified. Perhaps the operation of additon is replaced with subtraction. What about multiplication? Such simple investigations may lead to extraordinarily rich mathematical dialogue. Conjectures, patterns, and extensions abound.

Errors offer another rich source of mathematical learning. The errors may be direct examples from our own classes. Alternatively they may be curious truths that one would not be wise to generalize. Consider the fact that $3 + 1.5 = 3 \times 1.5$. One needs to be weary about concluding that the sum and product of two numbers are always equal. Trying to figure out when this property holds provides a wonderful starting point for investigation.

Mathematical definitions play a significant role in the teaching of our courses. Commonly these definitions are offered so that the ground rules are clearly laid out. It is these definitions that usually form the foundation for developing content in mathematics. Borasi (1993) makes a compelling case for broadening the net. She uses the idea of "taxigeometry" to reexamine the definition of a circle. Select a point (say an intersection of two city streets) on a square grid map. Mark off the points that are all a fixed number of blocks away as the taxi rides. It seems that this should satisfy the definition of a circle. Curiously enough, this circle has right angles! We may expand our horizons to allow such curiosities to evolve in our classrooms. The development of dialogue and questioning is fundamental to the development of mathematical researchers and community citizens.

Sometimes we need something placed in an awkward way to get us to rethink our basic assumptions. The routine may need a tune-up. As a teacher, it has been valuable to expose prospective elementary teachers to mathematics in different bases. The routine algorithms do not flow so smoothly. It calls basic mathematical concepts such as place value to the forefront. Indeed it challenges them to consider what it is like to confront multiplication or subtraction for the first time.

University of Waterloo's introductory algebra course offered a somewhat parallel experience in my studies. I fondly recall the course entitled "Classical Algebra". There were no vectors or matrices. Instead, terms like modulo arithmetic, the Euclidean algorithm, divisibility rules, and number systems were featured. It felt as if mathematics was being presented in an entirely different way than in high school. Whereas, the calculus course seemed to build directly upon (then) Grade 13 calculus, this algebra course appeared like a whole new avenue that had been off limits. I share this example because it seems important to consider travelling down entirely new roads at times. Similarly it is valuable for students to see mathematics presented in a different light. Modifying the context, the means, or the content may serve to broaden the horizon that much further.

The evaluation of mathematics is another critical piece of the big picture. We need to develop evaluative models within our courses (departments) that may support the pedagogical practices. An unconditional acceptance of final exams as end products frequently pose institutionalized obstacles to many initiatives. This is particularly true when performance on the multiple section common exam is equated with mathematical success. Projects may be incorporated into courses. For instance, poster presentations have been used in some of my classes. Students are required to prepare posters that they will preside over while classmates and interested others may ask questions. This idea is one of a wide range of activities that appears in a recent publication of the Canadian Forum for Education in Mathematics (Heinrich, 1995).

The title "Reshuffling the Baggage" acknowledges that the baggage may be neatly packed. It is reasonable to expect strong resistance from some students and colleagues for changes that others will readily embrace. The value of mathematical learning extends beyond a set of technical skills. Students need to know that there is much more to math than may meet the eye. Creating an awareness of this is a big step to take. The next challenge is to invite the rest of the performers, the students, behind the stage. That is where most mathematics takes place. Why not make your classroom part of the backstage?

ACKNOWLEDGEMENT

I wish to thank Peter Taylor for his contribution to the topic group. Peter played an active role in the development of the ideas. Unfortunately he was unable to attend the meeting in London.

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Topic Session C1

ENACTING A CHAOS THEORY CURRICULUM THROUGH COMPUTER INTERACTIONS

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This paper is a summary of a portion of my dissertation which is entitled "Human-Computer Enaction" (Barnes, 1994). The research focus of the dissertation is human-computer interaction. This field of study is often linked to cognitive science which sees as its central tenet a view of cognition as information processing. Some in cognitive science view computers as appropriate models of human thought processes. However, not all members of the cognitive science community are comfortable with this limited view. The scope of the field of cognitive science has failed to consider people interacting with people and their environment, peoples' personal history, and culture as aspects which might be worthy of inclusion in studies in this field (Norman, 1980; Putnam, 1989; Searle, 1992). My dissertation and more recent work responds to this call for a broader understanding of human-computer interaction through more qualitative methodologies.

The conceptual underpinnings of this work are based on what Varèla, Thompson, and Rosch (1991, p. 140) describe as "enaction." Let us consider the notion of computer interface to exemplify the enactivist stance. The interface may be considered a barrier between the user and the computer software, a frontier between the two in which both have vested interest, the place where the two parties communicate, or the place where the two coemerge. This last consideration is from an enactivist stance. From this stance the software is available to its user through the interface. The users' actions are occasioned through the interface. The user also acts upon/within the software through the interface. Thus they act upon each other and codetermine each other through the interface.

From the enactivist stance "the organism both initiates and is shaped by the environment" (Varela, Thompson, and Rosch, 1991, p. 174). Thus consideration of the environment in which the person takes action becomes important. Several computer-enhanced environments were studied in which high-school and university students engaged. These included: 1) an exploration of the properties of chaos theory (non-linear dynamics) through interaction with a computerized mathematical microworld, and 2) the use of an authoring system integrated with multimedia for the generation of students' creative expressions. The views of human-computer interaction emerged through descriptive excerpts drawn from audio-taped and video-taped sequences in which students interacted with the computer. Students' written and computer records also provided significant sources of data. The central research question for the dissertation was: What is the experience of computer usage and how can we conceptualize human-computer interactions within mathematical and educational contexts? The focus of one portion of the dissertation, the chaos theory study, was the enacting of a mathematically rich learning environment with high school students and computer software.

Study Overview - The Chaos Theory Context

A fuller description of the paper summarized here can also be found in Barnes (in press). In this study seven high-school students chose to be involved as an extension of their grade 11 mathematics class. Students attended 6 one-hour after-school sessions in six weeks in the computer laboratory. Students made records of their activities using a chart I provided. They recorded their input to the computer and its graphic or numeric output. They also noted observations, predictions and generalizations. Students also wrote responses to questions I posed at the end of each session. Video and audio tape of students' conversations and actions were also recorded. The 6 one hour sessions included:

<u>Day 1</u> Students' introduction to the concept of iteration.

Student explorations based upon an instructional simulation exploring interation of the function $f(x)=\sqrt{x}$.

Student explorations using the cobwebber computer program (Figure 1) which iterates the logistics equation $y = \mu x(1 - x)$, $0 < \mu \leq 4$.

<u>Day 2</u> What is a cobweb (the left side of Figure 1)?

<u>Day 3</u> What is the effect of μ in the logistics equation?

<u>Day 4</u> What is the effect of μ as it increases from 2 to 3.6?

<u>Day 5</u> Exploring Chaos.

<u>Day 6</u> Period doubling and Feigenbaum's number.

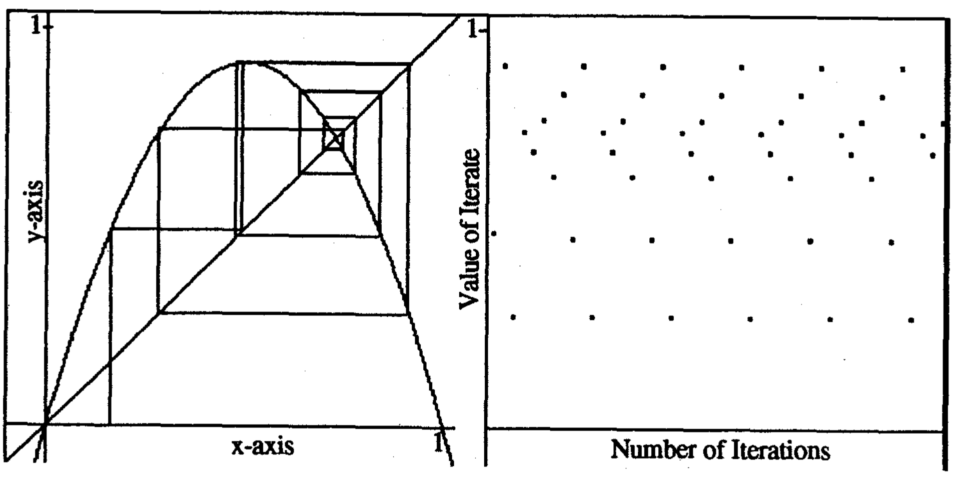

Figure 1

A Cobweb Graph (left) and an Iterate Graph (right) of the logistics equation for the function $f(x) = 3.66x(1 - x)$ in which an 8-Cycle is visible. These graphs were produced in the cobwebber computer program.

Discussing the Mathematical Activities

Students enacted a view of this mathematics which expanded beyond their interactions with the computer environment. They had not yet begun to determine what the mathematics might be beyond the confines of the software but had realized that they could think about this mathematics and perhaps even generate it. Students' generation of the bifurcation diagram involved extending their understandings beyond the visualizations available in the software and beyond their direct experiences with the mathematical software. Within their explorations they experienced mathematics by recording observations, creating diagrams, and producing generalized statements. They explored cycles through procedures they devised. Students realized that the computer generated functions approached the intersection of the parabola and the $y = x$ line. They solved a system of equations ($y = \mu x(1 - x)$ and $y = x$) by factoring the resulting expression.

Students did not see this mathematics microworld in a static unchanging way. They interacted with the cobwebber computer program and saw different patterns in their interactions as they continued to explore. Early in the study students used the cobwebber to explore chaotic functions. They drew the cobweb in exploring the procedure of cobwebbing. In later interactions with chaotic functions, students questioned patterns in chaos, noted the 'near' patterns in some cases, and the 'breaking' of these patterns. Early use of the software led students to question the graphics they saw and how they were produced. Later these graphics seemed to be understood. Students focused upon the kinds of patterns that were evident when exploring particular functions and in seeing different functions act similarly to other functions. Students may have seen the same graphics in both early and later explorations but how they perceived them was different. Through their interactions with the exploration environments, students generated changing understandings of the mathematics. These meanings were generated in the ever-changing context of interplay between students and exploration environment. Through the structure of the computer environment, it responded to students in predefined ways. It shaped students' understandings. However, students shaped the exploration environment by the choices of values and thus the explorations and the perceptions of the environment were brought forth by the students. The students' actions and the exploration environment codetermined each other.

Implications for Education and Research

Students in context codetermine the meaning of instruction. 'Information' in the context is codetermined by students in the setting.

> If the patterns revealed through this study characterizes human-computer interaction in this context then the role of the teacher, the student, the environment, the curriculum, and the computer changes. The effect of each is codetermined by the context in which activity takes place. The teacher's role changes from provider of information to facilitator of exploration becoming a learner and explorer with the students. The students' activities take on new importance as these are the ways that students experience and generate new meanings. The environment mediates the activities of students so it must be considered in educational decision making and educational research. The curriculum becomes that which is enacted by students immersed in a context (Barnes, in press).

New pedagogic concerns will undoubtedly emerge from student and teacher involvement in computer-enhanced contexts. These become places for further research.

Mathematical/computer research can focus upon students' explorations, activities and findings within mathematical environments (like Mathematica, the Internet, Cabris Geometry, etc.). Within a research agenda focused upon these kinds of environments we might question the environments and how they can be constructed so they are perceived by students to be elastic enough to engage in exploration while still simple enough to explore without being overwhelmed. Another question worthy of study seems to be the interplay between students' activities and the expansion of environments given these activities. This is a focus of my continuing research.

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Topic Session C2

WRITING TO LEARN HIGH SCHOOL MATHEMATICS AND CONCEPTUAL GROWTH

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INTRODUCTION

My thesis topic was inspired by the essays in Connolly and Vilardi (1989). These essays demonstrate how journal writing can be applied with mathematics students at all levels of schooling to achieve improved depth of understanding, to provide a means for improving equity of learning outcomes, and to relieve math anxiety (Tobias, 1989; Powell and López, 1988). In reading through the literature on writing-to-learn mathematics (WTLM), I was impressed by the notion that writing is *ipso facto* a form of problem solving (Kenyon, 1989). Connolly (1989) points out that when mathematics and science students write informal journals to discuss their thought processes, they are obliged to slow down their thought processes so that they deepen their own understanding through reflexive reflection. Powell and López (1988) point out that metacognitive and meta-affective writing are related to problem solving.

The works of Vygotsky are frequently referred to in the WTLM literature, particularly his notion of real conceptual learning. I draw the connection between Vygotsky's notion and WTLM in the Conceptual Framework section of my thesis:

> If we accept Vygotsky's notion that speech (oral, internal, written) is a "tool of thinking", i.e., a mental instrument that can be used to shape and transform cognitions, then it is a reasonable theoretical and pragmatic stance to consider the use of writing to learn mathematics as a thinking tool towards enhancing conceptual growth. Plausibly, the writing should serve to create understanding for the student because, as a tool, it is being used to mediate relational meaning (Confrey, 1993:26) through the act of manipulating concepts symbolically. Tool-using activity helps a wielder of a tool to understand both the purpose and nature of an object upon which the tool is acting and transforming through the interplay between action and reflection. In like fashion, a student writing about a mathematical concept mental tool-using activity is giving the concept an articulate form and the teacher's reaction in the journal dialogue serves to provide the vehicle for interactive reflection. Vygotsky wrote, "Our analysis accords symbolic activity a specific *organizing* function that penetrates the process of tool use and produces fundamentally new forms of behaviour" (cited in Confrey, 1993:27). Extrapolating this notion to an image of the high school mathematics learner, we may look forward to seeing that the use of writing as a thinking tool, in the domain of the student's zone of proximal development (ZPD), will begin as verbalization to influence another, i.e., the teacher, and gradually develop as a means of influencing the student's self. In the interactions of journal dialogue, the teacher seeks to create the conditions of a student-generated, powerful inner dialogue that leads to conceptual construction and conscious reflection that perfects it. (Doctorow, 1995:11-12)

VYGOTSKY'S NOTIONS OF REAL CONCEPTUAL KNOWLEDGE

The following is a point-form summary of the criteria laid down by Vygotsky (1986) for thinking that is characterized as real conceptual knowledge:

- entails coordination of systems of concepts and generalizing to form new concepts
- also entails metacognitive self-monitoring (echoed in Gardner (1991) and the constructivists)
- affects are assigned an important significance as triggers or engines of cognitive development
- Vygotsky argues that real conceptual knowledge attainment provides ready transferance across the curriculum.

In my thesis, I explain how Vygotsky provides a theoretical means by which one can measure advancement toward real conceptual knowledge:

> Vygotsky has constructed an index, called the zone of proximal development, which quantifies the improvement of conceptual learning by a student through interaction with a teacher. The ZPD was devised to provide a method of measuring intellectual progress through the intervention of formal instruction. It is the difference between the mental level of problem-solving possessed by an unaided student and the level of achievement of the student in solving the same problems through cooperation with a teacher. Consequently, Vygotsky proposes that the school curriculum should challenge students with problems that require her/him to go beyond what s/he can do alone through mere imitation. Working in the student's ZPD should lead to real conceptual growth. (Doctorow, 1995:28)

LITERATURE SEARCH

My literature search drew on related literatures in the areas of constructivism, hermeneutics, educational psychology in mathematics, journal writing in the classroom, Gardner's multiple intelligences, and social activism (active citizenship). Within the corpus of literature on WTLM, I looked at techniques that focused on math stories, formal and informal writing (Britton et al., 1975), and oral mathematical discourse (Pimm, 1987).

THE RESEARCH PROBLEM

The purpose of this study was to examine the relationship between reflective, informal writing about and for mathematics learning and cognitive growth toward real conceptual knowledge. How can Vygotsky's (1986) notion of conceptual development through working in the zone of proximal development using writing as a learning tool provide a sufficient theoretical link between mathematical thought and language about and for mathematics? The goal of this study is to help us gain insight into the following questions:

1. Does reflective, informal, dialogic journal writing by high school mathematics students improve conceptual knowledge in mathematics learning?
2. What criteria should be established for journal writing, e.g., frequency, focus, types of entries, for the purposes of establishing conceptual growth in mathematics learning?
3. What kinds of responses should a mathematics teacher be using in dialogic journal writing to

foster conceptual growth in mathematics learning in the high school student?

METHODOLOGY

I utilized a qualitative research approach in this study. Fischbein (1990) argues that quantitative psychological research has produced few results in practical mathematics education. He says that psychology has attempted as precisely as possible to isolate variables to get quantitative results, but this isolation has tended to be reductionist and applied to low-level cognitions and superficial features. He concludes that, therefore, there is a need to embark on qualitative research to combine precision with more pertinent problems.

The ethnographic character of the study was determined by my entering the study as a reflective practitioner (as both teacher and researcher of the students being studied). I was immersed in the culture of the setting that I was simultaneously studying.

I employed a case study analysis of three students (of the original four, one dropped out). My students were selected for variations in: gender, previous math achievement, previous writing achievement, and being "at risk".

> Case studies for qualitative research by their very nature can make no claim about population representativity either in terms of sample size or selection. However, contextual information is best elicited when effort is made to obtain some clear variety. One female and two male participants participated in the study to completion, while a second female departed too early to provide useful data. I sought to have represented in this group of students at least one who was comfortable with and adept at writing, at least one who seemed to dislike writing or found it difficult, and at least one who represented gradations between these poles. Of those who remained throughout the study, one had been consistently successful at school (Rachel), another (Ralph) had had a more checkered career yet retained a strong interest in school and in performing well enough to gain admission to university studies. The remaining one was a high risk student who had a history of dropping out of school (Mike). (Doctorow, 1995:48)

DESIGN AND COLLECTION OF DATA

Journal-writing approach

The study took place in the context of an alternative school from October, 1993 to May, 1994. The three students were studying mathematics at the OAC level (pre-university entrance maths for Ontario students — roughly grade 13 courses). In this alternative school environment, the students were provided with formal classroom instruction usually once a week for 90 minutes. They were given course outlines, textbooks and notes, and they were required to pursue their work at a self-determined paced, mostly independently. Normally, at this school, the students can avail themselves of individual, one-on-one consultations with their teacher — in this case, me. For the purpose of the study, I substituted oral consultations with interactive journal writing for all but a few interactions.

Writing was required to be carried on continuously and handed in on a weekly basis. The students were asked to comment on their learning and their feelings, understandings, insights, and difficulties as part of their normal routine. Such writing was submitted to me for commenting on the understanding that I would respond within 24 hours of submission (frequently within the same day). Student journal writing which has teacher responses is known as *dialogic journal writing.* At the three-quarter mark of the study, they were asked to utilize the "double-entry" (developed by Tobias) format to comment in writing on their understandings and feelings about their own progress. Teacher commentary was constrained in the sense that there were no evaluations made of the writing—either relating to grammar and spelling or to the quality of mathematical reasoning. Teacher comments were on the level of solutions, partial solutions, probes, suggestions, affirmative comments, enlargements, and questions.

As an incentive for doing this extra work, the students were excused from carrying out an independent study project normally assigned to each mathematics student for 10 per cent to 15 per cent of their overall mark. In lieu of receiving marks for independent study work, they were given a credit based on 15 per cent for participating in the study.

Data Collection

The following data items were collected in the course of the study:

1. Responses to a pre-treatment questionnaire from each student surveying their attitudes toward mathematics, their competencies and experiences in mathematics, and motivations toward mathematics.

2. Monthly interviews on audio-tape with each participant regarding observations, analysis of the work, and probes for unexpected or discrepant information.

3. A videotaped discussion with all the participants in April, 1994 in order to draw together reflections and ideas on observations and insights about the study and to generate new ideas.

4. Photocopies of all the student writings.

5. Photocopies of the students' academic records as far back as elementary school with a focus on their abilities and progress in writing and mathematics.

6. Analytic notes on the above data items created during the course of the study. Such notes included reflections and comments that came to mind at various points in the research process. Some observations were made in other contexts in which I encountered the students. For example, all of them were taking two or more courses with me, and I was able to draw on these other contexts for relevant data.

DATA ANALYSIS

A set of indicators of cognitive growth was developed (culled from criteria which surfaced in the literature search) and was used to help analyze the stages of cognitive development through which the students were proceeding. The quality of journal writing itself was placed within a typology. The data from the interviews, journals, tests, and exams were subjected to analysis via the indicators.

The WTLM and related literature are replete with affirmative references to Vygotsky's notions of real conceptual knowledge and examples of what kind of student learning activity demonstrate movement towards such knowledge. In the discussions, conceptual knowledge and growth are viewed from a variety of perspectives which often overlap. My effort in constructing indicators was not to eliminate these overlaps, but rather to be as comprehensive as possible, and to use the accumulated indicators as a means of triangulating the case study data to provide a useful and adequate means of assessing real cognitive growth.

Specifically, we are focused on evidence gathered from five sources: the students' journal writings; the students' tests and assignments; interviews between me and the students; school records of the students' attainments in math and English; and observations that I made as their classroom teacher. What follows is a categorization of various rationales and notions of what conceptual knowledge and development in mathematics looks like in the learner's behavior. The major indicators are briefly described followed by the sub-indicators and their descriptions as follows:

- Major indicator: description
  - Minor indicator: description

- *Meaning-clarification activity*: The student used journal writing to construct meaning for him/herself in order to clear the path for problem solving. The constructivists argue for the principle of meaning-making as essential to genuine problem solving. Pimm (1987) brings to our attention the problem of meaning-making within the framework of communicating in the "mathematics register". The linkage between meaning-clarification activity is explicitly made throughout the WTLM literature.

  - *Rewriting text/ideas*: the student makes a habit of rewriting in his/her own words.

  - *Defining and explaining concepts*: The student provides definitions and explanations, using correct vocabulary and accurate descriptions.

- *Powerful Conceptual Structures*: The student used journal writing to achieve integrative power of concepts and engages in metacognitive self-monitoring in solving mathematical problems.

  - *Integrative power*: The student constructs ideas that are internally consistent and that are effective concepts which entail the coordination and convergence of multiple forms and contexts of representation in his/her mathematical problem solving (Confrey, 1990). This idea is echoed in the conceptual framework of Gardner (1991) in which he argues for deep understanding.

- *WTLM is at a High Level of Communication*: The student functions: at the lowest level when s/he merely writes descriptive mathematics (unrelated thought); at the middle level when s/he summarizes his/her mathematical work (i.e., efforts towards conceptual construction are evident but the writing is not necessarily critically evaluated); at the highest level when s/he uses math as a thinking tool to communicate in a mathematical fashion with dialogue that is critical, reflective, and adaptive, i.e., in the student's ZPD (Waywood, 1992).

FINDINGS

Each student's data were tabulated into the above indicators scheme and a summary table of all three students was used to compare the outcomes. In addition a tabulation of the video-taped roundtable discussion responses was produced to draw the comparisons together.

The following indicators showed similar results for all three:

- *Meaning-Clarification Activity*—frequently engaged; *Catches Own Mistakes*—very good efforts resulting in growing success and improvements.

- *Displays Relational (Logical) Understanding of Mathematical Ideas*—showed considerable or very good improvement and success over time.

- *Uses Powerful Conceptual Structures*—demonstrated increasing ability to use powerful structures and to function both metacognitively and meta-affectively.

- *Vygotsky's Criteria for Real Conceptual Knowledge*—made major strides in working within their ZPD's to achieve significant gains in real conceptual knowledge.

The following indicators showed differences in results:

- *Assumes Responsibility for Own Learning*—Ralph, a bright student with a checkered school career, took considerable initiative but inconsistently. Rachel proved to be very diligent on a mostly consistent basis. Mike, our at risk student, was very inconsistent but did make great efforts at several critical points.

- *Asks Reflective Questions (Degree of Deliberate Metacognition)*—Ralph and Rachel displayed considerable effort and good results. Mike, on the other hand, made rare efforts in this respect.

- *Waywood's Writing Hierarchy*—Ralph and Rachel often wrote at the high levels of the writing hierarchy (summary, critical reflection). Mike most often wrote at the descriptive and summary levels (also engaged in critical reflection in his affective writing).

Relation to Previous Findings

The results of my research reflected a great deal of that which had been reported in the WTLM literature, namely the attainment of:

- critical thinking (Powell and López, 1988; Rose, 1989)
- alleviation of math anxiety (Tobias, 1989)
- student-teacher rapport (Connolly and Vilardi, 1989)
- feedback to the teacher for improved instruction (i.e., working within the students' ZPD)
- raising the students' levels in the writing hierarchy (Waywood,1992)
- transferance of improved cognitive abilities attained in mathematics through writing activity to other curricular areas (Tobias, 1989)
- successful intervention for high risk students (Powell and López, 1988).

Contribution to New Knowledge

The thesis study has contributed to new knowledge in the field of WTLM. In particular, I developed a set of cognitive indicators that are useful for assessment of students' journal writings in a classroom setting. This type of assessment is called for in the NCTM's new *Assessment Standards for School Mathematics* (1995). Moreover, the accounts in the study demonstrate techniques for teachers to reflect on how to work within a student's ZPD.

CONCLUSIONS

WTLM in its dialogic form is effective for student cognitive development and growth towards real conceptual knowledge. It is also useful for reflexive analysis for teachers toward the goals of pedagogic and curriculum change.

The study has three important limitations. The cognitive indicators need to be validated (both quantitatively and qualitatively). The sample representativity is limited by the nature of case studies. The frequency of journal writing for sufficient classroom-based assessment needs to be refined for practical classroom implementation.

RECOMMENDATIONS

The cognitive indicators need to be developed into better practical assessment instruments through the construction of appropriate rubrics (scales of attainment). Such an effort should be aligned with the NCTM's assessment standards.

The implementation of journal writing should be considered in focused contexts, in particular in conjunction with mathematics projects (group or individual) in order to promote the cognitive development process in a coordinated activity.

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Topic Session D1

USING ANALOGIES TO OVERCOME PROBABILITY MISCONCEPTIONS

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ABSTRACT

The existence of probability misconceptions at various levels and their resistance to change has been well-documented in previous research. Analogical reasoning has been successfully utilized by John Clement and associates in overcoming physics misconceptions by basing the knowledge reconstruction process on problems which draw out students' beliefs which are in agreement with accepted theory. Such problems are referred to as anchoring situations. A similar approach was attempted in this study in the area of probability.

Anchoring situations conceptually analogous to misconception-prone/target probability situations were generated. The target situations were placed in Version A of the WDYTTCA (What Do You Think The Chances Are?) instrument and the analogous anchoring situations were placed in Version B. The instrument was given to 41 senior high school students and 24 secondary mathematics student teachers. Version A revealed that probability misconceptions were common whereas Version B showed that anchors for overcoming these misconceptions could be generated. Follow-up interviews indicated that the anchoring situations could be effectively utilized in overcoming probability misconceptions when the participants were engaged in a process of analogical reasoning.

INTRODUCTION AND PURPOSE OF THE RESEARCH

Although the study of probability is highly relevant for understanding numerous everyday situations, it is also one of the topics in mathematics which is most prone to misconceptions (Shaughnessy, 1981; Jacobsen, 1989). Consequently, it becomes necessary to identify and understand these misconceptions, their sources, and how they can be overcome. This study attempted to provide further insights to these problems.

In particular, the purpose of the research was to investigate high school mathematics students' and secondary mathematics student teachers' conceptual understanding of situations involving probability for the ultimate purpose of overcoming the misconceptions which exist in this area. More specifically, the first objective was to generate anchoring probability situations which are conceptually isomorphic to misconception-prone/target situations. Anchoring situations are problems which are designed to draw out beliefs held by individuals which are in agreement with accepted theory and which are therefore expected to receive correct responses (Clement, 1987b). The misconception-prone/target situations used in this study were similar to those identified by numerous other researchers as frequently receiving incorrect responses thus indicating mathematically-incorrect concepts. The second objective was to utilize the anchoring situations which had been generated in attempts to overcome the probability misconceptions displayed in the target situations.

THE SIGNIFICANCE OF THE STUDY

Overcoming probability misconceptions is an essential component of acquiring a mathematically-correct understanding of probabilistic situations as they are studied in the classroom and as they occur in everyday situations. Consequently, finding a means of accomplishing this task is highly significant.

Attempts to overcome various mathematics misconceptions in general, and probability misconceptions in particular, have met with limited success. Clement (1987b), however, documented remarkable results in overcoming misconceptions in physics through the use of anchors and analogical reasoning.

The use of anchors and the analogies approach to overcoming probability misconceptions had not been attempted in previous research. Because of Clement's success in using analogies in physics, it seemed worthwhile to attempt his approach in the area of probability.

THEORETICAL BACKGROUND

Much of the original work on probability misconceptions was done by Kahneman and Tversky. They showed that these misconceptions are very common even amongst college students who have a statistics background. Two of the most common types of misconceptions are the result of relying on the representativeness heuristic (Kahneman and Tversky, 1972; Tversky and Kahneman, 1977) and the availability heuristic (Tversky and Kahneman, 1973). These heuristics are constructed by individuals as a general procedure for responding to various probabilistic situations. It is not appropriate, however, to apply the heuristics in many of these situations. For example, the representativeness heuristic misleads the student to believe that even a small sample should be representative of the population from which it is taken. Thus, if a coin is tossed a number of times and the result is a series of "heads," it is commonly believed that the probability of obtaining "tails" on subsequent tosses is greater than obtaining "heads" so that the sample will represent the theoretical population.

The approach which the present study utilized is closely related to the research conducted by Clement (1987a, 1987b, 1989) in the area of physics misconceptions. Clement describes a tutorial which uses the analogies approach to bridge the gap between anchoring examples which the students understand as illustrating a certain law or principle and the target examples which the students do not perceive as showing the same idea. Bridging examples which have some of the characteristics of the anchoring example and some of the characteristics of the target example are used as supports in conceptually bringing the two analogous examples closer together.

To illustrate, consider the common misconception, contrary to Newton's Laws, that a table does not exert a force on a book lying on it even though it is believed the book exerts a force on the table. The anchoring example which would likely activate the correct schema would be using your hand to push down on a spring. You are exerting a force downward on the spring but you can also feel the spring exerting a force upward on your hand. Clement (1987b) found that almost all students believed in the anchoring example that a spring can push back on one's hand, but many were still unconvinced that this is a valid analogy to the target example which is the book on the table. This is because the anchoring example is too far removed from the target example and so intermediate examples which serve as bridges must be given. These might be the book on the spring, then the book resting on a piece of foam, and then the book resting on a flexible plank. In this progression, we are gradually moving away from the anchor and towards the target.

In summary, the present research is based on five theoretical assumptions:

1) probability misconceptions are prevalent in students' knowledge structures;

2) students learn by constructing their own knowledge which often contains conflicting schemata -- some which are mathematically-correct and others which are faulty. Constructed knowledge, whether it is correct or faulty, is highly resistant to change;

3) knowledge construction must take into consideration an individual's prior knowledge. It is effective to begin the construction or reconstruction process with the mathematically-correct schemata the individual already possesses;

4) the specific representation of a problem is a key determinant in the student's ability to solve the problem. Some representations activate schemata which are appropriate for solving the problem whereas other representations activate schemata which are based on misconceptions;

5) students can be assisted in constructivist learning, particularly in overcoming misconceptions, through supportive frameworks such as a series of anchoring situations.

THE METHOD

The Instrument

A variety of situations similar to the ones extensively quoted in the literature as misconception-prone probability situations were matched with researcher-generated anchoring situations conceptually isomorphic to the misconception-prone situations. It was hypothesized that the anchoring situations were more likely to activate mathematically-correct schemata than the misconception-prone situations due to the various techniques which were utilised in generating the anchoring situations. These techniques included presenting the problem from a different perspective; utilizing concrete or familiar situations; changing the numerical quantities in order to present an extreme case, etc. Finally, ten misconception-prone situations were compiled as Version A of the WDYTTCA (What Do You Think The Chances Are?) instrument. Their ten analogous counterparts were placed in Version B of the WDYTTCA instrument. Questions regarding the likelihoods of events were posed in a multiple-choice format. Justifications for the likelihoods were requested in constructed-response format. Confidence lines were included so that the respondent could indicate his/her degree of confidence in the responses to the multiple-choice questions on a continuum from 0 to 3 with demarcations 0 (just a guess); 1 (not very confident); 2 (fairly confident); and 3 (I'm sure I'm right). An example of a question in the instrument with reasons for its inclusion is given below.

Question 3 (Version A)

Your sports team finishes first in its league at the end of the season and so you consider it is the best team. However you must compete in a playoff series against the second place team in the league to determine the champion. Would a 5 game series or a 9 game series give you a better chance of winning the championship, or doesn't it make any difference?

a) a 5 game series gives you a better chance;
b) a 9 game series gives you a better chance;
c) it makes no difference.

Those who rely on the representativeness heuristic believe that both the larger sample 9 game series and the smaller sample 5 game series will reflect the population (i.e., that your team is better) equally well in all cases. They will probably respond that it makes no difference disregarding the fact that the larger sample, the 9 game series, would be more likely to show which team is actually better.

Question 3 (Version B)

Your sports team finishes first in its league at the end of the season and so you consider it is the best team. However you must compete in a playoff series against the second place team in the league to determine the champion. Would a sudden-death 1 game playoff or a 5 game series give you a better chance of winning the championship, or doesn't it make any difference?

a) a sudden-death 1 game playoff gives you a better chance;
b) a 5 game series gives you a better chance;
c) it makes no difference.

Experience tells you that you do not win every game therefore a sudden-death 1 game playoff is very risky whereas a 5 game series gives you a better chance to show your talent. It is the extremity of the numerical quantity "1" in the 1 game series which makes the correct choice more obvious. Consequently, this situation provides an anchor for the statistically-correct concept that a larger sample is more likely to reflect the characteristics of the population from which it is taken than a smaller sample.

The Participants

The final version of the WDYTTCA instrument was tested with a group of 24 secondary mathematics student teachers and 41 senior high school students enrolled in a mathematics course. All participation was on a voluntary basis.

The Procedure

The participants began by first completing Version A of the WDYTTCA instrument. When they were finished they immediately went on to Version B. They were instructed not to go back to Version A after beginning Version B. The objective was to determine the effectiveness of the situations in Version B in eliciting correct responses and appropriate justifications especially in regard to those questions whose analogous counterparts in Version A had received incorrect responses and misconception-revealing justifications.

The interviews were conducted individually within a week of completing the written instrument. The purpose of the interview was firstly to establish whether or not the possible misconceptions which had been revealed in the WDYTTCA instrument were true misconceptions. Secondly, and most importantly, the interview attempted to determine the effectiveness of using analogies to overcome the probability misconceptions which had been revealed. The participants were presented with the situations in Version A of WDYTTCA to which they had responded incorrectly (possible misconceptions) when at the same time they had responded correctly and with confidence (anchors) to the analogous counterparts in Version B. They were engaged in a process of analogical reasoning by the interviewer in an attempt to guide the knowledge reconstruction process which hopefully would ultimately result in the participant changing incorrect responses to correct responses in the Version A situations. This change from an incorrect to a correct response combined with an indication of being at least "fairly confident" in the correct response was deemed as evidence of overcoming the misconception.

THE RESULTS

Potential Anchors

Of greatest interest were those situations in which an incorrect response was given in Version A indicating a possible misconception, and a correct response was given in the analogous question in Version B indicating a potential anchor. For the high school students, of the 232 possible misconceptions in Version A, 166 were matched with potential anchors in Version B. For the university students, of the 114 possible misconceptions in Version A, 90 were matched with potential anchors in Version B. These results provided evidence that the questions in Version B were serving their intended purpose.

Anchors

The students were also asked to indicate their confidence levels in their answers. When a student indicated that he/she was at least fairly confident (a rating of 2 or greater) in a situation deemed as a potential anchor, then this situation was referred to as a true anchor or simply an anchor for that individual. Using this definition, 113 of the 166 potential anchors could be referred to as anchors for the high school students. Thus the overall Instrument Anchoring Rate (anchors / possible misconceptions) for the high school students was 0.49. For the university students, 74 of the 90 potential anchors could be referred to as anchors resulting in an overall Instrument Anchoring Rate of 0.65.

Interviews

Seventeen of the 41 high school students volunteered to be interviewed. Each volunteer was interviewed regarding those situations where anchors were produced in Version B for the possible misconceptions which had been revealed in Version A. The results were categorized as Success, Partial Success, or Unsuccessful. For analysis purposes, these categories were assigned scores of 1.0, 0.5, and 0.0 respectively. Of the 46 situations attempted, 24 were categorized as successes, 18 as partial successes, and 4 as unsuccessful. This resulted in an overall Interview Success Rate of 0.72 for the high school students.

Fifteen of the 24 university students were interviewed. Of the 50 situations attempted, 30 were categorised as "Successes," 15 as "Partial Successes," and 5 as "Unsuccessful." This resulted in an overall Interview Success Rate of 0.75 for the university students.

CONCLUSIONS

The results of this study showed that anchors can be generated for misconception-prone/target probability situations. The results also showed that the analogical approach for overcoming probability misconceptions is indeed quite effective. In view of the great difficulty in overcoming misconceptions as indicated by previous research, these achievements are quite remarkable.

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Topic Session D2

THE EFFICACY OF AN ELEMENTARY MATHEMATICS METHODS COURSE IN CHANGING PRESERVICE ELEMENTARY TEACHERS' MATHEMATICS ANXIETY

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INTRODUCTION

The purpose of this study was two fold. The first purpose was to further an understanding of mathematics anxiety by placing it within a psychological model. The second purpose was to determine if and how an intervention for mathematics anxiety implemented within an elementary mathematics methods course, could reduce levels of mathematics anxiety.

Mathematics anxiety has been under investigation for several years and yet there is still a lack of understanding about how it arises and why it persists once it has arisen. Mathematics anxiety continues to be a research concern because of the perceptions that it threatens both achievement and participation in mathematics.

It was found that Bandura's (1986) social cognitive theory provided an explanation for the development, perpetuation and reduction of mathematics anxiety. Mathematics anxiety is a negative emotional and cognitive reaction to mathematics which interferes with a person's ability to do mathematical tasks. The development and perpetuation of mathematics anxiety may be explained through an understanding of social cognitive theory. People evaluate situations cognitively and then respond. Individuals are not automatically shaped by what happens around or to them, but rather, their response depends upon how the situation is interpreted by them. It is this interaction of cognition, behaviour and environment that shapes human behaviour.

A social cognitive perspective hypothesizes that it is the perceived inefficacy to cope with potentially averse events that makes them fearsome. It is this feeling of inefficacy that permits anxiety to develop and remain. Highly anxious individuals tend to internalize events and self-depreciate over stressful situations which takes their focus off the task at hand and results in misinterpreting or neglecting information. Anxiety arousal may result in a narrowing of attention. People who believe that they cannot perform a certain task dwell upon their coping deficiencies, magnify the severity of possible threats, and worry about situations that may never happen. Mathematics anxiety appears to develop through such negative cognitive distress. After a negative experience with mathematics, some people may develop mathematics anxiety by continuing to anticipate that they will do poorly; they focus on such negative thoughts as "I can't do this", "I am stupid", or "Why am I bothering, I never get these right." The amount of attention that is given to these negative thoughts diverts the amount of attention that can be focused on the problem at hand. By dwelling on these negative thoughts or on their perceived inefficacy to solve the problem, individuals will not be able to focus on the problem, will not be able to solve the problem, and will generate more anxiety by proving to themselves that these negative thoughts must be correct. There is only a finite amount of attention that can be allocated to a task. The amount of attention focused in one area limits the amount that can be focused in another. If anxious individuals dwell on internal thoughts such as "I'll never get this," there is not much attention left to be focused on the problem at hand.

The experiences that were recounted by the preservice elementary teachers as contributing to their mathematics anxiety are consistent with such a social cognitive explanation of anxiety arousal through perceived inefficacy. The preservice elementary teachers "knew" that they were going to do poorly in any class that involved mathematics. Mathematics anxious individuals debilitate their efforts by self-doubting and other self-defeating ideations. This self-inefficacy promotes a cycle of inadequate mathematics learning and anxiety because individuals do not allow themselves to be placed in situations where they might be required to do mathematics. The observation of negative behaviour towards mathematics by models in positions of trust and authority, such as elementary school teachers who possess mathematics anxiety, may also promote the development of mathematics anxiety. It is the cognitive interpretations of such behaviour and the environment in which it occurs that cause mathematics anxiety to develop and remain.

METHODOLOGY

The study examined three concurrent years of an elementary mathematics methods course and utilized an institutional recurrent cycle research design as the basis for collecting information on mathematics anxiety and attitudes toward mathematics. Questionnaires were administered at the beginning and end of the elementary mathematics methods course. Interviews and observations were conducted concurrently with the elementary mathematics methods course during the second year of the study.

Each year of the study involved preservice elementary teachers enrolled in the first year of a two year post undergraduate degree Bachelor of Education program. In total 112 (18 male, 94 female) preservice elementary teachers participated over the three years of the study. The questionnaire data were collected from six different administrations of the Fennema-Sherman mathematics scales, two pretests and four posttests. There were 163 questionnaires administered and completed during the research. The interview data were collected during the second year of the study. Of the 31 preservice elementary teachers participating in the second year of the study, 13 volunteered to also participate in a series of interviews while taking the methods course. These individuals were each interviewed three times during the course: at the beginning, in the middle and near the end. The professor who taught the course was also interviewed.

There appeared to be for the preservice elementary teachers a relationship between lower mathematics anxiety and improved confidence in learning mathematics. The preservice elementary teachers did reduce their mathematics anxiety levels after completing the elementary mathematics methods course. At the beginning of the methods course the preservice teachers were anxious about doing mathematics and were not looking forward to having to take a course "in mathematics." For the majority of the preservice elementary teachers their experiences with mathematics had been negative ones. Mathematics was disliked as a subject and consequently many of them had avoided situations that involved mathematics. Most of the preservice elementary teachers were able to explain how they thought their own mathematics anxiety had arisen. Some of the major explanations given for mathematics anxiety were:

- unsuccessful attempts to learn mathematical concepts and subsequent failures,
- timed mathematical experiences,
- mathematics used as a punishment for misbehaviour, and
- the way mathematics had been taught, e.g., repetitive drills.

As the course progressed and as it neared completion the preservice elementary teachers reported feeling less anxious about mathematics and more confident in their ability to learn mathematics. All sources of data, questionnaires, interviews, and observations indicated that there was a reduction in the mathematics anxiety levels of the preservice elementary teachers after completing the mathematics methods course. Improved self-efficacy was necessary for the reduction of anxiety to take place. The fact that an improved confidence in learning mathematics was required before the preservice elementary teachers could reduce their mathematics anxiety supports the placement of mathematics anxiety within a social cognitive perspective.

At the beginning of the mathematics methods course, the preservice elementary teachers had very low confidence levels in their ability to learn mathematics. They were also not confident in their ability to "do" mathematics, and saw the mathematics methods course as another "mathematics" course that they had to take. But as the course progressed and neared completion, the preservice elementary teachers were able to improve their level of confidence in their ability to learn mathematics. What remained a concern though for some of the preservice elementary teachers, was their ability to be able to teach mathematics. So while they reported feeling confident in their ability to learn mathematics, there was not that same level of confidence in their ability to teach mathematics, especially at the upper elementary grades.

PERCEPTIONS TOWARD ONE AS A LEARNER OF MATHEMATICS

The results concerning the perception of the teacher toward one as a learner of mathematics were not conclusive. There may have been some confusion for the preservice elementary teachers in answering this scale since they have had so many teachers over the years. But generally the preservice elementary teachers' remembrances of past teachers were not positive, in that they did not think their public school teachers had perceived them as successful learners of mathematics. All of the posttest scores did show improvement in the perception of the teacher toward one as a learner of mathematics, but the level of improvement was not statistically significant in the majority of the instances. Their improved confidence in learning mathematics and reduced mathematics anxiety might have allowed the preservice teachers to believe that the professor teaching the methods course did perceive them as successful learners of mathematics.

Another dimension of an attitude toward mathematics was the perception of mathematics as a male domain. The results from the interview data and the questionnaire data were not consistent. While the results from the questionnaire data indicated that the preservice elementary teachers did not perceive mathematics as belonging in the male domain, the results from the interview data indicated some definite ideas towards mathematics and gender. When asked to describe someone who was good at mathematics, the preservice elementary teachers invariably described someone with what has been traditionally labelled masculine qualities, or even stated a "he." The further feeling was that careers that required individuals to be successful in mathematics would be "dry and boring." The preservice elementary teachers also indicated that they possessed strong stereotypical beliefs about male and female roles and career choices. The general attitude seemed to be that elementary school teaching was a feminine pursuit, and that males were more suited for secondary or specialist teaching.

For the dimension of attitude toward success in mathematics, the preservice elementary teachers do desire to be successful in mathematics, but the nature of success to these individuals may be quite different from the level of success as intended by the Fennema-Sherman Scale. The preservice elementary teachers want to be successful enough in mathematics to feel confident about teaching it at the elementary level. All of the preservice elementary teachers do want to know how to "do" mathematics, yet they do not consider themselves as "successful" in mathematics if they only desire to teach mathematics at an elementary school level.

REDUCTION OF MATHEMATICS ANXIETY

The preservice elementary teachers were able to identify those components of the elementary mathematics methods course that aided them in reducing their mathematics anxiety. The components that enabled the preservice elementary teachers to feel more confident in their ability to do mathematics and actually showed them how to successfully learn and do mathematical tasks, were the same components that reduced mathematics anxiety.

1. Acknowledgement of Mathematics Anxiety as Real

   It was important for the preservice elementary teachers to realize that they were not alone with their feelings and anxieties towards mathematics. It is impossible to reduce any anxiety until you are aware of what it is, and you acknowledge that you possess such anxiety.

2. Utilization of Small Group Work

   Working in small groups enabled the preservice teachers to gain confidence in their ability to do mathematics. By working in small groups some of the preservice teachers were able to solve problems successfully for the first time. The emphasis is on cooperation rather than competition.

3. Use of Concrete Materials as Manipulatives for Mathematics

   By using concrete materials to learn mathematical concepts, the preservice teachers were able to understand mathematics rather than rely on the memorization of rules. It was not just the use of concrete materials but rather the method in which they were presented as well that enabled the preservice teachers to reduce their anxiety. The preservice teachers were able to experience learning mathematics in a fun and enjoyable manner.

4. Videotapes

   The use of videotapes provided an opportunity to see the modelling of appropriate teaching behavior. The videotapes also provided a means for the preservice teachers to practice working with concrete materials while observing an actual elementary class also learning the same mathematical concepts.

5. Remediation of Mathematics Skills

   There must be some provision made for the preservice teachers to learn how to do mathematics. The use of content mastery tests may not be the best avenue through which to remediate basic skills since they tended to create anxiety in the short term. The preservice teachers did express the need to learn mathematical concepts as a necessary component in the reduction of anxiety.

6. Modelling

   The professor herself was an important component in the reduction of mathematics anxiety. By modelling appropriate behavior and teaching strategies the professor really "practiced what she was promoting" to reduce anxiety. The preservice teachers also observed behaviour of the teachers in the videotapes, and by teachers in classrooms that they observed.

CONCLUSIONS

First, increasing self-efficacy expectations are important in the reduction of mathematics anxiety. Teachers of mathematics should pay as much attention to students' self-evaluations of competence as to actual performance, especially for girls, since girls tend to have lower perceptions of mathematics self-efficacy. Educators must not only assess these self-evaluations, but must also be prepared to modify these self-efficacy expectations. Continued and strong steps need to be taken to encourage women to participate in mathematical courses.

Second, an appropriate site of intervention for mathematics anxiety is within elementary mathematics methods courses for preservice elementary teachers. Teachers need to be free of mathematics anxiety themselves before their students can also be free. Such methods courses should include components that will improve/modify the self-efficacy expectations of preservice teachers toward mathematics. This study showed that influences on self-efficacy expectations are: performance accomplishment; vicarious learning; encouragement, support and information; modification of self-efficacy expectations and reduction of anxiety.

The reduction of mathematics anxiety as designed by this methods course included the modification of self-efficacy expectations. The professor was a positive female role model and was conscious of modelling behavior that would provide encouragement and support for the members of the class in mathematical situations. Mathematical performance was improved through the remediation of mathematics skills and through the use of concrete materials especially in small groups.

Thus the use of small groups to promote a risk taking atmosphere, the use of concrete materials to learn and understand mathematical concepts, vicarious learning through positive role models (real or symbolic), the acknowledgement of mathematics anxiety as real, and the remediation of mathematics skills, are important features of an intervention designed to reduce mathematics anxiety and improve self-efficacy expectations toward mathematics of preservice elementary teachers. Further research is necessary in the area of gender and career choice. Furthermore, gender awareness should be included as a topic in all teacher education programs.

Mathematics anxiety is real and needs to be reduced and prevented. Through the reduction of preservice teachers' own mathematics anxiety, and the learning of strategies to teach mathematics in a positive environment, mathematics anxiety may be prevented in future individuals.

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Topic Session D3

LISTENING TO REASON: AN INQUIRY IN MATHEMATICS TEACHING—A REPORT OF MY DOCTORAL RESEARCH

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From the first months of my doctoral studies, I began visiting different mathematics class rooms and audio-recording a number of lessons. My research interest originally lay in developing a better sense of the interactive structure of mathematics classes—or, more specifically, in examining the questions and answers, the topics of discussion, and the patterns of interaction in different settings.

When I first began these activities, I had no real sense of how I would use, analyze or interpret the verbal interchanges that I had recorded (and that I was spending much of my life transcribing). And then, one evening, as I was replaying the latest of the tapes, a colleague from English education stopped by. After attending for only a moment, he demanded that I turn off the machine ... and when I did so, he casually assumed the role of the teacher and proceeded to "re-enact" the unheard remainder of the lesson. He did so with an uncanny accuracy, imitating not just the structure and the rhythm of the lesson, but also capturing the voice, the manner, and the bodily aspect of the teacher.

At first I found his parody a bit shocking. How could this person, two decades removed from his school mathematics experience, present such a precise impression—and cutting critique—of this phenomenon? A little thought, however, brought me to the realization that all he was doing was announcing his familiarity with what might be called "typical" or "conventional" mathematics teaching. That is to say, there is a broad familiarity with mathematics teaching within our culture, albeit largely on an unformulated level. Everybody *knows* what mathematics teaching *is*—everybody, that is, who can recall that part of their youth spent at a desk that was not designed for an active body, hovering over a set of "exercises" that were not constructed for an active mind, sitting in a room that was intended for surveillance rather than for interaction.

The incident with my English education colleague prompted a critical re-orientation to my research. To that point, I was struggling to characterize some of the "realities" of the mathematics classroom; I sought to define and to reduce the phenomenon of mathematics teaching, and I might have described what I was about as attempting to answer the question, What *is* mathematics teaching?

But my friend made it clear that we already know the answer to that question. It is part of each of our histories. That is why, when I announce the question that oriented the research reported in my dissertation—that is, What *might* mathematics teaching be? —the response is often one of surprise: Don't we already know that?

I don't think we do. Moreover, in our never-ending quest for reductive certainty, it is not easy for us to appreciate the importance of a question that is framed around the tentativeness of "might". We want to know what something *is*, not to waste our time exploring alternatives and possibilities that lie outside our current interpretive frames.

In effect, my study was an investigation into an alternative to popular conceptions of mathematics teaching. It begins with the premise that, in spite of its seemingly sedimented and static qualities, we do not (and can not) know the precise nature of mathematics teaching. And even if we could provide an unambiguous account that is true for some moment in history, it would profit us little. Both mathematics *and* teaching are moving forms: mathematics history reveals it to be anything but the epitome of fixed knowledge; the place of teaching and the role of teachers have never been issues of widespread agreement. To suggest that we can know once-and-for-all what they are is to succumb to the pervasive and thoroughly modern tendency to define ... and in that definition to reduce ... and in that reduction to mechanize ... and in that mechanization to lose what is fundamentally human about the endeavor.

This is just as it seems we have done in countless mathematics classrooms, where the seating arrangements and the patterns of interaction reflect the same impersonal and rigid structures that are embodied in a fragmented and objectified conception of the subject matter.

A major part of my study has involved trying to understand how it is that the highly structured, rigidly-controlled, quiz-on-Friday, think-as-I-think conception of mathematics teaching has become so pervasive in the modern setting. What assumptions are being made? Which "theories" or "orientations" are being enacted by students, teachers, and teacher educators? Simply put, I am not interested in knowing how well we're doing whatever it is we're doing, nor am I eager to do it better. I am trying to understand what we might be up to in the first place.
And to inquire into alternatives.

THE STRUCTURE OF MY WORK

An inquiry into mathematics teaching must occur at the meeting place of several debates—including those that deal with the nature of mathematical knowledge, the place of formal education in our culture, the processes of cognition, and conceptions of what it is to teach and to communicate. I have structured my writing around investigations of these four dynamically intertwining issues. In a nutshell, what I have attempted to do is to characterize the current debates by first identifying the varied perspectives that are represented and then by attempting to uncover their origins. This analysis serves as a starting place for proposing alternative patterns of thinking and acting.

Regarding the nature of mathematical knowledge, for example, the current debate seems to be focused around trying to resolve whether the truths of mathematics are discovered or created—that is, are they pre-given, external, objectively real, or are they the imaginative creations of subjective consciousnesses? On the topic of formal education, is it a vehicle of transmission that is placed between past and future, or does it create its own place as a critical and transformative enterprise, affecting culture and individual alike? Regarding cognition, is thought a matter of developing accurate representations of an external real world, or is it about creating viable theories of that world? And as for teaching, is that a matter of disseminating truths or of facilitating the construction of personal theories?

By way of foreshadowing, I believe that each of these tensions is illusory. Put differently, the "opposing" perspectives that give rise to the tensions are really more alike than different, for they are founded on the same objective-subjective mode of thought that can be traced back through Descartes to many ancient Western civilizations. As such, there is little hope for resolving them—and little purpose in attempting to do so. What we have to do is delve into the origins of this sort of dualistic thought and then to seek out alternatives that do not begin with the same premises.

The four issues that I have identified as being subsidiary to our understandings of mathematics teaching are precisely the topics that we have endeavored to include in our pre-service teaching programs. The nature of mathematical knowledge is the realm of the subject area specialists and it is to be covered by the professors in the Faculty of Science. The place of formal education is the domain of the Foundations people; the processes of learning belong to Educational Psychology; the role of the teacher falls on the plate of those interested in curriculum and instruction. Education students are led through their fragmented programs and, as we know, many (if not most) have difficulty bringing together the diverse topics into understandings of teaching that differ much from the models presented to them in their grade eight mathematics classes.

In my discussions with education students—both within courses I've taught and in their practicum settings—it is clear that, while the topics have been addressed, it has been at an uncritical level. Most obviously, the interpretive frames brought to discussions of knowledge, education, and cognition have often been taken-for granted: unspecified, unnoticed, enacted. A pervasive example of this uncritical attitude is our frequent use of computer metaphors to describe the brain. This line of thought has become so entrenched that many have forgotten it is only a metaphor—that is, the brain is certainly not a computer. Our forgetting has led to all manner of assumptions and has had a profound impact on our thinking about thinking. Just try to talk about cognition without using computer-inspired terms such as "processing," "input/output," "storage/retrieval," and "computational power."

My study has focused on excavating these sorts of details—the metaphors that have somehow become literal, and in that literalness have formed the basis of our action. In terms of my direct work with practicing and pre-service teachers, this has taken the form of articulating the unarticulated—that is, giving form to the unformulated "theories" that we announce in our un-reflective speech and that we enact in our moment-to-moment conduct. That is our starting place for altering those forms along with the patterns of acting that they support.

THE RESEARCH ORIENTATION

My research has thus drawn heavily from the traditions of hermeneutics, a field of inquiry that is concerned with interpretation. In its conception as a discipline, it was oriented toward excavating the Truth of sacred, historical, and legal texts. It has, however, now shed this quest for unchanging certainties; its object of analysis has broadened from the written text to the text of our lived experience. One might say that hermeneutics is currently concerned with identifying those historical, contextual, and linguistic conditions that make particular understandings possible. It asks not just, What is it that we believe?, but, How is it that we came to think and act in the ways we do?

It is important to note that the hermeneutic inquiry is not a solitary endeavor. It is founded on the premise that knowledge and truth, although temporally and contextually specific, emerge in the relationships that we have with one another. There are thus two dynamically intertwining strands to the hermeneutic inquiry: first, an exploration into the broader historical and cultural conditions that frame our conceptions and, second, a participation in the shaping of those conceptions amid the immediate contingencies of our day-to-day living with others.

The hermeneutic inquiry occurs from within the phenomenon it investigates. That is to say, we cannot step outside of our experiences and understandings of mathematics teaching as we endeavor to study it. The place to ask about the phenomenon is not in a philosophy class, but within one's own mathematical, learning, and teaching experiences. Moreover, the hermeneut cannot maintain the detached objectivity of the scientist, for we are thoroughly implicated in the social phenomena we study. The essential point here is that in the simple act of investigating mathematics teaching, I and my collaborators in research are helping to shape the phenomenon. As our ideas change, our actions change. What mathematics teaching *was* when we began our study is not the same as what it *is* now, whatever that might be. Perhaps another way of saying this is that hermeneutic research is educational research—that is, research that educates. This also suggests an ethical dimension to this sort of inquiry; it makes particular transformative demands on the participants.

My study, then, did not—and could not—follow a scientific or quasi-scientific format. (Before I go any further, I wish to make clear that I have no brook with the application of scientific methodologies to studies in education. The scientific method simply did not suit the issues I sought to address. I tested no hypothesis; I gathered no empirical data; I reached no conclusions.) As a result, my dissertation does not follow the structure of a formal research report. Rather, it is more along the lines of a *speculative essay*; and instead of marking the endpoint to an inquiry, it announces a beginning.

One Strand of the Writing

What I would like to do is to provide a better sense of the "alternative" to mathematics teaching that I've used to structure my work. As you may have guessed, my research does not lend itself to quick summary, so I've decided to focus specifically on one of the four strands—that is, the nature of teaching—and to sketch it out in slightly greater detail. (For those of you wondering about the title of the dissertation, I might mention here that my current project is to move toward an understanding of teaching that is framed in terms of "listening.")

To begin, the model of teaching that is enacted in most mathematics classrooms, I believe, might be succinctly characterized to be about "telling." The pedagogic concerns are with dissecting knowledge into suitably-sized 45-minute gulps, for appropriately sequencing these info-bits, for delivering them in clear and unambiguous terms, and for ensuring that some pre-specified level of mastery is attained. If "meaning" is an issue at all, it is something that is thought to come *after* (not prior to or during) one's acquisition of mathematics concepts. In this very *telling* approach, the teacher often tries to anticipate every possible contingency and to explain away problems before they arise. Among the *telling* signs of such teaching are the physical and relational distances between teacher and learner, between one learner and another, and between all present and the subject matter.

There have been many critiques of this "transmission" model, so I won't beat it to death. What I would like to emphasize is that the teachers who teach this way are not thoughtless, unreflective, insensitive persons. They are merely enacting what they know, engaged in the patterns of acting that are implicit in the popular—albeit unformulated and uncritical—conception of mathematics teaching.

Enter constructivism with its condemning critique of transmission models of teaching and communication. Because Given that I have virtually no control over the sense you are making—your meanings are determined by your own histories and biological constitutions as prompted, but *not* as caused, by my words—then my efforts to teach you cannot be about telling. Teaching, in this frame, is more toward "facilitating" and "orchestrating experiences."

But there are some problems here. Constructivism is an inadequate basis on which to re-structure our teaching actions. Ernst von Glasersfeld said as much at the 7th International Congress of Mathematics Educators in Quebec City. To paraphrase him (because I cannot find the precise quote): Constructivism is not a philosophy because it does not have an ontology [that is, it does not address questions of existence, identity, and our relationships with others]. So it cannot tell us how to teach. At best it can tell us what we, as teachers, cannot do.

When he made that statement, I believe it was directed at those persons who have attempted to erect an entire philosophy of mathematics education on the insights of constructivism. Von Glasersfeld was, in effect, saying that we can't do that because constructivism deliberately limits its scope to the matter of individual cognition—which, while critical to any discussion of mathematics teaching, is simply not enough.

There are other difficulties arising from constructivism as well. It shares, for example, many of the modernist assumptions as the "acquisition" or "representation" models of cognition that it has supplanted. For example, the separation of cognizing agent and the pre-given world is implicit. These are distinct, and the individual learner is isolated, insulated, and autonomous. Moreover, when it comes to teaching, constructivism just can't help us much with such phenomena as collective knowledge, our tremendous communicative capacities, or the moral and ethical dimension of the educational endeavor.

Please note that I am not saying that constructivism is wrong or misguided. It just can't inform teaching the way that we might want it to. I am thus saying that we need to move beyond discussions of how people learn—the realm of constructivism—if we are going to have any effect on classroom practice.

A Middle Way

In my own work, the theoretical and philosophical underpinnings are drawn from several areas, including cognitive psychology (and the contributions of constructivist thought figure prominently here), continental philosophy, biology, ecological thought, and literary theory. Growing numbers of thinkers in these fields, true to the prevailing spirit of post-modern criticism, are offering some serious challenges to the analytic, reductionist, and totalizing—in a word, *modern*—discourses that we so commonly associate with the work of René Descartes. More importantly, they are offering alternatives that start with the dynamic and intertwining evolutionary metaphors of Charles Darwin, thus focusing on interconnections—not the separations—of individual and environment, self and other, knowledge and knower. In addition, they do away with the very control-fetishing, goal-oriented understandings of human agency that pervade modernist endeavors. This point is critical as we approach discussions of teaching, for it challenges one of the defining features of modern education: the desire to predetermine learning outcomes. Because we have traditionally taken-for-granted that we can *cause* particular learnings to occur, the recent shift among theorists from "teaching as *telling*" to "teaching as *facilitating*" has amounted to little more than a renewed attempt to prescribe and to control the thinking that happens in a classroom. That is, in spite of the insights offered by constructivist theorists, recent developments have served to bolster the modern desire to dictate outcomes, although the recommended means of achieving such ends are now framed in indirect rather than direct terms.

Enactivist theory challenges these practices on several levels. To begin, we simply cannot select a learning objective and then proceed to map out—often in absence of learners—the best and quickest route to reaching it. Engaging in such practices amounts to trying to predict and to control the future. An individual's learning, like a species' evolution, is not smooth and cumulative, nor is it linear and directed toward a particular goal. It is thus that the notion of "curriculum," for the enactivist, is better thought of in terms of a path that is laid while walking. It is a path that—while it might (and should) be anticipated—can only be described in retrospect, rather that one for which every footfall is predetermined (as many teachers and texts try to do).

Another level to the enactivist challenge is to point to the dynamic and mutually-affective nature of any interaction. In popular conceptions of education, we tend to think of learning as unidirectional, occurring as the mutable learner "adapts" to the pre-given and static world. But enactivists argue that this conception is founded on a troublesome separation of organism and environment, and suggest that the environment is in fact "brought forth" by the learner. This *does not mean* that the learner is creating a subjective reality; it points to, rather, the fact that in learning and acting, we become complicit in shaping the world. It is not static and pre-given. With an event so mundane as a thought changing, the universe changes because that thought is not just located in the universe, it is a part of the universe.

The ontological status of teaching is hinted at here. We are not merely "transmitting" a pre-interpreted world. We are participating in its transformation—always and inevitably. And the persons we are teaching, like ourselves, are not static creatures who maintain some mysterious integrity we call "identity" or "self" through their learning experiences. They are, rather, shaped by such experiences. When we say we are teaching, then, we are announcing that we are participating in a profoundly moral endeavor.

Enactivist theory also offers a challenge to the popular conception of subject matter. In the classroom, there is an almost exclusive focus on *formulated knowledge*. Charles Taylor draws a provocative distinction between formulated knowledge (that which we put into words as we narrate our experiences) and unformulated knowledge (that which we enact every moment, and that which allows us to converse freely and effectively without considering the meanings of the words we use). He argues that most of our knowledge, while it may have once been formulated, has slipped into unformulation. It has become part of the way we act—part of our enactment of ourselves. In terms of formal education, it is easy to argue that this unformulated knowledge should be our particular focus. It is, instead, unnoticed and ignored in the current setting.

Teaching as Listening

So, what might mathematics teaching be? To begin, it can be neither a matter of *telling* nor of *facilitating*, for these separate the teacher and the learner and they ignore the somewhat chaotic and happenstantial nature of any human endeavor.

From an enactivist stance, such endeavors cannot be predetermined. As regards teaching, one might say that rather than seeking to control learning outcomes, the teacher provides occasions for appropriate actions, recognizing that the students' meanings and understandings are developed through and revealed in those actions. To repeat, actions are not dependent on or subsequent to understandings, they *are* understandings.

These actions/understandings are "occasioned" by the teacher. That is to say, the teacher presents the occasions for learners to act. The notion of "occasioning" is used in contrast to "causing," for, like constructivism, enactivism acknowledges that our meanings arise from long and diverse histories, not merely from immediate experiences. This is a "truth" available to all who have had occasion to examine the diversity of student interpretations to the most straightforward of ideas. Such diversity could never be *caused* by what the teacher says or does; it is *occasioned* by her action.

A few weeks ago, for example, Isat in on a grade 7 classroom where students were using "fraction Kits" consisting of \variously sized pieces of paper to work on the question:

$$\frac{3}{12}+\frac{2}{24}+\frac{1}{6}$$

The range of answers was remarkable, Some students answered in terms of twenty-fourths, a few used twelfths as the common denominator. One rearranged the pieces into fourths. My favorite answer was four eighths, arrived at when one student built on another's answer. To begin, Elaine demonstrated tha the unknown sum was one half by gathering together the specified pieces and arranging them atop a half-piece, drawing a diagram to represent her actions to her classmates (see Figure 1.a, below). Van then announced that the sum could also be expressed as four eighths.

When asked by the teacher how he might get that answer using the specified pieces, Van explained, "Easy. One fourth of a sixth plus one twenty-fourth plus a half of a twelfth is an eighth," drawing a dotted line across Elaine's diagram (see Figure 1.b). "You do that twice, so you can trade that for two eighths. Then, across the bottom, you have one fourth of a sixth and two halves of twelfths, twice, so that's two more eighths," adding more dotted lines and then simplifying the diagram by erasing the unwanted marks.

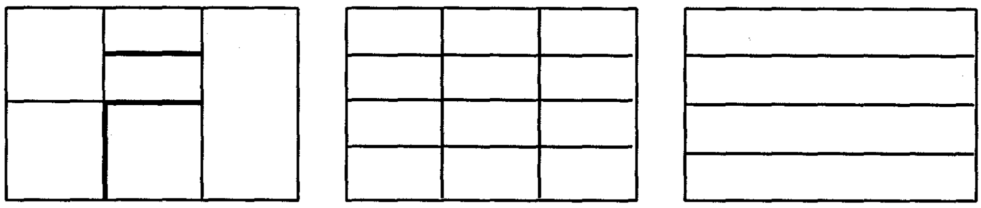

Figure 1

Elaine's and Van's diagrams

In essence, Van cut up and re-arranged the pieces, arriving at a sum through applying a sophisticated series of relationships with which he was familiar. In formal arithmetic terms, his reasoning might be expressed as:

$$2[\frac{1}{2}(\frac{1}{12})+\frac{1}{24}+\frac{1}{4}(\frac{1}{6})]+2[\frac{1}{2}(\frac{1}{12})+\frac{1}{2}(\frac{1}{12})+\frac{1}{4}(\frac{1}{6})]=2[\frac{1}{8}]+2[\frac{1}{8}]=\frac{4}{8}.$$

The richness of the mathematical thought in this setting is astounding. The teacher, thankfully, was not caught up in assessing the rightness or wrongness of particular answers; she was more interested in listening to the reasoning of the learners. She thus took the time to attend to the student's explanation, and in that attending to participate in the unfolding of his and other students' understandings.

There are three points here. First, there is no way that this student's response can be thought of as having been *caused* by the teacher. Second, by offering a more explorative setting, this teacher occasioned some profound insights into fraction relations. These students were not applying an algorithm. Many in fact were quite unaware that one existed. They were thus able to bring to bear a wealth of knowledge of relationships, equivalences, and processes. Third, Van's understanding is caught in a web that includes the physical setting, the teacher's actions, and the contributions of the other students—elements which cannot be extricated from one another. So enacted, mathematical understanding is a truly complex (and I use the term in the sense that complexity theorists use it) phenomenon in which the teacher is complicit.

This example is, I think, a powerful illustration of the roles of complexity and happenstance in teaching—two elements that I have endeavored to bring out with the notion of teaching as listening. In her attentiveness, the listening teacher can forego the desire to have others think as she does. Learning is not thought to occur in the duplication of the teacher's actions, but in engaging in appropriate actions. The teacher's task is to occasion such actions and then, once occasioned to listen to what is happening. Her actions both shape and are shaped by the events of the classroom.

The notion of "teaching as listening" is offered both figuratively and literally. Figuratively, listening presents a powerful alternative to metaphors for teaching which focus on a monological authority (such as "transmission," "telling," voice," or "empowerment"). Listening is necessarily dialogical, involving at the very least the intermingling of another's words with the text of my own experience. Similarly, a listening orientation denies the possibility of rigid subject-object distinctions—like those shared by Cartesian and constructivist epistemologies—reminding us that the issue of who we are is not separate from where we are, what we are doing, who we are with, and what we know.

In literal terms, listening can serve as a basis for teaching action—an idea that has been recently confirmed by Tom Carpenter and Elizabeth Fennema in their research into cognitively-guided instruction. After investigating the relative effectiveness of particular teachers, they concluded that "listening to their own students was the critical factor." It is my contention that a listening orientation helps to liberate us from the desire to control outcomes while acknowledging the fundamental role of the teacher in affecting learning. Put differently, in suggesting that the teacher's role involves occasioning of students' acting, I am not in any way arguing that the teacher must forego all hopes of promoting understandings of particular concepts (i.e., those mandated by curriculum documents). But I am saying that this is not a simple matter of telling or facilitating, for the teacher is necessarily implicated in all that she teaches. Moreover, there is an ethical dimension to the teacher's task which is implicit in the notion of listening, but which is either obscured by or absent from many other orientations to teaching.

I should perhaps be more explicit about my conception of listening. Briefly, listening is not merely an attending to the words or actions of others. It is, rather, a responsive and imaginative participation in the formation and the transformation of the world we co-inhabit and co-create. (This description emerges from a phenomenological inquiry into listening that formed a central element of my research.)

THE PLACE OF SUCH RESEARCH

My research is about teaching. Despite the impression I might have given thus far, I believe it to be thoroughly practical. My whole purpose is to find a way of talking about teaching differently, and my belief is that a different way of talking about teaching will contribute to different ways of interpreting and different patterns of acting.

I have spent much of my research time immersed in the reading and thinking. I've spent just as much time in classrooms. In particular, I have worked intensely over an extended period of more than two years with one junior high school teacher. Our collaboration began with my sitting in on her classes and discussing with her some of the issues surrounding teaching. We've since co-planned and co-taught. The final stage of our in-class collaboration also implicated my advisor, Tom Kieren, in an incredibly interesting 5-week fractions unit in a grade 7 classroom.

The math lesson that my English education colleague had parodied was recorded in this teacher's classroom. That event, as I said, occurred two years ago. The fraction addition example that I just presented was drawn from a lesson that this teacher taught in this fractions unit. Needless to say, her teaching has changed dramatically, as has the way she and I talk about teaching. Her concerns for clear explanations and for avoiding any possibility of ambiguity have given way to an embracing of happenstance and a noticing of the richness of learners' actions in a mathematically rich space. To capture it in a phrase, she is now teaching by listening.

We both have a long way to go. Re-framing teaching as listening hasn't been a panacea; like any metaphor, it obscures some details while it illuminates others. Nevertheless, it has served as a powerful starting place for the interruption and the interrogation of what we have long taken for granted. As might be expected, my research has also had a profound effect on my own teaching of pre-service teachers, and this impact might be summed up by saying that I am now unable to separate teaching about teaching from research into teaching. ... Which is to say that, if I have anything to offer prospective and practicing teachers, it is not about the technicalities of teaching. Teaching about teaching, for me, is not about providing tips or elaborating on techniques; that technical approach, I believe, serves to perpetuate existing practices, for those techniques are founded on particular modes of thinking. Rather, I believe, my particular strength is in engaging others in thinking differently about practice. The issue, for me, is not how to get teachers to implement particular principles, but how to invite them into the research process. Learning about teaching is the same as inquiring into teaching; the possibility for transforming teaching practice lies, I believe, in an inquisitive attitude toward that practice. It is an attitude which I am convinced that most teachers have, albeit one that is frustrated by institutional constraints, public expectations, and the momentum of habit. Such factors are not barriers to be overcome, but constraints to negotiate.

A critical aspect of our role as members of a university community is to help pre-service and practicing teachers toward an alternative framework for analyzing what we and they are doing. We will never accomplish this by standing outside of the process and pointing. Quite simply, we university-types see different things because we are looking through a different set of lenses. Teachers and pre-service teachers can only see what they see—which is to say that, if we want them to see otherwise, we have to participate in shaping their experiences at the level of the phenomenon to be investigated, and that is best done in the immediacy of collaboration and dialogue. Again, what I am arguing here is that teaching about teaching is not distinct from inquiring into teaching.

CLOSING COMMENTS

I am interested in understanding what mathematics teaching might be. That is an issue that frames both my research and my teaching—two facets of my professional life that I cast in terms of the immediate, intimate, and dialogical structures of *listening* and *learning* instead of the more static, distancing, and monological frames of *looking* and *disseminating*.

I believe we must work to foster this attitude of listening—to others, to our situations, to ourselves. It is by listening that we find those places where teaching and learning might happen.

SOME PUBLICATIONS

Should anyone be interested in reading more about this study, the following publications either derive from or are directly related to my doctoral research.

Davis, B. (1994). Mathematics teaching: Moving from telling to listening. *Journal of Curriculum and Supervision 9*(3), 267-283.

Davis, B. (1995). Why teach mathematics? Mathematics education and enactivist theory. *For the Learning of Mathematics 15*(2), 2-8.

Davis, B. (in press). *Teaching mathematics: Toward a sound alternative.* New York: Garland Publishing.

Davis, B. (in press). Thinking otherwise and hearing differently: An alternative enactment of mathematics teaching. *JCT: An Interdisciplinary Journal of Curriculum Studies.*

Davis, B. (forthcoming). Listening for differences: An evolving conception of mathematics teaching. *Journal for Research in Mathematics Education.*

Davis, B., Sumara, D., & Kieren, T. (in press). Cognition, co-emergence, curriculum. *Journal of Curriculum Studies.*

Kieren, T., Davis, B., & Mason, R. (in press). Fraction flags: Learning from children to help children learn. *Teaching Mathematics in the Middle School.*

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Carpenter, T. and Fennema, E. (1992). Cognitively guided instruction: Building on the knowledge of students and teachers. In W.Secada (ed.) *Researching Educational Reform: The Case of School Mathematics in the United States*, (a special issue of The International Journal of Educational Research), pp. 457-470.

Taylor, C. (1991). The dialogical self. In D. Hiley, J. Bohman, and R. Shusterman (Eds.). *The Interpretive Turn: Philosophy, Science, Culture*. Ithaca, NY: Cornell University Press.

AD HOC SESSIONS

Ad Hoc Session 1

COMPUTER EXPLORATIONS: MATH COURSES AT SIMON FRASER UNIVERSITY

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"Computer Explorations in Calculus I (II), supplement to calculus I (II) gives students the opportunity to explore and investigate the underlying principles of differential (integral) calculus using leading edge computer software currently used in mathematical and scientific research and industry. Previous experience with computers would be beneficial, but it is not required. Corequisite: Calculus I (II) or register by special permission."

This is how Simon Fraser University introduces the two new courses that involve calculus with computers in its Calendar. The word "computer" in the title was originally chosen because we wanted to allow the teacher a free choice of the computer algebra system, that is, any symbolic software package such as Derive, Mathematica, Maple and so on, may be used.

I was first introduced to Maple during a workshop given by Dr. Stan Devitt, in 1989 during the CMESG meetings at Brock University, St. Catharines. After that I taught my calculus students Maple through projects that I have prepared and which involved explorations of concepts from calculus and linear algebra. In 1990-91, during my sabbatical year at Harvard University, I developed a series of units with projects and assembled them together as manuals. These are:

1. *Maple Reference Card: A Beginner's Course*,
2. *Maple Explorations for Differential Calculus*,
3. *Maple Explorations for Integral Calculus* and
4. *Maple Explorations for Linear Algebra.*

After my sabbatical I continued teaching Maple through these projects for another year.

In the fall of 1993, a colleague, Dr. J. Hebron and I team taught Computer Explorations in Calculus I and Computer Explorations in Calculus II soon after these courses were approved by the University Senate. Over the past three years the Mathematics and Statistics Department of Simon Fraser University has offered and is offering Computer Explorations in Calculus to students who had taken or are concurrently taking introductory calculus. These courses are one credit each and they consist of a number of projects from differential and integral calculus. They provide an optional computer complement to our traditional calculus courses for the many students who are interested in learning computer algebra systems to enhance their skills.

The objective of these courses is exploration of calculus using computers. They are not intended to introduce new material or the concepts of calculus but are mainly centered on application, investigation and expansion of concepts by the students. The students are expected to apply and explore rather than drill and memorize. Finally the students are encouraged to experiment and expand problems to general ones, make their own problems—problems where the use of computers is advantageous over a "paper and pencil" solution.

The computer algebra system that we mainly use is Maple. I personally choose it because I believe that Maple is a powerful package and easy to learn. Its strength and power extend beyond first years of university science courses and its commands are simple and screen friendly. There is a simplicity associated with it which I like, as a small example: I prefer parentheses when commands embrace mathematical symbols or equations. Of course there are many other computer software systems, which are powerful and with fascinating plotting and animation capabilities, in a way that it becomes hard to compare or choose one. Students who know one system can easily learn another, also by learning one software package well enables students do advanced work in the future.

The book which we recommend our students to buy is *Maple V First Leaves: A Tutorial Introduction To Maple V* by Bruce W. Char et al. (1993). Many students find the manual that I composed, *A Maple Reference Card*, helpful, when they are getting started. I also use projects from my other manuals in my classes.

The first stage of these courses is to get students acquainted with the computers, the network system, electronic mail and Maple. They learn the Maple "help" function and Maple's general characteristics. Projects 1 and 2 consist of a quick tour of commonly used Maple functions along with brief explanations and examples that use these functions. This introduction gives a picture of mathematics through Maple, demonstrates to students the power of Maple and displays its limitations. If the students already know how to use a microcomputer then Maple becomes an easier activity.

The next stage is working the units and doing the projects. Each series has up to twelve units and each unit has four different parts: a review of the concept, discussion of new Maple commands, solved examples with Maple input and output, and the project which is a set of problems that expand beyond the given examples. The students are expected to solve these projects using Maple. Each unit is distributed to students during class or posted on the local network (Math Lab Server). During the weekly meetings we try new examples which involve new Maple functions and discuss strategies for the project problems.

Another stage in the course is when the students work on their projects in the Workshop for Computer Aided Tutoring (WCAT). Here teaching assistants provide help to students on an individual basis, answer Maple or other questions when the students are doing their projects on the computers. Many students use the WCAT regularly. Students are told that good project solutions are those which give an interpretation of the output and discuss or derive conclusions to problems. The answers are to be essay-like. The students are expected to hand in a printed form of their work when the project is due or drop it in the course folder via the network. Our experience showed that projects on paper are much faster to read and evaluate. Usually a teaching assistant will evaluate and write comments, which are mainly suggestions that refer to the explanations and conclusions. The project evaluation is really work-intensive.

Generally the projects are designed to give students a deep and proficient understanding of the mathematical concept under investigation as well as the ability to use Maple software to explore these concepts.

The students are encouraged to use Maple for other science classes, to solve their math problems and to think of Maple as a tool whenever they do mathematics. Also, the students are encouraged to try the solutions using "paper and pencil", to work in groups, to share methods with classmates and to learn to use Maple's "help" function at an early stage.

The outline for the differential calculus course includes among other concepts the following: tangent line approximations, continuity of functions and limits, derivatives, polynomials, maxima and minima, inflection points, investigating functions and their graphs, implicit differentiation, Newton's method, and proofs by induction.

The outline for the integral calculus course includes among other concepts the following: investigating the integrability of functions, approximate integration, problems on areas, volumes and arc length, improper integrals, centroids, harmonic series, Taylor series and polar coordinates.

As a specific example of a unit I will discuss the unit on areas which includes examples on using Maple functions to find tangents to curves, points of intersections of two curves and areas of regions between curves or between a curve and a line. These examples include the Maple functions and its output. The questions in the project combine these experiences and moreover they require some exploration to discover other possibilities. To illustrate this, below are the problems associated with the above examples.

EXAMPLE:

At x = -1 is a point P on the graph of the polynomial $f(x)= 0.5x^3-1.5x^2-2x+6$. Let PO be is a secant line from P to O, where O is an inflection point of f. Find the area between the curve and the line PO. Plot the graph and its secant.

Input

Here f is defined as an expression.

```
f:=(1/2)*x^3-(3/2)*x^2-2*x+6;
```

The functions below give the x and y-coordinates of the point P.

```
>x1:=-1; y1:=subs(x=x1,f);
```

Below are the first and second derivatives of f.

```
>Df:=diff(f,x);  D2f:=diff(Df,x);
```

Solving the second derivative for x and substituting x=1 in f gives the coordinates of the inflection point O.

```
>infx:=solve(D2f=0,x);  infy:=subs(x=1,f);
```

PO is the y-coordinate of the secant line to the curve.

```
>PO:=3+((y1-3)/(x1-1))*(x-1);
```

C is the area between the curve f and PO, the integral from x1=-1 to 1.

```
>C:=int(f-PO,x=x1..1);  plot({f,PO},x=-2..3.5,y=-4..8);
```

Output

```
>f:=(1/2)*x^3-(3/2)*x^2-2*x+6;

        f := 1/2 x^3 - 3/2 x^2 - 2 x + 6

>x1:=-1;
              x1 := -1

>y1:=subs(x=x1,f);
              y1 := 6

• Df:=diff(f,x);
          Df := 3/2 x^2 - 3 x - 2

>D2f:=diff(Df,x);
            D2f := 3 x - 3

>infx:=solve(D2f=0,x);
             infx := 1

>infy:=subs(x=1,f);
             infy := 3

>PO:=3+((y1-3)/(x1-1))*(x-1);
           PO := 9/2 - 3/2 x

>C:=int(f-PO,x=x1..1);
              C := 2

>plot({f,PO},x=-2..3.5,y=-4..8);
```

PROJECT PROBLEMS:

(i) Let $y = x^3$ and let P be any point on the graph of y. The line tangent to this curve at P, with coordinates (x1,y1), crosses the graph of y at another point (x2,y2), called Q. Let A be the area between the curve and this tangent line PQ. Now, the line tangent to $y=x^3$ at the point Q meets the curve again at R (x3,y3). Let B be the area between the curve and this new tangent line QR. Show that B=16A.

(ii) Area B equals 16 times area A is true for any point P on the curve $y=x^3$. Is this property true for every cubic curve? To do this take $y := x^3 + bx^2 + cx + d$, and let the coordinates of the point P be (x1, y1).

(iii) Do the same problem but this time use functional operators and other Maple routines to achieve a simpler version of the solution.

Such are the examples in the units and the problems for the projects. When an abstract concept is replaced by a computer image I frequently hear students say "now I understand."

Finally the students write final examinations for these courses which represents 20% of their course grade. The rest of the grade comes from the projects and attendance.

All students reacted positively to the course. Here are some of the students' comments when the course was finished:

Rather than simply remembering Calculus, I feel that I now understand it.

The Maple material definitely strengthened my grasp of the calculus.

Many times in producing graphs on the computer, my understanding of the calculus concepts was enhanced.

Math is the kind of subject that requires examination from many angles in order to make sense. By working the Maple workshop, I feel that my understanding is better than it would have been if I had just worked at it within the requirements for Math 151.

My research to complete my M.Sc. will require mathematical modeling and I feel that being familiar with how Maple can solve equations and produce graphs will aid in professional presentation of my thesis.

Rather than simply remembering Calculus, I feel that I now understand it.

I really found the manual helpful. It was organized and easy to follow.

I found the lectures with hands on more helpful.
One credit is minimum for this course.

I enjoyed it, it was enrichment.

Here are some of the students' comments before taking the course:

I hope that this program will help me to a greater knowledge, in both in the Maple program and in the course in general.

I feel that the skills obtained would help me in future math and engineering courses and in industry on my co-op terms (Maple and MATLAB are both pretty common in industry).

I heard that industries also use Maple. I will be participating in the co-op program in next Spring. Having Maple skills will be very useful.

Not much formal evaluation of the courses has been done but we feel that these courses have been both useful and successful. One of our students recently wrote to us that she was using Maple for her job in business. Other students use it to check answers to their homework problems and others use it for their research. Three of our students received Maple awards of $500.00 each, given by Waterloo Maple Software for excellence in using Maple.

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Ad Hoc Session 2

A LINGUISTIC VIEW OF MATHEMATICAL WORD PROBLEMS

Susan Gerofsky
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Mathematical word problems can be viewed as problematic from a number of different points of view. Although they are often treated as if they were practical, real-life applications of abstract mathematical ideas, studies in ethnomathematics (Lave, 1988; Lave, 1992; Schliemann & Carraher, 1992) have revealed that people who are successful and efficient mathematical problem-solvers in "real-life" may be unable to solve school word problems with pencil and paper, even when these word problems appear to be similar to real-life problems that the person is quite capable of solving.

Word problems are very old. Examples of word problems formally similar to those in use today have been found in Babylonian clay tablets and the ancient Egyptian Rhind Mathematical Papyrus. Yet their continued use in school mathematics raises questions both for students, who often have great difficulty making sense of these problems, and teachers, who often have a hard time explaining to students what such problems are about and what they are for. Students are taught to "translate" problems from story form into arithmetic or algebra and to throw away the story—so why cloak the mathematics in story form in the first place? Students are also typically given a set of ten or more word problems to solve quickly at the end of a textbook chapter, but the solutions are of no consequence to anybody, so why work so hard to find them? A few researchers (Borasi, 1986; Pinder, 1987; Brown and Walter, 1990) have begun to question the prevalent view that "problem-solving" means exclusively the solution of word problems.

In my doctoral thesis-in-progress, I use an analysis based on linguistic pragmatics and philosophy of language to look at word problems as a literary genre, and to find linguistic underpinnings for their problematic nature.

I consider the problem of *deixis*, ("pointing with language" or reference) in word problems. These problems use referring terms, the names of places and people for example, yet they seem to refer neither to real places nor real people nor to fictional ones. Word problems are not usefully regarded in terms of most fictional genres because of their paucity of features like plot, character development, dramatic tension, elaborated setting, and so on; if they are to be considered stories, they are very poor stories indeed. On the other hand, even when word problems refer to existing people, places and things, it is clear that they are not really "about" those people, places and things. Lave writes that "word problems are about aspects of only hypothetical experience and essentially never about real situations" (Lave, 1992, p. 78). Here are some examples from school textbooks currently in use in British Columbia:

> *A rock dropped from the top of the Leaning Tower of Pisa falls 6 m from the base of the tower. If the height of the tower is 59 m, at what angle does it lean from the vertical?* (Ebos et al., 1990, p. 354)

The tricky part of the story problem above is the "if" (my emphasis). Certainly the Tower of Pisa has been measured. Why use the conditional form here? Is it intended to indicate that the vertical height of the tower is not stable? (This may be true — it was recently closed to visitors because increases in its "lean" had made it dangerous...) Or is it a way of indicating that the referent for the words "the Leaning Tower of Pisa" is not the actual structure in northern Italy, but the hypotenuse of a hypothetical right-angled triangle whose length could be set at any value (say 59 units) and whose slope could be calculated using the Pythagorean Theorem? The writer of the problem seems to be taking pains to say, "Here is a story, ignore this story."

> *Each elephant at the Young Elephant Training Centre in Pang-ha, Thailand,* ***eats*** *about 250 kg of vegetation in a day. How much* ***would*** *43 elephants* ***eat*** *in 1 day? 1 week?* (Alexander et al., 1989, p.35)

Again, assuming that the Young Elephant Training Centre in Pang-ha, Thailand, does actually exist, why pose the question at the end of the word problem in the conditional form ("how much would they eat")?

Existing word problems can be reworded to emphasize their hypothetical nature—for example:

> *Every year Stella rents a craft table at a local fun fair and sells the sweaters she has been making all year at home. She has a deal for anyone who buys more than one sweater. She reduces the price of each additional sweater by 10% of the price of the previous sweater that the person bought. Elizabeth bought 5 sweaters and paid $45.93 for the fifth sweater. How much did the first sweater cost?* (Ebos et al., 1990, p.72)

The above could be reworded as follows without changing its truth value (although it would be a rather odd-looking word problem, highlighting as it does one of the implicit features of the genre):

> Every year (but it has never happened), Stella (there is no Stella) rents a craft table at a local fun fair (which does not exist). She has a deal for anyone who buys more than one sweater (we know this to be false). She reduces the price of each additional sweater (and there are no sweaters) by 10% of the price of the previous sweater that the person bought (and there are no people, or sweaters, or prices)...

The problem of reference in word problems is also seen in their use of verb tense and modality. Where a straightforward narration would use verb tense in a way consistent with the ordinary grammar of English, word problems mix tenses and modes in a way that would not be acceptable in ordinary conversation or explanatory writing. For example, the following problems shift back and forth between past, present, future and conditional verb forms in an arbitrary way:

> *Carman's Jewellers* ***had*** *a sale on watches. The Caravan* ***was*** *originally priced at $79.80 but it* ***had*** *a tag on it saying "25% off". a) What* ***is*** *the regular price of the watch? b) What percent* ***is*** *the discount? c) How much* ***could*** *you* ***save*** *buying the watch on sale? d) What* ***is*** *the sale price of the watch?* (Bye et al., 1984, p. 212)

> *Bruce* ***buys*** *sunglasses at $35.88 per dozen. He* ***sells*** *them at the ball park for $5.99 a pair. a) Find his rate of mark-up. b) Find his rate of margin. c) How much profit* ***does*** *he* ***make*** *on a day in which he* ***sells*** *25 pairs? His expenses for the day* ***were*** *$6.80.* (Ebos et al, 1990, p. 400)

In terms of linguistic pragmatics, this can be seen as a non-referential use of verb tense; the verbs do not refer to any "real-life" time which can be placed in relation to the time of the writing or the reading of the problems. Since these are not real places, people or situations there is no absolute need for logical consistency in the use of tense (and tense is often used in ways that would be considered selfcontradictory in standard expository English prose). Rather, word problems propose hypothetical situations with certain given conditions and ask for hypothetical answers. Most word problems could be rewritten in the form: "Suppose that (some certain situation A existed). If (conditions B, C, D, ... held), then (what would be the answer to E)?"

The very inconsistency and seeming arbitrariness of tense choices in word problems points not only to their tenseless and non-deictic nature, but also to an "understanding" between writer and reader that these supposed situations do not have truth value, and that the writers' intentions and the readers' task are something other than to communicate and solve true problems. (Otherwise the meaning of these problems in terms of a true situation would be very difficult to decipher.) This lack of truth value can be otherwise expressed in terms of the philosophy of language as "flouting the Gricean maxim of quality."

The philosopher of language, H.P. Grice, looked for a set of assumptions underlying the efficient co-operative use of language (Grice, 1975, 1978), and the five principles he found, including a general "co-operative principle" and four "maxims of conversation" are listed below:

1) The co-operative principle

Make your contribution such as is required, at the stage at which it occurs, by the accepted purpose or direction of the talk exchange in which you are engaged.

2) The maxim of quality

Try to make your contribution one that is true, specifically:
- i) do not say what you believe to be false.
- ii) do not say that for which you lack adequate evidence.

3) The maxim of quantity

- i) Make your contribution as informative as is required for the current purposes of the exchange.
- ii) Do not make your contribution more informative than is required.

4) The maxim of relevance

Make your contributions relevant.

5) The maxim of manner

Be perspicuous, and specifically:
- i) avoid obscurity
- ii) avoid ambiguity
- iii) be brief
- iv) be orderly.

Grice's point was not that all speakers must follow these guidelines exactly, since it is obvious that no one speaks this way all the time. Rather, he says that when an utterance appears to be non-cooperative on the surface, we try to interpret it as co-operative at a deeper level.

It is my contention that a key feature of the word problem genre is a consistent flouting of the Gricean maxim of quality, which is to say that, as a genre, word problems have no truth value. This feature is intimately linked with, or perhaps a result of, their deictic indeterminacy—their lack of reference to real people, places and times.

So what are word problems? Should we continue to use them in mathematics education, and if so, how? In my doctoral research I am addressing two aspects of these problems. First, I am writing versions of word problems that are non-standard in some of their linguistic features, and asking students and teachers to "think aloud" as they work with them, trying to determine whether or not they are still word problems. Secondly, I am working with David Pimm's suggestion to view word problems as parables (Pimm, 1995), and using this suggestion to find alternate ways of using word problems in pedagogy.

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Ad Hoc Session 3

RIGOR IN CALCULUS AND TEACHING OF CALCULUS

Marina Padamadan
Concordia University, Montreal

The dictionary meaning of "rigor" is "strict precision" or "exactness." In mathematics, we begin with the definition of our objects and develop ideas and implications following the rules already accepted. A statement that defines a mathematical object should be precise. It should be interpreted in its exactness for what it is meant. Defining a mathematical object in language form creates several difficulties. In the history of humankind, perhaps the first time we were confronted with such a problem was when there was an attempt to give a definition of a point. Greek mathematicians introduced the concept of proof in mathematics. To illustrate the concept they chose the concept of geometry. At the outset, a "point" is to be defined. The definition should be self contained in its explanation. It should communicate exactly what it is meant for. Euclid, in his celebrated work *Elements*, defines a point as "A point that has no path," (Heath). Does this negative statement communicate the essence of that geometrical object? Before Euclid, Pythagoras held that, that which is indivisible and has position is a point. Plato thought "a point is the beginning of a line." At one stage, Aristotle remarked that a point is a geometrical fiction. In all these attempts, there is a reflection of the shortcoming of language to communicate the exactness that would define a point, though in our mind we understand what it is.

In the work of Archimedes, it may be said there is a lack of rigor when he considered the weight of a cone as the sum of the weight of infinitely thin circular slices of which it is composed. Here, the rigor takes a different form. The discussion goes through a process which is not properly defined. An infinite sum requires a definition, an explanation; it lacks precision (rigor). The result obtained is correct, but the method itself is not sound.

The story of rigor in mathematics picks up its momentum with the discovery of Calculus. The lack of rigor was not so intensely realized before this achievement. In 1667, Newton defined the derivative:

> Let there be an increment h in $y=x^m$ and $(x+h)^m - x^m$ be the corresponding increment in y. Consider $\{(x+h)^m - x^m\}/h$, put (after simplifying) the increment h=0 to get mx^{m-1}, the derivative of x^m.

Not only does it lacks rigor, there is obviously enough ambiguity in the definition of the derivative when first revealed to (or given by) Newton. Though Calculus was used extensively over the next hundred years to study the relation between variables in nature, there was concern about its foundation. In 1734, Berkeley criticised the definition and pointed out the fallacy in taking the increment and then letting the increment be zero (Boyer). By the Fundamental Theorem of Calculus, we get results but there is no rigor in the definition. The process is not reflected in the method. The method had to be explained.

The discussion on the definition of the derivative spread over more than a century. There is an evolution of the concept in arriving at a correct and flawless definition. In the beginning of the nineteenth century, Cauchy rigorously defined limit. The derivative as the limit of the quotient as *h* tends to zero turns out to be a good definition. After this rigor in the definition of the derivative, there emerged a mushroom of new ideas that could be catalogued under the title "rigor in the proof." In Calculus, we came occasionally to a situation when the conclusion is obvious if one appeals to geometry but requires further filling of arguments if rules are strictly observed. For example, one can cite the Intermediate Mean Value Theorem:

> Let f be a function continuous on the interval [a,b] with f(a) > 0 and f(b) < 0, then there exists a c in [a,b] such that f(c) = 0.

In 1870, a new era of rigor emerges when Bolzano proposed to prove the above theorem. He emphasizes that geometrically, the theorem is obvious but pictorial representation is not entertained in the definition of a continuous function; the proof should depend on underlying assumptions. A rigor in the proof led to a definition or irrational number. The construction of real numbers by Dedekind and Cantor is in response to a required rigor in the proof of the Intermediate Mean Value Theorem. Several other theorems, geometrically obvious, depend on the least upper bound property for its proof which in turn is a consequence of the construction of real numbers. The properties of a continuous function on a closed and bounded interval frequently used in Calculus can now be established rigorously. The rigor once treated independent of Calculus is responsible for the origin of Analysis, a discipline emerging out of Calculus but different from Calculus. Following all the rigor and meshed with Calculus a logical order of the topics should enlist the content as:

Real Number

Limit

Derivative.

It is interesting to note that the logical order is inverse of the historical order of their discovery.

By the middle of the century, there is a tremendous increase in student population learning Calculus. It is no more a required course for Physics and Engineering students. Students in all branches of knowledge including Biological Science, Social Science and Commerce take a Calculus course in their program. But at the same time, the contents of the Calculus course underwent a change by absorbing more and more topics from Analysis. It made the subject Calculus rigorous. There was a hope that a rigorous approach being logical would be better suited for understanding the subject. With this aspiration, text books on Calculus were written. But the contents of Calculus, once it included the topics from Analysis to promote a rigorous approach, enlarged and the text books became thicker and thicker. Morris Kline once wrote jokingly that the prerequisite for Calculus is weight lifting. Instead of making the subject pedagogically easier, this approach made the subject more difficult to learn. In an attempt to cover the topics within a specified period and teach Calculus for practical use, teachers in the classroom circumvent various ideas dealing with rigor.

In this century, particularly in the second half, rigor has become a soul of mathematics. It is unbelievable to present mathematics concealing rigor. However, our experience tells us that rigorous Calculus is hard to teach. What level of rigor is pedagogically feasible is an open question. We should recall the spirit and reasons along which the subject has developed and has acquired rigor. Perhaps, human understanding follows the historical development of the topics. While designing a course on Calculus, the objective may remain limited and rigor could be introduced on its own in a different course on Analysis. At least, a history of mathematics justifies this approach.

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APPENDICES

APPENDIX A

WORKING GROUPS AT EACH ANNUAL MEETING

1977 Queen's University, Kingston, Ontario
- Teacher Education programmes
- Undergraduate mathematics programmes and prospective teachers
- Research and mathematics education
- Learning and teaching mathematics

1978 Queen's University, Kingston, Ontario
- Mathematics courses for prospective elementary teachers
- Mathematization
- Research in mathematics education

1979 Queen's University, Kingston, Ontario
- Ratio and proportion: a study of a mathematical concept
- Minicalculators in the mathematics classroom
- Is there a mathematical method?
- Topics suitable for mathematics courses for elementary teachers

1980 Université Laval, Québec, Québec
- The teaching of calculus and analysis
- Applications of mathematics for high school students
- Geometry in the elementary and junior high school curriculum
- The diagnosis and remediation of common mathematical errors

1981 University of Alberta, Edmonton, Alberta
- Research and the classroom
- Computer education for teachers
- Issues in the teaching of calculus
- Revitalising mathematics in teacher education courses

1982 Queen's University, Kingston, Ontario
- The influence of computer science on undergraduate mathematics education
- Applications of research in mathematics education to teacher training programmes
- Problem solving in the curriculum

1983 University of British Columbia, Vancouver, British Columbia
- Developing statistical thinking
- Training in diagnosis and remediation of teachers
- Mathematics and language
- The influence of computer science on the mathematics curriculum

1984 University of Waterloo, Waterloo, Ontario
- Logo and the mathematics curriculum
- The impact of research and technology on school algebra

Epistemology and mathematics
Visual thinking in mathematics

1985 Université Laval, Québec, Québec
Lessons from research about students' errors
Logo activities for the high school
Impact of symbolic manipulation software on the teaching of calculus

1986 Memorial University of Newfoundland, St, John's, Newfoundland
The role of feelings in mathematics
The problem of rigour in mathematics teaching
Microcomputers in teacher education
The role of microcomputers in developing statistical thinking

1987 Queen's University, Kingston, Ontario
Methods courses for secondary teacher education
The problem of formal reasoning in undergraduate programmes
Small group work in the mathematics classroom

1988 University of Manitoba, Winnipeg, Manitoba
Teacher education: what could it be
Natural learning and mathematics
Using software for geometrical investigations
A study of the remedial teaching of mathematics

1989 Brock University, St. Catharines, Ontario
Using computers to investigate work with teachers
Computers in the undergraduate mathematics curriculum
Natural language and mathematical language
Research strategies for pupils' conceptions in mathematics

1990 Simon Fraser University, Vancouver, British Columbia
Reading and writing in the mathematics classroom
The NCTM "Standards" and Canadian reality
Explanatory models of children's mathematics
Chaos and fractal geometry for high school students

1991 University of New Brunswick, Fredericton, New Brunswick
Fractal geometry in the curriculum
Socio-cultural aspects of mathematics
Technology and understanding mathematics
Constructivism: implications for teacher education in mathematics

1992 ICME-7, Université Laval, Québec, Québec

1993 York University, Toronto, Ontario
Research in undergraduate teaching and learning of mathematics
New ideas in assessment
Computers in the classroom: mathematical and social implications
Gender and mathematics
Training pre-service teachers for creating mathematical communities in the classroom

1994 University of Regina, Regina, Saskatchewan
Theories of mathematics education
Preservice mathematics teachers as pruposeful learners: issues of enculturation
Popularizing mathematics

1995 University of Western Ontario, London, Ontario
Anatomy and authority in the design and conduct of learning activity
Expanding the conversation: trying to talk about what our theories don't talk about
Factors affecting the transition from high school to university mathematics
Geometric proofs and knowledge without axioms

APPENDIX B

PLENARY LECTURES

1977	A.J. Coleman	The objectives of mathematics education
	C. Gaulin	Innovations in teacher education programmes
	T.E. Kieren	The state of research in mathematics education
1978	G.R. Rising	The mathematician's contribution to curriculum development
	A.I. Weinzweig	The mathematician's contribution to pedagogy
1979	J. Agassi	The Lakatosian revolution
	J.A. Easley	Formal and informal research methods and the cultural status of school mathematics
1980	C. Cattegno	Reflections on forty years of thinking about the teaching of mathematics
	D. Hawkins	Understanding understanding mathematics
1981	K. Iverson	Mathematics and computers
	J. Kilpatrick	The reasonable effectiveness of reasearch in mathematics
1982	P.J. Davis	Towards a philosophy of compuation
	G. Vergnaud	Cognitive and developmental psychology and research in mathematics education
1983	S.I. Brown	The nature of problem generation and the mathematics curriculum
	P.J. Hilton	The nature of mathematics today and implications for mathematics teaching
1984	A.J. Bishop	The social construction of meaning: a significant development for mathematics education?
	L. Henkin	Linguistic aspects of mathematics and mathematics instruction
1985	H. Bauersfeld	Contributions to a fundamental theory of mathematics learning and teaching
	H.O. Pollak	On the relation between the applications of mathematics and the teaching of mathematics
1986	R. Finney	Professional applications of undergraduate mathematics
	A.H. Schoenfeld	Confessions of an accidental theorist
1987	P. Nesher	Formulating instructional theory: the role of students' misconceptions
	H.S. Wilf	The calculator with a college education
1988	C. Keitel	Mathematics education and technology
	L.A. Steen	All one system

1989	N. Balacheff	Teaching mathematical proof: the relevance and complexity of a social approach
	D. Schattsneider	Geometry is alive and well
1990	U. D'Ambrosio	Values in mathematics education
	A. Sierpinska	On understanding mathematics
1991	J .J. Kaput	Mathematics and technology: multiple visions of multiple futures
	C. Laborde	Approches théoriques et méthodologiques des recherches Francaises en didactique des mathématiques
1992	ICME-7	
1993	G.G. Joseph	What is a square root? A study of geometrical representation in different mathematical traditions
	J Confrey	Forging a revised theory of intellectual development Piaget, Vygotsky and beyond
1994	A. Sfard	Understanding = Doing + Seeing ?
	K. Devlin	Mathematics for the twent-first century
1995	M. Artigue	The role of epistemological analysis in a didactic approach to the phenomenon of mathematics learning and teaching
	K. Millett	Teaching and making certain it counts

APPENDIX C

PREVIOUS PROCEEDINGS

Tlhe following is a list of previous Proceedings available through ERIC.

Proceedings of the 1980 Annual Meeting .... ED 204120

Proceedings of the 1981 Annual Meeting .... ED 234988

Proceedings of the 1982 Annual Meeting .... ED 234989

Proceedings of the 1983 Annual Meeting .... ED 243653

Proceedings of the 1984 Annual Meeting .... ED 257640

Proceedings of the 1985 Annual Meeting .... ED 277573

Proceedings of the 1986 Annual Meeting .... ED 297966

Proceedings of the 1987 Annual Meeting .... ED 295842

Proceedings of the 1988 Annual Meeting .... ED 306259

Proceedings of the 1989 Annual Meeting .... ED 319606

Proceedings of the 1990 Annual Meeting .... ED 344746

Proceedings of the 1991 Annual Meeting .... ED 350161

Proceedings of the 1993 Annual Meeting .... Not yet assigned

Proceedings of the 1994 Annual Meeting .... Not yet assigned

There was no Annual Meeting in 1992 because Canada hosted the Seventh International Conference on Mathematical Education that year.