CANADIAN MATHEMATICS EDUCATION STUDY GROUP

GROUPE CANADIEN D'ÉTUDE EN DIDACTIQUE DES MATHÉMATIQUES

PROCEEDINGS
1996 ANNUAL MEETING

Mount Saint Vincent University
May 31-June 4 1996

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CMESG/GCEDM 1996 Annual Meeting
Mount Saint Vincent University, May 31-June 4, 1996

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EDITOR'S FORWARD

I wish to thank all those who contributed reports for inclusion in these Proceedings. The care taken in preparing a hard copy and disk file of the report, together with camera ready figures made my work as editor a pleasant task. The value of these Proceedings is entirely the credit of the report authors.

These Proceedings will serve to revive the memories of those who participated in the meeting and hopefully will help generate continued discussion on the varied issues raised during the meeting.

Yvonne M. Pothier
Mount Saint Vincent University
August, 1996

ACKNOWLEDGEMENTS

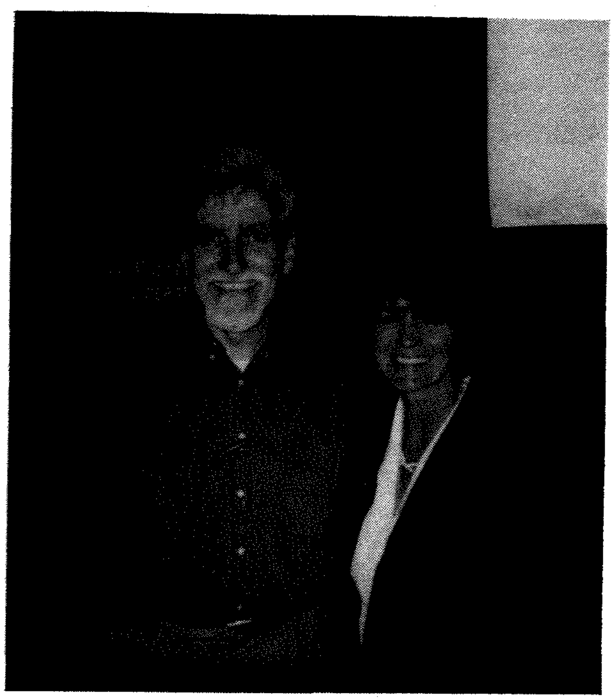
We would like to thank Mount Saint Vincent University, Halifax, for hosting the meeting and providing excellent facilities. Special thanks are due to Yvonne Pothier, Mary Crowley, LaJune Naud, Education Department and Suzanne Seager, Mathematics and Computer Studies Department, for their time and work prior to and during the meeting to make the experience pleasant and enjoyable for all participants.

Finally, we would like to thank the guest lecturers, working group leaders, topic group and ad hoc presenters, panelists, and all participants. You are the ones who made the meeting an intellectually stimulating and worthwhile experience.

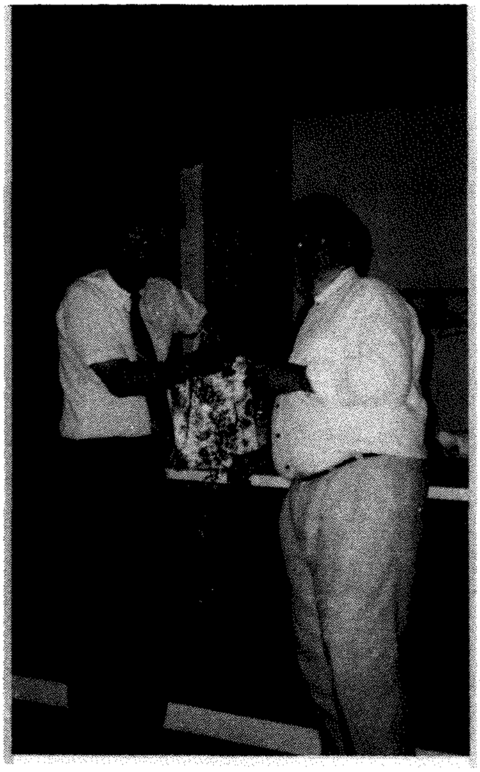
SCHEDULE

Friday May 31	Saturday June 1	Sunday June 2	Monday June 3	Tuesday June 4
1700-1800 Opening Plenary General Introductions ----------- 1800-1930 Dinner	0900-1200 Working Groups ----------- 1200-1330 Lunch	1000-1300 Working Groups ---------- 1300-1430 Lunch	0900-1200 Working Groups ---------- 1200-1330 Lunch	0900-1000 "Who drives the curriculum?" panel ------------ 1000-1030 Response to panel presentation ------------ 1045-1145 Closing Plenary 1200-1300 Lunch
	1330-1350 Continued small group discussion Plenary I ----------- 1350-1445 Questions from small groups to speaker --------------- 1450-1545 Topic Groups (Part 1) --------- 1600-1700 AGM	1430-1540 Plenary II D. Henderson ------------ 1540-1600 Initial small group discussion Plenary II ------------ 1600-1645 Ad Hoc presentations ------------	1330-1350 Continued small group discussion Plenary II -------------- 1350-1445 Questions from small groups to speaker ----------- 1450-1545 Topic Groups (Part 2) ------------ 1600-1655 New PhDs presentations	
2000-2110 Plenary I C. Hoyles ________ 2110-2130 Initial small group discussion Plenary I 2130 Reception	1730 Dinner Cruise on Harbour Queen I	1730 Excursion to Peggy's Cove	1730-1900 Banquet ------------- 1930-2030 Special presentation George Escher Post -Lecture Reception	

Four CMESG Presidents
Sandy Dawson, Tom Kieren, David Wheeler, Claude Gaulin



Plenary Session Speakers
David Henderson, Celia Hoyles



Guest speaker George Escher,
President Sandy Dawson

Opening reception

INTRODUCTION

These Proceedings cannot capture the spirit of what takes place at our annual meeting. They do, however, provide a record of the results of our work and discussions. The keynote addresses are included, and in some instances, the Proceedings is the only place these appear. It is clearly the case that the only printed record of the deliberations of the Working Groups and Topic Groups is contained in the Proceedings. It is important to have such records, both as a marker of what the various groups have accomplished in their study in any particular year, but also as a bridge for other groups in future years to build upon the work already done. For those reasons alone, the Proceedings are valuable.

But alas, the Proceedings cannot tell the whole story, so this introduction is designed to give those new to the organization, or those who have not had the opportunity to attend an annual meeting, a taste of what the meeting is really all about. During one of our early meetings at Queen's University, where four of the first six annual meetings were held, Izzie Weinzweig joyously sampled the many varieties of ice cream offered at the Queen's residence hall. Many of us were amazed at the prodigious amounts he could consume, but while he ate his ice cream discussion was lively, funny, critical, and searching about important happenings in the arena of mathematics education. Izzie's ice cream escapade is notorious in CMESG/GCEDM circles. An event such as this is also part of the annual meeting. This introduction will try to capture aspects of that spirit, items which are not recorded in the printed reports of working and discussion groups.

It was in the fall of 1977 that a couple of young mathematics education scholars, Claude Gaulin and Tom Kieren, gave two of the three keynote addresses at the inaugural meeting of what was to become the Canadian Mathematics Education Study Group—Groupe canadien d' étude en didactique des mathématiques (CMESG/GCEDM). Both went on to become presidents of the group. Though he wasn't a keynote speaker at that meeting, the group's first president, David Wheeler, was one of its organizers. As much as any set of mathematics educators in Canada, these three epitomize the spirit and diversity of the Group. And what is that spirit they so ably display, and how did they and their colleagues in the Group encourage and foster that diversity for twenty years?

The spirit is embodied in the intellectual playfulness of the Group. This manifests itself in a host of ways. It is there in the serious though not somber ways in which current issues and topics in mathematics education are *studied*, that is, where time is given during the three full working days of the annual meeting for an opportunity to *go deep*, in the vernacular of the day, to listen carefully, attentively, and without pre-judgement to the interests, ideas and experiences of one's colleagues. These study periods, called Working Groups (WGs), meet three hours each day, and provide opportunities for elongated discussions, not just the oftimes brief, superficial conversations one experiences at conferences where there are presentations hourly. Though the leaders of the WGs do extensive advanced planning, they really are the *provocateurs* of the study once the group is assembled. It is the working group members themselves who determine the particular pathway the group takes during its deliberations. In the words of David Wheeler, the philosophy behind this structure was that "...people [could] work collaboratively at a conference on a common theme and generate something fresh out of the knowledge and experience that each participant brings to it."[1] And members are diligent about guarding against a WG

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[1]Wheeler, D. (1992). The origins and activities of CMESG/GCEDM. In Kieran, C. & Dawson, A. J. (eds.). *Current Research on the Teaching and Learning of Mathematics in Canada: Les Recherches en Cours Sur l'Apprentissage et l'Enseignement des Mathématiques au Canada*, p. 6. Montréal, QC: CMESG/GCEDM.

becoming the platform for a particular point of view, or being dominated by the leaders. In fact, though leaders make detailed preparations and plans, provide extensive and elaborate occasions for group and group member thinking, it is usually not long into the deliberations that their plans fade into the background, and the discussion goes in directions not anticipated by the leaders. This is not to say that the groups operate in a random or disorganized manner. Rather, the *orders* of the working group are what Bohm would call implicate rather than explicit, and arise from the discussions occasioned by the interactions among group members.

There are other times as well when members are given opportunities for extended periods of study. Topic Groups (TGs) are sessions where individual members present work-in-progress, and invite and solicit feedback from their colleagues. These sessions are not meant to be one way informational sessions—indeed, efforts to have such a format would be frowned upon by the group—but rather are opportunities to present 'three-quarter baked ideas'[2] and have them critiqued in a supportive and caring environment. Ad Hoc groups serve a similar function, but these are events which are so current that it was not possible to include them in the Program prior to its printing and circulation. Nonetheless, it is important to note that the organization of the annual meeting provides the time and space for Ad Hoc groups to occur, and they invariably do.

The in-depth study of questions and issues in a conference setting does seem to be the prime characteristic of CMESG/GCEDM, what Wheeler has called 'its study-in-cooperative-action,'[3] and the heart of the intellectual playfulness of the Group. The scope of the topics discussed can be seen in the listing of the focus of WGs over the years which are listed in Appendix A. What is noticeable about that list is the central concern the Group has had "...with teacher education and mathematics education research, with subsidiary interests in the teaching of mathematics at the undergraduate level, and in which might be called the psycho-philosophical facets of mathematics education (mathematization, imagery, the connection between mathematics and language, for instance)."[4]

The Group's playfulness also has a social aspect. Over the years the excursions have developed a reputation for inventiveness and surprise and wonderment. We have toured the plains of Saskatchewan and Manitoba, sipped wine under the waterfalls on the Sea-To-Sky highway of BC's west coast, enjoyed Shakespearean and Shaw festivals in Ontario, toured the Plains of Abraham in Québec, sampled galleries and markets in New Brunswick, sailed around Halifax harbour, and hiked up and around St. John's. Marty Hoffman makes the rather dubious claim to fame of being the one who initiated midnight 'pizza runs', a tradition which has grown in frequency, size and inclusivity as the years pass. One night in Fredericton saw almost all conference registrants crowded into one very small, and overwhelmed, pizza parlour. In Regina, there were so many people prepared to wander the town in search of the 'perfect pizza' that the run took place over two nights.

The diversity which the Group achieves is accomplished in a number of ways. First, and perhaps most importantly, the Group has always sought to attract mathematicians as well as mathematics educators to its gatherings. The Group has been relatively successful in this venture with roughly a third of the Group's membership being drawn from the ranks of professional mathematicians. Moreover, recent years

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[2] To use a phrase Uri Leron is fond of and has written about. Uri was a most welcome non-Canadian visitor to the Group's Annual meeting in Regina in 1994. He argues that many ideas are more than just 'half-baked'. Some are better than that and are at least 'three-quarters baked'.

[3] Op. Cit., p.7.

[4] Op. Cit., p.5.

has witnessed a greater involvement by college and CEGEP mathematics instructors, a move widely applauded within the organization. Concerted efforts have also been made to have school people involved with the annual meeting, but typically this involves just teachers and provincial association representatives for the region where the conference is being held. It is a sad truism that not many teachers can obtain travels funds to attend conferences. Unfortunately, university, college and CEGEP instructors may soon be facing the same funding difficulty, if they aren't already. Nonetheless, the Group attracts a broad spectrum of the mathe-matics and mathematics education community across the country, something no other organization in Canada accomplishes.

The shifting location of the annual meeting is also a source of diversity. Since education is a provincial responsibility in Canada, it is difficult to get 'a fix' on what is occurring in all parts of the country with respect to mathematics education. Moreover, it is difficult to comprehend and understand the diversity which exists across the country, dictated by local settings, without actually visiting and living in, however briefly, particular regions of the country. We have been fortunate to have been hosted by universities all across the land, at incredibly reasonable costs, in ways which allowed us to experience the richness and diversity of Canada as few others in the general population ever have the opportunity of doing.

In their own way, the four presidents have brought their experiences of the west coast, the prairies, French speaking Québec, and English speaking Quebec, to bear on the focus and direction of the organization, and thereby fostered an understanding of the diversity of our country. It will not escape note, however, that all the presidents have been male. Over the past decade, however, the Executive itself as well as the cast of plenary speakers, working and topic group leaders at the annual meetings have been gender balanced. The increased participation of women in the Group has also led us to make changes in the programme components, such as the small group discussion format after the plenary talks, aimed at making our deliberations richer and more inclusive.

Keynote speakers also contribute to the diverse points of view to be examined by the Group. While the Group is Canadian, with only a small handful of members coming from outside the country, it was always foremost in the minds of those planning the conferences that the organization should not become parochial in its viewpoint. Efforts were made, therefore, to ensure that the keynote speakers were (1) foremost authorities in their areas of interest, those at the so-called 'cutting edge' of thinking in their field, and (2) brought a non-Canadian viewpoint to the Group. A quick perusal of the list of past speakers included in Appendix B will be sufficient to convince even the most skeptical that the Group has been successful in attracting leading mathematicians and mathematics educators to attend its meetings. These speakers don't just come, deliver their lecture, then leave, but they stay with us for the entire conference, participating in the WGs, the TGs, and the social events which embellish the 'headier' aspects of the meeting. They are active participants working right alongside our members. Moreover, one keynote speaker typically represents the mathematics education field, and the other the views of professional mathematicians. And for financial reasons, one is typically from a location 'close' to the site of the annual meeting, and one from some distance removed from that site. In all of this, the attempt is to invite individuals who will stretch our thinking, who will challenge our home-grown ideas, who will broaden our educational horizons. Sometimes these efforts are successful, sometimes not, but it still seems worth the effort.

Though rich in tradition and perhaps wedded to a particular format and way of working, the Group nonetheless continues to evolve. The face of the organization is gradually changing as individuals new to the field make their presence felt. They have begun to lead working and topic groups bringing with them perspectives and experiences new to the field. Discussion formats which are more inclusive for both new and long term members are being tested, adapted based on experience, and then adopted. Recognition, and a place on the program, is being given to those who have recently completed doctoral studies. The format of topic groups is being modified to give them greater exposure and opportunity for fuller discussion. As with most of life, some things about the Group change, while others stay the same.

But "study—étude" remains the central focus of the Group and perhaps it greatest strength and defining characteristic. This is as it should be if the Group is to be true to its origins. And if the ice cream is being kept cold in case Izzie comes along, and if local pizza parlours are stocking extra supplies, then you will know you are in a location where the annual meeting of CMESG/GCEDM is being held.

A. J. (Sandy) Dawson
President, 1993-1997

PLENARY LECTURES

Many studies have classified students' approaches to proving along various dimensions: from pragmatic involving recourse to actions, to conceptual arguing from properties and relationships, (van Dormolen, 1977; Balacheff, 1988); from weak to strong deduction (for example, Bell, 1976; Coe and Ruthven, 1994); according to different modes—enactive, visual and manipulative (Tall, 1995), or proof schemes (Harel and Sowder, in press). Despite differences in emphasis, this corpus of research evidence points to the fact that if the meaning of proof is taken only to be some kind of logical verification, proving in school mathematics is likely to be fraught with conceptual difficulties. Many students have a limited awareness of what proof is about. On the one hand, they show a preference for empirical argument over any sort of deductive reasoning and seem to fail to appreciate the crucial distinction between them: for example, many students judge that after giving some examples which verify a conjecture they have proved it, yet, on the other hand, students tend to assume that deductive proof provides no more than evidence with the scope of the proof's validity being merely the diagrams or examples in the text. Finally, other findings point to the difficulty many students have in identifying the premises of a proof and following through a logical argument from these premises to a conclusion (for evidence on all these points see for example Williams, 1979; Fischbein and Kedem, 1982; Balacheff, 1988; Martin and Harel, 1989; Porteous, 1990; Chazan, 1993; Finlow-Bates, 1994).

A common interpretation of these findings has been to argue that students' understandings of proof are organised along a hierarchy: with empirical 'proof' or procedural validation by action at the bottom and rigorous deductive argument or relational validation based on premises and properties at the pinnacle. But are there other, equally plausible, interpretations? In order to open an alternative window on to the situation, I will sketch out a fictional study in mathematics education that focuses on its potential limitations.

The study sets out to investigate students' understandings of proof and the proving process in mathematics. The sample of students is drawn from a school local to the researcher or from a class of students in the researcher's university or college. Usually, the mathematical background and experience of the students are briefly described but rarely is this description used as an explanatory variable in the interpretation of the results or in any discussion of how 'representative' the students might be. The empirical core of the study comprises the identification and analysis of students' written responses to a range of questions concerning proof. The meaning of what is required as a proof is not made explicit; neither is it clear what students have been taught, what has been emphasised and what forms of presentation have been deemed to be acceptable. The influences of the content and sequencing of the curriculum are ignored in an analysis which takes the individual student and their constructions of proof as the object of attention—an analysis that leads almost inevitably to some kind of hierarchical classification.

This uniformity in the research methodologies employed in the international mathematics education community stand in stark contrast to the huge variation in *when* proof is introduced and *how* it is treated in different countries—as evident from even a cursory glance at textbooks and examination questions. In some curricula, the nature of mathematical proof is discussed explicitly in terms of premises, definitions and logical deductions and the acceptable forms of presentation of proofs are made apparent. In others, definitions and criteria for proving are either implicit or negotiated during the activity. Informal discussions with teachers in one country (the U.K.) reveal a multitude of opinions about how proof should and would be introduced and judged—with some teachers declaring that they would be comfortable with an informal explanation to others who would require a formally presented logical argument.

These considerations lead me to question the existence of a universal hierarchy of 'proving competencies'. My argument elsewhere (Noss and Hoyles, 1996) has been that hierarchies of this sort (e.g., concrete/abstract or formal/informal) are largely artifacts of methodology—if we restrict our terms of reference simply to the interaction of epistemology and psychology, and ignore the social dimension, then it is inevitable that mathematical learning will be perceived as the acquisition of context-independent knowledge within a hierarchical framework. Thus starting from a position of epistemology/psychology locks research and its findings into a tautological loop.

There seem to be two ways out of this dilemma. One is to search for patterns of reasons for differences in student response that stretch beyond the purely cognitive—encompassing considerations of feelings, teaching, school and home factors. Another is to ensure that the goals for including proof in the curriculum and how these are operationalised are clarified and taken into account. Clearly proof has the purpose of verification—confirming the truth of an assertion by checking the correctness of the logic behind a mathematical argument. But at the same time, if proof simply follows conviction of truth rather than contributing to its construction, if proof is only experienced as demonstrating something already known to be true, it is likely to remain meaningless and purposeless in the eyes of students (see, for example, de Villiers, 1990; Tall, 1992; Hanna and Jahnke, 1993). I argue therefore that it is just as important, maybe more so at the school level, for proof to provide insight as to why a statement is true and to throw light upon the mathematical structures under study. Hanna has argued for this alternative approach based upon what she calls explanatory proofs—proofs that are acceptable from a mathematical point of view but whose focus is on understanding rather than on syntax requirements and formal deductive methods (Hanna, 1990, p. 12). Another related option for proof which also aims to encourage student engagement and ownership of the proving activity is again to emphasise explanation but in a social as well as a mathematical sense—the need to explain one's argument to a peer or a teacher as well as to convince oneself of its truth. It is this sense that has been taken up in the U.K. and it is to this innovation that I now turn.

PROOF IN THE U. K. NATIONAL CURRICULUM

In the U.K., the main response to evidence of children's poor grasp of formal proof in the 60's and 70's was the development of a process-oriented approach to proof. Following Polya, (1962), many argued (for example, Bell, 1976; Mason, Burton, and Stacey, 1982; Cockcroft, 1982) that students should have the opportunity to test and refine their own conjectures in order to gain personal conviction of their truth as well as to present their generalisation and any evidence of their validity in the form of a proof.

Clearly there are huge potential advantages of this approach in terms of motivation and the active involvement of students in problem solving and proving. Indeed many prominent researchers at the present time (see, for example, de Villiers, 1990) are arguing for just such a shift in emphasis, suggesting that students develop an inner compulsion to understand *why* a conjecture is true if they have first been engaged in experimental activity where they have '*seen*' it to be true. But before other countries follow this route it would be useful to learn some lessons from what has happened in the U. K. What the mathematics education reform documents failed to predict was how teachers, schools and the curriculum would act upon and re-shape this 'process' innovation: in fact, the deliverers of the innovation ignored just the same potential influences on student response as alluded to earlier in my description of a fictitious mathematics education research study. How will the goals and purposes of the different functions of proof be conceived and how will these functions be organised when they are systematised and arranged into a curriculum? What will be the implications of this choice of organisation? How will the changes be appropriated and moulded by teachers and students?

Answers to these questions can be sought by an analysis of the present situation in the U.K. following the imposition of the National Curriculum. The National Curriculum in Mathematics for children aged 11-16 years is organised into four attainment targets (Department for Education and Employment Education, 1995).[2]

AT1	**Using and Applying Mathematics**
AT2	**Number and Algebra**
AT3	**Shape, Space and Measures**
AT4	**Handling Data**

Rather strangely, communication and proving is to be found in AT1, the target named Using and Applying Mathematics. However, one major implication of the curriculum organisation does not derive from the naming of the targets but from the fact that almost all the functions of proof are separated from other mathematical content. To tease out all the reasons why this compartmentalisation happened would be a fascinating story of the demise of geometry intertwined with political intrigue—but unfortunately this is beyond the scope of this paper. But this separation has already had consequences. Many text books written for the National Curriculum are now divided into sections according to attainment targets. Rather than construction, justification and proof[3] working together as different windows on to mathematical relationships, students are expected to *use* results of theorems in, for example, Shape, Space and Measures but not to prove them; Pythagoras' theorem will be stated and students asked to apply it to calculate a length of a side of a triangle. Additionally, the work under the banner of AT1, has become transformed into an 'investigations curriculum' dominated by data-driven activity during which students are expected to spot patterns, to talk about and justify them. Rarely, if ever, are students required to think about the structures their justifications might illuminate. What also has not been considered is the inevitably ambiguous status of any justification or proof given its disconnection from other mathematical content. For example what is likely to be the reaction of students to proving a formula if it has already been used elsewhere as a fact?

The second major consequence of the organisation of the National Curriculum is the division of all attainment targets into eight levels of supposed increasing difficulty. In AT1, the sequence in the proving process is given below.

NATIONAL CURRICULUM: ENGLAND AND WALES

Attainment Target 1: Using and Applying Mathematics[4]

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[2] The national curriculum has been through several changes each time with a different number of attainment targets. Nonetheless the basis of its organisation has remained unchanged. The structure described here was put in place in 1995 where some attainment targets only appear for children of certain ages.

[3] In the remainder of this paper I will take justifying to mean an explanation which convinces oneself and is communicated to others. I will leave the term 'proving' to convey the more formal sense of logical argument based on premises.

[4] Children age 11-14 years should be within the range of levels 3 to 7. Level 8 is available for very able pupils.

Level 3 Students show that they understand a general statement by finding particular examples that match it.

Level 4 They search for a pattern by trying out ideas of their own.

Level 5 They make general statements of their own, based on evidence they have produced, and give an explanation of their reasoning.

Level 6 Students are beginning to give a mathematical justification for their generalisations; they test them by checking particular cases.

Level 7 Students justify their generalisations or solutions showing some insight into the mathematical structure of the situation being investigated. They appreciate the difference between mathematical explanation and experimental evidence.

Level 8 They examine generalisations or solutions reached in an activity, commenting constructively on the reasoning and logic employed, and make further progress in the activity as a result.

Exceptional Performance

Students use mathematical language and symbols effectively in presenting a convincing reasoned argument. Their reports include mathematical justifications, explaining their solutions to problems involving a number of features or variables.

First, it is worth noting that the division into levels and the stipulation of eight as the number of levels applies to all subjects in the National Curriculum. This decision was undoubtedly not to do with progression in any subject area but rather emanated from the need to impose a uniformity on the curriculum as a whole in order that levels could serve as a mechanism to measure and compare the achievement of students, teachers and schools. What was not anticipated, however, was the far-reaching implications of this levelled classification on individual knowledge domains— on the disciplines themselves and how they are experienced by students. Of relevance here is that the majority of students engage in data generation, pattern recognition and inductive methods while only a minority at levels 7 or 8 are expected to prove their conjectures in any formal sense. The imposition of this hierarchical organisation has therefore meant that most students have little chance to appreciate the importance of logical argument in whatever form and few opportunities to engage in formal discourse requiring any linguistic precision.[5] In a nutshell, it is now official that proof is very hard and only for the most able.

Clearly the shift in emphasis to a process-oriented perspective is an understandable attempt to move away from the meaningless routines that characterised what was largely geometrical proof in an earlier period. While some students managed to undertake the routines of Euclid correctly, far fewer understood more about geometry as a result. But in trying to remedy one problem, others have come to the surface. The meaning of 'to prove' has been replaced by social argumentation (which could mean simply giving

[5] A similar trend in North American has been noted by Hanna (1995) who has argued that the gradual decline of the position of proof in school mathematics and its relegation to heuristics can be attributed partly to the 'process orientation of much of the reforms in mathematics education since the 1960s'. She also suggests that another contributing factor is the persuasiveness of constructivism—or at least the way it is operationalised in the classroom.

some examples) or, at the very least, separated from it; justifying is largely confined to an archaic 'investigations curriculum' separated from the body of mathematics content; and proof is labelled as inaccessible to the majority.

But what are the consequences for student attitudes to and understanding of proof following this massive change in the treatment of proof? What are the consequences for student learning of a curriculum that now contrasts sharply with that adopted elsewhere—in the United States, France, Germany, and countries on the Pacific Rim to name but a few. Some recent research (Coe and Ruthven, 1994) into the proof practices of students who have followed this curriculum, suggests rather unexpectedly that nothing appears to have changed and students remain locked in a world of empirical validation. Even more surprisingly given the emphasis in the curriculum reforms, the researchers also report that students show little attempt to explain why rules or patterns occur, or to locate them within a wider mathematical system

How far are these findings generalisable? As yet this is not known but far more influential than any research study is the pervasive belief amongst influential groups in the U.K. that students' understanding of the notion of proving and proof in mathematics has deteriorated. There has been a huge outcry, mainly amongst mathematicians, engineers and scientists in our universities, complaining about the mathematical incompetence of entrants to their institutions. The argument is that the National Curriculum only pays lip-service to proof with the result that even the more able students who go on to study mathematics after 16 years fail to grasp the essence of the subject. The debate cumulated in 1995 in the publication of an influential report by the London Mathematical Society, a powerful group of mathematicians, known as the LMS Report (London Mathematical Society, 1995). Points 4 and 5 of its summary are reproduced below:

4. Recent changes in school mathematics may well have had advantages for some students, but they have not laid the necessary foundations to maintain the quantity and quality of mathematically competent school leavers and have greatly disadvantaged those who need to continue their mathematical training beyond school level.

5. The serious problems perceived by those in higher education are:
   - a serious lack of essential technical facility—the ability to undertake numerical and algebraic calculation with fluency and accuracy;
   - a marked decline in analytical powers when faced with simple problems requiring more than one step;
   - a changed perception of what mathematics is—in particular of the essential place within it of precision and proof (p. 2).

The message of the LMS report is clear. Students now going on to tertiary education in mathematics and related subjects are deficient in ways not observed before the reforms: students have little sense of mathematics; they think it is about measuring, estimating, induction from individual cases rather than a rational scientific process. Clearly we might argue that the evidence of 'decline' is not sound or that its putative causes are hard to pinpoint given the complexity of the educational process—not least the massive expansion in the university population in the U.K. But this argument is difficult to sustain in the absence of systematic evidence. In fact, the conclusions concerning proof are eminently plausible: given that there are so few definitions in the curriculum, it would hardly be surprising if students are unable to distinguish premises and then reason from these to any conclusion. But rather than pointing out what students 'lack', it would seem to be more fruitful and constructive to find out what students *can* now do and understand following the reforms of the curriculum and the different functions of proof they have experienced. What is needed is a comprehensive study of students' views of proving and proof and the major influences on them. Having followed the new curriculum, what do students judge to be the nature of mathematical proof? What do they see as its purposes? Do they see proving as verifying cases or as convincing and explaining? Do they forge connections between the different functions of proof or do these functions remain fragmented and isolated? What are their teachers' views? Although we have a national curriculum, are there variations in how it is delivered and experienced and if so why and what are the implications for student learning?

These questions take me to a discussion of a research project, Justifying and Proving in School Mathematics, which I have been undertaking with Lulu Healy at the Institute of Education in London since 1995. In this research, we aim to answer some of these questions by surveying student views of proof and trying to explain these against a landscape of variables and influences that extends beyond a simple description of students' mathematical competencies. In the next section, I will describe some aspects of the project in more detail.

THE RESEARCH PROJECT

The project set out in 1995 to probe the conceptions of justification and proof in geometry and algebra amongst 15 year old students.[6] Our aim is to open a range of windows on to students' conceptions of proof in order to find out what they think it involves, what they choose as proofs and how they read and construct proofs. We also want to tease out all the influences that might be brought to bear on these conceptions—the curriculum, teachers and schools. Given that we wanted to investigate students who had gained at least some familiarity with proving in our curriculum, we were forced to sample only high attaining students. We are only mildly interested in discovering what students cannot do, but rather seek to identify profiles of student responses in order to tease out how they might have been shaped and to identify their strengths as well as their weaknesses. Our findings from the survey are to form the basis for thinking about how we might introduce students to proof in the future—to capitalise on any positive outcomes of the reforms in the curriculum while seeking to reduce any limitations. In fact the survey is only the first phase of our project. In the second phase, following our analysis of student and teacher responses, we will design and evaluate two computer-based microworlds for introducing students to a connected approach to proving and proof.

We spent many months reviewing existing literature and discussing with teachers, advisers and inspectors in order to come up with a student questionnaire.[7] We wanted the mathematical content to be sufficiently straightforward for the proofs to be accessible, familiar and in tune with the U.K. National Curriculum, yet sufficiently challenging so there would be differentiation amongst student responses. In our questionnaire, proofs and refutations were to be presented in a variety of forms—exhaustive, visual, narrative and symbolic and set in two domains of mathematics—arithmetic/algebra and geometry.

The questionnaire was pre-piloted by interviews with 68 students in four different schools aiming to find out how far the questions were at an appropriate level and engaging for students. Following the pre-pilot, items were removed that were too easy or modified if too hard. We also wanted to be able to make comparisons between responses in algebra and geometry so we revised the format so its presentation in each domain was completely consistent.

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[6] The project is funded by the Economic and Social Research Council, Grant number R00236178.

[7] We also organised a small international conference on proof in order to share frameworks and present our first ideas for the questionnaire (Healy and Hoyles, 1995).

Simultaneously with the development of the student questionnaire, we designed a school questionnaire to obtain information about the schools—the type of school, its organisation generally and the hours spent on mathematics, the text books adopted and examinations entered and specifically the school's approach to justification and proof. We also sought teacher data to provide information on their background, qualifications, their reactions to the place of proof in the National Curriculum and the approaches they adopted to proof and the proving process in the classroom.

We piloted both questionnaires with 182 students in eight schools after which we were able to iron out any remaining ambiguities and to specify the time required to complete the survey (70 minutes) and the instructions for its administration. The questionnaires were completed between May and July 1996 by 2459 students in 94 classes from 90 schools in clusters throughout England and Wales. We had originally planned to use 75 schools but more requested to take part in our survey—a reflection we believe of the interest teachers have in this topic, their recognition of its importance and their concern about the changes that have taken place. The questionnaires were administered by members of the project team or mathematics educators in different parts of the country who volunteered to help us. This process ensured consistency in administration procedures and a 100% return of questionnaires. While the students answered their questionnaires, their teacher filled in parts of the student questionnaire (see later) as well as completed the school questionnaire.

Schemes for coding the questionnaires were devised and all the coding undertaken and checked during July and August 1996. We are now producing descriptive statistics based on frequency tables and simple correlations as well as modelling student responses against all our teacher and school variables using a multilevel modelling technique (see Goldstein, 1987). The purpose of this paper is not to report the findings of this statistical analysis but rather to provide a flavour of how students in the U.K. now see proof through the presentation of a selected sample of questions together with some student responses.

A WINDOW ON STUDENT VIEWS OF PROOF

The first question of the student questionnaire asks students to write down everything they know about proof in mathematics. A rather typical answer is given below:

> *All that I know about proof is that when you get an answer in an investigation you may need some evidence to back it up and that is when it is proof. You have to prove that an equation always works.*

Another student wrote:

> *All I know is the proof in mathematics is that, if say you are doing an investigation, and you find a rule, you must prove that the rule works. So proof is having evidence to back up and justify something.*

These responses clearly echo our curriculum structure where it is the 'investigation' which requires proof—or at least the presentation of some sort of evidence. Following this open-ended question, the questionnaire is divided into two sections, the first concerned with algebra and the second with geometry. The first question of each sections is in a multiple-choice format as illustrated in Figures 1 and 2.

The purpose of having a multiple-choice question at the beginning of each section is to introduce students who may not be acquainted with the meaning of 'to prove' to a range of possible meanings—remember that our students are not introduced to definitions nor generally required to produce logical deductions in mathematics. Almost all the student responses used as options for this question were derived from our pre-pilot and pilot studies or from school text books, so we could be fairly sure that

A1. Arthur, Bonnie, Ceri, Duncan and Eric were trying to prove whether the following statement is true or false:

When you add any 2 even numbers, your answer is always even.

Arthur's answer

a is any whole number

b is any whole number

$2a$ and $2b$ are any two even numbers

$2a + 2b = 2(a + b)$

So Arthur says it's true.

Bonnie's answer

2 + 2 = 4	4 + 2 = 6
2 + 4 = 6	4 + 4 = 8
2 + 6 = 8	4 + 6 = 10

So Bonnie says it's true.

Ceri's answer

Even numbers are numbers that can be divided by 2. When you add numbers with a common factor, 2 in this case, the answer will have the same common factor.

So Ceri says it's true.

Duncan's answer

Even numbers end in 0 2 4 6 or 8. When you add any two of these the answer will still end in 0 2 4 6 or 8.

So Duncan says it's true.

Eric's answer

Let x = any whole number, y = any whole number

$x + y = z$

$z - x = y$

$z - y = x$

$z + z - (x + y) = x + y = 2z$

So Eric says it's true.

From the above answers, choose **one** which would be closest to what you would do if you were asked to answer this question. [ ]

From the above answers, choose the **one** to which your teacher would give the best mark. [ ]

Figure 1: The First Algebra Question

G1. Amanda, Barry Cynthia, Dylan, and Ewan were trying to prove whether the following statement is true or false:

When you add the interior angles of any triangle, your answer is always 180°.

Amanda's answer

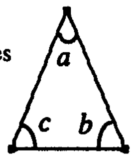
I tore the angles up and put them together.

It came to a straight line which is 180°. I tried for an equilateral and an isosceles as well and the same thing happened.

So Amanda says it's true.

Barry's answer

I drew an isosceles triangle, with c equal to 65°.



Statements — Reasons

$a = 180° - 2c$....... Base angles in isosceles triangle equal

$a = 50°$............... $180° - 130°$

$b = 65°$.............. $180° - (a + c)$

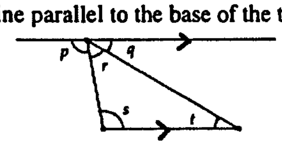
$c = b$.................. Base angles in isosceles triangle equal

$\therefore a + b + c = 180°$

So Barry says it's true.

Cynthia's answer

I drew a line parallel to the base of the triangle



Statements — Reasons

$p = s$.......................... Alternate angles between two parallel lines are equal

$q = t$.......................... Alternate angles between two parallel lines are equal

$p + q + r = 180°$........ Angles on a straight line

$\therefore s + t + r = 180°$

So Cynthia says it's true.

Dylan's answer

I measured the angles of all sorts of triangles accurately and made a table.

a	*b*	*c*	*total*
110	34	36	180
95	43	42	180
35	72	73	180
10	27	143	180

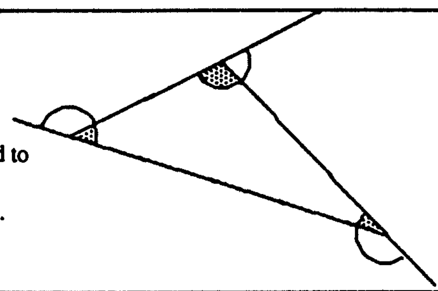
They all added up to 180°.

So Dylan says it's true.

Ewan's Answer

If you walk all the way around the edge of the triangle, you end up facing the way you began. You must have turned a total of 360°.

You can see that each exterior angle when added to the interior angle must give 180° because they make a straight line. This makes a total of 540°.
540° - 360° = 180°.



So Ewan says it's true.

From the above answers, choose **one** which would be closest to what you would do if you were asked to answer this question.

From the above answers, choose the **one** to which your teacher would give the best mark.

Figure 2: The First Geometry Question

some at least would be familiar. These questions (and others with a similar multiple- choice form) were designed to help us ascertain what students *recognised* as a proof. These responses could then be compared and contrasted with what students actually *produced* as proofs later in the questionnaire. Clearly these two processes are related but not identical—constructed proofs require specific knowledge to be accessible. As well as presenting different meanings for proof, the choices in the questions ranged over different forms to enable us to tease out how far students are influenced by the form as well as the content of a proof: the 'proof types' shown in Figures 1 and 2 are categorised as empirical, enactive, narrative and formal with two examples of this last case, one correct and one incorrect.[8]

We are seeking to investigate the influence of the teacher in various ways—through their responses in the school questionnaire but also through the eyes of the student in the student questionnaire. In the last part of each multiple-choice question, the student is required to choose the proof to which they think their teacher would give the best mark. Responses here will help us to see how students interpret what is rewarded by their teacher. The teachers are also asked to complete these same questions—to write down what *they* would choose as a proof as well as what they think their students will choose as the one given the best mark. The analysis to date reveals a picture that is by no means simple. What is interesting is the sizeable minority of students whose personal choice bears no resemblance to the one they believe will receive the best mark (only 21% overall made the same choice for both[9]) and the small group who choose, for the latter, a formal proof that is incorrect!

Following each multiple-choice question in both Algebra and Geometry are questions seeking to find out how students evaluate each of the choices previously presented. Do they think it is correct? Do they believe that the proof holds for all cases or simply for a specific case or cases? Do they judge it to be explanatory or convincing? An example of the format used as it applies to Bonnie's 'proof' is shown in Figure 3.

Bonnie's answer

$2 + 2 = 4$	$4 + 2 = 6$
$2 + 4 = 6$	$4 + 4 = 8$
$2 + 6 = 8$	$4 + 6 = 10$

So Bonnie says it's true.

Bonnie's answer:

Has a **mistake** in it	1	2	3
Shows that the statement is **always true**	1	2	3
Only shows that the statement is true for **some** even numbers	1	2	3
Shows you **why** the statement is true	1	2	3
Is an easy way to **explain** to someone in your class who is unsure	1	2	3

Figure 3: Student Evaluations of Bonnie's Answer

[8] Before the students started to respond to the questionnaire, it was pointed out to them that for this type of question several options could be 'correct'.

[9] In fact there is a significant relationship between these two choices but the correlation is low.

By analysing all the responses to this question, we will find out if students are convinced of the truth of a conjecture by a list of empirical examples. Do they judge that these examples help them to explain the result? Do they recognise that it only shows the conjecture is true for some numbers even though they might have chosen it as their response or as the one to which the teacher would give the best mark?[10] We are analysing whether it makes a difference if the proof evaluated was the one chosen by the student. Preliminary analysis in algebra suggests that students tend to choose for themselves 'proofs' that they evaluate as general *and* explanatory while the proofs they think will be assigned the best mark are evaluated as general but not necessarily explanatory. Formal presentation (correct or incorrect) is highly favoured for the best mark while narrative is the favourite for individual choice.

Following the set of multiple-choice questions—one of which in each section concerns an incorrect conjecture which is proved by some choices and refuted by others—the students are asked to construct some proofs. Great care was taken to choose proofs that would be either familiar or at least accessible to most students who had followed our curriculum. The influence of the investigations curriculum is again very evident in their constructions, as illustrated in Figure 4:

The same phenomenon is illustrated by another student's response to the second, rather harder, algebra proof construction in Figure 5.

Even in geometry, rarely the site for a school 'investigation', the discourse of investigations is evident in the form of many of the student responses and the explanations given, as illustrated in Figure 6.

What is evident from these responses is that students have learned a format for presenting and 'proving' an investigation. They have appropriated some structures to help them to make sense of a situation and to assist in developing a language for proof. But, limitations of these approaches are very apparent. Students appear to be imposing a 'type' of proof on a question; for example, a proof must involve data. All too easily students seem to have shifted their notion of proving from one ritual to another—from a *formal* ritual to a *social* ritual—something added on to the end of an investigation. This new ritual is likely to be equally meaningless, empty of mathematical illumination and missing any mathematical point unless ways can be found to connect it to a sense of the function of proof (in any of its forms) and to the constructive activity of the investigation itself.

Clearly I have selected student responses to illustrate my point and it is important to guard against the danger of over-generalising. Our survey shows huge variation in student response. We are in the midst of generating complex and sophisticated models using both student and school questionnaires to tease out how variables interrelate and to describe the range of contributory 'causes' for any differences in the student response profile. Is it curriculum, text book, examination board, school or teacher that shapes response or a combination of these? Are students' responses consistent in any domain or across domains? If so, how do their different answers correlate?

Despite this cautionary note, it has been salutary to identify the extent of the influence of the curriculum (either intended or unintended) and how it is delivered by teachers. These phenomena and their behavioural manifestations cannot be 'blamed' on the student. Their behaviours cannot simply be ascribed to properties of age, ability or even individual interactions with mathematics. We cannot and must not assume the students are learning an object whose meaning corresponds to that assigned to it in mathematical discourse. Its meaning has been radically changed by the way the curriculum has been

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[10] Preliminary analysis suggests this latter choice may be subject to interesting gender differences.

organised. For example, responses in geometry are very different—and much worse from a mathematical perspective—from those in algebra. This finding is unsurprising given the almost complete disappearance of geometrical reasoning in the curriculum, but nonetheless it casts doubt on how far proof can be con-

A4. Prove whether the following statement is true or false. Write down your answer in the way that would get you the best mark you can.

When you add any 2 odd numbers, your answer is always even.

My answer

Statement -
When you add any 2 odd numbers your answer is alway even.
Hypothesis - I believe that the above statement is absolutely correct.
My aim - I will prove the statment is correct by conducting some further work and calculations :-

1 + 1 = 2
3 + 3 = 6
5 + 5 = 10

There is clearly a pattern bedween the 3 addictions I have just carried out. Now letts see If I can use algebra instead to find the true answer to the statement

n = x + y the n th number = x - the first number plus y the second.

Conclusion -
I can see that the statement is correct from the working I have carried out.

Figure 4: An 'investigations' response to adding 2 odd numbers

sidered as a unitary mathematical 'object' separate from its domain of application. Many mathematics educators have shown how we must take seriously the influence of the teacher—and our teacher data will shed light on this. But surely, we have now to look for reasons which include this curriculum organisation and in the case of the U.K., its separate targets and the straightjacket of its levelled statements?

The starting point of our research was the belief that proving in mathematics need not be restricted to 'the exceptional' and that the organisation of the U.K. national curriculum seriously underestimates the potential of our students. Responses to the survey are proving to be rather promising in this respect. Alongside the ritualised responses described above and the all too numerous solutions that simply resort to empirical examples, there are some fascinating and ingenious proofs which provide a pointer to the wealth of resources that might be tapped and built upon in the process of building a proving culture.

Take, for example, the following proof of the first algebra example which combines a visual approach to show the structure of the problem with a narrative to indicate the generality of the argument (see Figure 7).

Even in geometry where responses are disappointing, we find several examples of creative proofs as illustrated in Figure 8.

Both of these responses are likely to be influenced by school factors which we intend to investigate through interviews with teachers in the schools concerned. It may be, for example, that the geometry proof above has roots in prior experimentation with dynamic geometry software. But note how both point to an iterative or inductive approach to proof where the starting point is not data but rather a specific and special case where the conjecture is known to be true from which a road to the general case is suggested—in the former case by language and image and the latter by means of 'adding a bit in one place' and 'taking the same from another'. This calls into question the whole notion that students' development of mathematical justification *has* to proceed from inductive to deductive processes. Clearly this assertion needs careful investigation.

CONCLUSIONS

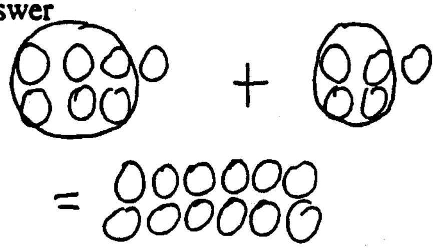
We have a long way to go to unpick all the factors that go together to underpin student conceptions of proof in the new scenario we now face in the U.K. It is almost certain that many of the influences on student responses, interactions in classrooms and institutions were not anticipated. Nonetheless we can learn from this experience. The main message of this paper is that mathematics educators can no longer afford simply to focus on student and teacher if we are to understand teaching and learning and if we seek to influence practice. We cannot ignore the wider influences of curriculum organisation and sequencing if we are to avoid falling into the trap illustrated by the U.K.'s ubiquitous 'investigation'.

The challenge remains to design new situations that motivate students to prove in *all* its functions and that help students to forge connections between them at *every* opportunity. We must resist the temptation of assuming that situations that engage students with proof *must* follow a linear sequence from induction to deduction. To do this we simply have to keep our goals clearly in mind. As Goldenberg (in press) has argued, we must aspire to developing 'ways of thinking' not their 'products' and use these as guides to curriculum organisation—but at the same time not neglect to recognise how these ways of thinking are deeply connected with content domain. To do this effectively, we must exploit all the resources at hand: our collective knowledge from research much of it undertaken under the influence of a very different set of curriculum restraints; the findings of our present survey; and the opportunities opened up by *new* tools now available—tools that will change the landscape of assumptions underpinning proof as well as the strategies open for undertaking proving. If we fail in this endeavour, there is a real

A4. Prove whether the following statement is true or false. Write down your answer in the way that would get you the best mark you can.

When you add any 2 odd numbers, your answer is always even.

My answer



This is because if you take away 1 from both the odd numbers you will get an even number (as circled). So if you add the 2 '1's left over from each side you will get 2. So it will effectively become even + even + 2 which is (odd − 1) + (odd − 1) + 2

Figure 7: Developing a language for proving

G4. Prove whether the following statement is true or false. Write your answer in a way that would get you the best mark you can.

If you add the interior angles of any quadrilateral, your answer is always 360°.

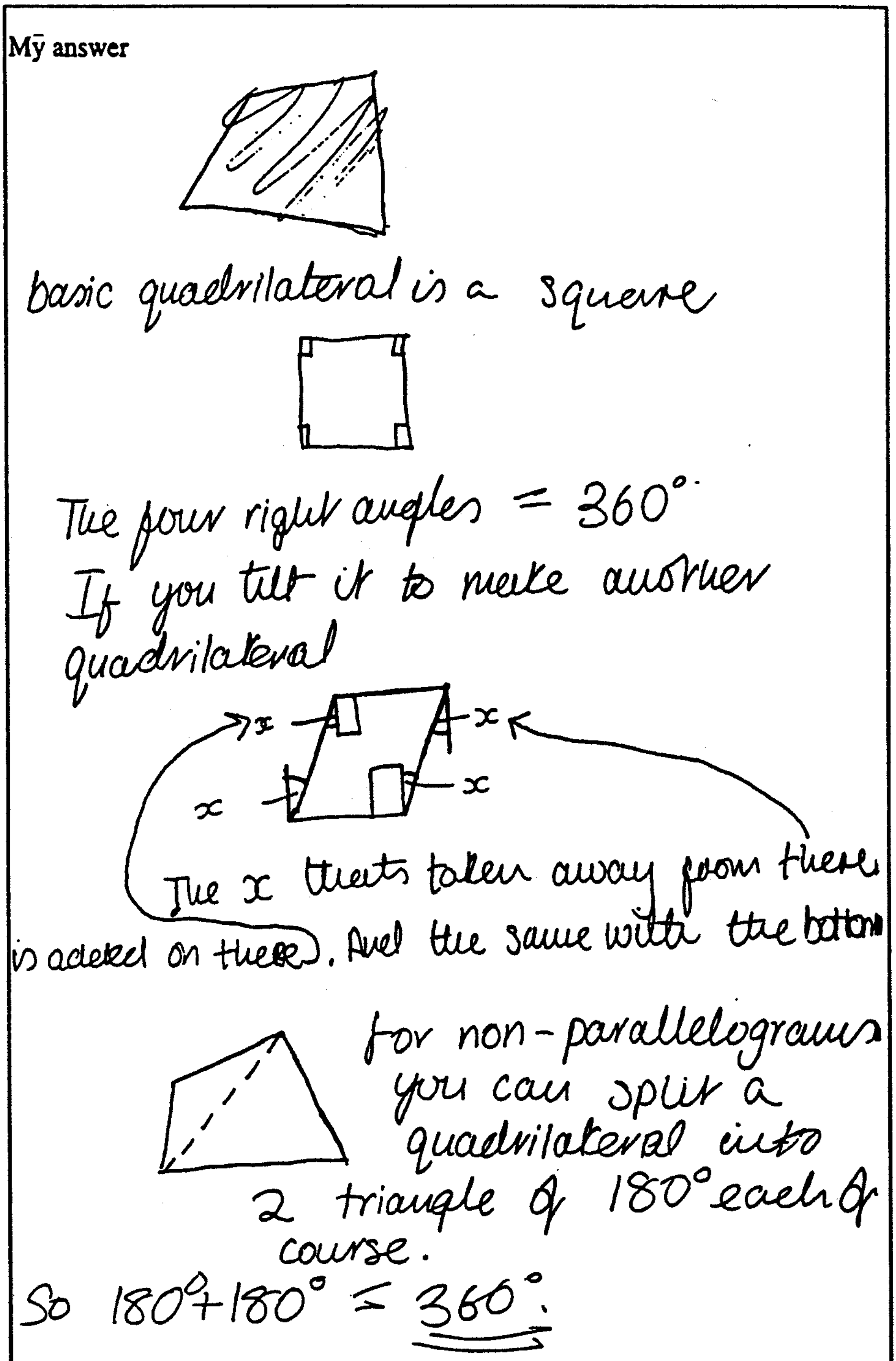


Figure 8: From a special case towards a generalisation

danger that the pendulum will simply reverse and we will return to the failed approaches of the past. In the U.K., we now stand at this turning point.

Will the curriculum 'swing backwards'? Or will we be able to seize the opportunity opened up by all the discussion around proof, to take a step forward and begin to induct all students into a negotiable but also mathematical proving culture where they can derive a sense of purpose in proving and come to see a proof as generative and not merely descriptive.

ACKNOWLEDGEMENTS

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Plenary Lecture II

ALIVE MATHEMATICAL REASONING

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INTRODUCTION

Cecilia Hoyles in first plenary told us that whenever we examine someone's conceptions of proof we should learn something about their background—what have they been taught about mathematics. So, I thought it was appropriate that I start by telling you something of my background, since I am going to talk with you for the next hour about my views of proof as gleaned from my experiences as student, teacher, mathematician and general experiencer of the world.

I have always loved geometry and was thinking about geometric kinds of things since I was very young as evidenced by drawings that I made when I was six which my mother saved. But I did not realize that the geometry, which I loved, was mathematics. I was not calling it geometry—I was calling it drawing or design or not calling it anything and just doing it. I did not like mathematics in school because it seemed very dead to me—just memorizing techniques for computing things and I was not very good at memorizing. I especially did not like my high school geometry course with its formal two-column proofs. But I kept doing geometry in various forms in art classes, out exploring nature, or by becoming involved in photography. This continued on into the university where I was a joint physics and philosophy major and took only those mathematics courses which were required for physics majors. I became absorbed in geometry-based aspects of physics: mechanics, optics, electricity and magnetism, and relativity. On the other hand, my first mathematics research paper (on the geometry of Venn diagrams for more than 4 classes) evolved from a course on the philosophy of logic. There were no geometry courses except for analytic geometry and linear algebra, which only lightly touched on anything geometric. So, it was not until my fourth and last year at the university that I switched into mathematics because I was finally convinced that the geometry that I loved really was a part of mathematics. This is not an uncommon story among research geometers. Since high school, I have never taken a course in geometry because there were no geometry courses offered at the two universities which I attended. So, in some ways I may have had the advantage of not having taken a geometry course!

But I was educated in a very formal tradition—in fact, mathematics was, I think, the most formal that it has been about the time that I was studying at the university in the late 1950's and early 1960's. One of the evidences for this was the number of geometry courses offered at colleges and universities—there were practically none anywhere at that time except for a few geometry courses for prospective school teachers and still at many institutions such courses are the only geometry courses offered.

I am the same generation as most of the faculty now in mathematics departments in North America (most of us are 50-65 years old) because we were hired to teach the baby boomers in the 1960's. So now my generation is clogging up most of the tenure faculty positions all over North

America, and in the USA we will not be required to retire because the Supreme Court has ruled recently that it is unconstitutional to have mandatory retirement ages. Almost all of the mathematicians in my generation had a very formal training in mathematics. This has affected us and affected mathematics and will continue to affect mathematics because my generation now has the positions of authority in mathematics.

I want to mention specifically one mathematician in my generation and that is Ted Koscynski, the suspected Unabomber. He had very much the same kind of university mathematics education that I had. Both of us at the time were socially inept and it was difficult for us to get to know people. In fact, in some ways, this was encouraged in our training all the way through graduate school and certainly was not a hindrance in any way. My thesis advisor talked to me and a few of the other men graduate students and said to us that it would be very important for us to find wives who would take care of all of our social responsibilities, so that we would not have to deal with social things and could put all of our energy into mathematics. Fortunately, one of the things that helped save me was that I did not take his advice—I got married but not to such a wife. The suspected Unabomber talks about similar things which happened to him.

Both Ted Koscynski and I accepted tenure track professorial positions at major research mathematics departments (Berkeley and Cornell). And we were initially both successful with professional mathematics. Then, in the early 1970's, both Ted Koscynski and I quit mathematics. I got angry with mathematics—I got very furious about what mathematics had done to me. It is too complicated to go into all my feelings then (even if I could retrace them accurately)—but if someone came up to me at that time and called me a mathematician I felt strongly like punching them in the face! The evidence indicates that Ted Koscynski had a different more violent reaction but his writings express feelings very similar to the ones I had at that time. He and I both went into the forest and built a cabin and lived there alone and we isolated ourselves. But, there was a huge difference—I made a constructive positive breakthrough and Ted Koscynski didn't.

It was geometry, the many friends I made, and my family that brought about this breakthrough and in many ways saved my life. I got back into geometry. Before this, I had not been teaching geometry—I had been teaching geometric topology and such courses but all of my teaching up to then had been very formal. There was one geometry course at Cornell at the time—the one for prospective secondary school mathematics teachers. It was not considered to be a *real* mathematics course and I considered myself to be a *real* mathematician so I did not have any interest in teaching it. But at that time, when I thought I was quitting mathematics, I needed to teach a little in order to have enough money to survive in my cabin, so I took a leave without pay and then occasionally came back and taught for some money. (Fortunately I did not burn any bridges.) So I started teaching this geometry course for prospective teachers. In the first three years that I taught the course, while living in the forest, three mathematics educators familiar to most of you were in the class, David Pimm (Open University), Jere Confrey (Cornell), and Fran Rosamond (National University, San Diego). This geometry course was essentially all that I was teaching for a few years. A lot of what I am going to talk about are my experiences with that course and what happened since then. This course (and my new friends) pulled me out of the fire that Ted Koscynski never got out of.

FORMAL DEDUCTIVE SYSTEMS

Global formal deductive systems can be very powerful and are important in certain areas (for example in the study of computer algorithms and in the study of questions in the foundations of mathematics). Local formal deductive systems can be important and powerful in many areas of mathematics (for example group theory.) But many people hold the belief that mathematics is only the study of formal systems. These beliefs are wide-spread especially, I find, among people who are not mathematicians or teachers of mathematics. Let me give some descriptions of formal mathematics. For example, in *FOCUS: The Newsletter of the Mathematical Association of America* a professor of computer science wrote:

> ... one of the most remarkable gifts human civilization has inherited from ancient Greece in the notion of mathematical proof [and] the basic scheme of Euclid's *Elements* ... This scheme was formalized around the turn of the century and, ever since ... mathematicians have rested assured that all their ingenious proofs could, in principle, be transformed into a dull string of symbols which could then be verified mechanically. One of the basic features of this paradigm is that proofs are fragile: a single, minute mistake (e.g., an incorrectly copied sign) invalidates the entire proof. (Babai 1992)

This is the kind of view of mathematics that I learned when I was in school and the university.

Here's a more recent description that just appeared in the past year in the *American Mathematical Monthly* in an article (by another professor of computer science and member of my mathematical generation) about a new reform teaching technique and text for discrete mathematics which is based on a "computational" formal approach which uses uninterpreted formal manipulations which have been stripped of meaning:

> ... most students are troubled by the prospect of uninterpreted manipulation. They want to think about the meanings of mathematical statements. Having meanings for objects is a "safety net", which students feel, prevents them from performing nonsensical manipulations. Unfortunately, the use of the "meaning" safety net does not scale well to complicated problems. Skill in performing uninterpreted syntactic manipulation does. (Gries 1995)

He literally means to get rid of the meaning. He takes literally the formalist view of mathematics that the meaning is not important. He goes further to say that the meaning actually gets in the way. I was at one of his talks when he was explaining his new teaching method and he gave a proof of some result in discrete mathematics and I tried to follow the meaning through from the hypothesis to the conclusion, because the hypothesis and conclusion did have meaning. I tried to follow that meaning through the proof in order to see the connections, but I failed to do so. I brought this up at the end and he said something close to: "*Yes!* That's precisely the idea! We have managed to get the meaning out of the way so that it doesn't confuse the students so they are now better able to do mathematics."

Now let me give another description of mathematics. This was written by Jean Dieudonné in an article which was written in response to an article by René Thom in which Thom was talking about intuition and how it was important to bring in and foster intuition in the schools.

> I am convinced that, since 1700, 90 per cent of the new methods and concepts introduced in mathematics were imagined by four or five men in the eighteenth century, about thirty in the nineteenth, and certainly not more than a hundred since the beginning of our century. These creative scientists are distinguished by a vivid imagination coupled with a deep understanding of the material they study. This combination deserves to be called "intuition." ...
>
> In most cases (the transmission of knowledge) will be entrusted to professors who are adequately educated and prepared to understand the proofs. As most of them will not be gifted with the exceptional "intuition" of the creators, the only way they can arrive at a reasonably good understanding of mathematics and pass it on to their students will be through a careful presentation of their material, in which definitions, hypotheses, and

arguments are precise enough to avoid any misunderstanding, and possible fallacies and pitfalls are pointed out whenever the need arises. (Dieudonné 1973)

Both of the first two quotes were from computer scientists who down-play the role of meaning and intuition in mathematics. Now, Dieudonné who certainly *is* a mathematician and a very good one, pointed out that intuition and imagination are very important but that there are only a few people (apparently, men) who have that intuition and that for the rest of us it is necessary for mathematics to be put down in a very precise formal way. Dieudonné has two claims to fame that are connected to this. One is he was the founder of the Bourbaki movement which was an attempt (which was never finished) to formalize all of mathematics. The other which is more significant for this gathering is that about the time of this article he was chair of the ICMI (International Commission on Mathematics Instruction) and chair of it at the time that the "New Math" was being spread around the world. He has always been involved in education.

Another example of descriptions of mathematics: *Mathematica*©, the computer program, when it first came out was advertised as a program that can do all of mathematics— remember those early ads? If you believe a strictly formal view of mathematics then that claim was believable and many people did believe it.

CONFINING MATHEMATICS WITHIN FORMAL DEDUCTIVE SYSTEMS IS HARMFUL

Now I want to talk about how I see the view (which I take as starting roughly a hundred years ago) that mathematics is just formal systems has been harmful. I see this view as harmful because:

— it **encourages what I think are incorrect beliefs.** For example, those beliefs mentioned above that mathematics is only the study of formal systems. Of course, people can have disagreements as to what mathematics is, but I think that most of the people in this room do not believe that mathematics is just formal systems. And let me make it clear that I believe that formal systems *do* have a place in mathematics and that they are very useful and very powerful in many ways. Formal systems are very important in computer science because that is what a computer does —deal with formal systems. So it is not surprising that it was professors of computer science who made the statements that I have put here. Formal systems have certainly been very important in various parts of algebra and analysis and topology (which was my area of research) which flourished in this century. But geometry virtually disappeared as evidenced by the fact that there were almost no undergraduate geometry courses in 1970. That trend has now reversed. For example, now at Cornell there are eight undergraduate geometry courses and of those eight there is only one-half of one of them that deals with axiomatic systems. So things are changing.

— **much interesting and useful geometry is either not taught at all or is presented in a way that is inaccessible to most students.** For example, spherical geometry was in the high school and university curriculum (or, at least in the textbooks) of 100 years ago. Of course, high schools in those days were more elite institutions than they are today, but spherical geometry is almost entirely absent from our courses and textbooks now. Why is it that it disappeared? It is *not* because it is not useful: Spherical geometry is very applicable—navigation on the surface of the earth, the geometry of visual perception, the geometry of astronomical observations, surveying on a scale of several kilometers. Spherical geometry is a very useful geometry, but we do not teach it anymore—why? I think the reason is that spherical geometry is very difficult to formalize—there is no convenient axiom system for spherical geometry. There *is* an axiom system for spherical geometry (Borzuk did it just before the Second World War)—it is in a book that is in many mathematics libraries but it rarely has been used because it is not a useful axiom system. "Non-Euclidean" geometry has been often taught in undergraduate geometry courses, but it has always been "the" non-Euclidean geometry, hyperbolic geometry, which has a relatively simple axiom system and which has only been around for about 160 years. Spherical geometry which is very old (the Babylonians and Greeks studied it) is rarely taught. I can not think of any reasonable explanation for why spherical geometry disappeared except that it does not fit into formalism. This is one of the reasons that I have it in my geometry course. When freed from the confines of formal systems it is possible to present spherical geometry in ways that are based on geometric experiences and intuitions. (See Henderson, 1996a)

— **important notions in mathematics are formally defined in ways that separate them from the students' experiences**. For example, the new Chicago Mathematics Curriculum for American secondary schools (which has many good things in it and is now the fastest growing curriculum in the USA) *defines* a rotation as the product of two reflections. Now that is an interesting fact (theorem) about rotations. But what does a student think when he or she comes to that as the definition of what a rotation is? It is very difficult to relate the product of two reflections with our experiences of rotations such as opening a door or riding a merry-go-round. One of the problems is that our intuition of rotations is a dynamic thing—it is actually a motion. Whereas to think of rotation as the product of two reflections is a static thing—it is the result of the rotation motion that is equal to the product of two reflections. If I were a student and saw this definition in the textbook I would say that this geometry is not relating to what *I* know geometry is and I would feel that the text is telling me that my experiences and intuitions are not important. It appears that the main reason for using this definition is that it is convenient formally in the deductive system in which the geometry in the text is confined. Also, differential geometry (the geometry of curves and surfaces, the geometry of the configuration spaces of mechanical systems, the geometry of our physical space/time) has extremely difficult formalisms which make it inaccessible to most students and even, I suspect, most mathematicians are uncomfortable with the formalisms of differential geometry. Some people have called it the most complicated formalism in all of mathematics. But, yet, differential geometry is basically about straight lines and parallelism—very intuitive notions. When we insist on formalizing differential geometry then it becomes inaccessible—even more so because there is no agreed upon formalism. My second book, (Henderson, 1996b), is an attempt make differential geometry accessible by basing it on geometric experiences and intuitions, as opposed to basing it on standard algebraic and analytic formalisms.

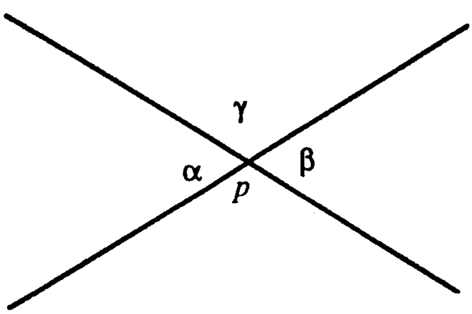
— **many important and useful questions are not asked**. This was something that really surprised me when I started teaching this geometry and started listening to the prospective teachers who were taking the course. There are a lot of questions that students have that we never ask in mathematics classes. For instance, the reliance on a formal Euclidean deductive system rarely allows for questions such as "What do we mean when we say that something is straight?" We normally don't ask that in any classes, even though we talk about straight lines all the time. We just write down some axiom or we just say "everyone knows what 'straight' is." In differential geometry the formalism has attempted to get at what the meaning of straight is, but in a way that is not accessible. But one can ask the question about what it means to be straight; you can ask that of students. I've done it with first graders—they can come up with good discussions. One of the results of this is that when spherical geometry or other geometries are talked about, usually they are just presented with some statement like: "We will define the straight lines to be the great circles on the sphere." But that is ridiculous, the great circles *are* the straight lines on the sphere, you do not have to define them. If you have a notion of what straightness is, then you can imagine a bug crawling around on the sphere and ask how would the bug go if the bug wanted to go straight. You can convince yourself that it is the great circles. But we cannot even ask those questions in a formal context. Also, the connections between linear algebra, geometric transformations, symmetries, and Euclidean geometry are very difficult to talk about in a formal system (in fact I don't know if I want to say impossible or not), but it is not conveniently done and often not done at all in a formal system. Remember the example above of defining a rotation to be the product of two reflections. (Other questions which we ignore include: "Why is Side-Angle-Side true on the plane (but not on the sphere)?", "What is the geometric meaning of tangency?", and "How do we experience the connections between ?")

— **mathematicians are being harmed.** I have already talked about how mathematicians are being harmed—I was harmed by the over emphasis on formalism—so was Ted Koscynski. And I'm sure that you know of examples (at your own university or around your own university) of mathematicians, roughly my generation, who have more or less dropped out of society. There are a lot of them around—people who have been good mathematicians, who had been successful in the system back in 1950's and 1960's. So it has been harmful to mathematicians.

— **students are being harmed.** When a student's experiences lead her/him to understand a piece of mathematics in a way that is not contained in the formal system, then the student is likely to lose confidence in her/his own thinking and understanding even when it is backed up by what I will call alive geometric reasoning. Deductive systems do not encourage alive mathematical reasoning (which in my experiences with students and teachers is a natural human process) and thus they serve to deaden human beings whose thinking and understandings are forced to reside in these systems. We now have machines that can do the computations and formal manipulations of deductive systems: we need more alive human reasoning.

Here are some examples:

One of the things that I clearly remember from the beginning of my teaching of the geometry course is the following: I was teaching the Vertical Angle Theorem and its standard proof:



$$\gamma + \alpha = 180 \text{ degrees}$$
$$\gamma + \beta = 180 \text{ degrees}$$

$$\alpha = 180 - \gamma = \beta$$
$$\alpha = \beta$$

I can still remember one of the students who was very shy and wouldn't speak up in class, but I was having the students do writing. She wrote on her paper something like the following:

> All you have to do is do a half-turn. Take this point here (p) and rotate everything about this point half of a full revolution. We have already discussed that straight lines have half-turn symmetry and so each line goes onto itself and α goes onto β.

I don't know what your reaction is now but my reaction then was "That's not a proof" and I told her so. Fortunately, though she was shy, she was persistent and stubborn and she kept coming back and insisting that that was a proof. She worked on me for about two weeks and I kept listening to her and struggling with the question, Is that a proof?, because it did not seem like proofs that I had been accustomed to and that I would accept. Finally, she convinced me and now I think it is a *great* proof and much better than the standard proof which is in most of the textbooks. The standard proof has a lot of underlying assumptions that need to be cleared out and many formal treatments do that—they put in the "Protractor Postulates" which state the appropriate connections between angles and numbers and then you can do the standard proof. But the proof with the half-turn is just connected to a certain symmetry of straight lines. You can use other symmetries of straight lines to prove this result also, but this proof is the cleanest, the simplest. And this proof is not possible in a formal system and it is particularly not possible in a formal system if (because you want to insist on putting everything in a formal system) you define a rotation as the product of two reflections. That particularly won't work here because, if you take one of the lines and reflect through the line and then reflect perpendicular to the line that is equivalent to a half-turn, but there is no pair of reflections that will simultaneous do that to both of these lines, but yet a half-turn clearly preserves both lines. I do not see any reasonable way for that to have been included in any kind of formal system. So, if I had been insisting on formal systems, I would have missed out on the half-turn proof and not learned this bit of mathematics. I almost missed out anyhow and it was only because she was very persistent.

After that experience I started listening more to students and expecting that when they would say things that I didn't understand, that maybe they really did have something (and something that I could learn). I took the attitude that we are not working in a formal system, but that we are doing mathematics the same way that mathematicians mostly do mathematics. (In geometry, mathematicians do not stick inside any particular formal system, we use whatever tools might be appropriate: computers, linear algebra, analysis, symmetries. Mathematicians use symmetries a lot!) As I listen to students I have been learning more and more geometry from the students. I used to be surprised at that and thought it was just because I had not been teaching the course for very long. I thought that after I have taught it for a while then I will know it all and I will not see anything new. Well, what happened is that I have been teaching the course for 22 years now and now 30-40% of the students every semester show me some mathematics that I have never seen before! (These students are in different programs—some are mathematics majors, all the prospective secondary school teachers, and most of the mathematics education graduate students.) I would miss out on most of this new geometry if things were being done inside a formal system.

Another example: There are many properties of parallel lines in the plane (for example, any line which traverses two parallel lines will intersect those lines at the same angle) whose proofs depend on the parallel postulate. When we get to that point in the course I let the students come up with their *own* postulate—whatever it is that they think is most important to assume that will separate the plane from the sphere. There is a difference between the plane and the sphere and there is some difference that has to do with parallel lines. The students come up with all kinds of different postulates, many of which I think would be much more reasonable to assume than the usual parallel postulates. By the way, Euclid's parallel (fifth) postulate is true on the sphere—Euclid's parallel postulate is *not* what distinguishes spherical geometry from plane geometry, contrary to what many books say. I take that as evidence that people who have written such mistakes about spherical geometry have never really looked at a sphere—they have just been looking at the situation formally and thus made the mistake.

Let me give another example to show that I can think about something that is not just geometric. Here is the proof that is usually given to American secondary school students that $0.99\cdots 9\cdots = 1$:

$$x = 0.99\cdots 9\cdots$$

$$10x = 9.9\cdots 9\cdots$$

now subtract both sides to get

$$9x = 9.0\cdots$$

and thus

$$x = 1$$

This proof embodies a very useful technique for figuring out, when you have a repeating decimal, what fraction is equal to that repeating decimal. It is a very useful technique in that context. But I claim that it is not a proof in this context. I claim it is something that is masquerading as a formal proof: It looks like a formal proof, it has steps and x's and all that stuff. I started asking my calculus students at Cornell what they thought, and some of the best high school students in North America come to Cornell. They mostly know this proof, because they learned it; but only about half of them believe it, because they do not believe that $0.99\cdots9 = 1$. To show you why *I* think that this is masquerading as a proof and really isn't a proof, let us consider the following: Let us try to make this a little more precise as to just what it is we mean by $0.99\cdots9\cdots$ (that is part of the problem here). Well, to most students what $0.99\cdots9$ means is, 0.9, then 0.99, then 0.999, ...— a limit of a sequence (at the time they are expressing this, they might not even know what a sequence is)—you keep putting on one more 9, you go on for ever—that is the way that they talk about it. It fits in nicely with calculus to do it that way and to think about $0.99\cdots9\cdots$ as the limit of a sequence:

$$0.99\cdots9\cdots \equiv \lim \{0.9, 0.99, 0.999, \cdots\}.$$

If you think of it as this limit and then follow the formal rules for subtracting sequences and multiplying sequences and so on, you come out with the amazing conclusion that:

$$x = 0.99\cdots9\cdots \equiv \{0.9, 0.99, 0.999, \cdots\}$$

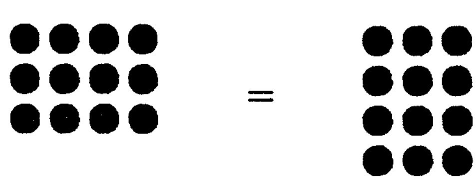
$$10x = 10 \times \lim \{0.9, 0.99, 0.999, \cdots\} = \lim \{9, 9.9, 9.99, \cdots\}$$
$$9x = \lim \{9, 9.9, 9.99, \cdots\} - \lim \{0.9, 0.99, 0.999, \cdots\} = \lim \{8.1, 8.91, 8.991, \cdots\}$$
$$x = (\lim \{8.1, 8.91, 8.991, \cdots\}) \div 9 = \lim \{0.9, 0.99, 0.999, \cdots\}$$
$$x = 0.99\cdots9\cdots!$$

This is true—not very useful, but it is true. And it *has* to be that way, because there is an assumption being made here—the Archimedian Axiom. Way back, Archimedes knew that in talking about numbers it was possible to talk about ones which we now call infinitesimal, and then Archimedes had an axiom or principle which rules out these infinitesimals. The Archimedian Axiom (or Principle) gets stated in various different ways but is rarely mentioned these days in the North American undergraduate curricula—most textbooks (if they mention it at all) relegate it to a brief mention in a footnote or exercise. The usual approach these days is to subsume the Archimedian Axiom under Completeness in a hidden way so that you do not even notice that it is there. I think it is important for the students to know that this is an assumption. They can understand why it is convenient to assume that $0.99\cdots9\cdots = 1$ and understand that there are a lot of reasons for making that assumption. But we should tell them that it is an assumption—and it really is.

Another example—here is a theorem:

For natural numbers, $n \times m = m \times n$.

Now, the usual formal proof which I learned for this theorem is a complicated double mathematical induction. I dutifully learned this proof and was dutifully teaching it when I first started teaching. But here is the prop for another proof (I do not want to say 'another proof' but only 'a *prop* for a proof'):



Here we think of 3 x 4 as three 4's or four 3's (it seems that most mathematicians think of 3 x 4 as three 4's, but many of my students think of 3 x 4 as four 3's). I find a proof based on this schema as more convincing than the one with the double induction. And this proof can be visualized with having arbitrary numbers of dots, because the whole point is that you do not have to count the dots to know that this is true—there is symmetry. But it is hard to express in words and put down in a linear fashion on a piece of paper and all that kind of stuff.

— **mathematics is being harmed**. Historically, most current-day mathematics was based on geometric explorations, geometric reasonings, and geometric understandings. The developers of our current deductive systems in algebra and analysis explicitly attempted to weed out all references and reliances on geometry and the geometric intuitions on which the algebra and analysis was originally based. When we confine mathematics to these formal systems we teach the students to distrust mathematics, not to value it, and not to use their intuitions in understanding mathematics. Many, many students who have a natural interest in mathematics are lost to mathematics by this process—I almost was.

HOW SHOULD WE DESCRIBE WHAT IS MATHEMATICS?

David Hilbert is considered to be "the father of formalism" so I checked what he had to say. In 1932, late in his career he wrote in the Preface to *Geometry and the Imagination*:

> In mathematics, as in any scientific research, we find two tendencies present. On the one hand, the tendency toward *abstraction* seeks to crystallize the *logical* relations inherent in the maze of material that is being studied, and to correlate the material in a systematic and orderly manner. On the other hand, the tendency toward *intuitive understanding* fosters a more immediate grasp of the objects one studies, a live *rapport* with them, so to speak, which stresses the concrete meaning of their relations. (Hilbert's emphasis)

As to geometry, in particular, the abstract tendency has here led to the magnificent systematic theories of Algebraic Geometry, of Riemannian Geometry, and of Topology; these theories make extensive use of abstract reasoning and symbolic calculation in the sense of algebra. Notwithstanding this, it is still as true today as it ever was that intuitive understanding plays a major role in geometry. And such concrete intuition is of great value not only for the research worker, but also for anyone who wishes to study and appreciate the results of research in geometry. (Hilbert 1932)

The last sentence in the first paragraph ("On the other hand ...") is a very nice description of what a lot of us are trying to do and he goes on to say how important this is in mathematics. I even went back to his paper "On the Infinite"—he does not say that mathematics is formal systems or that all of mathematics should be formalized. He, in fact, says very explicitly there that mathematics is based on intuition and that intuition is an appropriate basis for what he calls "ordinary finite arithmetic." He wanted to introduce the formalization in order to take care of various paradoxes that were coming up in dealing with the infinite, because there seemed to be some problems with intuition around infinite things. He never claimed that mathematics was formal—that was his followers.

Here is a more recent view expressed by William Thurston, who is director of the Mathematical Sciences Research Institute at Berkeley and one of the most prominent American mathematicians. Thurston rejects the popular formal definition-theorem-proof model as an adequate description of mathematics and states that:

> If what we are doing is constructing better ways of thinking, then psychological and social dimensions are essential to a good model for mathematical progress. ...
>
> ... The measure of our success is whether what we do enables *people* to understand and think more clearly and effectively about mathematics. (Thurston 1994)

I will now give a description of mathematics that is what I think Hilbert and Thurston are talking about. I call it "alive mathematical reasoning" where I take the word "alive" from Hilbert's quote.

WHAT IS ALIVE MATHEMATICAL REASONING?

Alive mathematical reasoning includes both abstraction and intuitive understanding as Hilbert says in the above quote.

Alive mathematical reasoning is paying attention to meanings behind the formulas and words—meanings based on intuition, imagination, and experiences of the world around us. It is not memorizing formulas, theorems, and proofs—this is again something that computers can do. We, as human beings, can do more. As Tenzin Gyatso, the fourteenth Dalai Lama has said:

> "Do not just pay attention to the words;
> Instead pay attention to meanings behind the words.
> But, do not just pay attention to meanings behind the words;
> Instead pay attention to your deep experience of those meanings."

Alive mathematical reasoning includes "living proofs", that is, convincing communications that answers—Why? It is not formal 2-column proofs—computers can now do formal proofs in geometry. If something does not communicate and convince and answer "why?" then I do not want to consider it a proof. What we need are *alive human proofs* which

— *communicate:* When we prove something to ourselves, we are not finished until we can communicate it to others. The nature of this communication depends on the community to which one is communicating and it is thus, in part, a social phenomena.

— *convince*: A proof works when it convinces others. Proofs must convince not by coercion or trickery. The best proofs give the listener a way to experience the meanings involved. Of course some persons become convinced too easily, so we are more confident in the proof if it convinces someone who was originally a skeptic. Also, a proof that convinces me may not convince my students.

— *answer 'Why?'*: The proof should explain, especially it should explain something that the listener wants to have explained. As an example, my shortest research paper [Henderson 1973] has a very concise simple proof that anyone who understands the terms involved can easily follow logically step-by-step. But, I have received more questions from other mathematicians about that paper than about any of my other research papers and most of the questions were of the kind: "Why is it true?" "Where did it come from?" "How did you see it?" "What does it mean?" They accepted the proof logically but were not satisfied—it was not alive for them.

One of my colleagues at Cornell was hired directly as a full professor based primarily on a series of papers that he had written even though at the time we knew that most of the theorems in the papers were wrong because of an error in the reasoning. We hired him because these papers contained a wealth of ideas and questions that had opened up a thriving area of mathematical research.

Alive mathematical reasoning is knowing that mathematical definitions, assumptions, etc., vary with the context and with the point of view. Alive reasoning does not contain definitions and assumptions that are fixed in a desire for consistency. It is an observable empirical fact that mathematicians and mathematics textbooks are not consistent with definitions and assumptions. We find this true even when the general context is the same. For example, I looked in the plane geometry textbooks in the Cornell library and found nine different definitions of the term "angle". Also, calculus textbooks do not agree on whether the function $y = f(x) = 1/x$ is continuous or not continuous; and analysis textbooks have many different axioms for the real numbers that have different intuitive connections and necessitate different proofs.

Alive mathematical reasoning is using a variety of mathematical contexts: 2- and 3-dimensional Euclidean geometry, geometry of surfaces (such as the sphere), transformation geometry, symmetries, graphs, analytic geometry, vector geometry, and so forth. It is not Euclidean geometry as a single formal system. When a mathematician is constructing a proof that needs a mathematical argument she/he is free to use whatever tools work best in the particular situation. Mathematicians do not limit themselves in this way. Also, those who use geometry in applications, do not feel restricted to a single formal system.

Alive mathematical reasoning is combining together all parts of mathematics: geometry, algebra, analysis, number systems, probability, calculus, and so forth.

Alive mathematical reasoning is applying mathematics to the world of experiences.

Alive mathematical reasoning is using physical models, drawings, images in the imagination.

Alive mathematical reasoning is making conjectures, searching for counterexamples, and developing connections.

Alive mathematical reasoning is always asking " WHY?"

BUT WHAT ABOUT CONSISTENCY AND CERTAINTY?

— **Formal deductive systems do not gain consistency**. For example, is the function $f(x) = 1/x$ continuous? Look in several calculus books. They give different answers! Differential geometry is another example where there is no consensus as to which formalism to use, but yet everyone thinks they are talking about the same ideas. Why?

— **Formal deductive systems usually do not gain for us the certainty that we strive for.** Formal deductive systems are useful and powerful in some circumstances, for example, in deciding which propositions can be logically deduced from other propositions and whether certain processes or algorithms will always produce the expected result. But, these deductive systems only give us certainty that certain steps (that can in principle be mechanized) can be carried out. They usually do not gain us certainty for the human questions of "Why?" or the human desire for experiencing meanings.

ALIVE MATHEMATICAL REASONING BRINGS BENEFITS TO MATHEMATICS

In my experiences, students with alive geometric reasoning are the most creative with mathematics. These are also the students who can step back from their individual courses and see the underlying ideas and strands that run between the different parts of mathematics. They are the ones who become the best mathematicians, teachers, and users of mathematics.

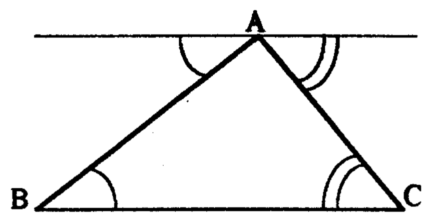
There is research evidence that successful learning takes place for many women and un derrepresented students when instruction builds upon personal experiences and provides for a diversity of ideas and perspectives. See, for example Belenky et al. (1986), Cheek (1984), and Valverde (1984). Thus, alive mathematical reasoning in school classes may contribute to increasing the numbers of mathematicians who are women and persons from racial and cultural groups that are now underrepresented.

In my own teaching, I encourage students to use alive mathematical reasoning and observe how their thinking and creativity is freed and their participation is opened up. (See Lo et al.,1996.) As I listen to the alive mathematical reasonings of my students I find that 30-40% of the students show me mathematics that I have not seen before and that (percentage-wise) more of these students are women and persons of color than white men. (Henderson, 1996)

CLOSING EXAMPLE

I will conclude with a proof that I learned from a student in a freshman course which is taught in the same style and using some of the same problems as the geometry course. The course was for "students who did not yet feel comfortable with mathematics" and who were social science and humanities majors. There, a student, Mariah Magargee, who was an English major, had been told all the way through high school that she was no good at mathematics and she believed it. I want to share with you her proof that *the sum of the angles of a triangle on the sphere is more than 180 degrees.* We had previously, in class, been talking about the standard proof that on the plane the sum of the angles of a triangle *is* always 180 degrees:

Standard planar proof: Given a plane triangle ABC, draw a line through A which is parallel to BC. The sides AB and AC are transversals of these parallel lines and therefore there are congruent angles as marked. We see now from the drawing that the sum of the angles is equal to 180 degrees.

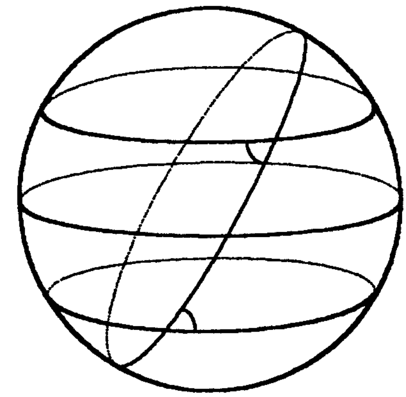


In class I stressed that the students should remember that latitudes circles (except for the equator) are not geodesics (straight on the sphere) and I urged them not to try to apply the notions of parallel to latitude circles. Mariah ignored my urgings and noted that two latitude circles which are symmetric about the equator of the sphere are parallel in two senses—first of all they are equidistant from each other and:

Note that:

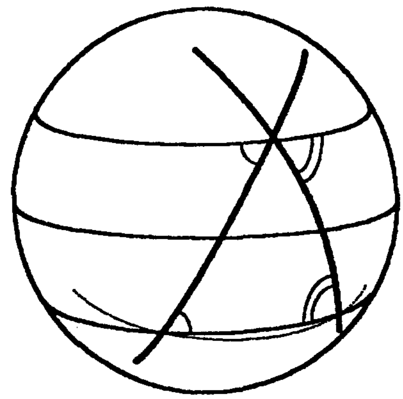
Two latitude circles which are symmetric about the equator have the property that every (great circle) transversal has opposite interior angles congruent.

This follows because the two latitudes have half- turn symmetry about any point on the equator.



Now we can mimic the usual planar proof:

We see that the sum of the angles of the "triangle" in the figure sum to a straight angle. This is not a true spherical triangle because the base is a segment of a latitude circle instead of a (geodesic) great circle. If we replace this latitude segment by a great circle segment then the base angles will increase. Clearly then the angles of the resulting spherical triangle sum to more than an straight angle.



You can check that any small spherical triangle can be derived in this manner.

Nice proof! I like it. That is Mariah's proof. This is a student who believed that she was no good at mathematics and was told she was no good at mathematics, but she taught me a really nice proof.

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QUESTIONS AND ANSWERS

Question: *How might we encourage our students to experience the passion that they have for mathematics—that passion can include joy, fear, excitement, however you want to interpret the word.*

Answer: I only know ways that I have tried and I do not think there is only one way. It seems to me that the most important thing is to expect it to happen and I do that and try to convey that I expect to see that from the students. The other thing that I started noticing when I started to have the students write is that a lot of the students' ideas and their passions are very fragile. There are a lot of students who do not dare to speak it in class, but they will write something. Maybe what they write isn't even directly what they really want to say but they will hint at it. So I have them write and then I respond to their writing and then they respond to my comments and so there is a dialogue that goes on in the writing. That I found to be the most powerful because then if someone just starts having an idea about something that is very tentative and very fragile, I can encourage it. I can encourage it easier in the written dialogue than it can happen in a class situation. That together with just expecting it and encouraging it whenever it happens and validating it is what I find that works for me.

Question: *What is formalism good for?*

Answer: One of the areas for which formalism is clearly good for is in computer science, in studying the algorithms and proofs in computer science. A huge area in computer science now is how to prove that a program does what you want it to do—it's a formal proof because that is what computers do, they are formal systems. The other place in which formalism is very powerful is in any situation like with groups. Studying groups is a good place to have axioms and build it up formally because there are a lot of different *models* for what a group is. So you can prove certain results that work for anything that satisfies these particular axioms and there are some examples and you can apply it across all the examples. There are a lot of areas like that in mathematics where that can happen. I think that Euclidean geometry is a particular *bad* place to apply formalism, because there is essentially only one model of Euclidean geometry and it is not a question of building these things up and then you can apply it somewhere else. Those areas where there actually are different models for a particular axiom system are areas where formal systems are powerful tools. I would also say that sometimes it is useful to use formal systems in areas where you actually have several different axiom systems—which we do not usually allow in courses as, we usually stick with only one. This happens in differential geometry.

There are very complicated formalisms for differential geometry but there are a lot of different ones and to play the different ones off each other can be powerful. If you try to stick within one then you lose the geometric meaning, but if you go across them then the only thing which ties them together is the geometric meaning; and that is one way to get at what the geometric meaning is.

Question: *How is the way that you teach geometry dependent on whether you have preservice teachers or have mathematics majors?*

Answer: The main course that my book is based on has mathematics majors and preservice teachers (who are also mathematics majors) and mathematics education graduate students and then there are miscellaneous people (teachers, artists, or other members of the community who are interested in geometry)—I do the same thing with all of them. Because most of the feedback is based on the writing that they do, they can respond in different ways, so I can have a different dialogue going on for different students depending on where they are coming from. I have not been able to do this as well in other subjects, but geometry is particularly suited to this because most people do not have much background in geometry, and that evens them out. Also geometry is more accessible concretely and through the intuition. I should tell you about one workshop that I did that was very powerful, one of the most powerful workshops that I have lead. It was in South Africa and they had gotten together a group of about 50 people which included elementary school teachers (many of whom had not finished secondary school, so had very weak mathematics backgrounds and virtually nothing in geometry), secondary school teachers, mathematics education people, and research mathematicians (including the chair of the mathematics department)—the whole span. I had them work on the same problems in small homogeneous groups, the elementary school teachers worked with each other and the research mathematicians were working with each other. Of course, what they were doing in their small groups was very different, but I then had them report back to the whole group what they had found. Then the research mathematicians had to express it in a way that made sense for the elementary school teachers and the elementary school teachers were able to express what they had found and see that they had found things that the research mathematicians hadn't seen. It was very powerful—I try to have as much diversity as possible in my class, but I have never had that kind of diversity before or since.

Question: *Can or should geometry be taught separately from algebra?*

Answer: When I teach geometry, I encourage students to use whatever they find useful to use and, if that is algebra, then—great. Is that the kind of thing you were meaning by the question?

Question: *Should geometry be taught before algebra?*

Answer: My own feeling is that geometry comes before algebra—much of algebra developed out of geometry, historically, and that should not be lost. Mostly I would like to turn the question around: Should algebra be taught separate from geometry? And there my answer would definitely be NO. Whenever I teach linear algebra, geometry is there a lot. And that is true historically—much of linear algebra was developed to help with the description and study of the geometry of higher dimensional spaces.

Question: *Could you elaborate on your comments that formalist mathematics is harmful and destructive? Is this perhaps a function of generation, because Leslie Lee seemed to be really able to identify with your comments because she too wanted to build a cabin?*

Answer: Several people here came up to me afterwards and said they had had similar experiences, not just Leslie, and they were all about my age. Our generation was the generation before the baby boomers—I think the baby boom generation (the ones that went into mathematics and the ones that didn't) was just different and maybe that is part of the reason why my generation is affected. Also I think that mathematics was most formal when we were in school. There seems to be something about our generation; and we are now in charge so we are more visible and that puts more pressure on us. Also, this formalism is only a product of this century, so for a long while the leaders in mathematics who were doing the formalism also knew that it was not all of mathematics—like the quote I have from Hilbert and that was in the 1930's. But then somehow after the war in the late 1940's and 1950's, most of the mathematicians who had had direct personal contact with what was going before formalism came in had died off and so that may have affected our generation.

Question: *How has the harmfulness or destructive nature of formalist mathematics manifested itself beyond the urges to live in the woods?*

Answer: Well, I do not think that the urge to live in the woods is harmful! I think that it was mainly that when I was going through school I never grew up socially and I was not effectively encouraged to grow up socially. I was a 'brain' (or 'science whiz') and that particularly substituted for growing up socially. Being immersed in formalism sort of fit that and encouraged that. That's part of it and I think that with the baby boom generation there were lots of external forces that started drawing people out, that was just not around in my days. And people who were more alive and more socially active than I was, tended not to go into mathematics, or if they were in mathematics then they dropped out.

Question: *Could you picture a world where mathematics majors and graduate students could be extroverted and emotional and deal with people and still do formalism? Do think there is something inherent in formalism?*

Answer: I do not know for sure. I look at the graduate students now at Cornell and there are some of them who are like I was but there are also significant numbers of them now who are not like I was and who are alive in lots of ways. They are surviving at Cornell and Cornell is basically still a very formal place, but they are also interested in teaching and we have mathematics graduate students who are taking the initiative to do some educational reform. So, yes, I think it makes sense that it fits in with formalism. I should say more about formalism, there is the part of mathematics, the foundations of mathematics, which is specifically studying formal systems and now in lots of places, in particular at Cornell, there is almost no distinction between it and a part of theoretical computer science. That is an active area of research where there is a lot of exciting things going on. I am not talking about that, that is a part of mathematics. I don't know ... Let me tell you one observation that I had that is less true now but it used to be true 10 years ago or so. Almost all of the women graduate students did not go into geometry or geometry related areas, but instead went into very formal areas. I talked with some of these women trying to catch why this was true, because in most of these cases these women were very active outside of mathematics—they were involved in various social movements, political movements, feminist activities, and other such things going on 10-20 years ago. They expressed to me that they had to separate their lives—when they were doing mathematics they had to separate it off from the rest of their life and it was easier to do that when they were doing formal mathematics. That seems to fit in with what I am saying. I see that happening less now, for both the men and the women.

Question: *We are interested in the resurgence of geometry at Cornell. Can you give us a sense of the constellation of the geometry courses that are available at Cornell and the population that they are for?*

Answer: As far as I can tell, Cornell has more geometry courses than anywhere else in the world. The undergraduate geometry courses are:

1. Euclidean and spherical geometry (the course that the book is based on) that is required for prospective teachers but it is taken by lots of other majors also.

2. Hyperbolic and projective geometry.

3. Geometry and groups—tessellations, transformation groups, etc. Cornell has a strong geometric group theory research group and the course grew naturally out of that research group.

4. Differential geometry.

5. Geometric topology.

6. "From space to geometry"—A freshman course which is based on writing assignments.

7. "Mathematical explorations"—for first and second year students who are humanities and social science majors. I use a lot of the same problems that are in my book—Mariah's proof about the sum of the angles of a triangle on the sphere was from that course.

8. Applicable geometry—sometimes computational geometry, sometimes the geometry of operations research (such as convex polytopes), and other applied topics that vary from year to year.

Question: *Isn't the issue more one of HOW you run the course rather than the subject matter of geometry?*

Answer: First of all, only one of those eight geometry courses deals extensively with axioms and formalism and that is the hyperbolic and projective geometry course and it does not deal with axiom systems totally. But I agree with you that the important thing is, How? I think that as long as you can start with something that is a concrete contextual situation and use that to start building the area of mathematics, then it can be done with any subject; and I think all parts of mathematics have such grounding. Geometry is easier to get into because there is not a tradition of having a long string of prerequisites, this linear sequence of courses and so on, in geometry, so it is easier to jump in different places. But I have done it in an abstract algebra course where, because it was me doing it, I started with symmetries of polyhedra and ended up with Galois theory. The main thing that I try to do is to have a concrete contextual situation where the students are able to experience the meaning of what is going on and so, in that way, it can be more constructive. The other part of it (what I mentioned in my answer to the first question) is eliciting from the students their ideas and their thinking, so that if there is some way that their imagination and intuition can latch on, then I just start pulling out of them what the ideas are and guiding them in the right ways and giving suggestions and writing challenging problems.

Question: *Are the other geometry courses at Cornell also not so prerequisite bound?*

Answer: None of the geometry courses have any of the other geometry courses as prerequisites.

Question: *It is perhaps true that the phenomena is not so much a growing interest in geometry but rather a growing interest in something mathematical that can engage the students in less formal ways?*

Answer: I think you may well be right and that geometry is just a particularly convenient or easier area to do that in. There is a debate going on in our department as to whether or not students who want to take my courses are wanting to take it because of me or is it something about the course. It is hard to gauge that but it seems to be what you say, that they are really looking for a different way in which to engage with mathematics— something that is less formal, that is not just lecture and exams.

Question: *Have you had any resistance to your approach from colleagues, other mathematicians, or from students?*

Answer: Yes, all of the above. The geometry courses, fortunately, are not required for mathematics majors except for prospective teachers, so students who do not want to do the course just do not take the course and I have had no complaints from the prospective teachers. But, I and some graduate students are trying to put some of these ideas into the calculus now. We have had a few cases of students getting up and stomping out of the room when we introduce small group work and other activities to engage the students. It seems that this happens because they are not there to learn calculus; they are there because they are required to take calculus and they want to do it with a minimum amount of effort to get their passing grade so they can go on to do whatever it is that it is required for. The students who want to learn calculus, they seem to love the new approaches. Sometimes I am teaching one section and there are other sections of the calculus course taught in traditional ways. I am required by the department for my students to have the same assignments and exams, so I am giving them stuff in addition to that. I tell them this right up front and I tell them why—I think they will learn it better and with more understanding, but they will have every week more assignments than the other sections. Typically, what happens when I announce that in the beginning, a few students drop out and other students hear about it and come in. At the end of the semester the students report that the extra stuff is the best part of the course.

WORKING GROUPS

Working Group A

REFLECTIONS ON TEACHER GROWTH: PRE-SERVICE AND IN-SERVICE PERSPECTIVES

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Participants:

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INTRODUCTION

Never before in education has there been greater recognition of the need for ongoing professional development. In-service training and other forms of professional development are a crucial component in nearly every modern proposal for educational improvement. Regardless of how schools are formed or reformed, structured or restructured, the renewal of staff members' professional skills is considered fundamental to improvement (Guskey and Huberman, 1995, Introduction).

Leaders in mathematics education are calling for significant revisions to today's mathematics programs at all levels. How to *effect* these changes has become of major import for many educational stakeholders. It is widely accepted that *the majority of* classroom teachers do not have the mathematical backgrounds that *will permit* them to implement the required changes without assistance. Nor *can it be assumed that we will* be able to depend on future teachers to bring in sweeping changes. Like the experienced teachers, *many* pre-service teachers lack sound mathematical backgrounds and most current mathematics methodology courses have *neither* the time *nor* the mandate to assist these novices in attaining the necessary *mathematical skills* or understanding.

Current in-service and pre-service programs are being scrutinized by many educational partners. Classroom teachers are often unhappy about the fragmented approach to in-service, politicians complain about the cost of professional development, and math leaders feel that little change has occurred as a result of the 'training' (Ball, 1989; Aichele, 1994). Matthew Miles, a leading researcher in the field, pulls no punches when he surveys the field:

> Let's frame the issue in extreme terms. A good deal of what passes for "professional development" in schools is a joke—one that we'd laugh at if we weren't trying to keep from crying. It's everything that a learning environment shouldn't be: radically under-resourced, brief, not sustained, designed for "one size fits all," imposed rather than owned, lacking any intellectual coherence, treated as a special add-on event rather than as part of a natural process, and trapped in the constraints of the bureaucratic system we have come to call "school." In short, it's pedagogically naive, a demeaning exercise that often leaves its participants more cynical and no more knowledgeable, skilled, or committed than before. And all this is accompanied by overblown rhetoric about "the challenge of change," "self-renewal," "professional growth," "expanding knowledge base," and "life-long learning" (Guskey and Huberman, 1995, Foreword).

Several research studies (Creemers, 1994) have provided evidence of effective classrooms and effective teacher behaviour but have provided "limited evidence about the design and implementation of a good mathematics teacher education program(s)" (Brown & Borko).

This working group, facilitated by Susan Stuart and Bill Higginson, set out to explore the answers to two questions through the eyes of both pre-service teacher educators and in-service teacher planners:

> What should the teacher coming out of our "best" programs look like?
>
> What are the exemplary strategies that we use (or that we could use) to assist developing teachers to demonstrate these characteristics?

The participants represented a wide range of concerns and backgrounds, but they all expressed an interest, as one said, in finding ways "to change teacher behaviour—not just mimicking—but true change." Some began by indicating a belief that thorough reform was needed to teacher education programs and others wondered how elementary and secondary math teachers could be encouraged to continue to learn and grow. New questions were added to our list.

> If we restructure pre-service programs, how much math and of what sort should there be, and where does it fit in?
>
> What relationships with the field would be most helpful, and how can such relationships be developed?
>
> How can we look, not only at the B.Ed. program, but all stages of teacher growth, including career-long development?
>
> Who are the players in the various teacher development models?

A tall order for nine hours, but we were anxious to explore and see where the questions would take us. The following account is simply a summary of many of the ideas that came from the discussions. Some are seeds from other sessions, others are ideas that will continue to grow and flourish long after these three days.

SESSION ONE

Before investigating effective strategies to develop mathematics teachers of the highest quality, we needed to explore our vision of the characteristics of a "developed" teacher of mathematics. Our first assumption was that a "developed" mathematics teacher could be described. We could have simply adopted, or adapted, the criteria for "good mathematics teaching" as found in the NCTM's (1990) *Professional Standards for Teaching Mathematics*, but we felt it was important for us to establish a common foundation. And so, working in small groups, the participants brainstormed, with the resulting characteristics compiled into a list. The characteristics listed below are thought-provoking and extensive, but *probably not* exhaustive.

In the collective opinion of the group, a 'developed' mathematics teacher would:

- Be knowledgeable
    - knowing how students learn
    - knowing appropriate math
    - having pedagogical knowledge
    - having math skills, competency
    - having technological expertise
    - having a broad vision of math
- be pedagogically effective
- see errors as valuable
- sensitive to student needs
- flexible to learning styles
- richness of math experiences
- provides supportive environment
- aware of student competencies
- confident pedagogically
- able to consult resources
- appreciative of diversity
- aware
- a believer in equity
- a good listener
- a good observer
- f lexible
- possess good communication skills
- confident in children's ability
- respectful of students
- active
- a critical thinker
- curious
- creative
- enthusiastic about the subject
- teaches with enthusiasm
- open-minded
- a problem-solver/model for learners
- share
- versatile
- be mathematically competent
    - competent and confident
    - capable of designing appropriate teaching/learning experiences
- help children view themselves as mathematicians
- view themselves as mathematicians

- open to seeing math from different perspectives
- connected with the math community
- promote aesthetic appreciation of math
- appreciates own 'mathematization'
- reflective
- interested in classroom-based research.

After placing the results of each group's brainstorming on the chalkboard, we attempted to find commonalities and then to classify, using fairly general headings. The group felt that the descriptors could be *grouped* as Knowledge (content, pedagogical and math), Dispositions or Attitudes, and Beliefs (appreciations, confidence, etc.). The teacher behaviours that are important encompass not only those that are related to quality of instruction (Brophy & Good, 1986), but those which are more 'teacher-as-person' characteristics. These are the behaviours which, although difficult to measure reliably (Creemers, 1994), can have lasting effect on learners. When asked about former teachers, adults will more often refer to personalities than teaching strategies or content to which they were exposed. Being aware, a good listener, and sympathetic, can go a long way with a student. These are generic qualities that we value in any teacher, in any subject. However, the group felt that they were qualities that were urgently needed in the mathematics classroom. Changing the commonly held view of mathematics from "a boring and meaningless act" (Franks, 1995, p. 78) to mathematics as dynamic and worth studying will require teachers who are curious, creative, and enthusiastic.

Susan presented several student-teacher quotes from personal mathematics journals, and one said:

> As a teacher I want to remind myself of what it is like to learn new things and the frustration which arises when instructions are incomprehensible and you simply do not understand (Bev)

Our image of a "developed" teacher was complex. It encompassed a depth and breadth of knowledge and skills including knowledge of self and students, knowledge about mathematics and mathematics teaching, and knowledge of generic educational research and theory. Interpersonal skills seem to go hand-in-glove with teaching skills. Several group members indicated that knowledge of mathematics and mathematics teaching went beyond skill, requiring teachers to not only have the conceptual understanding but the ability to transfer this understanding into conceptual approaches to teaching mathematics in the classroom. Not an easy task.

This led us to our next question. How can this be accomplished? Teacher development takes two forms. The first begins at the universities in our mathematics for prospective teachers and mathematics methodology courses. The other is a career-long commitment of professional development.

SESSION TWO

On day two to facilitate possibilities for discussion, the group divided into two sub-sections for the first part of the session according to relative interest in pre-service or in-service activities. The mandate for the two sub-groups was to share examples of effective teacher-development activities, and to return to the large group to report briefly on these activities and to be prepared to look for common patterns across the range of examples.

The pre-service sub-group, in particular, were looking for suggestions for designing or improving a mathematics methods course or a content course which would address student-teachers' assumptions, feelings and knowledge about the subject and about themselves as teachers and learners of mathematics, as well as their beliefs and assumptions about the roles of students and teachers in the classroom (Ball, 1989). Like most elementary and secondary math courses, most math methodology courses in Canada face problems of time and coverage. The numbers of hours allotted to our courses was mentioned by several group members. We were aware that course organization and time lines varied greatly among provinces, universities and even among courses within the same faculties (Stuart, 1995).

Among the "creative strategies" shared in the sub-groups were: The idea of using community schools as the learning environment for pre-service candidates. Elaine Simmt arranges for the prospective teachers whose major is secondary mathematics, organized in groups of 4 to 5 to a classroom, to teach a series of lessons to small groups of students. Each pre-service teacher begins by meeting with the classroom teacher, then prepares lessons on a specific math topic collaboratively with his/her peers who have been assigned to the same classroom. After teaching the initial lesson to their assigned group of 4 or 5 students, the pre-service teachers meets again for sharing, consultation and further planning. This process continues throughout the length of the assignment. Each pre-service teacher is required to keep notes and write a student profile of one of their learners. Elaine explained that this experience was quite different from the full practicum that was a requirement of the program, of which these students had already completed four weeks. Unlike teaching a full class, with the accompanying pressures of dealing with classroom management and presentation strategies, this teaching experience allowed prospective teachers to focus on teaching math. They had opportunities to interact with secondary school math learners individually to find out how they were learning and what they were thinking. Elaine described the rich conversations that took place during the collaborative planning time, telling us about several pre-service teachers who, for the first time, excitedly realized that teaching did not necessarily translate into learning or that all students learned different "things" at different times. As one group member observed, this task would give pre-service teachers a chance to move beyond the stage of thinking only of themselves in the teaching process and move to the stage of seeing the learners dealing with the skills and concepts.

Ann Kajander told us that one aim she has for her mathematics content course for elementary teachers is to build their math confidence. She has the teachers form small groups, with a rotating chairperson. Each person chooses a math topic, designs a lesson, and teaches this lesson to the other members of their group. They are encouraged to choose topics that are unlike any they have encountered or used before and which they feel "might not work." Each group then selects one of the lessons to be taught to the whole class (the teachers who agree to do this on behalf of their group receive bonus marks). We found it very interesting that Ann was using a teaching strategy which might be found more commonly in a math methodology course. This double experience—learning new math and teaching new math to others—requires these teachers to think about the mathematics in depth.

The students in Ann's course also explore mathematics through an active assignment which asks them to "Find something about math that you find interesting" and then demonstrate their learning at a Math Fair, through posters or model presentations. One of our group members wondered about grading these types of assignments and an interesting discussion ensued about maintaining math quality while supporting math confidence. Although, as a few said, we want our students to gain confidence in themselves as learners of mathematics, we also do not want to sacrifice the math standards. Would grading the knowledge of the mathematics delay or derail the confidence building?

Roberta Mura wants to encourage her pre-service students to have confidence in children's ability to learn mathematics and to encourage risk-taking in the way they approach mathematics teaching. Roberta has developed a series of videotapes which feature classroom teachers in elementary classrooms, using appropriate manipulative materials. The videotapes do not, she said, depict 'special' lessons, but demonstrate everyday lessons and common teacher behaviours. Before viewing each video, the pre-service teachers are given the opportunity to work on math questions using the featured manipulatives. After the videotape, the students discuss what they have seen, focusing on both learners' and teacher's behaviours. Unlike observing teachers during regular practicum experiences, this is a controlled and shared activity. Student reaction to the math and to the manipulatives can be discussed, focus can be drawn to such things as questioning strategies and teacher-student, student-student interactions. Roberta feels that the pre-service students see that manipulatives have a place in the math classroom and that using manipulatives is "doable." She also stressed that the pre-service teachers highly enjoy the videotapes and the discussions.

Doug Franks asks his senior division pre-service math teachers to maintain a journal of their experiences in working their way through Mason's Thinking Mathematically. They turn in their journal four times during the one-year program and he responds to their writing. They also discuss the book and their journalling experiences in class. Doug asks the students to give the problems in "Mason" a good 'go', but he also tells them that this is not primarily a problem-solving activity, so it not necessary to stick with a problem until they 'solve' it. They are asked to note their thinking and their emotional responses while working their way through the problems/chapters. They are asked to try to apply, and react to Mason's RUBRIC. As the year goes on, and they gain classroom experience, they are asked to think about how their experiences with Mason (often trying) cast light—if any—on the possible struggles students might have in secondary school math classes. They also are asked to reflect on how "Mason" might have applicability in their math classes, even though Mason says that school structures do not permit the development of such mathematical thinking. Most fundamentally, Doug says, the process of "masoning" and "journalling" is to experience 'stuckness', and to explore their responses to being in that state, and ultimately, to come to know themselves better as mathematical persons and mathematics teachers.

An assignment borrowed and adapted from Rena Upitis at Queen's University was described by Susan Stuart. Susan asks her elementary pre-service math teachers to choose a 'new learning' for the year. This can be anything at all, as long as it is something that "they have never done before, something that they have wanted to learn or do." The teachers must put together a plan for their learning and keep a journal as they work on the task. Topics have ranged from quilting and complex needlepoint, to re-measuring and re-drawing the map for the university's hiking trails, from learning American sign language to refurbishing a bathroom (tiles to tub). Throughout the year the projects are discussed in class in small groups, focusing on the 'ups and downs' of the learning and the implementation of their plan. As the projects progress, the teachers begin to focus on the mathematics that was part of the task they had chosen. At the end of the year, they submit a summary of their project, discussing in detail the mathematics that they encountered. The assignment allows the students to evaluate themselves as learners and makes them aware of the fact that mathematics, in some form, is an integral part of almost everything they do.

> If students cannot see what they 'know' reflected around them, then they cannot 'know' in any important or meaningful sense. As a teacher, I realize it is critical that I be aware of my own learning so that, by extension, I begin to see how children learn and attempt to manage their growth and learning (Student quote (David)).

Other activities shared included ones by Gord Doctorow (having students write letters to friends explaining mathematical concepts), Rita Janes (inservice programs for the implementation of new curricula), LaJune Naud (a 'Second City' "Freeze!" technique—"Why did I just do what I did in this lesson?"), Douglas McDougall (tasks for teachers involved in 'upgrading' courses), Richard McKinnon (an assessment technique of having students correct their own papers), and Vicki Zacks (an example of a rich problem for the promotion of language and thinking skills— "How many squares?").

The group took some time discussing the commonalities among these teaching strategies and assignments. Perhaps these examples are pointing us toward math courses and math methodology courses which look different from those we have experienced in the past. First, David Robitaille drew our attention to the fact that these strategies seemed to take away from the time that we "instruct," changing the nature of the pre-service classroom to one that models the theories that we are teaching rather than "lecturing about the theories we want them to know about." Doing, rather than talking about how to do. This led the group to realize that all the strategies or assignments focused on mathematics teaching and learning, including situations in which the pre-service teacher could observe math learners without being impeded by pressing issues such as whole class management.

The strategies also highlight our belief in the importance of reflective teaching and learning. Perhaps that is why many of the examples find prospective teachers interacting in small groups on tasks that are experiential or manipulative. Not only do the tasks help them to learn about mathematics teaching, but they require the teachers to demonstrate and discuss their learning (learning how to learn?). We also noted that the instructional opportunities went beyond the standard "walls" of the math methodology classroom, making use of a wide variety of people, places and materials.

The examples indicate to us that our programs need to model several things:

1. Beginning (or extending) the habit of taking responsibility for personal learning

2. Learning to reflect on action and on learning

3. Taking mathematics beyond the traditional classroom setting, brining the learning of mathematics teaching closer to situations in which students are learning and people are using mathematics.

Our discussion has allowed us to explore some creative approaches. Ball (1989, p. 7) describes math methodology courses as "the launching stage for learners of math teaching that will serve them well in continuing to learn on their own." These strategies, perhaps, could serve as the "launching" of long-term changes in many of our courses.

SESSION THREE

At the end of the second session—partly in an attempt to move the group somewhat closer to a tangible result by the end of its nine hours of deliberation—Bill distributed a draft of a framework (LeMaistre and Higginson, 1995) for teacher growth and development that had been generated at the Second Queen's-Gage National Mathematics Education Institute held at Kingston in the summer of 1995. Rather than outlining an implementation plan for staff development, the framework describes levels or stages that mathematics teachers might move through in a career-long developmental journey. The framework recognizes that each teacher enters the profession as a novice, and with each professional act, be it attendance at professional development activities, planning and implementing workshops or courses for other teachers, or curriculum writing, moves toward "master teacher" level. The stages of growth also take place in different environments or "spheres of influence"—from classroom participation, to school and board leadership, and eventually provincial and national involvement in mathematics education. The framework postulates (see Table 1) six levels of developmental achievement: Candidate, Associate, Mentor, Master, Fellow, and Wizard (well, it is in 'draft' stage). Each of these six levels is described in terms of six different categories:

Characteristics, Mathematical Skills, Pedagogical Skills, Sphere of Influence, Input From, and Adjudication By. The 'Characteristics' of a successful 'Mentor', for example, would include: professionally active, shows initiative and support for others and an interest in the development of others. Applicants for certification at this level would be adjudicated by representatives of local and provincial organizations.

TABLE 1

LEVEL	CHARACTERISTICS (DESCRIPTORS)	MATHEMATICAL SKILLS	PEDAGOGICAL SKILLS	SPHERE OF INFLUENCE	INPUT FROM	ADJUDICATION BY
Wizard	promotes effective teaching inspiration	actively researching an area of developing mathematics continuing curiosity	innovative applications large view of pedagogy	international national	National Institutes International conferences	National, international organizations
Fellow	leadership, humility,persistence, good communicator	research in mathematics or teaching modalities	actively promotes innovative techniques	provincial curriculum and examination committees, presentation at provincial conferences	publishers, Ministries, national conferences	provincial organizations
Master	leadership, resourceful,innovative, risk taker, makes progress despitecircumstances	committment to enrolling in math-related post-grad work, view of big picture in mathematics	critiques instructional modalities	school board curriculum and exam committees, presentation at local conferences	universities, provincial coordinators, provincial conferences	provincial organizations
Mentor	initiative, action, support for others, interest indevelopment of others	sufficient to instruct peers in new topics	presents workshops on particular techniques	Department Head, coaching others, serve on p.d. committees, sharing with school colleagues	S.B. coordinators and consultants, peers, provincial conferences	provincial and local organizations
Associate	involvement, excited, high expectations, optimistic,participation	strong understanding of major math concepts, reads math journals, books	demonstrates comfort with and openness to different approaches	classroom research, liaison with feeder schools, supervision of student teachers	administrators, Dept heads, peers, local conferences	local organizations
Candidate	committment to long-term development, career path, programmatic improvement attendance, interest,	basic skills for levels being taught,	understanding of a variety of strategies, learning styles, child development	attendance at p.d. days, local conferences, classroom*	conference attendance, peers	local and in-school organizations
All levels	reflection, desire to learn, committment, enthusiasm		attention to learner characteristics	*classroom function maintained throughout, peers, and level below current level	peers and levels above current level	At every level, there should be a blue ribbon panel with input from parents, students, colleagues, administrators, universities, professional organizations at increasingly national levels

The intention of the draft framework was to stimulate discussion about possible structures to assist teachers in thinking about career growth. Reaction to the framework at the third session was spirited and varied and resistance was, in many quarters, quite high. One set of objections was essentially that such a procedure violated one of the fundamental principles of Working Groups, that of independence. The Study Group President in his Introduction to the 1995 Proceedings (Dawson, 1996) stated this clearly, "members are diligent about guarding against a WG becoming the platform for a particular point of view, or being dominated by the leaders." A few members had concerns which were more a function of the particular model, especially with its hierarchical nature and the 'excessively masculine' language in which it was couched.

Other participants expressed the view that the framework could be a helpful mechanism for developing more substantial professional development programmes. There was general consensus, however, that the content factors (the first three columns) needed to be more clearly distinguished from the contextual or process factors (last three columns).

The session ended with the generation of a series of questions which related the proposed framework to some of the issues raised in the earlier sessions. Among these were:

Could we use a descriptive example like Vicki's "How many squares?" problem when illustrating the framework?

We are in control of what we do—how can we promote growth for ourselves?

NCTM documents have a 'mid-80s' feel to them. Our list of descriptors is more 'mid-90s'. How can we build in the attempts to revise our own thinking about teachers and teacher education?

What is the growth plan for this growth plan?

So ended another CMESG Working Group. More questions than answers - some frustration and a sense of incompleteness, but also many new insights, and a growing awareness of the richness and depth of different perspectives on fundamental issues.

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Working Group B

FORMATION À L'ENSEIGNEMENT DES MATHÉMATIQUES AU SECONDAIRE: NOUVELLES PERSPECTIVES ET DÉFIS

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Les objectifs que nous proposions pour le groupe de travail sur la formation des enseignants au secondaire qui devait prendre place à Halifax en juin 1996 étaient les suivants:

- échanger sur nos contextes respectifs de formateurs intervenant auprès des futurs enseignants en mathématiques au secondaire afin de comprendre les orientations globales qui sont retenues en divers lieux, les pratiques mises en place, leurs fondements et les questions et défis qui se posent;
- à partir d'exemples de pratiques de formation que l'on considère fructueuses, tenter de dégager les diverses composantes d'un modèle viable de formation des enseignants en mathématiques au secondaire.

Ce groupe de travail n'ayant pu avoir lieu, nous rendrons compte essentiellement dans ce texte de nos propres contextes de formation des enseignants en mathématiques au secondaire et d'une expérience qui est mise en place.

I. Orientations générales des programmes de formation des enseignants au secondaire au Québec.

I.1. Cadre global orientant la formation des enseignants en mathématiques.

L'énoncé par le gouvernement du Québec en 1993 d'une politique d'agrément des programmes de formation à l'enseignement, dans la perspective de contribuer à la qualité de la formation professionnelle menant à l'exercice d'une profession, ici celle d'enseignant[1], et les nouvelles orientations ministérielles qui accompagnent cette politique d'agrément[2] conduisaient en 1994 à la mise en place dans les différentes universités du Québec de changements importants dans les programmes de formation des maîtres.

Quelles sont ces orientations auxquelles doivent se plier les programmes de formation?

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[1] La mise en oeuvre d'un processus d'agrément des programmes dès maî est une activité nouvelle au Québec, voir à ce sujet l'énoncé des politiques d'agrément des programmes de formation à l'enseignement. CAPFE (comité d'agrément des programmes de formation à l'enseignement). Gouvernement du Québec, 1993.

[2] Voir La formation à l'enseignement secondaire général. Orientations et compétences attendues. Québec, MEQ- Direction générale de la formation et des qualifications, 1992, 35 p.

L'énoncé de politique situe d'emblée *l'orientation professionnelle* générale qu'on souhaite donner à ces programmes de formation qui deviennent maintenant des programmes de quatre ans. Cette visée s'exprime ainsi: "La formation à l'enseignement doit être considérée comme *une formation à caractère professionnel* orientée vers la maîtrise de l'intervention pédagogique dans les matières enseignées" (énoncé des politiques d'agrément, p.10).

Un certain nombre de principes directeurs découlent pour le CAPFE (comité d'agrément des programmes de formation des maîtres) de cette orientation globale impliquant entre autres une meilleure intégration de la formation dans les disciplines et de la préparation à enseigner, une meilleure intégration de l'ensemble des cours théoriques et des activités pratiques. Nous reviendrons tout d'abord sur certains de ces principes en identifiant à leur propos les nombreuses questions qu'ils soulèvent.

a) Une formation qui se veut polyvalente :

L'idée de polyvalence est associée pour le ministère à une formation à l'enseignement dans au moins deux matières inscrites au régime pédagogique de l'enseignement secondaire. Conçue pour préparer les enseignants à intervenir en fonction des grands objectifs de formation intégrale de l'école secondaire, cette interprétation exclusive que donne le ministère de la polyvalence apparaît quelque peu questionnable.

Elle force en effet une prise en compte de la formation à l'enseignement dans deux disciplines, et ce quel que soit le champ d'intervention visé. Elle ne prend nullement en compte par conséquent la complexité plus ou moins grande que recouvre une telle préparation en fonction du champ visé, de l'importance de celui-ci dans le curriculum scolaire (l'enseignement moral est ici par exemple traité sur un même pied que l'enseignement des mathématiques ou l'enseignement du français) et de la réalité scolaire (l'enseignant de mathématiques est appelé le plus souvent à intervenir non pas dans une autre matière mais à différents niveaux scolaires, auprès de divers types d'élèves, élèves en difficultés, classes d'accueil, classes multiethniques...). Ce n'est pas tant pour nous l'idée de polyvalence du futur enseignant au secondaire que nous questionnons, celle-ci apparaît en effet tout à fait pertinente, mais son interprétation. On aurait pu laisser place à d'autres interprétations possibles de cette idée de polyvalence, en regard par exemple du champ disciplinaire d'intervention (dans notre cas l'enseignement des mathématiques) et de ses liens avec d'autres matières enseignées au secondaire (sciences physiques, géographie...) contribuant à une ouverture, à un enrichissement et à une meilleure intégration de la formation au secondaire (ce qui est différent d'une formation dans deux disciplines); ou encore à une polyvalence préparant le futur enseignant à mieux intervenir dans ce domaine auprès de diverses clientèles (élèves en difficultés, tripleurs, décrocheurs, élèves réguliers, classes multiethniques...) et à ainsi affronter la réalité scolaire telle qu'elle se présente sur l'île de Montréal.

On semble aussi tenir pour acquis que le fait de préparer les étudiants à enseigner deux disciplines contribuera à approfondir la culture générale de ces futurs enseignants. On pourrait toutefois être confronté à une simple juxtaposition de deux formations dans des disciplines différentes sans pour autant contribuer à une mise en lien et en continuité des matières enseignées; ce qui semble d'ailleurs le choix retenu par plusieurs universités.

Aucune allusion n'est faite dans cette politique d'agrément aux deux matières spécifiques qu'il est possible de considérer, laissant la porte ouverte à diverses interprétations possibles (les matières qui seront retenues ne présenteront donc pas nécessairement de liens sur un plan épistémologique). On pourrait ainsi fort bien jumeler par exemple, même si cela peut paraître absurde, enseignement des mathématiques et enseignement moral ou encore enseignement des mathématiques et enseignement du français au sein d'une même formation.

D'autres arguments sont mis de l'avant, sans doute plus déterminants dans les choix qui ont été posés; on retrouve ainsi des motifs d'ordre économique associés aux possibilités d'affectations et réaffectations du personnel enseignant, l'enseignement des mathématiques pouvant alors être donné par un enseignant formé dans un autre champ. On retrouve également des raisons renvoyant à l'encadrement des élèves, un même groupe pouvant être suivi dans deux matières par un même enseignant.

b) Une formation qui se veut intégrée :

Les programmes de formation proposent de mettre de l'avant une formation davantage intégrée pour contrer l'éclatement et la fragmentation de celle-ci. Cette intégration devra se manifester notamment dans l'aménagement et le contenu des cours et des activités, montrant des liens organiques entre ceux-ci. Ainsi notamment la formation pratique (nous reviendrons par la suite sur celle-ci), la formation psychopédagogique et la formation dans les disciplines devront s'articuler.

Il est donc désormais impossible d'opter pour une formation disciplinaire suivie d'un certificat d'une année en éducation générale; le ministère pose là des balises importantes en regard de ce qu'on retrouvait dans le passé au sein de certaines universités, il faudra toutefois respecter ces balises à tous les niveaux (de multiples exceptions ont hélas eu cours).

c) Une plus grande place réservée à la formation pratique:

Les nouveaux programmes de formation mettent l'emphase sur la formation pratique du futur enseignant en proposant entre autres un contact avec le milieu scolaire survenant tôt dans le cheminement et des stages d'une durée totale d'au moins 700 heures.

Quelles sont les recommandations plus spécifiques par rapport aux diverses composantes de cette formation?

La formation pratique met l'accent surtout, on l'a vu précédemment, sur une présence dans le milieu scolaire par le biais de stages, même si celle-ci peut prendre, souligne-t-on par ailleurs, d'autres formes (ateliers, cliniques, laboratoires d'études de cas, d'expérimentation, de simulation...).

Pour favoriser une meilleure articulation entre formation pratique et théorique, un partenariat entre les universités et les commissions scolaires est favorisé.

Enfin, le programme fait montre d'un équilibre entre la formation réservée aux matières d'enseignement et la formation touchant les aspects pédagogiques et sociaux de l'éducation. Ainsi, le nombre de crédits consacrés aux disciplines ou aux champs d'étude d'une part, et le nombre de crédits affectés aux stages, aux aspects psychopédagogiques ou sociaux de l'éducation et de la profession d'autre part, sont à peu près équivalents.

Ces recommandations font a priori ressortir *une certaine conception de la formation pratique, avant tout associée à une présence sur le terrain* . Dans un tel contexte, quelle sera la place de la réflexion sur l'action conduite en classe en regard de l'enseignement dans un champ d'intervention donné? où sera t-elle conduite et par qui sera-t-elle supportée? Ces recommandations soulèvent aussi des interrogations sur *la place de la didactique dans cette formation*; la formation semblant faire appel avant tout, pour les concepteurs, à des composantes disciplinaires et psychopédagogiques qui n'apparaissent guère intégrées dans
les faits.

Quel type de professionnel veut-on former ?

L'énoncé des politiques d'agrément reprend un certain nombre de compétences attendues des personnes diplômées. Nous reprendrons certaines de ces compétences professionnelles qui concernent plus spécifiquement *l'enseignement d'une matière* :

- avoir une connaissance approfondie de deux disciplines d'enseignement, contenu et fondements épistémologiques;
- être initié aux méthodes et à l'histoire des disciplines ou champs d'études, de même qu'à leurs limites;
- avoir la capacité de situer ces disciplines les unes par rapport aux autres et d'établir des liens entre elles;
- avoir une attitude positive à l'égard des disciplines enseignées, ainsi qu'une bonne connaissance de l'ensemble des programmes d'étude;
- être en mesure de planifier des activités d'enseignement et d'apprentissage en lien avec les objectifs des programmes d'étude du secondaire et pouvoir utiliser à cette fin des ressources didactiques appropriées;
- être en mesure de procéder à l'évaluation formative et sommative dans les matières enseignées;
- avoir une maîtrise de la langue d'enseignement;
- être en mesure de déceler les besoins pédagogiques des différentes catégories d'élèves.

L'énoncé de politique reprendra également des compétences attendues à l'égard de la *maîtrise de l'intervention pédagogiquee* (conçue ici de manière générale, indépendamment du champ d'intervention spécifique où elle se déroule, on réfère par exemple à l'apprentissage de l'élève, à la gestion de classe). Or, peut-on vraiment parler de gestion de classe et de manière de créer un milieu propice à l'apprentissage des élèves sans prendre en compte le domaine spécifique dans lequel se fera l'intervention? Plusieurs études montrent en effet le rôle déterminant que joue le rapport au savoir dans l'élaboration de stratégies pédagogiques en classe, dans la manière de cadrer le savoir, de gérer l'activité avec les élèves (Bauersfeld, 1980; Voigt, 1985; Schubauer Leoni, 1986).

II. Comment cette formation s'actualise-t-elle dans chacune de nos institutions ?

Nous parlerons ici plus spécifiquement de deux modèles de formation des enseignants en mathématiques au secondaire, l'un graduellement mis de l'avant à l'Université du Québec à Montréal et l'autre, plus récemment, à l'Université de Montréal.

II.1. Quelques caractéristiques du modèle de formation graduellement mis de l'avant à l'UQAM.

La prise en charge de la formation des enseignants en mathématiques au secondaire remonte dans le cas de l'UQAM à plus de 20 ans. Elle fut prise en charge dès le début par une équipe de didacticiens et didacticiennes des mathématiques. L'intervention de ces didacticiens dans les cours de mathématiques, de didactique des mathématiques, les stages d'enseignement des mathématiques, les cours d'informatique, etc. a conduit à une articulation progressive des activités du programme sur la *maîtrise par le futur enseignant de l'intervention pédagogique en mathématiques.*

Le programme élaboré vise ainsi le développement, à travers l'ensemble de ses activités (voir le tableau qui suit), de compétences professionnelles à l'intervention en mathématiques au niveau secondaire (on permet à une compétence d'intervention de se développer et de gagner, au fil de la formation, une certaine maturité).

Baccalauréat en enseignement secondaire: concentration Mathématiques #7954
Options: Informatique, Physique et Initiation à la technologie

1ère année		2e année	
A-95 Ses. 1	H-96 Ses. 11	A-96 Ses.111	11-97 Ses.IV
TC1 3 cr. EDU 1000 Introduction et initiation à l'intervention pédagogique	TC2 3 cr. PSY 2010 L'adolescence	TC4 3 cr. EDU 2102 Relation éducative et gestion des situations d'apprentissage *(intensif)*	TC6 3 cr. EDU 5010 Évaluation des apprentissages au secondaire
MAT 1024 3 cr. Atelier d'exploration de l'activité mathématique	TC3 2 cr. EDU 2005 Fondements de l'éducation et enseignement au secondaire	MAT 2026 2 cr. Didactique de l'algèbre	MAT 3035 3 cr. Géométrie 11
MAT 1030 3 cr. Géométrie 1	6 cr. MAT 2024 Didactique des	MAT 2226 3 cr. Raisonnement proportionnel et concepts associés *(intensif)*	DIN 2200 Programme 3 cr. PHY 1050 Mécanique PHY 2000 Mécanique
MAT 1801 3 cr. Programmation dans l'enseignement des mathématiques	mathématiques et laboratoire	DIN 2812 3 cr. Applications pédagogiques de l'informatique dans l'enseignements des maths *(intensif)*	DIN 3200 Multimedia 3 cr. PHY 2230 Électronique PHY 2005 Électronique
4 cr. MAT 1005 Algèbre – algèbre linéaire		**STAGE 2** 3 cr. ESM 3202 Math. Initiation à la pratique de l'enseignement *(24 jours)*	3 cr. Choix Histoire ou société PHY 2270 Optique EDU 2085 Archit.
	STAGE 1 2 cr. ESM 2000 Immersion milieu scolaire *(7 jours)non disciplinaire*	ESM 3201 1 cr. Pré stage	
14 crédits	**15 crédits**	**15 crédits**	**15 crédits**

3e année		4e année	
A-97 Ses. V	H-98 Ses. VI	A-98 Ses. VII	H-99 Ses. VIII
MAT 3225 3 cr. Didactique de la variable et des fonctions	TC 13 3 cr. Culture générale *(intensif)*	TC 9 2 cr. EDU 6002 Problématiques interculturelles à l'école secondaire *(intensif)*	TC 10 3 cr. EDU 6005 Éducation, épistémologie et métacognition
MAT 2700 3 cr. Structure numériques	MAT 3224 3 cr. Didactique des maths II *(intensif)*	TC 5 3 cr. EDU 3060 L'adolescent en difficulté d'adaptation et d'apprentissage *(intensif)*	TC 8 3 cr. HIS-SOC-PHI Formation fondamentale
DID 3200 Programme 3 cr. PHY 1770 Ondes EDU 2080 Dém. technique	MAT 1085 3 cr. Probabilités et stratistiques *(intensif)*	MAT 6221 3 cr. Histoire des maths *(intensif)*	TC 11 2 cr. EDU 6025 Profession enseignante en milieu scolaire
DIN 3300 Inforoute 3 cr. PHY 2500 Thermod. DIN 3300 Inforoute			TC 12 2 cr ESM 6210 Séminaire de synthèse
DIN 5200 Did. info. 3 cr. PHY 2270 Simulation MIC 3000 Sys informatique	DIN 6200 Proj. 3 cr. ESM 3511 Did. ESM 3515 Did. *(intensif)*		DIN 4300 Int. art. 3 cr. FSM 3500 Histoire FSM 3500 Hist.
	STAGE 3 3 cr. ESM 4200 Inf. ESM 4501 Physique ESM 4502 Init. T..	**STAGE 4** 7 cr. ESM 6201 Math.	Choix info, 3 cr. PHY 5260 Phys. mod. PHY 2020 Matériaux
15 crédits	**15 crédits**	**15 crédits**	**16 crédits**

TCX:Cours du tronc commun en psychopédagogie => 30 crédits –
MATH: conc.:Cours disciplinaires et de didactique => 47 crédits –
INF., PHY., INIT. Tech Option => 27 crédits –
STAGEX Stages en milieu scol. => 16 crédits TOTAL: 120 crédits oct.95

Cette préparation ne se fait pas à vide, elle s'articule sur une réflexion à propos de concepts mathématiques précis, sur leur apprentissage et leur enseignement. La formation y est donc dès le départ *intégrée*: on retrouve ainsi une intégration constante des dimensions théoriques et pratiques dans l'ensemble de la formation (l'implication des didacticiens dans la supervision des stages et la prise en charge parallèle des cours de didactique et de mathématiques rendent possible cette intégration).

Ainsi les compétences qu'on cherche à développer sont reprises à travers plusieurs cours.

- Apprendre à observer, questionner avec pertinence, tirer partie des interventions et des productions des élèves, etc. sont des compétences touchées par plusieurs cours.
- Le cycle propre à l'intervention pédagogique (planification, réalisation, analyse et réinvestissement) est repris par chaque étudiant à plusieurs reprises.
- La réflexion sur l'activité mathématique, son enseignement, son apprentissage (entrevoir différents moyens de mettre en pratique des situations à propos de concepts précis) fait l'objet de la quasi-totalité des cours.
- Enfin un modèle de partenariat entre intervenants universitaires et praticiens du milieu scolaire, mis en place à travers certaines activités du programme (nous reviendrons sur celui-ci par la suite), contribue au développement d'une compétence professionnelle chez l'étudiant.

II.2. Quelques caractéristiques du modèle de formation mis en place à l'Université de Montréal.

À l'Université de Montréal, le programme de formation des enseignants du secondaire relève de la Faculté des sciences de l'éducation et ce quelles que soient les disciplines concernées. Les trois départements composant cette faculté (département de psychopédagogie et d'andragogie; département d'études en éducation et administration de l'éducation; département de didactique - regroupant toutes les didactiques spécifiques, didactique des mathématiques, du français, de la physique, de la biologie, de la géographie, etc.) se partagent la formation en éducation. La gestion de ce programme est assurée par le Centre de formation initiale des maîtres créé par la faculté.

Avant laréforme des programmes de formation des enseignants, l'université offrait un certificat en enseignement, lequel faisait suite à une formation disciplinaire, généralement un majeur ou un baccalauréat, reçue dans le département disciplinaire concerné (maths, français, etc.). Dans le nouveau programme, les formations disciplinaires et en éducation se veulent intégrées. L'importance disciplinaire en termes de crédits est décroissante au fur et à mesure du programme de 4 ans tandis que celle de la formation en éducation est croissante. Le tableau suivant donne l'organisation de l'ensemble des cours et des stages dans le programme de formation des maîtres au secondaire.

La formation mathématique disciplinaire se fait par des professeurs du département de mathématiques. Les contenus de cours sont les mêmes que les étudiants soient inscrits au programme de formation des maîtres ou à un programme de mathématiques, à l'exception de deux cours conçus plus spécifiquement pour les étudiants en formation des maîtres (mathématiques fondamentales et géométrie euclidienne). Le choix des cours de mathématiques à suivre s'est fait en collaboration entre les didacticiens des mathématiques et les professeurs du département de mathématiques. Quant à la formation en didactique des mathématiques elle se fait au département de didactique.

Dans la perspective d'une formation polyvalente, les étudiants sont formés pour enseigner deux disciplines. Si les mathématiques sont choisies comme discipline principale les disciplines complémentaires peuvent être la biologie, l'informatique, la physique, la chimie, ou l'économie. Les seules

Baccalauréat en éducation Option enseignement secondaire U. de M.

Année	Stages de formation pratique à l'enseignementen milieu scolaire	Pédagogie – Didactique – Fondements – Formation pratique à l'université (Faculté des sciences de l'éducation)	2 disciplines enseignées au secondaire (F.A.S. et Fac. de théologie)	Total
1ère	**1 crédit** EDU 1010 *Stage de familiarisation à l'école secondaire* (équivalent à une (1) semaine) **35 h.**	**5 crédits** ETA 1121–Finalités de l'éducation et l'école secondaire 2 cr. PPA 1210–L'adolescent et l'expérience scolaire 3 cr.	**24 crédits** Discipline principale	**30 cr.**
2e	**3 crédits** EDU 2010 Stage d'assistanat (équiivalent à quatre (4) semaines) **140 h.**	**10 crédits** DID 2101–Intro à la didactique 2 cr. ETA 2510–Introduction à l'évaluation des apprentissages 2 cr. PED 4005–Séminaire d'intégration: le dossier professionnel (début) PPA 2000–Labo d'enseignement 3(2)cr. PPA 2220–Apprentissage scolaire au secondaire 3 cr.	**18 crédits** Discipline principale 9 cr. Discipline complémentaire 9 cr.	**31 cr.**
3e	**4 crédits** EDU 3015 *Stage d'enseignement I* (5 semaines) Discipline complémentaire **175 h.**	**12 crédits** DID I de la discipline principale 2(1)cr. DID 1 de la discipline compl. 2(1) cr. ETA 3730–Technologie de l'information et de la communication en éduc. 2 cr. ETA 3920–Analyse du système scolaire du Québec 2 cr. PED 4005–Séminaire d'intégration: le dossier professionnel (suite) PPA 3230–Pédagogie en milieu urbain 2(1) cr. PPA 3236–Modèles de gestion de classe au secondaire 2 cr. ou PPA 3238–L'enseignant et la relation éducative au secondaire 2 cr.	**15 crédits** Discipline principale 6 cr. Discipline complémentaire 9 cr.	**31 cr.**
4e	**9 crédits** EDU 4005 *Stage d'enseignement II* (10 semaines) Discipline principale **350 h.**	**16 crédits** DID II de la discipline princ. 2 cr. DID II de la disilpline compl. 2 cr. DID 4000–Labo. de didactique 2(2)cr. PED 4005–Séminaire d'intégration: le dossier professionnel (fin) 3 cr. ETA 4500–Pratiques de l'évaluation en milieu scolaire 1 cr. ETA 4005–Pratiques éducatives et recherche 2 cr. ETA 4105–Fondements théoriques de l'éducation 2 cr. PPA 4400–Difficultés d'adaptation et d'apprentissage au secondaire 2(1)cr.	**9 crédits** Discipline principale 6 cr. Discipline complémentaire 3 cr.	**34 cr.**
Total	**17 crédits de stage** **(700 heures à l'école)**	**43 crédits** **240 heures de laboratoire d'enseignement**	**66 crédits**	**126 cr**

disciplines principales acceptant les mathématiques comme discipline complémentaire sont la chimie et la physique.

Jusqu'à la réforme, la formation pratique avait surtout un caractère psychopédagogique. Avec le nouveau programme, une place plus grande est accordée à la didactique dans la formation pratique. Cette importance s'exprime de diverses façons:

- par des cours de didactique comportant des crédits pratiques, permettant le travail en groupes restreints avec l'aide d'un assistant;
- par la prise en compte explicite des aspects didactiques dans les stages avec une supervision didactique alliée à la supervision psychopédagogique;
- par la mise en place de laboratoires de didactique durant le dernier stage où peut se faire l'analyse des situations d'enseignement et de leur déroulement effectif.

La formation pratique ne se restreint donc pas à une présence en stage.

Par ailleurs, peu de crédits sont consacrés aux cours de didactique de chacune des deux disciplines. Au total 12 crédits sont réservés à la didactique pour les deux disciplines:

- Un cours (2 crédits) d'introduction à la didactique par secteurs (ici, le secteur mathématiques et sciences) en deuxième année. Celui-ci devrait permettre, entre autres, de voir les liens entre les deux disciplines choisies par les étudiants et aussi de développer un questionnement par rapport à l'enseignement qu'ils recoivent en mathématiques et en sciences et par rapport à leur propre apprentissage de ces disciplines.
- Deux cours de didactique des mathématiques (2 crédits chacun), soit un en troisième et un en quatrième année;
- Deux cours de didactique de la discipline complémentaire (2 crédits chacun);
- Un laboratoire de didactique (2 crédits) dont nous avons parlé plus haut.

Ce programme de baccalauréat en enseignement secondaire «*vise la formation d'enseignantes et d'enseignants professionnels, capables d'analyser et de comprendre la complexité de l'intervention éducative et de faire preuve d'autonomie dans la prise de décision.*» (Programme de B. Éd. option enseignement secondaire de l'U de M, 1995, p.1)

II.3 Formation pratique: un exemple d'intervention réalisée dans le programme de formation des enseignants en mathématiques au secondaire de l'UQAM.

Nous parlons ici d'une formation à l'intervention en mathématiques au secondaire plutôt que d'une formation pratique, dans un souci de refléter la conception plus large que nous en avons, qui dépasse, et de beaucoup, la simple présence en classe autour des stages.

Cette formation à l'acte d'enseigner va être visée tout au long du programme, elle est ainsi déjà présente dès la première session à l'intérieur d'un atelier d'exploration de l'activité mathématique, dont l'objectif est avant tout de questionner les idées qu'ils ont développées tout au long de leur scolarisation antérieure à l'égard des mathématiques, de leur apprentissage et de leur enseignement et de les rendre réceptifs au questionnement didactique qui suivra lors de leur formation. Il s'agit ici de sensibiliser les étudiants aux structures cachées d'un champ complexe, tout particulièrement en regard de leurs propres expériences en tant qu'élèves (en rétrospective) et des limites de celles-ci, et ce, pour continuer à développer un habitus mathématique alternatif (Bauersfeld, 1994, p.182).

Confrontés à diverses formes de questionnement, de situations-problèmes, ils se retrouvent eux-mêmes dans le rôle de l'élève et se questionnent ainsi sur l'apprentissage. Ils ont à expliciter leurs solutions aux autres, à verbaliser celles-ci pour les communiquer, à argumenter sur la validité de solutions présentées par d'autres...(pour plus d'informations sur ce cours, voir Bednarz, Gattuso, & Mary, 1995).

Ce premier contact avec une manière différente de voir l'apprentissage et l'enseignement des mathématiques sera repris dans les différents cours de didactique (didactique 1; raisonnement proportionnel; didactique de l'algèbre; didactique de la variable et fonction; didactique 2...). Dans ces diverses activités, on ne parle pas d'intervention, de gestion de classe au sens large, celle-ci est toujours ciblée, elle s'articule sur un certain contenu mathématique à enseigner. Nous expliciterons plus précisément cette formation à travers un exemple d'intervention, apparaissant en début de formation (2ème session, 1ère année, avant les stages), qui illustre bien la pédagogie mise en place et le modèle de formation que nous privilégeons.

Le cours didactique I et son laboratoire (6 cr)

Cette activité est un premier pas important dans le sens d'une préparation du futur enseignant à l'intervention en mathématiques.

Les préoccupations de l'enseignant dans sa classe sont présentes dans ce cours, plus particulièrement sous les aspects suivants:

L'élève, avec ses difficultés, ses raisonnements, ses conceptions: on travaille dans le cours à partir de vraies productions d'élèves; on dispose de vidéos de leurs propos, de leurs actions en situation; les étudiants iront aussi interroger des élèves et reviendront avec le compte rendu de leurs observations dont ils rendront compte aux autres (différents thèmes sont ici abordés à ce propos, exposants, algèbre, moyenne, fractions, décimaux...)

La formation au diagnostic dans l'action (Schön, 1983, 1987): les étudiants sont appelés à faire de l'analyse d'erreurs, de raisonnements d'élèves à partir de traces de leurs productions ou de ce qu'ils disent en classe et à élaborer des stratégies d'intervention possibles face à l'erreur, ou prenant en compte les différents raisonnements élaborés (les fractions, les décimaux et les opérations sur ces nombres seront par exemple une occasion de réfléchir sur ces interventions).

Des situations sont proposées, utilisées, analysées, les objectifs sont questionnés: on travaille ici à faire en sorte que les étudiants soient en mesure de faire un choix de situations pertinentes face à l'enseignement d'un sujet mathématique donné (par exemple, en travaillant la résolution de problèmes en secondaire 1, les étudiants doivent composer eux-mêmes des problèmes à contexte impliquant une multiplication ou une division en respectant certaines contraintes; conjointement l'analyse de séries de problèmes déjà composés améneront à établir des critères qui permettront de juger de la complexité de problèmes proposés aux élèves).

L'utilisation de la langue dans l'enseignement: l'étudiant est appelé à verbaliser constamment son raisonnement ou une idée en mathématiques, on travaille ici sur un aspect particulièrement déficient de l'enseignement des mathématiques (Bauersfeld, 1994) en amenant les étudiants à pouvoir jouer avec différents niveaux de langage (pouvoir expliciter une idée, un raisonnement, un symbolisme en utilisant une verbalisation accessible aux élèves).

L'utilisation de matériels, de représentations, de contextes pour appuyer cette verbalisation et la construction de sens est aussi présente. Ce travail de verbalisation, de contextualisation et de représentation sera effectué par exemple lorsqu'on aborde les fractions. On verbalise entre autres les fractions équivalentes dans un contexte qui réfère au sens partie d'un tout, au sens quotient ou au sens rapport de la fraction.

On forme enfin à une première *réflexion sur l'action* (Schön, 1983, 1987), lors de l'élaboration et réalisation d'une intervention pédagogique sur un sujet mathématique donné, dans le laboratoire jumelé au cours: préparation d'une suite de 3 leçons consécutives sur un sujet donné (introduction à l'algèbre et construction de formules, résolution d'équations, constructions géométriques-bissectrice, médiatrice..., introduction à l'addition de fractions, à la multiplication de fractions, etc.), réalisation d'une des leçons devant le groupe, analyse et réinvestissement dans une nouvelle leçon.

- *Les thèmes retenus* sur lesquels va s'articuler ce travail sont tous des thèmes qui s'enseignent par la suite dans la période des stages (décimaux, fractions, opérations dans le cadre de la résolution de problèmes, introduction à l'algèbre...).
- Tout ce travail est supporté par un partenariat bien articulé autour des étudiants débutants (cf. figure 1).

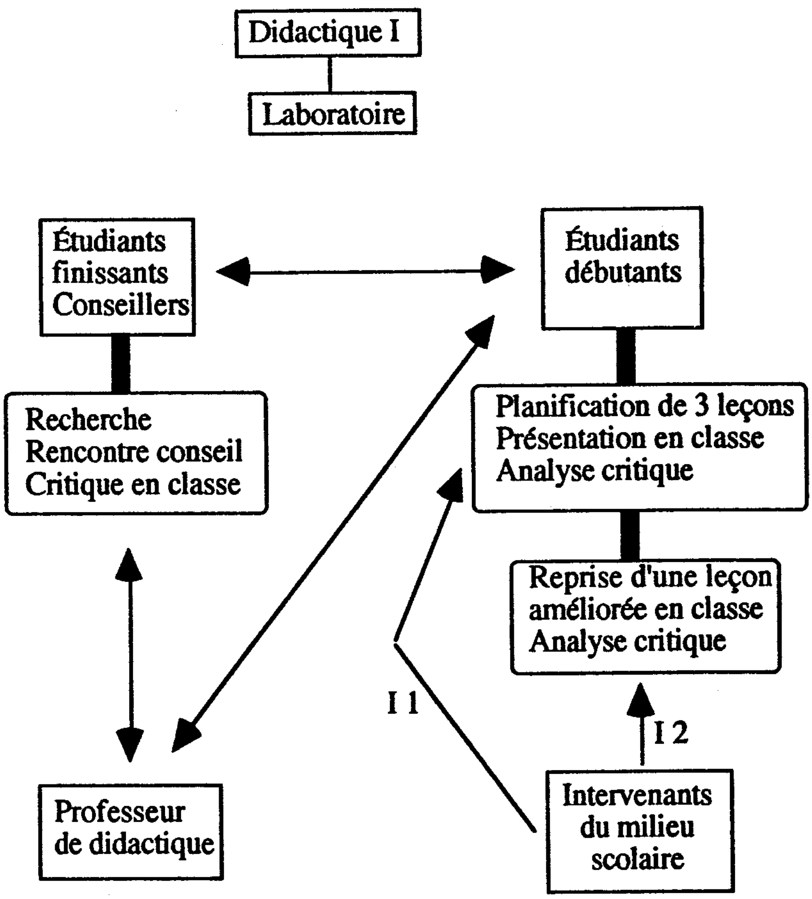

Figure 1: Le cours de didactique I et son laboratoire

Ainsi une équipe formée *d'étudiants finissants* (jouant le rôle de conseillers auprès des étudiants débutants dans la préparation des leçons), de *didacticiens* des mathématiques (responsables du cours et supervisant eux-mêmes les conseillers dans le cadre d'une activité qui leur est créditée), et *d'intervenants du milieu scolaire*, enseignants en mathématiques au secondaire, conseillers pédagogiques...(responsables de la partie laboratoire associée au cours) va travailler à supporter l'étudiant en formation dans ce cheminement.

Les étudiants débutants ont à préparer une séquence de 3 leçons consécutives sur un sujet précis. Ils sont aidés dans cette préparation par un conseiller, étudiant finissant, qui agit alors comme tuteur auprès de lui. Ces étudiants présentent devant le groupe de leurs camarades une des leçons qu'ils ont préparées. Ils sont alors critiqués par leur conseiller, leurs pairs, qui jouent le rôle d'élèves lors des présentations, un praticien du milieu scolaire (ces critiques ont trait à la maîtrise du sujet, à la gestion de classe, à la participation des élèves dans la classe, au questionnement du professeur...). À l'occasion, le professeur de didactique réinvestira dans le cours lui-même les observations et discussions qui ont surgi de ces présentations.

À travers l'avis de ses pairs, d'intervenants du milieu scolaire et de didacticiens, l'étudiant se construit ainsi progressivement un répertoire d'interventions pédagogiques possibles lui permettant d'avancer sur la voie d'une réelle formation.(pour plus de détails sur le contenu de ce cours, voir Bednarz, Gattuso, & Mary, 1995).

On peut entrevoir à travers cet exemple les visées de ce programme de formation qui s'éloigne des modèles dominants de formation des maîtres, et notamment du cadre ministériel qui associe essentiellement formation pratique à présence en stage, et qui différencie fortement, même s'il souhaite une intégration de ces diverses composantes, les composantes disciplinaires et psychopédagogiques de cette formation. Ce programme met au contraire de l'avant une formation intégrée visant constamment à la fois une appropriation des contenus à enseigner et la préparation à l'enseignement.

III. Quels sont les fondements sous-jacents à ces programmes de formation?

Les diverses activités élaborées reposent sur une certaine pratique éducative mise de l'avant de la part des formateurs, cohérente avec les buts que nous visons. Il s'agit ici de développer chez le futur professionnel de l'intervention en mathématiques une autonomie, une capacité de prendre des décisions, de s'organiser, de faire des choix appropriés, de mettre en place une culture de classe alternative en mathématiques, de développer dès la formation initiale une réflexion critique en regard de sa propre pratique, une attitude à la recherche, qualités essentielles à l'exercice de sa future profession.

Ce modèle privilégie ainsi la participation à une certaine culture dans laquelle le questionnement, l'explication des points de vue, les interactions entre étudiants et professeurs vont jouer un rôle important. Nous organisons la formation des enseignants comme une "culture" qui actualise elle-même les caractéristiques désirées. Nos pratiques de formation vont s'articuler en ce sens sur les raisonnements et idées développées par les étudiants maîtres en situation à l'égard des mathématiques, de leur apprentissage et de leur enseignement et essayer de les faire évoluer.

Les théories de la didactique des mathématiques ne sont pas exposées comme telles. On ne tient pas, on l'a vu précédemment, un discours sur l'action mais on fait en sorte que l'étudiant construise dans l'action (appuyé par les discussions avec les autres étudiants et les interventions du professeur) les connaissances qui lui permettront d'aborder sa future profession. Il ne s'agit pas d'exposer la didactique mais de former par la didactique. En fait nous cherchons la cohérence entre notre discours et nos propres pratiques d'enseignement. Il ne suffit pas à notre avis d'exposer des concepts pour que ceux-ci soient développés par les étudiants (autant élèves en mathématiques que futurs maîtres en formation). C'est à travers l'action, les interactions et la réflexion sur celles-ci que se construisent les connaissances et c'est sur ce modèle nous nous appuyons. Aussi, la didactique sert de cadre de référence à nos interventions comme formateurs. Par exemple, lorsque nous travaillons l'introduction à l'algèbre avec les étudiants maîtres, nous avons en arrière-plan toute une réflexion didactique qui nous permet de choisir les situations propices à une réflexion et à une discussion de la part des étudiants; dans un autre domaine, le choix de problèmes additifs ou multiplicatifs soumis aux étudiants pour fins d'analyse repose aussi sur un cadre de référence qui nous permet de juger de leur complexité relative.

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Working Group C

WHAT IS DYNAMIC ALGEBRA?

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PREAMBLE

Action and expression: what a considerable distance separates these two activities in the mathematical domain. I do, I reflect, I try to understand, I express the state of my understanding. Can I express my understanding of what I have done? Do I have a language to do it before I have understood or expressed myself formally? Is formalisation an end result of activity: or can judicious uses of new technologies draw together formal and informal in productive ways? Our idea in the workshop was straightforward. Let's engage with some card tricks; try to reproduce the actions on the cards; build a formal model (using Logo) of the tricks, by re-presenting them in the form of programs; and try to use the model to understand, perhaps generalise the trick. What do we gain from building and running the model? Does the construction of a program, a kind of (dynamic) algebra in itself, uncover the structure of the trick, or its underlying mathematical structure? Perhaps David Wheeler is right; maybe algebra "never performs an explanatory function". These are difficult questions, and I think the contributions below show that our workshop engaged with some of them. I cannot pretend that we came near to providing answers. But everyone, to the best of my memory, succeeded in some or all of the modelling tasks. It was certainly fun: but was it mathematics?

STATEMENTS OF THE PARTICIPANTS

Tom Kieren

On Two Experiences With Modeling

The two "problems" as I saw them were veneers of the two "card tricks" with our task being to reason from the veneers to the tricks underneath. With respect to "trick one" I joined a group in progress at the machine. Because the machine/software were available and we had the "trick one" task before us, we mostly acted and reflected on those actions almost independently of the problem or task itself. This led us in the end to have an alternate model for accomplishing the trick; an alternate representation for the veneer, but little insight into the "mathematics" beneath that veneer. We had mathematized the performance, but had not yet at least pried beneath that; rather like modeling an action as the copywriter.

With "trick two" in one trial we got the trick to work. We repeated and talked about several alternate tricks, each which worked. In this case we got an image of the trick (e.g., what happened to the order of cards under moves) for specific cases. My partners generalized a notation: EOEOEOEOEOEO which gave us a method for doing the trick without reference to the cards. This led us to try to see all the possible different cases (we found four, I think) which led us to "prove by exhaustion" that the trick worked. We felt that we knew that the mathematics worked, but not how. Marty offered us an alternate version notation (an alternate formalization), which allowed him to at least show us how the mathematics worked, and offered us a proof that the process worked and not just that it worked. We still needed to work this for ourselves and certainly had not generalized, but we felt little need to computer model anything at this point.

Hassane Squalli

I preferred the second activity to the first. For me, in the first activity, we went too rapidly to the computer. We did not take enough time to "appropriate" the problem, and then be able to understand what happens and to try some conjectures. In the second activity, on the contrary, we were able to appropriate the problem, make conjectures, try to validate them, and try to make proofs. Later, when we were satisfied with the solution we had found, we tried to find some interesting generalizations.

Have we done any mathematics? I really don't know if I had in the first activity. In the second activity, I had the impression that my mathematical experience helped me in understanding the problem and making conjectures and generalizations. I find it interesting to try to "modelize" concrete actions in a language of "codes" (symbolism?); focusing only on the "pertinent variables", and translating the study of the problem to paper. But, where is algebra in all of that? In the first activity, I did not have time to see it. In the second, although we resolved the problem and made some generalizations, in my sense, I haven't done any algebra.

Sandy Dawson

Working with Tom Kieren and Judy Barnes, we studied the second card trick, and by chance found an immediate solution. We did not, however, know why it worked. We undertook an exhaustive examination of cases, and convinced ourselves that the solution we had stumbled upon did in fact work in all instances when beginning with 2n cards.

However, I still did not know why it worked. We had a proof, we were convinced that the proof was valid, and that there were no counter-examples to it, but still I did not know why the solution worked. It was only when Marty came by, suggested a straight shift in terminology, and visually showed us his solution that I got a glimmer that a "flip plus slide" left matters unchanged as to the sequence of correct and incorrectly facing cards. Then I understood.

Moral of the story:

1. You can have a complete and convincing proof, but still not have meaning and understanding, and

2. When understanding is achieved, further action may cease, at least for a time.

It was also clear to me that working with others was a distinct advantage for me.

Jacqueline Klasa

"What is dynamic algebra?"

This workshop was devoted to problem solving and consequently, "Dynamic Algebra". Few discussions ran on to determine if we really worked with algebra. Algebra was expressed here as a model of a real situation with card games.

During the first morning, it was proposed that we model the problem with Logo. As many of us were unfamiliar with Logo, some frustrations arrived. We were handicapped by Logo and felt that we were asked to communicate a complex message in a language that was neither our mother tongue or even a language that we could be fluent in.

Before the second morning, I decided to use a few of my own tools of modelization: tables(triangular), and modular arithmetic. My model was clear to me and so my frustration disappeared.

From that situation where I did not feel at ease, I learned that sometimes I may impose too much on my own students. I may be too restrictive of the mode of expression for a given problem; (sometimes we are in computer labs, sometimes we work in clinical ways).

A good modelisation or algebrisation may take some time and use various tools of communication, back and forth (e.g., Do pencil work first, set up tables for small cases, discuss with team members. Observe some patterns and infer a structure. Then check your claim with more powerful tools: Graphical calculators, use of mathematical software, etc. At some time, you may take again your pencil and sketch a "proof".). I understood "dynamic algebra" as this composite behavior and communication.

I thank Martin Hoffman and Richard Noss for this very interesting and stimulating working group. When I go back to my college, I intend to propose activity I (and II, later) to the students visiting our Mathematics Resource Center as a competition. They will be allowed to use any mode of work and communication: pencil, blackboard in groups, computers with Maple and maybe the Windows version of Logo as Richard Noss proposed to give to me. Of course they will also have the night to search through the Internet if they may find some references of such games! This activity could be over a few months. (I could report back this experiment!)

Ralph Mason

Authority and Authoring

LOGO: Does it serve as author support or authority?
Computer: Does it serve as author support or authority?
Algebra Curriculum: Does it serve our students to support their authoring of their words/world, or do they answer to its authority?

My glasses help me see the world (be in the world) better, because they are so transparent and convenient while changing how I view things (do things). Is LOGO (the computer, algebra) like a pair of glasses ultimately, or does it remain the object of my students' immediate attention, blocking my students' autonomy?

Barbara Jaworski

Working on the second cards problem was different to the first. It was not at all clear what we had to do with it, whereas in the first problem the processes seemed clear. We just had to find a way to express them in LOGO in order to get an answer. The challenge was in creating the Logo program, but this was slow because we did not have the tools ready at hand.

In the second problem, the task was to find some way of representing visually what occurred when the pack was cut and the cards turned over. After three or four manual attempts with the cards and recording by hand the resulting sequence, we needed some way to produce this record quickly and reliably. The computer was clearly a tool which we could use for this. Thus, creating a program was a necessary step in being able to have enough data to analyze in order to see a way through to understanding the problem.

In the second problem the way we programmed the computer provided transparency to the problem. In the first we had not done so, although on reflection it might have been valuable if we had, in order better to understand the mathematics involved.

We talked in the larger group about where the mathematics lay in our activities. One possibility is that we were doing mathematics when we were writing programs. Clearly we were engaged in algorithmic thinking. We were specifying variables and establishing relationships between them. The computer then performed the manipulations.

However, in order to see the mathematics behind the problems we were using the computer as a tool. The process here was one of modeling the situation in order to gain insights which would then lead to further mathematical thinking. Because we did not finish our programs, we did not reach this stage. However, it seems clear that the thinking involved in theses activities was multi-layered, and that different parts of it could be justified in mathematical terms.

Richard seemed aware that the time of his inputs and the effects they might have on our decisions and progress. It was not always clear to us what actions from him would be most helpful, and he had to take responsibility for what he offered or didn't offer. I should have preferred some more overt input (maybe in written form) of the Logo primitives which were available and possibly helpful in these problems.

Tasoula Berggren

For both tricks I was very interested to find out how and why they worked and enjoyed the hands on activity with the cards, computers, and working with people. The first day, however, the use of Logo created negative feelings in me and I thought of my students, how they feel sometimes when they are asked to explore problems using software without knowing the necessary language.

Because I did not know how to use Logo the challenge for me was mainly to figure out the new language while expanding on the solution of the trick. I received some insight into the trick from the computer interaction but I saw no direction, no way leading to a mathematical solution, so I went back to the paper and pencil—a sure way.

The second day, the trick was easier to explore using the cards so I did not want to use the computer to build a model. Through observation, elimination and simplification the trick was clear. David Lidstone and I worked together and we discovered that the ascending order was not important and in fact the odd and even could be replaced by red and black and that the cutting of cards made no difference. I do feel that the use of the computer will add an experimental approach to problems and tricks; especially when the manipulation requires many computations.

Ed Barbeau

First Card Trick (16 Cards):

The computer played the role of a flunky to provide the data from which the pattern could be inferred. It also served as a catalyst; figuring out what to get the computer to do helped organize one's thinking about the problem. However, I essentially solved the problem without the computer.

Second Card Trick (10 cards):

The trick became completely transparent once I had my own deck of cards in hand and I could work through the effects of cutting and reversing the consecutive cards, followed by reversing alternate cards. The computer played no role whatsoever.

Some Issues:

The use of LOGO was ancillary to the whole exercise, but my unfamiliarity with LOGO distracted from the task at hand. This raises the general question of how prerequisites are handled. Does one separate them out and secure them in advance, with the danger that this separation may not allow for any context for them? Or does one weave prerequisites into the situation at hand, with the danger that focus is lost? This is related to the "back to the basics" movement.

Is this algebraic? There certainly was mathematical thinking going on, but I would not describe it as algebraic in the sense of carrying out formal operations on symbols or dealing with a structure abstractly. The first card trick has scope for getting into algebra. There are some parameters to deal with and the formulation of the remaining card in terms of parameter. One can think of this in operational terms, as a functional relation. If $f(n,m)$ is the remaining card when we have an original stack of n cards from which m have been removed, establish when $f(n, m) = m$? In particular, establish this when $1 \leq m \leq 8$ and $n = 16$. In the second trick, the algebraic content was basic—looking at the effect of composing transformations of certain types. Giving this a symbolic form is a challenging task. The crux of the matter is revealed by looking at two consecutive cards.

Eric Muller

The How and Why

In activity 1 (the 16 card trick) I never reached the stage of exploring the why. By moving away from the actual situation to the computer environment I spent my time trying to model the situation in a different medium—simulating the trick in a different environment. I felt it was more an exercise in learning the language than one of exploring why the trick worked and whether it would work under different conditions. Was there any mathematics in this simulation exercise? Certainly there was some logic and ability to follow instructions, but mathematics? To me the mathematics would have started once the simulation was working to explore patterns and other properties. Certainly the simulation could have either hidden the display of patterns or enhanced it for larger displays. So what we simulated would be important in the end but was not an initial preoccupation as I struggled with the language. There was no AHA! or enlightenment on the why.

In activity 2—by staying with the actual card situation— the situation was manageable and I was able to get quickly to how the trick worked. Getting to the why did not take long as card patterns on both sides were explored and their relationship as you move from one side to the other. A shorter mathematical notation was developed and "what if" questions were explored both with the cards and with the mathematical notation—both were easily and often interchanged when looking for solutions and reasons.

David Pimm

With my companions, I worked most on two (numbers 1 and 3) of the three-card tasks. The first invoked a fixed procedure (after a random cut and discard) and claimed a predictable invariance in the result; the second asked us to discover a procedure to produce a given invariance (odds one way up, evens the other) after another procedure had been applied an indeterminate number of times to ten cards in a given order; the third offered simply a procedure (a 'perfect' riffle shuffle) to explore, though Richard suggested one question that also implied an invariance, namely which root of the identity the perfect riffle shuffle was.

There was also the not-unrelated tension between the task being one of confirmation/determination and that elusive/illusory goal of understanding. I had not seen so clearly before the dismissive nature of understanding, that once I feel I have understood 'why', there is nothing more to be done. So understanding too closes down the problem space dramatically, as much as any belief in there being one 'right' answer. In both tasks, I felt I was looking for a 'pivot' (a term Leron uses in his description of structuring mathematical proofs), a central fact, reason or awareness that forced each 'trick' to work in the manner claimed. For task 1, I needed a way of showing both that it did, and then exploring why it did. The pivot would allow me to think about it without doing the trick anymore and see into the mathematical structure of the task.

I find it unproblematic to acknowledge the power of the computer to generate data in the face of a mathematical investigation of some kind. I feel it is an open question as to for which tasks programming the active task itself helps provide insight into identifying the pivot. I felt a familiar tension as we worked at these tasks between the computer/LOGO as field and as ground. In conjunction with David Henderson's earlier
remarks about theoretical computer science being the home of formal algorithms, and Richard's acknowledgement of algorithmic thinking being somewhat distinct from mathematical thinking, I am still pondering why I am resisting recursion as a fundamentally mathematical notion. Because I do.

Certainly in task 1, trying to program the procedure, once again I came across my feelings about 'the LOGO aesthetic', privileging certain seeings as being computationally elegant (no 'make' statements, recursion over repeat, no 'do' loops). After all, if I only wanted to have the output, any way that 'worked' would be as good as any other. But if it mattered how I porogrammed it in terms of producing insight, then why is 'good LOGO' the same as 'insightful programming'? Pencil and paper are aids to making evident certain relationships among elements, as well as recording a sequence of results. Our inability to structure the LOGO-generated output in the riffle shuffle also highlighted the need for surveyable data.

But this work also allowed me to see similar processes at work in algebra. David Wheeler's comments about algebraic restlessness immediately taking you somewhere else rather than encouraging gazing reminded me how much and how rapidly I had averted my gaze from the cards themselves. And perhaps, because in mathematics 'we attend to the relationships themselves', we are never able to gaze at what we would like to, there is no mathematical 'thing' to contemplate, mathematics always involves a deflection, a looking elsewhere.

One of the highlightings that occured was seeing that the word 'dynamic' in 'dynamic algebra' meant more than "fast", and dynamic notation was an aim, one where the manipulation kept pace with the structure of the procedure applied to the cards. By putting in 'windows' into the recursive programs, more traces (albeit static ones) were available for subsequent consultation.

Marty provided the best instance for me of mathematical understanding at work. First by offering the Gattegno-film inspired heuristic/way of working of 'run the film backwards', in order to identify where the card must have come from in order to end right, then by developing a notation/description that showed why most of the apparently-salient variables in Task 2 actually were themselves indifferent to what we were attending to, highlighted for me the power of staying with the objects but converting them into a working symbolism.

I am left thinking about differences between 'mere' notation and a working symbolism (one where I can manipulate it dynamically for my own mathematical ends). And the transformation that occurs when the objects 'themselves' are used symbolically (which is to do with the placing of human attention, as much as 'marking' cards).

David Wheeler

I hook my remarks onto three key words: "algebra", "model", and "experiment". In the plan of the task for the Working Group, algebra was invoked as the form of mathematics that underlies the puzzles, or "tricks", with playing cards that we were given to explore; in our discussions the observation was made several times that we were working alternative models of the phenomena under consideration; and Richard suggested that programming the Logo software to represent the puzzle might have the pedagogical effect of encouraging an experimental approach to an investigation.

Algebra

The word "algebra" has had many historical shifts of meaning. With or without the advantages of hindsight, we are likely to find Al Khwarizmi's algebra substantially different from Boole's algebra. Looking in another direction, there are arguments of an ontological sort which distinguish between "developed", "genuine" algebra and "immature" forms: pre-algebra or proto-algebra. But "algebra" is also frequently used quite loosely in conversation as a virtual symptom for mathematics, and it seemed that this usage kept surfacing in the discussions, and perhaps in the groups' title.

"Algebra" performs this function (of standing in for mathematics in common parlance) because it is (a) patently esoteric, as mathematics as a whole is felt to be, and yet (b) well-nigh inescapable in the long-term elaboration of any mathematical context. Algebra is called on to extend arithmetic and to mechanise geometry, and ultimately it can't be prevented from invading every non-algebraic part of mathematics.

I happen to think that algebra never performs an explanatory function, so even if the behavior of the phenomena we were exploring could be expressed algebraically, that in itself would not lead to a "justification" of the behavior. Indeed, one can only give a mathematical reason in algebraic when one already knows what that reason is. The title "Dynamic Algebra" seems to me a tautology since the one property that is common to all forms of algebra (including traditional "school" algebra) is its dynamism, its restlessness. Once algebra is called in, it immediately tries to take the situation somewhere else. Algebra allows a user to set up an equation, but it wriggles desperately until the equation is solved. A function can be defined algebraically, but then immediately cries out for inputs. Algebra is quintessentially a go-getter; it doesn't stand and stare. It refuses to allow the user to gaze, to brood, to reflect.

Model

Some of our discussions suggested we saw three different models, or embodiments of the card puzzles:

- the cards themselves, with certain "natural" operations
- "pencil and paper" symbolic manipulations
- Logo procedures.

The status of these alternative models is very different. The first seems less like a model than a "universe" within which the puzzles are situated. The second is a general-purpose model, of a familiar type, and is invested with considerable power. Once we represent a deck of cards by a linear array of numerals we have immediately available to our perception a wealth of mathematical experience, drawn from our considerable familiarity with the relationships between these numerals, and amplified by the geometrical perceptions that are triggered as soon as we begin to repeat and transform the arrays on paper. It is doubtful a priori if any single model of the behavior of a deck of cards could be more powerful or comprehensive than this.

"Pencil and paper", endowed with arrays of numerals and the usual elementary arithmetical operations, seems to me to qualify as a microworld. How far the Logo microworld stacks up against this in terms of complexity and flexibility is an interesting question that I have not enough know-how to answer.

Experiment

I agree that learners should be encouraged to take an experimental attitude to mathematics. How will they acquire the important skills of making "thought experiments" (the heart of mathematical activity) if they don't first have plentiful experience of making trials in concrete investigations? But I didn't personally get close enough to fluency with programming to have much inkling whether this tactic would in fact facilitate experimentation with the puzzle situations. Indeed, for me, it was the programming that was the subject of experiment since the only way I could make progress with it was to "try something and adjust it", and this activity ate up most of the time available. I experimented very little with the puzzles, and not at all with them in Logo form.

Postscript

The experience of working with Logo (after a long enough interval to have forgotten most of what I once knew) made me aware how difficult it is to construct a procedure out of the available primitives of the software, especially when one has no clear picture why that particular collection of primitives has been made available. Representing a cut in a deck of cards with Logo requires one to construct an "action procedure" out of a set of "actions" that are not obviously related to the re-ordering of a linear arrangement of cards. I understand that one cannot expect a software to provide a primitive for every sort of "real" action one might want to model (and it would be unmanageable in its multiplicity if it did), but what I lacked was any sense of the general form of the mental model that guided the designers and made them provide "these" primitives rather than "those". Logo operates with lists, I know, but I don't recall, and I can't reconstruct, what the designers supposed people would variously want to do with lists. Without a better idea of the nature of the designers' model(s), I could only stumble blindly about trying to use the software to produce the highly specific models I needed for the puzzles.

Working Group D

THE ROLE OF PROOF IN POST-SECONDARY EDUCATION

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INTRODUCTION

Before the Working Group met the following questions were distributed in order to facilitate some thoughts on the role of proof in post-secondary mathematics.

1) What is the point of rigour in mathematics?

2) What is the connection between proof and understanding?

3) Can you do mathematics without introducing the idea of proof?

4) Is it necessary to introduce *proof* as a distinct topic or course within the curriculum? If so, what should be the mathematical context: abstract or linear algebra, a first course in analysis, or a course on the structure of number systems?

5) Should mathematics departments have streams (or majors) in which mastery of *proof* is minimized or omitted?

DISCUSSION

The session began with brief descriptions of two programs, one at Concordia University and the other at Simon Fraser University, which introduced the notion of proof in a separate course. A complete description of Concordia's program appears in Byers and Hillel (1994). The Working Group decided to divide into two subgroups - one concerning proofs and proving for pre-service high school teachers, and the other for students taking calculus, linear algebra, etc. The two groups met separately to discuss difficult questions and then shared their thoughts with each other at the end of the sessions.

We agreed that there should be a shift of emphasis from *proofs* to *the act of proving*. The act of proving refers to "habits of mind" (Cuoco, Goldenberg, and Mark, 1995) which involve questioning, anticipating, asking "what happens if", etc. More succinctly, proving has to do with the business of "inquiry with confirmation". Later, the notion of 'proving' was refined to include deduction. An act of proving is an inquiry with confirmation by means of deduction.

This line of discussion led to a flood of questions. Are such habits of mind unique to mathematics? Does writing a term paper in sociology or history not constitute an inquiry with confirmation? What is the difference between confirmation in mathematics and confirmation in, for example, history or sociology?

How can we tell whether a particular action of proving really involves deduction on the part of the person showing a proof? Could it just be a rote reproduction of an argument (even if logically correct)? Is the requirement of being able to communicate a result to another person a sufficient indicator of deductive reasoning?

Having shifted the focus from the originally proposed set of questions, the Working Group decided that they would rather consider the following questions.

1. What kind of results should we prove?
2. What would be considered as a convincing argument in a given context?
3. What is the difference between written proof and the students' understanding?
4. Are there technical differences between the terms validation, justification, confirmation, verification, proving, and convincing?

What Kind of Results Should We Prove?

The first question needed clarification as to who is *we*. *We* could refer to university mathematics teachers and teacher trainers, or to their students (mathematics majors, or to pre- and in-service teachers) or to elementary and high school students. The group realized that at certain times in our discussions, we were shifting from one set of "we" to another. Clearly, what we should prove depended on the people involved.

The proof of the Chain Rule in elementary calculus proved to be an excellent example. Clearly the audience is calculus students. Why do we bother with such a proof and do we need to make it rigorous? What's wrong with the naive version of the proof using $\frac{\Delta y}{\Delta x}$? We spent time "unpacking" the proof and pointing to some of its features that are worth highlighting. What emerged was a consensus that the question is not whether the Chain Rule should be proved but how to go about it. The proof can be a focus of discussions, experimentation and group activities that can last several lectures.

Are There Technical Differences Between the Terms Validation, Justification, Confirmation, Verification, Proving, and Convincing?

We didn't get very far with the fourth question - there was an attempt to distinguish the terms *conviction* and *justification* as they are used in philosophy of science, but this didn't help to clarify the issue. After some fruitless discussion, the question was dropped.

What Would be Considered as a Convincing Argument in a Given Context?

Question 2 was more central to the discussions. It was noticed that there is a social side to proving, which requires a shared repertory, and a personal side, which does not. Both of these aspects must be considered when discussing the act of proving for prospective teachers.

For high school teachers, one of the most important attitudes to foster is that of being curious and questioning. Therefore, high school teachers should have a strong background in mathematics so that they can *play* with the mathematical proofs in a *didactical* way. They should be able to give many different proofs of a result so that they are able to understand, justify, and explain high school material. They need to understand proofs, to be able to explain a result without betraying the proof.

Therefore, we should have prospective high school teachers approach proof in the manner of Lakatos (1976). Namely, have them attempt proofs of results and then enable them to gradually unveil the reasons why the initial attempts were incorrect. The aim of this approach is that teachers can detect where their proofs are not valid, can construct proofs on their own, can validate their reasoning or intuition, so that their students can validate or show as invalid, the results they obtain.

What, then, constitutes a convincing argument? The group decided that a convincing argument or proof is one such that the person should be able to produce a similar proof and explain it to others. As an example, consider the standard proof that $\sqrt{2}$ is irrational. Given a proof of this fact, the students should be able to produce a proof that $\sqrt{3}$ is irrational.

What if an algorithm is involved? One thought was that the person convinced should be able to explain it to someone else who would then be able to use the algorithm correctly. This view may still beg the question as to whether any deduction is involved in the explanation, or is it merely a repetition of something seen. We were left with the question as to whether this constituted an "inquiry with confirmation by means of deduction."

The discussion then moved to presenting "proofs by example" instead of giving very formal proofs. It was felt that in some cases, using a well-chosen example is more effective in convincing or in conveying a result or a technique than giving a formal, decontextualised proof. It was pointed out that some mathematicians who were known for their formal and pedantic writing of mathematics, behaved very differently in their lectures and when they supervised graduate students. Proofs by example raise the issue as to whether students can extract the generalized features of such proofs or whether they focus too much on the specific example

We asked the question—What would be a convincing argument in the context of mathematics for teachers? The answer depends on the teacher's background. What is "obvious?" What is the "acquired knowledge?"

As noted earlier, the act of proving has both a social side and a personal side and the former relies on a shared repertory of things which are taken for granted which, in turn, determines if a proof is convincing or not. This raised the question of whether software tools such as Cabrie and Logo bring about a change of what constitutes the shared repertory. For example, consider a Logo procedure for generating three consecutive integers and then checking that their sum is divisible by 3. Is this a precursor to a formal proof? These thoughts pose more general questions. Are commonly accepted facts changing with Logo? With Cabrie? Is what we consider to be *obvious* changing?

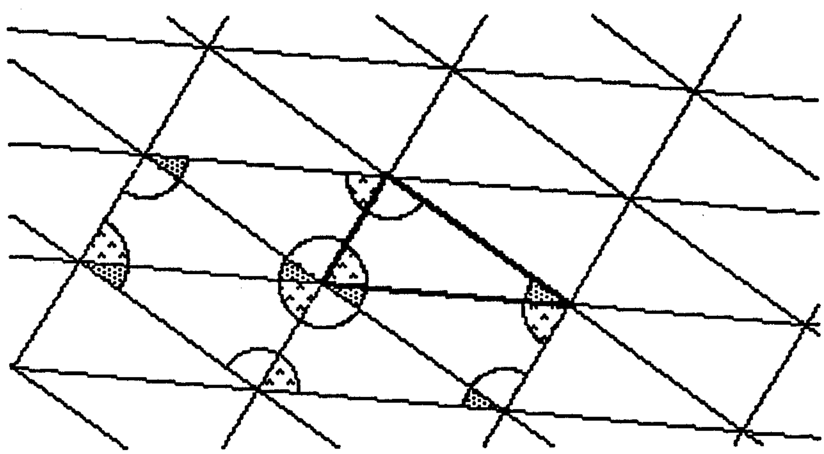
The Working Group recognized that many people *understand* a concept or proof formally before the concept or proof is properly *understood* or internalized. The discussion then turned to *formal* proofs. It was felt that arguments that are *formal* may be easier to accept by students because of their perception that these proofs are given by someone in authority.

We closed our work by trying to decide what constituted a convincing argument. Two examples highlighted the discussion. The first was the illustrative proof that the sum of two odd numbers was even (see C. Hoyles, Figure1, in this monograph). We all agreed that, given the appropriate audience, this would be a convincing argument.

There was a more heated discussion as to whether the following student's *proof* that the sum of the angles in a triangle is 180 degrees was acceptable as a convincing explanation. The proof relied on tessellation of the plane by triangles (Figure 1).

Yorath's answer

I drew a tessellation of triangles and marked all the equal angles.



I know that the angles round a point add up to 360°.

So Yorath says it's true.

Figure 1
Yorath's Answer

Some found it novel and convincing, others pointed out the circularity of the argument (being able to tessellate by triangles presupposes the result) and others argued that we have to suspend judgment until we know what the student's starting points were.

What is the Difference Between Written Proof and the Students' Understanding?

Time didn't permit our discussing this question.

CONCLUSION

The separation of the Working Group into two subgroups, one for proofs and proving for pre-service high school teachers, and the other for students taking calculus, linear algebra, etc., seemed natural enough. However, many of us worked with both kinds of students and had some difficulty choosing one of the subgroups to the exclusion of the other. More interesting, the two groups were in agreement almost all of the time, and even chose some of the same illustrative examples when they met separately. Finally, it was noted that many of the issues about proofs and proving that were discussed in the Working Group are hardly ever discussed with mathematics students or student-teachers, thus leaving a serious gap in their mathematics education.

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Groupe de Travail D

Le Rôle de la Preuve dans l'Éducation Post-Secondaire

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INTRODUCTION

Afin de faciliter la réflexion sur le rôle de la preuve dans l'éducation post-secondaire, les questions suivantes ont été distribuées avant le début de la rencontre du Groupe de travail.

1. Pourquoi la rigueur en mathématique ?

2. Quel est le lien entre preuve et compréhension ?

3. Peut-on faire des mathématiques sans introduire l'idée de preuve ?

4. Faut-il introduire la notion de preuve comme un sujet distinct ou encore un cours spécifique dans les programmes ? Si oui, dans quel contexte ? Un cours d'algèbre ou d'algèbre linéaire ? Le premier cours d'analyse ? Un cours sur les systèmes de nombres ?

5. Les départements de mathématiques devraient-ils offrir des cheminements ou des programmes de majeures dans lesquels la maîtrise des preuves serait minimisée ou même omise ?

DISCUSSION

La séance débute par une brève description des contenus de deux cours portant sur la notion de preuve: l'un à l'Université Concordia et l'autre à l'Université Simon Fraser. Une description complète du contenu du cours de l'Université Concordia est contenue dans Byers et Hillel (1994).

Le Groupe de travail se divise ensuite en deux groupes qui se penchent sur les preuves et sur *l'action de prouver*, l'un dans les programmes de formation des maîtres pour le secondaire, et l'autre dans les cours de calcul, d'analyse, d'algèbre linéaire, etc. Les deux groupes se réunissent séparément pour discuter de questions bien difficiles, et partagent par la suite leurs réflexions à la fin des séances.

Nous nous entendons pour mettre l'accent sur *l'action de prouver* et non sur les *preuves*. L'action de prouver est reliée à des habitudes mentales (Cuoco, Goldenberg et Mark, 1995) qui comprennent le questionnement, l'anticipation, se demander "ce qui se passe si", etc. En bref, prouver est relié à un questionnement avec confirmation. Par la suite, la notion de *prouver* est précisée pour inclure le raisonnement déductif. L'action de prouver a eu lieu lorsqu'il y a eu questionnement avec confirmation, et ce au moyen d'un raisonnement déductif.

Cette ligne de discussion amène plusieurs questions. Ces habitudes mentales sont-elles spécifiques aux mathématiques ? L'écriture d'un essai en sociologie ou en histoire ne constitue-t-elle pas une enquête, un questionnement avec confirmation ? Quelle est la différence entre confirmation en mathématiques et confirmation en histoire ou en sociologie, par exemple ?

Comment peut-on dire si une personne qui présente une preuve est vraiment en train de faire un raisonnement déductif? Peut-il s'agir d'une reproduction machinale d'un raisonnement (par ailleurs logiquement correct)? Être capable de bien communiquer un résultat à une tierce personne est-il un indicateur suffisant d'un raisonnement déductif ?

Ayant ainsi dévié des quatre questions initiales, le Groupe décide de se pencher sur les questions suivantes.

1. Quels types de résultats devrait-on prouver ?

2. Dans un contexte précis, qu'est-ce qui serait considéré comme un argument convaincant ?

3. Quelle est la différence entre la preuve écrite et la compréhension des étudiantes et étudiants ?

4. Y a-t-il des différences techniques entre valider, justifier, confirmer, vérifier, prouver et convaincre?

Quels types de résultats devrait-on prouver ?

Pour répondre à cette première question, on doit préciser de qui il s'agit. Parle-t-on de professeurs d'université, de ceux et celles qui enseignent aux futurs enseignants, des étudiantes et étudiants en mathématiques, de ceux et celles qui étudient en enseignement secondaire ou primaire, ou encore des étudiantes et étudiants du secondaire et du primaire ? Dans nos discussions au sein du groupe, nous changeons parfois de groupe-cible sans le dire explicitement. Il nous apparaît évident que ce qui doit être prouvé dépend de qui il s'agit.

La preuve de la règle de dérivation en chaîne s'avère un excellent exemple. Le groupe-cible est ici constitué des étudiantes et des étudiants d'un cours d'analyse ou de calcul. Pourquoi s'obliger à faire une telle preuve ? Doit-on alors faire une preuve rigoureuse ? Qu'y a-t-il de mauvais à donner une preuve naïve en utilisant f(x,y) ? Nous décortiquons la preuve de ce résultat pendant un certain temps et nous soulignons un certain nombre de points importants. En ressort un consensus: la question n'est pas de savoir si on doit donner la preuve de la règle de dérivation en chaîne, mais bien comment on doit le faire. La preuve de cette règle peut être un sujet de discussions, d'expérimentations et d'activités de groupe qui peuvent durer plusieurs cours.

Y a-t-il des différences techniques entre valider, justifier, confirmer, vérifier, prouver et convaincre?

Nous n'approfondissons pas la quatrième question. Après un essai pour distinguer *conviction* et *justification* tels qu'ils sont utilisés en philosophie des sciences, nous abandonnons la discussion.

Dans un contexte précis, qu'est-ce qui serait considéré comme un argument convaincant ?

La question 2 prend une place importante dans nos discussions. On remarque qu'il y a un aspect social à la preuve, qui requiert un répertoire de connaissances partagées, et un aspect personnel qui n'en requiert pas. On doit tenir compte de ces deux aspects lors des discussions dans le cadre de la formation des enseignants.

L'une des attitudes les plus importantes parmis celles dont il faut favoriser le développement pour les personnes qui enseignent au secondaire est celle d'être curieux, de se questionner. Afin de pouvoir jouer avec les preuves tout en tenant compte de l'aspect didactique, ces personnes doivent avoir un fort bagage mathématique. Elles doivent pouvoir donner plusieurs preuves d'un résultat de manière à pouvoir comprendre, justifier et expliquer les mathématiques du secondaire. Elles doivent comprendre les preuves de manière à pouvoir expliquer sans trahir la preuve.

Nous devrions donc favoriser, chez ces personnes qui enseignent ou enseigneront au secondaire, une approche de la preuve à la Lakatos (1976). Nommément, les amener à essayer de prouver des résultats et, par la suite, leur permettre de découvrir pourquoi leurs essais initiaux étaient incorrects. Cette approche vise à leur permettre de voir où leurs propres preuves sont erronées, de faire des preuves par elles-mêmes, de valider leurs raisonnements ou leur intuition; de cette manière, elles pourront valider ou invalider les résultats obtenus par leurs étudiants et leurs étudiantes.

Qu'est-ce qui constitue alors un raisonnement convaincant? Un raisonnement convaincant ou une preuve est un raisonnement tel que la personne convaincue peut alors produire une preuve similaire et l'expliquer à d'autres. Comme exemple de cela, on peut penser à la preuve habituelle de l'irrationnalité de la racine carrée de 2. Étant donné une preuve de ce résultat, les étudiants devraient pouvoir produire une preuve de l'irrationnalité de la racine carrée de 3.

Qu'en est-il s'il y a un algorithme ? Une idée émise est que la personne convaincue devrait pouvoir l'expliquer à une tierce personne de telle manière que cette personne puisse l'utiliser correctement. Cette idée nous amène cependant à demander si l'explication implique nécessairement un raisonnement déductif. Peut-il simplement s'agir d'une répétition ? La question demeure: s'agit-il d'un questionnement avec confirmation au moyen d'un raisonnement déductif ?

Nous discutons ensuite des "preuves à l'aide d'exemples" au lieu des preuves formelles. On pense que dans certains cas, l'utilisation d'un exemple bien choisi est plus efficace pour convaincre ou faire comprendre un résultat que ne l'est une preuve formelle et sans contexte. On fait remarquer que certains mathématiciens connus pour leurs écrits très formels se comportaient de manière fort différente dans leurs cours ou lors de leurs supervisions d'étudiants gradués. Les preuves à l'aide d'exemples permettent-elles aux étudiants d'extraire les principes généraux de ces preuves ou est-ce que ceux-ci concentrent leur attention sur l'exemple précis ? La question est posée. Qu'est-ce qui constituerait un argument convaincant dans l'enseignement des mathématiques aux futurs enseignants ? La réponse dépend du bagage de ces enseignants. Qu'est-ce qui est évident ? Quelles connaissances font partie de l'acquis mathématique ?

Tel que nous l'avons remarqué plus tôt, l'action de prouver comporte aussi bien un aspect social qu'un aspect personnel, le premier requérant un répertoire partagé de choses que l'on peut prendre pour acquises, ce qui, à son tour, détermine si une preuve est ou non convaincante. Les logiciels tels Cabri et Logo sont-ils en train de changer ce qui constitue ce répertoire partagé ? Par exemple, la procédure de Logo qui permet de générer trois entiers consécutifs pour ensuite vérifier que leur somme est bien divisible par 3. Est-ce là un précurseur à une véritable preuve formelle ? Ces idées amènent des questions plus générales. Est-ce que les faits communément acceptés changent avec Logo ? Avec Cabri ? Est-ce que ce que l'on considère évident est en train de changer ?

Le Groupe de travail reconnaît que plusieurs *comprennent* un concept ou une preuve avant que ce concept ou cette preuve soit correctement *compris ou intériorisé.* Parlant ensuite des preuves formelles, il nous apparaît que des arguments *formels* peuvent être plus faciles à accepter par les étudiants parce que ces arguments sont perçus comme étant donnés par quelqu'un en position d'autorité.

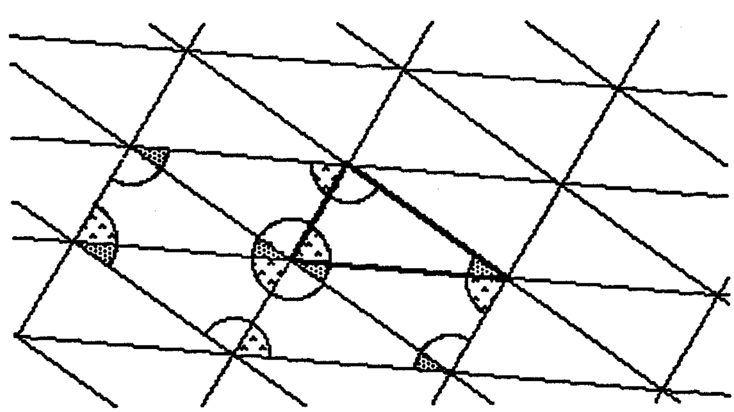
Nous concluons notre travail en essayant de décider ce qui constitue un argument convaincant. Deux exemples sont les points de mire de la discussion.

Le premier est l'illustration du fait que la somme de deux nombres impairs est un nombre pair (voir C. Hoyles, Figure 1, dans ce monogramme). Nous nous accordons pour dire que cela constitue un argument convaincant (avec un groupe approprié).

La discussion est plus animée pour ce qui est de savoir si la "preuve" donnée par un étudiant ou une tudiante du fait que la somme des angles intérieurs d'un triangle est de 180 degrés constitue une explication convaincante. La preuve est basée sur le pavage du plan par les triangles (Figure1).

Yorath's answer

I drew a tessellation of triangles and marked all the equal angles.



I know that the angles round a point add up to 360°.

So Yorath says it's true.

Figure 1
Preuve donné par un étudiant(e)

Alors que certains la trouve nouvelle et convaincante, d'autres soulignent la circularité du raisonnement dans cette preuve (le pavage du plan repose sur cette propriété des triangles). D'autres encore estiment que l'on ne peut en juger sans connaître le point de départ de l'étudiant(e) en question.

Quelle est la différence entre la preuve écrite et la compréhension des étudiantes et étudiants ?

Nous n'avons pas eu le temps de discuter de cette question.

CONCLUSION

La division du Groupe de travail en deux groupes, l'un pour les preuves et prouver dans les programmes de formation des maîtres pour le secondaire, et l'autre, dans les cours de calcul, d'analyse, d'algèbre linéaire, etc., nous est apparue assez naturelle. Cependant, plusieurs parmi nous travaillons avec ces deux types d'étudiants et avons eu une certaine difficulté à choisir l'un des deux groupes au détriment de l'autre. Il est intéressant de remarquer que les deux groupes étaient presque toujours en accord, et que certains des exemples choisis pour illustrer les propos étaient les mêmes.

Finalement, on note que beaucoup des questions discutées dans le Groupe de travail sont rarement l'objet de discussion avec des étudiants en mathématiques ou en formation initiale comme enseignant, laissant ainsi un sérieux manque dans leur éducation mathématique.

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TOPIC SESSIONS

Topic Session A

PROBLEM, PUZZLES, GAMES

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The more I get involved with introducing so-called "lighter" mathematical material to my students and the public, the harder I find it to draw the boundary line between serious and recreational mathematics. On the one hand, regular curricular material occasionally lends itself to more playful and exploratory treatment; on the other hand, recreational problems, puzzles and games often go to the heart of mathematics in a way that the regular curriculum does not. I would like to begin by listing particular benefits that may accrue from their use.

1. *Insight into mathematical structure and its significance.* It is the nature of good mathematical puzzles to be economical in their formulation and solution. School mathematics encourages the categorization of problems into types and systematically establishes techniques for dealing with these; often this inhibits the examination of a problem on its own merits so that its essence is lost. In recreational mathematics, one should assume as little formal background as possible, so a greater appeal to native ability is required. Mathematical tricks and puzzles often highlight some aspect of mathematical structure in their explication.

2. *Sensitivity to detail, pattern recognition.* One attribute of a good mathematics student is to be observant and solicitous as to detail. Games and puzzles, especially those exhibiting pattern or symmetry, foster these characteristics.

3. *Reasoning, analysis, use of strategy.* Some mathematical problems remain puzzles until one adopts an appropriate mental attitude. Once one begins to analyze the situation, it may be realized that in fact there are only a few ways to proceed and that systematically going through these must inevitably lead to a solution. This is so for example in traditional river-crossing problems or problems of transferring fluids among jugs.

4. *Motivation for skill development.* Some mathematical situations may give rise to conjectures, whose verification may motivate the use of computational and proof techniques.

5. *Linkages with other eras and cultures.* The best problems have a long history or arise in a particular set of circumstances. River-crossing puzzles appear in many ancient cultures, as do games with often simple rules but interesting mathematical structure. Some recent problems like the ``Monty Hall Car-and-Goats Problem" have become notorious. Thus, through mathematics, we can tap into the long and varied human saga.

6. *Surprising applications and insights.* Mundane facts often take on a new light when they are the basis of some mathematical trick or puzzle.

7. *Mathematical marvels and beauty (elevating taste).* It is often said that mathematics is not a spectator sport. This is not completely true, as mathematics does provide some occasions for showmanship. There is nothing wrong with putting on a ``concert" for our students occasionally.

8. *Examples of how a mathematician operates.* It is often outside of the curriculum that we can best demonstrate how mathematicians approach and solve problems, and how they formulate ideas. The public tends to see mathematics in terms of mechanical processes or presentation of data, and we need to find occasions to present both the aesthetic and power of analysis and argument.

Let us explore some examples in the light of these comments.

The rotating table problem

Symmetrically placed in the top of a square table are four deep wells, each containing a drinking glass. The glasses are not all the same way up and they cannot be seen. The table rotates and stops at random. The observer can thrust her hand into exactly two of the holes, feel the state (upright or inverted) of the glasses and change thestate of 0, 1 or 2 of them. When the hands are withdrawn, the table rotates and stops at random, at which time another move can be made. The task is to ensure, after a finite number of moves, that all the glasses are in the same state (at which point, a bell sounds).

One variant of the problem is to have a robot perform the moves as instructed by the observer (who, thus, does not know the initial state of the glasses). A second variant, used by Peter Taylor, is to generalize the situation to h holes and g hands, determining those situations for which the problem is solvable.

Indeed, it is not clear the problem is solvable in the present situation. For it is conceivable that some hole could always escape attention. However, a careful reading of the problem makes it clear that there are
two possible endpositions—all glasses upright, or all glasses inverted—so we just have to make sure we match the glass in any elusive hole. At first, we may be confounded by the randomness of the situation. Even so, there are two distinguishable alternatives; pick an adjacent or a diagonal pair of holes. From this point on, the analysis becomes quite straightforward because of the small number of options available.

This problem can be given to any high school student. The only requirements are to avoid facile conclusions, read the problem carefully, and then canvass the possibilities diligently using a small amount of reasoning.

The very divisible ten digit number

Construct a ten-digit number with all digits distinct for which the number formed by the left k digits is divisible by k for each k with $1 \leq k \leq 10$.

This problem has the advantage that it is easy for students to get into. Working from the left, by trial and error, one can make the number work without too much trouble up to the first five or six digits. While this strategy is not very productive, it does have the potential advantage of immersing the students into the situation and perhaps suggesting that one should go after certain digits first. It is not hard to see that the last digit has to be zero, so that the fifth digit from the left must be 5. The ninth digit can be left until last, since any entry forced at this stage must work (by "casting out nines"), and the first digit will be governed by considerations other than divisibility by 1. The number of possibilities can be greatly reduced by some preliminary reasoning. Since the even digits go in even positions, the remaining positions must be occupied by odd digits. The divisibility of a number by 4 depends only on the last two digits, by 8 on the last three. One can get at the third and sixth digit using the test for divisibility by 3. It is interesting that the solution is unique. As an extension, students could be asked to look at the analogous problem for other bases.

Late elementary students should be able to tackle this problem. One could simplify it initially by dropping the requirement that the digits be distinct, so as to guarantee widespread success. What is its value? Since the number of ten digit numbers is enormous, the students are forced to devise strategies to reduce the number of possibilities. Effective solution requires them to consciously address divisibility rules. Reasoning and systematic analysis and, if the students work collaboratively, communication are fostered. Finally, success is easily recognized without needing recourse to an outside authority.

The self-descriptive ten-digit number

Find a ten-digit number for which the $(k + 1)$th digit from the left indicates the number of occurrences of the digit k for $0 \leq k \leq 9$.

This is a much tougher problem, as its self-referential character can be confusing. It does illustrate the advantage of employing a notation to make the analysis more transparent. Generalize the problem to the following: Determine a vector $(a_0, a_1, \ldots, a_n)$ whose entries are integers from 0 to n inclusive so that the integer k occurs exactly a_k times. By adding the terms in two ways, we arrive at the crucial equation

$$a_0 + a_1 + a_2 + \ldots + a_n = a_1 + 2a_2 + 3a_3 + \ldots + na_n = (n + 1).$$

(To understand the right hand side, think of a shopkeeper counting the day's receipts by first sorting the coins into piles of coppers, nickels, dimes, quarters, halfdollars, loonies and toonies.) This yields

$$a_0 = a_2 + 2a_3 + 3a + \ldots + (n-1)a_n,$$

from which a careful delineation of cases will lead to all solution. When n is sufficiently large, there is a systemic unique solution, while for small values of n there are a few oddball cases.

If this were the only way to do the problem, one might quail at foisting it on a whole class. However, there is an alternative trial-and-error approach which mirrors iterative solution techniques useful in many areas of modern mathematics. Suppose we are given a ten-digit number. Consider a mapping that takes it to a new ten-digit number whose (k+1)th digit is the number of occurrences of k in the original number. For example, we have

$$8715300149 \rightarrow 2201110111.$$

The problem can be reformulated to finding a number that is carried by this mapping into itself. Let us try to find such a number by "successive approximation" through repeated iteration (as for example is used in the Newton-Raphson method for solving equations). Applying, this to the foregoing example yields

$$8715300149 \rightarrow 2201110111 \rightarrow 2620000000 \rightarrow 7020001000 \rightarrow 7110000100 \rightarrow$$

$$\rightarrow 6300000100 \rightarrow 7101001000 \rightarrow 6300000100 \rightarrow \ldots$$

As you can see, the sequence eventually cycles (as indeed it must; why? use the Pigeonhole Principle) and we do not get a solution. However, there are other starting points that will lead to a fixed

point for the mapping; it will not take long for some pupil to find one. The advantage of this approach is that it does not
rely on a sophisticated argument; the disadvantage is that it does not dispose of the question of uniqueness. But note how many essential mathematical issues can be dealt with for quite young children. Where in the standard curriculum is such a thing possible?

The standard magic square

Construct a magic square with three rows and three columns using each of the nine nonzero digits exactly once. (The sum of the numbers in each row, column and diagonal are to be the same.)

This problem has a long history and a wide amateur following, so it is worth doing for cultural reasons alone. But we can squeeze a lot more juice out of it. Again, we need some reasoning to cut down the plethora of possibilities that one might have to consider. Focus on the magic sum; one can argue that it has to be 15.
There are various ways of determining that the middle entry is 5. One way is to note that 5 is the only digit that figures in four different ways of obtaining 15 as the sum of three digits: $15 = 1 + 5 + 9 = 2 + 5 + 8 = 3 + 5 + 7 = 4 + 5 + 6$. More deduction will lead to putting the even digits in the corners and the other odd digits in the middle of the sides. The answer is (4 3 8 / 9 5 1 / 2 7 6).

Is this solution unique? Strictly speaking, no. However, we can engage in a discussion of what it means for two mathematical objects to be *essentially the same*, and finally agree that, up to certain rotations and reflections, the solution *is* unique. Having got the magic square, we can then show off some pyrotechnics. The three row products are $4 \times 3 \times 8 = 96$, 45 and 84, while the three column products are 72, 105 and 48. The row products and the column products have the same sum, $225 = 15^2$. This can be generalized. Let $\{ a_n \}$ be any second order linear recursion (i.e., it satisfies a relation of the sort $a_{n+1} = ba_n + ca_{n-1}$; the Fibonacci sequence is an example). Arrange nine successive terms $a_1, a_2, \ldots, a_9$ in a square array using the magic square to place the indices. Then we have

$$a_4 a_3 a_8 + a_9 a_5 a_1 + a_2 a_7 a_6 = a_4 a_9 a_2 + a_3 a_5 a_7 + a_8 a_1 a_6$$

so that the three row products and the three column products have the same sum. While the second order recursion may appear to be formidable garbed in the standard notation, I have found that it is not difficult to get the idea across to children through examples and getting them to extend sequences from the first two terms using "multipliers".

But there is more. Consider the following game between two players who play alternately. Each player selects from the numbers from 1 to 9 inclusive a number that has not previously been chosen. If any player finds among the numbers she has selected three (not necessarily picked consecutively) that add up to 15, then she wins. In this form, the game is unfamiliar to virtually everyone I have tried it on. However, it turns out to be isomorphic to (has the same mathematical structure as) noughts-and-crosses. To see the relationship, let the choices of the first player be indicated by an X in the corresponding position in the magic square and of the second player by an O. Three numbers sum to 15 if and only if they are in the same row, column or diagonal of a magic square. I have found this illustration of isomorphism to be meaningful to children and have made the point that the number game can be more conveniently analyzed when reformulated as a game of noughts-and-crosses. Indeed, I have used this example also with university students to convey an understanding of isomorphism that is hard to dig out of the formal definition.

The problem of the lion and the Christian

Into a large closed circular arena are introduced a lion and a Christian whose respective maximum speeds are u and v; both are tireless and very agile. The lion wishes to meet the Christian for lunch, but the latter demurs. Assuming that each adopts an optimum strategy, determine whether the lion will succeed in catching his quarry or whether the Christian will continue to evade the lion.

This is a very nice problem for class discussion and gets into some quite different mathematical issues, including the notion of a strategy. At its face value, it seems to be a tough pursuit problem, suitable only for upper year university mathematics specialists. But it can be pegged down to the school level. If the speed of the lion exceeds that of the Christian, it seems evident that the lion will soon be licking his chops. But how can this conveniently be argued? The simplest argument, giving a strategy that is successful but less than optimal, involves a blind lion who sniffs around until he picks up the trail of the Christian. He then follows the same track as the Christian, and, being able to run faster, will soon catch up. If the Christian can run faster, he can succeed by heading to the periphery of the arena. To catch his quarry the lion must simultaneously attain the same distance and same direction from the centre of the arena; unfortunately, trying to achieve one of these forces a loss of ground in the other. The most interesting case is when both have the same maximum speed. If the Christian sticks to the periphery, he will be caught by the lion starting at the centre in the time taken by the Christian to go quarter the way around the outside. However, the Christian can adopt a strategy to keep the lion at bay, although the lion will come arbitrarily close.

A card trick

A pack of 27 cards is dealt face up into three columns. A person is asked to think silently of a card and indicate to the dealer which column contains the card. The dealer collects the columns up, and then redeals the cards face up into three columns. The person indicates which column now contains the card. This is repeated for a third dealout, after which the dealer knows the card.

This is a nice trick to do for children, since the explanation is easy and they can learn how to perform it for others. It makes the point that with three choices among three options, one can isolate one of twenty-seven possibilities. The secret is in the dealing. After the first dealout, the dealer knows which of nine cards it is, and ensures that the nine cards are dealt evenly into three columns on the next round. This narrows the possibilities down to three, and for the final dealout, the dealer makes sure the three cards go into different columns. What makes the trick work is that most people are not prepared for 3^3 to be as large as it is.

A comparison of ages

Record the date of your birth and that of a friend or relative. Assume that both of you will live a long time. (a) When will the age of the older be exactly twice the age of the younger? (b) When will the age last birthday (a whole number) of the older be exactly twice the age last birthday of the younger? (Try this with several pairs of people and make a conjecture.) (c) What happens with twins?

The answer to (b) is striking: for any two people, the time over which one's age last birthday is twice the other's is one year, although this period will be in general broken into two parts. This is true even for twins, who will bear the required relation when both are 0 years old. Students looking at this situation will need to make a careful computation, devise a suitable wa y of formulating the situation, and be observant enough to arrive at a conjecture. This problem has the additional advantage that it relates easily to people they know, such as parents and siblings. I suspect that most children have a natural curiosity about matters of aging.

A reasoning question

You are told that on one side of each of four cards is written a letter of the alphabet and on the other side a positive whole number. The four cards rest on a table with one side of each visible; you read the symbols 1, 2, A, B. How many, and which cards, should be turned over to verify whether the following assertion about the four cards is true:

If a vowel is written on one side, then an even number appears on the other side.

If you want a hot discussion in a classroom situation, then this is the item that should deliver it. Most students will ask for the card marked A to be turned over. From there on, opinions will diverge and you will find that this is a good occasion to get into such issues as implication and logical equivalence. There is no point turning over the card marked B as the hypothesis is not satisfied; there is no point turning over the card marked 2 since it cannot falsify the assertion, but may support it. If a vowel appears on the other side of the card marked 1, then the assertion would be falsified; thus, this card should be checked. Out of this analysis, one can see that the given assertion is equivalent to the assertion that if an odd number appears on one side, then a consonant appears on the other.

Patterns

Below is given the first few terms of three sequences $\{ x_n \}$, $\{ y_n \}$ and $\{ z_n \}$. Given two successive terms of the x- and y- sequences, the following terms are obtained according to the recursion

$$x_{n+1} = 2x_n + x_{n-1}$$

$$y_{n+1} = 2y_n + y_{n-1}.$$

The third sequence satisfies $z_n = x_n y_n$

n	x_n	y_n	z_n
0	1	0	0
1	1	1	1
2	3	2	6
3	7	5	35
4	17	12	204
5	41	29	1189

Continue all three sequences for a few more terms. List all of the general relationships that you can detect among the terms of the three sequences.

The detection of patterns in modern pedagogical practice has a"flavour-of-the-day'"character to it, and many of the patterns given as exercises are arbitrary without any reason, apart what the teacher expects, for extending the pattern in a particular way. The ability to recognize patterns should be fostered. However, patterns should come up in a context. One good context is a succession of geometric figures, where there is a natural progession of generalization. For example, one could look at the number of matchsticks required to build a large equilateral triangle partitioned into unit equilateral triangles and ask pupils how many matchsticks are required if, say, there are 50 matchsticks along the side of a triangle.

The sequences given above arise in a number of contexts. For example, the x- and y- sequence terms occur as the numerator and denominators in the convergents for the continued fraction expansion for the square root of 2. I have chosen to take certain relationships as basic *in order that there is a well-defined* *way for all of the sequences to continue.* This is a marvellous trio of sequences, almost as good as the Fibonacci sequence, for harbouring a number of interesting properties. For a class, this has the advantage that some relationships are quite transparent, so that most students should be able to make some progress. However, there are others that take quite a while to discover, so the quicker students do not have to be bored. I invite the reader to spend some time with this trio. Senior students can be given the task of proving general properties of the sequences; in most cases, this can be achieved by induction arguments of greater or lesser delicacy.

The problem of the four couples

Four couples meet for dinner. As they greet, some individuals shake hands with others, although of course no individual shakes hands with either his (or her) self or spouse. After this is done, one of the husbands asks each of the other seven how many people that person shook hands with and gets seven different responses. How many people does the {\it wife} of that husband shake hands with?

This is a good warm-up problem for a group, and lends itself to role-playing. Occasionally, someone will even give the correct answer out of the blue based on some kind of intuition. Usually, the group realizes quite quickly that the seven different anwers must be 0, 1, 2, 3, 4, 5, 6. The key step is to focus on the person who shakes hands with six people and that person's spouse. The reasoning required to solve this problem is significant but probably accessible to most classes. Note that the solution proceeds by a "method of descent."

This provides a small sample items that would be useful in class. There are many puzzle books on the market that will provide both new and traditional material. In fact, it would be nice if many problems, puzzles and games, accompanied by a suitable analysis, were to become part of the middle school curriculum. Apart from adding some spice, students would derive a more authentic view of mathematics and through their play develop important skills and modes of thinking.

However, it is unfortunate that where games and puzzles are used in current teaching, they are often artificial and superficial. I have seen situations in which they are introduced without being followed up on and proper mathematical points made; this is counterproductive and leads students to the belief that mathematics is fragmented and inconsequential. It is important to introduce sound material with full awareness of the mathematics teaching that is to be done through it.

Topic Session B

{PARENTS}∩{CHILDREN}∩{MATHEMATICS}: RESEARCHING THE INTERSECTION[1]

Elaine Simmt, University of Alberta, Edmonton

Educators place a great deal of emphasis on parents and children reading together; but the same emphasis is not placed on parents doing mathematics with their children. Working Group C at CMESG 1994 noted that families were a potential target group for popularizing mathematics. Programs and activities such as Family Math, Math Packs, Math Trails and Math in the Mall (see discussion in Hodgson and Muller, 1994) are examples of the ways parents and children can interact together with mathematics. These various programs and activities point to some alternative conceptions of what it means for parents to do mathematics with their children, who teaches and learns mathematics, and where and how mathematics can be taught and learned. Unfortunately, there is not much research in the area of family mathematics with which educators can explore such alternative conceptions.

The purpose of my topic group was to bring forth and explore research being conducted in the context of parents and children doing mathematics together at the University of Alberta. For the topic group, I had proposed to do three things: describe the parent-child mathematics program; share some of the research that is being done in conjunction with this unique context; and consider what it means for parents to be involved with their children's mathematics education. In this paper I will discuss each of these in light of the interaction I had with participants[2] in the topic group.

THE PROGRAM AND THE PARTICIPANTS

Math Connections is an extracurricular mathematics program for parents and children. Once a week, for ten weeks, parents and children together engage in mathematical activity. The program was developed in conjunction with a local school board in response to a need expressed by parents to enhance their children's school mathematics. Math Connections was designed for children between the ages of 8 and 14. To date, the program has run three times. 15 of the 21 children have been girls whereas participation by the parents has been equally split between mothers and fathers—in some of the cases the mother and father took turns working with their child.

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[1]The research for this work was supported by a Social Sciences and Humanities Research Council of Canada Doctoral Fellowship and Elk Island Public Schools-Continuing Education.

[2] I would like to thank those members of CMESG who participated in the topic group. I am grateful to have had the opportunity to share my work with such an insightful group of educators.

There are a number of reasons children say they attend the program. Some children come because they like mathematics and want to be challenged, others come because they are having difficulty with school mathematics and want to be tutored, yet other children come because their parents bring them. The adults too have a variety of reasons for participating in the program. All of them express a desire to help their children with mathematics, but there are subtle differences in why they use this context to provide that help. Some parents express a dislike or fear of mathematics and they see the program as a way of helping their children with mathematics—something they believe they cannot help with on their own. Others, although they are comfortable with mathematics, claim they do not know how to help their children with it and want advice in this regard. Some of these parents really enjoyed mathematics when they went to school and expressed the desire for their children to have a similar positive experience in mathematics. Yet others just want to spend time with their children and felt doing mathematics with them was a unique opportunity to do this.

FACILITATING DIVERSITY AND ADDRESSING NEEDS

The program is designed to encourage and challenge children and their parents to engage in mathematical thinking and problem solving and at the same time give parents pointers for helping their children with school mathematics. The 1.5 hour sessions are composed of two parts. Each session begins with a short opener such as a number trick, a game of strategy, or a puzzle which takes the first 10 to 15 minutes and is followed by a single prompt which is intended to provide the context for the rest of the evening. The warm-up activity makes time for all the participants to arrive and it gives parents ideas for mathematical play with their children. The second part of the session might be thought of as problem-solving, although this is a simplification of what seems to happen when parents and children engage in this kind of activity together.

In order to facilitate the diversity among participants—in terms of their ages, background knowledge in mathematics, and experiences in mathematics—I use variable-entry prompts (Simmt, 1996b) as a means of triggering the main activities for each night. These are prompts which can be accessed at varying levels of mathematical sophistication and by a variety of actions. They do not require specialized background knowledge or specific mathematical skills; but the prompts must be interesting to persons with such knowledge and skills and they should lead to important ideas, concepts, and processes in mathematics. (See Appendix A for a selection of some of the prompts I have used in this program.)

The handshake problem is an example of a variable-entry prompt. I use it on the first night of the program. As each participant walks in I greet him or her with a handshake. Once the group is assembled I ask, "If all of us greeted each other with a handshake, how many handshakes would there be? Try to figure that out." Rather than saying anything more, I leave the parent-child pairs to begin talking about the prompt. Even the youngest child and the most insecure person in the room are usually able to enter into mathematical activity without much trouble. There are many different ways the parents and children engage the prompt and formulate the problem. The following are examples of the things that the parents and children have done when given this prompt.

- role playing - each person shakes hands with the other people and keeps track of how many handshakes there were;

- talking it out - "If I shake every body's hand, that makes 9 handshakes, if my mom shakes everyone's hand that is 9 shakes ..." (this is not an uncommon pattern of reasoning);

- making a picture of the people in the room and drawing lines between them to indicate the handshakes and then counting the number of lines;

- simplifying the given situation by considering the number of handshakes for 1, 2, 3 ... *n* people and then looking for a number pattern. This usually includes using a table or a chart of some form.

Of significance here is that the participants are acting in ways that are sensible to them. Clearly, there is range in the mathematics used and developed by the participants. Some children and some adults are unable to formalize their thinking and when asked the same question but given a different number of people these participants have to start from the beginning all over again. Yet there others (including children as young as 9 years of age) who can articulate a means of finding a solution for a general case but do not (or can not) use conventional (algebraic) notation. There have also been cases where parents use their children's pattern noticing as a springboard for developing an algebraic solution to the problem.

The forms of activity that parents and children engage in not only require that the participants act in ways which lead to resolutions for the given prompts but also provide the space for ample "practice" of basic skills, number facts, algorithms, and so on. This dual nature of the activity, to facilitate problem solving and practice, is important in light of the fact that many of the children who participate in the program do so because they are not doing well in school mathematics. But it is just as important for the students who come because they are good at mathematics, they are challenged with interesting prompts and are provided wih opportunities to develop alternative and efficient strategies for doing mental computations. Consider the following two examples. One night the students were given bingo chips and graph paper and were asked to find the triangular numbers. I demonstrated making a triangle from the chips; then I counted the chips and sugested that this was a triangular number. The participants were then instructed to carry on. As Steven (an 8 year old) worked with a handful of bingo chips arranging them into triangles, his mother kept track of the number of chips in the triangle. From viewing/reviewing the records his mother was keeping Steven noticed a pattern; take the number of chips you had before and add how many are in the new row. When I asked him if he could predict the number of chips if there were ten rows, he used his mother's chart and began to add, "1 plus 2 is 3 and 3 plus 3 is 6 and 6 plus 4 is 10..." He continued adding out loud until he got to 55. For a child that came because his mother was worried about how slow he was on speed tests, Steven was doing some much needed practice—but practice that was, for him, grounded in a meaningful context.

On that same night, Wayne, a 5th grade boy who was quite good at school mathematics, learned an efficient strategy for adding this sequence of numbers. I demonstrated to him that each number from the front of the sequence could be paired with a number from the back and when added together equaled 11. Since that happened 5 times, we could multiply 5 times 11 to quickly obtain 55. Wayne spent the rest of the session proving to himself this method would work for any sequence of numbers from one to an even number. He came up with the formalization, "take one more than the number and multiply it by the number divided by 2." Although Wayne was good at mental arithmetic he was very excited about having a more efficient strategy for adding the sequence of numbers. Given the same prompt, these two students with different backgrounds and needs were occasioned to act in ways that fostered growth in their mathematical knowing. These examples demonstrate some reasons why using variable-entry prompts in this setting has been valuable. Not only can all the participants find a meaningful way and context to act but in acting they lay down paths unique to their interests, actions, and past experiences—paths on which they further develop their mathematical understandings.

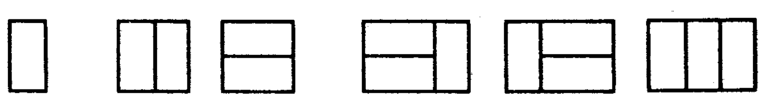
As I already mentioned, another dimension of the program was to help parents develop ways to interact with their children which would encourage and facilitate mathematical thinking in many contexts not just when helping them with homework. For many parents this meant that they had to start thinking mathematically. Participating in the program and doing mathematics with their children was a meaningful context to facilitate this. Throughout the sessions I took advantage of opportunities as they came up to talk to the parents about issues concerning the teaching and learning of mathematics and school mathematics: issues such as practice embedded in meaningful contexts, the value of inductive and deductive reasoning; and how to encourage children to reflect on their actions and to explain to others what they are doing. These pedagogical concerns are not new to educators however they are something that many parents do not have any experience with in terms of mathematics. The parent-child mathematics program was a useful site for addressing the parents' concerns.

RESEARCH INTO PARENT-CHILD MATHEMATICAL ACTIVITY AND INTERACTION

The parent-child mathematics program has become a research site for a number of researchers interested in a variety of issues related to mathematical cognition and mathematics education. Lynn Gordon Calvert (1996) has been studying the nature of mathematical conversations; Tom Kieren (1996) has been investigating mathematical cognition in the context of the parent-child interaction; and David Reid (1996) has considered the role of the need to reason when parents and children engage in mathematical activity. I have been thinking about the interaction between the parent and the child and with the environment (Simmt, 1996a); the way(s) in which the parent-child pairs bring forth mathematics with their interaction (Simmt, 1996b); and the reciprocal learning that goes on in this context.

In the topic group, we considered 1) how two parent-child pairs made sense of and developed mathematics given a variable-entry prompt and 2) the nature of the interactions within the parent-child pairs and the implications the type of interaction had on the nature of the mathematics that they brought forth. In particular, we explored the interaction and mathematics of two parent-child pairs—Dave (father) & Krista (10 year old daughter) and Robin (mother) & Casey (9 year old daughter)—who spent approximately 1.5 h working with the prompt:

> How many paths can you tile with a given number of dominoes (2x1 tiles) if the path must be two units wide. For example:



1 tile 1 way   2 tiles 2 ways   3 tiles 3 ways

Right from the beginning of the session, the two pairs worked quite differently with the prompt.

Dave and Krista immediately began by looking for a relationship between the number of tiles and the number of tilings that could be made from that set of tiles. "Maybe we should look for a pattern," Krista suggested. They worked together, Krista arranging the tiles and calling out the tilings to her dad (Transcript 1) who was keeping the records (Figure 1). The piece of transcript taken from the session reveals a high level of interaction between the daughter and her father. Notice the rhythm of their conversation (Gordon Calvert, 1996) and how they occasioned each other's actions (Kieren, 1996) (lines 7-14, 60-75).

Their highly interactive means of working with the prompt was key to the mathematical understanding they constructed. For example, it led to a need to find an efficient way to communicate the tilings to each other. Throughout the session Krista called the tilings out to Dave who recorded them by using vertical and horizontal ticks (as noted on the right side of Figure 1). Once Krista started calling the tilings for 5 dominoes Dave had trouble knowing to what she was referring and thus had trouble keeping up with her. Krista called out the vertical tiles as "one" and the horizontal tiles (which always exist as pairs) first as "going going," then as "2 sideways" (Transcript 1, lines 53-62). Her language was clumsy and the rhythm of their conversation was interrupted. However, it did not take long for this clumsiness to transform into a more efficient and smooth communication when they created a new word, "blip blip," to describe the move of two horizontal tiles (lines 62-69). Once they had this new word, we see an interesting transition in their actions. They began to treat the pair of horizontal tiles as an object and that changes the way they think about the task. For example, later in the session they diverted from finding all the arrangements for a given set of tiles to finding how many of the arrangements involve *x* "standing" (vertical) tiles (Figure 1, bottom right hand corner)—an interesting side issue which involved some deductive reasoning.

Transcript 1 - Dave (father) and Krista (daughter)

D: Okay. So 3, there was 3.
K: Next 4.
D: Four seems like a good number.
Good as any to do next.
K: If I am right about this, then, if 4 follows my theory -- Okay, let's see.
Do this.
D: Do that? Okay. 1, 2, 3, 4. Okay, got it. [Dave draws] ||||
K: Okay let's see. You can do that.
D: Okay. 1, 2, 3, 4. ||=
K: Do that.
D: Okay, 1, 2, 3, 4. =||
It's good that we are being consistent. Like if we are treating those as being different ones then. Okay, now what do we do -- Oh, I think that I can see another way.
D: Oh, Yeah, I didn't see that one. ==
That's not the one I was thinking of.
That's 1, 2, 3, 4. Yep, that's good.
K: What's the one you were thinking of?
D: Oh see if you can get it.
K: Hmm
D: Yeah, we have that one.
K: Yeah, I know
D: Oh, you're good. That's the one I was thinking of. |=|

Okay, so we have 1, 2, 3, 4.
K: Okay, I don't think there's any others.
R: [Researcher] I think that's a neat one. It's kind of a frame--picture frame one.
D: It's kind of symmetrical isn't it?
R: Oh, it's really nice.
D: So how many do we have?
[D&K say together] 1, 2, 3, 4, 5.

K: Shoot! That doesn't follow my theory.
D: No. So far it just blew it out of the water.
K: See I thought, 1 - 1, 2 - 2, 3 - 3, 4 - 5.
D: It looked pretty close. I guess we can't stop yet, but we might find a pattern yet. Okay -- one with five. Unless you want to try something else?
K: 1, 2, 3, 4, 5. [She counts out five dominoes.]
D: Okay.
K: Okay, you could do this one.
D: Okay. What is that one?
Okay, 1— |=||
K: 1, going, going.
D: 1, going, going, going, going. Okay.
D: Okay, you got your going, going. Yeah, that's the same as your other one, just turned around.
D: Okay, so what do you have?
K: 1, 1, 1 --
D: 1, 1, 1 --
K: 2 sideways.
D: Blip blip. |||=
D: Okay, what do you call the one sideways or something else?
K: Blip, blip.
D: Okay, the blip, blips.
K: Blip blip, 1, 1, 1.
D: Blip blip, 1, 1, 1, =|||
K: Comma.
D: Comma. Thanks. That's a good recording technique. What are we doing now?
K: 1, 1, 1, 1, 1.
D: 1, 1, 1, 1, 1, okay. |||||
K: I think that's it."

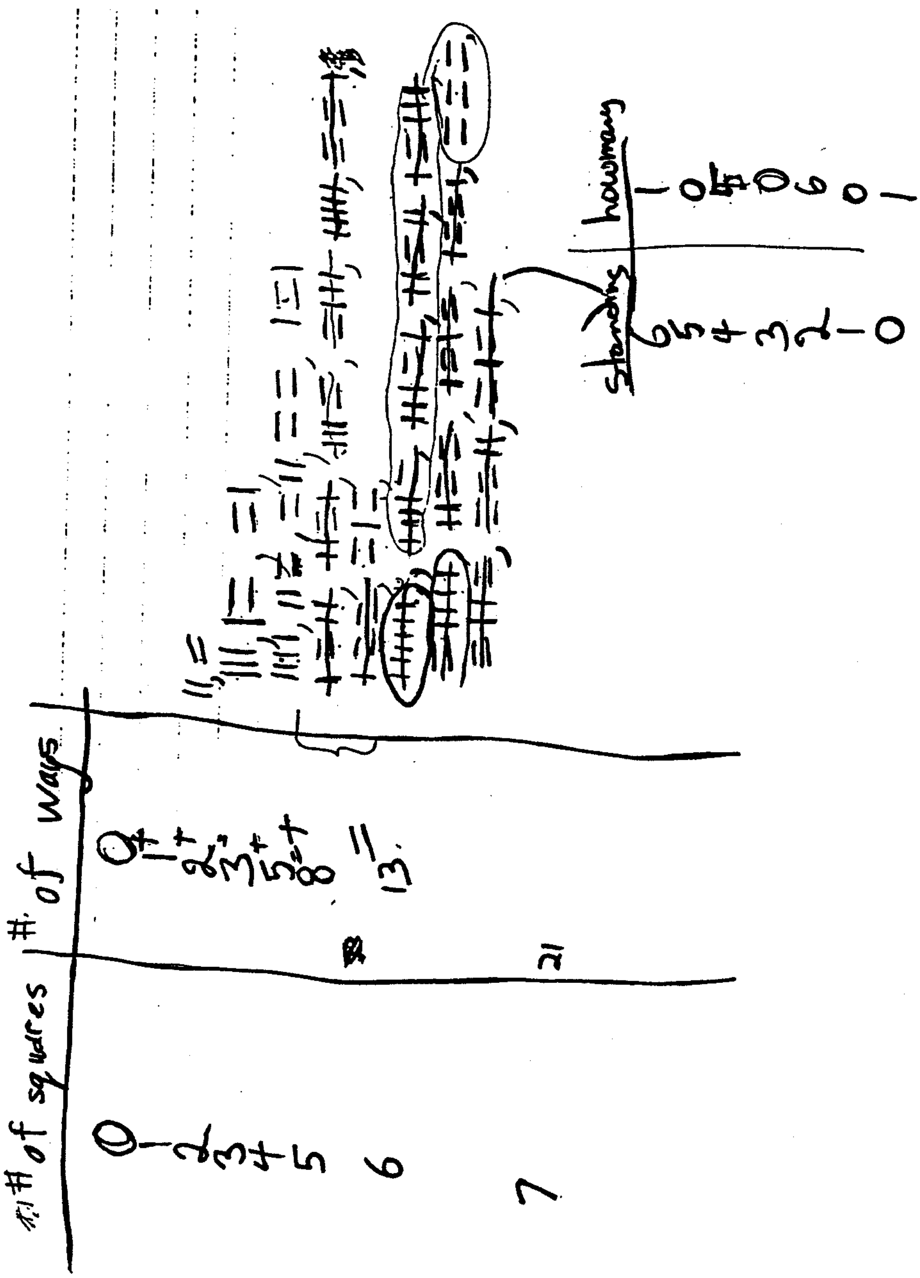


Figure 1 - Dave and Krista's Table

Robin and Casey, on the other hand, worked in parallel (Kieren, 1996) each finding arrangements and keeping records. Immediately after the prompt was given, Casey indicated she wanted to do 4 tiles (Transcript 2, line 4). Robin understood this to mean that *they* would look for patterns with 4 tiles and began doing so. A few minutes into the task Casey realized that she and her mom were doing the same thing (lines 20-30). Surprised by this, Casey asked her mom what she was doing. Once she confirmed her suspicion, she convinced Robin to do the patterns for 5 tiles instead of 4 so they wouldn't be doing the same thing. Robin and Casey, like Dave and Krista, shared the task but notice how differently.

Robin and Casey's misunderstanding is not surprising given their style of discourse. Their language was quite cryptic and a good portion of their talk was often not directed to the other. Robin (and to some extent, Casey) spoke to herself out loud as she worked. Since they did not have to interact given the way the task was divided, these utterances were important to triggering what little interaction they did have. It made it possible for each of them to follow along and participate in the other's activity. In a short piece of transcript it is difficult to demonstrate how their individual actions wove together for brief periods throughout the session (lines 35-49) and then apart again. Clearly, their discourse did not have the same conversational features that Dave and Krista's had; it lacked both the high level of interaction and the rhythm but it did allow them to bring forth mathematical understanding in the context of the prompt they were given.

Transcript 2 - Casey (daughter) and Robin (mother)

R: Okay, so we are going to work on this starting from 4?
E: [Researcher] Check 3.
C: I want to do 4.
R: You want to do 4?

C: Yeah, I want to do 4. [Casey turns to the paper in front of herself.]
R: [Robin talking to herself-] Okay you can actually take these out and see what you are doing. So, there's your 4. 1,2,3,4.
C: [Casey to herself-] Cool. This is obviously one way. It is obviously — [Turning to the researcher-] Are you taping us here?
E: Uhum.
C: Let's see.
R: It doesn't matter if I use a pencil or a pen, I guess. Okay so we will draw it on our graph paper. [She looks over at Casey's paper.] Follow Casey.
C: What are you doing mom?
R: Okay, well, I'm doing this one.
C: Which?
R: This one.
C: Four too!
R: Yeah. 1,2,3,4— see. Look two up and two this way.
C: Oh. so you are doing 4 too?
R: Oh. You want me to do 5 and you do 4?

C: Yeah, that's what I was thinking. So you are doing 5.
R: Okay. 5 tiles.
[Some time passes as Robin and Casey each look for more of the tilings. As they do they talk out loud to themselves.]
R: Okay, if you stay straight up, you've got—
C: [Casey looks over her mom's page and finds an arrangement her mom did not yet have.] Oh. How about this one?
R: Oh yes, right here.
C: Oh yeah. How about— No—
R: We have all five. Here's all five.
C: 1,2,3,4,5. Can you go like this?
R: No because you want it to be a— right there. That's the way. And then move it in here and see if you've got that one.
C: No you don't.
R: Right there.
C: Okay. You do.
R: Okay, now let's move it over one more.
[They continue to check Robin's tilings.]

The mathematics that the two parent-child pairs brought forth was as distinct as the nature of their discourses. Dave and Krista deliberately looked for a number pattern. My past experiences with them leads me to suggest they are pattern seekers. Given many different prompts, Dave and Krista have frequently begun by setting up a table in order to look for a function that describes their number pattern. Robin and Casey, are also pattern seekers but of a different kind. In this session they noticed the geometric patterns and were concerned with the way in which the tiles were arranged—the mirror images and the placement of the horizontal tiles in relation to the vertical ones. Instead of seeking out the relationship between the number of tiles in a set and the number of arrangements that could be made with them, Robin and Casey focused on ways of finding all the arrangements for a set by using geometric properties.

The differences in the mathematics is reflected in the records they kept as well as in the focus of their interactions. Whereas Dave and Krista's records had 3 parts: the number of tiles in the set, the number of arrangements possible from a set, and the tiling patterns (Figure 1), Robin's and Casey's records simply consisted of sketches that resembled the actual tiles (Figure 2). Not until they were asked to make a table did Robin and Casey even consider the sequence on which Dave and Krista's efforts were focused. Even then, Robin and Casey considered the table only briefly and then turned back to the geometry of the tilings. It is important to note here that this difference between the two pairs is not based on either pair's capabilities. Dave and Krista could understand what Robin and Casey were doing and vice versa; they simply seemed to be interested in different mathematics. Given a variable-entry prompt and a learning environment which encourages people to act in ways that they find meaningful, we should expect individuals to act differently, that is do different things, and not be surprised when their doing leads to different mathematics; after all these people each have different experiences, skills, aptitudes, and interests.

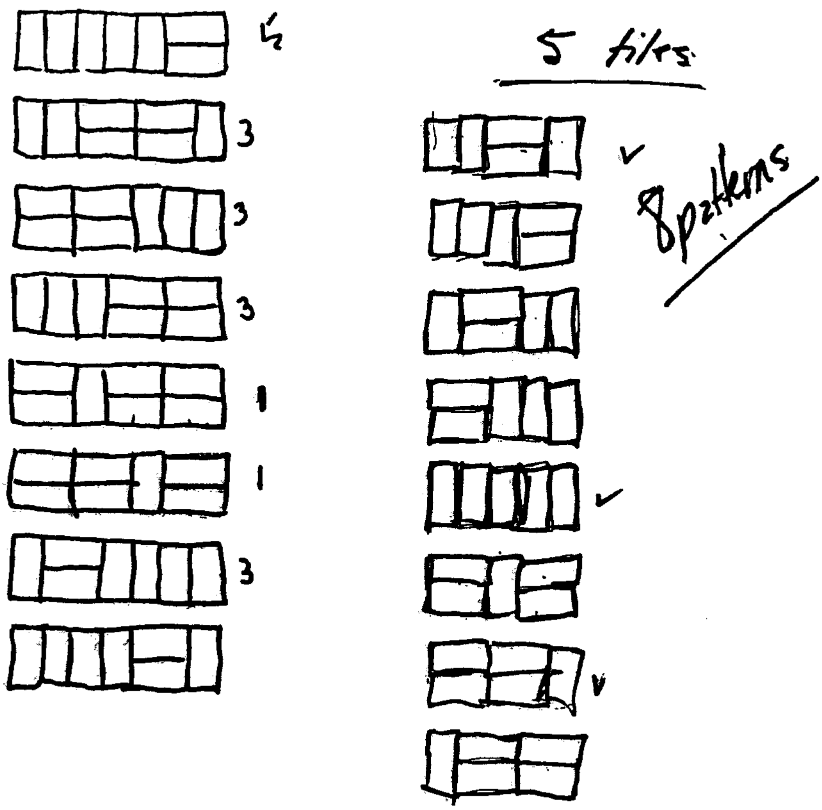


Figure 2 - Casey's (left) and Robin's (right) Records

PARENTAL INVOLVEMENT IN CHILDREN'S MATHEMATICS EDUCATION

At a time when math educators are looking for ways to popularize mathematics and at the same time parents are looking for ways to enhance their children's mathematics knowing there is a need to consider the parent's role in mathematics education. Mathematics educators might begin by recognizing what a valuable resource parents are—ask any literacy teacher. Further, we must acknowledge parental concerns about mathematics education.[3] As educators, we need to ask ourselves, what might we do to assist parents in their efforts to help their children. The work that my colleagues and I have been doing suggests a much more active role for parents than we have come to expect them to play in their children's mathematics education. It challenges the notion that helping a child with mathematics means simply helping with his or her homework. This research demonstrates the potential for fostering mathematical understanding when parents engage in complex and meaningful mathematical activities with their children rather than simply helping them memorize basic number facts or do long division.

Mathematics when grounded in shared activity and experience has the potential to become a "topic of conversation" (Gordon Calvert, 1996) between parents and children. It provides a unique opportunity for them to share an intellectual intimacy that is often neglected in an era when teachers are overwhelmed with the more public and social issues in education such as high failure rates on school mathematics exams, comparative international testing, funding cuts and large class sizes. Mathematics educators (from both the public and private sector) need to encourage parents to do mathematics with their children—to engage in purposeful, meaningful, and significant mathematical activity—as frequently as they read with their children. Thus, there is a need for programs and materials which could facilitate such activity. However, it is important that these programs and materials include the parents not in the role of monitor, proctor, or even teacher but as fellow learner—that is, a person whose thinking stimulates the child's and whose thinking is stimulated by the child's.

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Simmt, E. (1996b). Parents and Children Doing Mathematics: Bringing Forth a World of Significance. Paper presented at the International Council of Psychologists Annual Convention, Banff, AB.

Simmt, E., Kieren, T. and Gordon Calvert, L (1996). All-at-once: Understanding and

[3] Consider as evidence of this concern the growth in the private educational resource sector and its advertising which targets, not teachers but, parents.

inter-action in mathematics. Paper to be presented at the Psychology of Mathematics Education North America, Panama City, FL.

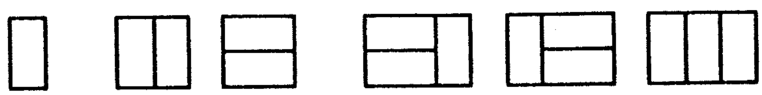
APPENDIX A

SAMPLE OF PROMPTS USED IN PARENT-CHILD CONTEXT

Handshake Problem: How many handshakes would there be if twenty people were in this room and each person shook hands with each of the others just once.

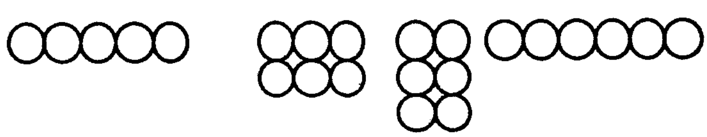
Mobius Bands: Take a strip of adding machine tape and tape the ends together. Now trace the path an ant would take walking along that path. How many sides does the band have? Cut the band along the ant's path. How many bands do you have now? Now do the same things but put a twist in the band before you trace and cut the ant's path. Can you predict what will happen. What if the number of twists changes?

Tiling Paths: How many paths can you tile with a given number of dominoes (2x1 tiles) if the path must be two units wide. There is one path for one tile, two paths for two tiles and three paths for three tiles.

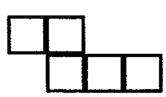


1 tile 1 2 tiles 2 ways 3 tiles 3 ways

Rectangular Numbers: Using bingo chips find the numbers which form rectangles. Example, 5 chips can only form a 1x5 line whereas 6 bingo chips can be arranged in a 2x3 or 3x2 rectangle as well as the 1x6 line. We will call those numbers of chips for which we can form a rectangle—other than the 1x *n* case—rectangular numbers.



Pentominoes: Using graph paper, you are to make as many shapes as you can using five squares. The squares must be touching another square on at least one edge. How do you know if you have them all?



This is an example.

This is a non-example.

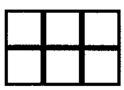
Common Letters: What do you think is the most common letter used in the English language. Using a book or newspaper or magazine, try to determine the most common letters.

Halloween Statistics: Without showing each other your candy bag, find a way to show the rest of us how much candy you collected on Halloween and the various kinds of candy you collected.

Square Take-away[4]: Cut a rectangle (not a square) from a sheet of graph paper. What is the largest square that can be cut from your rectangle? How many squares can you cut away from left over rectangles before you are left with two squares? Try this for a number of different rectangles. What do you notice?

Straw Polyhedrons: Given a handful of plastic drinking straws participants are instructed to construct a polyhedron. A set of data, including faces vertices and edges, is complied from the group. The parent-child pairs look for a relationships based on the data.

Diagonal Intruder[5]: Mark off a rectangle on a piece of graph paper. Draw in one of the diagonals. How many squares does the diagonal pass through?



Rosette: How many lines will you construct if you mark a set of points on a circle and join them to each other with straight lines.

Allowance: Which way would you like your allowance computed, \$1.00/week or beginning with 1¢ the first week and then after double what you received the previous week.

Rings of Pennies: Take a penny and around it place a ring of pennies. How many pennies did it take? Now place another ring around that first ring. How many pennies did that take? Predict how many pennies it will take to make the tenth ring.

Lego Towers: How many different Lego towers can you make if you have five different colours of blocks and you must use each colour just once in any particular tower?

APPENDIX B

RESOURCE LIST OF BOOKS FOR PARENTS AND CHILDREN

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[4] This activity was taken from Mason, Burton and Stacey (1982). *Thinking Mathematically*. Wokingham: Addison-Wesley Publishing Company.

[5] This activity was modified from one in Stevenson, F. (1992). *Exploratory Problems in Mathematics*. Reston: National Council of Teachers of Mathematics.

I have found the following resources to be suitable for parents and children. Although not all of the prompts, questions, problems, and concepts are what I have called variable-entry, these books are quite accessible to parents and children (especially if consider by a parent and a child acting together).

Thinking Mathematically, by J. Mason, L. Burton and K. Stacey. Published by Addison-Wesley, 1982.

Investigations for Your Classroom, by S.E.B. Pirie. Published by Macmillan Education Ltd., 1987.

The Joy of Mathematics and *More Joy of Mathematics*, by T. Pappas. Published by Wide World Publishing/Tetra, 1991.

Math for Smarty Pants (and other titles), by M. Burns. Published by Little, Brown an Company, 1982.

Family Math, by J. Stenmark, V. Thompson and R. Casey. University of California, Lawrence Hall of Science, 1986.

Exploratory Problems in Mathematics, by H. Stevenson. Published by the National Council of Teachers of Mathematics, 1992. (Look for other resources by the NCTM.)

Topic Session C

INTERNET AND MATHEMATICS EDUCATION

Linda Gattuso
Université du Québec à Montréal

The aim of this topic session was to look at the possibilities of Internet for mathematics education. First it was necessary to give an idea of what this new media offers to researchers, mathematics educators, mathematics teachers and finally mathematics students. Secondly, it was also important to examine the advantages and to discuss possible difficulties linked with new media. But first of all let us summarize and introduce Internet and the various kind of resources that are found on the Web.

To start with, some elementary information is needed. We first have to know how you get linked to Internet and what can be found on this "highway". The first step is to connect your computer (by modem or Appletalk or ethernet links) and load the necessary software: Netscape or Mosaic, Eudora or Pegasus, Telnet, etc... depending on your needs. Technical help is probably available through your university or a commercial server.

Most of you have already heard about electronic mail which has been in universities for many years. The e-mail, as it is called, was the precursor of the Internet. Initially communicating by mail was sometimes awkward and disorienting. It was necessary to learn bits of programming language but, nevertheless, it made it possible to keep in touch with researchers all over the world. Now the mail is, more and more, part of our daily life and its use is greatly facilitated by friendly user software such as Eudora which permit us not only to send messages to one, or a group of persons, but also to send all kinds of files that can be directly opened by the receiver at the other end of the line.

But let us get at the World-Wide Web. The "browser" is the software which permits you to "surf" on the Web. It is nowadays very accessible and this fact contributes to its popularity and its development. How can we describe the Web? It is in some way a huge library where you have access not only to written information, but also to graphics and pictures, to animations, to free software and to interactive connections. However, it is not simply a copy of what we could find as printed matter because of the new possibilities offered by hypertext and hypermedia links. A *hypertext* link creates a connection between any part of a text and any another part related to it. Afterwards, it is very easy to go backward and continue where we previously left off. This changes our way of reading. It is not as linear at it would be in a book. A *hypermedia* link joins the site actually visited to any other site in the world that could be of similar interest. This means that there is no end to your "visit" on the Web, and from one site you hop onto another and so on.

Because new sites are opened everyday, the resources offered on the Web are always increasing. Just as when you enter a library, it is essential to use some searching tools. The searching tools found on the Web operate like data bases and give you the opportunity to find sites starting with descriptors, or categories, as wide as "Education" or "mathematics". However, these searching tools are like a directory and each have different lists of sites and also different ways of classifying them. The first one in line is directly on your browser: for example, on Netscape you can start by looking at "Net Search" or "Net

Directory". "Yahoo" is one of the most popular searching tools (http://www.yahoo.com/) but you may have a particular aim and search, for example, for French sites by using *Francité*, (http://www.i3d.qc.ca) or *Outils de recherche*, (http://www.quebectel.com/gt/F000fr.htm). *Liszt: Searchable Directory of e-Mail Discussion Groups*, another searcher, will get you, as its name suggests, to e-Mail Discussion Groups (http://www.liszt.com). Each of these searching tools has its particularity, and will help you find the source you are looking for and connect you to it by a click of the mouse!

Before thinking about what can be e with Internet or how Internet can support or improve mathematics education, we must explore different types of sites found on the Web: references resources, mathematical sites, courses, statistical data and others. For example, literature research can be greatly facilitated by using Netscape. One of the most popular data base in the field of education, ERIC, is accessible on a number of sites. Not only can you look for references yourself and search with the usual descriptors by subject, author, type of publication, etc., but afterwards you will be able to download the results of your research onto your hard disk. You then have a list of references ready to print . Various other databases, like Current Content, are also available. Having the references is not enough, you subsequently need to find the book or journal listed. You might find it in your library but it is also very easy to consult the index of libraries all over the world, including your university's, to find the publication you want. Start by looking at the library index (Catalogues de bibliothèques: http://www.bib.uqam.ca/Catalogues.html). Most of the time, the connection with the library computer you want to consult will be made through Telnet which is a software you need to install at the beginning.

However, you can find sites where there is already a list of references related to subjects in math education (Paul Ernest's Page, Balacheff) or a list of sites related to mathematics and mathematics education (see Math-resources: annexed list). On the other hand, if you are looking for a particular researcher you might want to look at the Mathematics Education Directory (http://acorn.educ.nottingham. ac.uk//SchEd/pages/gates/names. html). Mathematics associations are also on the Web (*ATM* http://acorn. educ.nottingham.ac.uk//SchEd /pages/atm/welcome.html) and most of the conferences now have their sites (*ICME8,* http://icme8.us. es/icme8#ICME8-english). There you can find all the information on the papers presented and facilities related to the conference, etc.

Some journals like JRME have a Web page where they advertise their publications, describe their aims and summarize some recent articles (*JRME Links to Mathematics Education Sites,* http://www. indiana. edu/~jrme/MathEdSites.htm). Other journals are entirely electronic. One example is Journal of Statistics Education (*Journal of Statistics Education Home Page,* http://www2.ncsu.edu/ncsu/pams /stat/info/jse /homepage.html) where not only can you find and publish articles related to statistics teaching but also download some experimental software mentioned in the articles. Finally, you can also visit different schools and universities and get to know their projects, their personnel, and the courses they offer.

One of the innovations that comes with Internet is courses available to every one in the world. These courses come in different formats. You can find undergraduate or doctoral courses in statistics, for example. For some of them, you may enroll at the university offering the course but for others opening the page and reading is all you have to do (*Statistical Education Workshop,* http://www.stat.mq. edu.au/sew/). The type of support provided also differs. Some courses are very similar to the ones offered by mail, and consist mostly of written explanations followed by exercises or problems. On other pages, you can ask for help, or send your solutions by e-mail. More and more, the particular possibilities presented by the net are exploited. Dr. Kenneth Tobin, for example, has installed a Web page from which he conducts a doctorate course (Dr. Kenneth Tobin, http://garnet.acns.fsu.edu/~ktobin/). Visiting his homepage you will see that it uses various forms of interactions. The students are invited to comment on the text presented on the page. Their comments are registered and other students may reply. It is also possible to communicate by mail with the professor or the other students. There is also what we could call an «electronic class» where at some fixed time in the week there is direct interaction via electronic written conversation. Since it is only recently that vocal communication became possible on the Internet, we can already imagine the same class, with people from all over the world, discussing together.

However, it is not necessary to give a course on the Internet to use Internet for mathematics education. A simple option is to have the students look for information on the Web. For example, many pages present some statistical data that are related to everyday life. You can find some on the Statistics Canada page (http://WWW.StatCan.CA/Documents/Francais/Faq/Glance/Tables/d emo3_f.htm), or if you want something lighter you can look at sport pages (*Jacques Villeneuve Fan Page,* http://musicm.mcgill.ca/~marc/ http/jv.html). Many countries display geographic and economic data that can be very useful for a statistics class or for trying to find functional relationships! (*Population des pays,* gopher://gopher.undp. org/00/ungophers/popin/wdtrends/pop1994), *Cabri geomètre* is present on many sites in French and in English (*Cabri Geometry,* http://forum.swarthmore.edu/cabri/cab ri.html). At the "Cabri-thèque" you will discover geometry problems with their solutions that can be used in the classroom.

Mathematical problems of all kinds can be found on many pages. There are some of every level of difficulty. For some problems, you can see solutions given by students (*Richard Briston: Math Assignments,* http://forum.swarthmore.edu/briston/). Given the opportunities of the Web, it would be possible and very interesting to look at interactions between pupils of different schools and of different cultures.

The electronic highway also provides a vast documentation on the history of mathematics and the history of mathematicians (*Birthplaces of mathematicians,* http://www- groups.dcs.st-and.ac.uk/~history/Sensitive Maps/Euro_Map.html). On one of the pages listed, just by pointing at a spot on the world map and clicking, you will find a biography of a mathematician that lived there.

Teachers may rely on the Web to prepare new lessons. All kinds of ideas for lessons, activities and problems related to various topics in mathematics and in science can be found. The possibilities are unlimited. Another example is the CRAMS which offers on the page visited various example s of exams for secondary level mathematics in Quebec (CRAMS, http://www.synapse.net/~euler/crams.htm and also, http://cq-pan.cqu.edu.au/schools/smad/smad.html).

Some pages address the students themselves. They can find help with their mathematics problems (*Ask Dr. Math,* http://forum.swarthmore.edu/dr.math/dr-math.html) or they may play games. You can find a *Chess'n Math* page and also a quiz game about Pi, *The Pi Trivia Game.* Finally, there is free educational software that can be downloaded also for use by students.

If what you want is to get in touch with people who have interests related to yours, you might want to subscribe to a *listserv*. The *listserv* is similar to the mail but every message is sent to everyone on the list. The discussion usually focuses on a particular subject. If you want to find the listserv on a particular topic, you can look into a directory of listservs (*Listservs,* gopher://ericir.syr.edu:70/11/Listservs).

We limited ourselves to pages linked to mathematics and mathematics education and we can already see that there is an overwhelming variety of resources. The question is, now what can we do with the Internet? First, we have to accept, that like the FAX, the calculator, the vocal box and the video, it is here to stay. We must look at it positively and explore the development it can bring into math education. However, it is necessary to maintain a critical eye and examine closely some of the possible drawbacks.

The Internet brings together a great amount of information often in a very attractive format, with pictures, graphics and even animations. However, the sources are not controlled and almost anyone can install a Web page. As with paper publications, we have to rely on the reputation of the publisher, if not the author. For theWeb, the same thing must be done. Some people have expressed reserve and suggested that there should be some kind of guides. Certainly, it could help, but it is better to teach our students to be cautious about what they use. Another point is that it is very easy to download any text from the Web and forget to mention the reference. Still, all these objections, and others, must have been used at the time of Guttenberg. They are important, but they should be seen as cautions rather than negatives. There are opportunities that appear today, and probably more will come.

Internet is not only a source of information, it is also a source of communication. This aspect might be the one that will be more significant for mathematics education. Internet is a tool that can create new types of links. We have already seen that it facilitates communications between researchers all over the world. Contacts are easier because communicating by mail is more informal than a letter and very quick. We can now imagine undertaking collaborative research with teachers in schools even if they are at some distance. Material can be installed on the net, discussion and reflection can travel by electronic means. Everything can be recorded, and data about the development of thediscussion can be analysed. We can also think of situations where the students themselves could use the net and interact with their teachers or with students in other schools and countries to collaborate on a project. Installing Internet in the classroom could provide more material and easy access to problems or mathematics competitions or collaboration.

As teacher trainers, we can install electronic links between schools and universities particularly in the context of teaching practice. All the requirements and information needed for student teachers and their host teachers could be on the net. The person in charge of the practice at the university could communicate with the student teachers or their host teacher in an instant. More continuous support could be provided by the supervisor and reflection-on-action (Schön, 1983) could be fostered and so on. We have already mentioned university courses given on the Web; they are yet further possibilities. For example, sick children who have to stay in the hospital could be followed so that they can continue with their schooling. In fact, the uses of Internet are limited only by our imagination.

Is Internet a new fad that will slide away or will it insidiously slip into our daily lives. It is more probable that it will stay, and we have to ask ourselves if we want to take account of it, and moreover, if we want to manage this new tool to our advantage. We have already explored existing facilities and conjectured on its future usefulness; we must also use our critical faculties and sort out the good from the bad. There are many obstacles still to overcome. How do we define copyright? Does anything installed on the Web become universal property? How does one perceive the quality of information? Should some sort of authority, like a professional association, guide the surfers? In fact, as with books, journals or radio and television documentation, critics and surveys might help to provide some guidance, yet, we should be careful about extreme decision leading to censorship. Our role as educators is to look for the usefulness of this new media, to experiment, to prepare our student teachers adequately, and to support teachers and schools willing to benefit from our expertise.

BOOKMARKS[1]

Searching Tools

Yahoo	http://www.yahoo.com/

[1]The bookmarks can be sent by e-mail if you contact the author: gattuso.linda@uqam.ca

Francité	http://www.i3d.qc.ca
Outils de recherche	http://www.quebectel.com/gt/F000fr.htm
Liszt: Searchable Directory of e-Mail Discussion Groups	http://www.liszt.com
Recherche d'information sur le WWW	http://www.risq.qc.ca/info/table/rech/rech_01.html
Welcome to Magellan!	http://www.mckinley.com/
Inktomi Web Services	http://inktomi.berkeley.edu/

References

ERIC Query Form	http://ericir.syr.edu/Eric/
Welcome to AskERIC	http://ericir.syr.edu/index.html
Instructions	gopher://ericir.syr.edu:70/00/Database/Instructions
UTLink: University of Toronto and Local Libraries	http://www.library.utoronto.ca/www/librarylist.html
Catalogues de bibliothèques	http://www.bib.uqam.ca/Catalogues.html

Mathematics and Education

Yahoo! - Science:Mathematics	http://www.yahoo.com/Science/Mathematics/
Geometry Forum Web Site Index	http://forum.swarthmore.edu/~sarah/webindex.html
Forum Internet Resource Collection	http://forum.swarthmore.edu/~steve/
Mathematics Archives WWW Server	http://archives.math.utk.edu/
Math Forum Home Page	http://forum.swarthmore.edu/
Québec Science vous accueille (Mars 1996)	http://www.QuebecScience.qc.ca/
Mathematics Education Directory	http://acorn.educ.nottingham.ac.uk//SchEd/pages/gates/names.html
Mathematics education sites	http://acorn.educ.nottingham.ac.uk/Maths/other/
JRME Links to Mathematics Education Sites	http://www.indiana.edu/~jrme/MathEdSites.htm
DRD - Menu principal	http://www.eduq.risq.net/DRD/
Balacheff	http://leibniz.imag.fr/CABRI/CabriWeb/EIAH/EIAH.Balacheff.html
Paul Ernest's Page	http://www.ex.ac.uk/~PErnest/
ATM	http://acorn.educ.nottingham.ac.uk//SchEd/pages/atm/welcome.html

ICME8	http://icme8.us.es/icme8#ICME8-english
Canada's SchoolNet	http://schoolnet2.carleton.ca/francais/
La toile du Quebec	http://www.toile.qc.ca/

Journals

Journal of Statistics Education Home Page	http://www2.ncsu.edu/ncsu/pams/stat/info/jse/homepage.html
Yahoo! - Science:Mathematics:Journals	http://www.yahoo.com/Science/Mathematics/Journals/
E-Journals	http://www.lib.lehigh.edu/ejournals.html
Bienvenue sur le Web de La Recher-che	http://www.LaRecherche.fr/

Schools

School and Faculty of Education	http://acorn.educ.nottingham.ac.uk/
Adresses	http://cyberscol.cscs.qc.ca/Ecole/Ecole.csbd?fonction=general&c1=RESS&c4=SERE
WWW de l'UQAM	http://www.uqam.ca/
Département de mathématiques	http://www.math.uqam.ca/

Math Education

Math Forum - Mathematics Education	http://forum.swarthmore.edu/mathed/index.html

Courses

Dr. Kenneth Tobin	http://garnet.acns.fsu.edu/~ktobin/
Statistical Education Workshop	http://www.stat.mq.edu.au/sew/
excite NetDirectory: General/Education/Teaching/Stuff_for_Teachers/Teacher_Trainin g/	http://www.excite.com/Subject/Education/Teaching/Stuff_for_Teachers/Teacher_Training/s-index.h.html
Page Internet de Denis Hamelin	http://dimcom.uqac.uquebec.ca/~dhamelin/

Statistical Data

Assignment to Topic Groups	http://icme8.us.es/ICME8/conttg.html
pop1994	gopher://gopher.undp.org/00/ungophers/popin/wdtrends/pop1994

demo3_f.htm	http://WWW.StatCan.CA/Documents/Francais/Faq/Glance/Tables/dem o3_f.htm
demo4_f.htm	http://WWW.StatCan.CA/Documents/Francais/Faq/Glance/Tables/dem o4_f.htm
demo5_f.htm	http://WWW.StatCan.CA/Documents/Francais/Faq/Glance/Tables/dem o5_f.htm
heal3_f.htm	http://WWW.StatCan.CA/Documents/Francais/Faq/Glance/Tables/hea l3_f.htm
heal1_f.htm	http://WWW.StatCan.CA/Documents/Francais/Faq/Glance/Tables/hea l1_f.htm
heal2_f.htm	http://WWW.StatCan.CA/Documents/Francais/Faq/Glance/Tables/heal2_f.htm
Server Statistics @Where-no-one-has-gone-before.mit.edu	http://where-no-man-has-gone-before.mit.edu/WebStat.html
Server Statistics	http://mlab-power3.uiah.fi/usage/statistics.html

Lists

Listservs	gopher://ericir.syr.edu:70/11/Listservs
README	gopher://ericir.syr.edu:70/00/Listservs/EDNET-List/README

Problems

Richard Briston: Math Assignments	http://forum.swarthmore.edu/briston/
Secondary Mathematics Assessment and Resource Database	http://cq-pan.cqu.edu.au/schools/smad/smad.html
Erdös for Kids	http://csr.uvic.ca/~e4k/
Internet Center for Mathematics Problems	http://www.mathpro.com/math/mathCenter.html
Meeroh's Online Collection of Math Problems	http://where-no-man-has-gone-before.mit.edu/omap/competitions/

Cabri

Start	http://www-cabri.imag.fr/CabriWeb/Start.html
Cabri Geometry	http://forum.swarthmore.edu/cabri/cabri.html
TeleCabri.html	http://www-cabri.imag.fr/CabriWeb/TeleCabri.html

Lessons and Activities

Math	http://www.csun.edu/~vceed009/math.html#Lessons
Yahoo! - Science:Mathematics:Courses	http://www.yahoo.com/Science/Mathematics/Courses/
Yahoo! - Science: Mathematics:Statistics:Courses	http://www.yahoo.com/Science/Mathematics/Statistics/Courses/
Math Forum Internet Collection - mathlessons (Outline)	http://forum.swarthmore.edu/~steve/steve/mathlessons.html
Lessons by Susan Boone	http://www.cs.rice.edu/~sboone/Lessons/lptitle.html
Newton's Apple Lesson Plans	http://ericir.syr.edu/Projects/Newton/
Yahoo - Science:Mathematics:Numbers	http://www.yahoo.com/Science/Mathematics/Numbers/
Mathematics Archives - Lessons and Tutorials	http://archives.math.utk.edu/tutorials.html
Math Forum: Web Units	http://forum.swarthmore.edu/web.units.html

History

The Mathematical Museum - History Wing	http://elib.zib-berlin.de:88/Math-Net/Links/math-museum.hist.html
Canadian Society for the History and Philosophy of Mathematics	http://www.kingsu.ab.ca/~glen/cshpm/home.htm
History of Mathematics	http://www-groups.dcs.st-and.ac.uk/~history/
The History of Mathematics	http://www.maths.tcd.ie/pub/HistMath/

Math Help and Activities

Yahoo! - Science and Oddities:Math	http://www.yahooligans.com/Science_and_Oddities/Math/
Ask Dr. Math	http://forum.swarthmore.edu/dr.math/dr-math.html
Chess'n Math: Homepages	http://www.netgraphe.qc.ca/chess-n-math/index.htm
The Pi Trivia Game	http://cid.com/~eveander/trivia/index.cgi
EIAH.TeleCabri-CHU.html	http://www-cabri.imag.fr/CabriWeb/TeleCabri/TeleCabri-E/EIAH.TeleCabri-CHU-E.html

Software, etc...

The MathWorks Web Site	http://www.mathworks.com/

Mac Education Software: Chemistry Math Latin Greek French Spanish	http://www.sciedsoft.com/
Free Adobe Acrobat Reader Software	http://www.adobe.com/acrobat/readstep.html

Assessment

CRAMS	http://www.synapse.net/~euler/crams.htm

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Fristrup, J. A. (1996). *The essential Web surfer: survival guide.* Upper Saddle River, N.J.: Prentice Hall.

Glossbrenner, A. (1995). *Internet 101: a college student's guide.* 2nd ed. New York; Montreal: McGraw-Hill.

Hahn, H. and Stout, R. (1995). *The Internet: yellow pages.* 2nd ed. Berkeley, CA; Montreal: McGraw-Hill.

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Topic Session D

TEACHING FROM A PROBLEM-SOLVING PERSPECTIVE: A REPORT OF MY DOCTORAL RESEARCH

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I would like to thank the Canadian Mathematics Education Study Group for the opportunity to share my dissertation results (Seaman, 1995) and, in particular, Yvonne Pothier and her host committee for making the visit to Nova Scotia such an enjoyable one.

My talk today will center on three phases: How I arrived at my question; the results; and the discussion of the results.

PHASE ONE

When I undertook my study I had over 20 years of teaching experience in secondary and post-secondary mathematics instruction. I had, over those years, realized that teaching mathematics needed to emphasize more than knowledge-base skills, such as algebraic manipulations. Instruction of mathematical thinking skills was lacking, and needed more emphasis. I thought that using Pólya's four-step problem-solving strategy would help answer my concerns.

Our high school at this time was implementing an accelerated, internationally recognized mathematics curriculum called the International Baccalaureate. This program stretched the high school curriculum to post-secondary topics such as calculus, statistics, linear algebra and abstract algebra. I realized that because of time constraints we might have to rely on mastery learning and assign questions incrementally as John Saxon had done in his textbooks. That is, instead of assigning questions for only one lesson they could be spread out over two or three weeks. Now, instead of problems located at the end of the chapter they could be assigned incrementally. I was beginning to think that we could teach from a problem-solving perspective, introduce the knowledge base as needed, and assign homework questions incrementally.

If I were going to teach from a problem-solving perspective, the students might need an example of a cognitive strategy. Perhaps one bearing a similarity to Pólya's, but emphasizing the importance of representing a problem and then solving it (Figure 1). Aha! Maybe teaching from a problem-solving perspective, assigning questions incrementally, designing lessons that feature the representation of problems, and introducing knowledge base as needed to solve the problems would work. It would also be necessary to make the students aware of different representational strategies and where such strategies fit in the students' cognitive strategy.

Although the students would have a cognitive strategy to be used regularly and a representational strategy to help the students problem solve, it would be important to show them how to use it. This could be achieved by introducing a classification strategy that would have the students classifying the problems according to the underlying mathematical procedures needed to solve the problems (deeper structure). I

PROBLEM REPRESENTATION

I. DO YOU UNDERSTAND THE PROBLEM?
(Surface Structure)

READ QUESTION FOR UNDERSTANDING

DO YOU UNDERSTAND WHAT ALL THE WORDS MEAN?

UNDERLINE KEY WORDS THAT HELP YOU EXPLAIN WHAT YOU'VE JUST READ

TELL ME WHAT YOU HAVE JUST READ INCLUDING THE GOAL OF THE PROBLEM

II. FIND A PLAN (Deeper Structure)

READ QUESTION FOR ANALYSIS

CIRCLE KEY WORDS THAT HELP YOUR ANALYSIS

MAKE AN EXTERNAL REPRESENTATION

THINK OF A PLAN

PROBLEM SOLUTION

III. CARRY OUT PLAN

SOLVE

IV. EXAMINE THE SOLUTION

CHECK

FIND MORE SOLUTIONS

GENERALIZE SOLUTIONS

Figure 1
Cognitive Strategy

was getting closer to the formulation of my question, but how would I assess the effectiveness of my approach? I decided to make problem worksheets that used the cognitive strategy as a template to assess problem solving analytically.

PHASE TWO

Phase one led to a prescription for instruction that provided a cognitive strategy, representational strategy, classification strategy and an analytic scoring system based on these strategies. As part of the prescription it was intended that students would problem solve regularly and be taught knowledge base as necessary to solve their problems.

This prescription led to a quasi-experimental study that used a two-group comparison of intervention and comparison groups. The groups were made up of volunteer university students from two fall semester sections of an introduction to finite mathematics class that satisfied a degree requirement for the Faculties of Arts and Education. The comparison group was taught by a colleague in the mathematics department, and the intervention group was taught by me, and had the added element of systematic emphasis on representational strategies. Both classes had instructional blocks of 75 minutes plus an optional weekly mathematics lab. The instructional blocks for both groups were 26 lessons in length with two lessons taught per week.An evaluation task was administered upon completion of the study to both groups to analyze the problem understanding, representation and solution when solving word problems. A pretest/posttest was given to look at the difference in heuristics between the two groups.

The following hypotheses were tested:

HO (1): There will be no significant difference in problem understanding between the comparison group and intervention group when solving word problems.

HO (2): There will be no significant difference in mathematical problem-solving representation and solution between the comparison group and intervention group.

HO (3): There will be no significant difference in heuristics between the comparison group and instructed group.

In this study all three null hypotheses were rejected.

PHASE THREE

The main purpose of this study was to look at the effects of instruction from a problem-solving focus on problem understanding, representation, solution and heuristics. Cognitive, representational, and classification strategies were part of the regular classroom routine and they had implications for the teaching of mathematics.

One way to help students understand a concept is to provide an example. The automatization and regular applications of a cognitive strategy will give students an example of a thinking strategy and control over their problem solving. For the teacher, instruction from such a problem-solving perspective will provide a model from which to teach with representation and solution of problems the template for each lesson. Implied is that lessons may be constructed that emphasize the representation of problems on a concrete-to-abstract continuum with the knowledge base necessary for problem solution introduced as needed. Students will gain a better understanding of the usefulness of representational strategies in solving problems and in recognizing that problems may be represented in more than one way.

Classifying problems according to their deeper structure will give students a powerful strategy that will allow them to make mathematical connections concerning how mathematical ideas are related. Students will become aware of how isomorphic problems might be used as a vehicle to develop their analogical reasoning in mathematics and eventually across the curriculum. However, it would be helpful if a mathematics course were developed with this underlying concept as a basis.

Such courses might be developed more systematically with regard to deeper structure, with knowledge base introduced when necessary to solve the problem. Any textbooks used for such courses would have to support such a systematic development of representation and solution skills. A different view of assessing/evaluating mathematics, then, becomes necessary.

Analytical scoring may be necessary to assess these skills using a template based on the students' cognitive strategy. This evaluation might lead to different levels of evaluation where students receive a grade for their work on problems equivalent to those in their course. Evaluation would include comments made on problems that are isomorphic, similar and unrelated to those studied. This could make 'gotcha' questions on exams a thing of the past. Finally, students using feedback from the analytical scoring of problems might be able to locate weaknesses in their use of a cognitive strategy and could negotiate help from their teacher. Implied is the need to develop mathematical instruction through inservice or teacher education programs focusing on teaching from a problem-solving perspective. As well, teachers would have to experience the process of teaching from a problem-solving perspective using cognitive, representational, and classification strategies to gain a full understanding of what they are instructing. Romberg (1994) acknowledges such a distinction between "doing" and "knowledge about" by means of an analogy with basketball.

> ...when students learn to play basketball, they are always aware that their goal is to play the game. What is being argued is that in mathematics, the "game" is to solve non-routine problems. Basketball practice is important for skill development, learning strategies, and so forth. However, a coach would never get anyone to practice if they never played a game. Furthermore, practices are tailored to the needs of the team and the individuals. Today, in school mathematics all students practice skills, whether needed or not, spend almost no time learning strategies, and never get to play the game (p. 289).

This has implications not only for students and teachers but also teacher educators, curriculum planners, textbook writers and public perception.

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Topic Session E

EXPLORING THE THEORY OF MEASURE: ITS IMPLICATIONS FOR CLASSROOM LEARNING

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INTRODUCTION

Despite a long tradition of research, the ways in which children construct their own understandings of standard arithmetic remains elusive. A major difficulty for teachers could well be their own familiarity and ease with numbers and operations in arithmetic which obscures for them many of the issues that face children in their efforts to build their own arithmetical knowledge. Weinzweig's (1995) theory of measure provides a way of understanding the ideas and mechanisms involved in the practice of arithmetic, how they arise, and how they evolve as means and tools in mathematical practice. He argues that the generation and evolution of arithmetical knowledge in measurement situations has strong parallels to the building up of this knowledge by learners, and has used the theory to design a series of instructional modules. This paper outlines aspects of the theory that provide a framework for viewing the nature of experience in the construction of number understanding and the operation of addition. It explores how this view can be applied to understanding how a series of activities using a bead frame has been designed to facilitate knowledge building and provides an analysis of the experiences of children working with these activities.

Weinzweig's theory of measure provides an analysis of mathematical constructs related to number and the operations on number in arithmetic. It identifies a collection of mathematical ideas and mechanisms or, what he calls, "ingredients" operating in such situations. It defines what we measure when we measure something, why we measure and how units are introduced to facilitate measurement activities and how numbers are introduced to create quantities. It thus provides operational definitions of important and often poorly understood constructs such as measure or magnitude, number and quantity

Weinzweig's learning activities are carried out in specially designed contexts in which the ingredients arise and can be recognized. There are a series of tasks associated with each context that engage children in the use of the ingredients in problem solving. The study reported here (McCaul, 1996) was undertaken to explore how the theory could be used as a lens for viewing the nature of experience in instruction and in viewing the progress of knowledge building in children in the course of these experiences.

THE THEORY OF MEASURE

Aspects of the theory are discussed below in general terms to provide an overview of some of its key features. The discussion is limited to aspects of the theory dealing with the derivation of whole numbers and the operation of addition. The details of each ingredient are presented in mathematical terms in the theory and are not easily applied to an understanding of the nature of experience and the progress of knowledge building. One objective in the study was to identify a more general framework for understanding how ideas arise in specific experiences and how they subsequently evolve to higher levels of abstraction in the course of specific types of experience. The report briefly outlines this framework, applies it to one of the instructional contexts used in the study (the bead frame) and uses it to analyze some of the experiences of children engaged in activities in the bead frame context.

The theory is an account of the derivation of numbers from their origins in experiences in the manipulation of physical objects and thus addresses the learning challenge that faces young children. It is a key feature of the theory that the definitions are operational, that is, given in terms of how an idea arises in an action in a physical context and subsequently how this idea is built up and incorporated into mathematical operation at higher levels of abstraction. It is this type of definition that leads to the identification of specific experiences that can facilitate a similar or parallel building up of ideas by learners. An assessment of the progress of children's knowledge construction is made in these terms as well. A careful examination of how they deal with a task in a context indicates whether they have recognized an idea and whether or how well they have been able to use it as an effective means in dealing with a problem more efficiently.

The theory organizes mathematical practice into levels or spaces. Operation in a space is determined by the understanding of the object that is being used to deal with a problem. The theory defines how the understanding of an object used in each space leads to a new understanding of the object at the next level or space of operation. In this way the object at each level corresponds in some way to the object at the level below it and to the object that derives from it at higher levels. Similarly the ideas generated at each level find a correspondence at every level in the hierarchy of spaces. This idea is similar to the idea of reciprocal structure preserving systems discussed by Kaput (1989). The following discussion looks briefly at each system in the series to try to present an overview of how ideas arise and how they evolve to higher levels of abstraction.

SPACES OF OPERATION

The General Measurement Situation

The general measurement situation is a first level or threshold space of operations where the objects being manipulated are physical ones but where in the course of manipulating physical objects specific mathematical ideas with respect to them are generated. The collection of these mathematical ideas and mechanisms or "ingredients" is identified in the theory for the general measurement situation. The ingredients identified here are common to all measurement situations in all measure domains. The term domain refers to the set of objects which share the same measure property, for example, the domain of line segments and the measure length, the domain of sets and measure numerosity, the domain of regions and the measure area, etc. In the instructional contexts, children are introduced to a variety of these situations, especially situations involving line segments and the measure length, sets and the measure numerosity and regions and the measure area and recognize the common ingredients and how they are manifest in each.

Any level or space of operation should be considered first in terms of the understanding of the object that is manipulated in it. In the general measurement situation the object is a physical one that comes to be understood in terms of its mathematically relevant property, its measure. The idea of measure is difficult to define. Weinzweig's operational approach resolves the problem by considering measure in terms of the procedure used to determine what the measure of an object is. Thus the measure numerosity for example, arises in the one-to-one correspondence of the elements of two sets, the measure length in the congruent match of two line segments. Measure is the property shared by both objects in an equivalence relation generated in this procedure. All objects that can be included in a particular equivalence relation share the same measure and are grouped into the same class of measures. These measure classes become the new objects in the next space of operation, the measure system. In the measurement situation the idea of measure is a property of the physical objects being manipulated. In the measure space the idea of measure has been separated from the object and is understood as common to any object that can be placed in a particular equivalence relation.

Other ingredients identified and defined in a measurement situation, are procedures carried out on objects and properties of these procedures. The operational definition of each indicates an action or experience that gives rise to the idea that can be incorporated into an experience for learners. The procedure to establish equivalence (comparison), gives rise to an equivalence relation and properties of the equivalence relation, transitivity, reflexivity and symmetry. Dissection leads to the creation of partitions and unit partitions and consolidation leads to the addition of measures in the measure system and of numbers in the quantity system and the properties of associativity and commutativity that are associated with them.

The Measure System

The measure system is a next level of abstraction and, as noted above, the object manipulated in it is a measure class object. An example in the Cuisenaire rod context attempts to clarify what is meant. Cuisenaire rods provide an example in the domain for the measure length. The objects in this domain are the individual rods. Rods of the same length have the same colour and rods of the same colour have the same length. Children working with the rods soon cease to refer to individual rods but refer to the colour of the rods where in effect the colour designates the length of any rod. What has happened here is that the children have divided the rods into what we have defined as measurement classes and each class is denoted by a colour. They compare the length of a specific yellow rod to the length of a specific blue rod. However they describe this situation by saying the blue rod is longer than the yellow rod. Implicitly they are assuming that this statement refers not to individual rods but to any blue and any yellow rod, i.e., the class of blue rods and the class of yellow rods. All rods of the same colour have the same length and hence are equivalent and all rods of the same length have the same colour. The colour serves to designate the collection of all rods of a given length. When children begin to talk about the colour rather than the specific rods they have inadvertently started to use a new object, not individual rods but rather the collection or equivalence class of all rods of a certain length, an equivalence class measure object.

An equivalence class measure object is a conceptual entity. It is built up in the understanding of the equivalence of objects sharing the same measure and the idea of transitivity of that relation. For any equivalence class measure object in the measure system it is possible to find a particular measure object in the general measurement situation that belongs to the equivalence measure class. In this sense an object in the general measurement situation is said to correspond to an object in the measure system and vice versa. Similarly for any procedure or property of the procedure in the general measurement situation there is a procedure and property of that procedure that corresponds to it in the measure system.

The Quantity System

An object in the quantity system is a quantity, a composite entity comprised of a number and a unit. The generation of quantities is traced from the general measurement situation for numerosity. An object in the domain of sets is a unit partition, a set of unit segments. The measure of this unit is the unit component of a quantity and the count of this set is the numerical component of the quantity. If the units are the same it is possible to compare the measure of two measures by comparing two quantities and we can compare the quantities by comparing their numerical components. The unit partition of the object, the set, belongs to the measure class of all objects that can be matched to it. The number derived in counting the set is now used to identify the measure class to which the set belongs. The number, understood in this way, is the new object manipulated in the quantity system. As in the measure system, for every procedure and property of procedures in the general measurement situation for the domain of sets and numerosity their is a corresponding procedure and properties of procedures in the quantity system.

THE THEORY AS A LENS FOR VIEWING INSTRUCTION AND LEARNING

The account of the generation and evolution of mathematical ideas provides a view of what mathematical ideas children need to appropriate into their own practice in terms of how these ideas are generated in measurement situations, thus providing a rationale for the design of instructional experiences for children. The approach focuses on how experiences facilitate the building up of mathematical knowledge and as such is not interested per se in the changing cognitive structures of the mind. The theory describes how ideas arise in measurement situations and how they evolve as means and tools in mathematical problem solving. The building up of mathematical ideas by children is thought to have important parallels to this process. In setting a situation in a physical context of some sort, the conditions that originally led to the generation of an idea can be recreated. Tasks which directly call for the use of an idea can be presented to provide the necessary exposure and practice that are needed to ensure the idea is well established and "reversed", that is, used as a means in dealing with a situation. Progress is followed in terms of whether or not the idea was recognized and how well the child was able to develop an ability to use it as a means of dealing with a problem situation. In cases where a child is not progressing as expected, the theory directs attention to what idea might be missing and to the kind of experience that would be required to facilitate its development.

The theory of measure provided the rationale in the design of instructional modules that would facilitate the recognition and appropriation of specific mathematical ideas into practice. First a context was provided where specific ideas arise. Weinzweig designed six of these; bus, bead frame, the number track, Cuisenaire rods, count sheet/fact rectangle, and number balance. The contexts were constrained in the sense that they reduced as much as possible any physical or social aspects of the situations that are a distraction to children. They were however presented much like a game so that they would be interesting and engaging for children. Activities in the contexts provided focused experiences, that is, experiences which focused attention on a specific idea or ideas operating there. Tasks carried out in a context focused attention on using the idea as a means of dealing with the situation more efficiently.

The activities introduce a language to describe a situation in a context. The language is unique to each context and reflects the activity going on there. The language allows the children to communicate about the context in a way that is understood by everyone. It also focuses attention on what in the context is being communicated. Children engage in record keeping activities in which the language is used to describe a situation. In notation "reading" activities they learn to read a notation and recreate what is going on in the context.

A look at the bead frame context and some children's experiences with it, is used here to explore some implications of the theory of measure for understanding the nature of experience in instruction and the progress of children's mathematical knowledge building. The bead frame consists of 25 beads strung on a stiff horizontal wire between two supports with 5 yellow beads alternating with 5 red beads.

Building Measure Objects

Initial tasks in the bead frame context are designed to assist children in identifying a segment of beads with a number without counting. The context presents an object that can be considered in either the domain of line segments and the measure length or, the domain of sets and the measure numerosity.

Children are asked to identify segments of n (5, 7, 9, 10, etc.) beads without counting. Initially they count the number of beads in a segment but soon realize they can identify the segment with a number without counting. Segments of 5 and 10 are pushed over more often and became a base for other numbers so that 7 is 5 yellow and 2 red, 9 is 10 minus 1, etc. A number is always used to identify a segment of beads having a specific measure so a number begins to be associated with the object or measure it is used to identify, and its meaning is always considered in terms of this object. Once children can identify segments without counting they can focus on other aspects of the context without being distracted by the need to count.

In a second task children identify segments of *n* beads starting at different places on the bead frame. For example there are three different colour combinations for 7: 5 yellow and 2 red; 1 yellow, 5 red and 1 yellow; 3 yellow and 4 red. This exercise helps children when they begin to do consolidations of two segments. They can readily identify the first and third segments but usually have to count the second segment. Practice with task 2 allows them to identify different configurations so that they can identify a second segment without counting.

The tasks assist in building the idea of equivalence class measure objects for sets, an idea not easily recognized by children and which needs to be built gradually in the course of a variety of experiences. In the bead frame children are encouraged to identify a segment of beads visually by its colour configuration. Implicitly they recognize that any similarly configured segments are the same in the sense they share the same measure and therefore belong to the same equivalence class. They learn to identify the segment with a number. As an understanding of the segment as an equivalence class measure grows, the number used to identify it also incorporates this understanding as well and leads to an understanding of quantities for whole numbers in the quantity system.

In the study, four of the children in the group were able to identify numbers to twenty by the end of the sessions. They used this ability in subsequent record keeping and recreating tasks. These children also seemed to develop a good sense of a set as collective entity. In their actions they would directly identify a segment and think of it in terms of its color configuration. This was understood to indicate a recognition of the segment as a measure that could be identified by its colour configuration and the number determining it in counting. Two of six children progressed much slower than the others in the group to identify segments with a number. All the tasks presented greater difficulty for these children than for the others and more time was needed to assist their progress. Unfortunately the conditions of the study did not allow for this.

Understanding the correspondence between a context and a notation

A central feature of the theory is the definition of the way in which each ingredient in the general measurement situation finds its correspondence in any other system and vice versa. In the instruction, an understanding of this correspondence is built up effectively in the building up of a notation to record information about a context. Numbers in the notation for the bead frame first identify, and subsequently are understood, to correspond to specific bead segments. Numbers also encode and correspond to the relations of objects in the context and special features of the notation correspond to actions carried out on objects in the context.

When working in the bead frame context children are quickly introduced to a notation that allows them to communicate easily about what they are doing. A dice is rolled and the result recorded in the first wedge. A second is rolled and this result entered in the second wedge. The final result or consolidation of the two segments is recorded in the circle. The bead frame notation is designed to relate the action of pushing over the bead segments and recording the result of their consolidation.

The number in the first wedge indicates the number of beads in the first segment, the number in the second wedge indicates the number of beads in the second and the number in the circle indicates the number of beads in the consolidation. A child records the activity in the notation, and understands it in terms of the activity in the context, and can communicate with everyone else using it. They can "read" a record and recreate the action in the bead frame context. The relation between the action in the context and the notation used to record it, needs to be well understood. Later if a child has a problem understanding the notation they can return to the context and figure out what they need to know.

Once children are able to fill in the record sheet they are introduced to missing information tasks, a) missing sum, b) missing first addend, c) missing second addend. The sheets show a record of several dice trials but some of the information is missing and children are asked to find the missing information.

Understanding the correspondence between the context and the notation is critical to knowledge construction. Once children are familiar with the context and what to do there, they are free to explore other ideas operating in it and to notice how these are recorded or preserved in the notation. Gradually they begin to recognize that number combinations repeat and they just know them without needing to refer to the context. If they don't know a combination however, they know how to figure it out in the context. When children use numbers without reference to the context but where the meaning of the number and notation in the context is well understood they are said to be operating in the quantity system for whole numbers. There is much more they need to understand about the derivation of units in multiplication and fractions but that can be built up similarly in activities that focus on where these special units come from.

Recognizing Ideas: Some Examples

At the beginning of the instruction the children were at the stage of meaningful counting. They could use a number derived in counting an object to identify another object matched to it but did not readily use this skill in problem solving. In the instruction they learned to use a number to identify an object (a segment) in the context and to understand a number notation in terms of the context. When children recognized how an event in the context corresponded to the record of it in the notation they were free to notice certain relations and properties arising in the context. The following example demonstrates how Ali's understanding of the correspondence between the context and the notation, allows him to recognize the invariance of addition. The invariance of addition is the idea that the disjoint union of two specific measure objects for sets will always yield the same result. In the following example Ali notices that the result of consolidating 1 bead and 1 bead is the same on two consecutive trials. He notes that the notation repeats as well. The fact that the notation and the context correspond seems to both focus his attention on this idea and to reinforce his confidence in the ability of the notation to encode the context.

Ali and Suresh: (they have noticed that a roll of 1 and 1 on the dice gives a final result of 2 on two separate trials, Ali seems amazed that this is true)

Ali: Mrs. McCaul!, Mrs. McCaul! Look! 1, 1, 2 (points to one recorded sequence then another), 1, 1, 2!

That the two trials have repeated in exactly the same way is purely serendipitous but it has the happy consequence of directing attention to the fact that what was repeated in the context was also repeated in the notation and that the notation was a reliable means of recording the event.

In the following example John's understanding of the reciprocal relation between the context and the notation seems to trigger an exploration of the interesting features of both and a discovery of ingredients. He notices that a different ordering of two measures yields the same result. The following example shows how he is using his understanding of the correspondence between the context and the notation to confirm his intimations of associativity.

Trial: 7, 6, _

John: (pushes 7, counts 6 more, counts total) 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12.

Thandi: (starts to record 12, she does not question the result)

John: (checks beads visually) no that's 11.... It's wrong, it's one more than 12. It's 13. He redoes the series.

John: (pushes 7 and counts 6) Altogether makes 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13.

Trial : 6, 7, _

John: The next one is 6 and 7. 6 (pushes 6) and 7. Is that the same? Ya it goes 13. (jabs the sheet) 7, 6 (reads one series left to right) 7, 6 (reads next series right to left) is 13.

Thandi: No it's 6, 7...7, 6.

John: It's the same thing 13.

John is now convinced and he uses this knowledge from this point on. Thandi on the other hand is not sure of how the notation and context are connected and does not see what John has seen and is very annoyed at his determination to insist that it doesn't matter.

SUMMING UP

The exploration reported here was carried out to identify a general framework that could be used as a lens for viewing how the building up of mathematical ideas was facilitated in the instructional modules and to view the progress of mathematical knowledge building by the children in the course of using the modules. The exploration identified two features of the theory that seemed particularly useful in viewing this process, the changing understanding of objects manipulated in mathematical practice and the operating spaces they defined. The analysis highlighted how the design of the instructional contexts and the experiences associated with them, facilitated the construction of new objects, measures, measure classes and quantities and operations in the operating spaces defined for these objects. It further identified how the view could be applied to understanding children's progress in terms of a movement from one operating space to another as the understanding of the object changed and developed.

In the study, it became clear that understanding the way in which an event in the context corresponded to the language used to communicate about it, was a critical aspect in learning. The understanding allowed children to establish the truth of their record or, vice versa, to verify a situation recreated from the record. When a child noticed that an event in a context corresponded to the record of it in a notation, his attention was directed to the idea that the event was being expressed in both. For example, Ali noticed the invariance of addition. John discovered the associativity of addition when what he noticed in the context was confirmed in the notation. When the correspondence was recognized, the child could "use the context" to find out what he needed to know and later to verify ideas being formed.

It iss not possible within the limits of this paper to consider the building up of other ingredients identified for the measurement situation. In future it will be necessary to examine each of the ingredients identified for the general measurement situation to better understand how it develops in instructional experiences and whether some ingredients pose greater difficulty than others for some children. It will also be necessary to explore how the theory of measure can be used to understand the derivation of new units and the building up of knowledge of multiplication and fractions in instructional experience. Many of the ideas contained in the theory are not easily expressed and future efforts need to explore how to make these ideas relevant to teachers in classrooms.

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AD HOC SESSIONS

Ad Hoc Session 1

MATHEMATICS ENRICHMENT IN AN OUT-OF-SCHOOL SETTING

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The relationship between mathematical creativity and school achievement in mathematics is not a direct one (Kajander, 1985; 1989). Although students in writing or art class may be asked to write their own poem or draw their own flower, creativity is almost never addressed in school mathematics. This lack of creativity may be connected to the structured approaches of teachers and the lack of exploration in mathematics classrooms. Davis describes mathematics classes as "perhaps the least playful" (1996, p.212). The mathematical flowers drawn in school are often red with green stems.

Carter (1994) feels our aims should be "to give the young a feeling for the beauty and eloquence of mathematics and its profound relationship with the real world" (p.93). Students need "early to begin consideration about what they think is beautiful and creative [in mathematics]" (Keswani, 1996, p.3).

The result of a program that does not encourage deeper mathematical thinking is that talented students never practice thinking intuitively and creatively. Generalisations come easily to the students better in school mathematics, and hence they may later lack the experience, strategies and ability to "fiddle around" with mathematical ideas to achieve these generalisations. Their desire for immediate grasp is strong (i.e., it has always been achievable) and their lack of learned manipulative skill may cause them to give up easily later on (Davis and Hersh, 1981, p.283).

It may also be difficult for teachers to recognise (Kajander, 1989) and thus stimulate (Taylor 1985, p.1) mathematical creativity in a school setting. A bright child may have the germ of a new mathematical idea, but the teacher must be willing and able to follow it up. This requires both a reasonable mathematical knowledge on the part of the teacher, as well as time. The teacher must also feel comfortable saying "I don't know—let's find out." Not all teachers have the confidence to take such risks (McDougall, in press). In fact, the tendency may be to lead the student toward the "right" or standard answer instead of their own creative but not quite correct idea (Feikes, 1995). Also, if the student's idea is very different from a standard method (we could call this "highly creative"), it may be difficult for the teacher to accept these methods as being valuable (e.g., Henderson, 1996, p. xxiii).

Zack (1995) cites two reasons for her students' improved progress in her classroom, namely, her own exposure to deeper mathematical ideas, and her growth in awareness of "the mediating roles played by peers for each other, such as rephrasing, interpreting, resisting closure, and serving as a receptive audience" (p.106). However, facilitating such interaction may leave the teacher with a sense of loss of control of the class (McDougall, in press).

A problem for teachers of brighter students is that an environment rich in social interaction of an appropriate nature may be hard to achieve. A teacher at *University of Toronto Schools* (UTS), a Toronto school for the gifted, once said to me "if we do nothing else here, we at least bring these students together [to stimulate each other's minds]." Many enrichment programs in classrooms are 'pullout' programs—the students go off by themselves to work on an idea. How do we get these students together? If social interaction is to take place as suggested by the social constructivist perspective, an appropriate social environment with peers of similar ability must be provided.

Kindermath was designed in response to parental dissatisfaction with the lack of creative stimulation their children were receiving in school mathematics. Parents wanted a visual focus, a hands-on environment, social interaction amongst the brighter children, and most of all, they wanted to see their children excited about mathematical ideas. "My child is bright but isn't stimulated in school math" was something I heard often. Having failed a number of times to stimulate interest in enrichment programs through the school system, I decided it was time to create my own enrichment program.

A pilot project was conducted involving four 7-8 year olds. The children (two girls and two boys) worked with the teacher in a group or in pairs for two sessions of an hour and a half in length. At the end of each session parents were invited to participate. After a brief explanation of the day's activities and the philosophy of mathematics behind the activities, parents could work with their child on the computer, play a mathematical game with them, or look at their geometric creations. The response was very positive and an expanded program was planned for 1996/97.

Separate sections were offered for 7-8 year olds, 9-10 year olds, and 11-12 year olds, with up to eight children in each group and two facilitators. Children were encouraged to work in pairs both on one of the four computers and in other tasks.

Sessions were one hour and a quarter in length with a 15 minute "sharing time" at the end, during which the parents were invited to participate by trying some of the days' activities with their child. Weekly take-home activities are described during sharing time, many of which require some parental involvement. This also allows parents the opportunity to "talk about" mathematical ideas with their children.

Several criteria are used for choosing topics for Kindermath. Topics must be fun and interesting, and lend themselves to collaboration as much as possible. Visual elements are also kept in mind. An underlying goal is that the projects hint at deeper ideas in mathematics as well as its underlying beauty and structure, and an attempt is made to choose topics with this in mind. For example,drawing polygons with a larger number of sides until the polygon resembles a circle hints at the idea of limits and the formal definition of a circle. Cutting out fractal cards gives a sense of the inherent beauty and infinite patterning of structures in fractal geometry. Escher tilings allow children to create visual patterns of their own with some geometric beauty and symmetry.

The program has benefitted from great enthusiasm from my students in a mathematics course for elementary education majors, several of whom have asked to be able to observe and facilitate sessions. Since they have already been exposed to the social constructivism perspective in mathematical learning through my course, they are ideal partners in the venture. Not only is it helpful to the children involved to have additional facilitators, but it provides an ideal forum for these pre-service teachers to experience some alternative learning strategies in a supportive environment. In other words, Kindermath provides a forum for trying some alternatives to traditional learning methods without being evaluated by an associate teacher who may or may not agree with such ideas.

Kindermath may prove a more useful environment for studying mathematical creativity than the regular classroom. I have found two difficulties with encouraging mathematical creativity in classrooms in the past. The first is that once students have been shown the traditional method of doing something, they appear to be less likely to come up with novel alternatives than if they were shown no methods. The second is that students who know that they will be evaluated with traditional measures at the end of a program appear to be less likely to engage in exploratory activities than they are if the explorations themselves are evaluated (Kajander, 1989). Kindermath avoids both of these difficulties.

As with any enrichment program, it is hoped that the field tested ideas will become simply good mathematics learning environments, and that Kindermath will have some use and influence in an improved system of learning mathematics in school.

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Ad Hoc Session 2

STUDENTS' DIFFICULTIES WITH THE NOTION OF ISOMORPHISM: SOME PRELIMINARY RESULTS[1]

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The first abstract algebra class in the undergraduate mathematics curriculum confronts students with special difficulties, even though their failure rate may not be very high. Informal observations indicate that even those who pass the course may harbour an uneasy feeling of not really understanding what it was all about. Some claim that they were only able to make sense of it years later. Many students complain that the course is "abstract" (small wonder since it deals with *abstract* algebra!) and "difficult," and develop a dislike for abstract mathematics. Such a reaction is no doubt due to the fact that this type of course is among the first which deal with formal systems rather than numbers or geometric figures and which rely heavily on proofs rather than computations and constructions. Some might argue that the course acts as a test, serving to select the students who enjoy formal mathematics and have the potential for becoming mathematicians. As mathematics educators, however, we are curious to learn in greater detail what exactly students find difficult or unappealing about abstract algebra. We hope that such insights will eventually help teachers foster achievement and enjoyment among greater numbers of their students.

The research questions that we have set ourselves are the following:

a) What are the difficulties that students encounter concerning the first notions of group theory (group, subgroup, isomorphism, etc.)?; and

b) How can these difficulties be explained?

Let us clarify what we mean by "difficulty," for this word is used in two rather different ways. On the one hand, difficulty refers to the quality of things that are difficult, and in this sense difficulty belongs to (difficult) concepts. On the other hand, a difficulty may be something that a person has, as in "having difficulty in breathing" or "having difficulty in understanding a concept." Of course the difficulty of a concept is always relative to an individual: the idea of quotient group, for instance, may be difficult for a beginning student or may have been difficult for a 19th century mathematician, but it is easy for contemporary algebraists. However, we believe that the notion of difficulty is not totally subjective: some concepts are "objectively" more difficult than others. In spite of the fact that just about anything can appear easy once it has been grasped, there exists something like "intrinsic" difficulty. We consider these two viewpoints (difficulty residing with the concept or with the person) to be complementary, and we intend to integrate both within our research. Accordingly, our method must be twofold: we must carry out a theoretical conceptual analysis of the notions that we wish to study, and we must also make empirical

[1] Parts of this article have been published in French in Lajoie and Mura (1996).

observations of students confronted with these same notions. In the following, we provide a small sample of both approaches toward understanding the difficulty of the notion of isomorphism.

1. A BRIEF CONCEPTUAL ANYLYSIS OF THE NOTION OF ISOMORPHISM

The notion of isomorphism is central to abstract algebra. Most properties of interest in any algebraic theory are those which are invariant under isomorphisms, and in most situations isomorphic systems (groups, rings, etc.) are viewed as one and the same object. Historically, the idea of isomorphism was instrumental in the transition from "concrete" theories of groups (permutation groups, transformation groups, etc.) to a general abstract group theory, a change in perspective that took hold gradually—and not without resistance among the mathematical community—over the second half of the 19th century (Kleiner, 1986, p. 208-210).

Let us consider the definition of this important notion:

An isomorphism between two algebraic systems is a one-to-one correspondence between the underlying sets that preserves the algebraic operations.

On the basis of this definition we can already see that we are dealing with a difficult concept, combining the set-theoretical idea of one-to-one correspondence (which is rather difficult in itself, since such a correspondence must satisfy four distinct but easily confused conditions—namely, that each element of the first set is associated with one and only one element of the second set and that each element of the second set must likewise be associated with one and only one element of the first set) and the algebraic idea of preserving the operations. It should come as no surprise that some students make the mistake of concentrating on only one of these two aspects.

It should be noted that the term "isomorphism" is also used in a more general sense, as in the title of the present article, where it does not refer to any specific mapping. Or, if we know that two groups are isomorphic, we may speak of "their isomorphism"—meaning, "the fact that they are isomorphic"—without referring to any specific isomorphism between them.

Closely related to the concept of isomorphism is the concept of isomorphic systems, that was previously mentioned. It is defined as follows:

Two algebraic systems are isomorphic if there exists an isomorphism between them.

In less formal language, it is customary to say that two algebraic systems are isomorphic if they are essentially the same—i. e., if they have the same structure, although their elements and operations may differ in nature. While this description is probably a salient feature of an expert's image of the concept, we cannot assume that beginners will find it more instructive than the formal definition, for what does it mean to be "essentially the same?" What are the "inessential" features that one must learn to disregard? Such questions are at the heart of the process of abstraction.

The formal definition of isomorphic algebraic systems, compared to the definition of isomorphism, introduces a new layer of difficulty due to the conditional existential quantifier "if there exists...." Leron, Hazzan and Zazkis (1994, p. 153) have observed that some students act as if they took this phrase to mean "there exists a unique function" or "there is a canonical, algorithmic way to construct a function." We too have found evidence of this kind of misinterpretation, together with a few others as well, some of which were quite unexpected. Although our main data collection has not yet begun, our preliminary observations have already provided some fresh insights into the difficulty of these definitions.

The preliminary data that we have collected so far are of two kinds: exam questionnaires and recordings of pilot interviews. We shall present here an example taken from each of these two sources.

2. EMPIRICAL DATA: EXAM QUESTIONNAIRES

We have collected over 300 exam questionnaires from ten different exams given in three universities. All the exams belong to the first course in abstract algebra that must be taken by students enrolled in an undergraduate program in mathematics. Depending on the university, this may be a first- or second-year course.

We shall discuss here three different responses to one of the exam questions.[2] The question read as follows:

Are the fields (Q,+,.) and (R,+,.) isomorphic? Justify your answer.[3]

Fourteen out of 48 students (29%) answered correctly. The expected answer was that no isomorphism between the two fields can exist since the set of rational numbers (Q) and the set of real numbers (R) have different cardinalities. It may be remarked from the outset that the question is difficult on two accounts: first, one must know the result concerning the cardinalities of Q and R (it is not the kind of result one may hope to reconstruct during an exam!), and second, one must think of using it in a context (an algebra exam) where one would expect to perform algebraic manipulations rather than make use of purely set theoretical considerations. In fact, the students who were asked this question had been taught the result concerning the cardinalities of Q and R and should have been prepared to see it featured on the exam.

This question prompted several interesting responses. Let us look at three examples.

Answer #1
No, because it is not bijective.
injective: yes, surjective: no ($\sqrt{2}$, Π,...)[4]

Answer #2
The function is unknown, therefore it is not possible to say whether it [the function] is isomorphic or not.

Answer #3
(Q,+,.) and (R,+,.) isomorphic??
homomorphic + bijective = Isomorphic.
(Q,+,.) is not isomorphic because addition is not injective.

$f(2,4) = 6$

$f(1,5) = 6$ and $f(2,4)\ \pi\ f(1,5)$

(R,+,.) is not isomorphic because addition is not injective.

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[2] For further examples, see Lajoie and Mura (1996).

[3] The question and the answers, as well as the interview excerpts that follow, have been translated from the French original.

[4] We have strived to reproduce the answers as faithfully as possible, including their layout. Each answer is reported in its entirety, without any omissions.

What does "it," in the first line of answer #1, refer to? We may surmise that it is the mapping called "inclusion," by which each element of Q is assigned to the same element considered as an element of R.

However, the author of this excerpt did not bother to state this, as if he or she believed that inclusion was the only possible function between Q and R, or the only one that it was necessary to consider in order to decide whether the two fields were isomorphic. It appears as if the student concluded that the two fields were not isomorphic on the basis of the observation that inclusion is not surjective (for numbers like $\sqrt{2}$ and Π lie outside its image). The conclusion is correct, but the reasoning is not, since there are infinitely many other mappings from Q to R, and one has to prove that none of them is an isomorphism in order to conclude that the two fields are not isomorphic. This student's behaviour is similar to the one reported by Leron, Hazzan and Zazkis (1994) which we quoted earlier, and it may be explained by their hypothesis that the student believes that there is a unique, canonical function (in this case inclusion) responsible for making two systems isomorphic or not. The same behaviour may also be explained by the following alternative hypothesis: without necessarily believing that only one (canonical) function must be considered in order to decide whether two systems are isomorphic, the student may simply believe that the existence of a (reasonable) function that is not an isomorphism is sufficient to show that the two systems are not isomorphic. Unfortunately, the data at our disposal do not allow to test these hypotheses.

In a sense, answer #2 exhibits a point of view opposite to the previous one. Far from taking for granted the mapping to be considered, the author of this excerpt asserts that it is impossible to answer the question because "the function" is not given. Other students gave similar answers, for example: "it depends on the application *f* that is given," and "I cannot say whether these 2 fields are isomorphic because I do not have any function linking them together." These students act as if they believed that being isomorphic is not an intrinsic property of a pair of algebraic systems, but a property of two systems linked by a particular function. Thus the same two systems could be isomorphic or not depending on the function being considered. This conception in turn may be explained by a misinterpretation of the conditional existential quantifier in the definition of isomorphic systems: "<u>if</u> there exists an isomorphism" may have been replaced by "<u>when</u> there exists an isomorphism" ("when" implying that the two systems may <u>at times</u> be isomorphic and at other times not) and understood to mean "when an isomorphism is given."

Answer #3 is rather surprising. It gives the impression that its author was trying to decide whether each field was isomorphic in and of itself, independently of its companion, as if being isomorphic could be a property of <u>one</u> algebraic system, like being finite, cyclic or commutative. Is answer #3 simply the effect of a student panicking during an exam and writing phrases at random? Again, alas, we are unable to satisfy our curiosity on the basis of a written questionnaire.

3. EMPIRICAL DATA: INTERVIEWS

Interviews offer an opportunity to probe further into students' thinking. We shall present here three excerpts taken from an interview with a second-year mathematics student.[5] In order to preserve her anonymity, we shall call her "Brigitte." At the time of the interview, Brigitte had passed her required abstract algebra course with a final grade of C^+, and was enrolled in a further elective course in group theory. She said that she had found the algebra course relatively easy compared to her other courses.

[5] All interviews comprised two sessions of approximately one hour each. They were videotaped and transcribed. The excerpts presented here belong to the first session.

Excerpt #1

Caroline: How would you explain to a first-year student what "isomorphic groups" are?

Brigitte: I don't know what they are.

Caroline: It doesn't mean anything to you? Nothing at all?

Brigitte: Maybe, hold on... An isomorphic group... means that it is a homomorphism... which is bijective. This means that... It has nothing to do with homomorphism. The homomorphism is $f(a)$... with $f(b)$... it belongs to... it doesn't work, it's no use showing... homomorphism. You have to prove that it is bijective. Maybe if G, if your group is commutative, it is isomorphic. Is that a possibility? I don't know. I would put that out as a hypothesis.

Caroline: If your group is commutative, is it isomorphic?

Brigitte: Because of bijectivity, because an isomorphism is a homomorphism which is bijective, but in the group... Of course, a group is a homomorphism in itself, unto itself.

As with the author of answer #3 above, Brigitte spoke of "an isomorphic group," in the singular. For her, "being isomorphic" seems to be a property of one structure rather than a relation between two structures. Her remarks are as nonsensical as talking about one identical object or one parallel line. This exchange with Brigitte has made us realize that familiarity with the ways in which mathematical terms can or cannot be used linguistically is part of understanding mathematical concepts. "Isomorphic" belongs to a small family of relational attributes, like "identical," "equal," "equivalent," "similar," etc., which cannot be applied to objects by themselves, but only to objects in relation to each other. For experts, this is almost a linguistic reflex, but obviously it must be acquired, it cannot be taken for granted. Students who hear the expression "isomorphic groups" may not be aware that it must be understood "to each other," and may treat it as functioning in a way analogous to, say, "cyclic groups." Then, by analogy with "a cyclic group," they may (eventually) arrive at "an isomorphic group." In French, the difficulty may be compounded by the fact that in spoken language the singular and the plural sound identical, so that one may become accustomed to hearing the phrase *"groupe(s) isomorphe(s)."*

As can be seen in excerpt #1, Brigitte went on to advance a hypothesis as to what "an isomorphic group" might be: she suggested that a commutative group is isomorphic. Where does this astonishing idea come from? Notice that only prior to formulating this view, Brigitte had been struggling with the notion of homorphism and had come to the conclusion that she had to prove that something was bijective. It is precisely at this moment that she introduced the notion of commutativity. Possibly, she associated bijectivity and commutativity, since both notions imply some kind of reciprocity, symmetry or reflexivity.

The following dialogue took place earlier in the interview. It opens with the first question that we asked Brigitte concerning isomorphic groups. The student had three groups of order four in front of her.

Excerpt #2

Caroline: Are some of these groups isomorphic to others?

Brigitte: Isomorphic ...

Caroline: [Laughs] You gave me a funny look!

Brigitte: [Laughs] [...] I'll tell you, the definition of isomorphic ... It is a homomorph... An isomorphism, no homomorphic ... Now, does isomorphic mean that there are the same number of elements?

Caroline: Does isomorphic mean that there are the same number of elements?

Brigitte: I don't remember if that's what it is.

Caroline: A little while ago, you told me what an isomorphism was...
Brigitte: An isomorphism is a homo... a bijective homomorphism. Does it mean that isomorphic has something to do with an isomorphism?

Caroline: Yes.

Brigitte: A homomorphism is... I have to find a group that's going to be... homomorphic to the other one?

Caroline Ok. You don't remember how...

Brigitte: Isomorphism...

Caroline: ... how to verify that [two] groups are isomorphic?

Brigitte: No.

[...]

Brigitte: I'll tell you... I can tell you whether a homomorphism is an isomorphism, but I can't tell you...

Caroline: Really?

Brigitte: To do so, I'll check whether it's bijective, I will say it is bijective, so then it's an isomorphism, but I can't tell you what makes two groups isomorphic to one another. I don't know.

Like many students, Brigitte was baffled by the task of deciding whether two groups were isomorphic when no mapping was provided. "Having the same number of elements" seems to be the first idea that occurred to her, maybe simply because this property does not depend on a function, or maybe because she remembered this property as having been featured in some theorem or problem involving group isomorphism.

When Caroline mentioned the word "isomorphism," Brigitte asked whether "isomorphic" had something to do with "isomorphism," a very surprising reaction, since the two words sound so much alike! The next excerpt confirms that Brigitte indeed had not made the connection between the two ideas.

Excerpt #3

Caroline: Here is a suggestion that one of your fellow-students might offer a beginner to help him decide whether two given groups are isomorphic. I would like you to comment on it:

"If one of the groups is multiplicative and the other one is additive, then they are not isomorphic."

[...]

Brigitte: I don't know what it means to say that [two] groups are isomorphic. [...] You have two groups: does he give you a function to show... with them? When we saw homomorphism [in class], we always had a function and it's with that function that we checked whether if it was a homomorphism. Here, if you give me two groups, is there a function that goes with them?

Caroline: Let's say that here, you don't have any.

Brigitte: You don't have any. I wouldn't know what to do.

Caroline: You wouldn't know what to do. And if you had one?

Brigitte: Well, that may help... Isomorphic, the way I see it, it means 'almost the same,' 'almost... alike.' That's what isomorphic makes me think of.

Caroline: 'Almost the same,' you mean the groups?

Brigitte: Yes.

Caroline: Almost the same?

Brigitte: Now, maybe saying that they're... Since one is multiplicative and the other one is additive, it means that they are not the same, for sure they can't be... they don't have the same law and maybe they don't look alike...

Caroline: When you say 'almost the same'...

Brigitte: This is what isomorphic brings to my mind... Isomorphic doesn't make me think of isomorphism, it makes me think of... 'almost identical.'

In this exchange, Brigitte gave clear evidence of her discomfort with discussing isomorphism when no function is given between two groups, an attitude similar to the one expressed by the author of answer #2 in the previous section. At the same time she rather inconsistently claimed that for her the idea of isomorphic groups did not evoke the idea of isomorphism, but instead made her think of "almost identical." By itself, the last assertion could be interpreted as evidence that Brigitte has the right intuitive understanding of the concept; however, the rest of the interview, in particular the dialogue reproduced in excerpt #1, suggests the opposite.

4. CONCLUSION

One of the uncontested results of the research carried out over the past decades in mathematics education has been to document the extent and depth of elementary and secondary school students' misunderstanding of mathematics. Could a similar phenomenon be observed even among university students who have chosen mathematics as their field of specialization? The preliminary results that we have reported raise the possibility that the gap occurring between what is taught and what is learned in undergraduate algebra may be greater than most teachers suspect. It is sobering indeed to think that some students conceive of one isomorphic group alone or of two fields that may or may not be isomorphic depending on the occasion. We were surprised, to say the least. Was our surprise justified, or should we have anticipated that constructing new concepts does not get any easier with age or with experience, and that "elementary" concepts are such only from an advanced standpoint?

Over the course of exchanges with student participants, we were also able to verify another finding of mathematics education—namely, that concept images[6] are usually made up of contradictory components. Brigitte (and she was not exceptional) managed to verbalize several inconsistencies during an interview which lasted but one hour. Different ideas sprang to her mind in response to what we said or as the result of her own chains of association. Often she was unaware of the inconsistencies, or, if she was, they did not make her overly uncomfortable. Research might be easier if, at a given moment, each individual held one clear-cut coherent conception of a given notion, and if we could study and describe it as though it were a well-defined object. However, reality is not like that, and we have to live with the fuzziness of... life!

Finally, our preliminary observations have highlighted the complexity of the notion of isomorphism. In particular, both the exam questionnaires and the interview data indicate how much harder it is to determine whether two algebraic systems are isomorphic than to verify whether a given mapping is an isomorphism. In the latter case, one has to check whether the mapping is bijective and whether it preserves operations, a task which is essentially algorithmic. By contrast, in the first case, one must start by choosing between two possible goals: showing that the two systems are isomorphic or showing that they are not. Later, should one's efforts prove fruitless, one must decide either to persist or switch to the opposite goal. There is nothing to guide these decisions but intuition and heuristic considerations. Furthermore, whichever goal is chosen, the task is far from being algorithmic in nature. In order to prove that two systems are isomorphic, one generally (but not necessarily) must construct an isomorphism: a task for which no recipe is readily available. As to the opposite goal—i. e., proving that two systems are not isomorphic—it requires proving that no isomorphism can exist between them—i. e., that no bijection can exist or that no bijection preserves operations. This is usually achieved indirectly, by pointing to some difference between the systems that could not exist if they were isomorphic.

In the light of the foregoing analysis, it is not surprising that beginning students confuse the two strategies, try to make them fit some algorithmic mold (e. g., a canonical mapping, like inclusion), or reject the task completely, declaring it impossible and demanding that a function be given together with the two systems. We suggest that students would benefit from extended experience with this kind of task and from being made explicitly aware of its nature and intricacy.

ACKNOWLEDGMENTS

We would like to express our gratitude: to the faculty members of three mathematics departments who, on a basis of anonymity, allowed us to consult their students' exam questionnaires; to "Brigitte" and to all the other students who agreed to be interviewed; to the many individuals who, on various occasions, listened to our public presentations and shared with us their thoughts on our data; and to Donald Kellough for editing this paper.

[6] For a discussion of the notion of concept image, see Tall and Vinner (1981).

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Ad Hoc Session 3

WHEN SILENCE SAYS IT ALL: AN EXPLORATION OF STUDENTS' MATHEMATICAL TALK

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The issue of students' mathematical talk is addressed widely in the mathematics education literature (Richards, 1991; Pirie 1991). In the current climate of reform, emphasis is placed on encouraging students to discuss mathematics in pairs or groups (NCTM, 1989, 1991). Naturally, researchers hoping to collect data on students' mathematical talk may choose to focus their tape-recorders and video-cameras on students who are gregarious, or who, at least, can be relied upon to talk. While I have sometimes employed these same tactics myself in gathering video-taped classroom data, my attention has increasingly been drawn to the quieter students.

Noddings (1982) has pointed out that we know little about the thought processes of quiet children and that maybe they learn quite a lot in group discussions. Perhaps the time has come for us to look harder at such children. In this paper, I propose not only that we look harder at them, but that we listen harder to them, too. Davis (1996) is leading the call for a movement towards a more participatory, hermeneutic listening in the classroom, a stance which I suggest also extends to researchers who study those classrooms. In lending an ear to (rather than turning the spotlight upon) these quieter students we must consider their reasons for choosing to remain silent.

THE STUDY

The classroom episode described here was video-recorded as part of my research into students' understanding of mathematics. The data were collected in my own classroom in a British high school at a time when I was a full-time teacher of mathematics, simultaneously engaged in a study of my own practice. Three pairs of Grade 7 and 8 students (corresponding to Grades 6 and 7 in North America) were video-taped during several weeks of mathematics lessons, and each student was then video-taped in a one-to-one interview with me at the end of the series of lessons. During the research project, which focused on students learning to formulate and manipulate algebraic expressions (Towers, 1994), the students were introduced to the notion of formulating algebraic expressions within the context of perimeter and area. The episode described here focuses on the attempts of two students, Kayleigh and Carrie (North American Grade 6), to find the perimeter of the shape shown in Figure 1. Working together, these students had earlier demonstrated their ability to cope easily with such a shape (again with missing values) when all the given information was numerical, not algebraic. When presented with this problem, Kayleigh immediately noticed that there was a missing value (the horizontal section) and articulated her problem as "We need to take c and b off twelve". Their conversation continued:

C: What happens if we go ten...

K: We don't know that length so we don't know what to do.

C We do, 'cause that's twelve and then that's like what you said...oh...yeah, I see what you mean, see what you mean....oh, it's just...

K: Well, how are we supposed to know what c....what the value of c and b is?

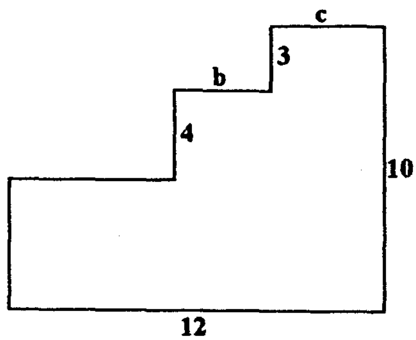

Figure 1

At this point both girls raised their hands to ask for help. Kayleigh articulated their difficulty to me as "We know we've got to take c and b from the value of twelve to get that [pointing to the horizontal line section with the missing value] but how do we write that 'cause we don't know the value of c and b to take away from twelve". Together we started to work on how that might be done:

K: So if we make all that twelve there... [pointing to the three upper horizontal sections]

JT: It certainly is. It certainly is all twelve, isn't it? The c and the b....

I believed at this point that Kayleigh had 'seen' that she didn't need to use the b and the c at all, and that she simply needed to add two lots twelve, but Kayleigh interrupted:

K: Wait a minute, if we did twelve add ten add four add three is what, is....sixteen....nineteen....

K&C: Twenty-nine.

K: So if we had twenty-nine add one...well, let's just say add twelve...no, that doesn't make sense does it? We need to make...add another twelve to...

JT: Twelve to where? Which bit's twelve?

I had two reasons for asking this question. Firstly, I wanted to clarify for myself that Kayleigh really had realised that the three upper horizontal sections together were equal to twelve units, but also I hoped that by insisting that she consider this relationship again, Kayleigh might this time *hear* the significance of my continued interest in this feature of her explanation.

K: Along there [pointing to the three upper horizontal sections]

C: Along there 'cause if you'd pushed it all up....

K: And then we're going to take away c and b.

At this point I realised that Kayleigh knew that the upper three horizontal sections totalled twelve, but was only using this information as a means to work out the missing horizontal value, so I did not push the notion any further and we continued to work together to find an expression for the perimeter of the shape.

JT: OK, yes that's fine, I'm happy.

K: Twelve and twenty-nine....and twelve and twenty-nine is going to give us our grand total which is....

C: Forty-one.

K: Forty-one....ah, and then you have to take away c and b. So, it's forty-one take away one c plus....[writing 41 - 1c + 1b]. Does that make sense?

JT: Well, you've taken away c but added on b at the moment. [Kayleigh adds brackets to her expression to give 41 - (1c + 1b)] Yeah, that's more like it. Can you see what she's done?

C: Oh, yeah, yeah.

At this point we had accounted for all of the sides with numerical values attached, and for the horizontal line section with the missing value. I then attempted to bring into play the vertical line section with the missing value which seemed to have been overlooked, and the girls quickly accounted for this value by adding three on to their current expression to give 41 - (1c + 1b) + 3. Things then became a little complicated. Kayleigh believed that everything had been accounted for, but there was still the matter of the sides marked b and c on the diagram.

JT: But you've not added on the c and the b yet, have you?

C: Yeah!

K: Yeah, but if we had....

JT: No. Hang on, let's think. All you've done is you've added all the number bits that you could....

K: Oh, right. I see....

JT: Including this three [referring to the vertical line section with the missing value] and then you've found that missing length [pointing to the horizontal line section with the missing value] which is the twelve take away the b and the c....

K: So, we need to add....and now we need to add....

JT: So we still need...

JT&K: The b and the c.

At last everything had been accounted for, and I moved away. The students' final expression for the perimeter of the shape was: 41 - (1c + 1b) + 3 + 1b + 1c. During the one-to-one interviews, both girls were presented with the following two diagrams (Figures 2 and 3) and asked to find the perimeter of each. In a manner consistent with the method she articulated in class, Kayleigh solved the first of these problems by finding the missing values on each of the sides and then totalling where appropriate to reach an answer of 14p + 20x. This solution method was inadequate on the second of these problems, and Kayleigh initially could not solve it.

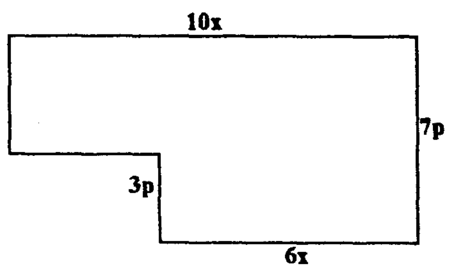

Figure 2

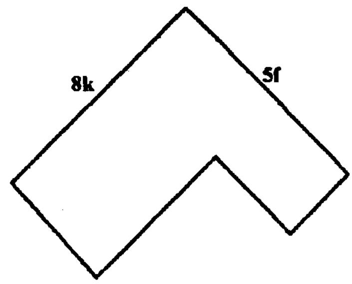

Figure 3

With help, she saw that it was not necessary to know the values of the separate line sections and was able to produce an answer of 10f + 16k. Carrie, on the other hand, wrote down a correct solution to the first problem (20x + 14p) in complete silence and so quickly that an observer is left with the distinct impression that she did not work out the missing values, but instead added two lots of 10x and two lots of 7p. This impression is further reinforced by her rapid solution to the second problem (16k + 10f), again completed in silence, with no hint that she might have found this problem in any way a challenge.

DISCUSSION

Issues concerning how I, as the teacher, rather than privileging the more articulate Kayleigh, might have listened differently in the classroom episode presented here, and the consequences of that altered mode of attending, are addressed elsewhere (Towers, in press). In that paper I critique my own listening, and, while acknowledging the difficulties raised for the busy teacher, suggest that teachers must adopt a more *hermeneutic* (Davis, 1996) orientation to listening. My claim is based on the notion that there were indications in the classroom interaction which should have alerted me, as the teacher, to the possibility of Carrie having an alternative strategy. This, in turn, is based on the assumption that Carrie may have been using a more efficient alternative strategy during the interview. As Carrie did not speak while solving the problems given in Figures 2 and 3, however, one might raise the question of whether it is valid to make (even tentative) assumptions about students' understanding based not on what they say, but on what they *don't* say. (It should be noted that although Carrie's silence is compelling, it is not the only evidence on which the assumption about her solution method is made. Body language, speed in reaching a solution, and other factors also play their part.) However, if we wish to assume that Carrie was using the more efficient alternative strategy in solving the problems given in Figures 2 and 3 (and it is difficult to see how she could have solved the problem in Figure 3 so rapidly in any other way), and that she was in the process of formulating that strategy while working on the problem given in Figure 1, we must consider why she might have chosen not to reveal that strategy earlier.

Though Easley and Taylor (1990) suggest that children seem to be able to communicate novel ideas to each other more easily than to the teacher, such communication is only possible if there is a "listener as well as a speaker" (Pirie, 1996, p. 105). The opening dialogue suggests that Carrie tried, and failed, to persuade Kayleigh to listen to her. Had Kayleigh been prepared to listen, rather than insisting she be heard, both students may have benefited from Carrie's emerging image.

Carrie may have been further frustrated in her attempts to formulate her method by my continued interest in helping Kayleigh to find the missing horizontal value. Though Carrie may have seen that search as superfluous, the fact that the teacher continued to show interest in it may have been enough to make her unsure whether her own method was valid, and therefore unlikely to present it as an alternative. Rather than *hearing* Carrie's silence as a reluctance to speak, however, perhaps, instead, we ought to ask ourselves whether she was being silenced by poor listeners. Davis (1994) refers to such listeners as those people with whom we are all acquainted who, when we talk with them, give us the uneasy feeling that even though we are within the interaction, we are not part of it. I suggest that this was Carrie's experience in the classroom episode presented here, a notion supported by observation of Carrie's body language on the video-tape. I am thereby fully implicated in Carrie's silencing. Clearly, my mode of listening was not hermeneutic (Davis, 1996). With no example to follow, it is little wonder that students like Kayleigh are unable to adopt this orientation themselves. It is, therefore, imperative that teachers model the kind of listening behaviour they wish to encourage in their students.

It is also interesting to consider Carrie's acceptance of the expression 41 - (1c + 1b) + 3 + 1b + 1c as a complete and correct solution to the problem shown in Figure 1. If, as I suggest, Carrie was in the process of formulating a more efficient solution method, which in this case would have involved adding together two lots of ten and two lots of twelve, she might have been expected to object to this complicated expression as a final answer. We should remember, though, that, for the reasons elaborated above, Carrie's belief in her method may have been on somewhat shaky ground by this point in time. Not only was I continuing to signify the importance of the missing horizontal value by taking a great deal of time to make sure it was included in the expression, but also, at the moment that the students produced their expression, I moved away. This action seemed to be interpreted by both students as indicating that the end had been reached, and they paid no further attention to the expression they had produced. It appears that rather than giving them the opportunity to discuss the expression further, my action closed down the conversation. If Carrie had any lingering doubts about the completeness of their answer, she seemed to take my leaving as a cue that the problem was finished to my satisfaction. It is important, therefore, to realise that what teachers *don't* say is often as strong a cue for students as what they do say.

CONCLUDING COMMENTS

Though I have been somewhat critical of my own practice in analysing this episode, I take heart from the fact that however poorly I listened, Carrie still appeared to hold on to her image strongly enough to be able to put it to use when left to her own devices in the interview. In doing so she has taught me a great deal, and my hope is that this paper will inspire others to listen to the quiet students, for they have a great deal to say.

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Ad Hoc Session 4

"WHY DOES A LETTER ALWAYS ARRIVE AT ITS DESTINATION?": OPENING UP LIVING SPACE BETWEEN PROBLEM AND SOLUTION IN MATH EDUCATION

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I am interested generally in opening up our culture's widely-held tacit preconceptions about mathematics education. I want to suggest new ways of making our view of mathematics broader and our teaching more inclusive. In particular, I am interested about opening up spaces for mathematical exploration.

Mathematics education, as much as it emphasizes the existence of problems, has traditionally entailed a race to the end of those problems—an emphasis on solution, on dissolution of the problematic. In most mathematics classes, students are presented with a great many problems and required to find a solution to each as quickly as possible, then to discard both problem and solution. The rhythm of many school mathematics classes, with their repetitive cycle of *problemSolution, problemSolution, ...* does not allow for a space to linger in the messiness of the unknown, the paradoxical, the problematic. There seems to me to be an impulse towards cleanliness; exclusion of the ambiguous, dreamlike, poetic or uncertain; and a wish for finality, closure, self-containedness, crystallinity. *Desire* in traditional mathematics education seems to be for an immediate closing down of messy living spaces, foreclosure on the unknown.

In this paper I would like to argue for the possibility of an alternative approach, one which allows for an opening of exploratory space between problem and solution, for allowing lingering in a space of uncertainty and ambiguity as a legitimate practice in math classes. I will frame my argument in terms of an idea taken from Jacques Lacan's psychoanalytic critique of culture.

Lacan's work is well known in France, where it has been an important source for much of the work of post structuralist and "new French feminist" philosophers like Hélène Cixous, Luce Irigaray, Jacques Derrida and others. His work has lately become influential in the rest of Europe and North America as well. Lacan's work is a reinterpretation of the most subversive and revolutionary aspects of Freud's writings, particularly the concept of the unconscious, and extends the scope of psychoanalysis to a general framework for understanding culture. His writing is notable for its densely-written, teasing style full of tremendously generative aphorisms, the most famous of which is "the unconscious is structured like a language."

In this paper, I will focus on another of Lacan's aphorisms, "a letter always arrives at its destination." This idea has multiple interpretations as elaborated by Lacan and those influenced by him. For Lacan, the idea of "the letter" was related to his reading of Edgar Allan Poe's story *The Purloined Letter*, in which a stolen letter takes on significance according to the relationships of the three people who obtain it, although the letter's content is unknown to all of them and to the reader. For Lacan, "the letter" came to mean the message with no fixed meaning, a purely formal signifier without a stable relationship to any signified, which could be related to the intrinsically meaningless letters of our alphabet as well as to the "writing" of the unconscious which does not attach in any fixed way to objects.

For my purposes, one of Slavoj Zizek's interpretations of "a letter always arrives at its destination" was most useful. Zizek, a researcher at the Institute for Social Sciences in Ljubljana, Slovenia writes about Lacan's theories in relation to film, popular culture and contemporary politics. In *Enjoy Your Symptom*! (Zizek, 1992) he offers three possible interpretations of Lacan's "letter which always arrives at its destination":

1) an approach pointing to "imaginary misrecognition," in which one misrecognizes oneself as the addressee of a message and accepts it as having uncannily arrived at the right place (rendered succinctly by Barbara Johnson as quoted in Zizek (1992) as "A letter always arrives at its destination since its destination is wherever it arrives." (p. 10))

2) a symbolic interpretation, in which "the sender always receives from the receiver his own message in reverse form," and "the repressed always returns" (p. 12). At this level of interpretation, the letter or message always says more than it intends to, and the unintended effects of a signifier cannot be known until its consequences are enacted—that is, "there is no repression previous to the return of the repressed" (p. 14).

3) a further symbolic interpretation related to the Lacanian concept of the Real—that the "letter which always arrives at its destination" can be interpreted to mean that "one can never escape one's fate," or "the symbolic debt has to be repaid" (p. 16). It is this third level of interpretation I would like to examine here.

Zizek says that the letter which always reaches its destination for all of us, the fate which we all must meet, is our own death, "the letter which has each of us as its infallible addressee" (p. 21). We are all aware of our own mortality from a young age, and in some way we are always aware of the closure, the end that no one can evade. Zizek points out the ambiguity of the English word "end", which indicates at once "goal" and "annihilation", and relates it to Derrida's emphasis on the lethal dimension of writing, where "every trace is condemned to its ultimate effacement" (p. 21). He quotes the story of the Iranian president Ali Hamnei, speaking about the death sentence on Salman Rushdie: "the bullet is on its way," he said," sooner or later, it will hit its mark." The same could be said for all of us—that as soon as we are born, the bullet with our name on it has been shot, and will eventually reach its mark. Yet it is in that space between the bullet leaving the gun and reaching its destination that we live.

I want to draw attention to this living space, between the birth of things and their end, in education generally and in mathematics education in particular. A space for living is a space of uncertainty, ambiguity and multiple interpretations. Dwelling in this space involves taking chances; it is messy, it means messing in with the stuff of life. This messiness can be horrific because it is amorphous. Zizek characterizes the Lacanian Real as a source of existential horror, a "grey and formless mist, pulsing slowly as if with inchoate life" (Zizek, 1991: 14). Yet contact with the messiness of the Real is also the source of possibility and renewal, inspiration and life.

The emphasis on rapid and repeated closure in mathematics education can be seen as a desire for the letter to arrive as quickly as possible at its destination—in Zizek's interpretation, as a wish for some form of death. The wish to solve or *dissolve* problems as quickly as possible comprises the death, at the very least, of the problematic, of our interest in the mathematical question, of engagement in making mathematics, and perhaps a kind of death wish or wish for closure in some stronger sense.

Robert Early (1992) has written about students' accounts of their school mathematical experiences in terms of their similarity to images from alchemy. Early uses these alchemical images as a basis for a psychoanalysis of student writings, based on the Jungian idea of universal themes in dream imagery. Early asked his undergraduate students to write about fantasy images which captured their feelings about a recent math problem which had challenged them. Their responses were emotionally highly charged, and included images that Early read as parallel to alchemical/psychological processes such as the *prima materia* (the amorphous primal matter, "worthless and despised, yet full of potential" (p. 16), corresponding to the Lacanian Real); *calcinatio*, the burning off of impurities in the *prima materia*; *mortificatio* , death, shattering or murder; *sublimatio*, an ascendence of spirit in which a new point of view becomes possible; *solutio*, a dissolution or drowning, sometimes also viewed as a baptism; and *coagulatio*, the coagulation of the *prima materia* into something hard, fixed and substantial. Both *solutio* and *coagulatio* were associated with finding "solutions" to a mathematical problem, as in these examples:

> I felt as though I knew everything, then the exam came. Everything changed, my mind went blank... The sweat began to flow out of me like I was a water faucet turned on full blast. Within minutes I was drenched and I felt as though I could swim in my pool of sweat like you swim in the pool at the YMCA. Then things changed, I started to panic, the water was getting too deep. I couldn't move, I felt as though I was chained to the chair, the water kept rising, it wouldn't stop. It was up to my neck and getting higher and higher, faster and faster. The water was now over my head and I couldn't breathe. It was like someone was smothering me. I couldn't do anything about it, no matter how hard I tried there was nothing I could do. Finally it was all over, I was at peace again. I thought I knew everything once more, but little did I really know because I was dead. (Early, 1992: 18)

The image of dissolution is clear here, although it is the student who is being dissolved by the problem, rather than the problem solved by the student. Nonetheless, the implication of "death by solution" is evident.

The *prima materia* of the Real, of the problem, is often seen in these student writings as formless, swirling clouds similar to Zizek's "grey and formless mist." Resolution of the mathematical problem often takes the form of *coagulatio*, coagulation of the formless into a hard, tangible object as in the following example:

> The final try at the problem brought me out of the clouds onto a fixed place and the equation with the correct solution was in my grasp. (Early, 1992: 19)

Early notes the ultimate futility in seeking an ultimate *coagulatio* (in a style reminiscent of Lacan's insistence that the signifier cannot ultimately attach to the signified):

> Only in the easiest problems would *coagulatio* seem to represent final solutions, by the way. To achieve *coagulatio* a problem must be taken literally on some level, its meaning fixed. This limits it, binds it in time. The solution today may not hold tomorrow—either a flaw will be discovered or an extension will suggest itself. *Solutio, mortificatio* or another process may follow. (Early, 1992: 19)

School math classes work at the level of "taking problems literally," fixing meanings and binding them in time, specifically to avoid the recurrence of the Real, the ambiguous, the messy space of living. The desire to solve or dissolve the problem without allowing a space for play involves shutting down the space to think mathematically, to struggle with the ambiguities of the Real, to have patience and courage, and to know as a mathematician that no problem is ever more than provisionally solved.

How could an idea like "dwelling with ambiguity" be translated in the practices of mathematics classes? I think there is no single method that will guarantee this—it is more a question of attitude and an approach. But a willingness to keep a problem alive, with all its irritations and uncertainties, should make a difference. Mathematical problems, like parables in philosophy, religion and literature, point to many possible human worlds, including worlds of lived experience and conceptual, culturally mediated worlds of human-made abstractions and metaphor. If we treated math problems as we do parables, for example, and attended to their multiple associations, resonances and paradoxes, we could live with the problematic and engage with it on many levels simultaneously, before invoking a provisional and temporary closure. I suggest that a mathematics education which provided such a living space would be far more inclusive and relevant, and allow many more of our students to live with mathematics as well.

I propose that we begin to play with the messiness that has long been suppressed in school math teaching, that we acknowledge that the answers are not usually of much significance to anyone, but that interesting questions lead to connections with other questions and to the nature of curiosity and inquiry itself. Rather than requiring students to quickly solve sets comprised of many similar problems, I suggest we take the time to live with a single problem for days or weeks, keeping open that space that lets us be curious and engaged with the problematic without requiring it to dissolve, reach a climax, die.

Talk about finding the Final Solutions, in mathematics or physics, evokes horrific images in terms of genocidal campaigns of contemporary history—and well it should, for such impulses spring from the same desire for sterility and ultimately a world of "everything always already dead."

The dream of an ultimate mathematical solution that would do away with life's messiness and uncertainty is at least several hundred years old and is intimately connected with a fantasy of an all-powerful rationality and determinism. It is perhaps best expressed by the eighteenth-century mathematician Laplace, who dreamed of "solving" the universe and time with a single mathematical formula:

> Given for one instant an intelligence which could comprehend all the forces by which nature is animated and the respective positions of the beings which compose it, if moreover this intelligence were vast enough to submit these data to analysis, it would embrace in the same formula both the movements of the largest bodies in the universe and those of the lightest atom: to it nothing would be uncertain, and the future as the past would be present to its eyes. (Laplace, quoted in Moritz, 1942: 328)

Laplace's dream is of a single mathematical formula which, taking account the state of the universe in a particular present moment, could extrapolate forwards and backwards in time and know all past and future states of the universe. This is the dream of determinism, a dream of a universe "always already dead" because it contains no uncertainty, no unpredictability. Mary Midgley, in her book *Science as Salvation* (1992) notes the futility and desperation felt by physicists who accept a completely deterministic model of the universe. "The Second Law of Thermodynamics is held to ensure that some day the success of the human race [and indeed the entire universe] will end, and this is found intolerable," she writes. "Without permanence, said [physicist] Stephen Weinberg ... 'the more the universe seems comprehensible, the more it also seems pointless.' [Another physicist, Freeman Dyson,] like Stephen Weinberg, thinks that the prospect of an eventual end to human life, however distant, is so awful as to deprive life now of all meaning." (Midgley, 1992: 148, 21) These physicists have felt driven to postulate a rather absurd technologically-achieved immortality for human intelligence, supposedly through transferring disembodied human mind to clouds of interstellar dust after the apocalypse. Midgley writes, " Instead of asking why Weinberg took the meaning of human life to depend wholly on its going on for ever, his colleagues therefore looked for ways of proving that in fact it *will* go on for ever." (Midgley, 1992: 148 - 149) A letter always reaches its destination, even a letter addressed to the whole of our universe. Yet it seems that to deny the living space between the sending of "the letter" and its destination, to fix one's efforts on reaching an end as quickly as possible, is to deny life in all its wonder and horror. Fixity on ends can become a kind of death wish, a suicide, if not literally than at least the death of interest, engagement, pleasure, vitality.

Some students express an appreciation of traditional mathematics classes for just these qualities—they are tidy and orderly, predictable and closed off from the world. One high school math teacher told me that she had gone into mathematics teaching precisely because there was a "right answer" to every problem in math, unlike those messy humanities subjects like English literature and history. Undeniably there is some comfort to students in escaping the messy, chaotic, often incomprehensible world for an hour of tidy rationality, right answers, clear rules and closure in math class. Such comfort is closely related to the satisfaction some people feel in completing crossword puzzles or jigsaw puzzles—although the puzzle may be meaningless in itself, and will be discarded as soon as it is done, there is a strong urge for its achievement, its completion, its closure. Without denying a human need for the occasional tidy moment or "properly" completed task, I question the identification of a truly protean field like mathematics, founded on paradox and dealing in such notions as various-sized infinities, multidimensional spaces and non-Euclidean geometries, with such a closed view.

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Ad Hoc Session 5

WHAT THE "FAILURE" OF THE WHOLE LANGUAGE MOVEMENT CAN TELL US ABOUT THE DESIGN OF A WHOLE MATH CURRICULUM

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I incorporate in this summary my ideas from my brief presentation, as developed and sharpened by the excellent discussion that followed.

Our starting point was an article in the current Globe and Mail by Robyn Sarah on the failure of the whole language curriculum. Two of the points made in the article were:

1) That the amount and the level of the student's reading was less than it might be, and consequently the quality of her writing suffered. In effect, the student was given pen and paper and told to write from the heart, and was praised for whatever she produced.

2) That her technical writing skills were poor, such skills as grammar, sentence structure, coherence, and flow were not given their due place in the classroom.

In short, these criticisms concern the importance of (1) a high artistic standard in the curriculum, and (2) attention to technique. These problems may not have been as serious or as widespread as the Globe article suggested, but they do serve as timely warnings to the redesign of the mathematics curriculum, and I will say more about each of them.

1) In both literature and mathematics, the curriculum should be based on what I might call works of art—problems which embody what is best in the subject—and the student must make a serious study of these. This does not mean that the material need by unnecessarily sophisticated, though I believe that a higher level of sophistication is possible than is currently found in mathematics curricula. [Here, literature has got it right—the students study works which are technically beyond most of them.] Indeed, "art" seems to me to be conspicuously missing from most school math texts. Fundamentally, art is really the only way to convince the student that what he is studying is important, and to encourage him to produce works of high quality himself. In the first instance, the material should be chosen without regard to technical skills, in that it is the art that must drive the curriculum.

2) Given (1), the art will provide a context for the technical skills, which will encourage the student (and the teacher!) to foster them with the care and reverence they deserve. They must be seen as gifts that will allow us to develop our power as artists. They should be developed as needed, but when they arise they must be thoroughly rehearsed. Technique is certainly important, as is memory work and drill, but it is not nearly as important as is commonly supposed to be comprehensive or to cover technical skills "in the right order." In an art-based curriculum, the skills get their context not from their place in a hierarchy of techniques, but from the art which engendered them. This gives them meaning and allows them to be effectively used in the future.

The idea of a whole math curriculum is quite radical, and we are in fact a long way from it, especially in the senior grades and introductory undergraduate years. The first requirement is perhaps for new teaching and learning materials which present a selection of problems (like an anthology) and interpret or "fashion" them in a way that allows them to "work" in a classroom. That presents a challenge to each of us individually, and to CMESG as a group.

CURRICULUM PANEL

WHO DRIVES THE CURRICULUM?

Organizer:

David Lidstone, Langara College, Vancouver

Panel Members:

LaJune Naud, Mount Saint Vincent University, Halifax
Harry White, Université du Québec à Trois Rivières
Gary Flewelling, Arthur, Ontario
Florence Glanfield, University of Alberta, Edmonton
Richard Noss, University of London, United Kingdom

Panelist: LaJune Naud

CURRICULUM CHANGES IN THE ATLANTIC PROVINCES

What forces are driving mathematics curriculum? In the Atlantic Provinces it is politics, politics, politics that is driving towards a common curriculum. There have been numerous statements from politicians about visions of a "world class" curriculum and careful avoidance of the realities of the complexities of developing and implementing curriculum. There have been many statements about the benefits of this common curriculum such as "the outcomes based approach improves learning", "common assessment in the Atlantic Provinces will improve learning" and "common curriculum will cost each province less."

Without discussion of these statements, I will take a few minutes to give an overview of what is happening in the move to a common mathematics curriculum in the Atlantic Provinces.

The Maritime Provinces have been part of many cooperative education projects since 1982 with the establishment of the Maritime Provinces Education Foundation (MPEF). Most of these projects were focused on the development of resources to support courses and programs—not on the development of a common curriculum. In 1993, the Ministers of Education of the Maritime Provinces recommended that the provinces develop common curriculum in core areas (language arts, mathematics, and science) as well as common assessment in these areas. This recommendation was approved by the premiers in April 1994. The clearly stated goal of the premiers was to develop "world class" curriculum. Newfoundland and Labrador became full members of the group in 1995. This group is now the Atlantic Provinces Education Foundation (APEF) with common curriculum in mathematics being developed at three levels (P-6, 7-9, 10-12). All three levels are under development at the same time with each level being developed by a different province. There are very short time lines given for program development, piloting and implementation. These time lines severely limit the opportunites for piloting and for curriculum changes.

At the same time as the beginning of the APEF mathematics curriculum development, there have been school board amalgamation, the elimination of the positions of board mathematics consultants and mathematics department heads in many areas.

For most teachers of mathematics in these provinces the APEF common curriculum development has added one more layer of confusion to the layers of piloted MPEF mathematics units, partially implemented recently published provincial mathematics guides and to the partially completed adoption of new mathematics textbooks and support materials.

Teachers report that they feel left out of the APEF common curriculum initiatives and that the rush to complete the mathematics curriculum development and start APEF assessment leaves no time for teachers to have input into the common curriculum.

The final straw for many teachers is the news of a Pan-Canadian common mathematics curriculum. Teachers report that they feel confused, discouraged and see little hope of help or support. Many have decided to close their classroom doors and wait till it all goes away.

There are several possible roles for CMESG in the move to a Pan-Canadian curriculum. But is CMESG ready to play a role in curriculum leadership? Do we want a leadership role, a seat at the table where the decisions about the Pan-Canadian mathematics curriculum will be made or do we want to respond to the initiatives with one strong CMESG voice? CMESG has a national face that can legitimately address issues related to the Pan-Canadian common mathematics curriculum.

Panelist: Harry White

THE QUEBEC CURRICULUM PERSPECTIVE

For this panel, I will concentrate my talk on the following points:

1. System of Education (Quebec)
2. Reform
3. Comments about the statement (see the program)
4. Alternate ways

1. System of Education (Quebec)

For the benefit of some persons who are not familiar with the system of Education in Quebec, I would like to give an overview of our system of Education. The length for the *Primary* level is six years. Afterwards, the *Secondary* level has a duration of five years. We call each year as secondary I, secondary II, and so on. It is possible for a student to go for a vocational training at the end of the secondary school (e.g., mechanics, hairdressing, ...). I would like to mention that mathematics is compulsory at the primary and secondary levels.

After the secondary, there is the *College* level (we say «cégep»). It is two years for the students who plan to go to university and three years for the ones who take a technical program. The *University* level requires a minimum of three years for a baccalaureate degree. A master's degree requires a minimum of 45 credits, and doctoral studies a minimum of 90 credits.

2. Reform

The primary level has the same programs (content) since 1980-81, however, programs are being revised and new programs should take place in the near future. At the secondary level, it is basically the same content since 1980-81. Modifications are in progress with revisions mainly concerning methods of teaching and learning (re: NCTM's Standards).

For the college level, there is no official (i.e., coming from the Ministry of Education) changes for the content but the calculus reform in many places is about application. At the university level, there is an important reform concerning the primary and secondary programs for pre-service teachers. The new programs require four years instead of three and the pre-service teachers for the secondary level must have two specialities (e.g., mathematics and biology). During their studies, students will do about one year of teaching practice in schools.

Essentially, the orientation of the school mathematics *reform* is based on constructivism, solving problems and new technologies.

3. Comments About the Statement (see the program)

"The agenda driving the development of the mathematics curriculum, K to post-secondary, all across the country, seems to be one generated by ...

a) *governments to meet election promises,*

It seems to me that the politicians in Quebec don't do any election promises about mathematics education. Maybe this is not a politically profitable issue. On the other hand, a vast public consultation about education is in process. We call this operation *États généraux*. Many groups are heard at this forum and it's quite difficult to say presently what will be the consequence on the development of the mathematics curriculum but it might have an influence regarding what it should be.

b) business and industry in the pursuit of the holy grail "applicability,"

This assertion might be true for some technical options at the college level and also in specific programs (e.g., engineering) at the university but it doesn't seem to be the case for school mathematics (primary and secondary).

c) *universities and colleges as an "elitist" selection mechanism of those capable of further studies in mathematics and technologies,*

In Quebec, the general tendency is the opposite of this *elitist* selection mechanism. There is some social pressure for fewer prerequisites in admission to programs. For example, in some universities the student population interested in studies in pure mathematics and in the sciences is stable or decreasing (it happens...),therefore, it becomes awkward to require more prerequisites in such a case.

d) *provincial mathematics associations as a means of protecting the jobs of secondary maths teachers."*

There are four mathematics teachers' associations in Quebec: APAME (Association des promoteurs de l'avancement de la mathématique à l'élémentaire), GRMS (Groupe des responsables en mathématique au secondaire), AMQ (Association mathématique du Québec), and QAMT (Quebec Association of Mathematics Teachers). These associations don't have any influence on jobs which depend on a provincial collective agreement. Their interests concern the improvement of in-service programs for teachers and the conditions of teaching and learning.

4. Alternate Ways

This brief analysis of each premise of the statement lets me believe that overall this statement does not apply very much to the situation in Quebec. But the context is not so simple. We are confronted with this choice concerning the agenda driving the development of the mathematics curriculum: we wait for administrative directives or we take an active part of our professional development in being more involved in the mathematics teachers' associations. For instance, in Quebec, there is a regrouping of the mathematics associations called CQEM (Conseil québécois de l'enseignement des mathématiques). This group is consulted by the government when a change is to be made and conversely, this group can do some lobbying when modifications are needed. Grouping strengths together is essential if we want to be involved in the decisions. What is the role for the CMESG / GCEDM ? This will be the focus of the general discussion.

Panelist: Gary Flewelling

WHO DETERMINES THE MATH AGENDA IN YOUR PROVINCE?

In Ontario, I believe it is illegal to possess the answer to that question. If it is legal then I'm pretty sure that trafficking in it is not. Quite honestly, I have no idea who determines the math agenda in our province (I asked my colleagues and they don't know either). I think the math agenda just happens and everybody is surprised when it does.

I have no idea, for example, just how influential the CMEC (Council of Ministers of Education) is in influencing the agenda in this province. Nor do I know how significant is the influence of indifferent or disappointing results achieved on large scale assessment initiatives like the Second and Third International Mathematics and Science Studies. Western and Atlantic curriculum initiatives have, or will have some influence on our math agenda. The NCTM Standards, at least the spirit of them, seem to be having a big impact on the mathematics reform agenda in this province.

Typically, the Ministry of Education gets the ball rolling with the (often hasty, often ad hoc) publication of a curriculum policy document, such as, *The Common Curriculum: Policies and Outcomes, Grades 1-9*, 1995, or the assessment document, *Provincial Mathematics Standards,* 1995 (insufficient time, planning, and resources in the construction and implementation of such documents blunting their effectiveness). In anticipation of such documents, provincial mathematics organizations, such as the Ontario Association of Mathematics Educators (OAME) and The Ontario Mathematics Coordinators Association (OMCA) play a significant role in setting the mathematics agenda with the proactive publication of documents, such as, *Focus on Renewal of Mathematics Education,* 1993. This document, for example, an Ontario interpretation of the NCTM Standards, has significantly influenced the contents of the *Provincial Mathematics Standards*. (More often though, these provincial organizations are asked by the Ministry to send experts, in ad hoc fashion, to help fix flawed, hurriedly-constructed policy documents.)

More and more the mathematics agenda in this province will be influenced by provincial assessment initiatives, especially those developed or managed by the Education Quality and Accountability Office. Hopefully, in the future, more and more of the agenda for math reform will also be set by groups, such as, the Fields Institute Mathematics Education Forum and The National Mathematics Education Institute.

Panelist: Florence Glanfield

THE ALBERTA CURRICULUM PERSPECTIVE

In the fall of 1993 the ministers of education from the four western provinces and two territories met together and signed the Western Canada Protocol for Basic Education. The purpose of signing this protocol was for the ministries of education in each province and territory to work together in developing and implementing curriculum. The mathematics curriculum was the first curriculum to be considered.

In August, 1994, a group of approximately 60 teachers and ministry representatives met in Regina to put together the first draft of a Western Canada mathematics program for kindergarten to grade 12. The intent was to look at content from a kindergarten to grade 12 perspective. Looking at the mathematics curriculum from this perspective was not what had been practiced previously in Alberta where there would be a curriculum development process for grades 1 to 6; one for grades 7 to 9; and then one for each of the three programs in grades 10, 11, and 12. The work of that committee included looking at how number and number operations; shape and space; patterns and relations; and statistics and probability were developed throughout the thirteen years of schooling. The final version of the kindergarten to grade 9 program was completed in June, 1995 and the grades 10 to 12 program was completed in June, 1996.

I was working at the Alberta ministry of Education at the time that this protocol came to be and was able to participate in the committee work in August, 1994. There was much negotiation around the tables that week—negotiation around what it means to teach mathematics, what experiences students should have to learn mathematics, and what outcomes students should have to demonstrate that they have learned the mathematics. I was reminded at that time of all the different voices that are involved in creating drafts of curricula documents. The direction of conversations in our groups were dependent on each participant's own experiences with teaching and learning mathematics and with mathematics itself. The manner in which topics were presented for inclusion in the curriculum again were influenced by each participant.

In Alberta, there were several other voices that provided input and critique to draft versions of the document. The voices of our colleagues in mathematics and mathematics education departments at post-secondary institutions, representatives of business and industry, parents, and of course numerous other teachers. Ultimately, although all the voices might have had a chance to provide input and critique in the development of the document, the ministry consultants from the six jurisdictions wrote it. So in the interpretation of the committee work, the ministry consultants brought their own beliefs about teaching and learning mathematics and about mathematics into the picture.

Although there may have been several voices in the development of this curriculum document, it is the voice of the teacher that is heard loudest by students. It is the individual teacher's perspective of teaching and learning mathematics and of mathematics itself that resonates in their voice—the voice that is shared with students.

The Western Canada Protocol mathematics program is currently being implemented in Alberta: grades 7 and 9 in the 1996 - 97 school year, kindergarten to grade 6 and grade 8 in the 1997 - 98 school, and grade 10 is scheduled for the 1998 - 99 school year. Implementation is another issue that needs to be considered.

APPENDICES

APPENDIX A

WORKING GROUPS AT EACH ANNUAL MEETING

1977 Queen's University, Kingston, Ontario
- Teacher Education programmes
- Undergraduate mathematics programmes and prospective teachers
- Research and mathematics education
- Learning and teaching mathematics

1978 Queen's University, Kingston, Ontario
- Mathematics courses for prospective elementary teachers
- Mathematization
- Research in mathematics education

1979 Queen's University, Kingston, Ontario
- Ratio and proportion: a study of a mathematical concept
- Minicalculators in the mathematics classroom
- Is there a mathematical method?
- Topics suitable for mathematics courses for elementary teachers

1980 Université Laval, Québec, Québec
- The teaching of calculus and analysis
- Applications of mathematics for high school students
- Geometry in the elementary and junior high school curriculum
- The diagnosis and remediation of common mathematical errors

1981 University of Alberta, Edmonton, Alberta
- Research and the classroom
- Computer education for teachers
- Issues in the teaching of calculus
- Revitalising mathematics in teacher education courses

1982 Queen's University, Kingston, Ontario
- The influence of computer science on undergraduate mathematics education
- Applications of research in mathematics education to teacher training programmes
- Problem solving in the curriculum

1983 University of British Columbia, Vancouver, British Columbia
- Developing statistical thinking
- Training in diagnosis and remediation of teachers
- Mathematics and language
- The influence of computer science on the mathematics curriculum

1984 University of Waterloo, Waterloo, Ontario
- Logo and the mathematics curriculum

The impact of research and technology on school algebra
Epistemology and mathematics
Visual thinking in mathematics

1985 Université Laval, Québec, Québec
Lessons from research about students' errors
Logo activities for the high school
Impact of symbolic manipulation software on the teaching of calculus

1986 Memorial University of Newfoundland, St, John's, Newfoundland
The role of feelings in mathematics
The problem of rigour in mathematics teaching
Microcomputers in teacher education
The role of microcomputers in developing statistical thinking

1987 Queen's University, Kingston, Ontario
Methods courses for secondary teacher education
The problem of formal reasoning in undergraduate programmes
Small group work in the mathematics classroom

1988 University of Manitoba, Winnipeg, Manitoba
Teacher education: what could it be
Natural learning and mathematics
Using software for geometrical investigations
A study of the remedial teaching of mathematics

1989 Brock University, St. Catharines, Ontario
Using computers to investigate work with teachers
Computers in the undergraduate mathematics curriculum
Natural language and mathematical language
Research strategies for pupils' conceptions in mathematics

1990 Simon Fraser University, Vancouver, British Columbia
Reading and writing in the mathematics classroom
The NCTM "Standards" and Canadian reality
Explanatory models of children's mathematics
Chaos and fractal geometry for high school students

1991 University of New Brunswick, Fredericton, New Brunswick
Fractal geometry in the curriculum
Socio-cultural aspects of mathematics
Technology and understanding mathematics
Constructivism: implications for teacher education in mathematics

1992 ICME-7, Université Laval, Québec, Québec

1993 York University, Toronto, Ontario
Research in undergraduate teaching and learning of mathematics
New ideas in assessment
Computers in the classroom: mathematical and social implications
Gender and mathematics

Training pre-service teachers for creating mathematical communities in the classroom

1994 University of Regina, Regina, Saskatchewan
Theories of mathematics education
Preservice mathematics teachers as pruposeful learners: issues of enculturation
Popularizing mathematics

1995 University of Western Ontario, London, Ontario
Anatomy and authority in the design and conduct of learning activity
Expanding the conversation: trying to talk about what our theories don't talk about
Factors affecting the transition from high school to university mathematics
Geometric proofs and knowledge without axioms

1996 Mount Saint Vincent University, Halifax, Nova Scotia
Teacher education: challenges, opportunities and innovations
Formation à l'enseignement des mathématiques au secondaire: nouvelles perspectives et défis
What is dynamic algebra?
The role of proof in post-secondary education

APPENDIX B

PLENARY LECTURES

1977	A.J. Coleman	The objectives of mathematics education
	C. Gaulin	Innovations in teacher education programmes
	T.E. Kieren	The state of research in mathematics education
1978	G.R. Rising	The mathematician's contribution to curriculum development
	A.I. Weinzweig	The mathematician's contribution to pedagogy
1979	J. Agassi	The Lakatosian revolution*
	J.A. Easley	Formal and informal research methods and the cultural status of school mathematics*
1980	C. Cattegno	Reflections on forty years of thinking about the teaching of mathematics
	D. Hawkins	Understanding understanding mathematics
1981	K. Iverson	Mathematics and computers
	J. Kilpatrick	The reasonable effectiveness of research in mathematics education*
1982	P.J. Davis	Towards a philosophy of compuation*
	G. Vergnaud	Cognitive and developmental psychology and research in mathematics education*
1983	S.I. Brown	The nature of problem generation and the mathematics curriculum
	P.J. Hilton	The nature of mathematics today and implications for mathematics teaching*
1984	A.J. Bishop	The social construction of meaning: a significant development for mathematics education?*
	L. Henkin	Linguistic aspects of mathematics and mathematics instruction
1985	H. Bauersfeld	Contributions to a fundamental theory of mathematics learning and teaching
	H.O. Pollak	On the relation between the applications of mathematics and the teaching of mathematics
1986	R. Finney	Professional applications of undergraduate mathematics
	A.H. Schoenfeld	Confessions of an accidental theorist*
1987	P. Nesher	Formulating instructional theory: the role of students' misconceptions*
	H.S. Wilf	The calculator with a college education
1988	C. Keitel	Mathematics education and technology*
	L.A. Steen	All one system

1989	N. Balacheff	Teaching mathematical proof: the relevance and complexity of a social approach
	D. Schattsneider	Geometry is alive and well
1990	U. D'Ambrosio	Values in mathematics education*
	A. Sierpinska	On understanding mathematics
1991	J .J. Kaput	Mathematics and technology: multiple visions of multiple futures
	C. Laborde	Approches théoriques et méthodologiques des recherches Francaises en didactique des mathématiques
1992	ICME-7	
1993	G.G. Joseph	What is a square root? A study of geometrical representation in different mathematical traditions
	J Confrey	Forging a revised theory of intellectual development Piaget, Vygotsky and beyond*
1994	A. Sfard	Understanding = Doing + Seeing ?
	K. Devlin	Mathematics for the twent-first century
1995	M. Artigue	The role of epistemological analysis in a didactic approach to the phenomenon of mathematics learning and teaching
	K. Millett	Teaching and making certain it counts
1996	C. Hoyles	Beyond the classroom: The curriculum as a key factor in students' approaches to proof
	D. Henderson	Alive mathematical reasoning

*These lectures, some in a revised form, were subsequently published in the journal *For the Learning of Mathematics.*

APPENDIX C

PROCEEDINGS OF ANNUAL MEETINGS OF CMESG/GCEDM

Past proceedings of the Study Group have been deposited in the ERIC documentation system with call numbers as follows:

Proceedings of the 1980 Annual Meeting	ED 204120
Proceedings of the 1981 Annual Meeting	ED 234988
Proceedings of the 1982 Annual Meeting	ED 234989
Proceedings of the 1983 Annual Meeting	ED 243653
Proceedings of the 1984 Annual Meeting	ED 257640
Proceedings of the 1985 Annual Meeting	ED 277573
Proceedings of the 1986 Annual Meeting	ED 297966
Proceedings of the 1987 Annual Meeting	ED 295842
Proceedings of the 1988 Annual Meeting	ED 306259
Proceedings of the 1989 Annual Meeting	ED 319606
Proceedings of the 1990 Annual Meeting	ED 344746
Proceedings of the 1991 Annual Meeting	ED 350161
Proceedings of the 1993 Annual Meeting	Not yet assigned*
Proceedings of the 1994 Annual Meeting	Not yet assigned*
Proceedings of the 1995 Annual Meeting	Not yet assigned*

There was no Annual Meeting in 1992 because Canada hosted the Seventh International Conference on Mathematical Education that year.

*These *Proceedings* have been recently submitted to ERIC.