CANADIAN MATHEMATICS EDUCATION STUDY GROUP

GROUPE CANADIEN D'ÉTUDE EN DIDACTIQUE DES MATHÉMATIQUES

PROCEEDINGS
1999 ANNUAL MEETING

Brock University
June 4 - June 8, 1999

EDITED BY
John Grant McLoughlin
Memorial University of Newfoundland

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1999

23rd ANNUAL MEETING
CANADIAN MATHEMATICS EDUCATION STUDY GROUP/
GROUPE CANADIEN D'ÉTUDE EN DIDACTIQUE DES MATHÉMATIQUES

Brock University, June 4 - 8, 1999

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EDITOR'S FOREWORD

I shall take this opportunity to acknowledge the assistance of others while commenting on the editorial process as a whole.

The process proved to be more challenging than I had anticipated. Indeed, I expected to learn much and on that note, there were few surprises. The proceedings reflect the meeting; hence, the plenary lectures shape the proceedings in an unusual manner. Previously my perception had been that the working group reports represent the core of the proceedings. The presence of five plenary lectures shifts this focus by giving the working group reports an appearance of another section, rather than that of being the core section.

Any suggestion that technology makes things easier is open to question. A total of 23 submissions in various forms complicated the preparation of the document. As the editor, I have made an effort to honour the writing styles and preferences of contributors; that is, consistency within contributions was given priority over consistency between contributors. Yvonne Pothier, (the previous editor) offered helpful information on other things such as justification and margins that needed to be consistent throughout the proceedings. Ann Kajander and David Reid submitted photos of the meeting. Eric Muller provided the cover photo of the host site. Ralph Mason is to be commended for making many of the photos possible. "The Mason Thing - 1999" (as described by Bill Higginson , p. 115) clearly caught the attention of photographers. The contributions of these individuals and the efforts of all those who have prepared submissions for the Proceedings are appreciated.

It would have been impractical to see this process through to publication without many helping hands. I am particularly grateful to Tammy Constantine, a graduate assistant, who has spent in excess of 100 hours on the project. Natasha Blanchard, Deanne Burton, and Renee Lynch also supported the process at different stages. The office staff (Carolyn Lono, Laura Walsh, Eileen Ryan, Wanda Bourne, Carolyn Bourne, and Debbie Connors) has offered valuable assistance particularly with attachments arriving in various forms. Also, Sandra Hiscock's technical advice on various matters is appreciated. Finally, I would like to thank Memorial University of Newfoundland for its funding of student assistants in addition to its infrastructural support offered through the Faculty of Education, Computing and Communications, and Printing Services.

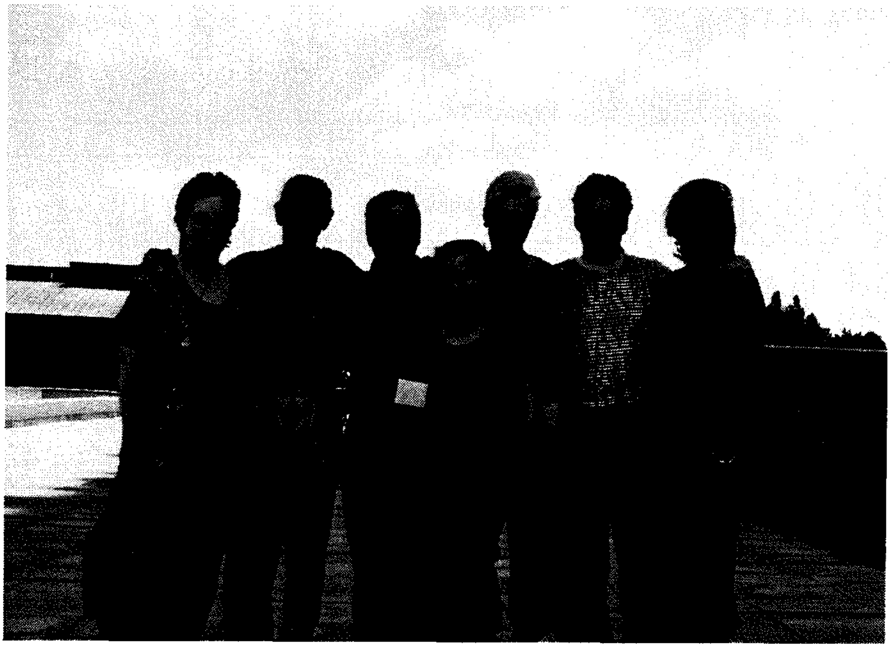
John Grant McLoughlin
Editor, 1999 CMESG Proceedings

ACKNOWLEDGEMENTS

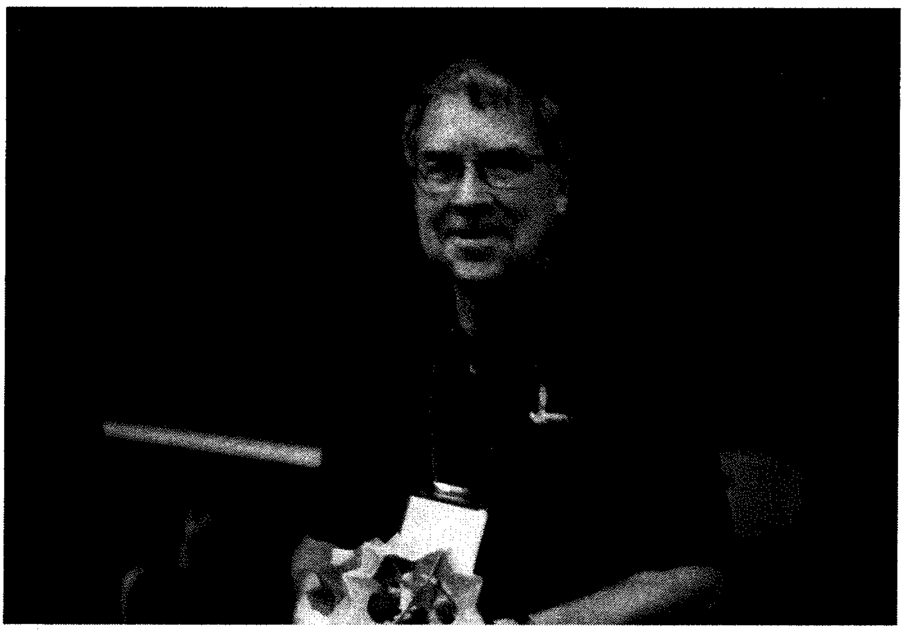
The executive and members would like to thank the people at Brock University, St. Catharines, ON for hosting the meeting and providing excellent facilities. Special thanks are extended to the organizing committee from the math department, namely, Bruce Cload, Tom Jenkyns, Dorothy Levay, Eric Muller, and Cathy Ugulini. Gratitude is also extended to those at the Conference Center, Wendy Laslo and residence staff, for their time and work prior to and during the meeting to make the experience pleasant and enjoyable for all participants.

SCHEDULE

Friday June 4	**Saturday June 5**	**Sunday June 6**	**Monday June 7**	**Tuesday June 8**
	0900-1015 Working Groups ---------- 1015-1045 Refreshments ---------- 1045-1215 Working Groups ---------- 1215-1330 Lunch	0900-1015 Working Groups ---------- 1015-1045 Refreshments ---------- 1045-1215 Working Groups ---------- 1215-1330 Lunch	0900-1015 Working Groups ---------- 1015-1045 Refreshments ---------- 1045-1215 Working Groups ---------- 1215-1330 Lunch	0900-1000 Topic Groups William Higginson Joel Hillel ---------- 1000-1030 Refreshments ---------- 1030-1115 Small Group Discussion Plenary Didactique II ---------- 1115-1215 Question Period on Plenary Didactique II ---------- 1215-1245 Working Groups Reports and Closing Session
1500-1620 Friends of FLM ---------- 1630-1725 Welcome and Introduction to Working Groups ---------- 1730-1830 Plenary Mathematics I *Jonathan Borwein* ---------- 1830-2030 Dinner ---------- 2030-2300 Reception	1330-1430 Plenary Didactique I *Jill Adler* ---------- 1430-1500 Refreshments ---------- 1500-1555 CMESG AGM ---------- 1600-1700 Plenary Mathematics II *Walter Whiteley* ---------- 1700-1900 Dinner ---------- 1900-1925 Ad Hoc Presentations ---------- 2000 Social	1315-1400 Small Group Discussion Plenary Didactique I ------------ 1400-1500 Question Period on Plenary Didactique I ------------ Dinner and Tour	1330-1430 Plenary Didactique II *Bill Barton* ---------- 1430-1500 Refreshments ---------- 1500-1550 Presentations by new PhDs ---------- 1600-1700 Plenary Mathematics III *William Langford* ---------- 1700-1900 Dinner ---------- 1900-1925 Presentations by new Ph. D's ---------- 1930-1955 Ad Hoc Presentations ---------- 2000 Social	



The 1999 -2000 Executive
Elaine Simmt, Recording Secretary; Joel Hillel, Member at Large;
Eric Muller, Conference Organizer; Malgorzata Dubiel, President; Mary Crowley, Vice-President;
Frédéric Gourdeau, Conference Co-coordinator; Olive Chapman, Treasurer/Membership Secretary.



INTRODUCTION

It is my great pleasure to write an introduction to the CMESG/GCEDM proceedings from the 1999 meeting held at Brock University in St. Catharines, Ontario, and to welcome our new editor, John Grant McLoughlin. Thank you, John, for agreeing to undertake such a challenging task!

A necessary part of the introduction to the CMESG/GCEDM Proceedings is an attempt to explain to readers, some of whom may be newcomers to our organization, that the volume in their hands cannot possibly convey the spirit of the meeting it reports on. It can merely describe the content of activities without giving much of the flavor of the process.

To understand this, one needs to understand the uniqueness of both our organization and our annual meetings. CMESG is an organization unlike other professional organizations. One belongs to it not because of who one is professionally, but because of one's interests. And that is why our members are members of mathematics and education departments at Canadian and other universities and colleges, and school teachers, united by their interest in mathematics and how it is taught at every level, by the desire to make teaching more exciting, more relevant, more meaningful.

Our meetings are unique, too. One does not simply attend a CMESG meeting the way one attends other professional meetings, by coming to listen to a few chosen talks. You are immediately part of it; you live and breathe it.

Working Groups form the core of each CMESG meeting. Participants choose one of several possible topics, and, for three days, become members of a community which meets three hours a day to exchange ideas and knowledge, and, through discussions which often continue beyond the allotted time, create fresh knowledge and insights. Throughout the three days, the group becomes much more than a sum of its parts - often in ways totally unexpected to its leaders. The leaders, after working for months prior to the meeting, may see their carefully prepared plan ignored or put aside by the group, and a completely new picture emerging in its stead.

Two plenary talks are traditionally part of the conference, at least one of which is given by a speaker invited from outside Canada, who brings a non-Canadian perspective. These speakers participate in the whole meeting; some of them afterwards become part of the Group. And, in the spirit of CMESG meetings, a plenary talk is not just a talk, but a mere beginning: it is followed by discussions in small groups, which prepare questions for the speaker. After the small group discussions, in a renewed plenary session, the speaker fields the questions generated by the groups.

As mathematicians, we understand that there is one more year of the second millennium, but at the same time that every day is the beginning of a new millennium and the end of one, which started a thousand years ago. And so we did join with the whole world's celebrations of the approaching year 2000. To celebrate the approach of the new millennium and the Year of Mathematics, three mathematics talks were included in the program of our meeting. They attempted to look at the past and the future of mathematics, and how its development may be affected by the tremendous changes in technology we have been witnessing.

Topic Groups and Ad Hoc presentations provide more possibilities for exchange of ideas and reflections. Shorter in duration than the Working Groups, Topic Groups are sessions where individual members present work in progress and often find inspiration and new insight from their colleagues' comments.

Ad hoc sessions are opportunities to share ideas, which are often not even "half-baked" - sometimes born during the very meeting at which they are presented.

A traditional part of each meeting is the recognition of new PhD's. Those who completed their dissertations in the last year are invited to speak on their work. This gives the group a wonderful opportunity to observe the changing face of mathematics education in Canada.

Our annual meetings are traditionally set on university campuses with participants staying in dormitories rather than hotels, both to make the meetings more affordable and to allow for discussions to continue far beyond the scheduled hours, at times ending in the increasingly famous evening "pizza runs".

The 1999 Annual Meeting was no exception. It was hosted on the beautiful campus of the Brock University, close to one of Canada's best wine regions, Niagara Falls, and many other attractions. Eric Muller's superb organization and attention to detail gave us welcome snacks waiting for us upon arrival, an excellent program, and great excursions. It was truly a meeting to remember. Thank you, Eric!

Malgorzata Dubiel
President (1999-2000)

PLENARY LECTURES

Plenary Lecture 1

THE IMPACT OF TECHNOLOGY ON THE DOING OF MATHEMATICS

Jonathan Borwein, Simon Fraser University

ABSTRACT

Technology has repeatedly promised to transform mathematics pedagogically. More recently it has made similar promises to the research community. That said, mathematics in 1999 looks a lot more like mathematics in 1939 than is the case with any of its sister sciences.

That this is about to change is inarguable. The confluence of ubiquitous computer power with new networking and collaborative environments will push the teaching and discovering of mathematics in conflicting directions often beyond our control. The burgeoning role of corporate edu-packages is hardly likely to diminish. Nor are battles over curriculum and its delivery about to stop.

In my talk, I intend to survey and illustrate some of the ways in which twenty-first century mathematics will be changed by these new technologies. I also intend to discuss how as mathematical educators we might best prepare for the coming storms. Finally, as a partner in a small educational technology firm, I will offer some modest prescriptions for living on both sides of the fence.

Intellectual issues, technological issues and commercial issues all bang up against each other.

TWO QUOTES

"The world will change. It will probably change for the better. It won't seem better to me." J. B. Priestley.

"It's generally the way with progress that it looks much greater than it really is." From *The Wittgenstein Controversy*, by Evelyn Toynton in the *Atlantic Monthly*, June 1997, pp.28-41.

The epigraph that Ludwig Wittgenstein (1889-1951) ("whereof one cannot speak, thereof one must be silent") had wished for an unrealized joint publication of *Tractatus Logico-Philosophicus* (1922) and *Philosophical Investigations* (1953): suggesting the two volumes are not irreconcilable.

INTELLECTUAL PROMISES

- Lively and realistic examples: learning by doing (Papert) - "we are all constructivists now"
- Math goes into colour: sliding down surfaces/virtual reality
- Background pattern-checkers and inverse calculations

- Speed & space ≡ insight (demands rapid reinforcement via micro-parallelism)
- Individually tailored learning: varied pathways for quick/slow and for distinct modes of thinking - algebraic, analytic, topological
- Promises students richer means to represent the fruits of their mathematical imagination
- Increased need to teach how to judge the results of computation (visual candy everywhere)
- Unifying research and teaching, theory and practice (jobs)
- Serious curricular insights from neurobiology (Dehaene et al., 1999)

INTELLECTUAL PITFALLS

- Wasted or wonderful enhancements ("Newton meets Java" or the " Idiot pivoter")
- Loss of focus
- Loss of control: student centered learning of hierarchical subjects
- Degradation of long-lived robust mathematical knowledge (unique to our discipline)
- Growing reliance on effectively closed architecture software ("total solutions")
- "Haves and havenots": class, race, gender
- Degeneration to machine-based rote learning ("buzzword compliant shovelware")

TWO MORE QUOTES

"Keynes distrusted intellectual rigour of the Ricardian type as likely to get in the way of original thinking and saw that it was not uncommon to hit on a valid conclusion before finding a logical path to it."

"I don't really start," he said, "until I get my proofs back from the printer. Then I can begin serious writing." From *Keynes the man* written on the 50th Anniversary of Keynes' death. (Sir Alec Cairncross, in the *Economist*, April 20, 1996)

TECHNICAL PROMISES

- Teachers abilities vs students demands
- Access to global data bases (free access to information not access to free information)
- Doing what is easy: machines don't think like us. - cognitive vs descriptive models
- What we learned earlier is not always easier

- Expert systems & belief revision
- Seamless workspaces: marriage of text and computation

TECHNICAL PITFALLS

- Legacy software
- Legacy hardware
- The weakest link determines the value
- Over promising layoffs and underestimating effort (reform calculus)
- Infinite time-sinks - especially in higher level courses
- Growing (unavoidable) reliance on commercial software

INTELLECTUAL PROPERTY ISSUES

- Different stakeholders often have wildly different views
  - supervisors and teachers
  - students (and parents)
  - professional societies (big and small)
  - publishing houses (big and small)
  - software companies (big and small)
- As job security disappears more students see IP as their future: (Ma vs Phong & Stein, non-disclosure, insider-trading, interleukin).
- The researcher as CEO: conflicts of interest are inevitable. They must be declared. They are rarely resolved.

COMMERCIAL ISSUES

- Can't make what you can't sell
- Can't sell what you can't make (market discipline?)
- Conservatism in the edu-software business: no R&D model
- Commoditization (macro-media everywhere)
- Machine closets versus kitchen cabinets
- Weaning from software: overloading the senses

TWO MORE QUOTES

" I have no satisfaction in formulas unless I feel their numerical magnitude." Lord Kelvin (William Thomson)

"The object of mathematical rigor is to sanction and legitimize the conquests of intuition, and there was never any other object for it." J. Hadamard, in E. Borel, *Lecons sur la theorie des fonctions*, 3rd ed. 1928, quoted in G. Polya, *Mathematical discovery: On understanding, learning, and teaching problem solving* (Combined Edition), Wiley, 1981.

FINAL QUOTES

"If you have a great idea, solid science, and earthshaking discoveries, you are still only 10% of the way there." David Tomei, LXR Biotechnology Inc. on the vicissitudes of startup companies, quoted in *Science*, November 7, 1997, p.1039.

"A truly popular lecture cannot teach, and a lecture that truly teaches cannot be popular." Michael Faraday: ' When Gladstone was British Prime Minister he visited Faraday's laboratory and asked if some esoteric substance called 'Electricity' would ever have practical significance. "One day, sir, you will tax it," was the answer.' (*Science*, 1994)

SUGGESTIONS/CONCLUSIONS

- Clearly identify expectations of technology
- Be realistic about the learning curve for advanced software (such as *Mathematica* or *Maple*)
- Commit to use of open architecture software
- Form (not for profit and 'pre-competitive') consortia
  sharing of expertise; access to markets; ability to compete with the big guys
- Opportunity to recapture computing from our sister sciences
- Realistic now to benefit from:
  -advances in cognitive neuroscience
  -advances in software design, and testing, interfaces, expert systems
- Good technology will never be cheap (*Malthusian principle* that 'expectations outstrip performance')

FINAL QUOTE

"... so long as we conceive intellectual education as merely consisting in the acquirement of mechanical mental aptitudes, and of formulated statements of useful truths, there can be no progress; although there will be much activity, amid aimless rearrangement of syllabuses, in the fruitless endeavour to dodge the inevitable lack of time." A.N. Whitehead, *The Rhythmic Claims of Freedom and Discipline*.

Plenary Lecture 2

THE DECLINE AND RISE OF GEOMETRY IN 20th CENTURY NORTH AMERICA

Walter Whiteley, York University

INTRODUCTION

While I will begin with my own evidence for the decline of geometry in this century and my own description on how such a decline has proceeded, my basic theme is hopeful. Geometry has not died because it is essential to many other human activities and because it is so deeply embodied in how humans think. With the introduction of computers with rich graphical capacities and the recognition of multiple ways of learning, our current situation offers an unprecedented opportunity for geometers and those who work visually. An independent, but related, description of this past decline and the present possibilities can be found in [55].

I. THE DECLINE OF GEOMETRY THROUGH THE 20th CENTURY

As a graduate student, I worked in an area of mathematics that was officially 'dead': invariant theory or a classical theory of the foundations of analytic geometry [18, 19]. From conversations with mathematicians and from reading the sociology of mathematics, I learned that a field of mathematics 'dies' when it is no longer viewed as an 'important' area of mathematical research. Geometry 'died' in this sense by the mid 20th century in North America. I now see that geometry in the education system then followed in a predictable (though not inevitable) decline. This decline proceeded from the graduate schools into the high school and elementary classrooms over the last decades. Knowing this path may help us plan tactics and strategies for accelerating the rise of geometry. We do not have a half-century to spare for a comparable, gradual 'rise' of geometry!

Let me begin with an 'indicator' that discrete geometry has declined as an 'important area' of mathematical research. At the turn of this century, David Hilbert delivered a famous lecture containing twenty-three problems that might shape mathematics in the 20th century [7]. How many of these were problems in discrete geometry? Three out of twenty three - about 13% of the problems! Hilbert also expressed his sense of geometry in the very readable book [26]. In 1976, a symposium was held on the mathematics arising from these problems [7]. A group of mathematicians gave twenty eight sets of problems - and none of these sets included discrete geometry (or its relatives, such as combinatorics) - although there were sets of problems in more 'current' geometry: algebraic geometry, differential geometry, geometric topology etc. Discrete geometry was no longer an important area. In case you think this was because all the problems were solved, an important part of Hilbert's eighteenth problem: proving that packing spheres in three space like the standard packing of oranges, is the best possible; was solved by Thomas Halles in 1998. See [13] for a recent collection of unsolved problems in discrete geometry.

Philip Davis has chronicled the rise and fall of a specific field of discrete geometry – 'triangle geometry' - over the 19th and 20th centuries [14]. Within a richer analysis of the sociology of this decline in the very geometry most often taught in high schools, he quotes E.T. Bell [5 p. 323] :

> " The geometers of the 20th century have long since piously removed all these treasures to the museum of geometry where the dust of history quickly dimmed their luster."

To summarize, my preliminary point is that discrete geometry virtually died as an 'important' field of mathematical research through the twenties and thirties and forties, at least in North America and parts of Europe. It survived in pockets (Hungary, Germany, Switzerland, Austria, Russia...) and through a few key people in other places (H.S.M. Coxeter, D. Pedoe, B. Grünbaum). In the Canadian context, this death was confirmed as Professor Coxeter retired at the University of Toronto several decades ago. The department followed a policy of not hiring in discrete geometry and shifted to the 'hotter' areas such as algebraic geometry. Here is a visual representation for this decline of discrete geometry as a field of research (Figure 1A).

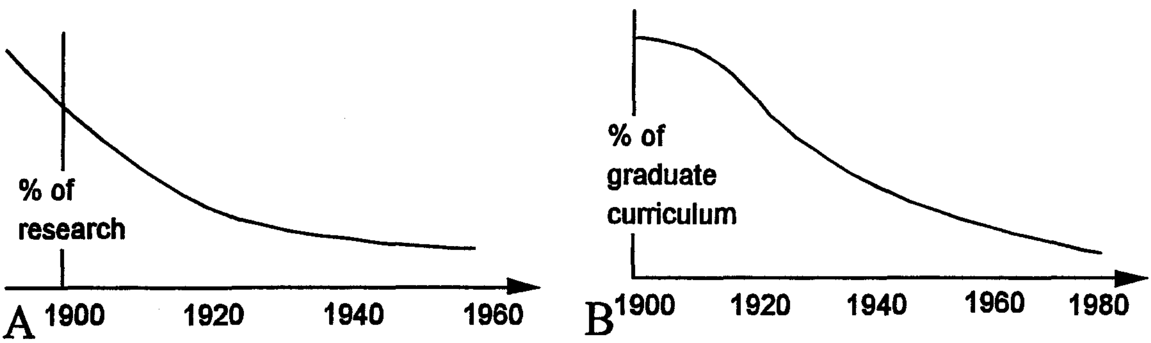


Figure 1: Geometry in decline: research (A) and graduate programs (B).

Here is my model of how this decline was transmitted down from 'research activities' to various levels of mathematics education. As research in geometry declined, the importance of teaching geometry in graduate programs also declined, as did the number of faculty proposing courses in geometry. More and more graduate programs contained no researchers in geometry. No graduate courses in geometry were taught - or if taught the topics were not on the core syllabus or comprehensive exams. Of course, there is lag in this and the previous curve of decline shifts over several decades (Figure 1B).

After a few decades more, we have a generation of people moving out to teach undergraduate mathematics who have not experienced discrete geometry as an important, lively field of current mathematics, and who may not have studied any geometry during their graduate studies. If geometry is then taught to undergraduates, it is taught by someone who is not a geometer and who does not work with visual forms - often by a logician or an historian of mathematics. As a whole, both of these groups would teach geometry as an important past accomplishment (often as an axiomatic study and an exercise in logical proofs) but not as a continuing source of new mathematics. Many undergraduate geometry courses wear a veneer of geometric language without any playful geometric and visual spirit in the problems, the solutions, or the presentation. Over time, this decline reached the point where algebra and analysis became the core areas in the undergraduate curriculum. Geometry was relegated to a service course for future high school teachers by the sixties and perhaps not even that by the nineties. So the curve of decline has shifted over again (Figure 2A).

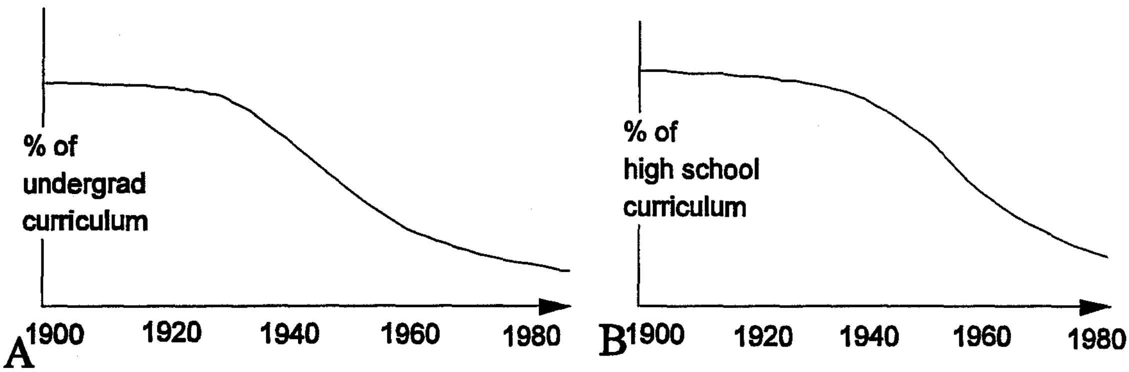


Figure 2. Geometry in decline: undergraduate (A) and high school curriculum (B).

After a few more decades, we have a generation of high school teachers who either had no geometry among their undergraduate courses or had a 'course for teachers'. This implicitly communicates that geometry is not a central part of modern mathematics. Consider the two questions I asked during my talk:

(i) How many of you had an undergraduate geometry course?
(ii) How many Faculties of Education require a course in geometry vs. requiring a course in calculus, linear algebra or statistics?

A few years ago, a group of graduating pre-service students in a geometry course asked me: "Why do we teach geometry in high school?" After some reflection, I realized what their question was about. They took geometry in high school but did not see any material that connected with it during their previous undergraduate program! They were asking: Why teach something in high school that is a 'dead end' for learning more mathematics?

The final stage of this decline is a group of teachers who may be uncomfortable with open-ended problems in geometry and who will leave geometry 'to the end' as something that is much less important to their students than core areas like functions, algebra, calculus.... The sense that geometry is an 'optional' topic continues to grow among the curriculum writers, the textbook writers, the tutors and the parents. The shift to the 'new math', with its emphasis on set theory and algebra, encoded this decline in geometry. The message that geometry is not important is embedded in the dominant culture in undergraduate mathematics departments and in high school mathematics curricula in North America today [46 p.184]. This fits a final shift in the curve of decline (Figure 2B).

This identification of mathematics with language and formulae is also characteristic of people working in the foundations of mathematics [8] and, to a significant extent, of people in algebra and analysis. For example, this is explicit in the Bourbaki tradition. For example, Dieudonné urges a "strict adherence to the axiomatic methods, with no appeal to the "geometric intuition", at least in formal proofs: a necessity which we have emphasized by deliberately abstaining from introducing any diagram in the book" [8 pp. 173-174]. I recall that my linear algebra text had one almost irrelevant picture in the entire book. Many students emerge from an abstract algebra course with no sense of a 'group' as the central feature of 'symmetry'. In fact, 'symmetry under a group' is the very definition of 'a geometry' – a point I will return to below.

The recent literature in educational psychology and cognitive science confirms this broad cultural (mis)perception that geometry is marginal within mathematics. Outsiders automatically associate 'mathematics' with formulae, algebra and maybe analysis. A recent (and very interesting) book on mathematical cognition identifies 'numbers' and the abilities based on them (e.g.algebra) with mathematics [9]. When a scholar of multiple approaches to learning, such as Howard Gardner, considers 'mathematical intelligence', he identifies mathematics with a single approach involving logical sequences of formulae and sentences [20]. In his description, 'mathematics' is cut off from the "visual intelligence" and the "kinetic intelligence". When a book for teachers [2] describes the theory of multiple intelligences and the associated careers, the 'mathematician and scientist' are associated with the logical /mathematical intelligence, while the visual intelligence is associated with 'artist and architect'. Similarly, in this description the culture values 'scientific discoveries, mathematical theories, counting and classification' from the logical/mathematical intelligence and values 'artistic works, navigational systems, architectural designs, inventions' from the spatial intelligence. Of course these outcomes associated with the logical/mathematical intelligence are valuable. Unfortunately, geometry, as associated with the visual intelligence, is presented as marginal in mathematics and in science.

In short, many mathematicians present a public face to their students and to other intellectuals that mathematics (at least higher mathematics) is essentially about the logical intelligence [8].

The popular culture sees mathematics as detached from the spatial (visual) intelligence. From this point of view, the visual and the geometric are not an essential part of mathematics. Where 'geometry' appears it is quickly made analytic and treated as a source of calculations to illustrate the 'important areas' of math like algebra and calculus. This public face for mathematics is an important cultural result of the decline of geometry.

Does this decline matter? Does the public and educational disconnection of 'mathematics' from 'geometry' matter? Perhaps the current state of undergraduate and high school geometry accurately reflects the value of geometry to the learning and the future of students. Clearly the current curriculum is crowded and we have to cut to make room for important new mathematics. Is geometry now part of what someone called the 'saber tooth curriculum'? Have we now 'got it right' that geometry *should* decline?

II. GEOMETRY IS RISING

Discrete geometry is already rising as an area of research inside and outside mathematics. Geometry is beginning to rise as an important area of learning and teaching. In this section I will focus on three distinct trends to support my assertion that geometry has more life now than two or three decades ago. In the following sections I will then offer some comments about what geometry is now and how we should teach it.

(A) Applications of geometry: new results and new problems. Geometry is again very active as a field of research in many disciplines and industries today. This work has generated new geometric problems, used new geometric results and even generated new areas of geometry. Sometimes this activity includes mathematicians and mathematics departments; often it is centered outside of 'mathematics'!

(B) Human abilities - visualization. Geometry is central to a basic human ability - visualization and reasoning with visual and spatial forms. For a variety of reasons, often associated with computers, this ability is playing an increasing role in learning, in memory, in communication, in problem solving and the practice of many professions.

(C) Suitable resources for learning. The development of dynamic geometry programs for teaching and for research is dramatically changing what researchers, students, and therefore teachers, do when they solve problems in geometry. Companion resources for teaching geometry in a rich way are accelerating the impact within the undergraduate and secondary classrooms of the rise in geometry at the level of research and applications.

(A) APPLICATIONS OF GEOMETRY: NEW RESULTS AND NEW PROBLEMS

Numerous current applications have a strong geometric component. In many cases, the problem includes getting 'geometric' information into a computer in a useful format, solving geometric problems, and outputting this solution as a visual or spatial form, as design to be built, as an action to be executed, or as an image to entertain. Solving these problems requires substantial geometric knowledge and people using the results of the research also benefit from a basic understanding of the geometry involved. Here, briefly, are a few illustrative examples.

(1) Computer Aided Design and Geometric Modeling.

A basic problem is to describe, design, modify, or manufacture the shapes we want: cars, planes, buildings, manufactured components, etc. using computers. The descriptions should be accurate enough to directly control the manufacturing and to permit simulation and testing of the objects, prior to making any physical models. For example, the most recent Boeing plane was entirely designed inside of the computer, without any physical models. Here are a few samples of the geometry involved in such work.

∑ Consider the hood of a new car. How is this described in the computer? We could input a bunch of points - but that does not give the surface, nor a 'picture' of the object during modifications of the design, nor instructions on how to manufacture the surface. Instead, the 'surface' is divided into regions and each region is described by some simple function that approximates, or even passes through, the initial points. These pieces have 'control points' to modify their shape that are combined to ensure that the pieces fit together in a geometric sense: continuity - no gaps in the hood, continuous derivatives - no sharp creases. Even continuous third derivatives are needed for display in the showroom where our eyes can 'see' such details. The standard mathematical objects are called 'splines' and they are a striking generalization of those strange problems in calculus in which we piece together parts of two polynomial functions to make a single continuous function (Figure 3) [22].

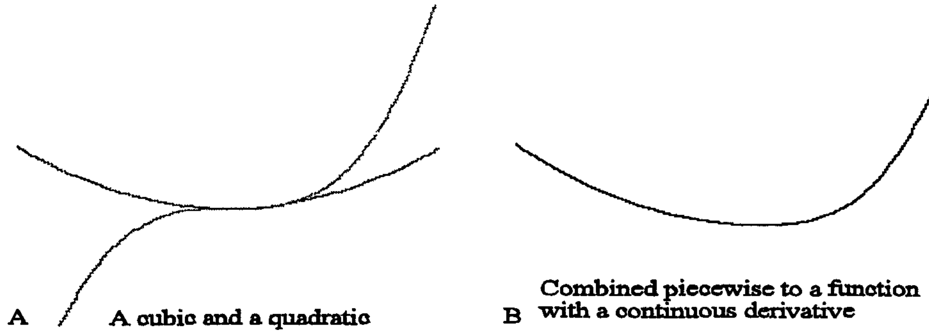


Figure 3

∑ Constrained CAD. Given a structural design or the 'measurements' of an object, how do these measurements determine the shape? Which measurements can be changed, without altering other measurements or constraints?

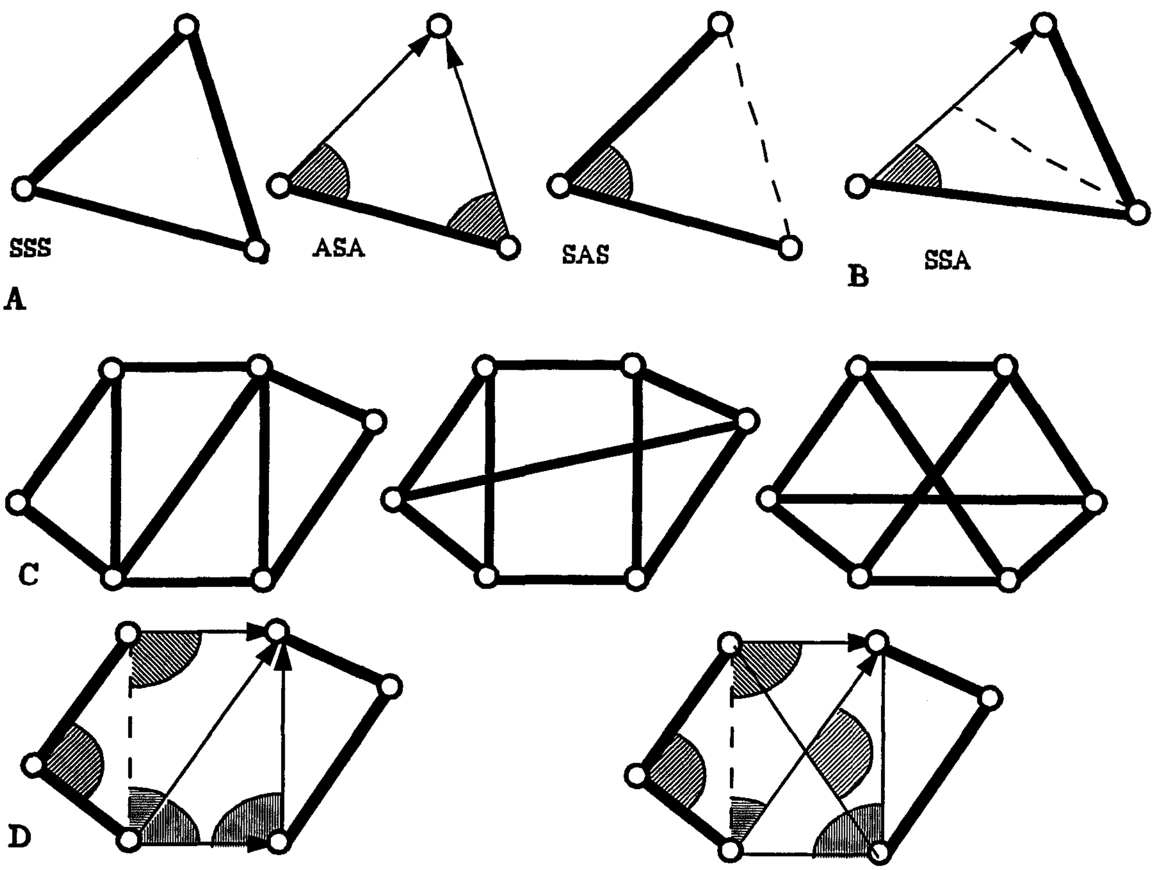
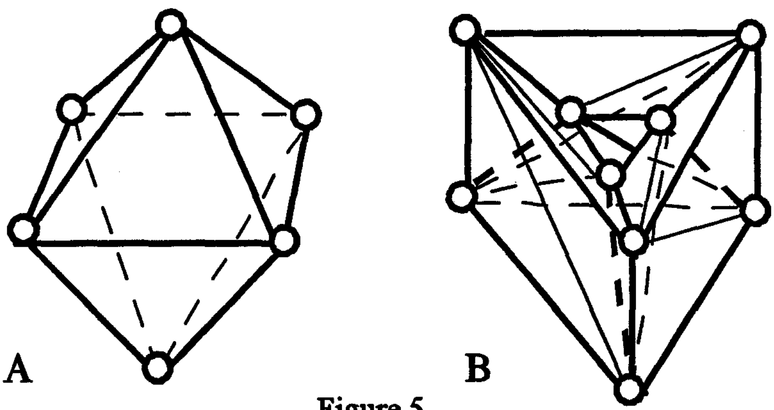


Figure 4

∑ Consider three points in the plane (Figures 4A, B). We can choose three measurements (angles, lengths) and then the others are determined. That is an important part of the content of the congruence theorems SSS, SAS, AAS (Figure 4A). What constraints will make these unique up to congruence, or at least up to congruence in a neighborhood as with SSA (Figure 4B)? What about six points in the plane? Figure 4C, D illustrates some patterns that guarantee local uniqueness, up to congruence. What about 1000 objects in the plane - and algorithms for handling these? (The basic count for such independent constraints in the plane on n points is 2n-3, see [22, 59].) What about the analogous theorems and questions on the sphere?



A B

Figure 5

∑ Consider a triangulated sphere in 3-space (Figure 5A). All convex triangulated spheres are rigid (locally unique up to congruence) in 3-space, as illustrated by domes and natural structures. All of the edges are independent constraints and we can change in one a small amount and the structure will not crack [58]. All of this is not the observation of Fuller but of Cauchy and is a property embodied in many naturally growing organisms. How about other patterns and other objects? What about the triangulated torus (the types of objects projected for space stations etc.)? That is a subject of current research (Figure 5B). In general, we ask: can the computer also generate the overall change in the structure from the small changes in the lengths?

How these kinds of questions are answered (exact symbolic expressions, numerical approximation, etc.) is a basic issue determining the 'competitiveness' of CAD engines which run inside AutoCAD or the programs at Boeing, GM, Mercedes, etc. [44]. The people working on this are generating numerous new geometric problems and are using everything we know about straightedge and ruler constructions, etc. Of course, the designer is not willing to wait for days to see the results of a change - so the responses must be efficient, often real-time.

(2) Robotics

To use a robot, we must input (cameras, sensors, prepackaged information) a geometric model of the environment. The whole issue of what vocabulary will be used (e.g. solid modeling – Boolean combinations of basic shapes, polyhedral approximations etc.), and how the information will be structured is a major area of research in a field called 'computational geometry' [43].

With this model in the computer, we must plan what can be moved, and along which paths (motion planning). Can we get the object from here to there with the robot? Using this 'motion plan', we must determine which sequence of actions by the manipulators and the locomotion will move the objects along the planned path (inverse kinematics). At every step, there are geometric problems to be solved - and they must be solved efficiently.

These problems alone have generated books of new results, new forms of old geometric techniques, and new questions [22]. There may be more graduate computer science courses in computational geometry than graduate mathematics courses in discrete geometry today.

(3) Medical Imaging

We want to use non-intrusive measurements (pictures) to construct an adequate three-dimensional image of parts of the body. For example, a series of projections or images from ultra-sound, or MRI from several directions or points are collected. How many measurements do we need to construct the full three dimensional image? What algorithm can we use to reconstruct the full image from the pieces? Again, lots of geometric problems, lots of research and some substantial new results in fields like geometric tomography [21].

(4) Computer Animation and Visual Presentations

How can the computer generate sufficiently rich images to fool our human perceptions of the static form and the moving objects? Experimental movies such as 'Gerri's Story' are exercises in substantial mathematics with a clear geometric component. The current version of the video 'A Bug's Life' contains this Academy Award winning short. One of the computer scientists/geometers who worked on this film described it as exercise in handling texture and modeling clothing with new levels of mathematics. New mathematics with a geometric base, such as fractals, are a piece of this work. So is geometric modeling; one of these key developers at Pixar moved from academic research working with Boeing on geometric modeling.

(5) Linear Programming

In business, a widely used 'new' piece of mathematics for scheduling and for decision making is linear programming - the optimization of some function (e.g. low cost or high profit) under constraints. The basic conceptual processes for linear programming, and a number of the innovations in linear programming, have a substantial geometric basis (polyhedra, higher dimensional polytopes, and duality). One of the big puzzles in this area is why the algorithm is so efficient - and how to predict the special situations where it will not be efficient. Sometimes the algorithms themselves are 'dumbed down' because a correct understanding requires familiarity with projective geometry and few people have that familiarity these days.

There are comparable or related geometric problems arising in chemistry (computational chemistry and the shapes of molecules), material physics (modeling glasses and aggregate materials), biology (modeling of proteins, 'docking' of drugs on other molecules), Geographic Information Systems (GIS), and most fields of engineering. As a reflective chemist said in a recent lecture: 'chemistry is geometry'. Some sources of information on these developments would include [3, 22, 43, 52].

In summary, geometry is out there and it is essential for application. Geometry will be practiced, with or without mathematicians, and with or without an education in 'geometry'. I believe this geometry would be done better if the future practitioners of geometry receive an appropriate preparation in geometry. The 'geometry gap' will haunt North America.

(B) A HUMAN ABILITIES – VISUALIZATION

Many sources confirm that human intelligence (collecting information, organizing and remembering it, reasoning and problem solving with it, communicating it) is a mix of many distinct interconnected abilities [20]. One package of these abilities, developed through our visual perceptions and our visual experiences of the 3-D world, augmented by our kinesthetic experiences and intelligence, I will call 'visualization' for short. Let me cite some examples and evidence.

∑ From our earliest months, our visual apparatus is one of our richest sources of stimulation and information. Vision and spatial perception is richly wired into the brain, with amazing capacities to process and interpret [28, 35]. Recent studies of our 'visual intelligence' in the sense of direct perception already demonstrate in a dramatic fashion that we construct what we perceive. I strongly recommend the recent book by Donald Huffman: Visual Intelligence: How We Create What We See [28]. A fascinating part of this analysis is the rich set of skills, with their deep, implicit mathematics (geometry and topology) which the child develops by about age one and continues to develop throughout life. By age two, given a pattern of shifting features, we 'create' a single rigidly moving 3-D image if that can fit the perceptual data [28]. I encourage you to check the associated web site for some illustrations of what we create! Visual work in general, and spatial (geometric) work in particular, build on one of our richest sources of information and highly developed set of cognitive abilities.

∑ Today, when people want to display rich sets of data in statistics or other sciences, seeking 'patterns' to understand the information, they use rich, carefully designed images. There are substantial efforts to encourage people to 'visualize' data for work in statistics and other related fields. The recent book Visual Revelations wonderfully illustrates the value of putting together pieces of information in overlapping visual pattern [56]. The books [53, 54] are other classic collections of information and data in visual form. Books on 'Scientific Visualization' are spreading the images of what is now possible and desirable in many fields of science [11].

∑ We speak of 'imagining' (imaging) ideas and experiences in our heads. Recent studies in neurology and cognitive science confirm that this internal imaging uses brain processes and 'spatial' search techniques in common with what we do when we 'inspect' an external diagram or object [35]. The two experiences can be considered as parts of a single whole.

∑ In problem solving, images and diagrams play an effective role in (re)organizing the information into associated parts and even coherent wholes in a 'gestalt' that are very different from how we work with language or formulae [37, 41].

∑ Much of our ability to use the objects and devices which augment our memory and our ability to control our environment (things that make us smart [42]) depends on good visual design of interfaces to indicate, without words, what can be done and what effect our actions are having.

∑ With good notation, steps in algebra are determined by 'appearance' in essential ways. What I do in the next step in a problem is based on what I 'see' and how the current step is presented. Much effort in algebra is spent changing appearances to evoke the correct next step. Of course, done correctly, these are controlled steps. As I tell my students, algebra is cosmetics, not surgery: change the appearance but not the substance.

There is a rich interdisciplinary field of research under headings such as 'diagrammatic reasoning', 'thinking with diagrams' and 'spatial reasoning'. (See for example [15, 37, 38]). These studies bring together work on cognitive science, artificial intelligence, design, history of science, pedagogy, human-computer interfaces, philosophy, and creativity, among others. There is now a rich literature about the role of diagrams and geometric reasoning in effective learning and creation in a variety of professions, including mathematics. Much less is known about how individuals actually use diagrams and about the roots of the substantial differences in ability among individuals.

Moreover we can change the way we 'see' things. I have recently been working to develop a course on visualization, as my report elsewhere in this volume describes [60]. I have been struck by the consistent messages from books such as 'Drawing on the Right Side of the Brain', Thinking Visually, and 'The VizAbility Handbook' [17, 38, 61]. Their goal is to 'change the way you see' and this can be done. This is an important message that I will return to below - most people can improve their visualization and spatial reasoning.

When we compare different individuals on their visualization skills, we find substantial variation. For example, tests confirm that people have a wide variety of skills with mental transformations, and with related problem solving skills involving spatial and visual reasoning [63]. The role of such transformations (learned ways of mentally modifying diagrams) and how these connect with specific skills developed from geometric experience and for solving geometric problems would be an important area to understand. Much of this research is currently being done by non-geometers.

It is tempting to assume that visualization, while valuable for some people, is not essential. However, for some people visualization is their essential mode of reason. Consider work by and about high functioning autistics which indicates that some of them are essentially visual learners. See, for example, the autobiographic: 'Thinking in Pictures: my life as an autistic', by Temple Grandin, a tenured university professor who designs facilities for handling domestic animals [23, 50]. There are people who make very little use of visual and geometric reasoning and there are people who rely on visual reasoning almost entirely.

What is the role of visualization in mathematics? Tommy Dreyfus [16] gives a good survey of related literature as well as the impact of computers on visualization. Everything we are finding out about the how the brain works, about the variety of people's 'intelligences', and about the different parts of the brain that are active for different approaches to a 'mathematical' problem, confirms how distinct our approaches to mathematics can be and how rich it is to combine multiple approaches. As an example, I mention something from the Science last April [10]. Experiments with bilingual people, and two types of problems confirmed that explicit numerical calculations involved the linguistic centers, but approximate estimations about the same numbers involved other parts of the brain (parts closer to the control of the fingers and perhaps to spatial reasoning) [9, 10]. The role of geometric and spatial cognition in how our brains 'do' mathematics (mathematical cognition) is a rich area for research and insight.

More often than most mathematicians admit in their public communication, creative mathematics is done in visual forms. For some classical descriptions of this, I recommend the book of Hadamard [24], in which this role of the visual in creative mathematics is a central theme. Moreover, many parts of mathematics have a geometric counterpart - a counterpart that may give an important sense of 'understanding' and 'insight' [8]. However, the public culture of mathematics has downplayed the role of the visual - as at best an analogy and at worst an inadequate substitute for the 'real mathematics' of theorems and proofs (done in formulae and language). Here is a quote from the mathematician J.E. Littlewood [8 p.xi]:

> "A heavy warning used to be given that pictures are not rigorous; this has never had its bluff called and has permanently frightened its victims into playing for safety."

Mathematics has many faces and needs people with many different approaches.

When I have spoken with reflective high school teachers about why geometry should be taught in high school, they often respond in terms like: "I need geometry and the activities it opens up in order to involve some of my students in mathematics. It is delightful to see how roles change and students who were struggling are now the leaders in certain activities." Mathematics needs to be taught in an inclusive way that helps the visually strong people to connect to the core of mathematics and also see themselves as empowered users and creators of mathematics. Geometry can play an important role in inclusive teaching of mathematics.

A key point is that many of these outcomes such as scientific discoveries, new mathematical theories, and problem solving in general have substantial visual/spatial components. In some patterns of dyslexia, visual strengths more than compensate for weakness in reading and writing words [57]. Both of these skills, and more, should be presented as core abilities to bring to science. Listening to my class on visualization for first year science students, it is clear that all subjects could do a better job of integrating and teaching the use of visual in science. It is also clear that, in general, mathematics classes do a particularly bad job of this.

In short, visualization is a rich ability for many people and can play an important role in fields. All subjects, including mathematics, should aggressively incorporate this ability into their public and private practices. My regret is that we have developed a curriculum which convinces many people who are 'algebraic' that they do math well and do not need other skills, and convinces many people who are 'visual' (geometric) that they do not belong in mathematics. I will return to that below.

So far, these observations lead me to two connected conclusions for students: the value of visualization in a variety of careers and professions; and the value and possibility of increasing their abilities in visualization and spatial reasoning. Geometry has a role to play in both of these.

(C) SUITABLE RESOURCES FOR LEARNING

In my talk, I had the liberty to include overheads from books and articles, some animated images from my computer, and even some objects to be seen and manipulated. I have described the great value of appropriate use of geometry and visuals. Why is there so little 'visual' in this paper and in mathematical presentations in general?

When I was drawing up this paper, I had still and moving images for almost every paragraph. The difficulties I faced represent some the restrictions of print as a medium and of my resources:

∑ the lack of simple tools for the preparation and display of some of these images;

∑ the conventions of copyright and 'fair use' for images compared to conventions for text;

∑ the fact that many visual conventions I have for myself are private – not part of the shared conventions of mathematics (this is a vicious cycle).

All these mean that I am doing something very perverse in this section: using text to describe the impact of visuals, rather than the visuals themselves. My students have reported similar experiences. Even when CDROMS are used in class and in studying, they face exams that are almost entirely text – and have their own expectation that answers will be essentially text. They have not learned (or been taught) the effective use of visuals for their communication. My very difficulties illustrate, in part, our standard text rich, visual poor presentations of mathematics including geometry.

The last decade has seen an increasing number of dynamic geometry programs for teaching and for research. Some of these programs, such as The Geometer's Sketchpad and Cabri, were originally designed for teaching geometry at the secondary level, but they quickly became tools for researchers as well [6, 31, 33].

People who wish to solve plane geometry problems, at any level, find these animated tools for explorations and investigation are invaluable in adding precision and dynamic transformations to what would otherwise be a static, possibly crude, external representation. Although we start with an unlimited supply of 'planes' (pieces of paper) to experiment with, we find that these new tools make us 'smarter' in geometry [42]. We (teachers and students) can now conjecture, generate a rich set of examples or counter-examples, and extract visual patterns and processes for reworking into proofs of various types, including visual proofs. The recent article of Philip Davis [14] highlights how dynamic geometry programs and other computer aids have transfigured the study of triangle geometry. Anyone reading recent exchanges on the geometry lists at the Math Forum [39] will have observed the depth, the passion, and the insights that are being reported from classrooms and researchers across the world.

A new generation of tools designed for geometry research, such as Cinderella [48], are spreading this impact from the classroom and plane Euclidean geometry to the classical geometries (Euclidean, hyperbolic and elliptical), with multiple models and broader transformations. At best, we start with a very limited supply of physical spheres or 3-D pieces and objects [51]. These new programs, and similar programs for polyhedra [45, 47], bring the playfulness and the precision of plane dynamic geometry to these other fields of geometry.

These tools for transforming the practice of geometry are spreading. However, effective programs for the full range of 3-D geometry are still not available. Given an algebraic formula, we can generate 3-D displays in Maple or Mathematica. We can even use a mouse to change the view of this displayed object. However, this remains a long way from our experience with object in the world – and the spatial cognition that goes with that experience. The struggle centers around input devices that capture what our hands do in space, and on control over displays which capture what our head and eye movements do with real objects. I anticipate that some of these difficulties will be solved in the near future.

As is richly illustrated by the recent book [33], these dynamic geometry programs can change what questions we ask and what methods we are likely to use. Increasingly, these programs permit easy display of dynamic images on the web, in machine independent Java. I know that when I do visual and diagrammatic reasoning, I often run 'animated movies' in my head. Such images are now accessible for communication with our students. They are also accessible for the students to use, without years of hit and miss learning or fumbling with inaccurate ruler and compass constructions where 'concurrent lines' never seem to meet. Moving these images from an internal image to an external form has a dramatic impact on the role they play:

- ∑ the precision and reliability is increased, particularly for the beginner;
- ∑ the range of examples experienced in a short time changes by an order of magnitude;
- ∑ students experience 'invariance' of properties over changing examples (a fundamental concept of geometry – see the next section);
- ∑ students move to a higher level of analysis and synthesis, as illustrated by the difference between a 'drawing' and a 'construction' in these programs.
- ∑ we establish both shared experiences and common conventions for the classroom community and the larger community;
- ∑ we have improved communication among the users, based on these common experiences;
- ∑ we even have more closely shared internal images - extracted from the shared external forms.

What we can do in the geometry classroom or laboratory has changed. This generates a dramatic shift in how geometry is practiced and in what we 'see' (and who sees it).

Changing our tools generates critical changes in our subject. The history of science offers an insight into the impact of a change in technology. In the history of science, 'instruments' play a critical role in the development of any experimental approach. Dynamic geometry, complete with measurements, constructions etc. plays the role of our new 'instruments' - replacing the earlier (and more limited) compass and straightedge instruments, or the transitional 'instrument' of origami and paper folding. We now have the basis for 'experimental geometry' at a very high level. While these programs primarily produce images (often moving images), they can also produce numbers, and tables of numbers (graphs). In this, they very much resemble instruments in physics or biology - extending our senses and our kinetic experiences to a level that becomes a qualitative change. As in experimental sciences, these instruments raise the possibility of 'moral certainty' that something is true, even though we do not have a mathematical proof (and therefore may be unclear about the exact assumptions being made) [25 pp.229-231]. This in itself is an interesting debate [8].

Companion resources, which embody active collaborative learning with open-ended problems, a rich variety of connections, and strongly visual presentations, including the appropriate use of manipulatives, have been developed for teaching geometry [27]. Dynamic geometry programs become one of the tools for collaborative learning. While these resources assume a limited class size and an engaged student body, the experience among those using the resources is that the students quickly become engaged, provided the other resources of space and limited class size are provided. The only catch is that these active collaborative methods are not common in other mathematics classrooms and some students find this transition hard. For the same reason, some of the graduates of these courses find that they cannot sustain these methods in their own secondary classrooms, under the pressure of a crowded curriculum and the lack of resources and support.

Effective changes in the geometry classroom depend on changes in how other parts of mathematics are taught, Changes in how all of math is taught becomes an associated goal. There are new programs to teach statistics (at the high school level) in a highly visual way, coming from the same people who developed dynamic geometry programs [32]. There are programs to 'visualize' what is happening in differential equations and dynamical systems. Their visual form and appeal drive the interest in fractals, for teaching and for many applications. Fractals have substantial applications in graphics for 'natural objects'. Fractals are also being used for image compression. The study of fractals and dynamical systems would not have evolved as rapidly without our powerful tools for visualization. Other fields are experimenting with the appropriate combination of diagrammatic and algebraic reasoning [64]. These, however, are the topic for another occasion.

There are now materials for a variety of geometry courses for non-mathematicians who are called on to use 'geometry' in their own fields. At a school such a Cornell, there are about eight different undergraduate geometry courses and the enrollment is strong. There are also courses taught in other departments, such as computer science, which contain substantial portions of geometry. As with many other parts of mathematics, if the need is there and the mathematics department does not meet that need and meet it well, parts of the teaching of mathematics will migrate into the other programs. That will be our loss and perhaps also a loss for our students.

Must there be a long 'lag' between the revival of geometry at the research level, in applications and in mathematical work, and the revival of geometry in the high school and elementary curriculum? It is possible that these new resources and our awareness of the trends to increasing use of geometry and visualization can shorten this lag.

This will only happen if these tools and this vision are in the hands and the minds of the next generation of teachers. The recent province wide purchase of the Geometer's Sketchpad for all schools and teachers in Ontario, and the companion requirement to use dynamic geometry from grade 9 on in the new curriculum, are encouraging signs. If supported effectively by Faculties of Education and Departments of Mathematics in our teaching and our approach, we can accelerate the shift towards geometry and the visual.

III. WHAT IS A GEOMETRY?

Having stated my case for a revival of geometry, I should be more precise about what geometry is, and some conclusions we can draw for the curriculum and the pedagogy of geometry.

The 'modern' definition of geometry, due to Felix Klein in 1870, states that a geometry is a space with a group of transformations into itself [34]. The geometer studies the properties which are invariant (unchanged) under these transformations. There are different 'geometries' connected by different groups of transformations, forming a hierarchy of geometries.

Consider the visual presentation of the hierarchy in Figure 6. This shows (via inclusion) the groups of transformations increasing from the restricted congruences of Euclidean geometry, through the shearing and parallel projections of affine geometry and the central projections of projective geometry to the general continuous maps of topology. As the group of allowed transformations gets larger, fewer properties are unchanged (invariant). The concepts we will study get fewer, the vocabulary gets simpler, and the theorems get more general. More and more objects are 'the same' up to the 'symmetries' induced by these transformations.

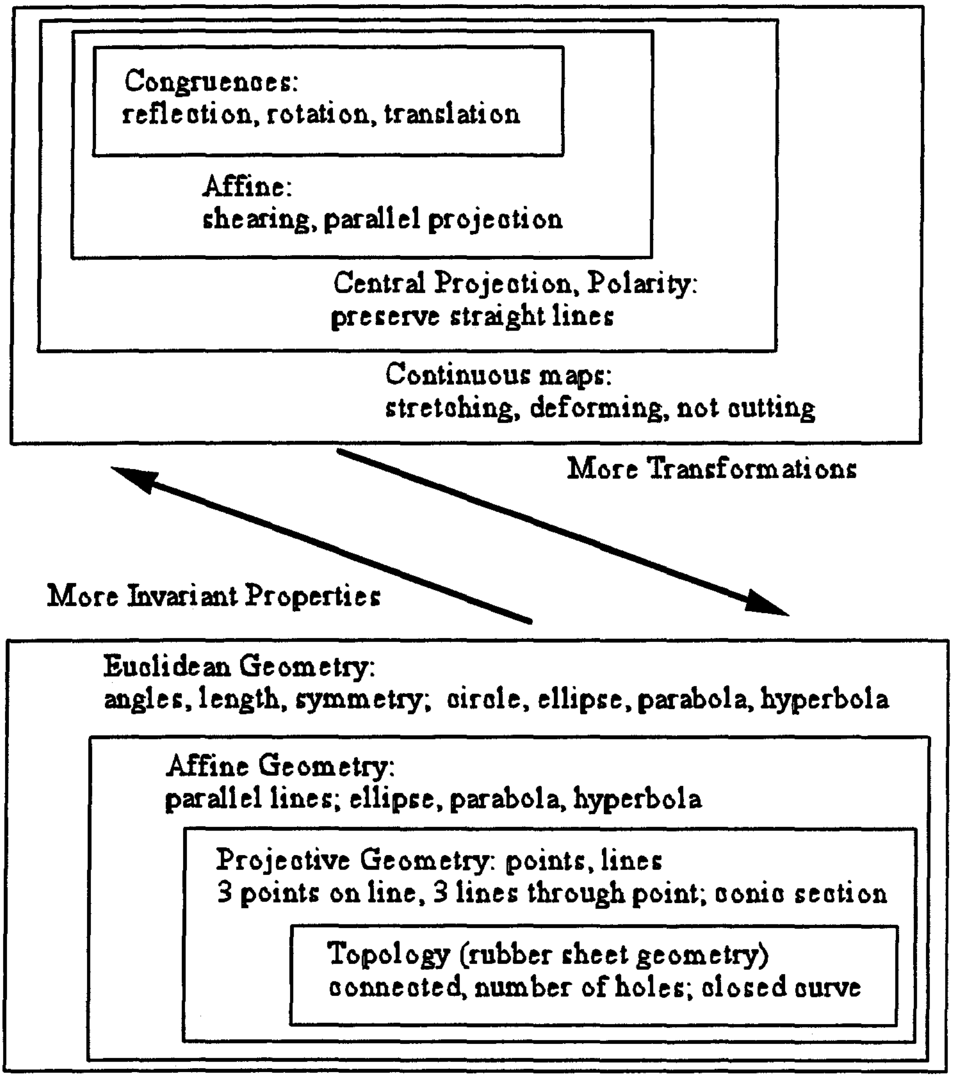


Figure 6: Klein's hierarchy of plane geometries

For example:

1. In topology, all simple polygons (even all simple closed curves) are 'the same'. For any pair of simple closed curves in the plane, there is a reversible continuous map (a topological homeomorphism) which takes one onto the other. A typical topological theorem is the Jordan Curve Theorem (Figure 7): a simple closed curve has an inside and an outside.

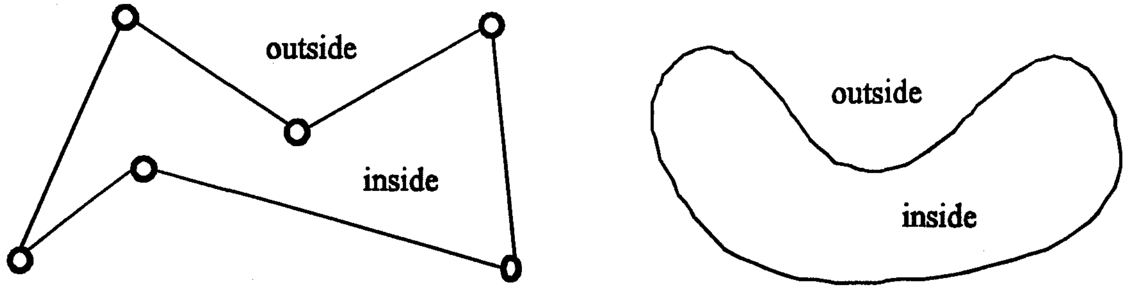


Figure 7: A topological theorem

2. In projective geometry, all quadrilaterals (except those with three points collinear) are 'the same'. That is, given any two quadrilaterals, with no three points collinear, there is a projective transformation (a collineation) which takes one onto the other. Straight lines, points of intersection, and associated properties (such as six points sharing a conic) are preserved. A sample theorem would be Pascal's Theorem (Figure 8) that a hexagon on any conic section has alternate pairs of sides meeting in collinear points. Once you prove this for a circle, projective transformations will prove it for any (non-degenerate) conic section.

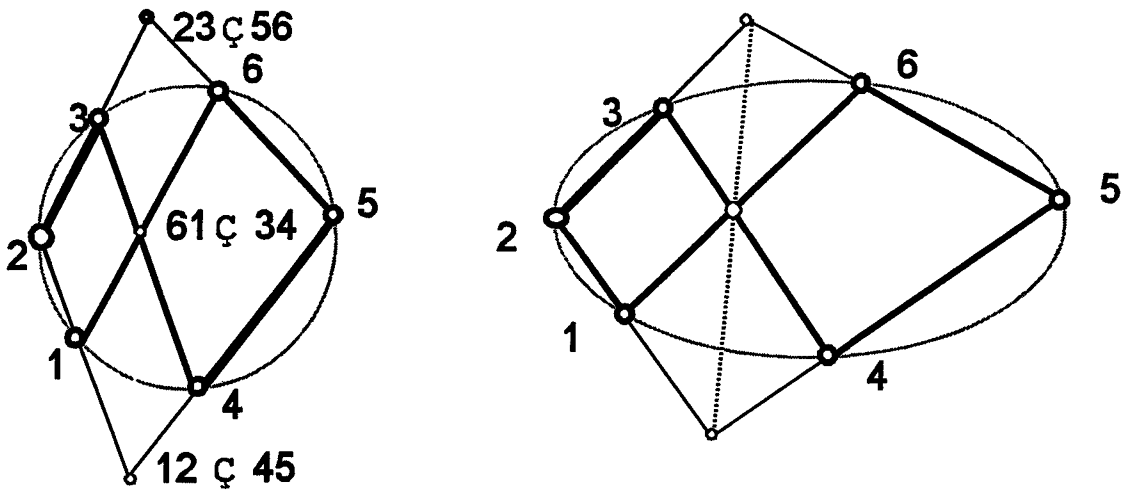


Figure 8: A projective theorem

3. In affine geometry, all non-collinear triangles are 'the same'. That is any non-collinear triangle can be taken onto any other such triangle: the equilateral triangle is the 'typical triangle'! A sample theorem would be that the medians of a triangle meet in a point (Figure 9). Since this is true

for the equilateral triangle, and medians are preserved by affine transformations, we have a simple proof for all triangles. All parallelograms are the same, so the square is a typical parallelogram. Thus another typical theorem would be that the diagonals of a parallelogram bisect one another.

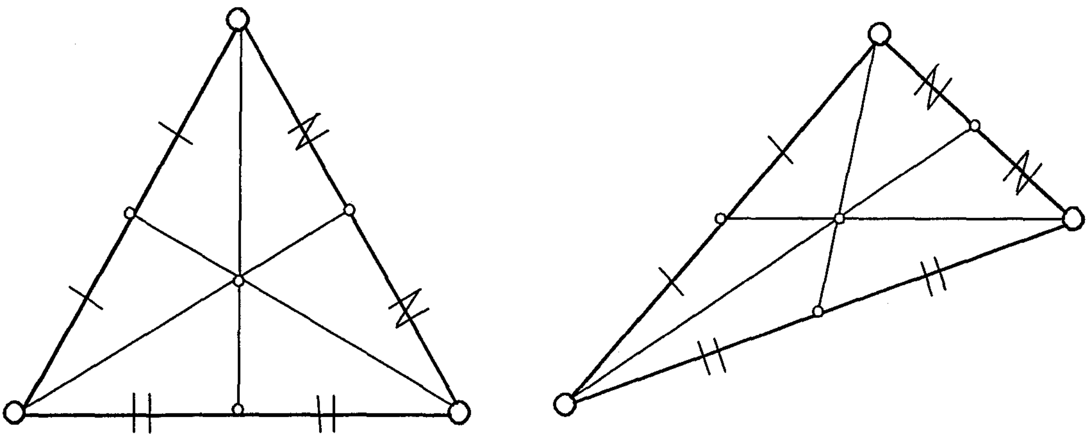


Figure 9: An affine theorem

4. Finally, in Euclidean geometry we distinguish triangles by the size of their edges and the size of their angles. A typical theorem (nicely proven by reflections) is that the right bisectors of the sides intersect in a point (the circumcenter) (Figure 10).

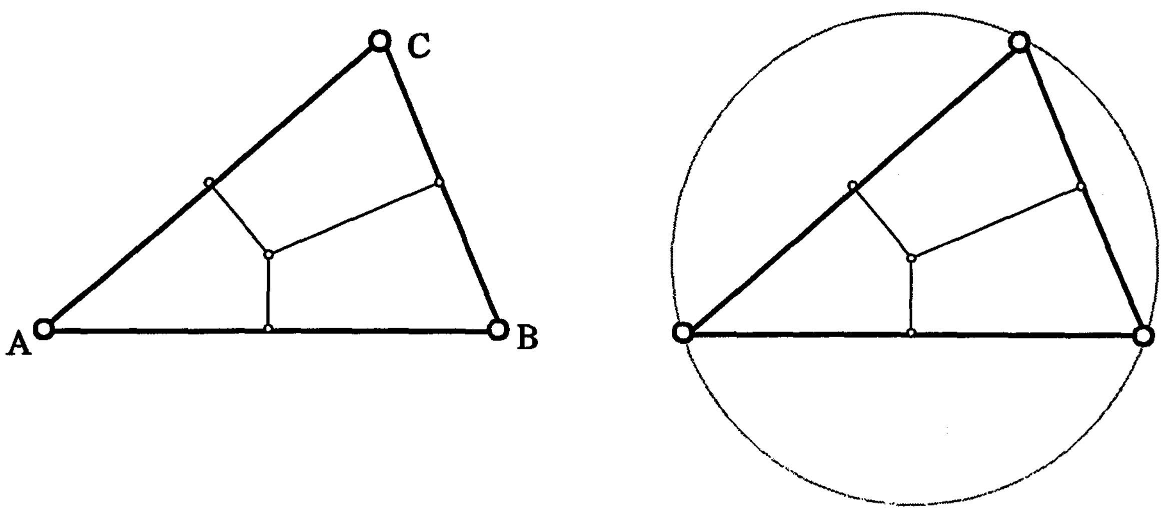


Figure 10: A Euclidean theorem.

There are several practical points that emerge from this hierarchy of geometries:

(i) 'Transformations' are the key concept of geometry. Reasoning with transformations should be a central theme of our learning of geometry [62]. Patterns, the very core of mathematics, are about invariance or sameness under certain transformations.

(ii) The more transformations you have, the more objects are 'the same' and the simpler the properties and vocabulary will be. If you can use topology or projective geometry rather than Euclidean geometry, the thinking and writing will be simpler. Knowing only Euclidean geometry is like having only one tool – a hammer. Everything becomes 'a nail to be hit' and many tasks cannot be done effectively. Geometers need to know, to 'see' the patterns of many classes of transformations.

(iii) There are many groups and many geometries - and adept people will choose which geometry to use for a particular problem with care. For new geometric problems, the first crucial task to is to decide: which geometry? Unless the problem is correctly placed within the geometric hierarchy, there is a substantial risk of:

∑ either burying the pattern in a mass of irrelevant detail by being too low in the hierarchy so that little effective can be done;

∑ or losing the pattern completely by being too high in the hierarchy.

On several occasions in my work in applied geometry, I have found people stuck 'too low' in the hierarchy, lost in a maze of irrelevant details and unaware of the level of invariance and the powerful tools that a better, 'higher' geometry would bring [59]. On the other hand, as Einstein said, "things should be a simple as possible and no simpler". I have also encountered people working too high in the hierarchy, where nothing correct can be worked out because the properties studied are not invariant under all the transformations. The quality of the answer to this fundamental problem of 'Which Geometry?' will shape the entire study.

(iv) Plane geometry, spherical geometry and other geometries can be studied, compared and connected to see different and comparable forms of geometry [27]. This too is part of the geometric hierarchy. For example, many of the common theorems of Euclidean, spherical, and hyperbolic geometry lie in the shared projective geometry that lies above them all in the larger hierarchy. This common projective geometry is also important for application [22].

(v) People learn basic skills in various geometries at different stages of development. There are a number of indications that, roughly, children develop 'down' the hierarchy from topology (first) to Euclidean geometry last, as was indicated by Piaget [30, 63]. Certainly by age four, children know in a practical sense, what is connected and what is not (what they can reach, or whether a new space includes a 'race track' they can run around and return to their starting point).

(vi) Children learn 3-D transformations before they learn 2-D transformations [28] and they learn the 3-D geometric hierarchy before they learn the 2-D hierarchy [4 p.6]. Experiments in the former Soviet Union on spatial reasoning indicated that even elementary students have substantial abilities with projections and 2-D representations of 3-D objects, and can learn more than is usually taught, if we think it is important [63].

(vii) Our teaching of geometry, from the plane up, disguises and even blocks that knowledge [63 p.200]. In short, we teach geometry in the reverse order of children's development, from the bottom of the hierarchy up and from 2-D to 3-D, through most of the K-10 curriculum.

As you will see in the next section, an understanding that geometry is about transformations is central to my view of how geometry should be taught.

IV. REFLECTIONS ON TEACHING GEOMETRY INTO THE NEXT CENTURY

Which would you choose?

A geometry class is like:
(a) a trip to the dentist or the doctor;
(b) a trip to your favorite restaurant.

From the student comments I see on the internet, the common answer would be (a). However, from what I also see on the internet, with a different environment, geometry is an area of mathematics that is highly engaging and can generate high quality learning for a wide variety of people. A basic issue is how geometry is taught.

I offer a brief summary of some conclusions about the teaching of geometry that I draw from the previous discussion.

1. The overall curriculum should teach geometry 'down the hierarchy' - from topology, through projections and finally to Euclidean geometry, making visible the connections that the students themselves have already learned.

We ignore the rich talents of young students at the upper levels of the geometric hierarchy and disconnect from this experience at our peril.

2. Start 'geometry' early - students can do a lot more than we credit them with.

At a very early age, children develop a very rich 'visual intelligence' in terms of perception and experiences. They have questions and lots of these questions and explorations can be connected to geometry if we use the right types of physical and visual presentations. They have developed advanced visual skills for which the precise vocabulary is 3-D differential geometry and differential topology [28, 36]. I would not propose we use this vocabulary but I would propose we do not ignore, even suppress these visual abilities. Instead we should connect with these abilities.

3. Teach visualization, transformations and spatial reasoning using manipulatives and graphic representations in a systematic way. An important and achievable goal is to expand the way that students 'see'.

We often don't teach the use of visual tools and visual reasoning in any systematic way [1]. We just test it as something obvious – 'the students will see what I see' - or else we avoid the use of visual tools because we (correctly) predict that a substantial number of students will not 'get it'. The van Hiele model of geometry education reminds us of the basic pedagogical lesson that students learn from their experience up and we cannot lightly skip over basic stages and basic connections [29]. Even at University today, we cannot assume that the basic levels of experience, vocabulary and communication are in place. This imperative that we work through all the van Hiele levels in any new geometry, including the use of appropriate software and manipulatives, applies to the undergraduate geometry classroom. It is important both for the learning of the students, and as a model of how other geometry can be learned and should be taught in their future classrooms.

4. Teach transformations with animated images and not just static images.

The tools are now available to bring this into the classroom and to bring these home, on the web or the home computer. Transformations and change within geometry are central to understanding geometry, as we experience it [36] and as we apply it (see point 9 below).

5. Use 3-D from the beginning, along with representations and transformations.

Children and adults live and see in three dimensions [49]. Neglecting this in early geometry education actually interferes with appropriate transfer from this experience into working with geometries. To quote from the NCTM standards:

> "In grades 5-8, the mathematics curriculum should study the geometry of one, two, and three dimensions...so that students can visualize and represent geometric figures with special attention to developing spatial sense."

3-D is our primary experience and 2-D representations are highly 'conventional' and difficult to 'read'. We need a rich curriculum on representing, analyzing, and constructing 2-D images for 3-D objects and 3-D objects from 2-D images [1, 4, 40]. However, I would start earlier than grade 5.

6. Include visual forms throughout the mathematics, at all levels, to include more of the students within the tent of 'those who are good at math' and to enrich the range of approaches of all students.

Mathematics needs to be taught in an inclusive way that helps the visually strong people to connect to the core of mathematics and also see themselves as empowered users and creators of mathematics in the sense of geometry and information that can be encoded in geometric forms.

7. Include applications of geometry and integrate consistent visual forms throughout the curriculum [12].

Visualization and the use of geometric forms and geometric reasoning runs across many subjects: geography, science, technology, and computer science. Just as it is important to use consistent notation for numbers and words, it is valuable to use consistent visual forms in a variety of situations and years. Without this, there will be very limited visual communication.

8. Geometry should return as a central part of the undergraduate curriculum, both for future mathematicians and as a service course for a variety of fields.

Geometry has an important role to play for mathematics majors and for future professionals in may fields. Every undergraduate program should include appropriate geometry courses aimed at these varied groups. Such courses should be rich in the use of dynamic geometry, manipulatives, transformations, and connections with applications. Universities that have many geometry courses find that enrollment is strong - there is the student interest.

9. Teach the geometric hierarchy within the core geometry curriculum.

Although the geometric hierarchy is not central to most current teaching of geometry, it is already central to working with geometric applications where invariance under transformations is a key issue. Any geometry course taught to people who will apply geometry must include an effective introduction to deciding where a problem lies in the geometric hierarchy. Any course taught to future geometry teachers should also include a solid working introduction to using the hierarchy.

10. An appropriate geometry course is an essential part of the training of future teachers.

All teachers of mathematics at the secondary and primary levels need to be comfortable with visualization and exploration in geometry. They need to be comfortable with multiple approaches to geometry, including dynamic geometry programs, and thinking with diagrams. In order to practice these approaches in their own classrooms, they must experience undergraduate courses that are rich in the use of dynamic geometry, manipulatives, transformations, and connections with applications. Teachers who have been cut off from the sources and practice of geometry cannot nourish the imagination and make the connections to the rich experience of the students.

This is something I have quietly believed for some time. Writing this article has encouraged me to be public with this proposal. I know of no Ontario Faculty of Education that makes geometry part of the required background even for a mathematics specialist, let alone someone with mathematics as a teaching subject. Like many of these 'requirements', the key may be the availability of 'appropriate' geometry courses. That is a responsibility of mathematics departments.

Geometry is alive. Let's join in a celebration and in a conversation about how we support and enjoy this lively part of mathematics.

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Plenary Lecture 3

INDUSTRIAL MATHEMATICS FOR THE 21st CENTURY

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ABSTRACT

As we enter the new millennium and our civilization passes from the "Industrial Age" to the "Information Age", the mathematical sciences are playing an increasingly important role in the economy of Canada and the world. Dramatic advances in applied mathematics have transformed old industries and created new ones. Mathematics is an engine of innovation on the information highway. Computing technology, in turn, has transformed the way mathematics is used in the workplace, and is beginning to transform mathematics education. Using anecdotes, data and concrete examples (drawn from several different industries) it is shown that today in industry, mathematics is embraced, and mathematicians are employed, as never before. This trend is accelerating. A serious shortage of mathematically-skilled professionals is forecast, for early in the next century. A recent demographic analysis shows that this problem will be compounded by a shortage of mathematics teachers at all levels. Action is required to avert these threats. The good news for our students is that exciting career opportunities will abound, for those who develop their mathematical skills appropriately. The dual challenge facing mathematics educators is to attract more students to mathematics, and to develop new curricula which will better prepare students for the career opportunities of the 21st century.

INTRODUCTION

The invitation to present a plenary lecture at CMESG'99 on industrial mathematics and mathematics education has presented a welcome and timely opportunity. As I complete my term as Deputy Director of the Fields Institute and return to teaching duties at the University of Guelph in September, it is natural to reflect upon the unique experiences I have enjoyed through my work at the Fields Institute, for example involving liaison with industry, and to consider how I may wish to revise my own teaching strategies to better prepare my students for their life and work in the 21st century. I am happy to be able to share these reflections with you. I wish to emphasize that the views expressed here are strictly my own, and are not official positions of the Fields Institute, the University of Guelph or any other organization with which I am associated.

MATHEMATICS EDUCATION IN THE 1960's

In order to put into perspective the recent dramatic changes I have seen in the world of mathematics, I wish to begin with a brief review of the attitudes towards mathematics education which were current in the 1960's, when I was a student at Queen's University. As it happens, several of my classmates are present in this audience, and may wish to expand on my remarks.

Canadian mathematics education in the 1960's was strongly influenced by the legacy of G.H. Hardy in England, the Bourbaki school in France and the "New Math" movement in North America.

In 1940, Cambridge University pure mathematician G.H. Hardy published an influential essay titled A Mathematician's Apology.[1] You may ask, "whatever did Hardy think mathematicians had to apologize for?" He wrote that mathematics was seen in the public perception of his day as a set of useful and practical tools, which helped engineers to build bridges, generals to win wars, and the like. In his essay, Hardy put forward the idea (apparently novel in its day) that mathematics was a human creation worthy of being appreciated for its own sake, like art, independent of any practical applications. The proper role of mathematicians, therefore, was the creation of new mathematics. Only mathematicians themselves could judge what new mathematics was worthy of their attention. In his essay, Hardy begged the indulgence of society, to support the work of mathematicians pursuing their abstract endeavours. This idea took root, especially among mathematicians, and by the 1960's "pure mathematics" was king, while applied mathematics was relegated to second class status in most university mathematics departments.

Similarly in France, the famous Bourbaki school developed its program of systematically organizing the body of modern mathematical knowledge into a logical structure, built up from first principles. In their view, mathematics exists independently of the real physical world, and has its own inner truth. Any appeal to physical arguments was strictly forbidden; even the use of diagrams was considered dangerously subversive by the Bourbaki school. I remember using the Bourbaki volume, ALGÈBRE, Chapitre 1, Structures Algébriques[2] as a reference at Queen's, and revelling in its elegant and austere beauty. The influence of this Bourbaki school on shaping the attitudes of my generation of mathematicians was enormous.

The most visible consequence of this view of mathematics, on education in North America, was the development of the "New Math" curriculum in the public schools. Now widely viewed as a dismal failure, at the time it had strong support of mathematicians and educators, many of whom hoped that all students could experience the excitement they had felt on first seeing mathematics the way Hardy and Bourbaki did. Instead, it caused great confusion among students, parents and teachers, and this grand experiment was soon abandoned.

In many university mathematics departments, however, the Hardy/Bourbaki view of mathematics became and remained the reigning paradigm. This led to a flowering of a golden age of pure mathematics in the second half of the 20th century, such as the world has never seen before. Yet, towards the end of this century, some negative consequences began to appear on university campuses: Mathematics professors came to be viewed as living in an ivory tower, irrelevant to other disciplines. Schools of engineering, economics and other fields started to teach their own mathematics courses to ensure relevance for their students. Average research grants for mathematics fell in size to be the smallest of all the sciences – perhaps Hardy's wish that mathematicians be accorded the autonomy of artists led to the conclusion that they should also be funded like "starving artists". A few mathematicians began to wonder if mathematics would go the way of philosophy in the university curriculum. Students began to demand more real-world relevance in their mathematics courses. The pendulum had begun to swing back.

CHANGING ATTITUDES TO MATHEMATICS IN THE 1990's

There is ample evidence, at the end of this last decade of the 20th century, that the austere Hardy/Bourbaki view of the role of mathematicians is in full retreat. A few press clippings will make this clear. In The Toronto Star, January 3 1999 article, How math is suddenly cool to the nth degree[3], examples are given of changing attitudes towards mathematics. Author John Allen Paulos (Innumeracy[4]) is quoted as saying that the phrase "popular math books" was once seen as an oxymoron; now they sell very well. Books on Andrew Wiles' proof of Fermat's Last Theorem have sold beyond expectations. The financial success of Bill Gates and other "nerdy" types has helped make mathematics attractive, even sexy. Haute couture perfume maker Givenchy has introduced a new fragrance for men named Pi (= 3.14159...), promoted as "the thinking man's scent" in full-page ads in exclusive men's magazines. Recent Hollywood movies such as Jurassic Park, Good Will Hunting, Contact and Pi have featured mathematicians in leading roles, and even as sexy heros. The Globe and Mail, January 23 1999, in the article Days of restraint and cutbacks appear to be over[5], lists the ten hottest occupations and undergraduate programs, with the job of mathematician listed first among the hot occupations and the study of mathematics listed in the top 5 hottest programs.

The Globe and Mail, March 5 1999, Report on Business article Bankers grapple with math gap[6], reports that mathematics has become a stumbling block to career advancement in the ever-changing banking industry, as employees are being required to do more than just add and subtract. One upwardly-mobile Royal Bank employee, asserting that she is "math challenged", sums up the situation as follows: "There are computer programs [...] that figure all these things out, but if you don't understand the whole principle, the math behind it, then you can't explain it and advise your clients." In the same newspaper, Report on Business, July 26 1999, it is reported that among university graduates, those in mathematical sciences have one of the lowest jobless rates and highest median incomes[7].

In the Canadian research community, mathematics has achieved a newly elevated status in the past two years. Canadian mathematicians have had an impressive run of major victories. The First China – Canada Mathematics Congress, in Beijing, August 1999, is a "Team Canada"-like initiative to develop future collaborations in the mathematical sciences between these two nations. It is strongly supported by NSERC and the Canadian Embassy in Beijing, and by the Chinese National Natural Science Foundation, and is expected to pave the way for similar initiatives in other disciplines. The Canada Gold Medal, Canada's highest honour in all of science, was awarded for the first time to a mathematician in 1999, Jim Arthur[8] of the University of Toronto.

In October 1998, for the first time in the mathematical sciences, a new federal Network of Centres of Excellence was announced. Called MITACS (Mathematics of Information Technology and Complex Systems)[9], it will support joint academic/industrial research in the mathematical sciences across Canada, involving hundreds of researchers and students in universities and industries. It is expected to bring $25 million in new funding to mathematics over seven years, the largest award ever in the history of Canadian mathematics. Most of this money is allocated to the training of young Canadians in fast-expanding fields of the mathematical sciences. A key tenet of the NCE philosophy is that building bridges between academic researchers and Canadian industries will lead in the end to a stronger economy and greater prosperity for all Canadians.

Also in 1998, the NSERC Reallocation Exercise increased base funding for research in mathematics relative to other disciplines. An NSERC-funded Review of Mathematics in Canada[10] reported that the health of Canadian mathematics is excellent. The 1990's have seen the emergence of the three mathematics institutes (CRM[11], Fields[12], PIMS[13]) as a dominant force in the advancement of mathematics in Canada, and in 1998 the funding envelope of the math institutes was increased substantially. Internationally in 1998, the IMU[14] (International Mathematical Union) elevated Canada to the top rank of its member nations, from its former second tier membership. Truly, the past two years have been unprecedented in the recognition and honours bestowed upon Canadian mathematics and mathematicians.

VIGNETTES IN MODERN INDUSTRIAL MATHEMATICS

In contrast to G.H. Hardy's "ivory tower" view of mathematics, many modern mathematicians work with at least one eye clearly focussed on real-world applications. A few sample vignettes are presented here; enough to reveal that there exist models which are starkly different from that of Hardy, for the role of a mathematician in contemporary society. Further information may be found in the references, including world wide web sites, at the end of this article. (We interpret the word "industry" here very broadly, for example to include the financial industry, communications industry, etc.) These examples show that, in today's increasingly sophisticated and quantitative knowledge-based economy, often the key to progress comes from the world of mathematics. The necessarily limited mathematical training of experts in other fields can be an obstacle to progress, when new situations arise. Paradoxically, the more abstract training of the mathematician often is the key to transcending such practical difficulties.

In his own time, Hardy clearly overshadowed his young contemporary at Cambridge, Alan Turing[15] (1912-1954), who worked then in relative obscurity. Today, Turing is known as a father of modern computing, and as the creator of the Turing machine, Turing test and Turing patterns. Few individuals have had greater influence on late twentieth century life than Alan Turing, through the subsequent development of the computer, according to his principles. Although he did not live to see the impact of this work, it seems likely that he did appreciate its practical significance, and thus did not at all fit the mould of Hardy's Apology. During the Second World War, Alan Turing worked in secrecy on a project that succeeded in breaking the German U-boat communications code, giving the Allies an advantage which Winston Churchill later claimed shortened the War by two years.

In the public consciousness, no one can challenge Bill Gates[16] for the title of "math nerd who made good" – the teen who enjoyed pushing computers to their limits, now the adult who has amassed the largest personal fortune in the history of mankind. His Microsoft software is familiar to people of all ages, in all walks of life, around the globe. Putting aside the accusations[16] that his profits derive from monopolistic marketing strategies, and that his products are poor quality implementations of others' original ideas, it is clear that his influence is enormous. His unparalleled success has inspired countless imitators, and no doubt has much to do with the recent "math is cool" trend in our society.

The name of Bob Merton[17] does not register as highly in the public consciousness, but his impact also has been great. As a new grad student at Caltech in the 1960's, Bob Merton began by studying the applied mathematics used by Caltech's rocket scientists. Soon he decided that he would rather apply mathematics to financial matters, so he left Caltech to study mathematical economics, a field which barely existed at the time. His classmates (including yours truly) bid him farewell, expecting never to hear of him again. In 1997, Robert Merton was invited to Sweden to receive the Nobel Prize in Economics, together with Myron Scholes, for their invention of the Black-Scholes-Merton partial differential equation (Black was no longer living). This formula and its generalizations have revolutionized mathematical finance. (The slang term on Wall Street for those who use sophisticated mathematics is "rocket scientists"; perhaps Bob Merton is the original Wall Street rocket scientist.)

In October 1998, Dr. Ron Dembo was honoured with Ernst & Young's Ontario Entrepreneur of the Year Award for Financial Services.[18] Ron Dembo holds a PhD in Operations Research from the University of Waterloo, along with several other degrees, has held appointments at Yale University and MIT, and has published over 50 papers on mathematical finance and optimization. In 1989, he founded Algorithmics Inc., a leading provider of innovative enterprise-wide risk management software.

Algorithmics now maintains 13 offices around the world, and is based in Toronto with Dembo as President and CEO.

Professor Scott Vanstone is one of the founders of Certicom[19] where he holds the position of Chief Cryptographer, and he also holds an NSERC Industrial Research Chair at the University of Waterloo. Certicom is a leading provider of cryptographic technologies for computing and communications companies. Certicom's success is based on the efficient implementation of the elliptic curve cryptosystem (ECC), pioneered by Vanstone and his colleagues in mathematics at Waterloo. Thus, Certicom represents a significant Canadian breakthrough in transferring new knowledge from the world of pure mathematics to the world of digital communication.

Waterloo Maple[20] is a world leader in the commercial development of software for the symbolic manipulation, graphical display and numerical computation of mathematical objects. Its success is due in part to the solid mathematical foundations on which Maple algorithms are based. To this end, Maple employs outstanding mathematicians, and maintains a network of mathematical contributors around the world. Maple was started in the early 1980's by a small group of mathematical scientists at the University of Waterloo, and today not only enjoys financial success but also has won respect for Canadian mathematics, worldwide.

Generation 5[21] is a Canadian company that is a world-class leader in both developing and implementing the next generation of market-analysis technologies. Founded in Toronto by Dr. Milorad Krneta, a leader in advanced statistical and econometric analysis, Generation 5 provides customer and market-analysis services to an international clientele including major banks and retailers. Currently, 75% of Generation 5 employees are mathematical scientists with at least a Master's degree; six have Ph.D. degrees.

These vignettes provide evidence that the world of mathematics has changed substantially from the view advanced by Hardy and Bourbaki at mid-20th century. While no one is denying the fundamental importance of research in pure mathematics (most of the successes profiled above were based on strength in pure mathematics), increasingly many of the best mathematicians are turning their talents to solving real-world problems. In so doing, they often find substantial financial rewards and job satisfaction. Perhaps most significantly, these new-generation applied mathematicians now receive also the respect of their academic colleagues.

ELLIPTIC CURVE CRYPTOGRAPHY

Elliptic curves are mathematical objects which once dwelt in relative obscurity, but only recently have made dramatic appearances in the daily newspapers, on both the science pages and the financial pages. One of the big news stories of twentieth century science has been the heroic achievement of Andrew Wiles[22], who finally proved the famous conjecture known as Fermat's Last Theorem, which had remained open for 300 years. This is not the place to explain Fermat's Theorem or Wiles proof, except to make the following point. The key to Wiles proof was that he was able to translate Fermat's statement about number theory, into an equivalent statement about elliptic curves. Wiles was then able to prove the elliptic curve version of the statement.

On the financial pages, one reads that one of the biggest problems of the information age is data security. Industries are accumulating more and more data, which must be protected against theft or alteration. Electronic commerce must be secure against electronic forgeries and eavesdroppers. This need for security must be balanced against the need for speed and transparency of communications for legitimate users. Obviously, a security system that took hours to transmit credit card information would be unacceptable. Cryptography is the mathematical science which deals with encoding information into a form which is unintelligible to all except those authorized persons who have the key to decode the information. No cryptographic system is perfect; always there is a trade-off between security and speed.

Canadian mathematicians are partly responsible for a recent major advance in cryptography, which shows promise for improving the security/speed factor in data communications. This new technology is provided by Elliptic Curve Cryptography (ECC). Requiring less computational overhead than conventional security systems, it has already been implemented in smart cards, pagers and personal data assistants. As described above, the Canadian company Certicom is in the forefront of this work.

What are elliptic curves? Since this is a mathematical audience, let me give a brief mathematical definition. They are curves (loci) defined by polynomial equations in two variables which can be written in the form, with real constants a and b

$$y^2 = x^3 + ax + b$$

i.e. that are quadratic in y but cubic in x. It is required that the cubic polynomial on the right has three distinct roots. Thus, in the following two examples the first is an elliptic curve, shown in Figure 1, but the second is not.

$$y^2 = x(x-1)(x+1) \quad y^2 = x(x-1)^2$$

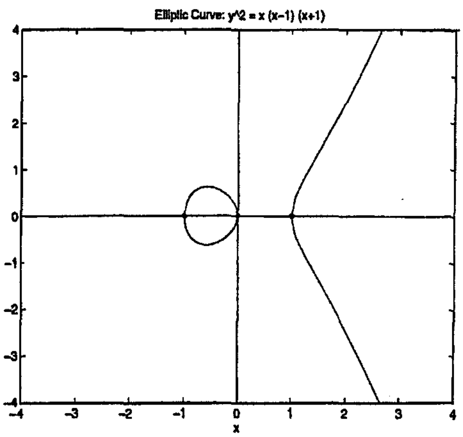

Figure 1

To learn more about the very interesting properties of elliptic curves, and their relation to cryptography, a good starting place is the lecture of V. Kumar Murty on the SIMMER website[23]. For more serious study, a complete course in Elliptic Curve Cryptography (ECC) can be found on the Certicom website[24].

MATHEMATICS EDUCATION FOR THE 21st CENTURY

There is a growing consensus that future growth of the Canadian economy will come primarily in the high-tech and knowledge-based industries. To prepare for this future, our society must invest now, in education and training for this employment sector. The Province of Ontario launched in 1998 the Access to Opportunities Program (ATOP)[25], which recognizes this need and responds with an investment of $150 million over three years, to double enrollments in Ontario university programs in computer science and high-demand engineering fields. It is important to note that all of the fields targeted by ATOP for expansion are mathematically-intensive. Mathematics may be viewed as a resource that fuels the growth of the knowledge-based industries which create new jobs for Canadians. Thus, ATOP implies a significant expansion of mathematics education in Ontario universities. The same arguments as presented for ATOP in Ontario apply at all levels of education, and in other provinces across Canada. The Canadian mathematics education community must prepare itself for a dramatic expansion in demand for mathematics education in the coming decade.

In July 1997, The Fields Institute for Research in Mathematical Sciences and Nortel (Northern Telecom) joined forces to host a workshop for discussion of the global and technological challenges facing secondary mathematics education for the twenty-first century. The outcome of the workshop was the Fields-Nortel White Paper Mathematics Education for the 21st Century[26], which presents a model of mathematic education for the twenty-first century based on a profile of the mathematics learner as a creative, thinking, learning citizen with the knowledge and skills to compete and succeed in a global economy. The recommendations presented in the Fields-Nortel White Paper result from a sharing of ideas, research, visions, exciting achievements, concerns, and challenges expressed in presentations from the industry, business and education sectors and in working group discussions. These recommendations represent a model of mathematics education that is intended to help Ontario secondary school mathematics students develop into successful graduates who match this profile of the learner and are equipped to meet the challenges of life in the twenty-first century. Copies of this White Paper are available free of charge from The Fields Institute for Research in Mathematical Sciences (Deputy Director's Office, 222 College Street, Toronto, ON, M5T 3J1). The full Proceedings of the Fields-Nortel Workshop is available from the same address, for the nominal price of $10 per copy. Some key observations and recommendations from the White Paper are summarized in the following paragraphs.

The Fields-Nortel White Paper begins with the recognition that we live in an era of unprecedented change in knowledge-based industries and communication technologies that are transforming the workplace, schools and everyday life. A strong foundation in mathematics and numeracy is essential for our students' success in a competitive global economy. As Tony Marsh, CEO of Canadian Microelectronics Corp. and Chair of the Education Council of the Conference Board of Canada, stated in his Opening Address to the Workshop, "A shortage of quantitative skills affects our competitive advantage in the global economy." Too many students drop mathematics early, and then are unqualified for post-secondary programs leading to rewarding careers where skilled applicants are in short supply. In the words of American engineer Robert M. White, "Mathematics ... must become a pump instead of a filter in the pipeline of education"[27]. Ontario's economic health in the twenty-first century depends on increasing the flow of creative, highly skilled graduates.

Another impetus for reform of mathematics education is the changing nature of mathematics itself. As illustrated in the vignettes earlier in this lecture, mathematicians in the twentieth century have developed whole new branches of mathematics and invented powerful new tools for problem solving in the real world. Here are just a few more broad examples: Game theory and optimization methods are now used to improve both service and profitability of airlines and express couriers; mathematical modeling has become indispensable to designers of automobiles and aircraft; advances in differential equations and dynamical systems theory have improved robotic controllers and weather forecasting; probability theory is essential to the design of telephone networks and the testing of new drugs. The revolutionary developments in mathematical sciences that have occurred in this century remain unappreciated by many citizens, and are rarely taught in the classroom. There is no other major subject of study in high school for which the present curriculum reflects so little of its modern development and practice.

While fully recognizing the importance of mathematics education for the growth of our knowledge-based industries, the White Paper expresses even greater concern for the mathematics education of the majority of high school students who are not headed for research or high-tech careers. Mathematical literacy (or "numeracy") has become every bit as essential as language skills for survival in our society. As we pass from the industrial age to the information age, many traditional jobs are disappearing, and the workplace is changing to a digital world requiring higher levels of numeracy and problem-solving skills. Nineteenth century "shopkeeper arithmetic" no longer suffices in this world of bar-codes and digital cash. We educators must raise the general level of numeracy in the population. We cannot afford the continued waste, in both human and economic terms, of tragically high youth unemployment rates. Numeracy involves more than familiarity with numbers; tomorrow's citizens must deal confidently with many mathematical concepts, for example: chance, graphs, logic and the dynamics of change. A citizen lacking in numeracy can hardly make valid judgements on issues ranging from interpretation of opinion surveys to predictions of global climate change. The model of high school mathematics education presented in the White Paper is intended for everyone. If we can raise the quality of mathematics education for all students, then more will be enabled to advance to higher levels.

THE LOOMING TEACHER SHORTAGE IN MATHEMATICS

Compounding the crisis implied by the surging demand for mathematics education described in the previous section is a projected shortfall in the supply of mathematics teachers. A demographic study conducted by the Ontario College of Teachers[28] shows that the profession must prepare for a massive turnover in the province's teaching population, due to a wave of retirements of teachers in the "baby boom" generation. Of the 1998 total of 171,500 teachers in Ontario, an astonishing 41,000 will retire within five years and more than 78,000 will reach retirement age in 10 years. These numbers do not include those who will leave the teaching profession for reasons other than retirement, such as family responsibilities, greener pastures or death. Since the baby boom demographics affect all regions of Canada, it is reasonable to assume that the other provinces in Canada will face a similar turnover. In mathematics and some science and technology subjects, the numbers of teacher candidates in the pipeline fall far short of the minimum required for replacement. One reason is that more and more mathematically-talented students are aggressively recruited for rewarding careers in computing and high-tech industries. The problem is compounded by programs such as ATOP, which can have the effect of increasing the demand for mathematics teachers while at the same time reducing the supply.

A concerted effort is urgently required, to recruit more mathematically-inclined students into teaching careers, and to increase the supply of mathematics teachers for the next century. If this is not done, there is a real danger that these teaching positions will be filled with whoever is available, and then our next generation of students will face mathematics education as outlined in this lecture, it is important to consider not only numbers of teachers, but teachers who would in fact rather be teaching some other subject. It would be tragic for students to miss out on having teachers with enthusiasm for mathematics. Because of the changing needs of mathematics also the quality of preparation of the next generation of mathematics teachers. In this sense, the dramatic turnover in the teaching profession presents an opportunity for a significant update of the teaching of mathematics in a short period of time, provided that our faculties of education are able to seize this opportunity. The danger is that they will be so overwhelmed with the problem of producing sufficient numbers of teacher graduates that the opportunity for progress will be missed. Clearly, mathematics education in Canada is on the verge of a deep and multi-faceted crisis. There are many things which can be done; a Working Group formed by the Fields Institute Mathematics Education Forum has already made a study of this problem and reported its recommendations to the Ontario Ministry of Education and Training in May 1998.

There is, however, a bright ray of hope shining over these dire predictions. An outstanding opportunity has been presented to us by the fact that the United Nations Educational, Scientific and Cultural Organization (UNESCO) and the International Mathematical Union (IMU) jointly have declared the year 2000 to be World Mathematical Year (WMY-2000)[29]. Countries around the world will be celebrating their mathematical achievements and promoting mathematics as a key to development. The timing of this special international focus on mathematics could not be better, to help increase Canadians' awareness of the strategic importance of mathematics to the future health of the Canadian economy, and to encourage young Canadians to prepare themselves for mathematically-oriented careers in industry and in education.

COMPUTING TECHNOLOGY AND MATHEMATICS EDUCATION

The twentieth century is remarkable for the dramatic advances made in science and technology, which have transformed almost every aspect of human life. In the second half of the twentieth century, perhaps no advance has had greater effect than the development of the computer. At mid-century, computers were the preserve of a few richly-funded researchers and military planners. Today, at the end of the century, computers play a role in almost everything we do. The new Sega Dreamcast video game has more computing power than did the entire university computing system when I was a student at Queen's. My generation of students arrived proudly equipped with tools required by their programs of study, including typewriters, slide rules and drafting kits. Now these tools have been rendered obsolete by computers, which have vastly superior capabilities in all three areas of writing, calculating and drawing, respectively. Textbooks now come with an interactive CD-ROM in a pocket at the back. (Some say the CD-ROM will replace textbooks, but I do not believe this.) The exponentially increasing power and falling cost of computers (Moore's Law) have made them ubiquitous. It is shocking to me that one of the few places left today where one rarely sees a computer is in the typical mathematics classroom of a Canadian school.

Much has been written about the use of computers in education, and many studies have been carried out with students. Most of these studies show that the use of computer-based instruction in schools has little if any beneficial effects on student learning, after the initial enthusiasm and novelty wears off. The recent book of A. Armstrong and C. Casement, The Child and the Machine[30][31] makes the case that an over-emphasis on computers in education can actually harm our kids. Most of their study relates to elementary schools, and to Integrated Learning Systems (ILS), where computers act like electronic workbooks in part substituting for teachers. Their main argument is that computers are resource-intensive (in both financial and human resources) and thus they divert scarce resources away from other more creative modes of education. My own limited experience with interactive Computer Guided Learning (CGL) software for university calculus and high school mathematics (such as that produced in Canada by ITP Nelson[32]) is that it has the potential to add a new and positive dimension to the learning experience. However, I will leave this debate over the value of computers for ILS to the experts.

Instead, I wish to address my remarks to two potential uses of computers in secondary school mathematics education which have received very little attention until now. These are based on two observations, both of which have already been made in this article. One is that the principles on which computers operate at the most fundamental level are mathematical, and were developed by twentieth century mathematicians (such as Alan Turing and John von Neumann). It is true that the realization in hardware of modern computers was made possible by breakthrough discoveries in solid state physics and electrical engineering, and this fact is well-known to the general public. However, even in the secondary mathematics curriculum, the role played by mathematicians in inventing the computers which have revolutionized twentieth century life is overlooked. The typical high school graduate can not name a single twentieth century mathematician, nor can he cite any contributions made by mathematicians, which have changed our life in this century. Contrast this with other secondary school subjects: every graduate knows something about twentieth century authors (in English), Einstein (in physics), the discovery of DNA (in biology), etc.

I propose that the basic mathematical ideas on which computers operate can and should be an integral part of the secondary mathematics curriculum, for the 21st century. The "basic computer math" which I have in mind here lies in finite (or discrete) mathematics, and includes topics such as mathematical logic, Boolean algebra, search algorithms, binary arithmetic, and the like. These topics require little background preparation or mathematical sophistication: they are much more accessible to high school students than the Calculus, for example. Furthermore, their relevance to the computer revolution (and to good jobs in information technology) would help to sell mathematics to more students. If we fail to build into our mathematics curriculum this fundamental link between mathematics and computer science, then students will have to turn elsewhere to learn about computing, for example in technology courses. These courses can provide useful computing skills, but they do not give the deeper understanding which mathematics can provide. Thus students lose out, in at least three ways: they miss learning about computing as one of the great intellectual achievements of this century, they fail to develop their skills in abstract logical reasoning which are so necessary to serious work in computer science, and they are left with the impression that mathematics has little to do with the exciting world of computer science. It is up to us, the mathematics educators, to restructure our mathematics curriculum so that it integrates computer science with mathematics to the benefit of both disciplines of study.

I should add at this point that I have discussed these points with university computer science faculty members, who make the following observations. Secondary school mathematics is very important in the preparation of students for study of computer science in university, but the present secondary mathematics curriculum is far from ideal. They would like to see students come to university with more highly developed logical reasoning skills. There is too much emphasis in the present mathematics curriculum on calculus, which is largely irrelevant to comp students' logical thinking skills. (A rigorous presentation of calculus would go far beyond the capabilities of almost uter science because computers are finite state machines. Furthermore, high school calculus is necessarily presented intuitively, rather than rigorously, which does little to develop all high school students.) On the other hand, the computer programming taught in high school technology courses is often memorized without true understanding. Thus, the curriculum proposed here would be a great improvement in preparing students for university computer science programs.

The second observation, which has implications for the secondary mathematics curriculum of the 21st century, is that computers have completely transformed the way mathematics is used in the workplace. In all areas of science, engineering, technology and finance, powerful computer software has been developed which dramatically increases the productivity of the worker. Yet, the typical high school mathematics classroom shows no evidence of this transformation. We are still teaching 19th century "horse and buggy" mathematics to our students. Can this prepare them for the high-tech workplace of the 21st century? Let me emphasize that this is not a question of using ILS programs which teach students about mathematics; rather it concerns access to computer-based tools which enhance the student's ability to do mathematics. These tools include spreadsheets, statistical packages, symbolic manipulation software such as Maple (and the related MathResource Math Dictionary), geometry software such as Geometer's Sketchpad, mathematical drawing software such as Mathcad and numerical computation engines such as Matlab. All of these tools are now in common use by university students and in the workplace. These modern packages share a common user interface, so that it is easy for users to move from one package to another. On today's university campus, computers are as natural a part of the learning experience as are textbooks. (There is a rumour circulating among teachers that university mathematics professors are against the use of computers by students. This is completely untrue. There are circumstances in education at all levels when computers are not appropriate. As an illustration: we all give "closed book" tests where use of textbooks is forbidden, but this does not mean that we are against textbooks! In fact, the vast majority of university faculty are strongly in favour of appropriate use of computers.) The time has come for the secondary mathematics curriculum to embrace the computer as a normal part of "doing math"; just as natural as pencil and paper and the blackboard. Students should not have to wait until they enter university or the workplace before they experience the way people actually use mathematics in the modern real world.

Finally, I wish to make a clear distinction here between computers and graphing calculators. Many recent reports concerning secondary mathematics education advocate the increased use of "technology", without distinguishing between different classes of technology. At the same time, there are vocal interest groups advocating graphing calculators as the key to modernizing the curriculum. They cite the low cost of graphing calculators compared to computers, which, they say, makes it possible to place one in the hands of every student, so they can have the kind of "real world math" experience I have advocated in the previous paragraph. I strongly disagree. Here are a few reasons why I believe that graphing calculators are no substitute for computers in mathematics education, and that they could even do more harm than good to our students. Graphing calculators have very limited screen resolution, rendering them virtually useless for any serious graphing. Modern computers have high resolution colour displays and even higher resolution printers, which allow full visualisation of graphical details of interest. Graphing calculators have roughly the computing power which personal computers had a decade ago. No one would advocate equipping our classrooms with 10 year old computers. All the mathematical scientists I know think that graphing calculators are great toys; they could play with them for hours. However, as soon as they have real work to do, they throw the calculator in a drawer and turn to their computer. Graphing calculators actually cost more, not less than computers, when one does a full cost-benefit analysis: Graphing calculators have a very limited range of uses, while a computer is a universal tool, as valuable in business, English, science or technology classes as it is in mathematics. Thus school computers can be used by many students throughout the school day, while calculators may spend most of their time locked in a closet. Because of their universal application to all areas of education, technical support for computers in schools should be provided centrally; whereas with graphing calculators, mathematics teachers will have to provide their own technical support. This can be a major burden; for example, security against theft is a bigger problem with hand-held calculators than with computers fixed to a desktop. The prices of today's calculators and computers are not directly comparable, since, as noted above, today's calculator is about as powerful as a 10 year old personal computer (which today is worthless). Thus, based on actual computing power, the calculator is overpriced. Graphing calculators will not support the "real world" mathematical software packages described in the previous paragraph, except in stripped-down versions. Thus they deprive students of the experience of using the tools they will face in university and the workplace. Finally, consider calculators from the point of view of school board finances. After an investment is made in graphing calculators for a school mathematics department, it would become extremely difficult to convince the board to spend additional money on computers for mathematics education. In other words, the purchase by a school board of graphing calculators, which may spend most of their time locked in a closet, will have the net effect of blocking students' access to what they actually need: real computers. In this sense, purchasing graphing calculators for secondary education may do more harm than good.

CONCLUSIONS

During the final decades of the twentieth century, the former compartmentalization of mathematics into "pure" and "applied" has largely broken down. Outstanding Canadian mathematicians are now contributing, not only to the advancement of their own discipline, but also to the solution of real-world problems and to the growth of new industries. Strength in mathematics has become an imperative for Canadian competitiveness in the global economy. The demand for mathematically-skilled graduates will explode in the early years of the new millennium, putting unprecedented pressures on all levels of mathematics education in Canada.

The challenge of meeting the increasing demand for mathematics education in the next decade will be compounded by a wave of retirements of baby-boom generation teachers. Ontario demographics show that half of those now teaching mathematics in Ontario schools will leave in the next 10 years. Similar departures are expected in universities and in other Canadian provinces. It is essential that we mathematics educators take action now to attract more students to study more mathematics, and to interest our ablest students in mathematics teaching as a career. The coming celebration of World Mathematical Year 2000 is perfectly timed to raise public awareness of mathematics and to help confront this looming crisis in mathematics education.

At the same time, we must change our mathematics curriculum to bring mathematics education in line with the real-world mathematics of the new millennium. Our students' education surely should include some of the exciting 20th century developments in mathematics itself, and the fundamental contributions that mathematicians have made to our changing world. The development of computers, perhaps the most influential development of the twentieth century, has profound implications for mathematics education which have yet to be realized. The role of mathematics in computing and information science may soon surpass in importance the traditional role of mathematics in physical science and engineering, which has driven the content of our mathematics curriculum for decades. This will lead to a shift in emphasis, away from calculus and towards discrete mathematics. The use of computers in universities and the workplace to expand human mathematical capabilities far beyond traditional pencil-and-paper calculations must be reflected in the secondary classroom. In the new millennium, computers in the mathematics classroom should become as accepted and commonplace as textbooks and pencils are today.

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Plenary Lecture 4

WHAT COUNTS? RESOURCING MATHEMATICAL PRACTICE IN THE SOUTH AFRICAN SCHOOL CLASSROOM

Jill Adler, Department of Mathematics, University of the Witwatersrand

INTRODUCTION

My major premise in this presentation is that there is a growing gap, certainly in South Africa and other countries in the developing world, between curriculum reform in school education and the realities of schools and classrooms in which the majority of teachers find themselves. Curriculum reform is driven by the dual imperatives of globalisation and social justice, and there are inevitable tensions between managing and encouraging increasing diversity when there is limited systemic capacity. In my analysis, teacher education programs are unintentionally reinforcing the gap between the goals of curriculum reform and on the ground realities. We are not doing enough, or the appropriate things, to uncover what would really count in diminishing this gap.

One reason for our reinforcing the gap is the contradiction we find in teacher education and professional development discourses. In the same breath we talk about the 'teacher as professional', 'teaching as a craft', with the teacher as the 'key' to, but a 'significant obstacle', in the reform process. This contradiction is a serious challenge in the South African context. The majority of the current teaching corps would have had all their schooling and initial teacher education in apartheid institutions. We have had a National Audit which vividly describes the impoverished education that was offered in most such institutions (Hofmeyr and Hall, 1995). The science and mathematics focus of the audit provides a clear description of the crisis in utilisation, supply and demand of teachers in these fields. Over 50% of secondary mathematics and science teachers have studied mathematics or science for at most one year after leaving secondary school, and they have between one and three years teaching experience. There is a shortfall (about 3000) in the provision of new mathematics teachers, and over 8000 who need in-service professional development (Arnott et al, 1997).

At the same time as delivering important information about where we are, the public message in the audit is intensely demoralising for individual teachers, as well as for the profession as a whole in the country. The message is: "You are there, you are the teachers, you have to deliver the promise of transformation, but your knowledge-base is inadequate. Your competence is in question". This is not too different from the kind of report in the Educational Researcher recently which describes the extent of out of subject teaching in the USA, particularly in mathematics and science. The report goes further to suggest that this places significant constraints on reform since the teaching of 'big ideas' requires depth of subject knowledge (Ingersol, 1999).

Our challenge as mathematics teacher educators is to turn a demoralising message into realistic and pragmatic inspiration. In South Africa, the rhetoric of curriculum reform is inspiring many teachers. My concern is that such inspiration will be short lived. The gap between the promise of reform and the constraints in real classrooms threatens to intensify demoralisation in the longer term.This argument arises out of experience in a teacher development programme in South Africa. I was involved in the conceptualisation and initial implementation of the programme and have co-ordinated a three year research project related to the programme. In this presentation I will first describe aspects of the development programme and the research project. I will then tell two stories that emerge from research data to draw out and illustrate some of the major findings in the research. I will then return to the argument presented above. My purpose in drawing on the research project is not the research per se. I am using it as a vehicle to open up debate on the growing gap between reform and on the ground realities.

THE FURTHER DIPLOMAS IN EDUCATION (FDE) PROGRAM

In 1996, the University of the Witwatersrand (Wits) launched a Further Diplomas in Education Programme (FDE) in Mathematics, Science and English Language teaching. The context nationally and provincially at that time can be characterised as one of rapid change, at least at the level of policy, and of significant past neglect. The FDE programme was concerned to simultaneously address quality and inequality. It provides access to further formal teacher development to those previously denied such; and it sets out to address quality by providing a transformatory orientation to knowledge and classroom practice.

The FDE programme is school-focussed and delivered in mixed mode. Each participating teacher has to complete five inter-related courses that are offered through a combination of self-instructional learning materials and quarterly residential workshops. Mathematics teachers on the programme take two courses in Mathematics. The central aim of these two courses is the deepening and broadening of teachers' subject knowledge. They take one course on the Theory and Practice of Mathematics Teaching where the focus is on deepening pedagogical subject knowledge. And they take two courses in general Education aimed at deepening teachers educational knowledge. The courses offer an integrated approach through explicit linkages between them, as well as emphases through all on reflective practice and the development of the extended professional. The ultimate goal of the programme is the improvement of classroom practice in schools, and not simply a certification process for teachers.

There were clearly articulated intentions in the curriculum, at least at the level of the FDE development team. Activity was focussed on classroom practice. Classroom activities encouraged pupil interaction, and the programme activities for the teachers encouraged them to work with a partner, preferably a colleague in their school. The broadening of teachers' subject knowledge was aimed at through connecting mathematics maths to history and to applications; a deepening of subject knowledge and educational knowledge was provided by direct engagement with mathematical and educational conceptional frameworks. Improvement of pedagogical knowledge was sought through offering new and different teaching strategies, thus expanding the pedagogical imagination, and through encouraging reflective practice. To further assist teachers in a re-sourcing of their practice, aspects of the various courses included attention to accessing and making additional resources; to using learners' main languages as resources for learning; and to building on what learners bring, their orientations to learning and their conceptions, as cultural resources.

THE RESEARCH PROJECT

A three year research project (a base-line study in the first year and two years of follow-up study) was carried out by the programme and an extended research team between 1996 and 1998[1]. The research project set itself three 'big' goals: to provide systematic feedback on program and so opportunities for programme improvement; to inform policy and implementation nationally and to contribute to and

[1] For detailed reports on the research project (the team, its processes and products) see Adler et al, (1997, 1998)

broaden the debate on teacher education internationally. The central research question we set out to explore was: What is the dynamic interaction between a formal in-service programme and teachers' classroom practices?

As a research team, we were constantly aware that we were working in a "developing country" context and thus in quite specific classroom conditions: they were multilingual; there was a limited English infrastructure in rural areas (and yet English remains the language of instruction); the teachers' pre-service qualifications were of dubious quality; there were limited material resources available to teachers in their schools; and overall, there was tremendous flux and instability.

The design of the research was qualitative with some structured observation. We used a range of instruments (school inventories, semi-structured principal/teacher interviews; structured and classroom observation schedules that provided for simple quantitative observations and a qualitative commentary; classroom videotape; structured analysis of pupils' class work books; some pupil testing; and field notes) to illuminate: the school context; teaching and learning practices including the availability and use of resources for learning and teaching; the content knowledge made available and task demands on learners; the mediation of knowledge; assessment practices; and teachers' intentions and reflections on learner performance.

The research sample was purposefully selected across subjects and school contexts. It comprised 25 of the 140 teachers enrolled in their first year of study in 1996, and located in 10 schools. There were 11 maths, 7 Science and 7 English teachers in the sample. Our unit of study was the "teacher in context". The 10 schools were all historically black schools, located in three different 'contexts'. There were 8 teachers across four urban township schools, and 17 teachers in six rural and semi rural schools. Eight teachers in three of these latter schools had access to additional teacher development support through an active non-governmental educational organisation near their schools.

In the research we set out to see what happened to teachers who chose to come on a programme like the FDE. What did they take up? How did they take it up? What were the effects? We hoped that through this kind of interrogation we might come to understand what counts for teachers as they re-source their practice. I am going to illustrate the FDE programme and research project through a story of two of the mathematics teachers, one primary teacher in a 'can-do' semi urban township school, and one secondary teacher in a functional urban township school.

MRS THEMBI SHONGWE (pseudonym)

Mrs Shongwe is an experienced Grade 7 primary mathematics teacher. She passed Grade 10 mathematics at school, and completed a Foundation Mathematics course during her initial teacher training. She attended a number of mathematics in-service workshops prior to joining the FDE programme. She graduated with her FDE (Mathematics Teaching) in December 1997.

Mrs Shongwe teaches in the Northern Province in South Africa, in a relatively poor, small township. The area where she lives and works was a homeland or apartheid bantustan. In the new political dispensation, the Northern Province is one of the poorest of the nine provinces. Her school is, nevertheless, a 'can do' school with a good reputation. It has *basic* physical infrastructure, a qualified, stable and collegial staff, and reasonable teacher:pupil ratios (1:35). There are, however, no extras e.g. no staff room, no photocopier, no science laboratory. The main language of most the teachers and pupils is TshiVenda. English is an additional language, but also the language of instruction. Two of Mrs Shongwe's colleagues (one science teacher and one English language teacher) studied for their FDEs at the same time, and were also part of the research sample. Their work receives ongoing support from their principal who has a reputation in the area for vision and good leadership.

Data from the base line study reflected that Mrs Shongwe adopted a fragmented and procedural approach to mathematics, with an emphasis on calculations. Mediation forms were exposition accompanied by recording procedures on the chalkboard, followed by practice by pupils, with emphasis on repetition. This description obscures Mrs Shongwe's enthusiasm. She is a motivated teacher who then took a number of risks as she recruited ideas from the FDE programme and tried out new tasks and new approaches in her classroom. These did not become the major focus of her teaching, but were rather add-ins. Her teaching remained predominantly focussed on mastering procedures for calculations. Nevertheless, the atmosphere in her classes was one of engagement and participation.

There were three interesting dimensions to the new tasks she introduced. They signified attempts to make connections with pupils' everyday world (for example, she had them bring in garbage from their homes and the school grounds and then construct a bar graph of different kinds of waste found), and she drew on material resources available in the environment. She nevertheless struggled to sequence and grade the mathematical demands in the tasks. In the graphing tasks, for example, pupils could not understand how to draw up a scale and in the end copied her construction off the board.

The most challenging data for the project was the poor performance of her pupils, both on her own assessments (tests and examinations), and on a standardised Grade 7 test we administered ourselves in all her four Grade 7 classes in the third year of the study. Mrs Shongwe spoke at length about the difficulties she has covering the required syllabus for Grade 7, and the constant didactic tension between enabling learners' understanding and preparing them for secondary school.

The table below (on the opposite page) provides a brief description of our observation and understanding of Mrs Shongwe's teaching over the three years of the research project. Not only is it brief, but it is also inevitably a selection. It cannot convey her practice in all its complex dimensions. It summarises her practice in relation to her professionalism (how she acts as a teacher in a development programme), her approach to mathematical knowledge (as reflected in her selection of tasks, and in how she described her selections in interviews and conversations), the resources in use in her teaching, and the way she mediated mathematical knowledge in class. The final column places a + or - judgement of the take-up from the programme and its intentions.

Mrs Shongwe	1996 base line data	1997 and 1998 follow-up	take up
Professionalism	motivated, openness	takes risks, demonstrates increased confidence	+
Approach to mathematical knowledge	fragmented, procedural	connections attempted - mainly procedural	+ o
	low or reduced task demands	new tasks, low cognitive demands	o
		textbook tasks improved level of demand	+
	math is 'calculations'	maths is 'calculations' and 'connections'	+, o
Resources in use	chalkboard	disappears with new tasks	-
	textbook - exercises predominate	not available for new tasks, still dominates for exercises	o
	pupil-pupil talk in TshiVenda	pupil-pupil talk in TshiVenda	
		disappears (as group work dominates)	o
	whole class chanting and chorusing in English	new additional materials brought in for new tasks	-/+
			+
mediation	exposition	reduced	+/-
	some concrete demonstration	draws on everyday knowledge and objects	+
	repetition	stays	o
	invites articulation of procedural steps	stays	o

Reflecting on this summarised account of her practice, we can make some comments about Mrs Shongwe's take-up from her participation in the FDE programme. She undoubtedly took risks, and in her interviews reported tremendous self-benefits in terms of her own understanding of mathematics, and in how she felt she had improved her teaching. At the level of professional practice, she increased her confidence, and was insightful in her reflections about the content she was attempting to teach and her own knowledge of that content. More specifically, in terms of re-sourcing her subject knowledge-base, we saw her work with new and different content and more open tasks. She extended her view of mathematics as 'calculations' to include mathematics as connected to the real world. In areas of mathematics where she was already knowledgeable, such as fractions, she resourced her pedagogical subject knowledge and demonstrated far greater flexibility and engagement with pupils' understandings and productions in this area. In contrast, when she worked with new mathematical areas, the tasks tended to be quickly closed down, learner meanings were skipped over, and she reverted to a focus on product over process.

In terms of resource use, she constantly brought in additional materials resources, and extended her use of the chalkboard to include demonstrations by pupils. She encouraged pupils to use ThsiVenda in their discussions. One of the unintended consequences here is that in this more discussion-based class, chanting disappeared and with it all opportunity, it seemed, for pupils to verbalise mathematics in English. She made very little use of pupil report back on their group tasks.

And she faced a range of new challenges. The new tasks presented her with difficulties clarifying her mathematical purposes and linking these to sequencing and graded cognitive demands. Limitations to her own conceptual frameworks came to the fore as pupil responses presented her with unanticipated mathematical ideas. Moreover, while she was innovative and brought in additional materials, these were firstly at her own expense and thus a practice unlikely to be sustained, and secondly, there were insufficient for each pupil.

Mrs Shongwe enables us to see that despite her take-up, motivation and increased professionalism, breadth and depth coverage and overall poor performance persisted. This assists us in coming to understand that take-up, or re-sourcing of practice from formalised in-service programmes is ongoing, partial, uneven and contradictory.

MR TSEPHO MANGANYE (Pseudonym)

Mr Manganye is a very experienced secondary mathematics teacher, and he has taught extensively across Grades 8-12. He is currently the HOD for Mathematics in his school. Since completing his three year teacher diploma, and before joining the FDE programme he completed Mathematics I and II at the University of South Africa. He completed the FDE (Mathematics Teaching) at the end of 1997.

Mr Manganye teaches in Soweto, in the Gauteng Province, a large township well known for its turbulent past and present. His school is in a relatively poor part of Soweto, but also that part where the Soweto revolt of 1976 was at its most intense. Relative to the schools in the Northern Province, this Gauteng school has adequate resources. There is an administrative block, a school secretary, a photocopier, HODs have their own offices, and the teacher: pupil ratios are reasonable. There are a range of main African languages in the school, and its pupils are largely drawn from its low socio-economic surrounding areas. The school has a volatile and politicised history. In contrast to the supportive collegial environment in which Mrs Shongwe works, high levels of collegiality, vision and leadership were not visible in Mr Manganye's school.

From our first visit to Mr Manganye at his school it was quite clear that he was a confident, experienced and competent mathematics teacher. His approach to mathematics can be described as 'reasoned procedures'. He offered clear and well structured exposition, with appropriate meta mathematical commenting. He provided opportunity for pupils to ask questions, and for them to practice their procedures. He switched between English, Sesotho and IsiZulu in his exposition and he encouraged pupils to use their main languages, particularly when they asked him questions. He made extensive use of both the chalkboard and the available textbook.

In our follow-up visits we witnessed Mr Manganye involve himself in professional activity in the school circuit while he remained largely inactive as an HOD within the school. As an already experienced and competent teacher it was interesting to observe how he worked on and improved his metamathematical commenting, how he explained to, and engaged with, his pupils. We observed him listening carefully to pupil-pupil interactions. He expected and encouraged procedural justifications from pupils, and based his mathematical scaffolding on pupil productions. The textbook and chalkboard remained central to his teaching, but they were used to serve new functions. The chalkboard was a focal point for both his explanations and for pupil productions; the textbook was a source of activities for learners to do in groups. He made increasing use of learners' main language, and there was increased pupil-pupil discussion.

And yet challenges remained. He did not recruit any new tasks into his teaching. He explained that as a senior secondary teacher he did not have time for this. And time was also not taken to encourage pupils report back on group work. A consequence here was that there was no opportunity during the lessons for pupils to produce mathematical English in the public domain. And like Mrs Shongwe, he continued to battle with coverage and performance. Many of his pupils also performed poorly on his own assessments (regular tests), and coverage of sections was limited from the point of view of breadth (not many of the required sections had been done) and depth (exercises completed by learners in the class work books did not range in level of demand and tended to be restricted to introductory level tasks for particular concepts).

Tsepho Manganye	1996 base line data	1997 and 1998 follow-up	take up
Professionalism	motivated, confident	outside (not inside) his school pedagogical content knowledge	+ / o +
Approach to mathematical knowledge	reasoned procedures well structured exposition coverage	reasoned procedures well structured exposition coverage	o o o
Resources in use	chalkboard textbook code-switching	shared group activity pupil-pupil public English productions	+ + + -
Mediation	metamathematical comments questions exposition	good scaffolding listening and building same	o

So, like Mrs Shongwe there was partial, uneven and contradictory take-up from the programme. We can see the potential for both positive and negative consequences.

And this was so across and within the other teachers in the research project.

TELLING STORIES ABOUT TEACHER EDUCATION

As I draw out these descriptions (and in these lie implicit interpretations and explanations) I am acutely conscious that both stories are not only in my voice (and not their own voices) but also that they might do just what I was warning of at the beginning of this presentation: that this description unintentionally sets the teachers up as simultaneous agents for, and obstacles to, change. I have struggled to capture in both teachers, their uneven professionalism, the take-ups and practices that create possibilities for quality mathematical learning, and their continual battle with pupil performance. What does it mean that both these capable teachers, neither in dysfunctional schools, are struggling to make curriculum headway in ways that improve learner performance? How do we tell good stories (by which I mean constructive and productive) in teacher education research and practice, without obfuscating harsh realities? How do we shift from what seem like inevitably demoralising 'discourses' towards pragmatic inspiration that does not smack of naive realism nor empty rhetoric?

As a start, I believe that one way to tell good stories about teacher development in relation to curriculum reform is that we stop talking about *change*, and the *need to change*, as if somehow change is not happening, as though teachers and classrooms were static. Change is a constant in schools and classrooms as everywhere else. Teachers are subject to these changes, and they are themselves changing. The point is, how is this happening? How do we talk about consequences of change?

What I have done through this presentation is to talk about our concern in the research project with what it is that teachers are doing, with coming to understand how, in the context of constant change, and a direct, selected experience in a formal in-service programme, teachers *re-source* their mathematical practice. What I am signalling here is that we need an intentional discursive shift. We need to build a language for talking about teacher development and change over time. What I offer here is the beginning stages of this process.

I have argued elsewhere that in discussion about 'resources' in and for the teaching of school mathematics, we need to shift our attention off the resources per se and onto their use in context (Adler, 1998a). This could enable us to tell stories about teachers' changing *resourcefulness*, about how they use what they have, and how they create anew, in context. We also need to describe their take-up, their selections for such re-sourcing. As the teacher brings ideas from a teacher development programme into the classroom, they are inevitably *recontextualised.* As they emerge in classroom practice, they shape, and are shaped by, its context.

What then of the teachers themselves, and their take-up? I have argued further (Adler, forthcoming) that teachers' take-up can be described as *appropriations*, as learning in context and as an ongoing process with advances and reversals, rather than a change from one state to another. Re-sourcing, recontextualisation and appropriation are descriptive categories, with explanation implicit in the description. Re-sourcing practice is a function of personal biography, and contextual constraints and enablements.

So what *is* happening? What *does* count for teachers as they re-source the practice? I will now move from illustration and some theorising to overall comments and findings that provoke the title of this presentation: "What counts?" and to substantiate my opening claim that there is a gap between the goals of reform and on the ground realities.

WHAT COUNTS?

I would like to assert that a key and overarching observation in the research is that effective re-sourcing is a function of both *stability* (maintenance) and *change*. This is not a new idea, and underpins a great deal of organisational development theory. Yet our conception of policy and practice in teacher education is still one which emphasises change, and is silent on stability. Whether intentional or not, the hidden assumption is that the new can be cast without the old. What then are some of the elements of stability or maintenance in school mathematics practice?

First, key to any form of school mathematics practice is the formulating of clear mathematical purposes, and the selection, sequencing and grading of related tasks. With all its limitations, a well designed textbook offers such selections, and this provides an infrastructure for teaching. Across the three years of the research we saw some teachers who seemed to not use textbooks, reasons for this ranging from "they are poor quality", "they are old fashioned" to "there are not enough for the whole class" and to "I tell them (Pupils) to keep them at home as they get stolen at school". Yet, they did not draw on other structured learning materials. This was particularly acute in primary maths classrooms.

In the programme, we made assumptions about teachers' prior learning through experience, and what they would have learnt in their initial training and so there was little explicit attention in the program to reinforcing the importance of purposes and related tasks; of sequence and progression; and of maximising the use of available text books.

Second is that as school lessons unfold, we need to ensure that there is some inscription of the processes of teaching and learning, be they oral and/or written. We need to ensure some collective memory, pointing to that which needs noting. Somewhere, processes need to be captured - both for immediate reflection and for later revision. The major technological resource for written inscription for the teacher in the ordinary classroom in South Africa is the chalkboard (for pupils it is their few exercise books provided). This is particularly so where paper is in short supply in the school, where there is no surplus for rough work, and where photocopying of worksheets is not a possibility. Again, the programme did not pay explicit attention to inscribing process, assuming (as with textbooks) that in an in-service programme, teachers come with such teaching knowledge. We could do a lot more with how the chalkboard could be optimised to serve more open and complex goals. This, in fact, was how a number of the secondary teachers did re-source their practice.

I make these points about textbooks and chalkboards and their functions because the discourses around reform and progressive practice polarise choices and constitute both as 'bad', as 'old' practice. Teachers are urged to let go of chalk and talk, as these are assumed to be necessarily teacher-centred. New curriculum policy in South Africa advocates laudable goals of the production of flexible, critical and knowledgeable citizens on the one hand, and teacher development through an integration across foundational, applied and reflexive knowledge on the other. And as it does so, it characterises textbooks as prescriptive and narrow, as if somehow the form necessarily and always dictates the content and the substance. Instead teachers are to work with illustrative learning materials, guidelines and frameworks from which they will create their own texts, lessons, sequences and progression.

I want to pause here as perhaps there is an interpretation that I am moving to a "back to basics" position as it is taking shape in the USA. I want to distance myself from the conservatism and anti-democratic ideology of that debate, and insert myself quite firmly in the progressive education movement where I am implicated in the emergence of polarising discourses. What I am suggesting is that here is where the tensions between development and democracy (quality and inequality) become acute and visible. In the developing world context, the presence of textbooks and chalkboards have neither a long term history, nor a permanence which renders them invisible as I would imagine is the case in the developed world.

This presents both an opportunity and a constraint: we may be able to bypass the text book, but we can't bypass its function, that is, the presentation of a carefully thought out teaching and learning sequence. The contradiction in the developing world is that we are removing textbooks through progressive discourse but are not able, materially/financially and in human capacity terms, to deliver an alternative set of materials. The effect of this is that teachers are left to draw on their common sense knowledge, and inevitably, the poor get poorer. We have had a glimpse in our research programme at how competent teachers understand the key functionalities of textbooks and chalkboards and continue to rely on them.

Third is that learning mathematics in school involves both learning *from* talk and learning *to* talk. For most learners in South Africa, learning from talk means using their main language. Learning to talk means learning mathematical English. Through the research project, in both primary and secondary classrooms, group work and increased pupil-pupil interaction appeared in all classrooms. We can understand this as a re-sourcing of practice through a harnessing of both exploratory talk and a harnessing of main language as a learning resource. Chanting disappeared from primary classrooms. Teacher talk dominated the public speech channel, and this talk was in English.

One consequence of these shifting practices was that they diminished opportunity for public pupil talk or expression of mathematical English. These shifting practices and their consequences are a manifestation of teachers managing dilemmas of the dilemmas of code-switching and language transparency (Adler, 1998b, 1999). If teachers don't switch languages and engage with their learners in their main languages, then learners struggle to understand the concepts being dealt with. At the same time, if teachers continually teach through learners' main languages, then learners' access to English (the language of power, access and assessment) is denied. Furthermore teachers know that they have to pay explicit attention to mathematical English, particularly in multilingual classrooms. But at the same time, they see that if they pay too much attention to language per se, it becomes too visible and the mathematics can be obscured. Both these dilemmas are acute in areas with restricted English language infrastructure, where teachers face the enormous dual task of teaching both the language of instruction itself (English) and a subject like mathematics in the language of instruction.

The FDE programme, and the discourses of reform in general, place emphasis on encouraging interaction, exploratory talk, and main language as a resource. What we did not attend to enough was the continuing importance learners' public productions in English. I point to this language issue because we are a long way from understanding just how to manage the dual task of teaching mathematics and English at the same time, particularly at the primary level. The politics and pedagogy here are even more complex than at the time of my earlier research referred above. All main African languages now have official status. Multilingual learning and teaching practices are not only sanctioned but encouraged. Ironically, at the same time, the "market" (parent choice) is opting for schools where a 'straight for English' policy and practice is in place.

Fourth is that basic levels of coverage and performance are necessary to progression up the school. We did not set out in the research to investigate coverage, and in the planning phases of the research, the team grappled internally, and with others in the field, with establishing pupil performance. Our initial concern was that we were under pressure from funders to relate pupil performance to programme impact. There is a double inference here: from pupil performance to teaching, and from teaching then inferences about the teacher development programme. There has been enormous expenditure in South Africa, through non-government organisations prior to 1994 and since then in various partnership programmes, on teacher development. These programmes and their support have been motivated by 'teachers are key to change' arguments. Despite this injection of resource into teacher development, pupil performance in school is still poor and possibly even deteriorating. Hence the current demands from funders of teacher development programmes for pupil performance assessments as indicators of the quality of the programme impact.

Despite serious reservations in the research team about the validity of the demand for pupil testing to infer programme impact, we tested some classes in the third phase of the project. In both follow-up years we also carefully examined pupils' class work and test books. We are still working to make full sense of what we found. But we cannot escape what we saw: despite teachers' efforts and across the range of schools and teachers, there is a spiralling inwards of less and less content coverage, and with this poor performance, not only on the independent tests that we administered, but on tests the teachers themselves carried out. The didactic tension is acute: covering vs uncovering content. On a daily basis, teachers face pupils with holes in their knowledge-base. Their complaints about pupils' background knowledge are well-founded. This is a difficult issue - but one teacher education programmes like the FDE need to acknowledge and address.

CLOSING THE GAP

What I have described so far is a re-sourcing of practice through a teacher development programme where there was partial and uneven take-up in relation to intentions of the programme. Yet significant problems persisted at the classroom level, in particular poor coverage, low cognitive demands, and ongoing poor performance.

I then highlighted how the programme has not paid explicit attention to key aspects of school teaching and learning and hence to stability in the face of ongoing change. We made inappropriate assumptions about in-service teachers' professional knowledge of articulating mathematical purposes for the lessons, of matching purposes and tasks and setting up progression and grading. As a result we did not pay enough explicit attention to the significance of structured learning materials and of inscribing process. We did not pay enough attention to the realities of multilingual settings, and the reality of pupils' poor prior learning.

These identifications might help us tell stories about what can and does count for teachers in appropriate ways. Our challenge in South Africa is to turn a demoralising situation into an inspirational challenge. Blaming Apartheid is necessary but not sufficient. We have to acknowledge the damage we know was done, and work with the realities on the ground. There, unfortunately, is no short cut, and no other way forward.

As teacher educators, the more we learn about what **does** count as teachers re-source their practice, the more effective we will be at closing the gap and realising the ambitious goals of curriculum reform.

IN CLOSING

I need to close by emphasising that what I have presented here is work-in-progress. It is not yet refined. When I was invited to do the talk nearly a year before the conference, I thought we would be further along the road in the research project. Alas, the wheels of research analysis in our collaborative project are turning slowly. This is partly because of the pressures we are all under academically and professionally in the university. But moreso because we are struggling to make constructive sense of our empirical work, in our rapidly changing context. While we can see programme take-up, we are deeply concerned with what else we have seen. I am uncomfortable with what sounds like a back to basics message in this presentation. Yet, as a teacher educator who squarely positions herself with progressive change in South Africa, I have to confront where and how I might be implicated in widening a gap between desire and reality.

I decided to stay with this research for this talk, despite its in-progress state. This partly because it is current, and largely because I feel so passionately about it. I believe that some of what we are coming to understand is crucial to progress in mathematics teacher education in South Africa, and then possibly too in other contexts, and not only the developing world. In addition, so many people have told me how the processes in this conference are different. That it is a working conference. I thus look forward to the discussion groups and following interaction.

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Plenary Lecture 5

Une Archéologie des Concepts Mathématiques: Tamiser le langue pour des significations mathématiques

AN ARCHAEOLOGY OF MATHEMATICAL CONCEPTS: SIFTING LANGUAGES FOR MATHEMATICAL MEANINGS

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Résumé/Abstract

Si les mathématiques sont définies de manière plus large que d'habitude, nous pouvons alors faire des recherches dans des idées mathématiques qui étaient auparavant peu familières. C'est ainsi qu'une considération des manières qu'ont les gens de se représenter les quantités, les relations et l'espace pourrait nous amener à de nouvelles mathématiques et pourrait ainsi impliquer une nouvelle pédagogie. Une grande partie de l'ethnomathématique a jusqu'à présent focalisé sur les artefacts culturels. Cet exposé focalisera sur la langue comme endroit où les concepts alternatifs des mathématiques sont enfouis.

Des résultats des recherches initiales sur les langues basques, hawaïennes, inuktitut, maori, ojibway, et gaéliques vous seront présentés. Il apparaît alors que l'examen des langages mathématiques dans ces différentes langues (plus particulièrement pour les langues qui n'ont pas de racines communes) montre que des concepts qui étaient sensés être universels sont plutôt définis en fonction de la culture et varient donc suivant celle-ci. Un exemple sera donné d'une différence qui est en train de se produire en anglais contemporain.

Les conséquences de ce travail pour les mathématiques sont importantes, particulièrement dans le domaine de la créativité mathématique. Les implications pour l'éducation mathématique ainsi que les politiques qui pourraient en résulter sont beaucoup plus équivoques mais méritent d'être débattues.

If mathematics is defined more broadly than usual we open up the possibility of investigating previously unfamiliar mathematical ideas. Thus a consideration of the ways in which people make sense of quantity, relationships or space (QRS systems) may lead to new mathematics, and may imply a new pedagogy. Much ethnomathematical work to date has focussed on cultural artefacts. This talk focuses on language as the place where alternative concepts are buried.

Results from initial excavations in Basque, Hawaiian, Inuktitut, Maori, Ojibway, and Welsh languages will be presented. The evidence suggests that an examination

of mathematical 'talk' in different languages (especially languages with no common roots) will show that supposedly 'universal' concepts may be culturally defined. An example will be given of a difference-in-the-making within contemporary English.

There are important implications of this work for mathematics, especially in the area of mathematical creativity. The implications for mathematics education (and the resulting politics) are much more equivocal, but need to be debated.

PLANIFICATIONS / PREPARATIONS

Lake Titicaca. Three thousand years ago 200 000 acres in this area were highly cultivated. Canals were dug through the swamps and the soil placed on 'raised fields'. After the Inca conquest 500 years ago these fields were destroyed. Since then there have been attempts to cultivate the area using high-tech equipment, imported crops and chemicals. All had failed until the archaeologists teamed up with local farmers and rediscovered the raised fields, the technology of their cultivation, and the traditional crops. The result has been highly successful, and it is fully adapted to the drought/flood propensities of the area. Such contemporary payoffs for archaeological work are few and far between, but they teach us renewed respect for ways of doing things that are different from the conventional wisdoms of today's technology. Today's methods are assumed superior because they have served us well in many respects, but we may be blind to their deficiencies in new surroundings.

L'Archéologie / Archaeology

I wish to use this metaphor to describe my own 'archaeological' adventures in mathematics. Perhaps they, too, will have contemporary spinoffs within mathematics or mathematics education. But, like archaeology, they are interesting in themselves and lead me to look at my subject area in a new light.

The work described here is archaeological in the sense that ideas are being generated from fragments of evidence. The danger is that these ideas might be taken as reconstructions - but that is not possible, there is not enough evidence remaining nor can we return to the context of the past. We must remember that the Stone-Age is so-named because stone is all that remains to us from that time: it probably should be called the "Old Wood Age" as that material was likely to have dominated the technology of the time. Similarly it is dangerous to read too much into the evidence. For example, Stonehenge is not a celestial computer as some researchers have imagined. That is just wishful thinking. Perhaps it is best if we consider the mathematical ideas that follow as possibilities - but even possibilities have the power to open our minds to new ways of thinking. Archaeology involves other dangers, of course, notably the post-modern criticism that we are bound to approach our discoveries with present-day assumptions about what they represent. But let us note these caveats and proceed.

So, an archaeological search for mathematical edifices. What are the shards that are to be dusted from the sands of time? If mathematical ideas are edifices of the mind (as I believe) then the shards which remain to us from other worlds will be the words which were used to communicate those ideas. And to find shards which might be unexpected or new the search begins amongst those languages which are not regarded as 'normal' for mathematics, i.e. in languages which are not Indo-European in origin. Hence I recently found myself going to various corners of the world searching for tiny shards of evidence of worlds which I believed must have existed, but for which there is no contemporary knowledge.

Like all good explorations there is a background. For example, how did I become interested in this search. Well, like amateur archaeologists before me, I accidentally stumbled upon a fascinating little artefact in my own back yard, and thereby learned a lot about the history of my own country.

Le Développement du Vocabulaire Maori / Maori Vocabulary Development

Since 1985 a small group has been working in New Zealand to develop the indigenous Maori language so that it can be used to teach mathematics to senior secondary levels. The group involves teachers, mathematics educators, linguists, and Maori elders and language experts. The initial collection of vocabulary items extended to the development of both new vocabulary under strict guidelines laid down by the Maori Language Commission, and to the exploration of the syntax implied by this vocabulary and its mathematical context. The process has been characterized by a cycle of finding out what is being used in bilingual and immersion classrooms, taking that back to communities for their comment, and presenting this material to the Maori Language Commission for decision. The cycle has been repeated three times over 15 years, and the process and results have been published in a series of papers and dictionaries (Barton et al, 1995; NZ Ministry of Education, 1992).

In 1993 a new national mathematics curriculum was produced, and the political environment of the time made it possible to argue that a separate Maori mathematics curriculum should also be produced. This document is not a translation of the English-language one, (indeed, neither has been translated into the other language), but is similar in format and objectives (NZ Ministry of Education, 1994).

So, has the Maori language successfully been adapted to the teaching of Maori? The answer is yes, ... and no. Although the numbers are small, there is increasing evidence that students taught mathematics in Maori are doing at least as well as parallel compatriots, and possibly even better (Aspin, 1995). There is certainly a positive response from students and staff in Kura Kaupapa Maori (Maori Immersion Schools). But those of us involved in the language development became increasingly uncomfortable with some aspects of our work. Somehow it did not feel completely right, but we were unable to put our finger on why. We came to talk about this as the "Trojan Horse" phenomenon: mathematics education seemed to be a vehicle which led to the subtle corruption of the ethos of the Maori language (Barton et al, 1998). It took many years before we found our first shard of evidence.

Bonnes Vacances / Happy Holidays

From the experiences in New Zealand, the question arose as to whether this phenomenon had been experienced elsewhere. What strains developed in other languages which had to be adapted to teach mathematics? What were the effects on mathematics of being taught in languages not usually associated with the subject in its academic form? Is there a possibility of different mathematical expression, or even different mathematical concepts? These are not new questions. Benjamin Whorf, the American linguist of the first half of this century who is associated with ideas of linguistic relativity, was originally an engineer. Perhaps that mathematical background helped him to recognize that (Whorf 1956, p245):

> ... an important field for the working out of new order systems, akin to, yet not identical with, present mathematics, lies in more penetrating investigation than has yet been made of languages remote in type from our own.

Thus it is suggested that a potentially useful study would be where mathematics is taught in languages as different as possible from the Indo-European tradition in which academic mathematics has mostly developed. This includes Pacific, American First Nation, South American Indian, African and isolated European languages. A sabbatical leave that took in North America, the Pacific, and Basque country in Spain provided an opportunity to begin such a study.

La Linguistique / Linguistics

An understanding of linguistics was clearly going to be important for this archaeological expedition since this provided the material which was to be sifted for mathematical shards. While there is now a considerable body of writing about the language of mathematics and how it differs from other types of discourse (Halliday, 1975; Dale & Cuevas,1987), this study was looking, rather, at the mathematics in languages not the language of mathematics.

There is an on-going debate in linguistics about the issue of linguistic relativity, that is, the extent to which languages differ from each other in construction, meaning, and underlying concepts or world view (Foley, 1998). This debate is relevant since, if languages do only express the same ideas in slightly different forms or if humans are "hard-wired" to understand quantity, say, in only one way (e.g. Dehaene, 1997), then a search in different languages for mathematical difference will find only trivial examples. If, on the other hand, different languages are incommensurate and can never fully be translated into each other, then a search like the one proposed might find interesting new mathematical concepts. Of course this begs the question of what 'mathematical' means, but let us pass on that for the moment.

Important in this debate is the ongoing work of George Lakoff and colleagues at Berkeley. Lakoff is interested in the underlying metaphors which guide our classification systems. If these metaphors affect our language construction and use, as he suggests (Lakoff, 1987), then clashes will occur where subjects developed in a language using one metaphor environment are expressed in languages using another metaphor environment. As the study began this seemed to be a good way of thinking about some of the evidence which was emerging: more on this later.

LES EXCAVATIONS ET LES ARTEFACTS / THE EXCAVATIONS & THE SHARDS

Whereabouts in the different languages were productive sites for mathematical excavations, and what evidence did the digging throw up? It must be said that this study is still in its beginning stages. The few shards that have been uncovered can still be interpreted a number of ways. As one linguist working in Basque country commented during discussion, it is almost always possible to find isolated constructions in any language which, taken alone, would suggest strange conceptual formulations. Before inferences can be drawn it is necessary to undertake much broader analysis of language types. Any final conclusions would need to be expressed in broader terms than are used below, where the conventional Indo-European grammatical categories are used to express features of quite different languages. These caveats having been stated, some examples are now given.

Qu'est-ce que c'est un nombre? / What is (a) number?

The original shard from the Maori mathematics vocabulary was to notice that the number words in Maori often carry verbal indicators. In everyday English numbers act like adjectives: compare 'there are three bottles on the table' and 'there are glass bottles on the table'. In mathematics discourse, numbers are used like nouns, as things which can be, for example, added, multiplied, or which can have characteristics like primeness. But what about numbers acting like verbs?

In Maori, the number words are tahi, rua, toru, ... But when using them to count you often say: ka tahi, ka rua, ka toru, ... 'Ka' is a verbal marker indicating future tense. Thus what is really being said is: "becoming one, becoming two", or, as the number is actually the verb, it is more like: "one-ing, two-ing, three-ing".

It turns out that four of the five verbal markers can be used with numbers, and, indeed, in normal discourse you would often use one of them, despite the fact that in modern Maori it is usually assumed that numbers are adjectives as in English (Trinick, 1999). It was only in our created mathematics discourse that verbal markers were ignored. There is further evidence that, in the Maori spoken before European contact, quantity was expressed verbally. For example, in the area of negation:

Number sentence

E wha nga kina	=	There are four sea-eggs Il y a quatre oursin
Kaore e wha nga kina, e toru ke	=	There are not four sea-eggs, there are three Il n'y a pas quatre oursin, il y en a trois

Verbal sentence

E haere tatou ki Te Kaha	=	We are going to Te Kaha Nous allons à Te Kaha
Kaore tatou e haere ki Te Kaha, e hoki mai ke	=	We are not going to Te Kaha, we are returning Nous n'allons pas à Te Kaha, nous retournons

Adjectival sentence

He pouaka nui tenei	=	This is a big box C'est une grande boîte
Ehara tenei i te pouaka nui, he pouaka iti ke	=	This is not a big box, it is a small one Ceci n'est pas une grande boîte, c'est une petite

It turns out that this verbal form of quantity is a feature, not just of Maori, but of many Pacific languages (Samoan, Hawaiian), and also of some First Nation languages in North America. Alternatively, some First Nation languages have a noun-like usage of number. Peter Denny (1986) has written on the Ojibway (which is also verbal) and Aivilingmiut Inuit languages which has the following noun-like structure.

one	atausiq	(none)	(singular noun)
two	marruuk	-uk	dual noun ending
three	pingasut	-t	plural noun ending
Compare:	pingasut		a group of three
with	pingasuit		three groups
Hence:	pingasut tuktuit		three caribou
is actually:	a three-group of caribou, or		a caribou group-of-three
but:	pingasuit tuktuit		three groups of caribou

What is the significance of all this? First of all let us notice that previous investigations of language and mathematics have mostly looked at number words, and focussed on different bases of counting systems, evidence of doubling or tripling, and so on. The actual nature of number itself has never been questioned. Even when the evidence in Maori of the verbal nature of numbers is now so very obvious, it was essentially ignored by most modern speakers, probably because of the dominant English-speaking environment - it was an English missionary who first made Maori a written language, and the need for translation would have encouraged conforming to English patterns of speech. Other languages in the region have similar histories.

However, it is possible that the verb-like usage does not affect mathematics, which is (partly) the system of numbers in their abstract form. This is to ask the question the wrong way round. What we should be asking is "is it possible to have a formal abstract system of verbal types"? As far as I know, no such system exists. But the question is really a hypothetical one: could one exist, what would it be like?

S'objecter aux objets / Objecting to Objects

It is not just numbers which are expressed in different parts of speech. In Euskera (the Basque language) mathematical qualities are expressed using words with verbal roots. For example: 'the continuity of a function f / la continuité de la function f' is expressed in Euskera as 'f funtzioren iraunkortasuna'. The word 'iraunkortasuna' is constructed as:

iraunla verbe "continuer" / the verb "continue"
kor la propriété "faire changer" / the property "to make change"
tasunindiquer un qualité / indicating a quality
a l'article "la" / the article "the"

As an interesting aside to this expression, notice the Euskera word for function: funtzioren. The suffix -ren indicates the possessive, i.e. nouns are declined (as in Latin). When a phrase like "the continuity of f' is used, the question arises as to how to decline f? In fact this is an example of Basque mathematics teachers bending their language: they say "f-ren"!

The objectifying tendency of mathematics has been commented on before in relation to Navajo geometry (Pinxten et al, 1983, 1987) and in general terms by Bishop (1988).

Les Formes Privilegiées / Privileged Forms

It is not just making mathematical concepts into objects (as opposed to actions) that is privileged by the Indo-European languages of its development. Other mathematical forms are reinforced by the usual language of mathematics.

For example, in English, multiplication can be expressed as, say: ' five times two' or 'two times five' where the 'five' and 'two' can be interchanged without altering the word forms, the grammar or the sense. In other words, commutativity is part of the language of multiplication. Not all languages are like this. In the Kedang language of Indonesia the words udeq, sue, tèlu, apaq, leme are one, two, three, four and five respectively. However there are also the nouns munaq (one unit), suen (two units), ..., lemen (five units), etc (Barnes, 1982). Hence, multiplication is expressed abstractly as lemen sue (two lots of five units), which is different from suen leme (five lots of two units). Thus the grammar of the language is non-commutative. (Note that this is not to say that Kedang speakers cannot understand or express commutativity if they wish to do so)."

Another example of the language embedded in the mathematical form is with the expressions: "for all x there exists ∂ such that $(w - x) < \partial$' - usually written in symbol form. When expressing mathematics through the Basque language the symbol sequence is written in the same way, but it is unnatural to speak it in that order. In that language the sentence order is determined by the most important idea in the sentence, in this case $(w - x) < \partial$.

The previous example seems to be a case of the mathematical form (symbol order) following the language form. An example which may have arisen through the language following the mathematics is the privileged place given to vertical and horizontal when considering orthogonality. It is convention to draw graph axes vertically and horizontally, and it is usual to orient right angles that way. This is reflected in our use of 'vertical' and 'horizontal' to describe axes, and our use of 'right' and 'normal', which are derived from building terms meaning vertical. Furthermore, we 'drop' a perpendicular from a point, or 'erect' one on a line. There are contexts where lines at right angles are more appropriately oriented in a way English-speakers would describe as diagonal. Basket and mat weaving are such contexts. Unlike cloth weaving, to construct a basket or mat both sets of strands are set up along one edge, one off to the right, one to the left, and then they are woven together. Designs from these items are properly 'seen' as oriented diagonally. In European books about such weaving it is common to find the designs 'incorrectly' turned so that they are vertical and horizontal.

La Classification du Designe / Pattern Classification

A final example of a linguistic shard of evidence for different mathematical conceptions also comes from the world of weaving. When discussing a series of basket patterns with a Maori weaver, she gave the same name to each of six patterns which looked, to me, quite diverse. In my terms, four of the patterns had rotational symmetry of order 2 (of which 2 had lines of symmetry and two did not), one had rotational symmetry of order 4, and one had rotational symmetry of order 1. In other words, four different types of pattern analysed using symmetry. The explanation was that all six patterns required the same original set-up of black and white strands (namely: black, white, black, white, black, white, white), with the pattern emerging from the way in which they were woven over and under each other. To a weaver, it is essential to set up the strands in the right order to produce a desired pattern, so patterns are analysed in this way.

This is not to suggest that such a method of analysis has such extensive spheres of application as symmetry (although other spheres of application have not been investigated), but it does highlight the idea that patterns may be systematically categorised in different, non-equivalent ways. Symmetry has become a dominant way, but that does not mean that it is the correct or only one.

LIRE ENTRE LES LIGNES / READING BETWEEN THE LINES

Having found some pottery shards or building foundations or food middens, an archaeologist must set about trying to interpret these traces of evidence, and develop a picture of the original objects and their roles and relationships within society. Doing this often involves guess-work and reference to the thoughts and experiences of others, linking the finds to other pieces of related information: weather conditions, demographics, known technologies of the period, and so on. What picture of mathematical concepts can we develop from a few pieces of linguistic information? The quick answer is, of course, that we cannot be certain of anything. However there are some other related developments which may shed light on what we have found.

Whorf, Lakoff & Linguistic Relativity

The idea that language and thought are inextricably linked is not new. Benjamin Whorf has already been quoted. He also raised the idea that languages carried embedded assumptions (Whorf, 1956, p244):

> ... but to restrict thinking to the patterns merely of English, and especially to those patterns which represent the acme of plainness in English, is to lose a power of thought which, once lost, can never be regained. It is the "plainest" English which contains the greatest number of unconscious assumptions about nature. ... Western

> culture has made, through language, a provisional analysis of reality and, without correctives, holds resolutely to that analysis as final.

George Lakoff and others have explored the nature of these assumptions, and found them much more deeply embedded than we realise. In particular, he writes about the way in which classifications are made (Lakoff, 1987; Lakoff & Johnson, 1980), and shows that the accepted 'classical' model does not correspond with the ways we use concepts in language. For example the concept of a 'table' is not clear-cut. We do not simply look at an object and decide whether it has particular 'table' characteristics, and if it does then we call it a table. Some objects can be more table-like than others, some things are more 'red' than others. Our conceptual categories are relational, blurred, and linked in chains of association. Lakoff suggests that the classical 'container' model of categories is deeply embedded in English (and other Indo-European languages) so that we talk as if that is how categories are determined, when in fact they are not.

Another way of describing this insight is to say that, whatever we are talking about, we are talking through metaphors. These metaphors are so embedded in the languages we speak that they become unconscious. The Fields Medallist Rene Thom expressed this as (Thom, 1992):

> I think it is, more or less, philosophically an illusion to distinguish between reality and metaphor. In fact, analogy is, to some extent, a deep phenomenon of our thinking and if we want to understand what analogy is, then we are led to very fundamental philosophical problems.

The mathematical shards of language which we have found may be able to be interpreted in this light.

Language and Topological Concepts

Another development which bears on this work is a research project on the way in which topologists understand the concepts of their subject. This work is aimed at determining whether topologists who work in different languages understand their subject in different ways. Although still in its data-gathering stage, some interesting anecdotal evidence is emerging. For example, a discussion about the origin of the term 'open' to describe 'open sets' led to an awareness that the four people in the discussion interpreted the word 'open' in four fundamentally different ways.

One person regarded open simply in relation to closed, and said that any complementary pair could have been used to name the concept: yin/yang, black/white, male/female. Another person regarded open in the sense of a border guard: an open border is one that admits aliens, i.e. interpreting the word spacially. A third member of the group understood open as it is used in describing an open field, that is, one with no boundaries at all. And the final member of the group understood it like an open door, in opposition to an open door.

Such variation raises the question of how such diversity arises in a mathematical concept which all regarded as well-defined and mutually understood. What effect does this variation have on topological thinking? Can such variation arise through different languages, or is it a matter of different teachers, different texts, or different experiences?

Linguistic Corruption & "Snapping to Grid"

Since we are dealing with essentially linguistic evidence, perhaps the shards we are finding are simply evidence of a natural process of language change? If this is the case, then the hypothesis of linguistic relativity might be weakened because it could be argued that all languages are developing towards a 'natural', universal mathematical expression. What signs are there that mathematics forces change on language because of conceptual-linguistic structures? It happens that there are some examples which have arisen in the development of mathematics vocabulary in indigenous languages.

In Euskera, the Basque language, nouns are declined. Hence the translation of "the continuity of the function *f*" is "*f* funtzioren iraunkortasuna". Now, in mathematical discourse, we also say "the continuity of *f*". But how do you decline a symbol? In fact Basque mathematicians do decline the symbol, and say "*f*-ren iraunkortasuna" - clearly a corruption of language brought about by the requirements of mathematics.

When developing Maori mathematical vocabulary there were problems expressing negative numbers. Eventually the Maori Language Commission approved the use of the adverbs 'ake' and 'iho' which have the general meaning of 'upwards' and 'downwards' as in "e heke iho te ua" (the rain falls downwards). But to operate in mathematical discourse these adverb had to become adjectives: "tau iho" (negative number). This was accepted until a Commission member overheard a child talking about "he tangata iho" (a negative/bad man), i.e. the adjectival use had become part of everyday language. A furore ensued, as a result of which the adjectival use in mathematical discourse was officially withdrawn so as not to corrupt the spoken language.

In the Polynesian language of directions there is possible evidence of other mathematically-forced changes. In Maori the accepted word for 'south' is 'tonga'. In Hawaiian the equivalent word is 'kona' which has the meaning 'leeward' - the Kona Coast is the leeward, coffee-growing coast of Hilo. In Hawaii this is a south sou-west direction. Perhaps, as the migration brought people, and the language, to New Zealand, this word continued to mean SSW, but, when the European contact arrived and the NSEW compass became dominant, the word for the direction closest to south got adapted to due south? A similar thing may have happened with the term 'muri' which means 'north' but derives from the word meaning 'behind', and, in particular, 'the stern of the canoe'. The migrating canoes came from north nor-east, not due north, and 'muri' may well have had the original meaning of 'where we came from'. As the NSEW compass came into use, the term "snapped to grid" and adopted a meaning corresponding to that pre-determined system, just as freehand constructions on a computer can be snapped to a pre-determined grid at the click of a mouse-button.

If we accept that there are different ways of seeing the world, then these examples are evidence of the imperialistic tendencies of languages of dominant groups. We talk mathematical objects concepts into existence, but we can also talk them out of existence by not speaking about them. If ways of thinking are not expressible in the dominant language, or even if systematic structures of quantity, relationships and space are not represented in our language, then those structures and ways of thinking will die out.

Puis Après (Mathematiques) ? / So What (Mathematics) ?

What message does this archaeological expedition have for mathematicians? In what ways does this new understanding change mathematics? What difference does it make? Is there a Lake Titicaca-that can be usefully recovered through mathematical concepts not recognized in our languages?

Verbal Numbers & Lakoff Equivalence

Any mathematical benefits from this kind of investigation come from thinking mathematically in new ways, which may lead to new mathematics. For example, what can be made of verbalizing numbers and other mathematical objects? In Maori, if one is counting using numbers verbally, one is saying the equivalent of: "Becoming one, becoming two, becoming three, ...". This is a continuous expression of quantity, and might raise the idea of formalizing quantity in a continuous way. What would be the mathematical consequences of such a system? As another example, take the mathematical 'objects' of a circle and an ellipse. The container metaphor embedded in our language makes us talk about the differences between these objects in terms of properties of the objects: centres, length of radii, and so on. If we tried to adopt another metaphor such as the metaphor of 'doing something' common in indigenous languages, then these objects would become actions: circling and ellipsing. Differences between them would focus on what one does when moving in these ways – possibly leading to a new categorization of geometrical shapes.

There is an interesting parallel between the nature of categorization which Lakoff has described in linguistic concepts, and the new branch of topology dealing with Fuzzy Sets. Lakoff shows that even a concept as 'clear cut' as a table is actually a network of related ideas which allows some objects to be more table-like than others, rather than definitively a table or not. Fuzzy set theory allows 'partial' membership of sets: membership is described by a number between 0 (not a member) and 1 (a full member). It is interesting that, many years after this theory was first developed , there are many mathematicians who reject fuzzy topology as not mathematics. It is as if they cannot step outside the conceptual boundary erected by classical categories. There are parallels with the way first negative numbers, then complex numbers were both rejected by many eminent mathematicians for many years after they were established mathematical concepts.

Alternative Foundations

At a deeper level, the metaphoric underpinnings of language make us want to question some of the more fundamental structures of mathematics. If it is true that mathematics is the formalization of the linguistic concepts by which we make sense of the world, then there are alternatives to the formalization which is presently in place. It was suggested above that the 'container metaphor' is part of Indo-European language. It is therefore no surprise that set theory has been the basic tool for describing the foundations of mathematics. What other foundations might there be? Such a question has been asked by many mathematicians. Hermann Weyl has said:

> The question of the ultimate foundations and the ultimate meaning of mathematics remains open: we do not know in what direction it will find its final solution or even whether a final objective answer can be expected at all. 'Mathematizing' may well be a creative activity of man, like language or music, of primary originality, whose historical decisions defy complete objective rationalisation.

The one other foundation that has been seriously attempted is the Category Theory of Saunders Mac Lane, a theory which uses functions. This could be interpreted as the mathematical formalisation of the 'path metaphor' which is dominant in many indigenous languages. Nor did Mac Lane consider this to be the only alternative (Mac Lane, 1981, p469):

> The set-theoretic approach is by no means the only possible foundation for mathematics. Another approach is to formulate axioms on the composition of functions. This ... probably gives better insight into the conceptual form of mathematics than does set theory. There may well be other possible systematic foundations different from set-theoretic or categorical ones.

Weyl's 'historical decisions' imply that there have been branchings in the history of mathematics where, for some reason, a particular direction has been followed, and not another. In this sense at least mathematics is relative – it could have been otherwise. Such branchings are not difficult to find. Within analysis there is the Cauchy/Weierstrauss debate, which is described by Lakatos (1978, Chpt 3) as two mathematicians talking past each other because they were talking about different mathematics. Within statistics there is a contemporary debate between Frequentist and Bayesian paradigms based on differing understandings of the concept of probability. On a more applied level, it is possible to discern different approaches to navigation: one derived from European roots based on position on an imaginary grid; the other from Pacific roots based on pathways across an imagined seascape (Kyselka, 1987). Both are formal systems, although one has had a large technological investment and come to dominate.

Web-Math

At the conference on Technology in Mathematics Education prior to the one at which this paper was presented, there was a plenary talk given about developing the protocols for mathematics on the internet. Such protocols are necessary because of the need for a common standard for the multitude of mathematical environments (present and future). It makes sense for there to be means of cross-communication between such environments.

> Such protocols are an obvious pragmatic response. What was interesting was the lack of any acknowledgment of their role in defining mathematics as it will be performed and communicated. On the contrary, there was considerable assurance that the protocols being developed were sufficiently general and broad that no limitations would be experienced. It is of concern that the producers of the protocols, which will define a large part of mathematical activity for the foreseeable future, regard their concept of generality, and their vision of the breadth (and hence boundaries) of mathematics, as definitive. This paper has suggested that there are unconscious limitations embedded in the world's most versatile languages, particularly English, let alone in the restricted domain of web-language.

PUIS APRES (EDUCATION) ? / SO WHAT (EDUCATION) ?

This work on language does not mean that different peoples are limited by their language to the concepts expressed in that language. This talk is an example of the way that we can consider ideas which have arisen in other language structures. Thus the mathematical ideas which emerge from any language are potential concepts for mathematics for speakers of other languages. This applies, for example, from Spanish to Euskera as it does from Euskera to Spanish.

There is an important consideration for people from those cultures which experience cultural estrangement when studying mathematics (which has been developed through a different world view). Overcoming this estrangement is no easy task, but acknowledging the problem is essential. Such acknowledgment must be given by teachers, but also in the curriculum. One attempt at this is described by Lipka talking of an Alaskan programme:

> The pressure behind developing a Yup'ik mathematics is three-fold:
> 1) to show students that mathematics is socially constructed;
> 2) to engage students in a process of constructing a system of mathematics based on their cultural knowledge;
> 3) to connect students' knowledge of "their mathematics" through comparisons and bridges to other aboriginal and Western systems

In other words, access to the conventional, widespread field known as 'mathematics' must come through the world-view in which it is expressed. If your world-view is different from this, then it is first necessary to understand the role of your own world-view in making sense of quantity, relationships and space, so that you can appreciate another one.

Such an educational task seems to place an added burden on anyone who is starting from a different world-view than that of conventional mathematics. This is true, but there are two important points to be made. Cummins has produced evidence that bilingual learners, provided they are fluent in both languages, have a cognitive advantage in any educational task. I interpret this to mean that the sort of knowing which results from having two (or more) world-views is a deeper, more aware, sort of knowing than that which results from having only one. Hence people learning mathematics from a different world-view have to do more, but they reach a different, deeper understanding.

The second point is that mathematics learners from the same the world-view as that of conventional mathematics also have an added task if they wish to reach this deeper level of understanding. It is a feature of many education systems, especially mono-lingual English-speaking ones, that such a different level of understanding is not even recognised. It behoves us as mathematics teachers to create this awareness in our students. I think of this as putting more emphasis on mathematics as a humanity than on mathematics as a science - and particularly to avoid teaching mathematics as an unquestioned series of results and techniques. At the very least it means that we have a duty as mathematics educators to teach something <u>about</u> mathematics, not just to focus on mathematical methods and results.

Conseil Municipale / Town Planning

Where has our archaeological investigation taken us? Hopefully it has challenged mathematicians to view the edifice of mathematics as more restricted than had been realised, just as a new (or old) building can break the boundaries of accepted norms, and in doing so transforms the way in which existing buildings are viewed. This has happened in Bilbao, where the Guggenheim Museum has not only broken the architectural boundaries (to the delight of some and horror of others), but has also transformed the way the city works: there is now a shortage of accommodation as the number of visitors to the city has grown exponentially.

Is it possible to have mathematics which has been developed in different ways? I believe that it is possible, but that this involves an openness to changes in mathematics itself, and a high level of linguistic awareness, i.e. an understanding of mathematics and its symbolism, and how this relates to the language in which it is expressed.

It is an intriguing thought that, internationally, mathematics is being communicated and conducted increasingly in the English language. Is it being fossilized into modes only expressible in that language? Are new avenues of mathematical thought being cut off by not doing mathematics in new languages? Is it possible that a new generation of mathematicians, brought up through an indigenous language, may be some of the most creative mathematical thinkers of the next century?

I believe that the idea of linguistic difference is particularly important for mathematics because conceptions of quantity, relationships and space are fundamental to our understanding of the world. Furthermore, a good place to look for diversity in mathematics is in indigenous languages. To quote Benjamin Whorf again (Whorf, 1956, pp244-5):

> ... to restrict thinking to the patterns merely of English ... is to lose a power of thought which, once lost, can never be regained. ... I believe that those who envision a future world speaking only one tongue ... hold a misguided ideal and would do the evolution of the human mind the greatest disservice.
>
> ... an important field for the working out of new order systems, akin to, yet not identical with, present mathematics, lies in more penetrating investigation than has yet been made of languages remote in type from our own.

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WORKING GROUPS

Working Group A

TECHNOLOGY AND MATHEMATICS EDUCATION

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(Gila Hanna was to have been the leader of this working group but was called away to committee duties for ICME 2000. I would like to thank Lynda Colgan for her gracious help in taking notes and providing feedback.)

INTRODUCTION

In order to provide a focus for this rather broad topic, we intended to look specifically on the role of the Internet in mathematics education, drawing on our own experiences in and out of the classroom using both older and newer technologies. Our initial plan was to highlight three interrelated modes of the Internet: as a medium for communication, as a resource for learning and teaching mathematics in the classroom, and as a tool for doing mathematics.

Given the wide range of agendas and experiences, of pressures and opportunities brought by the members of the working group, this plan soon gave way to a more organic journey through an immense landscape.

It became apparent that we couldn't broach the issues emerging from the use of the Internet in mathematics education without first addressing more fundamental questions about pedagogy and technology. As David Pimm asked: "how can we successfully implement "technology" when we have not articulated a pedagogy of technology and a technology of pedagogy?" Others expressed concerns about using the Internet in preparing future mathematics teachers when faced with the inevitable lack of support and equipment in schools. Can the Internet really be a tool for teaching and learning when it is still inaccessible in most classrooms? Still others raised the issue of intellectual property: does what you create and publish on-line remains yours? Do we run the risk of "buying" on-line professors? Those veterans of computer technology in mathematics education wondered whether we had learned our lessons from experiments such as LOGO, and felt unable or unwilling to keep up with the constant barrage of new technologies purported to enhance the learning of mathematics.

The brief introductions by members of the working group provided starting points for many different paths we would follow. A majority of the group reported hardly ever using the Internet in their teaching, reflecting not only on its unwieldy size but on the tendency of educational materials on the Web (when you can find them) to control experiences rather than open the doors to exploration. Many also felt that there was a lack of vision as to how the Internet can interact with mathematics learning and teaching: when or why does it get used; where is it; how do we make the transition from the highly structured system that is successful for many teachers to a more fluid state; how do we help students make that transition?

This report will necessarily zoom in and out, from our focussed presentations, to our more wandering discussions and explorations. The presentations can be seen as snapshots of Internet usage in mathematics education; they helped structure our journey and provided us with a common experience from which we drew many insights.

FIRST PRESENTATION: ON-LINE HELP

There has been a great deal of research activity on the Internet as a place for building learning communities, as a way for students and teachers to communicate with each other sharing resources, expertise, and ideas. Technologies such as email, listservs, web pages, and bulletin boards allow individuals and groups to interact with each other without the usual geographic and time constraints. Dragana has been using the Internet in this capacity for her Math courses at Humber College in Toronto. She shared with us her experiences and aspects that she has found both promising and problematic.

Through her webpage at the College, Dragana provides on-line support for her students. These include administrative services such as courses of study, calendars, assignments, sample tests, and glossaries. They also include on-line help, links to quality resources, and tools for doing mathematics.

Aside from the extraordinary investment in time these services require on the part of the instructor, Dragana found that communicating mathematics through the Internet was fraught with difficulties: text-based email is ineffective in communicating the symbol-laden language of mathematics and completely inconducive for displaying visual representations of mathematics. This makes understanding students very frustrating and thus curbs the students' inclinations to seek on-line help or advice.

ENSUING DISCUSSION

Dragana's site is an example of a "manager" model as opposed to a shared community of learners model. In the former, the manager is responsible for creating and maintaining the site, choosing appropriate materials, and addressing the needs of the users. Sites such as TAPPED-IN and Connect-ME are examples of shared communities of learners (in this case mostly teachers) where the community members themselves participate in the building, maintaining and communication of materials and expertise. These not only have the advantage of distributing responsibility and workload, but in shifting the locus of authority from the "manager" to the learners. Although these have had success in professional development initiatives, their role in the classroom is not clear.

This question prompted an exploration of what exactly the Internet is; group members expressed a desire to categorize what is "out there" in order to understand how and where it can be used. In our quest to classify this always open and ever growing library, we identified five categories: resources and information, tutorials, help and support, demonstrations, and mathematical sandboxes. Thus teachers can use the Internet to locate and gather teaching materials, to demonstrate concepts, to provide enrichment opportunities and to communicate with their students and their peers. Students can use the Internet also to locate and gather learning materials, to ask questions and get help, and perhaps to explore and build in the mathematical sandboxes.

From the teacher's point of view, the time requirements for locating or producing on-line materials are immense. Moreover, with the constant changes in location and content, web pages demand ongoing maintenance. Whereas teachers might use the Internet at home to find lesson plans, they are less likely to use it to demonstrate concepts in the classroom, and even less likely to have their students use it in an activity setting. Some teachers use the Internet to publish their courses of study, lessons, and tutorials. However, these are often designed to support out-of-classroom learning and to address enrichment or remedial needs.

From the student's point of view, the Internet offers learning opportunities outside of the traditional classroom environment. Sites such as the "Math Forum" offer homework help, provide problems of the week and point students to interesting mathematics on the Internet. Students will most certainly be accessing the Internet and teachers should be aware of what is out there. We should also be helping students distinguish good websites from bad; we might even establish criteria for ourselves and ask ourselves what it is a site is doing to the student.

Do we view the Internet then as an electronic library, a stimulus or a set of resources that can be woven into the preparation and delivery of lessons? Certainly it can, but can it also, as a technology, facilitate the doing and understanding of mathematics? Is it both an entity and a learning environment? These questions precipitated the need to see examples of how others are doing with the Internet in their teaching. Indeed, Margaret's presentation on the second day was designed to do just this.

SECOND PRESENTATION: USING SOFTWARE IN THE CLASSROOM

Margaret has been making extensive use of spreadsheets, dynamic geometry packages, and the Internet in her high school classrooms. She cited the following possibilities for use of the Internet in a classroom: as a research tool, as a contact tool, as a source of enrichment or remedial ideas, as a place to display outstanding work and as a source of technological information and software. Which of the many possible uses of the Internet will improve the mathematical learning environment?

Margaret also pointed out the advantage of the Internet as a generic tool that is platform independent and the most accessible piece of software in schools. Although there are still many equity issues to resolve around the Internet, it seems to hold more promise then other modular, specialized and expensive computer learning tools.

Using web pages that she had prepared for other teachers and for her own students, she identified the following ways in which she used the Internet to bring "life" into her classes: it gave them some sites to visit to get extra help with homework; it pointed them towards some of the best math sites around; it allowed her to communicate with them in an easily updatable format even at home; it provided a place to display their work (in the future); it acted as a location for her to load math, both static and interactive.

Margaret's web pages are used extensively by other teachers, often as a template; this greatly reduces the time and expertise requirements. She has experimented with using web pages to help guide students for activities and found that it was not enough for them to see a graph or diagram; they needed to create them from scratch using other tools such as Spreadsheets, Maple and Geometer's Sketchpad. That is, it's necessary for students to actually input rather than merely manipulate. However, they enjoyed using the Internet as a place to "publish" their findings and solutions.

In terms of email communication, Margaret reiterated Dragana's concern about the difficulty of writing mathematics on the Internet. There are a variety of tools available for web page makers to include equations, graphs and diagrams but these are not readily available to students, nor are they conducive for synchronous communication.

ENSUING DISCUSSION

Margaret raised the question of how the use of computer units in the math classroom affect assessment, and suggested that there has been little research in this area, something which might prove to be an insurmountable gap for many teachers. Of course, this raised the issue of what kind of mathematics the students are doing on the Internet and whether we can or should assess them in the same way we usually do.

The notion of extending the classroom became a topic of discussion. Can we get students to do things outside the classroom now? Do we reconceptualize the traditional classroom? Does it give up its role as the central locus of learning (if indeed it ever was)? Perhaps the unstructured time outside the classroom is more akin to the unstructured explorations afforded by the Internet.

Having introduced other software tools such as Sketchpad and Maple, it became important to distinguish using the Internet from using software. We also decided to discuss the ways in which educational software in general can facilitate mathematics learning, what it does that couldn't be done without it, and how it changes the kind of mathematics we can do. We decided that these questions might be best approached by working through one specific topic, identifying when a specific technology would be useful for teaching, learning, or gaining a window on children's mathematical meaning-making, and discussing how we would design a computer-based activity to achieve our goals.

One group chose grades 7 & 8 probability as their topic area, using the Ontario curriculum standards from the Internet. They identified the "big ideas," listed the tools that could be used in a classroom, and the activities that could be developed around these tools. This provided them with insights into what exactly appropriate mathematics software could offer, as well as the added complications that this software might introduce. They agreed that the role of the Internet was unclear, except as a means of sharing data with other classrooms, or locating data about a specific population. Although there are several random number generators on-line as well as dice-throwing, coin-tossing, needle-dropping simulations, none of these were deemed appropriate or constrained enough for pedagogical purposes. They are "black-box" type programs that don't allow students enough opportunity to be active: to interact and explore. However, it was noted that teachers could use these simulations as departure points for activities, and as a means of giving children a "sense" of the phenomenon.

Another group focussed on the timing and coordination of different tools in the learning process. They noticed the difficulty in moving from computers to traditional mediums, not only in a very physical sense (different classroom? different desk?) but in terms of mathematical representations. Although some software programs (e.g., LOGO, Geometer's Sketchpad, etc.) allow students to express mathematical relationships explicitly, the Internet, as noted above, is more of an impediment. With the advent of communication protocols such as MathML, it will become easier - when browsers adopt them - to communicate mathematics on the Internet, both visually and semantically. Programs such as WebEQ enable the creation and manipulation of mathematical objects, including equations, on-line through the use of a plug-in. However, compared to undergraduate or research mathematicians, the needs of school children are relatively minor; they do not need to write integral signs or matrices. Teachers then might find these types of solutions overkill. There is also the problem of communication between programs. Ideally, we would want children to be able to exchange mathematical information between, say, spreadsheets, graphing tools and calculators. Some research initiatives in this direction will be discussed in the following section.

The last group, after having looked at a selection of applets (these are smaller programs that run on an internet browser, written using the Java computer language), discussed the potential for on-line microworlds. They invoked the customer vs. creator metaphor to draw attention to the importance of putting the tools in the hands of the users to manipulate and create. They noted that in order for applets to help us understand children meaning-making, they should keep track of the student's actions.

In order to facilitate communication, teachers and students should be able to annotate them. This means they are shared over a network, can be saved in different states, and retrieved at a later date. There is currently much work being done to satisfy these demands; it is now possible to share an applet with someone else, whether that someone is in the same classroom or in a different country. This depends however on cutting-edge technology and high internet speeds which have yet to become pervasive in schools.

THIRD PRESENTATION: JAVA AND COMPONENT ARCHITECTURE FOR EDUCATIONAL SOFTWARE

This presentation was intended to give the participants a sense of new directions in software development for mathematics education, notably those designed for the Internet.

The market for software for mathematics education is dominated by static, modular, feature-laden, and expensive packages. Although many of these are high quality educational products (Geometer's Sketchpad, Function-Probe, Maple, Fathom), they have not generally been able to "crack" the school walls. They encounter problems such as limited accessibility, narrowness of focus (teachers have several strands to cover; they certainly won't buy and learn a different package for each one), inability to communicate with other packages, and high cost. The kind of money and time required to develop polished, complex packages is typically out of reach for most mathematics educators interested in computer-based tools. These researchers will often develop small programs to answer specific research questions, without any hope of any kind of wide-spread implementation.

The proliferation of Java has stimulated the creation of thousands of "interactive" applets on the Internet. Until now, the creation of applets required expertise in programming, which few mathematics education researchers and teachers possess. Thus many (but not all) of the applets currently on-line are not sensitive to the learning needs of children. And, although applets are interactive in the sense that you can change parameters, drag vertices, and press buttons, they cannot be easily modified: they are consumer rather than creator tools.

Recently, there has been a focus on finding sustainable and scalable solutions for mathematics software. One initiative has been to use component-based technology to create software. Components that can run on the Internet become accessible to more students at a lower cost. Researchers at different universities can design components specific to their needs (or the local needs of students or teachers) and, because such components "speak" the same language, they can interact with other components. Economically, this makes the development of quality pedagogical programs more feasible. Socially, it allows researchers, programmers, teachers, and students to work closely together to fulfill their needs, whether or not they share geographic proximity. The potential then exists for a growing "toolkit" to emerge, comprised of tools for various purposes (plotting, calculating, drawing, programming, etc) that can be re-used, integrated, and combined in multiple ways.

I showed some of the applets that I had produced to teach transformational geometry at the middle school level using a 'micro' toolkit of components. I was able to build and modify (according to the needs of my students) a set of fifteen scaffolded applets that allowed us to work with basic transformations, as well as tesselations and wallpaper patterns. For example, by combining a plotting tool with a programming tool (a crude version of LOGO), students could instruct shapes to reflect, rotate or translate through a maze. This put me, as the teacher, in a creator role, as well as the students, to the extent that I could design the activities to allow for it. Students were able to share a single applet and work together to navigate their shapes through the maze. As one participant noted, there was much in common between these applets and the LOGO based microworlds. If teachers are able to use such a toolkit to create applets, then students might be able to as well. We could thus have constructionist environments where children solve problems by constructing an appropriate applet; they could build games and puzzles, model mathematical problems, or create learning tools for their younger peers. In principle, this is feasible; however, the technology is as of yet too early in its development to pursue these possibilities on any larger scale. They might however, provide a vehicle for research in student understanding.

ENSUING DISCUSSION

Technology solutions do not automatically provide pedagogical solutions. We drew interesting parallels between the LOGO phenomenon and the promise of this new technology. We identified areas of continued concern, which include: lack of attention to assessment issues, the need for changes in curriculum and instruction, the limitations of current educational policies, and the need for more research on cognitive aspects of learning with technology (role of visualization, multiple representations, and dynamic notations), and integration of computer-based tools with traditional classroom teaching methods and tools.

At the end of the third day, we ended up with more questions then we had started with. Our discussions proved perhaps overly eclectic, indicating that the topic was too broad to be able to do justice to it. On the other hand. the diversity of both the participants and the issues involved was a strong reminder of the complexity of our discipline.

FINAL DAY: (UNCONVENTIONAL) WORKING GROUP REPORT

Inspirée par Frédéric et sa poésie en français j'ai decidé de vous raconter notre travail en couplets.

Like the new Guggenheim(1), our topic proved unwieldy so our anglo-canadian(2) moved we approach it tangentially.

Though we thought and shared, excavating our assumptions the best I can do is share with you some questions.

From Eric(3): "The WWW puts me in a liquid state, alas! "What is our vision? Are we solid or are we gas?"

We have some concrete foundations to stand from: Monday morning, probability, the grade 8 curriculum.

Start with dice, add spinners, markers and graphers, stir a little, did they understand the law of large numbers?

Spreadsheets, sketchpad, simulations and a java applet they felt it, sensed it, can they now communicate it?

Perhaps if not to us, then at least to each other, especially on some futuristic two-personal computer.

But wait, have we even agreed on what is this math education? Gosh I hope, Joel exclaims, after 20 years of conference deliberation!

Mais dites, en passant, qu'est-ce que c'est cet Internet? Une salle de classe et une bibliothèque, toujours ouverte?

On the Net, websites come and go, talking of Michelangelo(4) Are we moving forward, clicking backwards or revisiting Logo?

Lessons plans, links, tutorials: they're multiplying. Is there room for mathematical castle building?

Perhaps in microworlds and sandboxes, where the tools are correct for E. M. Forster mathematics: Only Connect!

If we build them, will they, our teachers and students, come or is learning the software ultimately too cumbersome?

Press this button, press that one, now turn the page textbooks like this, open mouths say, have become the rage.

Mais il faut quand-même que les profs puissent se servir de ces outils promettants qu'on ne cesse d'introduire.

But wait! cried the Englishman(5) not far from York. What's the technology of pedagogy? Let's get to work!

So we did for many hours and here's the scoop: we definitely know what to discuss in next year's working group.

NOTES

1. Bill Barton talked about the Guggenheim at Bilbao in his plenary
2. John Mason... he was always adding "tangential" remarks.
3. Eric Muller
4. adapted from t.s. eliot, and david pimm
5. David Pimm

Working Group B

APPLIED MATHEMATICS IN THE SECONDARY SCHOOL CURRICULUM

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Dalene Swanson, University of British Columbia
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INTRODUCTION

A significant factor in motivating the presenters and attracting the participants to this working group was the fact that "applied mathematics" courses have been proliferating in the secondary school mathematics curriculum landscape. The western provinces and territories have developed a common mathematics curriculum with an Applied Mathematics stream for grades 10 to 12. In Ontario, the new curriculum framework specifies an Applied Mathematics stream for grades 9 and 10, College preparation courses (applications-focused) for grades 11 and 12 and practical, workplace destination courses. Certainly curriculum interventions towards applied mathematics, as the bibliography indicates for example, are not new. What is new is the rapid and unquestioned mandating of courses termed to be applied.

In the course of discussions, the working group examined possible meanings and intentions that inform the categories of pure (academic) mathematics and applied mathematics. We took it upon ourselves to review current curriculum changes embedded in documents from Ontario and the western provinces. We took on the role of students learning some applied mathematics using graphing calculators and we reflected on (struggled with) that experience afterwards.

The working group leaders provided a set of questions and perspectives to help launch the group's work (Appendix A). We, of course, addressed only some of these and considered others that came out of the participants' thinking.

DAY1

Overview of the Ontario Curriculum Approach to Grade 9 Applied and Academic Mathematics:

Gord Doctorow reviewed the grade 9 Applied and Academic Mathematics document for the new Ontario high school mathematics curriculum. The Ontario curriculum document begins with an emphasis on meaningful mathematics:

> Mathematical knowledge becomes meaningful and powerful in application. This curriculum embeds the learning of mathematics in the solving of problems based on real-life situations. Other disciplines are a ready source of effective contexts for the study of mathematics. Rich problem-solving situations can be drawn from closely related disciplines, such as computer science, physics, or technology, as well as from subjects historically thought of as distant from mathematics, such as geography or art. It is important that these links between disciplines be carefully explored, analyzed, and discussed to emphasize for students the pervasiveness of mathematical knowledge and mathematical thinking in all subject areas. (Page 4)

The document goes on to focus on the exploration of applications and the effective use of technology for both applied and academic (pure?) courses. However, the academic course emphasizes abstract reasoning while the applied calls for "extended experiences with hands-on activities".

Overview of the Western Consortium Curriculum Approach to Applied Mathematics:

Elaine Simmt provided an overview of the changes that had taken place in high school mathematics in Alberta. She mentioned that Alberta used to have a highly streamed curriculum. The western provinces and Territories had come together in the last few years to provide a unified curriculum approach which provides a uniformity of outcomes grade by grade. This has led to the development of "applied" and "pure" mathematics streams that are meant to provide a high quality mathematics program for both university and non-university bound students. These two streams do have some common outcomes - statistics, consumer math, and trig measurements. However. they vary in implementation from the applied to the pure programs, but the critical skill of using mathematics to find solutions to real life situations is developed in both programs.

Elaine summarized the intentions of the Applied Mathematics program:

- gives students a clearer picture of why they are learning the mathematics
- motivates them in learning
- helps students to understand that mathematics is much more than theory emphasizing a set of algorithms
- gets students to understand that mathematics is a powerful set of processes, models and skills that can be used to solve non-routine problems, both in and out of the classroom.

The approach to teaching applied mathematics, consistent with the intentions, was summarized as follows:

- data driven
- numerical and geometrical problem-solving techniques
- data collections in experiments and activities and development of math concepts from analysis of the data
- exploration of connections among other mathematical areas, other school subjects and real-life objects.

Historical Overview of Curriculum Movements Toward Applied Mathematics:

Claude Gaulin began his presentation by asking what forces are pushing for an applied mathematics curriculum as evidenced by a North American trend. He questioned whether the terms applied and pure mathematics are adequate.

Claude pointed out that applied mathematics curriculum is an old issue. Applied mathematics is seen as useful math. It is a vehicle for transferring learning of tasks to demonstrate understanding. It is seen as a way of motivating students.

Claude set out to describe some characteristics of applied mathematics. Applying mathematics is seen as a process. Applications may be internal (related to mathematics itself) or external (other curriculum areas). Applications include both exercises and problems. The use of modeling in applications is an ambitious goal to introduce modeling and problem defining.

Are "real life" problems the only interesting ones? Here we are into the real vs. artificial debate. What makes a problem a good one? Is it the realism of the problem or is it the question (especially for the students): do we care about the problem?

Claude proposed an interesting trichotomy to describe teaching applied mathematics:

- teaching for applications (after theory)
- teaching via applications (to develop new knowledge/skills, understanding)
- teaching about applications.

Claude ended with a discussion of the methodological issues. The traditional approach to teaching applications involves teaching the theory first, then applying it. "Word problems" are artificial or are "ready made" providing a very confined context. When embarking on learning applications through modeling, the context itself requires efforts at understanding in addition to the mathematical skill set to build the model. So, we have to consider modeling opportunities that appeal to students, not just real life ones.

Claude displayed a number of books on applied mathematics in the curriculum which have developed out of many efforts in the past to enhance the mathematics curriculum. A bibliography including some of these books is provided at the end of this report.

DAY 2

This session began with a hands-on graphing calculator lesson on Mathematics of Finance problems to illustrate how an applied mathematics curriculum could be implemented. Gord provided a set of problems (Appendix B) and instruction on how to use the features of the TI-83 graphing calculators to solve problems. The participants investigated compound interest problems through multiple representations (or models) to accommodate different learning styles and to reinforce understanding.

The participants followed up a lot of explorations with a discussion/analysis that combined some of the issues raised in the Day 1 discussion with observations based on their experiences in the hands-on activity.

Applied mathematics was characterized as being intuitive, a new way of thinking what mathematics is, having implications for technology, concrete as opposed to the more abstract nature of pure math - but applied math could also occupy other roles in the curriculum.

Rick Seaman introduced the idea of "the gap". He argued that teachers don't do enough between the exploration and representation stages to help the students transfer the knowledge. How to fill the gap? Teachers need time. Time needs to be made to link manipulation activity to representation schemata - time to explore.

Technology and applications were seen as symbiotic in the current context.

Jill Adler raised the issue of how to make math more inclusive. There is a tension between inclusiveness and math empowerment. It becomes a problem of resources - money, supplies, teachers' willingness.

Related to these comments were questions about the availability of technology which is called for in the applications curriculum. How necessary is the technological component?

What about assessment? How are we to deal with differentiated learning and assessment?

Elaine Simmt spoke eloquently of the need to make mathematical experience "good in the moment of the day". She made an existential appeal to our minds and hearts to see learning experiences as meaningful life experiences.

DAY 3

A plenary meeting of the Working Group decided to conclude our discussions by breaking out into two subgroups which would focus on particular issues and select a recorder to report back. We ended with a round-table discussion. Subgroup 1 dealt with teach issues. Subgroup 2 dealt with content issues.

TEACHING ISSUES REPORT - Doug Franks & Elaine Simmt (reporters)

What are we trying to do?

A new way of teaching - intentions, power and control.

- what are new applications with technology?
- how to teach - is this a forced change?
- what is the thinking behind the "trend"? - viewed as progressive
- what is driving the applications?
- is the applications curriculum career directed?
- contrast between general/advanced distinction and applied/academic distinction
- government labels were for public consumption
- calculus reform - public image is poor; so the decision-makers respond to the public image and in comes applied
- students looking forward in life and wanting mathematics that is related
- new method, using technology and manipulatives
- what applications are useful at which time for the topic?
- issues of cost of training: inservice and preservice
- is math being used to promote technology outcomes? - what are the math outcomes? what are the technology outcomes?
- computer labs - the mathematics comes in the report, can't lose the mathematics
- sometimes teachers test the students' ability to use the calculator
- applications and mathematics brought into the situation
- "gap": moving from the particular to the general, the concrete to the abstract - situated knowing, e.g. carpenters' knowledge of mathematics

- who are the kids that take applied mathematics?
- what about the gap for our younger children?
- do we leave kids working with the mundane - how do we move to the general?
- is applied mathematics for the workplace? what are the utilitarian work forces? is there a capacity for abstraction?
- what is mathematical thinking?
- view the forest from afar, approach the forest, see the trees, then back out and view the curriculum
- mathematics as a continuous spiral
- new mathematics is not hierarchical but we can put students in a context - the mathematics then uses what students know
- need to address student desires and needs - need for explanation, trying to understand in the other's terms, mathematics arising in the interaction

CONTENT ISSUES REPORT - Ed Barbeau (reporter)

The spirit of a good applied problem involves the student constructing data, performing experiments, studying the data and asking questions. But there is more than just getting the answer. Issues of judgment arise - representing the data, choosing the approach, assessing the mathematics and relating it to the situation at hand. The selection of problems used should reflect different types of mathematics and mathematical processes. Sometimes one can get an exact answer and make predictions (as in physics or the finance of annuities). Other times mathematics enables us to focus on core issues or clarifies the situation, without being prescriptive. But they should epitomize mathematical thinking and encourage reflection.

But how do applied problems become accessible to students? It is neither necessary nor sufficient that they be "interesting", but they should be meaningful and connected to their experience. Careful orchestration is needed to ensure that appropriate psychological and mathematical prerequisites are in place. Both the teacher and the text need to illustrate through examples how to model. A good book should guide student and teacher by giving a detailed path through some problems. In this way, students can in stages acquire mathematical and expository skills and a perspective on the modeling process.

We need a supply of problems of different intensities, from those that can be handled in one or two sessions to longer investigations that can span several weeks. There should be enough gradation to enable students to get to possible strategies and make necessary connections and achieve what is appropriate from their backgrounds.

NCTM-MAA book on criteria for good problems

- data real
- students gather data
- unknown in problem should be plausibly unknown
- solution not intuitively obvious - mathematics is necessary
- avoid ad hoc formulas

Peter: have students do things - construct data, perform experiments

Ed: judgmental aspects, more than getting answer; example: height and reach measurements - get students to study data and ask questions

Claude: Pollack's article - spirit of applied math - explore/find good questions

Peter: also spirit of math

Rick: how will student represent data? acknowledge student contribution

Ed: multiple entries - compare, evaluate approaches

Claude: context - familiar? interesting? meaningful?

Olive: role in curriculum, Alberta textbook: teach via problem-solving - but applications require students to have facility with math concepts and formulation. What about projects being done by students who do not know what to do? They need a sense of mathematical thinking.

Claude: in other countries, students will not learn to model from the end of the chapter - must show students how to model through detailed examples

Malgorzata: need some sort of reflection - what sort of modeling

Peter: what the model will bear? need a sense of the class, know when to move on, what can be reasonably done in time

Rick: variety of ability in a class

Rina: when does it come? what is prior? orchestration

Rick: proportional reasoning, goal - think how to represent, lesson plan - exercise/support knowledge base

Olive: teacher's way of thinking is important

Malgorzata: what is a good problem is independent of the teacher - capacity of teachers is a different question

Claude: what helps students learn is that it be significant (i.e. student can understand) - interest not enough if student does not have resources. Imagine writing a textbook - how detailed should activities be?

Peter: good book should have detailed path through problem - student needs model and teacher needs guidance. Understand how/why - make distinction.

Rick: why do we teach mathematics?

Claude: student must learn about modeling process - choice of good activities, math used in different areas - learning progressively

Ed: different sorts of applications to illustrate information. Discussion of Alberta and Ontario processes.

Rick: what relation does problem have to what students have done before? Student needs to develop retrieval strategies.

Ed: good problem - point of strategy and connections become manifest

Claude: need to pass through several problems to get to general idea - keep continuity, connect with curriculum, material to be covered

Peter: dilemma of detail of curriculum constraining choice of problems

ROUND -TABLE COMMENTS

Aside made during the break: Here's an aside that was recorded. In the compound interest exercise, should we be providing students with a compound interest function ready-made on the calculator or is it better to build up a procedure using the technology?

Olive: commented on the use of the term "modeling". Felt that the discussion around this approach has given her a new perspective to help her with her graduate students.

Rina: stated that it was important that the students feel that the problems presented are meaningful - this relates to the issue of having sufficient background knowledge. It is also important that the teacher feel the importance of the problem.

Rick: argued the need to give students some control. Need to provide students with expertise and knowledge as background, i.e. explicit teaching. Need meaningful problems in the "context of the student". Questions the need to create a separate applied category for mathematics. When he teaches, he makes plans to clear "the gap".

Patricia: how does a teacher orchestrate modeling of a problem? How to generalize an approach?

Peter: the distinction between applied and pure doesn't matter. What is important is meaningful mathematics or caring about the problem. How do you make a problem meaningful? A teacher needs to convince the students to suspend their disbelief and get them to come along with him/her. Teachers need to systematically ask questions all the time (inquiry-based model) in order to create an atmosphere of curiosity and wondering (why? how?)

Doug: need to recognize that the language of Applied is a political reality. However, we need to go back to the classroom to create meaningfulness.

Jill: enjoyed the dialogue between mathematicians and teachers. Worthwhile to focus on the tension between from and substance.

Dalene: are we looking at situated knowledge to abstract meaningful math or are we relegating people to a vocational state? Mathematical culture is an important issue. Tension: what am I seeing? formulating with my mind or just another prescription?

Susan: addressed the idea of desire in the mathematics classroom. What is exciting is seeing irritation affecting students so that they focus on problems out of an intense desire - therefore, not a burden. This is what is meant by importance. Sometimes problems can be like that. Notion of how you approach mathematics: accessibility - strings of symbols vs. drawing a picture; multiple approaches that lead to the same end (isomorphisms); a wide range of approaches that converge on problem solutions. Mathematics as a phenomenon - not a thing but really an interaction of forces (Brent Davis' notion of distributed sites of math actions).

Malgorzata: agreed with our group's early discussion which started with the notion that a good problem is something we care about. Doesn't like "mathematics for the workplace" label - limiting, doesn't recognize the potential of math deriving from the workplace. Doesn't want applied math to be treated as a barrier to doing serious math.

Ed: doesn't object to Applied/Academic separation. Gave an example of a Russian puzzle - is it pure or applied? But boundaries between pure and applied should be left fuzzy.

Jacqueline: dealt with "experience". Experience (Dewey's sense) is necessary to help students learn about math. Applied math stories are a good idea. It is also a good idea to link applied mathematics with technology. Students can experiment with mathematics and then, at the end, the teacher can offer a final proof.

Medhat: sees an artificial division of mathematics. Math should be treated as a spectrum rather than applied vs. pure.

Claude: the applied mathematics trend is merely a continuation of a trend that goes back decades. Why would we now want to mention applications separately? "Situated learning" as a frame of reference is becoming more and more popular. What are the sociological reasons for this? The putative benefits of Applied Mathematics, such as providing problems with many solutions, aren't limited to the domain of Applied Math.

There is a shift in the political winds. It is evident in the discussion around new NCTM standards. Discrete mathematics has been deleted - to be redistributed into topic areas. Seems like they don't know where to put the applied part.

We're still making mistakes for political reasons:

- teachers aren't ready
- textbooks aren't ready
- progressives seem obliged to help - raises the issue of our responsibility as math educators: how should we critique the trend even if we feel obliged to participate?
- what can we do individually? The movement is so strong.
- the choice of applications will pose a problem - if you use them at the wrong place, then we won't get the benefits

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APPENDIX A

1. What is the dichotomy between applied and pure (academic) mathematics? What is the value of applied mathematics curriculum in relation to the goals of mathematics education for a democratic, pluralistic, and increasingly technologically driven society?

2. What intentions or purposes lie behind an applied mathematics approach? For example, there are statements of purpose on applied mathematics in curriculum documents that target relevance, motivation, and meaningful involvement in doing mathematics. [Refer to the Curriculum Documents of Ontario, the Western Consortium, NCTM Standards, the Harvard Model, the Atlantic Provinces.] What are the images of the learner that are being portrayed? What are the perceptions of mathematics that are being built up? [For example, is mathematics a catalog of skills, an approach to problem solving, an exercise in algorithm making?]

3. Should applied mathematics education focus on applied mathematics per se or on the applications for which one uses mathematics to solve problems? Should the main focus be on teaching about the nature of mathematical problem solving? Should we centre on the process of mathematical modeling, particularly in conjunction with technology?

4. Here are some thematic organizers for discussion:
   - mathematics as culture
   - mathematics as intellectual tool
   - mathematics as a philosophy and discipline
   - mathematics as a life skill

5. What are the implications of the current initiatives toward Applied Mathematics curricula at the secondary level:
   - for students?
   - for teachers?
   - for Boards of Education?
   - for professional associations (teachers federations, mathematics teachers groups)?
   - for technology?

6. This working group is confronted with making a review and critique of recent reform initiatives which could be usefully examined to formulate a balance sheet to meet the goals of public education. Beyond that, the working group might seek to designate vectors of investigation to provide a timely response to a dynamic situation of curriculum change and renewal.

APPENDIX B

Due to the length of the set of problems, an editorial decision was made to remove the contents of this appendix from the formal Proceedings. The problems may be obtained by contacting the working group facilitators.

Working Group C

ELEMENTARY MATHEMATICS

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INTRODUCTION

Welcome readers to our working group on elementary education! As other working group chairs have indicated, when we use the printed format to capture the dynamic conversations of CMESG working group sessions, we set ourselves a formidable task. In particular, we will find it impossible to capture the laughter and good feelings that the 11 of us shared over the three days at Brock University. As co-chairs we had planned to use the sessions to ascertain our collective knowledge of elementary education in Canada and to discuss three curriculum issues pertaining to young children's mathematics learning. We will try therefore to re-present those three working sessions for the reader as they seemed to evolve.

SESSION 1: SATURDAY

PART 1: COMING TO KNOW WHO WE ARE

Most working group sessions begin with introductions 'around the table' and we were no exception to this. However we encouraged our participants to extend the usual, 'name, affiliation and interest in Working Group' to try and capture who we were as individuals and what our current roles in elementary education were. In addition, each participant completed an index card profile, indicating grade levels or age groups for research and teaching, actual research topics and so on. A summary of these profiles appears at the end of the report. So, what did we discover about this group of Canadian mathematics educators interested in elementary education?

Many of the participants were just beginning or preparing to begin their careers as full time faculty in universities across the country. Most of us had 'trained' as secondary teachers; one of us was a psychologist. Many of us had young children enter our professional or personal lives and we were intrigued by their thinking. Many of us have been involved in some research with children in elementary grade levels. Few of us personally 'know' what it's like to prepare for teaching as a generalist or to be the enrolling teacher for an elementary class. Most of us are currently educating prospective elementary teachers: some through mathematics education courses only, others through participation in varied aspects of their students' program, including case-based teaching seminars, and supervision of practica. Many of us are attracted to elementary classrooms and teachers as we sense an openness and willingness to change and a setting that permits flexibility and yet we struggle with some of these teachers' low selfesteem with regard to our subject area. Many of us seek opportunities to be in elementary classrooms and to carry out lessons in collaboration with practicing teachers to inform our practice and our research.

PART 2: CHILDREN'S MATHEMATICAL ENGAGEMENT?

The second half of Session 1 opened with a video clip from Louise Poirier's research in grade 1 classrooms. The video excerpts captured young children involved in story telling sessions which lead to solving and representing 'missing addend' problems. In particular, the video provides episodes of teacher(s) sharing various stories of a school bus and children getting on and off the bus at different stops. For instance, an early clip has the bus with a known number of children stop at a bus stop where some children and the teacher are waiting, but a truck pulls up and blocks the vision of the teacher (i.e. the story teller) so she is unable to see how many children get on the bus. When the truck pulls away there are now another given number of children on the bus. "How many children got on the bus?" is the question posed. Children are given a short time to find a solution, and hands begin to rise frantically. When acknowledged, a young boy announces, "That's not allowed. It's not safe". He is of course referring to the practice of a truck parking alongside a bus which has stopped to pick up passengers. Of course, the discussion which follows is animated but does not focus on the 'number of children boarding the bus'. Such an excerpt reminds us of similar incidents when, either in research or in teaching. 'Our adult view' of a context or issue did not take into account children's honest awareness of the 'ways of their world'.

In another excerpt, a child stands behind a screen where she is to represent a story consisting of a series of arrivals and departures and solve the posed problem, in any manner she/he wishes so that a second child, waiting outside the classroom door, can use the representation to re-tell the story appropriately. In other excerpts we see different variations of the children's written recordings of various school bus story problems.

The discussion that arose from the video re-affirmed the interest of this group of mathematics educators in children's mathematical thinking and doing. We were excited by the mathematics these children shared and their development of representations that communicated effectively the problems at hand. We were intrigued by the method of 'hiding a child' from view to record the problem. We wondered aloud about the action behind the screen and how and why we might wish to see the "symbol making" as it was being created. We queried Louise about the duration of the children's sense making with respect to symbolism and how it corresponds or 'meshes with conventions'. We were intrigued by the development from physical drawings (i.e. hands together for addition) to less specific symbols (i.e. arrows) as means for these children to represent their thinking. We spoke of 'school mathematics' obsession' with generalities and the 'perceived need' for all learners to use the same symbol for subtraction, say the conventional minus sign. When Louise shared how these children's intermediate teachers questioned "what's this about school buses, my children keep bringing them up as they do operations", our confidence increased as we realized that years later, the school bus context continued to hold meaning for these children and was used to support further understanding (of symbols) in more structured environments. In addition to discussing the video excerpts in our working group, many of us felt compelled to find occasions to try this problem situation with our groups of interest. Indeed, upon our return from Brock, many of us responded enthusiastically to Ann K's child's response to a 'School Bus' problem. It was intriguing to see this young child respond with a two step solution, rather than the 'expected' missing addend.

SESSION 2: SUNDAY

PART 1: COMMONALITIES/DIFFERENCES FROM PARTS OF CANADA

To encourage participants from different areas of Canada to share what is the "State of Curriculum" in their jurisdictions and what roles they play, Louise opened this session with her rendition of the Quebec scene in mathematics education and beyond. As we looked around our table, all regions of Canada were represented and we availed of this opportunity to get a sense of curriculum developments in elementary mathematics education across the country, from these personal perspectives.

It became very clear that across the country we are in flux. Most provinces have recently implemented or are in the process of introducing 'revised' curriculum documents. In addition, we heard about the Western Consortium and the Atlantic Consortium attempts to develop and implement common curricula across provinces. Not surprisingly, the NCTM Standards figure prominently in most revisions. In Quebec, there is a tendency to look to Europe and a trend toward integration across subject areas was more prominent here.

In many of the revised curriculums, there seems to be a move toward introducing mathematics topics earlier. This prompted a discussion of 'bringing down' versus 'bringing in' topics of study. It was felt that for many teachers who see "fractions" recommended for grade 1 children think of 'operating on fractions usually reserved for grades 5-7' and believe that this is what was intended. These teachers need support to recognise that aspects of the study of fractions are very appropriate and make sense for this young age and it is these aspects of fractions that are meant. For many jurisdictions, curriculum enhancements are seen as 'more' in an already 'stuffed' curriculum and cries of "how do I cover all this" are not uncommon in most provinces.

The role of assessment in curriculum change also arose and in particular, Ontario was reported to be using assessment as a tactic to evoke changes. Because of widespread testing and very direct reporting requirements, topics in mathematics are now getting done in Ontario classrooms. Finally, there seemed to be a consensus that most curricular documents in all provinces are 'outcome' based strands and that assessment is either at the forefront or is being developed after the fact. Our roles in curriculum development varied and although some were directly involved in the development or evaluation stages in their provinces, many of the mathematics educators attending the working group were involved as respondents, and some spoke of limited communication between provincial ministry personnel and themselves.

PART 2: STATUS QUO CURRICULUM?

Our conversations of curriculum changes in our various provinces and states continued after the break but more with respect to the 'trickle down' aspects of change. We began to ponder what mathematics is appropriate for primary age children and the role teachers should play in curricular change. The role of textbooks, in curriculum change or maintenance of the status quo, arose; as did issues of trust with respect to teacher judgement. There was recognition that even in the 'new' curricula we are dealing with topics and methods from the 1500s and shopkeepers' arithmetic is still a mainstay in our curriculum. This seem to extend into issues of appropriate resources and we acknowledged the irony in "expecting a book to somehow teach teachers how to teach". And yet, we recognised that textbooks and curriculum documents seem to more or less focused on how to teach (i.e. using manipulatives, working in small groups). We deliberated on the role(s) of preservice education and what we do in terms of supporting change. We challenged ourselves to examine our current practices.

In certain B.Ed. programs the mathematics content is under the jurisdiction of the educators and the program contains multiple sites to visit mathematics teaching/learning issues. In other programs, we are limited to one course in which mathematics learning/teaching is a focus and in others we are experiencing problem-based case approaches in which mathematics per se is not necessarily explicit. We spoke of linking preservice teachers, practicing teachers and children in holistic ways so they can inform one another.

One revelation seemed to be how we want (expect) elementary teachers to think like us...like mathematics specialists... and yet most elementary teachers are not. We wrestled with our perspectives and our knowledge of the subject area being very different from our clients. We questioned how we might bridge the gap while honoring both sides; this discussion brought us to collaborations with others in which participants are valued for individual expertise. We seemed caught in between believing in our teachers and their abilities as professionals and in their perceived lack of subject matter knowledge. "Hats off to Multiplying" is a classroom activity Ann A. has used in classrooms to alert practicing teachers and their students to other multiplication algorithms (i.e. Russian peasant method; Scottish lattice) besides the North American convention followed in most schools. This anecdote, among others, reminded us of the challenges we face in helping our prospective teacher 'unpack' the limited perceptions and knowledge they have of mathematics.

SESSION 3: MONDAY

PART 1: INNOVATIONS IN THE CLASSROOM [ROLES IN THE BROADER COMMUNITY]

Because of audio visual equipment problems on Day 2, we planned to reverse today's sessions and using another video from Louise's research, begin with a focus on children. We would move to broader issues and bring closure to the sessions in the second half. Once again the technology failed us and we settled for oral descriptions of what the children were doing in the video excerpt. From this, we progressed toward "what is mathematics anyway and why would we teach it?". This was not a pre-planned topic. Rather, our previous discussions had raised so many issues for us, as individuals and a collective, that trying to state aloud what we believed math was or could be seemed like a reasonable place to begin again, to re-visit some of our musings. Our personal descriptions moved us toward mathematics as "making sense of the sense that's been made". This pays tribute then to the conventions of the discipline but honors the need of any learner to make sense of it. We continued to spin this web of understanding toward " mathematics educators making sense of the teachers' sense of us making sense of the sense mathematics makes" and so on. Iterations which included "teachers making sense of the sense children make" provided images of mathematics teaching and learning that were recursive and could not be reduced to "explaining".

We queried as to what are we preparing the child for through math and further through schooling. Again the outdated shopkeepers image leapt to the fore. We began to see math as enabling and limiting; we discussed how the 'labelling of something' as mathematics could both be constraining and liberating and we ventured that "mathematics lies between contexts".

We mused further about the label mathematics..sometimes it legitimatizes, when we label what we do; in other cases, it constrains the exploration because we have named it so. Where resides mathematics had us pondering ... is there math in it?... I can do math with it?...is mathematics a formalized process?...is math about relationships and generalizations which allow us to articulate competencies in one setting to another..thereby mathematics is something greater.

Alternative programs, such as Waldorf, were explored briefly and used to illuminate some of the ideas with which we were playing. The mathematics of 'mouse houses' and 'skipping' kept us busy on many levels, trying to decipher what we accept as mathematical. We spoke of formulated and unformulated knowledge; of formal and informal mathematics. We spoke of the need for 'surprise', of an un-settling, to alert us to our enculturated practices.

Metaphors were generated...such as the suitcase metaphor. This led us to play around with perceptions of mathematics as tools or a toolkit and we were of the mind that we still have not provided a case in which to carry the tools of mathematics. We spoke of "curriculum" as a neatly packaged "suitcase" which needs some shaking. We perused images of general purpose tools (i.e. hammer) and specific tools (i.e. a mallet) and how you would want both in your mathematical toolkit. This metaphor helped some see how important context is to determining which tool to use; and the parallel to mathematics seemed strong.

PART 2: INNOVATIONS IN CLASSROOMS

We now moved from our broader discussion of what is and why mathematics toward our roles in the broader community ... at parent meetings, on talk shows, with local associations. Here we tried to reflect on the different ways we may influence and participate in debates occurring in the community concerning mathematics learning and teaching. Some of us had prior experiences with the media at different levels and shared both good and less favorable reports based on interviews. We discussed who we represent when we are involved in more public debates...personal and professional views as an individual mathematics educator may differ from the institutional stance or the stance of other colleagues. When we read or hear public debates about say 'back to basics' should we and do we respond with balanced rebuttals? In the end, we felt most comfortable with the mantra: "Act locally, think globally". Here many of us recounted events in which we may have worked with a small number of others, and yet the enthusiasm of those few proved influential with others whom we did not contact directly. Collaborative research was pointed to as ways and means to enhance support and dialogue among professionals. We seemed to agree that we must acknowledge that individuals bring different strengths and gifts to the collaboration. We spoke strongly of the wisdom in supporting obsessions, passions, and personal interests. "Why must everyone be doing everything same?" This question has impact for children in classrooms, teachers in schools or districts and educators.. We spoke to a need for both diplomacy and advocacy as new resources, often viewed as problematic for teachers and parents alike, venture on to the scene. In closing, we spoke of the effects of spilling over from the drips we make.

A PARTING WISH

We are not convinced that readers who were not present are able to "make sense of the sense we were beginning to make" as our sessions drew to a close. Working Group C bonded easily around the desire and need to find ways to have children experience mathematics differently from the conventional mathematics of our youths and more sadly, the continued traditions that some of our daughters and sons are experiencing. With support from one another, we hope to find mechanisms within our own locales to 'make a difference'. Usually this will translate into reaching out to specific classroom teachers (on a small scale) and joining with them to extend the possibilities. Our parting wish is that through future CMESG gatherings we might construct a position statement so that such local action will spill over into more pervasive differences for children's mathematics learning in Canada.

Working Group D

TEACHING PRACTICES AND TEACHER EDUCATION

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Tom O'Shea, Simon Fraser University

PARTICIPANTS

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Stéphane Cyr, Université Laval
Sandy Dawson, Simon Fraser University
Norma Evans, Simon Fraser University
Kgomotso Garegae-Garekwe, University of Manitoba
Linda Gattuso, Université du Québec à Montréal
William Higginson, Queen's University
Kanthi Jayasundera, Simon Fraser University (Sri Lanka)
Isabelle Jordi, Université de Montréal
Gerard Kuntz, IREM Strasbourg
Lesley Lee, Université du Québec à Montréal
Ralph Mason, University of Manitoba
Doug McDougall, OISE (day 1)
Tom O'Shea, Simon Fraser University
Pat Rogers, York University (day 1)
Geoff Roulet, Queen's University
Susan Stuart, Nipissing University
Dominic Voyer, Université Laval
Walter Whiteley, York University
Ananda Wijaweera, York University (Sri Lanka)

INTRODUCTION: THE TASK

The task of this working group as described in the program was as follows:

This working group intends to explore the area of teacher professional development, with a particular emphasis on identifying experiences that nurture the craft of mathematics teaching. We would like to build on the work done during previous CMESG sessions (e.g., Reflections on teacher growth: Pre-service and in-service perspectives (Stuart & Higginson, 1996); Professional development for preservice mathematics teachers (Bednarz & Gattuso, 1998). We hope to begin our discussion with an identification of the qualities, knowledge, dispositions and habits of mind that effective mathematics teachers possess. We then propose to examine the types of professional development experiences which might best nurture the development of these qualities at various points in a teacher's career (during initial preservice education programs as well as during subsequent teacher inservice programs).

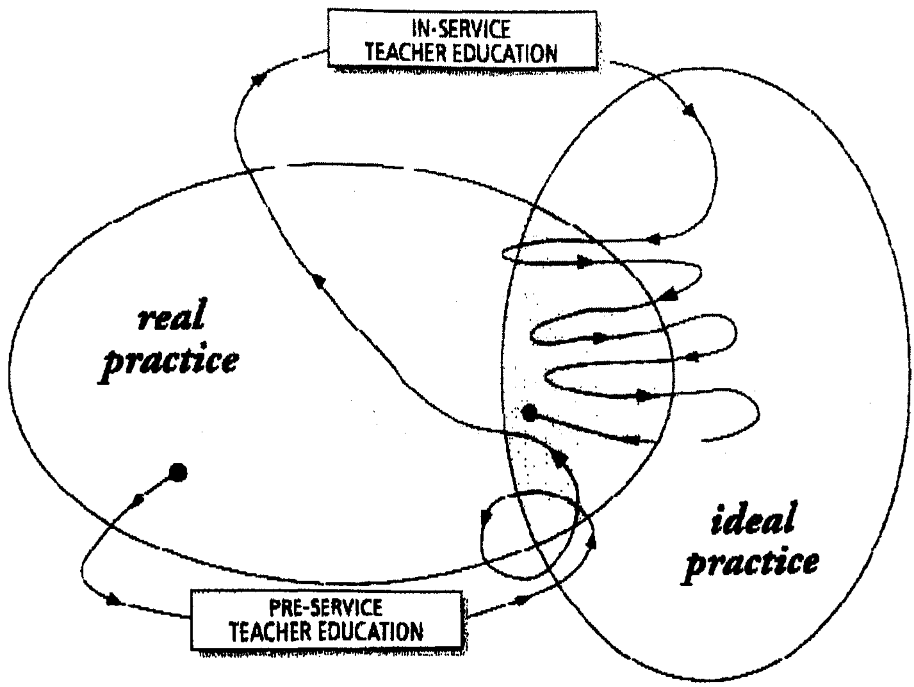
The report that follows describes our struggle with the task.

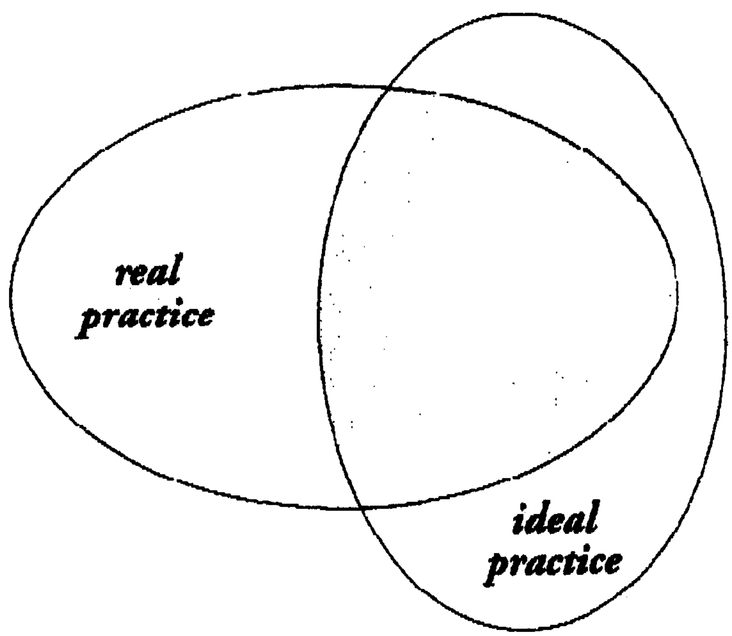
DAY 1: WOULD WE KNOW IT IF WE SAW IT?

...ways in which new thinking and practices might develop and become the reality of students' experiences in mathematics lessons are neither agreed upon, nor, possibly yet, well understood. At its simplest level, there is considerable common theory about the direction of change in teaching mathematics, but it is in the complex practical manifesting of this theory where issues arise. These issues lead to a necessary questioning of the associated theory. (Jaworski &Wood, 1999, p. 127)

Members of the group introduced themselves and gave a brief statement as to why they were interested in the session.

Tom O'Shea began the discussion by presenting a sketch of model that might guide thinking about a teacher's professional development. The sketch ended up looking something like this:





To interpret: Tom suggested we might consider two worlds—the first the world of teaching practice as it exists and the second a world of ideal practice that might reflect what we as mathematics educators would wish. Those two overlap to varying extents depending on the system, the classroom, the students, and the teacher. A teacher then might be thought of as following a twisting path in and out of those two worlds as he or she moved from initial pre-service teacher education into the world of the classroom and then through a series of in-service experiences during the course of his or her career.

In the beginning the teacher comes out of the real world and into a pre-service model that likely includes an initial exposure to ideal practice as espoused by the teacher education institution. That may be followed by a practicum in which those ideas are tested in a classroom situation. This is followed by one or more interactions, perhaps culminating in an extended practicum where the teacher seeks a match between the ideal and real. Once licensed and with some teaching experience the teacher may engage in a series of inservice activities that will allow him or her to expand his or her repertoire to more fully realize the ideal in the real. One might hope that the effect of such experiences over a large group of teachers would increase the overlap between the real and ideal and make the problem of preparing the next generation of teachers more tractable.

This model was presented solely to provoke thinking about teacher education and a number of question were posed to and raised by group members. What are [the] ideal teaching practices? Do they exist? Is it possible to describe them? What are the goals of Mathematics education? Outcomes? Do we know where we want to go? How can we help teachers discover what's effective? What is the knowledge base in content discipline of mathematics?

To address such issues the group was divided into five sub-groups and given a two-part question: (1) If you were to observe a mathematics classroom in which "ideal" teaching was taking place, what would you expect to see?, and (2) Why would you believe such behaviour reflected your ideal?

The groups deliberated and reported back to the main session. Their reports reflect the difficulty all had with the task and the varying degrees to which the groups grappled with the task.

Group 1

Group 1 began with the caveat that it is difficult to imagine one particular classroom because there are different cultures. Nevertheless, there might be a number of activities that transcend the specific constraints of a given classrooms. These consist of

-evident mathematical activity
-mathematical culture-building
-teacher and kids doing mathematics
-a variety of activities, not single voice
-exposure to mathematics and mathematical activities
-demonstration of mathematical "confidence"
-the teacher plans activities

Group 2

Group 2 reported they would like to see
-students engaged in doing mathematics
-teacher posing questions beyond "what is the answer," e.g., "what would happen if --- ?"
-students asking mathematical questions as often as the teacher

-the teacher taking on the responsibility to
- choose good tasks to allow this activity to take place
- focus on concepts rather than just procedures or algorithms

-teaching not bound by curriculum
-interconnections made across and between mathematical ideas
-teacher choosing tasks that allow links to be made
-participants making sense of activity

Group 3

Group 3 struggled with what an "effective classroom" is and felt better about judging outcomes by which they would be able to generate a number of questions. They saw mathematics as social control, that is, as providing a gateway to opportunities. They commented on the complex political dilemmas in judging effective teaching . They wanted to see student involvement in decisions about what is done and how they might effectively be carried out.

Group 4

Questioned the epistemological assumptions of the assigned task. The group struggled with the notion of whether it is possible--or desirable--to establish a list of "ideal" teaching practices. It was suggested that the activity necessarily implied a positivist view of knowledge, one not shared by all members of the group. They did agree that teaching is a highly contextualized art, dependent upon the personal constructs of the individual teacher, the knowledge, assumptions and dispositions of the individual learners, and the nature of the interactions that occur in the classroom. Teaching practices deemed to be effective in a particular situation--or cultural context--are not necessarily directly transferable or reproducible in another. By the same token, practices deemed to be ineffective in a particular context may be highly effective in another. It was suggested that teachers need to be presented with a range of models of classroom practice--reflective of diverse views of learning and teaching--and that they have the freedom to select those models that best work for them in their particular contexts.

Despite the recognition of the necessarily contextual nature of effective teaching practices, the group was able to agree on some overall goals for mathematics classrooms

-that students become aware of themselves as do-ers of mathematics;
-that teachers recognize what is going on with students, i.e., that two-way conversations occur between students and teachers;
-that teachers listen to students' thinking and respond in ways that acknowledge and extend that thinking; and
-that students ask questions about the mathematics concepts being explored.

Their discussion ended with an acknowledgement of a fundamental dilemma: if teaching and learning are so contextualized and individualized that it is not possible--or desirable--to identify generic "effective" teaching practices, how do we (or should we?), as mathematics educators or curriculum leaders, provide direction to teachers? What is our role?

Group 5

Group 5 reported a number of things they would like to see in a mathematical class:

-opportunity to really explore content rather than follow a lesson
-the teacher allowing student involvement in decision making
-the use of effective questioning techniques
-a clear philosophy of teaching

-the use of communicative ideas through mathematical language
-the use of appropriate mathematical context
-the use of a variety of instructional and assessment strategies
-the use of cooperative learning to get students involved
-an openness of questions
-a spirit of inquiry
-flexibility in lesson planning and a willingness to investigate
-use of age-appropriate materials (manipulatives)
-evidence of reflection by teacher in considering alternatives
-the use of action research for systematic inquiry/reflection

Large-group discussion

The ensuing discussion served mainly to raise a number of more general observations and questions about the task.

- To what extent is mathematics teaching discipline centered?
- Can student teachers develop their own image of effective ideal math classroom, perhaps by making use of the TIMSS videotape showing how teaching geometry differs between the USA and Japan.
- You can make sense of your path only when you get to the end, that is, "you make the path by walking."
- Growth is uncomfortable.
- How would we recognize constructivist activities in the classroom?
- How do we construct a teacher and system that values/supports/elicits the outcomes we value?
- Is teacher behaviour learnable/changeable? How?

To complete the work for the day and to provide a bridge for further discussion, the group was asked to consider the following question to start the second day: If you had no limitations, what would your "ideal" teacher education program look like?

Day 2: "IF WE BUILD IT, WILL THEY COME?"

> Teachers are in a constant state of "becoming". Being a teacher implies a dynamic and continuous process of growth that spans a career. (NCTM, 1991, p. 125)

In the second day we attempted to identify experiences which might nurture the development of professional practices and disposition identified in Day 1. Participants were presented with three scenarios, each designed to reflect a particular demographic and/or professional context. One more scenario was suggested during the preliminary discussion.

<u>Scenario 1</u> - Preservice Programs

School District A has an ageing teaching population and anticipates that a significant percentage of elementary generalists and secondary mathematics teachers will be retiring within the next 4 to 5 years. The superintendent of the school district, Ms. Linda Gattuso, is interested in working collaboratively with university representatives to design a new, innovative four-year pre-service program which would nurture "effective" mathematics teaching in beginning teachers (elementary and secondary).

<u>Scenario 2</u> - Credited Inservice Programs

School District B has a relatively stable teaching force, with a median age of 43. The majority of teachers have reached the top of the salary scale in terms of years of service. The district has begun to implement the new mandated mathematics curriculum (K-12) and has purchased new resources to support this implementation. Unfortunately, teachers are struggling with the overall vision presented in these documents and as a result the documents remain "on the shelf." The district would like to put in place a two-year credited inservice program in mathematics education (at either the post-baccalaureate or graduate level) to help teachers understand and implement the new curriculum. Such a program would allow teachers to increase their certification--which would be financially beneficial to them--while at the same time engaging them in discussions of "effective" mathematics teaching. The Superintendent of the school district, Mr. Tom O'Shea, is interested in working collaboratively with university representatives to design an innovative inservice program which would result in teachers 1) gaining a better understanding of the rationale behind the proposed changes, and 2) implementing the new teaching practices suggested.

<u>Scenario 3</u> - Non-credited Inservice Programs

School District C has a relatively stable teaching force and does not anticipate a large number of retirements in the next five years. The majority of teachers have reached the top of the salary scale in terms of certification, hence further university credit is not necessarily a motivator for participation in inservice programs. Although the district recognizes that new mathematics programs and curricula have created a need for teacher inservice in the area of mathematics, budget cutbacks have forced the elimination of the position of mathematics helping teachers/consultant. As a result, the Superintendent of the school district, Ms. Norma Evans, is interested in working collaboratively with university representatives to design a non credit inservice program for elementary and secondary teachers that will help them understand and implement changes proposed in the new programs.

<u>Scenario 4</u> - None of the Above: Build Your Own Scenario

Participants who might not be enthralled with any of the preceding three scenarios were asked to design their own scenario that best reflected their current professional interests and to work with other faculty members and the Superintendent of schools to design a corresponding teacher education program. No limitations, other than those specified in the individual scenarios, were placed on the program design and participants were to assume unlimited financial resources.

Participants were given all of the second day and the first part of the third day to develop a program. On the third day, each group presented the fruits of their labours.

Participants self-selected to each scenario as follows:
Preservice programs: Linda + Dominic, Isabelle, Gerard, Peter
Credited inservice programs: Tom + Walter, Bill, Lesley, Susan, Wije
Non-credited inservice programs: Norma + Sandy, Geoff, Stéphane, Kanthi, Ralph, Kgomotso
None of the above: None

DAY 3: WHAT WILL IT LOOK LIKE?

> [Teachers'] growth is deeply embedded in their philosophies of learning, their attitudes and beliefs about learners and mathematics, and their willingness to make changes in how and what they teach. (NCTM, 1991, p. 125)

On the third day each group presented its proposal and a general discussion of the models took place. Each scenario will be presented in turn.

Scenario 1: Preservice program

Priority: to foster effective mathematics teaching
Pre-requisite: pre-university diploma with mathematics
Motivation: entry into the profession

The mathematics courses can be grouped in two sections: fundamental mathematics and the mathematics related to the teaching program (elementary, secondary...). It is critical that mathematics be taught by faculty who model good teachings practices and are aware of the school mathematics curriculum so as to make connections

.

The teacher education curriculum should consist of courses in mathematics teaching (pedagogy) and general education, and the teaching practicum.

The Teaching courses consist of three types: Methodology (more precisely Didactique), Pedagogy, and Technology. The main distinction made between the didactique courses and the pedagogy courses is that the latter are subject related, in this case, related to mathematics while the others are not. Pedagogy consists of courses in learning theory, psychology, teaching techniques (cooperative learning, etc...). Courses concerning the structure of the school system, the social problems encountered by teachers in schools, for example, were grouped under the title "sociology of schools". Finally, we found it essential to add courses on the use of new media for teaching (Technology). New teachers should feel confident with a computer, a graphic calculator, and the web and be able to use them to enhance their teaching.

The teaching practica are very important. They should allow the preservice teacher to experiment with different teaching experiences at different levels and in different types of schools. The level of responsibility required should be progressive. It is essential to insure that the supervision be the responsibility of persons trained in mathematics education and for the high-school level under the direction of mathematics educators. The supervising teachers should at least be required to have completed the same training as the students and be experienced in teaching mathematics.

Nowadays, the school context changes rapidly and it is important for the teacher to be prepared in various ways. That is one reason that courses such as general history were suggested. Courses in language or multiculturalism aim at preparing teachers for the new diversity in the classrooms. Other courses could be added in this section depending on needs. The structure would look as follows:

Mathematics (30%)	**Teaching** (35 %)
-fundamental (15 %) -math related to teaching (15 %) including history of mathematics -taught by faculty who model good teaching practice and are aware of the school math curriculum (to make connections)	-Methods (Didactique) -Pedagogy -learning theory -psychology -teaching techniques -sociology of schools -technology -Technology
Practicum (20 %) -different experiences, level, school -progressive responsibility -trained supervision (math ed)	**General** (15 %) -multiculturalism -language -history

Most important of all, everything should be related. It is important to have unlimited resources, to keep the classes small so as to ensure mentoring throughout the whole training. There should be close cooperation between faculty and teachers. Faculty should go into the schools and teachers should be present in the university (or college). In that way, everyone would take responsibility for the training of the preservice teachers. Also, this would permit faculty to benefit from the everyday experience of school teachers and the teachers would profit from their contact with mathematics educators and innovative practices experienced in research contexts.

But this describes an ideal situation. In our actual context we face a number of problems that present obstacles to full realization of ideal practice:

1. Resources. Limited budgets have many consequences. One is a limit to the number of faculty or mathematics educators. Another is that the increasing size of the classroom limits interactions between the educator and the preservice teachers. Less monitoring is provided.

2. The nature of instruction during mathematics courses. It would be necessary for the mathematicians to be conscious of the impact of the model they provide on the preservice teachers and also to be aware of the school mathematics so they would be able to link them to the content of their courses.

3. More courses of didactique. In many universities, preservice teachers get only one course in methods, and this is too limited.

4. Students' limited preparation in mathematics. This is obviously true for elementary preservice teachers but also for secondary. Often, the latter have completed college courses in science and mathematics but the accent is more often placed on procedures then on the meaning of the mathematics learned. For the most part, entering students have a very technical experience of mathematical activity.

5. The conditions of the practicum. Teachers in school are often enthusiastic and "de bonne volonté" but have little training and almost no support. There should be incentives for teachers to take student teachers and it should be considered an honor for an experienced

teacher to have preservice teachers in her/his classroom. Authorities should provide a meeting place where teachers and mathematics educators can meet and work together. Release time should be provided so that sponsor teachers can meet with student teachers outside of class time. Time is a problem: a problem for the students, who work and have families, and hence do not always have sufficient time for their studies. Time is a problem for teacher educators; most find they don't have enough meeting time with their students at the university and in the schools. It is also difficult for teacher educators to keep engaged in research work at the same time.

Finally, the discussion started with the idea that is was difficult to build one model for everyone because the needs are different but after some time, it was possible to see that most of the needs are not that different from one situation to the other, even though the application of such a program would probably differ from one place to another.

Scenario 1: Credited inservice program

Priority: mathematics teaching and learning through innovative programs
Pre-requisite: certified practicing teachers
Motivation: salary increase (most likely)

This program would be presented over three years (eight semesters) and preferably to closed groups, that is, cohorts of teachers. Each year is organized into Fall, Winter, and Summer semesters. The program consists of 30 credits of course work organized into 3- or 4-credit courses. The Superintendent indicated that the district would be willing to finance some of the program if it could be demonstrated that the activities are clearly defined and in line with district priorities.

Year 1

The program begins with a Summer entrance seminar. This involves all teachers in the program and consists of an introduction to issues and a survey/overview of the program. Teachers have the opportunity to think about and plan future growth, and to identify a focus for their work. Some attention is given to the task of setting goals and literature on teacher change.

In the Fall semester, teachers undertake introductory mathematical activities that encourage them to reflect on mathematics. At some point the group is split into elementary and secondary subgroups to extend their thinking about topics in the curriculum, e.g., number, geometry, and to establish links to classroom practice.

In the Spring semester, the emphasis is on technology. Teachers learn how to use and teach with calculators, graphing calculators, software such as spreadsheets, and the internet. Teachers are together for some things and split into elementary/secondary subgroups to develop specific skills and knowledge.

Year 2

Teachers meet in the Summer semester for an intensive workshop that looks at the history of mathematics and the place of ethnomathematics in which they explore mathematical topics through an historical or cultural perspective. Sample activities consist of the use of games and exploring the development of algorithms.

The Fall semester is given to examining policy and curriculum change. The role of international assessments is considered. Developments at the national (e.g., NCTM), provincial, and local level are examined. Teachers are not separated into subgroups and full advantage is taken of the internet for communication and discussion.

In the Spring semester, teachers undertake an individual study. They may wish to work as individuals or in small groups. The studies have a content or pedagogical focus and are designed to enhance classroom practice.

Year 3

Teachers meet in the Summer or Fall semester to plan and carry out an action research project based on their investigations in the previous Spring semester. This project consists of a major attempt to revise their teaching in light of what has been learned over the previous two years.

In the Spring semester, teachers consolidate their learning through self and peer evaluation. They have an opportunity to share their experiences through presentations and reports, and make further links to classroom practice. They reflect on their growth and plan for their future focus and growth. They now possess the skills to guide their own professional development.

Scenario 3: Non-credited inservice program

Priority: to help investigate teaching practices
Pre-requisite: certified practicing teachers
Motivation: teacher interest in program

The third group chose not design an inservice program, but rather focussed on identifying operational and pedagogical principles which might guide university-district inservice partnerships. The discussion centered less on identifying what teachers should learn and more on examining how learning needs to happen if teachers are to make changes in how and what they teach.

QUESTIONS THAT NURTURED THE DISCUSSION

1. What is it that university inservice programs have to offer to teachers that cannot be offered by others in the educational community (district personnel, other classroom teachers, teachers active in professional mathematics associations, private educational consultants etc.)? What do we have that sets us apart?

2. How do we design a program that empowers vs. depowers teachers--one that doesn't assume that what is currently happening in classrooms is deficient?

3. How does one move away from a deficit model of teacher inservice when working within a political reality of mandated educational reform and the required implementation of new curricula and programs?

4. What encourages teachers to attend professional development sessions offered outside of school time? What is it that will keep them coming back?

PEDAGOGICAL PRINCIPLES

1. Acknowledge the tensions between meeting teacher's immediate needs (short-term results) and having them engage in more transformational types of experiences (long term results).

> The goal of teacher education is to "light the path" for those who follow, providing directions on how to plan and teach mathematics. (NCTM, 1991, p. 125)

The above quote describes the general intent of most inservice programs currently in place in districts. They are generally short in duration (a one-day or half-day workshop), transmissive in nature and designed to demonstrate to teachers how to implement practices deemed to be desirable and effective. Such programs are generally well received by teachers, as they tend to meet teachers' immediate needs.

Participants acknowledged that one of the dilemmas in designing university- sponsored inservice programs is the tension that exists between the need to respond to teachers' immediate needs and the desire to have them engage in more transformational type activities--activities that encourage them to define (or redefine) for themselves the nature of mathematics as a body of knowledge and their roles as teachers of mathematics. Although it was generally agreed that the latter is more likely to result in shifts in classroom practices and beliefs, it was acknowledged that without the former, teachers are unlikely to continue attending professional development sessions which do not offer incentives in terms of either increases in salary or certification.

Participants also stressed the importance of discussing with district administrators the limitations and advantages of transmissive versus transformational inservice experiences.

2. Acknowledge the necessity to build on teachers' needs or wants.

University-sponsored inservice programs need to recognize the reality of the context in which teachers work and build from that. New programs and curricula are often prescriptive in their descriptions of which practices should be adopted and how particular topics should be taught. It is important that inservice programs acknowledge this and provide teachers with the tools to work within this reality. Asking teachers to identify their immediate concerns and build a program that incorporates these priorities can also give teachers a sense of having some control over program design.

3. Acknowledge that the reconstruction of current understanding of mathematics learning and teaching is most likely to occur as teachers engage in mathematical activities and reflect on their experiences as learners.

A university-sponsored inservice program should provide opportunities for teachers to examine fundamental mathematical concepts via investigations or problems. Problems should be significant and intriguing. As one of the participants suggested, the problems should "grab them in their soul." Encouraging teachers to record their thinking and to reflect on their experiences as learners can lead to a critical investigation of broader pedagogical issues.

4. Acknowledge that most elementary teachers are apprehensive about their mathematical abilities.

An atmosphere needs to be created in which teachers can openly and comfortably acknowledge their apprehensions, while at the same time providing opportunities for teachers to build their skills or recognize the skills and knowledge base that they do possess. This is important if teachers are to reconstruct their visions of themselves as mathematics learners (and teachers).

5. Acknowledge the importance of providing "new" ideas.

An inservice program which requires teachers to give up personal time either in terms of attendance at after school sessions or in terms of hours spent preparing for a substitute (supply) teacher needs to be viewed by teachers as current, innovative, and pertinent to their particular classroom situation. The sessions must offer them something that they cannot get--or haven't seen--elsewhere.

6. Acknowledge the importance of providing opportunities for extended conversations.

The dominant model of inservice delivery traditionally places teachers in a very passive mode as receivers of new ideas. If teachers are to reconstruct their understandings of teaching and learning, they must be provided with opportunities to identify issues which are pertinent to them in their particular teaching situation. They must also have opportunities to engage in extended conversations with teaching colleagues about these issues.

7. Acknowledge that teachers need support if they are to take risks and implement new pedagogical practices

Inservice programs need to encourage teachers to experiment with new teaching practices within an environment that supports and nurtures risk taking. Teachers need to see new teaching practices modeled and to identify those practices which they are most interested in incorporating into their daily practice. Time needs to be provided for teachers to work with colleagues to plan how to implement these practices in their classrooms. Time also needs to be provided at subsequent sessions for teachers to share what happened when they attempted to incorporate the new practices. This linking of sessions--and the subsequent attention to what happened in individual classrooms between the sessions--reinforces the theory-practice link and encourages the implementation of new teaching practices.

8. Acknowledge that change takes time.

Inservice programs which attempt to engage teachers in a reconstruction of their understanding of the nature of mathematics and mathematics learning and teaching must necessarily take place over time. Districts and teachers need to commit to ongoing dialogues. Themes addressed in individual sessions need to be connected in such a way as to produce a seamless and cohesive program.

OPERATIONAL PRINCIPLES

1. Acknowledge the political structures that exist within a school district.

University-sponsored inservice programs need to recognize that school districts are hierarchical political organizations with established decision-making processes and procedures--as well as established lines of communication--that need to be respected. It is imperative that the appropriate individuals be involved in various aspects of the design and delivery of inservice programs. It is also important that, if appropriate, other political organizations (teachers' union, local subject association leaders, etc.) be involved in the decision making process.

2. Acknowledge the importance of getting support from those in positions of power.

Once a program begins, it is important that those involved in the initial design phase receive feedback from participants as to the success of the program.

3. Acknowledge the need to work closely with someone familiar with the teachers' lived realities.

Universities need to recognize the importance of involving local district people who are familiar with current mathematics programs and curricula and their implementation at the classroom level.

The voice of a respected practitioner can lend credibility and currency to discussions of general pedagogical issues, and provide teachers who are anxious about proposed changes a sense that it is possible to successfully implement particular practices. The practitioner can serve as a cultural broker and provide a link between theory and classroom practice.

4. Acknowledge the importance of building a community of learners.

If the objective is to have teachers engage in sustained discussions over time, and on their own personal time, attention must be given to the social and personal elements of group work. Providing teachers with food, getting to know their personal/professional situations, providing opportunities for them to establish personal/professional relationships with others in the group, and allowing them to build a personal relationship with the instructor all contribute to the establishment of a sense of community.

5. Acknowledge that teachers' efforts need to be publicly and professionally recognized.

Non-credit programs carry no benefits in terms of financial rewards or increased certification. However, teachers who willingly engage in sustained inservice programs on their own personal time should be recognized publicly and professionally--either with a certificate of attendance or with letters of commendation from the university and/or school district.

6. Acknowledge the importance of delivering programs in a physical space where teachers feel comfortable.

Programs which are offered in a local school or library require that university instructors enter into teachers' physical--and social/cultural--space, thereby helping to shift a perceived power imbalance.

CLOSING REMARKS: BACK TO THE FUTURE...

The juxtaposition of the three models served to further illustrate some basic philosophical and epistemological differences identified on Day 1.

As one participant pointed out, implicit in each report are beliefs about learning and teaching:

> "It was in [the third) session that I felt a strong sense of difference, of difference among us. We all had similar intentions, all of us believers in the possibilities of university people making a significant positive difference in the practice of teaching mathematics. However, we were not united on some fundamental premises. Should we "design" programs and courses with generic potential clients in mind, and then use our authority, privilege, and position to implement those programs? Or should we engage in consultative relationships with actual clients--learners--and let the educational structure co-emerge with the education? The former is a traditional approach to university-level curriculum development ...Alternatively, we can make significant sacrifices in the structural logic of our curriculum plan, for the sake of altering the information flow and our adaptability within that flow. In part, the difference might be one of maintaining role (professor, expert, designer) or of establishing relationship with learners."

We ended up with descriptions of three qualitatively different teacher education programs, but without sufficient time to make explicit or critique the epistemological assumptions upon which each of the programs was based. An examination of the diversity of theoretical perspectives--and of the advantages and limitations of each perspective in terms of designing professional development experiences which will enhance teacher growth--would no doubt make for an enriching exchange. As Jaworski and Wood (1999, p. 145) point out, "[there is] a growing awareness of a need for mathematics teacher educators themselves to reflect on and reconsider their own practice, to engage in an exchange of ideas, and from this to create a common ground from which to communicate about their work." The discussions over the three days enabled participants to define their current vision of [ideal] teacher education practices or behaviors. There is a now a need to examine, collectively or individually, the beliefs upon which those behaviors are based. Such an examination may lead to a reconsideration of both our practice and our beliefs about learning and teaching.

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TOPIC SESSIONS

Topic Session 1

Amusing About Aesthetics And mAthematics

William Higginson, Queen's University

APOLOGY

The title perhaps deserves some comment. The conference organizers quite reasonably followed up my acceptance of their invitation to present a one-hour session in the 'Topic Group' section of the programme with a request for a title. I knew that I wanted to run some preliminary ideas concerning the relations between aesthetics and mathematics past this very special audience; my 'musings'. I had found the discovery of these connections to be interesting and enjoyable - 'amusing musings'. I had previously been struck by the fact that 'esthetics' is on some occasions used instead of 'aesthetics'. This playing/musing with or about these ideas would, I thought, most likely lead to the presentation and defence of different possibilities. 'Musing about' could, anagrammatically, become an 'amusing bout'. Et, as they say, cetera. I was to find that presenting this idea visually in a design for a title-page overhead with one large capital 'A' was a far from trivial. What I completely forgot, however, in my 'logomotions' was that this convolution would have to be translated. While the challenge was not quite of the magnitude of that facing the translator of Perec's "La Disparition" - that of taking a novel of several hundred pages in French and turning it into a novel of the same length in English without ever using the letter 'e' in either text - brilliantly executed in "A Void", - it was not an easy task. So, a public apology to a talented colleague who had to cope with a nasty problem.

BACKGROUND

The attempt, considerably after the fact, to document what 'proceeded' in this component of the 1999 Annual Conference is complicated by the reality of the session. Like all elements of its class, this invited presentation brought to a large, early-morning (by late-in-the-conference standards) audience a substantial claim, buttressed by statements from a wide range of sources, which, at least in the eye (brain/mouth?) of the presenter, supported the allegation. Less commonly for this type of session there was a significant 'hands-on' component which was alleged to illustrate some aspects of the argument. The range of sources cited and/or referred to in the talk was extensive and eclectic (see References/Bibliography below) and few participants managed to cope effectively with both the bibliographic excesses and the physical demands of the hands-on tasks. Despite this and the generally high degree of sleep deprivation, the session was a spirited one which led to a number of lively discussions at the end of session and during the remainder of the conference. One participant noted wryly that it was the first time he had ever seen a lecturer make a talk out of a reference list. To the extent that this observation was accurate I make claim here to have invented a new academic form, the 'animated bibliography'.

CHALLENGE

Audience members drifting into this first session of the day were given several small squares of paper and a number of strips. A 'warm-up' question was to fold one of the squares accurately into 9 smaller, identical squares. It was noted in later discussion that while subdividing the original square into 4 or 16 [with comments about how this pattern continued] smaller squares was quite easy, other values, including 9, presented more significant problems. The 'rip-off' solution generated by the 'pragmatists' - fold the square into 16 smaller squares and then 'tear off' the 'top'-four and then, subsequently, the 'side' three squares (the numerically nimble were deducting 7 from 16 and nodding enthusiastically) was deemed, by the 'purists' to be wasteful and not elegant. The more algebraically able in that subgroup did concede that the general case, with its insights into fundamental properties of the square of (n + 1), had considerable charm.

DIRECTIONS

One of the first claims made in the lecture was the cliche that the last years of the twentieth century were ones of significant and rapid social and cultural change and transformation. Somewhat more originally it was claimed that mathematics - largely through its influence on the devices of new information technology - was, simultaneously, a major contributor to this phenomenon and an area which had been greatly affected by it. In the general area of cultural change, of the books of Diamond (1999) and Dyson (1997) were recommended while the more mathematically-focussed Bailey (1996) was portrayed as being provocative and interesting but quite uneven

.

Elements of these patterns of transformation which were referred to were:

1) a growing sense that the classic forms of mathematics instruction were not meeting many of the most important needs of learners (Davis, 1996; Higginson, 1999) and that, in particular, the frequently oppressive and punitive characteristics of mathematics classrooms badly needed to be changed. Cited in support of this view was a statement by Alan Bishop from his opening remarks at a major international conference at Nottingham in September of 1998: "I am hoping that this conference gives me and all of us who attend, an opportunity to discuss our journeys and through our papers to develop some new perspectives in research with which to confront the meaningless and oppressive mathematics education which many students still have to suffer today around the world." Participants only too familiar with the recent events in schools in Colorado and Alberta seemed to find these views quite plausible;

2) a wide-spread distrust of pure, disembodied, decontextualized rationality (Saul, 1992) with a concomitant realization that many of our fundamental linguistic and mathematical processes and concepts are rooted in fundamental body actions and genetic structure (Dehaene, 1997; Lakoff and Johnson, 1999; MacLane, 1986);

3) increasing awareness of the central role of the aesthetic and the emotional in a number of different areas of human activity (Barrow, 1995; Chandrasekhar, 1987; Damasio, 1995; Gelernter, 1998; McAllister, 1996; Weschler, 1978; Wilson, 1984) was noted. The thesis of Dissanayake (1988, 1995) that humans are, at their core, aesthetic animals, was pointed to as a particularly powerful perspective in this area. The long-standing links among pattern, beauty, symmetry, and mathematical power were noted (Cole, 1998; Eglash, 1999; Flake, 1999; King, 1992). This union of the active, the tactile and the visual when yoked to the power of computer technology has had a particularly powerful impact on and has generated something of a creative renaissance in the field of geometry (Cromwell, 1997; Hilton, Holton and Pedersen, 1997; Martin, 1998; King and Schattschneider, 1997).

EMBODIMENTS

Following this rapid tour of a few highlights of recent publications in the area of science and philosophy, the claim was made that an examination of both formal and informal features of our own conference would reveal strong evidence of some of these trends. The keynote presentations of Professors Adler, Borwein and Whiteley could, for instance, it was argued be seen as perceptive and imaginative narratives of cultural transformation in the domains, respectively, of schools, technology and geometry. Even more compelling was the the phenomenon of the "Mason Thing - 1999". Regular participants at our annual gatherings will acknowledge the gift that Ralph Mason has for generating (what may be called for lack of a better term) 'rich learning environments'. The Pizza Hut in Thunder Bay was, as many of us will recall, the 'skunkworks' location for the "MasonThing 1997", 'Footprints'. The 1999 'Thing' was a complex, highly-symmetric, geometric object created from thirty folded squares of paper. Within the first two days of our gathering almost every CMESGer had had a bash at creating the intricate modules. Somewhat more surprising but quite significant was the enthusiastic involvement of individuals from several other conferences which shared the cafeteria space. It was interesting to note that the word 'icosadodecahedron' never - at least in my hearing - surfaced but that everyone who 'folded' for even a few minutes had learned, at an embodied level, a number of fundamental geometric ideas. [The interested reader can pursue some of the links between origami and mathematics in sources such as Engel, 1989; Fuse, 1990; Kasahara and Takahama, 1987; Lang, 1988 and Mitchell, 1997]. The 32-faced 'thing' - 12 pentagonal and 20 triangular faces - is known in the origami world as "Electra" and is built from a module designed by David Mitchell (see Jackson and A'Court, 1993; for example where "Electra" graces the cover). Strangely enough the paperfolders don't talk about icosidodecahedra much either. Still, it is a beautiful 'Thing' and illustrates the appeal that rich mathematical objects have for many individuals.

FORMALIZATION

In attempting to bring some of these insights together it was suggested that there is at the moment a significant opportunity to bring mathematical ideas and concepts to learners through experiences which are consciously and conspicuously aesthetic, social and constructive. Three examples of early-stage efforts to work from this 'M:ASC' [the logo of this initiative is the 'butterfly/mask' image of the Lorenz attractor from non-linear dynamics (Gleick, 1987; Lorenz, 1993)] perspective were noted. The first was the Gage/MSTE "Tomorrow's Mathematics Classroom" project (Higginson and Flewelling, 1997) with its tripartite vision of mathematics as a tool, a language and an art. The second was the call for the "Missing Standard", namely "Mathematics as Art" (Rogers, 1999), and the third a report on a "constructive aesthetic" approach to mathematics in a Grade 3/4 classroom (Upitis et al; 1997).

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Topic Session 2

REPORT ON THE ICMI STUDY ON UNIVERSITY MATHEMATICS

Joel Hillel, Concordia University

INTRODUCTION

I was asked to report on the recent conference, held in Singapore in December 1998, which was part of the ICMI Study on University Mathematics. I was both a participant in the conference and responsible for the Working Group on Trends in Undergraduate Curriculum, as well as a member of the International Programme Committee.

Not unlike CMESG meetings, Working Groups, together with plenaries and panels were at the core of the Singapore conference, though individual presentations also took place. The reports of the Working Groups are still being written and they are to become a prominent component of the forthcoming Volume. What I will do in my talk is highlight the issues addressed by each of the plenary speakers at the conference.

Some background, first. The International Commission of Mathematics Instruction (ICMI) conducts regular studies on themes that are of interest to the general mathematics community. The most recently completed Study is on the Teaching of Geometry; another on the Use of History in Pedagogy is near completion. As I am reporting on the latest Study on University Mathematics, a new Study on the Teaching of Algebra (at all educational levels) has just been announced. Most ICMI Studies follow the same critical path. Once a theme for a Study has been chosen, an International Programme Committee is struck and writes a Discussion Document (DD) which outlines the main issues and questions that it wants to address. The DD is widely circulated and reactions are solicited as well as proposals for contributions. Based on these responses, individuals are invited to a small international conference (about 80 people). A volume related to the Study is eventually published with contributions from some of the participants, as well as from other colleagues in the field. In our case, in addition to the Volume, there were also pre-proceedings ([1]) and an upcoming special issue of the International Journal for Science and Mathematics Teaching (iJSMT) will have selected papers from the conference.

The ten Working Groups (five running in parallel and meeting for 5 hours over two days) chosen for the Singapore Conference are pretty faithful indicators of the kind of issues that were raised in the Discussion Document. They were on:

- Secondary/tertiary interface
- Mathematics and other subjects
- Preparation for the profession
- Assessing undergraduate mathematics
- Trends in curriculum
- Practice of university teaching
- Mass education
- Preparation of primary and secondary teachers
- Policy issues
- Future of research in tertiary mathematics

Furthermore, the panels were chosen to deliberately coincide and inform three of the Working Groups, namely Secondary/tertiary transition, Mass education, and Technology.

THE PLENARY LECTURES

There were five plenary lectures. While the first four plenary speakers had plenty of time to reflect, prepare, and write up their lectures, Bernard Hodgson had the unenviable task of giving a plenary on the last day that was to be an overview of the whole conference[2]. I will summarize in my talk the lectures of Lynn Steen, Hyman Bass, Michele Artigue, and Claudi Alsina.

I. Lynn Steen: Rethinking undergraduate mathematics

Main points:

Enrollments in post-secondary education have increased from 13 million in the 1960's to 90 million in the 1990's.

The practice of mathematics has changed; there is an unprecedented penetration of mathematical methods into new areas of applications most of which are invisible in the undergraduate mathematics curriculum.

Most of the mathematics used is not learned in a course called "mathematics". Rather, post-secondary students pick mathematics invisibly and indirectly in their courses and internship, and use mathematics in the actual practice of their craft.

Consequently,

"... "real mathematics" as practiced by real universities now constitutes only a tiny fraction of post-secondary mathematics"

University courses fall into the following categories:
- Traditional math courses
- Context-based math courses
- Courses in other disciplines that employ significant (hidden) math methods.

There is a proliferation of non-traditional learning formats: Internet, corporate training centres, weekend courses, open universities, and for-profit universities (e.g. the University of Phoenix, whose stocks are traded in the Stock Market).

These non-traditional formats are, in fact, responding to students' needs more than mathematicians do in their "real mathematics" courses.

Mathematicians often design curriculum like proofs – there is a complete lack of feedback.

Steen concluded that there is, in effect, a "*stealth takeover of undergraduate mathematics by its entrepreneurial clients who are now setting the agenda*" and that "*we should welcome our new owners and thank them for propelling mathematics into the 21st century*"

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[2] I note with some satisfaction that aside from the CMESG members with major responsibility at the conference (B. Hodgson, J. Mason, and myself), two of the other plenary speakers (Lynn Steen and Michele Artigue) as well as one of the panelists (Celia Hoyles) were previous plenary speakers at our meetings.

II. Hyman Bass: Research on university level mathematics education

H. Bass, who is one of the most prestigious mathematicians in recent years to have rallied to the cause of mathematics education, outlined major problems in undergraduate mathematics which need systematic research, namely:

- Secondary/tertiary transition
- Use of new technologies
- University-level teachers and teaching
- University contexts and attention to teaching

Further elaboration on the kinds of research needed within each area included:

Secondary/tertiary transition

- Careful examination of the kinds of mathematical knowledge and skills that do emerge from the new secondary curricula.
- Systematic data on current freshmen; their understanding, skills, capacities, and beliefs about mathematics.
- Examining the enacted curriculum by means of fieldwork studies of the mathematical curriculum at strategically selected classrooms.

Use of new technologies

Investigations of:

- the diverse ways in which technology is used in secondary and tertiary levels
- how the use of technology is taught
- how technology affects students' engagements in mathematics and their mathematical development.

University-level teachers and teaching

The operative assumption has always been that "good teaching" requires expert knowledge as well as lucid, organized and rigourous presentations, rather than attending to students. Students' failure in mathematics was therefore assumed to be the fault of the students themselves, their previous teachers, or previous curricula. There is a need to survey academic mathematicians on their beliefs about and knowledge of teaching and learning.

University contexts and attention to teaching

There is a need to confer upon mathematics education work and scholarship recognition and reward as is done in other disciplines. Mathematicians tend to reason about complex phenomena of teaching, learning, curriculum, and policy from personal experience and anecdotes, even though they "*will never make this sort of epistemological error within their own discipline*".

There are possibilities for mathematicians to either become 'critical consumers' of mathematics education research, to make inquiry into their own practice, or to actually do research in mathematics education.

III. Michele Artigue: What can we learn from didactic research?

Main points:

- Mathematics education research is still far from being a unified research field. There are multiplicity of 'local' theoretical frames and methodologies.
- Results of research are not easily transformed into effective actions and "solutions" are neither easy nor cheap since they usually entail increased resources and expertise.

- Initial research findings had "negative aspects" since they mostly highlighted students' misconceptions, and the limitation of usual teaching practices. For example, research in the area of calculus showed that the students' notion of limits, differentiability, and continuity was a kind of "algebraic analysis" which reduced calculus operations to algebraic manipulations (a notion which was often reinforced by traditional assessment items).
- Research at the tertiary level has tended to focus on students of mathematics rather than students from other disciplines who take mathematics.

Examples of theoretical perspectives are those that are based on cognitive evolution, epistemological evolution, or reconstruction of knowledge. A typical example of the first is APOS theory (Action-Process-Object-Schema) which models the mental constructions at play in advanced mathematical learning. A salient finding of such research is that the transition from process to object (e.g. from thinking, say, that an eigenvector is something one gets via some operations on a matrix to thinking of an eigenvector as a vector which has certain properties vis-a-vis a linear transformation) is difficult for most students; a difficulty is underestimated in traditional instruction (where the shift from process to object can happen quickly and imperceptibly). At the heart of the second theoretical perspective is the notion of the necessities of 'epistemological obstacles' or 'gaps' in the construction of new knowledge. New understanding is stipulated to only emerge after the rejection of previous forms of knowledge, forms which have been stable, coherent and effective but can no longer account for some new phenomena. In a slight variation of this model, previous forms of knowledge are not rejected but are reconstructed. A typical example of the kind of reconstruction that students of mathematical analysis have to make is to replace their meaning of equality from that of $a=b$ to that of $|a-b| < 1/N$ for all $N > 1$. Here again, traditional teaching underestimates the kinds of reorganizations that students have to make and assumes that it is the students' responsibility to make them.

IV. Claudi Alsina: Towards a new paradigm of teaching mathematics at the university

Alsina, masterly intertwining humour, music, film, and slides, throughout his lecture nevertheless, made a serious analysis of the current practices of undergraduate teaching and a plea for change. First came a list of common 'myths' about mathematics teaching implying that:

- expert knowledge is both necessary and sufficient for good teaching
- good teachers are self-made so training is not necessary
- mathematics is context-free so one can have a core curriculum for everyone
- deductive organization of content is the best hence the theorem/proof syndrome
- 'perfect theory' presentation – no place for errors, false trials, zigzag arguments
- assumption of 'instant maturity' of students the moment they leave secondary school so there is no need to address transition problems
- best assessment is a long, written final examination

In proposing a new paradigm for university mathematics, Alsina argued for:

- redefining mathematics research so as to be more inclusive of a whole range of activities related to undergraduate and graduate education
- having mathematics professors engage in educational research
- instituting a 'teacher training' component at the university level
- introducing innovative technological tools, pedagogical styles, and assessment styles.

As a final note, the special issue of iJSMT of selected papers from the Singapore conference will be the first issue of the journal in the year 2000. The Study Volume is in preparation and it is hoped that it will be ready in time for the next ICME conference in Japan, August 2000.

[1]. Pre-proceedings, ICMI Study Conference: On the teaching and learning of mathematics at university level, 8-12 December 1998, Singapore.

AD HOC SESSIONS

A COURSE ON VISUALIZATION

Walter Whiteley, York University

I described a course which I will offer in the fall of 1999 to first year science students, on the use of visual and spatial information, both as an external artifact (diagrams, graphics, animated images) and as an internal process with images. I presented some of the information, examples, and resources, which I have collected, and asked about additional resources and commentary. In the discussion, a number of valuable points were made, including the necessity of 'selling' these ideas to students (they are not 'obvious' - yet) and some additional resources were suggested.

In brief, the course is: In The Mind's Eye: Information in Visual Form.
Text: Robert H. McKim: Thinking Visually: A Strategy Manual for Problem Solving, Dale Seymour.

"Diagrammatic reasoning is the only really fertile reasoning."
- C. S. Peirce (1839 - 1914)

Scientists use graphs, diagrams and other visual images to organize information, to solve problems, to remember, and to persuade. This course explores the nature of visual information and reasoning, the construction of good representations, how visual forms may mislead and how we can learn what we 'see' in visual forms and how we can change what we 'see'. A primary goal is to make our own thinking with visual forms more effective and more reliable. In all areas of science, important information comes to us as graphs, displays and diagrams. These visual forms play an essential role in communication among people, in our individual memory and in our internal problem solving. We will draw on the examples in texts, lab reports and notes brought by students. We will reflect on the practices in current courses, and in classic examples by leaders in many fields. The advent of CD-ROMs and web presentations of information have made the visual more important and more accessible for communication and for access to information and we will study examples from these sources.

Much of our thinking, particularly our creative thinking and problem solving centrally involves visual forms: graphs, diagrams, flow-charts, other displays, not formulae or words. However, with all our practice of algebra and writing, we have had little practice at 'reading' (viewing) diagrams or constructing images for our own use and for communication. We will consider:

- what information is in various diagrams and how to extract it;
- how we create what we see and how we can change what we see;
- what reasoning we now do, or can do, with information in visual or spatial form;
- how to build visual forms to organize information;
- how to build visual forms to making new connections;
- how to build visual forms which work for communication;
- how to convince (lie?) with visuals (visuals as propaganda);
- constructing visual forms for memory;
- what role visual forms, both static and dynamic, play on the Internet;
- how can we improve our own use of diagrams and visual forms.

Activities will include (a) in-class group work, both with and without computers; (b) readings in cognitive science (what we know about how the mind creates and thinks with images); (c) a visual journal of examples and reflections; (d) individual or group projects. More details about the course, as well as references, links and an annotated bibliography, etc. are available at http://www.math.yorku.ca/Who/Faculty/Whiteley/Visual.menu.html.
More information has arrived since the conference and more will be added during the year.

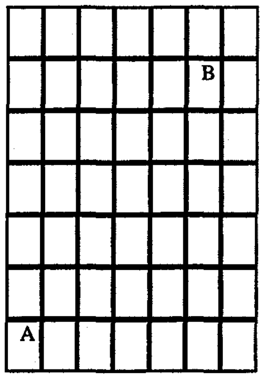
ENCOURAGING EXPERT-LIKE THINKING IN MATHEMATICS

Rick Seaman, University of Regina

This presentation, through the use of examples, will demonstrate one way that students can become more expert-like in their thinking while problem solving. As an expert the students will retrieve the deeper structure of a (source) problem to be used to help solve another (target) problem. The deeper structure of a problem is identified, as what the student believes is most relevant in helping them solve the problem. Students when analyzing problems will explicitly ask, "Where have I seen a problem like this before with respect to deeper structure?"

SOURCE (Pascal's triangle): *In the triangular display of numbers find as many patterns as you can among the numbers.*

TARGET: *In the grid below determine the number of different paths from point A to point B in the grid below. No backtracking is allowed you must only go up and to the right.*



TARGET: *The television show The Price Is Right has a game called Plinko on it. The contestant drops a circular disk down a board with nails that are arranged like the pattern below. If the disk is equally likely to go to the left or to the right at each nail which column would have the highest probability of winning $5000?*

[For more information on this problem and others see Haws (1995) and Lemon (1997).]

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THE IMPACT MATH IN-SERVICE PROJECT: CASE STUDIES AT ONE RESEARCH SITE

Ann Kajander, Lakehead University

The Impact Math Project used a train-the-trainer model to in-service grade 7 and 8 teachers on the new Ontario curriculum (see McDougall, 1998). An initial cadre of ten provincial trainers provided one to two sets of two day workshops to over 600 teachers in Ontario by April 1999. As well, five supporting print modules (one per strand) were made available to all teachers.

Four provincial sites were chosen for follow up. At each site, one pilot study as well as four detailed main study cases were conducted. Data including beliefs surveys, classroom observations, and extensive transcribed interviews were collected for each teacher. At the time of writing, qualitative software analysis of emergent themes was being done.

This discussion focused on the five teachers at the Thunder Bay site. A very wide range of teaching practices and levels of change were observed at this location.

While all five of the teachers at this site had stated preferences for collaborative learning and rich learning tasks, only three of the teachers were observed to be effectively using such strategies on the observation day, and little evidence was found of the other two using these techniques at other times. The same three teaches were also making excellent use of the print modules, and were especially appreciative of the content background and graded samples of student work provided with the exemplars. These three teachers exhibited many desirable characteristics of reformed classrooms, and were exciting to observe in action. Many transcribed examples of rich student discussions were noted in these classrooms.

In contrast, two of the teachers were not using the materials effectively or at all, and were uncomfortable with either classroom management issues or the new content. One teacher was very concerned about patterning, saying she couldn't learn algebra herself this way, so she couldn't teach it like that. Both had strong concerns with evaluation. Further opportunities will be needed for these two teachers to continue their change.

While a longer term may be needed in some cases for significant teacher change, the first year of this project does indicate that the train-the-trainer model can effectively support some teachers in their reform efforts, and that significant change has occurred in at least some cases.

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IMPROVING THE QUALITY OF IN-SERVICE TEACHER EDUCATION IN SRI LANKA

Kanthi Jayasundera, Simon Fraser University

PROBLEM

I have been a teacher, deputy principal, assistant director of education in a provincial education ministry and a teacher educator in a university for eighteen years experiencing many aspects of in-service teacher education system in the country. Policy makers are demanding more quality in teaching while teachers are complaining about the inadequacies of existing teacher training programs. Conflicts between policy makers, teacher educators and practicing teachers provoked me to search deeper into alternative ways of addressing theses issues in Iiservice Teacher Education (INSET).

BACKGROUND

There are two types of in-service teacher education programs in the country. The long term program (1 year) which is considered as the professional training is offered by some universities Open University, National Institute of Education and Teachers' Colleges. In Sri Lanka teachers enter the teaching profession without pre service training.[3] These teachers have different types of entry qualifications. They may be graduates from a university or a high school. They can go through a professional training program after a minimum period of three years teaching. However, due to limited vacancies in teacher education institutions, teachers have to wait more than the required minimum period of three years. Nearly 66% of the graduate teachers and 22% of the non graduate teachers in the service are untrained (Medium Term Education Development Plan 1990-1994). Finally, teachers with a degree can enter into a post graduate diploma program at a university, the Open University, or the National Institute of Education (NIE). The others may go to a special (according to their subject area such as Math, Science, Music, Dancing) or general (specially for primary school teachers) teacher training program at Teachers' Colleges.

The second type of in-service programs are short termed ones offered by the Ministry of Education & Higher Education (MOEH) each year, either through the National Institute of Education (NIE) or Provincial Education Departments. These programs are organized according to subjects. They are typically focused on introducing teachers to and familiarizing them with changes of curriculum. Therefore, they are few and far between. Usually when there are reforms in curriculum, the NIE conducts short term training sessions for Master Teachers. Master teachers are the practicing teachers selected through a written examination. Then at provincial level, subjects specialists according to the provincial budget allocation and criteria set by the education authorities of that province--conduct short term programs with the help of the Master Teachers. Certificates are issued at the end of each program either for performance, or just for participation, because teachers need these certificates for their promotions and salary increments. There are no follow-up activities to assess the impact of any of the programs conducted. Many in-adequacies have been identified with these short

[3] In 1985 Colleges of Education were established as pre service institutions for teacher trainees with minimum entrance qualification of GCE (AL), and a three weeks training program of all the other new entrants.

term programs such as one way (Top-bottom) communication and incompetence of Master Teachers Perera (1998).

One of the main goals of the new policy reforms in Sri Lanka is to improve the quality of existing teacher education of the country. To fulfill this goal the National Authority on Teacher Education (NATE) was formed in 1994 with the hope of improving the quality of preservice and in-service teacher education, providing adequate support for research, directing principals, teacher educators and teachers for systematic professional development, networking all the teacher education institutions, etc. All these goals are still at the planning level with a large amount of unutilized money burrowed from the World Bank. Lack of policy in teacher education and poor project planning has led to a wastage of resources the country can barely afford.

Several studies have been done on teachers and teacher education in Sri Lanka, but very few on the short term in-service programs. Findings on the long term teacher education programs raise many issues. Therefore, the main objective of my doctoral study at Simon Fraser is to develop a policy framework for in-service teacher education in Sri Lanka based on the research literature, research done on in-service teacher education, and some selected practices in British Colombia with the focus of teacher as an adult learner.

In this study I refer to *In-service teacher education* as credit and non-credit programs, workshops, seminars and the like that are offered to practicing teachers to support their long term learning and growth as teachers. I am hoping to address the following questions in my study.

What are the needs of Sri Lankan practicing teachers?

On what principles of in-service teacher education will the proposed framework based on? What are the ways of delivery/alternate forms? Why this framework will suit the Sri Lankan context /Benefits? How will I test the viability of the framework? What barriers/issues that need to be overcome to implement the framework in Sri Lanka?

I would like to share these ideas with you today with the hope of your rich feedback.

MATH CENTRAL

Vi Maeers and Harley Weston, University of Regina

Vi and Harley demonstrated Math Central, an Internet service for K-12 mathematics teachers and students. This project, which they run along with Denis Hanson at the University of Regina, began in September 1995 and has a number of features, including:

- The Resource Room - A searchable database of teaching resources
- Quandaries and Queries - A question and answer service
- Teacher Talk - A discussion list for mathematics teachers
- The Bulletin Board - Conference announcements, newsletters, links to mathematics teachers organizations, etc.

Participants were invited to use Math Central, submit resources for the Resource Room, subscribe to Teacher Talk and volunteer to help in answering the Quandaries and Queries questions.

Math Central can be found at http://MathCentral.uregina.ca/

LARGE SCALE PERFORMANCE-BASED ASSESSMENT IN ONTARIO

Mary Lou Kestell, Education Quality and Accountability Office

Yearly the Education Quality and Accountability Office (EQAO) assesses every student in grade 3 and grade 6 in a 5-day integrated Reading, Writing and Mathematics Assessment. I spoke about the assessment process used by EQAO and the fact that Ontario teachers develop, pilot, revise, field test, mark the field test, revise, administer and holistically mark the papers of about 150 000 students in each grade level. The assessment is developed from a blueprint created from the Ontario Curriculum. The assessment provides each family with information about the individual child. School, board and provincial data is also reported to the public annually.

WHAT IS MATHEMATICAL MEANING?

Olive Chapman, University of Calgary

Mathematical meaning is beginning to become a cliché in the current reform movement in mathematics education. In current literature, it seems to be a taken-for-granted construct of this movement. Teachers are now encouraged to teach with a focus on mathematical meaning. But what is mathematical meaning for the teacher in the context of the classroom? At the beginning of a mathematics course for elementary mathematics majors (i.e., have at least 2 full-course, university-level mathematics), I asked students to respond to "what is mathematical meaning?" They had difficulty doing this and most chose not to respond, claiming that it was a difficult question, they were not sure, maybe I would tell them. I did not explicitly discuss it with them, but at the end of the course, which involved a focus on meaning of the mathematical concepts, they were asked the question again. This time everyone responded, but their answers still reflected a lack of understanding of how to interpret "mathematical meaning" as a way of characterizing mathematics. Similarly, a group of inservice teachers I gave the question to found it difficult to articulate an understanding of it.

For inservice teachers I have worked with, two interpretations of mathematical meaning were reflected in their thinking. First, mathematical meaning referred to something inherent in mathematics, i.e., "things" with some objective, "out-there" existence. These teachers considered mathematical meaning as something you taught *about* and saw telling as a justifiable mode of teaching it. For example, one teacher explained to his students *why* a power with zero exponent is one. When the students were later asked to write about "why", the teacher was surprised that there was little or no effect on the incorrect meanings they had constructed when they learnt the concept by rote the previous year. However, he blamed this outcome on the students' ability. The second interpretation of mathematical meaning expressed in the teachers' thinking was described in relation to the learner, e.g., the interpretations or understandings the learner should acquire in learning mathematics. My ongoing research includes exploring relationships between teachers' understanding of the nature of mathematical meaning and their treatment of the content in teaching for meaning.

CLEANSING THE MATHEMATICAL PALETTE:
Metaphors as scaffolds for higher order thinking.

George Gadanidis, Durham District School Board

Language is essentially metaphorical in nature and metaphors play a central role in the organization and acquisition of conceptual structure.

Metaphors create cognitive *disequilibrium* as they communicate by misdirection. It is only when a sentence appears to be false that we accept it as a metaphor and start to search out the hidden implication. Metaphors also lead to cognitive *accommodation*. Metaphorical thinking involves making connections between two words or ideas that are not normally related to one another but do share some commonality. A metaphor changes our perception of both ideas it connects.

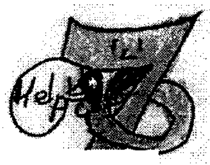
Metaphors can be useful in math education to help students move toward higher order thinking. Experience and research (Kamii's work, for example) indicates that students who are taught through traditional methods often find it difficult to think mathematically on their own and to trust their own thinking processes; and this becomes increasingly more difficult as we move up through the grades. I suggest that exploring math metaphors may have the effect of cleansing the mathematical palette as students move to higher order thinking in our math classrooms. The creative metaphorical exploration of mathematical concepts can help students be more willing to let go of the rules-based thinking associated with traditional teaching and learning. One example of what this may look like at the classroom level is shown here.

Session participants suggested looking into the work of Gregory Bateson, Dorothy Buerk, Olive Chapman, Lynn English, Mark Johnson, George Lakoff, Ralph Mason, John Miller and David Pimm. For an on-line encyclopedia of metaphor research go to http://www.compapp.dcu.ie/~tonyv/encyc/cyc.html.

The following poem was written by a student from Karen Beatty's class, Cadarackque P.S., Ajax, ON, after students explored mathematical metaphors.

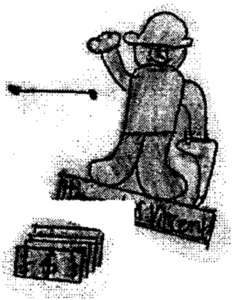
Numeration

Numeration is a murderer,
7 ate 9,
When he got into court,
He said, 'I had to dine!'



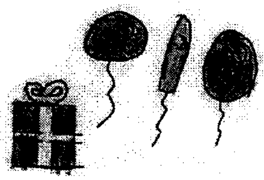
Subtraction is like ice cream,
They both disappear,
I know someone who likes them,
And he is a peer.

Subtraction is like geometry,
They both use line segments,
Line segments are used a lot,
They're on the monuments.



Subtraction is a casino,
You never come out with more,
When you do get some cash,
You'll use it at the store.

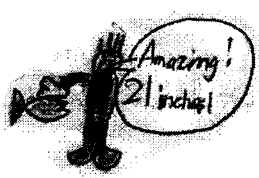
Addition is a birthday party,
You always get more,
You get a bit of money,
And presents galore.



Division is like friends,
You have to share with both,
Both are essential
For your childhood growth.

Multiplication is a herd of animals,
It's always getting bigger,
But when one set hits another,
I think they'll merge together.

Operations are really cool,
It's one of the things I like,
Math and fishing are also great,
We measured my caught pike.



Vincent Kong

BETWEEN-METHODS TRIANGULATION: IS IT POSSIBLE ?

Kgomotso G. Garegae-Garekwe, University of Manitoba

Rationale for mixing methods

There are several reasons for mixing methods. Amongst these are: (1) triangulation, (2) complementary, (3) development, (4) initiation, and (5) expansion. However, this paper focuses on the triangulation intent, particularly methodological triangulation.

There are four types of triangulation (Jick, 1979). First, data triangulation in which several data sources are used to study the same phenomenon. Second, investigator triangulation in which more than one investigator is involved in a research project. Third, theory triangulation where diverse theories are used to bear on a the common problem. Fourth, methodological triangulation in which different methods are used to measure the same phenomenon. Moreover, there are two types of methodological triangulation: (1) between-methods and (2) within-method (Holloway & Wheeler, 1996). With regard within-method triangulation, different techniques within the same paradigm are used to assess the same phenomenon. Similarly, between-method (or across-method) triangulation uses methods from two different paradigms (positivist and interpretive paradigms) to examine the same research problem. That is, "within-method' triangulation essentially involves cross-checking for internal consistency or reliability while 'between-method' triangulation tests the degree of external validity" (Jick, 1979:603). As stated earlier in this section, triangulation aims for convergence of results to a single proposition. I consider this convergence of results from different paradigms problematic. Below, I examine the alleged possibility of convergence of results when employing between-method triangulation.

Convergence of data from mixed-method designs: Is it possible?

In our discussions during the ad hoc presentation, we examined the possibility of obtaining true convergence of results by looking two factors: (1) assumptions of quantitative and qualitative methods about the nature of the world and (2) the influence of the research question. However, because of lack of space, I will cover only one factor: the assumptions about the nature of the world. In this paper, I define convergence as a process whereby data collected by different strategies (either from the same or from different methodologies) join or come together to form a single 'product' or "provide evidence that will result in a single proposition about the same phenomenon" (Mathison, 1988: 15).

Assumptions: Qualitative and quantitative paradigms are believed to be different in nature. As a result, they differ in the goals of investigation and the assumptions about the phenomenon (Buchanan, 1992). These assumptions, however, have direct methodological implications (Husën, 1988: Rossman & Wilson, 1985). That is to say, it is the paradigm that actually determines how a problem is formulated and methodologically tackled.

Furthermore, Creswell (1994) contends that assumptions provide direction for a designing all phases of a research study: the focus, paradigm, format, and the literature review. Hence, his conclusion is that, construction of a good statement of purpose is based on paradigm of the study. Greene, Caracelli, and Graham (1989) warn us that dat from different paradigms influenced by different set of assumptions may not converge to a single truth, which is sought in the process of triangulation. They assert that :

the attributes of a paradigm form a 'synergetic set' that cannot be meaningful segmented or divided up. Moreover, different paradigms typically embody incompatible assumptions about the nature of the world and what is important to know, for example realist verus relativist ontologies. So mixed-method evaluation designs, in which the qualitative and the quantitative methods are conceptualized and implemented within different paradigms (characteristically, interpretive and postpotivist paradigms, respectively) are neither possible nor sensible" Greene, et al., 1989:257).

The examination of the above arguments, leads me to ask the question: How can a biased, subjective piece of work [from the qualitative approach] confirm validity of unbiased, objective piece of work [from a quantitative approach] (Holloway & Wheeler, 1996)? Or put another way, How can data from different methods influenced by different paradigmatic philosophies "provide evidence that will result in a single proposition about the same phenomenon" (Mathison, 1988: 15) [emphasis added]. I do concur with Greene et al. (1989) that "the notion of mixing paradigms is problematic for designs with [between-methods] triangulation..." (p. 271).

Taken all together, the issue of convergence of results from between-methods triangulation to a single truth or proposition is problematic to me. Nevertheless, I do concur that data from within-method triangulation may converge because the research techniques used (e.g. classroom observations and in-depth interviews) are from the same paradigm, hence they share the same views about the nature of the world. Therefore, we cannot say with confidence that data from qualitative and quantitative methods can converge to a single proposition.

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LINEAR ALGEBRA AS A DISTANCE COURSE

Asuman Oktac, Concordia University

At the Instituto Tecnologico de Monterrey in Mexico, an introductory linear algebra course was offered as a distance education course as part of a masters degree program in education with specialization in mathematics. The students were in-service high school or university teachers.

The set-up of the course involved virtual groups with students from different campuses in Mexico. Participants were expected to read the material from the book, engage in discussion by e-mail or discussion groups on the internet to answer the assigned homework problems, submit their solutions that they had agreed upon as a group, before the topics were discussed during satellite sessions which took place every two weeks. To help the instructor follow the discussions and to assure guidance to the students when needed, every time a group member posted a message, a copy was to be sent to the instructor.

As to the nature of discussions that took place throughout the course, two points were observed:

1) The asynchronity of time changed the character of group discussions ,allowing the students to reflect more upon the comments of their groupmates as well as their own solutions before responding to the whole group, as a result of which some profound mathematical thinking took place in case of some students.

2) The written as opposed to the oral character of discussions forced the students, through time, to be clearer in their expressions and ideas when they were communicating mathematics.

One disadvantage was that there was no single compatible means to use to communicate mathematics over the internet. Students had to send their solutions as text files most of the time. In some cases this led to creative methods of communicating mathematics, sometimes forcing a different way of thinking about their solutions.

During this presentation examples were presented about the discussions and mathematical ideas that emerged as a result.

NEW FACULTY TEACHING AT YORK, 1999-2000

Pat Rogers, York University

Points from the topic group session

Needs to be individualized; identify areas of strength and areas for development
Up front training on who York is – teaching environment/culture; students and how they learn; students' lives; assessment structure.
Micro-teaching – in second year watch each other – peer-pairing ; needs/areas of concern to match; use pairing to plan workshops
Use micro-teaching to identify effective teaching practices
How to disrupt the "isolation" of the pedagogical environment
Why do we do what we do?
Issue of course release – 1/2 CD in the following year?
Role of Fac Assoc. to develop individualized contracts; organize micro-teaching
Plan for 100 hours – ie 12 x 8 hours because they get 1 or 2 Courses off?
Maintenance for change

New Faculty Teaching at York is a year-long program of teaching development for newly-appointed faculty members designed to enhance their teaching experience by:

* increasing understanding of the teaching and learning process;
* expanding repertoires of teaching and assessment methods; and
* promoting habits of informed classroom practice.

Since we anticipate that new faculty will have different levels of teaching experience and familiarity with York's diversity, we have structured the program to allow a variety of choices and options for engagement. Program components include:

- **Summer Institute, August 23-26, 1999**
  The Institute will provide an opportunity for new faculty to fine-tune one of the courses they will be teaching in the upcoming academic year. The program will be interactive, involving a wide spectrum of York faculty and modelling a broad range of approaches to teaching and learning throughout the week. Each of the four days will centre on a particular aspect of university teaching and curriculum, and will allow plenty of time for participants to work together independently or in small teaching circles on course planning. The week's events will lead to the formulation of individualized programs of teaching development to be pursued through the academic term.

- **New Faculty Teaching Circles**
  Cross-disciplinary teaching circles, including those formed during the summer institute, will meet regularly every two to three weeks throughout the year to discuss teaching-related challenges, develop teaching dossiers, and explore teaching and learning issues of interest to group members.

- **Teaching and Learning Workshops**
  A variety of workshops, based on needs identified by new faculty themselves, will be offered throughout the year during the months of September, October, January and February.

- **Mid-term check-in**
  Two mid-term sessions will be held for new faculty to meet again as a group and explore issues of mutual interest.

- **Feedback on Teaching**
  Feedback on teaching is available to all faculty on demand and is confidential. This may take a variety of forms including participating in a video-taping workshop or arranging for a classroom observation or student focus group discussion with CST personnel or other colleague.

- **Course (Re)Design Institute, Spring 2000**
  This week long institute will be offered to all faculty members in April or May.

- **Documenting Your Teaching**
  Workshops on documenting teaching accomplishments will be provided during teaching circle meetings and advice and feedback is also available to individuals by request.

Time commitment
This is, of course, up to the individual. For those who participate in the full program, we estimate that the time required would be equivalent to teaching a half course and is allocated as follows:

Summer institute (4 days @ 6h):	24 hours
Teaching circles (5 per term @ 1.5h):	15 hours
Mid-term check-in (2 @ 3 h):	6 hours
Workshops and other activities:	55 hours

NEW PhDs: RESEARCH REPORTS

PhD Report 1

LEARNING TO TEACH MATHEMATICS THROUGH MATH LETTER EXCHANGES WITH FOURTH GRADERS

Sandra Crespo, Michigan State University

Radical changes to the nature of teaching mathematics have posed serious challenges for mathematics teacher educators as they attempt to help prospective teachers learn to teach mathematics in ways they have not experienced before. One of the main challenges teacher educators face is that preservice teachers' past experience with school mathematics, more often than not, tend to promote a limited knowledge of and about mathematics, along with a negative view of the subject (Ball, 1990). In addition, their extended experience as students in traditional mathematics classrooms help them develop implicit and imitative pedagogical tendencies, such as habits of correcting, telling and supplying the answers (Feiman-Nemser, 1983).

Unfortunately, teacher education courses have been shown to be a weak intervention in preservice teacher education. Studies from the "teacher socialization" perspective highlight the potency of preservice teachers' early apprenticeship of observation in limiting the impact of the on-campus and field-based experiences offered in teacher preparation programs. In this literature there are many "examples of students interpreting the messages of teacher education courses in ways that reinforce the perspectives and dispositions they bring to the program, even when these interpretations involve a distortion of the intentions of teacher educators" (Zeichner & Gore, 1990, p. 337).

This research literature has made it clear that the traditional designs and instructional strategies used in teacher education courses, which rely on a "pedagogy of presentation," have not had much success in altering the traditional images prospective teachers bring to their teacher preparation programs. It has become evident that a different design and pedagogy is needed if initial teacher preparation is to have a greater impact on prospective teachers' mathematics teaching. As a response, leading mathematics teacher educators have called for a "pedagogy of investigation" which relies on problem solving and reflective inquiry to "help prospective teachers learn what it is *like* to teach rather than learn *how* to teach. ... [and] to *think and inquire* about teaching rather than to learn *answers* about teaching." (Ball, Lampert, & Rosenberg, 1991, p. 7).

As a mathematics methods course instructor I have been inspired by such calls for reform and have drawn from recent literature on inquiry-based teacher education, such as case-based pedagogies and teacher inquiry methods, to help me design meaningful learning experiences for my students. Yet while these ideas are argued convincingly in the published world, some questions still remain: How can such an inquiry-oriented pedagogy be integrated within teacher preparation courses, in this case, a mathematics methods course? How might such investigative pedagogy help preservice teachers reconsider their preconceived notions and redirect them towards alternative ways of thinking and acting?

CONTEXT AND DESIGN OF STUDY

This study explored the nature and substance of preservice elementary teachers' learning in the context of an innovative version of the mathematics methods courses typically offered in the Teacher Preparation Program at the University of British Columbia. This course engaged preservice elementary teachers in a weekly math letter exchange with fourth graders from a nearby school. These interactive experiences with students were meant to serve as a source for reflective inquiry for preservice teachers and as a context for their inquiry of mathematics and its teaching and learning. To bring their inquiries to closure, preservice teachers then wrote a case study of their learning experiences with students.

Provided with such an opportunity, what might preservice teachers learn? What would they make of such an experience? How would this experience contribute to their learning to teach mathematics? These were the questions that guided my research project. The main goals of this study, therefore, were to (a) understand and characterize preservice teachers' learning in this particular context, and to (b) get a sense of how such learning was encouraged and developed throughout the course.

Using the data of 13 participating preservice teachers' written letters to students, their journal reflections, and their final case study, I began the analysis of the data looking for incidents which indicated (either explicitly or implicitly) puzzlement, tensions, and difficulties raised by their interactions with students. This led me to notice patterns of similarity within and across the preservice teachers' data. Then, I grouped their tensions and puzzlements into three broad categories which spoke of preservice teachers' learning related to their: (a) posing of mathematical problems, (b) interpretations of the students' mathematical work, and (c) responding practices. Further analysis into these three categories revealed patterns of change in preservice teachers' discourse and practices. Next, I provide a snapshot of preservice teachers' learning in each of these categories by contrasting their earlier and later patterns of discourse, views, and practices. A more complete discussion of this work can be found in Crespo (1998).

LEARNING TO VALUE "PROBLEMATIC" MATHEMATICS PROBLEMS

Preservice teachers' initial selection and adaptations of problems revealed their preference for unproblematic problems—problems which could be solved easily and quickly. Their early problem posing practices could be characterized by their attempts to remove potential difficulties and avoid students' errors. Their journal reflections suggested that by choosing such unproblematic problems preservice teachers intended to provide a pleasant experience for their students. Working on difficult problems and not being able to solve them or getting incorrect answers, in the eyes of these preservice teachers, would only serve to frustrate and "turn off" students from mathematics. Not surprisingly, they could be seen avoiding to pose "problematic" problems and simplifying problems they thought would be too difficult for their students to solve. Later interactions with students and journal reflections, however, revealed changes to preservice teachers' views and practices, which suggest their beginning to value "problematic" or non-trivial mathematical problems. Some of the ways in which this learning was revealed in the data included preservice teachers beginning to: (a) make more adventurous and less leading selections and adaptations, (b) rethink fun, interest, and enjoyment in mathematics, and (c) reconsider the role of ambiguity and errors in teaching and learning mathematics.

Making More Adventurous and Less Leading Selections and Adaptations. Preservice teachers' pattern of problem selection and adaptation became more adventurous and less leading later in their letter writing experience. Preservice teachers' selection of problems became more "adventurous," as David Cohen (1989) would say, in that these were less traditional types of problems (puzzle-like, exploratory, open-ended), extending beyond topics of arithmetic, and requiring more than computational facility. Furthermore, their adaptations to problems became less imposing or leading, and were meant to be helpful to students "without giving away the answer," as one preservice teacher, "Marcia," said.

Rethinking Fun, Interest, and Enjoyment in Mathematics. Their journal reflections also showed changes in preservice teachers' notions of what would make mathematics fun, interesting, and enjoyable to students. The students' work and feedback to the problems they were given began to challenge preservice teachers' assumptions about what kinds of problems students would be able and willing to solve. Problems which preservice teachers thought would be too difficult or frustrating to students often turned out to be enjoyable and attainable. Problems which preservice teachers thought would be easy or interesting to solve, were not always so for students. Therefore, preservice teachers' ideas as to what problems students might find enjoyable and interesting ceased to be taken for granted and began to be more explicitly investigated in their journals.

Reconsidering Ambiguity and Mathematical Errors. The more exploratory types of problems introduced and worked on during our mathematics methods classes also began to extend preservice teachers' ideas about the types of problems that they could pose to students. They began to see that certain types of problems could "teach kids to think about math, not necessarily as numbers and correct answers, but math as ideas," as Marcia said. Preservice teachers' own mathematical and pedagogical explorations were also helping them reconsider their earlier assumptions. Some began to see ambiguity "as a valuable tool for further exploring mathematics" (Sally) and others began to realize, as Thea did, that becoming confused and making errors could lead to interesting and important mathematical investigations.

LEARNING TO SEE AND CONSTRUCT MEANING FROM STUDENTS' WORK

Preservice teachers' initial interpretations of their students' work focused mainly on its surface features. This was apparent both in their response letters to the students as well as in their journal entries. The few comments preservice teachers made about their students' work tended to highlight the merits of the students' work. They simply suggested their student "did a good job," "was successful," "made a careless mistake," "was almost right" without attending to the details and the meaning of the student's mathematical work. Initially their discussions about the students' work were brief and evaluative. In short, preservice teachers' initial interpretations tended to be at a surface level and to make no inference or speculation about the student's underlying understanding of specific mathematical concepts and procedures.

In contrast, preservice teachers in this study were quite willing to speculate about, and make inferences (though quite often unsubstantiated) from the students' attitudinal comments (about math, teaching, learning, textbooks, school), their questions, and from the surface features of their writing (e.g., length of response, handwriting, spelling, neatness). Using students' brief comments as evidence—comments such as: "math in this class is prity isey (sic)," "as for math I need more practice," "I like the teacher teaching me"—preservice teachers tended to make quick, evaluative, and sweeping generalizations about their student's mathematical attitude and ability. Rosa, Nilsa, and Linda, for instance, were convinced after receiving only one letter from their student that their particular student "(was) not very good in math", "(did) not enjoy math", and "(was) at the bottom of her class."

The introduction of an "interpretive tool" and their own mathematical investigations into the problems they were posing, however, began to help many of the preservice teachers to focus their attention onto, and delve into the meaning of, their students' mathematical work. Preservice teachers could be seen learning about students' work by: (a) raising questions about the meaning of students' work, (b) seeing beyond correctness, (c) making analytical interpretations, and (d) extending own understanding through interpretations.

Raising Questions about the Meaning of Students' Work. Preservice teachers' initial observations of their students' mathematical work focused on its surface features (e.g., correctness, spelling, and neatness). Initial journal entries scarcely and briefly referred to the meaning of the students' work. Students' unexpected and unclear responses, however, began to raise the curiosity in preservice teachers about what their students' work meant. For instance, receiving students' inexplicit, brief, and unclear mathematical work encouraged preservice teachers to raise questions about the meaning of such work. Preservice teachers, for instance, began to ask: "Does he know why he got 3 or why he added it to the 5?" (Rosa); "Do they know what showing your work means?" (Carly); "I am worried she did not understand the problem in the end" (Megan).

Seeing Beyond Correctness. Unexpected work from the students also encouraged preservice teachers to look for and see more than the correctness in students' work. They began to use the students' communication of their work as another source of evidence for students' mathematical understanding. Preservice teachers' comments about their students' work began to read: "She did a good job explaining," (Linda) "I would have liked to see his rough work," (Megan) "I can finally follow her thinking," (Carly) "She did not say why she did what she did" (Miriam). These indicate that preservice teachers had begun attending to the length, clarity, and explicitness of the students' written communication as other clues for mathematical understandings.

Making Analytical Interpretations. Preservice teachers' later interpretations of their students' work turned more toward the details and the meaning in students' work, even when such work was brief and not obvious. This means that students' correct work ceased to be assumed as evidence of understanding and students' incorrect solutions were not dismissed as careless mistakes or signs of confusion. Instead, students' mathematical work began to be closely examined and discussed at length in the journals with particular attention to the mathematical details and meanings. A "descriptive-interpretive" journal writing tool provided in our class provided a useful format for helping preservice teachers organize and focus their reflections upon the students' work. In addition, the introduction of unfamiliar and exploratory types of problems and the opportunity for collective and individual explorations of these problems began to help preservice teachers make more analytical interpretations of their students' mathematical work. Their own mathematical investigations of the problems they would pose to their students raised preservice teachers' interest, confidence, and resources to delve into the meaning of their students' work.

Extending Own Understanding through Interpretations. Close examination of their students' work also helped preservice teachers extend their own mathematical and pedagogical understandings. It helped preservice teachers become familiar with students' communication and explanations of their mathematical work. It also gave preservice teachers further insights into the problems they posed, that is insights into: the mathematical concepts involved, the students' initial interpretations of the problem, the subsequent questions and difficulties that could arise, and alternative solutions and problem solving strategies for the problems they had posed.

LEARNING TO INTERROGATE HIDDEN MESSAGES IN THEIR "TEACHERLY TALK

Preservice teachers' responses and feedback to students' mathematical work initially focused on the overall correctness of the student's answer. These responses were mainly evaluative of the students' work. For instance, preservice teachers tended to respond by praising the correct answers and by correcting the wrong ones. Interestingly, these kinds of responses, often associated with the immediacy of classroom interactions between teachers and students, also pervaded the slower paced letter writing interactions between preservice teachers and their student writers. In later interactions, however, preservice teachers' responses to their students' work became more deliberately constructed and to focus on more than the correctness of the students' answers.

Writing their responses to students encouraged preservice teachers to carefully consider a response to their students' work. At times writing out their response in their letters helped preservice teachers notice the underlying messages they were sending to students. Other times it was when revisiting and revising their letter exchanges that preservice teachers were able to problematize their responses to their students. For others, it was much later, when writing a case study of their learning experience that they identified and reflected on their potentially damaging responses to students. Some of the ways in which preservice teachers can be seen recognizing hidden messages in their responses to students included: (a) problematizing praise, (b) questioning the practice of correcting, and (c) questioning own lack of questioning.

Problematizing Praise. Preservice teachers who received correct work from their students became aware and began to problematize praising as a response to students' answers. Carly, for example, realized that indicating to students that their answers were right discouraged students from relying on their own sense making and from exploring other answers and solution strategies to problems. Linda and Megan also became aware of their tendencies to consistently respond with praising comments to one of the two students with which they were exchanging letters. They noted substantial differences in the quality and length of the responses they gave to the students they perceived as high and low achievers. The implicit and dangerous messages of such responses became apparent to Linda and Megan when they reflected on the effects that such responses had on their students' mathematical attitudes and performances. The practice of praising also became problematic to other preservice teachers when they found themselves praising students' effort and their incomplete and often incorrect work.

Questioning the Practice of Correcting. Their responses to students' incorrect work—correcting and supplying the answer—were a source of much deliberation and reflection for preservice teachers throughout their letter writing experience. Very early in their journals preservice teachers began questioning and investigating the effects of pointing out and correcting mistakes in their students' work. Furthermore, all the preservice teachers who responded by supplying the correct answers regretted having done so in their very next journal entries. Interestingly, the underlying messages of their corrective responses were more easily recognized and problematized by these preservice teachers than the implicit messages in their praising responses to their students' correct work.

Questioning Own Lack of Questioning. Imposing their answers and ways of solving mathematics problems, many preservice teachers could see, was disrespectful and discouraging to students' sense making. However, responding without pointing towards the solution was not an obvious choice for them. On many occasions, it was after they had already sent their responses to students, that they began questioning their lack of questioning of the students' work. These are some examples of preservice teachers' comments after supplying students with the answers: "I wished I had asked him how he did this, but I forgot," (Carly); "I'm not sure how she did this, ... I didn't think to ask her," (Sally); "Had I posed the following problem instead, John would have had to do the same math, but would have had to think more deeply about his answer and he may have been more certain about himself and his own knowledge" (Marcia). For others, like Linda, it was in retrospect that they realized "the importance of questioning beyond superficial levels," and regretted not having asked more "how" and "why" questions to their students.

ROLE OF INTERACTIVE EXPERIENCES WITH STUDENTS IN SUPPORTING THE LEARNING OF MATHEMATICS AND MATHEMATICAL PEDAGOGY

A math letter writing experience associated with a mathematics methods course offered a rare opportunity to preservice teachers in this study. They were engaged in a sustained interaction with school students while attending their on-campus classes. This sort of course-related field experience, while enthusiastically advocated in the literature, as Carter and Anders (1996) point out, has not been systematically studied. In fact, I found very few research articles on this topic and very few have been written since the 1970s. The present research study, therefore, contributes to this literature.

In particular, this study provides further evidence of the transformational influence that working closely with students has on the learning of not only experienced teachers (as the CGI studies have found), but on preservice teachers as well. Preservice teachers in this study took their students' data very seriously. They sought and worked very hard to understand their students' mathematical work and communication. The students' data, in turn, provided them with much insight and deliberation related to their understandings of mathematics, students as learners, and pedagogical practices. This demonstrates that preservice teachers' attention and concerns can be focused away from "self-concerns"— concerns with classroom survival, managerial, and disciplinary concerns—even in contexts which are closely related to classroom practice.

Different from traditional interactions between students and teachers, written letter interactions afforded preservice teachers no concerns for managerial or disciplinary issues and no school and curricular pressures. It, however, brought to the foreground concerns for students' abilities to communicate their thinking. The structural features of their written interactions with students, therefore, made it possible for preservice teachers to engage in mathematical and pedagogical inquiries and to focus their reflections on issues of mathematics teaching and learning. In particular preservice teachers in this study focused on issues related to: communicating mathematically, writing in mathematics, understanding students' mathematical thinking, posing good questions and practicing good questioning.

Written interactions with students provided opportunities for mathematical inquiry alongside and often intertwined with preservice teachers' pedagogical inquiries. Writing the solutions to the problems the students posed to them provided one such opportunity. These, in turn, were often taken as pedagogical opportunities for modeling to students how they might respond and make their mathematical work and thinking more explicit in writing. In addition, the students' responses served to, in some occasions, engage preservice teachers in further mathematical explorations of the problems they had posed. There were many occasions when preservice teachers' pedagogical inquiries—examining and trying to make sense of a students' response—led to further examination of the mathematics involved in a particular problem.

Students' responses were also very influential catalyst to preservice teachers' learning. Preservice teachers took their students' data very seriously yet oftentimes they could not make sense of it or had more questions about it than answers. Students' unexpected, unclear, and brief responses to their questions were a source of much deliberation for preservice teachers. They served to challenge some of the preservice teachers' prior assumptions about students and mathematics learning and helped them make new assumptions, some of which turned out to be false and were eventually revised.

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PhD Report2

EXPLORING STUDENTS' PERCEPTIONS OF MATHEMATICS THROUGH THE CONTEXT OF AN UNDERGRADUATE PROBLEM SOLVING COURSE

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Perceptions of mathematics are shaped by our perceptions in mathematical learning environments. A change in the nature of such an environment may lead to potentially different perceptions of mathematics. How mathematics is done will influence how it is perceived. Students' conceptions of mathematics may be restrictive. It is healthy to challenge these conceptions by engaging students in a different experience of doing mathematics. In doing so, students are invited to perceive mathematics in ways that are new to them. This may also foster an enhanced appreciation of math among less interested students. Concurrently it may challenge others to broaden the window through which they envision the subject. Of course, some may be threatened by such a shift. The student may be content with their representation; however it is worthwhile to broaden their experiential background.

In a study of teenagers' perceptions of mathematics, Schoenfeld (1989) reports two common perceptions: "Learning mathematics is mostly memorizing" and "Doing mathematics requires lots of practice in following the rules." (p. 344) Students are rarely called upon to think creatively and solve a problem for which rules are not readily available.

> Virtually all the problems the students were asked to solve were bite-size exercises designed to achieve subject matter mastery; the exceptions were clearly peripheral tasks that the students found enjoyable but that they considered to be recreations or rewards rather than the substance of what they were expected to learn. This kind of experience, year after year, has predictable consequences. (p. 348)

"Fullan (1982) wrote of educational change as affecting three dimensions: (a) the possible use of new or revised material, (b) the use of new teaching approaches, and (c) the alteration of beliefs." (Martens, 1992, p. 150) This study considers the latter two dimensions: pedagogy and beliefs. It is important to clarify that beliefs are not to be taught but rather challenged. [C]onceptions cannot be directly taught, but rather developed or formed (implicitly or explicitly) in the individuals on the basis of their experiences. (Borasi & Janvier, 1989, p. 75)

> Changing a belief system consists perhaps in introducing the seed for a new conception to emerge and that, as a result, the subject will be faced with a multiplicity of conceptions available. (Borasi & Janvier, 1989, p. 76)

The teacher is encouraged to introduce ideas or events that will clash with students' conceptions, thus challenging students to reflect upon learning in new ways. Civil (1993) expresses a need to challenge the conventional assumptions of preservice elementary teachers. She suggests that Lampert's (1988) description of an approach best reflects the spirit she had tried to capture in her own course. Lampert writes:

> I assumed that changing students' ideas about what it means to know and do mathematics was a matter of immersing them in a social situation that worked according to different rules than those that ordinarily pertain in classrooms, and then respectfully challenging their assumptions about what knowing mathematics entails. (p. 470)

The context for this study is an undergraduate problem solving course that was offered to a class of 12 students at Sir Wilfred Grenfell College in Corner Brook. The unfamiliar structure of the course invited subjects to reframe their own perceptions.

How did the problem solving course in this study differ from past mathematical learning experiences? The course featured group work, presentations, discussion, and journal writing. Although most of the time was spent solving problems, the emphasis was placed on process rather than solution. Students were expected to exchange ideas and solutions. Assessment did not place a major emphasis on a final exam unlike other math courses at Sir Wilfred Grenfell College that generally weight 60% of the course mark on the final exam. Alternative pedagogical practices are central to the study. It is through such practices that students' perceptions are most likely to be challenged. The most prominent of these practices is known as 'convening'. The convening process involves reviewing the work of one's peers for the specific purpose of preparing a presentation to the class. The presentation is intended to draw out the varied approaches employed in solving a problem.

PURPOSE OF THE STUDY

Specifically, three research questions are examined:

1) How does participation in an undergraduate mathematical problem solving course impact upon students' perceptions of mathematics as a learning process and as a discipline?
2) How does the convening process facilitate development in mathematical understanding?
3) How do students' impressions of problem solving develop through the course?

SIGNIFICANCE OF THE STUDY

The primary significance of this study is to extend our knowledge of the interplay between pedagogical practices and students' perceptions of mathematical learning. In reviewing the literature, it also appears that research on students' perceptions of mathematics is restricted mainly to children and/or prospective teachers. It is significant that this study focuses attention on undergraduate students' perceptions of mathematics in the context of a mathematics course.

A second significant feature of the study is that it examines the convening process. This process is presumably unfamiliar to the mathematics education community. The study offers it as an alternative pedagogical approach that appears to be aligned with current reforms. Further, the study goes beyond simply introducing the model to actually investigating its value in terms of developing beliefs and mathematical knowledge.

Finally, the social value of the course is significant. Students deserve to be given the opportunity to participate in an active mathematical learning environment. As a teacher, I share the sense of responsibility to service that is addressed in Schoenfeld's (1983) remarks. Perhaps the most valuable knowledge "we can offer our students, both our majors and the ones we will never see again, is to provide them with thinking skills that they can use <u>after</u> they take our final exams." (p. 7)

THE TEACHER AS RESEARCHER

The context for the study is a problem solving course that was taught by me. As the teacher and researcher, I am relating the story of what transpired through the course. Although some may be critical of this idea, Brouwer (1993) notes that as the teacher, one is allowed "to build upon the atmosphere of trust and cooperation within the class and to foster the idea of a learning community within the class." (p. 19) The interplay between 'what is learned' and 'how one is taught' impacts upon the educational experience. It would seem that acting as the teacher provides optimal insight into this relationship.

FINDINGS

Three dominant themes emerged from the data. These themes shall be referred to as frustration, changes of attitude, and unfamiliarity.

Frustration represents an emotional response to things within the course itself. The nature of the frustration often dealt with specific problems or problem solving situations. Much frustration stemmed from individual feelings of failure or incompetence. The second theme, changes of attitude, is self explanatory. It is important to note that the changes have generally been identified by the students themselves in some form of reflection. In fact, changes in attitude may appear to be rooted in large part with some aspect of unfamiliarity, a theme that crosses the discussions of the three research questions.. Students repeatedly expressed how this course was different from other math courses they had taken. This so-called "unfamiliarity" subsumes a number of related themes. Aspects of pedagogical practice and course structure such as group work, journal writing, and convening appeared novel. For instance, Rosemary's opening paragraph of her journal reads:

> Seeing as this is a math journal, I guess I'm supposed to write about math, but there is the problem; what do you write? To me, math was always numbers not words.

The shift of the focus from the "right answer" to the process was noted as a striking difference. Others noted how learning from other students rather than only from the professor made the environment unique. This idea of "student to student" learning is explored further with respect to convening. Sara's comments in the course evaluation offered an insight ful distinction between the roles of groups and the convening process in the learning of mathematical problem solving:

> I thought that working in groups helped me to see how other people approach a problem. So did convening, to a certain degree, it was more of how people solved a problem. In groups you could see how a person started a problem, what frustrated them, and so on, as it was happening.

Her comments reinforce the idea that the convening process offers a retrospective look on the processes in play, in sharp contrast to the process of working together form the outset of a problem.

Other aspects of unfamiliarity will surface in the discussion of research questions. Carl's commentary provides a window through which the reader may gain a better sense of the interplay among the themes. About five weeks into the course, Carl wrote:

> I haven't made an entry for awhile so I guess it's about time I get started. I am really getting frustrated with this course and I am questioning whether I should have taken this or not. Too late now I guess... I am coming away from classes with a feeling of emptiness, that I am not really learning anything... instead of doing more of the same and

familiarizing myself with a particular method, we are going on to another problem of a different nature altogether... This was the point where I had to stop, read over what I have written and question whether I should have expressed these views openly or just forgot about it and continue to work to do my best.

Carl had advised me that the tone of his comments was negative. He verbally expressed his reluctance to submit the journal at the time he handed it to me. Carl and I met after class and chatted in my office for about forty-five minutes. My written response to his journal entry began as follows:

I appreciate your honest reflections. I find that some people want to be too kind so that they write anything but negative comments. Your honest commentary is far more telling. I like the way you contrast the human experience (nice teaching style, personable, etc.) with the mathematical (frustrating, paranoid, etc.).

This subsequent entry in Carl's journal reads:

I think my problem why I find this course so frustrating is that when I am learning something new, I want to "know it all" and I realized that with different people's interpretations of our problems I will never understand everything that goes on in class.

Carl proceeded to express appreciation for the time taken in our conference to discuss his frustration. Also, he expressed his gratitude for the small class size that allowed people in the class to get to know one another better. It is Carl's final entry that captures the growth and change that had taken place during the course. He wrote:

I was reading over some of my earlier entries and boy has my attitude towards this course changed a lot over the last few weeks. I can clearly see now that this course was a real learning experience for me. It really took me all this long to feel comfortable with the fact that getting the right answer is not the most important thing... I really think that for anyone who is doing a lot of math courses this particular course should be compulsory for them. I think the change in attitude in approaching problems is something that a lot of people will benefit from.

PERCEPTIONS OF MATHEMATICS

Let us consider the umbrella question for this study:

How does participation in an undergraduate mathematical problem solving course impact upon students' perceptions of mathematics as a learning process and as a discipline?

The question guided the research at hand. How has it been answered by the experiences documented here? The evidence presented clearly suggests that taking a course in which mathematics is approached differently can effect a change in perceptions. The theme of unfamiliarity echoed loudly through the written and verbal voices of the participating students. Unfamiliarity refers to dimensions of the mathematical experience that appear different from prior experiences. What makes these things unfamiliar is that they are either unexpected in a mathematics course or new to a student's experience altogether. Indeed, this course offered a different dimension to the mathematical histories of these students.

Mathematical learning was perceived to be a collaborative process, one in which communication between students figured prominently. Students learned from other students. This represented a notable change in students' perceptions of doing and learning mathematics. The dominant theme in discussions of convening was the value of sharing work with one's peers. This surfaced in terms of the unfamiliarity with such an experience as well as its perceived value through its contribution to learning. The emphasis on learning from others in the class rather than only the professor and the opportunity to examine the differing processes resonated in the reflections of students.

Noreen's comments reinforce both the novelty of seeing the work of peers and the inherent value in such a situation:

> This is the first course that I have taken that has had convening. Actually, it is quite neat because the other students get a chance to have the answer told or explained to them in a way that they understand (well, not "understand" but from a peer's perspective and language). Actually, I will admit that I had a lot of fun and enjoyed it very much.

The nature of mathematics as a discipline came to be understood differently. The roles of answers and process figured in this reconfiguration. It is evident that the value of mathematical process has been enhanced in the students' eyes. The multiplicity of approaches to solving various problems brought with it a different perception. Such differences enabled students to gain greater appreciation for the potential contributions of their peers to their own learning. Further, it extended the perception of doing mathematics forward from simply attaining or verifying a result to the thinking which proceeds it. Actually the 'it' must be pluralized because students now recognize that multiple (or even no) solutions are valid.

Reflecting upon the results of the study, the aforementioned findings may seem somewhat predictable. Yet there is another point which is far more remarkable. The data unequivocally portrays a group of students who perceive mathematics very differently than they did just a few months earlier. While these differences provide a formal response to the research question, it is the magnitude of the differences that excites me. I am not speaking of a quantitative measure. Rather it is noteworthy to consider how relatively easy it was to reshape perceptions of mathematics in such a short time. The individual mathematical histories span a decade or two. However, one semester of differently formatted instruction can produce such changes. This is the message that speaks loudest to this researcher upon completion of the analysis. It is reasonable to assume that the relative similarities among the various course structures, particularly in high school and first-year university, had shaped these students' perceptions prior to the start of the Math 1031 course. Hence, this course appeared as such a striking contrast to this relative "sameness" that students had come to expect from mathematics' classes. Suddenly it was time to reframe their own perceptions of mathematics.

The "undoing" of mathematical perceptions seems paradoxically simple yet difficult. Students seem to have only been evaluated through the use of tests and problem sets. The mathematical experiences of this group of students appear to be relatively uniform in spite of the range of emotions and backgrounds that they brought with them. On the one hand, it is difficult to undo such deeply engrained perceptions. Conversely, the uniformity of the experiences ensured that there were a minimal number of competing perceptions; that is, this seemingly novel experience really had to compete with one(few) way(s) of looking at math. Changes in perceptions may have been tempered had the students arrived with a broader range of mathematical experiences. In fact, it could be argued that this may have been their first experience of "doing mathematics". Not all mathematics problems are solved in three to five minutes. It is not a failure to struggle with a problem. Nor is it the case that every problem has a solution. Such experiences in this course challenged the mathematical histories and expectations of these students.

CONVENING

The organization of the course made student to student sharing an integral aspect of the learning environment. The most evident role of such sharing focused upon convening. The value of convening represents the core of the second research question:

> How does the convening process facilitate development in mathematical understanding?

Convening contributed to students' understandings of mathematics in various ways. This value was enhanced by the multiple roles played by students. As convenors, students commented upon how this role provided them with unusual insight into the mathematical processes of others. Further, they noted how it gave them a better sense of teaching. The class' chemistry was an ongoing reminder of the limitations within which the class must progress. That is, sound algebraic skills and a fluency in mathematical language could not be assumed. Rather it was essential that the mathematics be presented in a form that was understandable to all. Rosemary remarked on how convening figured into this reality:

> By having the convening as a part of the course we were able to see how other people, from different mathematical backgrounds, attacked the problems. It helped me to understand some of the questions better and to acquire new ways to do a problem. Also by having to do convening I could get a new view of math. To me the teaching was harder than the learning. Even though you had the answer in front of you, you had to present it in such a way that others could understand it. But by doing the problem you could just work with it until the answer satisfies you, you don't have to worry about anyone else.

The challenge of understanding different approaches was stressed. Randy added that "we probably worked harder to make our answers more clear when it was one of our peers who would have to interpret them."

Carl expressed a sense of surprise with what he found in reviewing his peer's answers:

> From going over this problem it's funny how almost every person did something a little different in their workings but yet almost everybody came out with the same answer.

Elaborating on the process, he mentions how this diversity of thinking processes impressed him:

> I didn't mind doing this convening at all. I kind of liked it actually, especially to the point of having everyone's solutions to look over and see all the different ways of thinking that go into approaching a problem.

Deborah also valued this aspect of the convening process:

> I was able to see the various viewpoints from which people attempted finding a solution to the problem. This really helped me to open my mind to different perspectives and to open my mind to the methods and problem solving techniques of others.

Sara pointed to the value of seeing such differences on paper, as opposed to listening to them in class:

> I did like being able to see how different people approach a problem. Yes, I see it in class, but having it written down is different than just <u>hearing</u> people's ideas.

Lynn appreciated the opportunity to gain greater insight into the work of her peers:

> The idea of convening was a good idea. It allowed students to see the work of other students. The idea of analyzing and comparing someone else's work with your own was a good experience. In other math courses, we never had to compare students' work and get up in class to show the different ways people approached one problem.
>
> This was a good idea because it gave us a chance to see how students approached the same problem. I'm not saying that I would do the same with my students. But I will get them to go up to the board and do some problems on the board. That way students can see how their peers do the same work.

Barbara concurred:

> When collecting the people's math problems for my convening, it gave me the opportunity to see other people's work. That there are so many different solutions to one problem.

The contribution to other convenors' presentations is also significant. Prior experience with a mathematical problem combined with the fact that a peer was presenting the problem combined to produce a keen interest in learning. Observation is unlike that of typical observers of a person outlining a problem in class. Keep in mind that it is these same people who have provided the fodder for the convenor. The experience of having worked with the problem sharpens the observational edge of the student. It is intriguing to see how the convenor will draw out the ideas from the contributions; it can be exciting to watch someone develop your work with appropriate credit.

Convening contributed to the changing perceptions of problem solving. The presence of the convening process provided ample opportunity to see the multiple approaches to solving problems employed by one's own peers. It also opened a window for examining misconceptions which arose from errant solutions and/or judgments made by convenors. Some anecdotes are provided here to provide a flavour of the nature of the situations that arose through convening such problems.

Rosemary convened a problem in which the incorrect answer was shared. Excerpts from my response to Rosemary set the context:

> You had an unusual solution to deal with. The solutions submitted were all incorrect. However, the final answers of 18 minutes (by Noreen) and 21 minutes (by the majority) seemed reasonable. Therefore, it is not likely to strike the convenor - in this case, yourself - that there is an error... Anyhow, you did not observe that an error existed. That's o.k. You proceeded to present the solutions offered in a well organized manner... Let me speak more about the pedagogical value of the experience.
>
> Rosemary, it was intriguing to observe you present the solutions provided by your peers. They did not offer any protestations. I expect that they were comforted by the sense that their solution was correct - or so it seemed. I was surprised that nobody connected their prior experience with the farmer's problem (done in the first week of the course) to the problem solving situation. Anyhow, I felt that it was my role to redirect the discussion. You did a wonderful job of moderating the presentation by presenting the "correct" solution as suggested by my directions... Most importantly, you caught on quickly to the glitch in prior efforts that had been presented. Your sound mathematical skills and your ability to catch on allowed me to take a secondary role again...

> In a curious way, it may have been the most valuable learning experience associated with any of the convening presentations. It's fascinating to see how a group of people can feel so comfortable with what is being presented until... suddenly they are surprised.

Carl convened a problem in which the objective had been to minimize the number of two-person meetings required to pass information amongst a group of spies. The solutions produced a collection of approaches that showed how as few as 7 meetings would be sufficient. Carl presented a selection of these responses before delivering the surprise. Lynn's solution confirmed that only 6 meetings were needed! Responding to Carl, I wrote:

> I suspect that many were surprised to see that the information could be shared in 6 meetings instead of requiring 7. That aha! sort of realization is also a benefit that may be realized by sharing one's efforts with others.

This example provides a different insight. It is not that the students misunderstood the problem. Rather the problem invited them to seek a minimum number and their solutions had left room for improvement. This realization came about through the work of one of their peers rather than by checking the back of a book. The convening process invites this sort of affirmation for one's own work. Indeed, we've seen through Rosemary's example that this self-checking method is not flawless.

A more sophisticated form of misconception arose in a separate example. Kristen convened a problem which required students to present a convincing argument for a particular fact given a scenario. In essence, a proof was required. Again I outline my response to the convenor as a means of illustrating the point.

> Your presentation featured a different dimension than most others. It seemed as though virtually all students thought they had provided you with a complete solution. You did not seem to challenge these perceptions. Instead you presented their solutions as if they were all correct. Perhaps that was also your perception. Let me expand upon this notion.
>
> ... Diagrams were generally used to solve a visual problem that seemingly had so many possibilities to consider. The choices were limited by the assumptions imposed by the solvers. This is where the difficulty arose in terms of the completeness of solutions. If one assumes a particular setting of wine glasses, then they may provide a convincing argument for that <u>specific case</u>. However, this does not generalize to all possible cases without additional work. I gave full marks to only 2 students who convinced me that they recognized this distinction... Unlike others, they proceeded to explain why this scenario must take place somewhere at the table... I've explained this in such detail for two reasons. First, it was an omission in your presentation. The concept of a proof is not well understood. Second, I see this as an opportunity to extend the learning process beyond the actual presentation.

The aforementioned examples show how the breadth of mathematical knowledge may be expanded. The multiplicity of methods is the most obvious way in which convening may contribute, though the convenor is most likely to gain such an appreciation. It is incumbent upon an effective convenor to bring such diversity out during the in-class presentation of a problem.

PERCEPTIONS OF PROBLEM SOLVING

Thus far, the discussion has dealt with both the umbrella question and the secondary question concerning convening as a practice. The remaining question shall be addressed here:

How do students' impressions of problem solving develop through the course?

This question was answered quite directly by the students when they wrote their final exam. Three observations appear to summarize the changing perceptions. First, there is the importance of process in problem solving. This actually juxtaposes with the second issue of validation for a variety of methods in producing a solution. Finally, the definition of what constitutes problem solving has changed for these students. The essence of a problem seems to lie in the fact that it draws upon one to think creatively rather than to simply execute a readily available algorithm.

The most striking change appears to be an enhanced appreciation for the value of process. Staib (1981) effectively captures the essence of this change in the title of a paper, "Problem Solving Versus Answer Finding." That is, problem solving became more than simply finding an answer. Elements of this change in perception are identified in terms of correctness or what it means to be successful. For example, Deborah indicated how previously she would have perceived a correct solution to have been the benchmark of problem solving. Glenn also identified correctness as being at the core of his prior belief; in his words, "... not getting a correct answer was a failure." His experience in Math 1031 changed that perception.

Another significant change dealt with methodology. The idea of employing a multiplicity of approaches appeared new to these students. In fact, the idea of putting one's own mark on a solution presented itself as a notable change. Randy expressed how it was different to see problem solving as anything other than executing previously described methods to produce a solution. Cheryl affirmed this change in perception. Like Randy, she had previously defined the essence of problem solving as being "required to find a solution in one structured way." Others like Carl echoed this changing perception from problem solving as one method to it as a variety of approaches. It follows that such a change would connect to an enhanced appreciation for the role of process in the problem solving experience. This is certainly the case when one considers that the structured singular approach was understood by these students to be the one discussed prior to solving the problem. Hence, the role of thinking would surely have been diminished in prior experiences. It is this need to think and consider different possibilities that underlies the newly acquired definitions of problem solving.

Problem solving was not explicitly defined by many students; however, students did identify a number of common features of what they now understand problem solving to represent. It involves thinking as opposed to merely executing prescribed techniques. The methods are not uniquely determined. As Cheryl notes, it may draw upon "many mathematical concepts - algebra, probability, graphing, etc. - and even physics." The problem solving process may entail frustration yet culminate in "one of the most powerful feelings of satisfaction that I have ever experienced", according to Glenn. The blending of these various components suggests that problem solving has come to represent a rich experience that offers opportunities for learning through the process. Randy's words seem to encapsulate the spirit of the modified perceptions expressed by many:

> When I first began this course I thought that I already would have learned the knowledge to do the assigned problems. Instead I found that I had to find my own way to solve the problems. That is we were taught to think for ourselves, we had to learn how to do each problem by doing them.

CONCLUSION

What does it mean to do mathematics? The beliefs we bring to educational settings will profoundly influence the nature of mathematical learning that takes place. The goals of both the students and teachers will figure into this experience. The (in)consistency between what is being assessed and what is being learned will also factor into the proceedings.

In summary, the students in this study have taken from the course different mathematical views than those with which they entered. It is evident that the nature of mathematical learning presented itself differently. That is, the experience challenged prior perceptions of both mathematics and problem solving. Learning from one's peers proved to be a valuable part of the learning experience. Students emphasized this as a strength; it was acknowledged to play a critical role in the revision of their own perceptions. The multiplicity of approaches employed in solving problems was particularly enlightening for students to observe.

Perhaps the most significant contribution of the study pertains to convening as a method of teaching and learning mathematics. The practice of "convening" has never before been closely examined. The study suggests that it is a powerful tool for mathematical learning. It engages students in a valued peer to peer sharing that provides insight into the process of problem solving. Student feedback and my own observations suggest that it is a teaching practice worthy of using again in the future. Admittedly other factors such as my own beliefs as a teacher make it tenable within the mathematical culture I envision. While this may provide a rationale for its use in my teaching, it is hoped that the method will be utilized and examined by others.

The observed differences strongly suggest that one's perceptions are not etched in stone. Rather we may reasonably expect to see changes when the style of mathematical teaching differs considerably from that which has traditionally been modelled. "How is mathematics perceived to be doing?" Many people have been trying to answer that question through standardized exams and international comparisons. Let's flip the words around and ask, "How is doing mathematics perceived?"

ACKNOWLEDGMENTS

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PhD Report 3

TEACHERS' INTERVENTIONS AND THE GROWTH OF STUDENTS' MATHEMATICAL UNDERSTANDING[4]

Jo Towers, University of Calgary

BEGINNING THE JOURNEY

In 1990, Stevens, reporting on the deliberations of an International Congress on Mathematical Education discussion group of which he had been a member in 1988, touched upon the problem of "when it is appropriate for a teacher to intervene in order to redirect a child's thinking" (p. 231). Stevens reported that the group was unable to reach agreement, and suggested that the issue was unresolved. The dilemma of when to intervene is an enduring one for teaching, and continues to perplex and challenge practitioners and researchers alike. As a teacher I had long been challenged by this dilemma, and so it was from this perplexity that my journey towards this research project began. However, I shaped a study based on the recognition that there are questions that can and should be asked about teachers' classroom interventions before asking *when* teachers should intervene in students' learning. One such question is the one that framed my study. It asks: In what ways do teachers' interventions interact with and occasion the growth of students' mathematical understanding?

Framing a question is a defining moment in the life of a research study. As van Manen (1990) notes, in crafting a question the writer must pull the reader into that question in such a way that the reader cannot help but wonder about the phenomenon in the way that the writer does. In choosing to present only one orienting research question (rather than sub-dividing the question and thereby hinting at its answer and removing some of its mystery), I attempted to preserve the wonder that I experienced in my journey through the study. As Gadamer (1975) notes, "the essence of the question is the opening up, and keeping open, of possibilities" (p. 266). To reduce a rich and complex enquiry to a series of simpler questions is to close down some of its possibilities. I also intended my choice to reflect the complexity of the space (the mathematics classroom) in which my research was situated by preserving the complexity of the queries that oriented my data collection and analysis, and I remain convinced that I was best able to do so through the preservation of a single complex question.

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[4] **This paper is based on my doctoral work: Towers, J. (1998). Teachers' interventions and the growth of students' mathematical understanding. Unpublished doctoral dissertation, University of British Columbia.** **I would like to acknowledge my dissertation supervisor, Susan Pirie, as well as members of my committee, Ann Anderson and Gaalen Erickson, for their guidance during this study. I also wish to express my thanks to Brent Davis for his continued support and provocative ideas. I am grateful to Karen and the student participants who made my enquiry possible, and to friends and family who supported me throughout.**

THEORETICAL FRAMEWORK

The theoretical framework within which this study lies has its roots in constructivism and, to follow the metaphor, its new growth in enactivism. The enactive approach was first postulated by Varela, Thompson and Rosch (1991), and is drawn from a diverse collection of recent and ancient thought, including Buddhism, continental philosophy, biology, and neuroscience. Drawing on recent developments in evolutionary thinking, which place an emphasis on natural drift (with the guiding metaphor of viability) rather than the Darwinian notion of natural selection (with the guiding metaphor of optimality), enactivist theorists Varela, Thompson and Rosch (1991) situate cognition not as problem solving on the basis of representations, but as embodied action.

Enactivism, as a framework, offers a means of incorporating cultural commentary with discussions of individual cognition (Davis, 1995). It parts company with constructivism on the common assumption that constructivism's focus on the individual cognising agent is an adequate unit of analysis either for understanding thought, or for studying education. Enactivism challenges that "in focusing on the individual cognising agent, both the participation of that agent in the larger community and the fluidity of the context are obscured" (Davis, 1995, p.8). Enactivists view the individual and the context as co-emerging in mutual specification, rather than as one operating on the already existing other. Enactivism troubles the boundaries between knower and known, mind and body, individual and collective, self and other; thereby opening a space for discussions of understanding and cognition which recognise the interdependence of all the participants in an environment (such as a classroom). Significantly for my study, such a shift enables understanding to be seen as a continuously unfolding phenomenon, not as a state to be achieved.

TEACHER-STUDENT INTERACTION

In shaping my research question, and in refining a strategy for responding to it, I turned to the literature on teaching and learning, and specifically to that on the teaching and learning of mathematics, in order to discover what had been learned since Stevens' (1990) call for action. Progress had been made in such areas as the nature of teachers' talk in mathematics classrooms, and particularly their use of questions (Martino & Maher, 1994; Vacc, 1993), the nature of students' talk in mathematics classrooms (Ball, 1991; Wood, 1990), and the nature of students' understanding of mathematics (Cobb, Yackel & Wood, 1992; Confrey, 1994, 1995a, 1995b; Pirie & Kieren, 1994; Sfard, 1991; Sierpinska, 1990). In addition, some researchers had begun to view the classroom more holistically and many of the more recent works focused explicitly on the culture of the classroom (Cobb & Bauersfeld, 1995), sociomathematical norms (Yackel & Cobb, 1996), cultural tools and mathematical learning (Cobb, 1995), and taken-as-shared-understandings (Cobb, Yackel & Wood, 1992). Most recently, and of most relevance to my own work, researchers have begun to coordinate investigations of teachers' actions with students' learning (Cobb, Boufi, McClain & Whitenack, 1997).

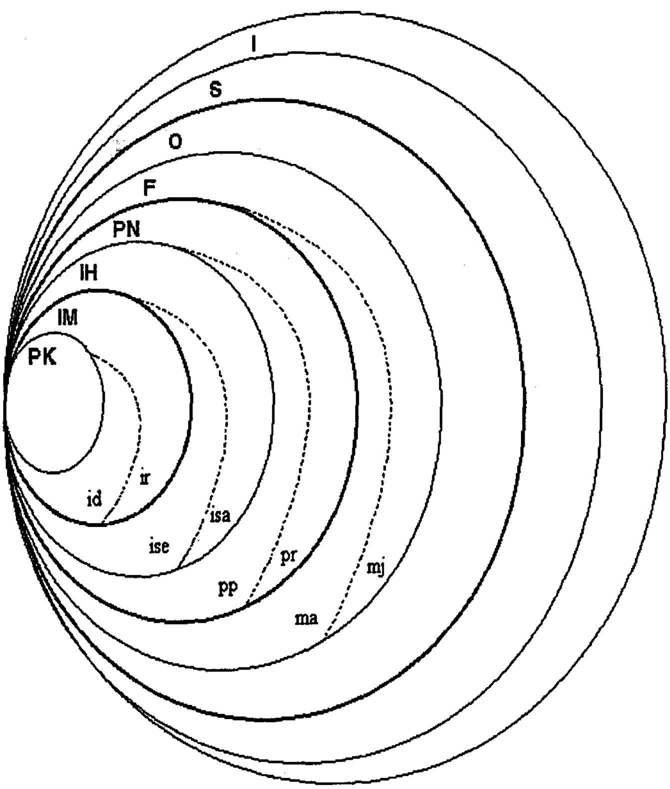
THE STUDY

I adopted a qualitative case study approach in order to provide a rich and detailed analysis, and a deep and comprehensive description and interpretation, of the processes of classroom interaction leading to the growth of mathematical understanding. As part of the study two 'cases' were documented, and these formed the two strands of my research. The first strand concerned data collected in my own classroom at a time when I was a full-time teacher of mathematics in a small, rural, British secondary school. Students of (Canadian equivalent) Grades 6 and 7 were participants in this strand. The second strand concerned data collected in a large, urban high school in

Vancouver, a single mathematics teacher and a group of her Grade 9 students being the focus of this strand. In order to develop an understanding of each classroom environment, I engaged in detailed analyses of video-recorded lessons, field-notes, copies of students' work, my own journal entries, and video-recorded interviews with the 'Vancouver' teacher, Karen, and with the student participants in both strands of the study.

MATHEMATICAL UNDERSTANDING

In order to understand the nature of the unfolding mathematical understandings of the students in my study I adopted the Dynamical Theory for the Growth of Mathematical Understanding (Pirie & Kieren, 1994) as a theoretical tool for analysis. Consistent with an enactivist perspective, the Dynamical Theory for the Growth of Mathematical Understanding considers mathematical understanding as an on-going process in which a learner responds to the problem of reorganising his or her knowledge structures by continually revisiting existing understandings. Pirie and Kieren have termed this process "folding back". The theory considers understanding in terms of a set of embedded levels or modes of knowledge-building activity. These modes are illustrated in diagrammatic form in Figure 1. Pirie and Kieren stress, however, that it is not the modes themselves that define the growth of mathematical understanding, rather it is the non-linear pathways of students' behaviours, which can be tracked through the modes, that illustrate dynamical growth. Pirie and Kieren maintain that growth in understanding involves multiple and varied actions of folding back to inner less formal understanding in order to use that "thicker" understanding as a springboard to the construction of more sophisticated outer level understanding.



Key:

PK Primitive Knowing
IM Image Making
IH Image Having
PN Property Noticing

F Formalising
O Observing
S Structuring
I Inventising
id image doing
ir image reviewing
ise image seeing
isa image saying
pp property predicting
pr property recording

Figure 1: The Dynamical Theory for the Growth of Mathematical Understanding

In videotaping children learning mathematics I tend to focus the camera on a pair or small group of students for an extended period of time. I do this in preference to videotaping a whole class so that particular students' growth of mathematical understanding can be traced. These traces, known as mappings, were created using an adaptation of a model described by Pirie and Kieren in their Dynamical Theory for the Growth of Mathematical Understanding. Appendix A shows one such mapping for one of the students in my study, Kayleigh. Readers will notice that I have moved away from using the embedded rings to represent the various modes of understanding, however, this is not a conceptual shift. I continue to recognise the modes as embedded within one another. This rather more linear representation began as a pragmatic move resulting from the difficulty I began to face as I tried to fit a representation of a student's understanding over the period of several months onto the diagrammatic form favoured by Pirie and Kieren. However, rather than restricting my analysis, I feel that this development enabled patterns to unfold that may not have been so evident on the conventional mapping diagram. For instance, on completing Kayleigh's mapping my attention was immediately drawn to the pattern of growth of understanding revealed in the fourth lesson in the sequence. I began to wonder what forms of teaching interventions might have occasioned such growth, and so I began to concentrate my attention on the teachers' interventions.

TEACHERS' INTERVENTIONS

In a manner consistent with Glaser and Strauss' (1967) constant comparative method, I developed fifteen intervention themes to describe the teachers' actions-in-the-moment[5]. In beginning to respond to my research question I drew together the students' mappings with the teachers' interventions as I had characterised them in terms of these themes. I paid particular attention to turning points in the pathways of growth of understanding revealed by the mapping diagrams – those moments when a student extended to an outer mode of understanding, or folded back to an inner one. I then attended to the teachers' interventions leading up to those turning points and began to search for patterns.

I was faced with a dilemma, however. How was I to comment upon the ways in which teachers' interventions occasion students' mathematical understanding without relying on simplistic analyses of cause and effect? I wanted my interpretations to reflect the complexity of the phenomena into which I was enquiring, and so I turned again to the literature on enactivism to help me to understand and explain the interactions I had documented.

Enactivism focuses on the dynamic interdependence of individual and environment rather than on their autonomous constitution (Davis, 1996), and sees the individual and environment as bound together in reciprocal specification (Varela, Thompson & Rosch, 1991). This position, which draws on the evolutionary metaphors of Darwin rather than the analytic and reductionist thinking of Descartes, recognises the futility of separating what we do (as teachers and learners together) from who we are and what we know. Reflecting on such thinking, Capra (1996, p.41), however, raises an important question. If everything is connected to everything else, how can we ever hope to understand anything?

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[5] The intervention themes I identified were: showing and telling, leading, shepherding, checking, reinforcing, inviting, clue-giving, managing, enculturating, blocking, modelling, praising, rug-pulling, retreating, and anticipating. Though I do not have space to define and describe each of these themes here, interested readers are welcome to contact me directly for more information.

I recognised that although enactivist theorists begin and end their analyses with an acknowledgement of the fundamental inextricability of all things (Davis, Sumara & Kieren, 1996), this is not to say that we cannot or should not reflect upon such notions as cause and effect. In fact, enactivism gives us a new language with which to explore these phenomena. Maturana and Varela, who have laid the groundwork for much of the current enactivist discourse, note that:

> the perturbations of the environment do not determine what happens to the living being; rather, it is the structure of the living being that determines what change occurs in it. This interaction is not instructive, for it does not determine what its effects are going to be....[T]he changes that result from the interaction between the living being and its environment are brought about by the disturbing agent but *determined by the structure of the disturbed system.* (1992, pp. 95-96, original emphasis)

It follows that the growth of students' understanding can be interpreted as being *dependent on*, but not *determined by*, the actions of the teacher (Davis, 1996). Traditional thinking placed teachers in a dominant role in classrooms, assuming when the desired end product of "understanding" was reached that the teaching *caused* the learning. Enactivism rejects this formulation and not only challenges the view of understanding as a possibly achievable end-state but also challenges the assumption that teaching *causes* learning. Davis, Sumara and Kieren (1996) suggest that Varela's analogy of a wind chime is useful in thinking about the issue of causality in learning. Varela asks the reader to imagine a wind chime made with thin pieces of glass dangling like leaves off branches:

> Clearly, how the [wind chime] sounds is not determined or instructed by the wind or the gentle push we may give it. The way it sounds has more to do with ... the kinds of structural configurations it has when it receives a perturbation or imbalance. Every [wind chime] will have a typical melody and tone proper to its constitution. In other words, it is obvious in this example that in order to understand the sound patterns we hear, we turn to the nature of the chimes, and not to the wind that hits them. (1992, p. 50)

I do not want to suggest that the teacher is rendered powerless in an enactivist formulation, for, like the wind hitting the wind chime, the teacher certainly has a part to play in occasioning students' understanding. My study suggested that what and how the students learned was indeed dependent on the teaching but not determined by it, evidenced by the fact that similar (though not identical) patterns of growth of understanding were seen to be enacted by a range of students in response to particular teacher intervention styles. This finding strengthened my conviction that studies of teachers' interventions and how they occasion students' understanding do not have to confine themselves to simplistic descriptions of cause and effect. I returned to my data with renewed vigour.

I noticed that certain intervention types appeared to occasion the growth of students' mathematical understanding whilst others seemed to inhibit growth. Whilst I do not have space here to record all of my findings in this regard, I would like to draw the reader's attention to three of the intervention themes. These three themes, *showing and telling*, *leading*, and *shepherding*, represent broad intervention *styles*, in contrast to the remaining twelve intervention *strategies*. Intervention styles, as I defined them, are broad practices that appear extensively within a particular teacher's activity. Karen and I tended to draw upon one of these three styles predominantly and the others less frequently. An intervention strategy, on the other hand, I characterised as a brief intervention, and Karen and I appeared to have a repertoire of many of these strategies upon which to draw. I defined *showing and telling* as an extended stream of interventions often involving the giving of new information but usually without the teacher checking that the students are following the explanation. I defined *leading* as an extended stream of interventions aimed at directing the student(s) towards a specific answer or position, often involving step-by-step explanations. Leading differs from showing and telling in that the teacher often attempts to involve the student(s) in the explanation through frequent questioning. I defined *shepherding* as an extended stream of interventions directing a student towards understanding through subtle nudging, coaxing, and prompting. The inclusion of the word 'understanding' in the definition of shepherding (and its deliberate omission in the other two definitions) announces a critical difference between shepherding and the other themes. Common to each of the shepherding episodes was a *search for understanding entwining the participants*. It is significant to note that of the three intervention styles I identified, only shepherding consistently appeared to occasion the growth of mathematical understanding.

TO TELL OR NOT TO TELL

It is also significant that showing and telling and leading were the dominant intervention styles in both strands of the data. These two styles involve patterns of interaction whereby the teacher is the predominant speaker and all exchanges are mediated through him or her. For the two teachers in this study, shepherding appeared to be a more unnatural and difficult form of teaching. For a variety of reasons, showing and telling and leading may appeal to teachers as safe options. Nevertheless, there was evidence within my study that for both teachers there was a desire not to tell, and this desire seemed to be at the root of both the leading and shepherding styles. The evidence I collected suggests that not telling is very difficult. Both teachers appeared unwilling to allow students to struggle for long, if at all. Clearly, then, the urge to tell tempered by the desire not to constitutes a dilemma for teachers.

As Chazan and Ball (1995) have noted, reform efforts too often exhort teachers to avoid telling without suggesting anything that they might do instead. Further, Chazan and Ball (1995) have called for a more complex, explicit, and contextualised characterisation of the roles teachers play in classroom interaction, and for a more precise means of describing the telling that teachers do. I offer these intervention styles and strategies as a response to Chazan and Ball's call for a new vocabulary, and as a suggestion to teachers for alternative teaching "gambits" (Mason, 1999).

To conclude, I draw on an appeal made by Chazan and Ball who say, "we hope that the development of vocabularies for describing the teacher's role...will enhance opportunities for sustained, critical, and insightful discourse among researchers, teachers, and teacher educators" (1995, p. 23). It is my hope that the implications to be drawn from my study will encourage such a discourse.

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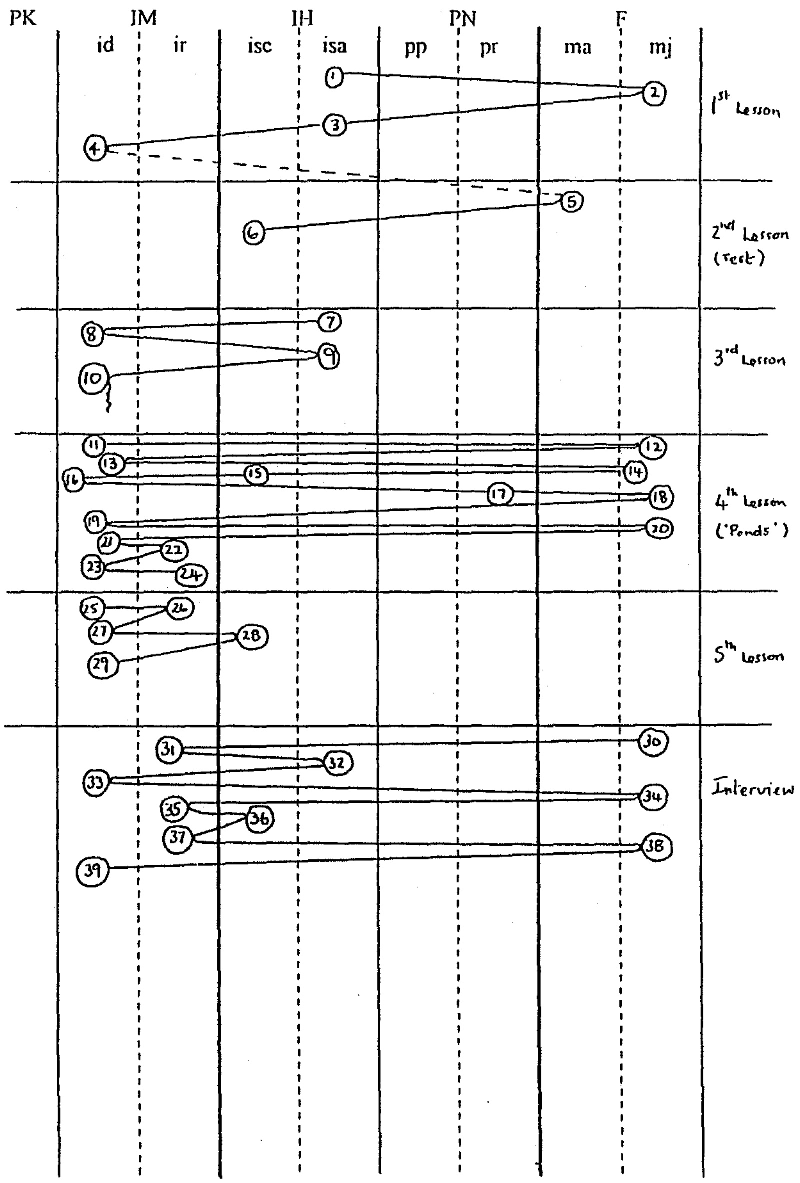
APPENDIX A

KAYLEIGH'S MAPPING DIAGRAM



APPENDICES

APPENDIX A

WORKING GROUPS AT EACH ANNUAL MEETING

1977 Queen's University, Kingston, Ontario
- Teacher Education programmes
- Undergraduate mathematics programmes and prospective teachers
- Research and mathematics education
- Learning and teaching mathematics

1978 Queen's University, Kingston, Ontario
- Mathematics courses for prospective elementary teachers
- Mathematization
- Research in mathematics education

1979 Queen's University, Kingston, Ontario
- Ratio and proportion: a study of a mathematical concept
- Minicalculators in the mathematics classroom
- Is there a mathematical method?
- Topics suitable for mathematics courses for elementary teachers

1980 Université Laval, Québec, Québec
- The teaching of calculus and analysis
- Applications of mathematics for high school students
- Geometry in the elementary and junior high school curriculum
- The diagnosis and remediation of common mathematical errors

1981 University of Alberta, Edmonton, Alberta
- Research and the classroom
- Computer education for teachers
- Issues in the teaching of calculus
- Revitalising mathematics in teacher education courses

1982 Queen's University, Kingston, Ontario
- The influence of computer science on undergraduate mathematics education
- Applications of research in mathematics education to teacher training programmes
- Problem solving in the curriculum

1983 University of British Columbia, Vancouver, British Columbia
- Developing statistical thinking
- Training in diagnosis and remediation of teachers
- Mathematics and language
- The influence of computer science on the mathematics curriculum

1984 University of Waterloo, Waterloo, Ontario
- Logo and the mathematics curriculum
- The impact of research and technology on school algebra
- Epistemology and mathematics
- Visual thinking in mathematics

1985 Université Laval, Québec, Québec
Lessons from research about students' errors
Logo activities for the high school
Impact of symbolic manipulation software on the teaching of calculus

1986 Memorial University of Newfoundland, St, John's, Newfoundland
The role of feelings in mathematics
The problem of rigour in mathematics teaching
Microcomputers in teacher education
The role of microcomputers in developing statistical thinking

1987 Queen's University, Kingston, Ontario
Methods courses for secondary teacher education
The problem of formal reasoning in undergraduate programmes
Small group work in the mathematics classroom

1988 University of Manitoba, Winnipeg, Manitoba
Teacher education: what could it be
Natural learning and mathematics
Using software for geometrical investigations
A study of the remedial teaching of mathematics

1989 Brock University, St. Catharines, Ontario
Using computers to investigate work with teachers
Computers in the undergraduate mathematics curriculum
Natural language and mathematical language
Research strategies for pupils' conceptions in mathematics

1990 Simon Fraser University, Vancouver, British Columbia
Reading and writing in the mathematics classroom
The NCTM "Standards" and Canadian reality
Explanatory models of children's mathematics
Chaos and fractal geometry for high school students

1991 University of New Brunswick, Fredericton, New Brunswick
Fractal geometry in the curriculum
Socio-cultural aspects of mathematics
Technology and understanding mathematics
Constructivism: implications for teacher education in mathematics

1992 ICME-7, Université Laval, Québec, Québec

1993 York University, Toronto, Ontario
Research in undergraduate teaching and learning of mathematics
New ideas in assessment
Computers in the classroom: mathematical and social implications
Gender and mathematics
Training pre-service teachers for creating mathematical communities in the classroom

1994 University of Regina, Regina, Saskatchewan
Theories of mathematics education
Preservice mathematics teachers as pruposeful learners: issues of enculturation
Popularizing mathematics

1995 University of Western Ontario, London, Ontario
Anatomy and authority in the design and conduct of learning activity
Expanding the conversation: trying to talk about what our theories don't talk about
Factors affecting the transition from high school to university mathematics
Geometric proofs and knowledge without axioms

1996 Mount Saint Vincent University, Halifax, Nova Scotia
Teacher education: challenges, opportunities and innovations
What is dynamic algebra?
The role of proof in post-secondary education
Formation à l'enseignement des mathématiques au secondaire: nouvelles perspectives et défis

1997 Lakehead University, Thunder Bay, Ontario
Awareness and expression of generality in teaching mathematics
Communicating mathematics
The crisis in school mathematics content

1998 University of British Columbia, Vancouver, British Columbia
Assessing mathematical thinking
From theory to observational data (and back again)
Bringing ethnomathematics into the classroom in a meaningful way
Mathematical software for the undergraduate curriculum

1999 Brock University, St. Catharines, Ontario
Applied mathematics in the secondary school curriculum
Elementary mathematics
Teaching practices and teacher education
Information technology and mathematics education: What's out there and how can we use it?

APPENDIX B

PLENARY LECTURES

1977	A.J. Coleman	The objectives of mathematics education
	C. Gaulin	Innovations in teacher education programmes
	T.E. Kieren	The state of research in mathematics education
1978	G.R. Rising	The mathematician's contribution to curriculum development
	A.I. Weinzweig	The mathematician's contribution to pedagogy
1979	J. Agassi	The Lakatosian revolution*
	J.A. Easley	Formal and informal research methods and the cultural status of school mathematics*
1980	C. Cattegno	Reflections on forty years of thinking about the teaching of mathematics
	D. Hawkins	Understanding understanding mathematics
1981	K. Iverson	Mathematics and computers
	J. Kilpatrick	The reasonable effectiveness of research in mathematics education*
1982	P.J. Davis	Towards a philosophy of compuation*
	G. Vergnaud	Cognitive and developmental psychology and research in mathematics education*
1983	S.I. Brown	The nature of problem generation and the mathematics curriculum
	P.J. Hilton	The nature of mathematics today and implications for mathematics teaching*
1984	A.J. Bishop	The social construction of meaning: a significant development for mathematics education?*
	L. Henkin	Linguistic aspects of mathematics and mathematics instruction
1985	H. Bauersfeld	Contributions to a fundamental theory of mathematics learning and teaching
	H.O. Pollak	On the relation between the applications of mathematics and the teaching of mathematics
1986	R. Finney	Professional applications of undergraduate mathematics
	A.H. Schoenfeld	Confessions of an accidental theorist*
1987	P. Nesher	Formulating instructional theory: the role of students' misconceptions*
	H.S. Wilf	The calculator with a college education
1988	C. Keitel	Mathematics education and technology*
	L.A. Steen	All one system

1989	N. Balacheff	Teaching mathematical proof: the relevance and complexity of a social approach
	D. Schattsneider	Geometry is alive and well
1990	U. D'Ambrosio	Values in mathematics education*
	A. Sierpinska	On understanding mathematics
1991	J .J. Kaput	Mathematics and technology: multiple visions of multiple futures
	C. Laborde	Approches théoriques et méthodologiques des recherches Francaises en didactique des mathématiques
1992	ICME-7	
1993	G.G. Joseph	What is a square root? A study of geometrical representation in different mathematical traditions
	J. Confrey	Forging a revised theory of intellectual development Piaget, Vygotsky and beyond*
1994	A. Sfard	Understanding = Doing + Seeing ?
	K. Devlin	Mathematics for the twenty-first century
1995	M. Artigue	The role of epistemological analysis in a didactic approach to the phenomenon of mathematics learning and teaching
	K. Millett	Teaching and making certain it counts
1996	C. Hoyles	Beyond the classroom: The curriculum as a key factor in students' approaches to proof
	D. Henderson	Alive mathematical reasoning
1997	R. Borasi	What does it really mean to teach mathematics through inquiry?
	P. Taylor	The high school math curriculum
	T. Kieren	Triple embodiment: Studies of mathematical understanding-in-interaction in my work and in the work of CMESG/GCEDM
1998	J. Mason	Structure of attention in teaching mathematics
	K. Heinrich	Communicating mathematics or mathematics storytelling
1999	J. Borwein	The impact of technology on the doing of mathematics
	W. Whiteley	The decline and rise of geometry in 20th century North America
	W. Langford	Industrial mathematics for the 21st Century
	J. Adler	What counts? Resourcing mathematical practice in the South African school classroom
	B. Barton	An archaeology of mathematical concepts: Sifting languages for mathematical meanings

*These lectures, some in a revised form, were subsequently published in the journal *For the Learning of Mathematics.*

APPENDIX C

PROCEEDINGS OF ANNUAL MEETINGS OF CMESG/GCEDM

Past proceedings of the Study Group have been deposited in the ERIC documentation system with call numbers as follows:

Proceedings of the 1980 Annual Meeting	ED 204120
Proceedings of the 1981 Annual Meeting	ED 234988
Proceedings of the 1982 Annual Meeting	ED 234989
Proceedings of the 1983 Annual Meeting	ED 243653
Proceedings of the 1984 Annual Meeting	ED 257640
Proceedings of the 1985 Annual Meeting	ED 277573
Proceedings of the 1986 Annual Meeting	ED 297966
Proceedings of the 1987 Annual Meeting	ED 295842
Proceedings of the 1988 Annual Meeting	ED 306259
Proceedings of the 1989 Annual Meeting	ED 319606
Proceedings of the 1990 Annual Meeting	ED 344746
Proceedings of the 1991 Annual Meeting	ED 350161
Proceedings of the 1993 Annual Meeting	ED 407243
Proceedings of the 1994 Annual Meeting	ED 407242
Proceedings of the 1995 Annual Meeting	ED 407241
Proceedings of the 1996 Annual Meeting	ED 425054
Proceedings of the 1997 Annual Meeting	ED 423116
Proceedings of the 1998 Annual Meeting	not available

There was no Annual Meeting in 1992 because Canada hosted the Seventh International Conference on Mathematical Education that year.

*The 1999 *Proceedings* have been submitted to ERIC.