CANADIAN MATHEMATICS EDUCATION STUDY GROUP

GROUPE CANADIEN D'ÉTUDE EN DIDACTIQUE DES MATHÉMATIQUES

PROCEEDINGS / ACTES
2002 ANNUAL MEETING

Queen's University
May 24 – 28, 2002

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26th Annual Meeting
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Acknowledgements

On behalf of the members, the CMESG / GCEDM Executive would like to take this opportunity to thank our local hosts for their contributions to the 2002 Annual Meeting and Conference. Specifically, thank you to Peter Taylor, Marg Lambert, Geoff Roulet, Morris Orzech, Bill Higginson, Lynda Colgan, Joan McDuff, and Lionel LaCroix.

On behalf of our membership, we would also like to thank the guest speakers, working group leaders, topic group and ad hoc presenters, and all of the participants. You are the ones who made this meeting an intellectually stimulating and worthwhile experience.

Supplementary materials to some of the contributions in these *Proceedings* are posted on the CMESG /GCEDM website (http://www.cmesg.math.ca), maintained by David Reid.

Schedule

	Friday May 24	Saturday May 25	Sunday May 26	Monday May 27	Tuesday May 28
AM		**09h00 - 12h15** Working Groups	**09h00 - 12h15** Working Groups	**09h00 - 12h15** Working Groups	**9h00 - 11h00** Anniversary Session 3 **11h30 - 12h30** Closing Session
Lunch		**12h15 - 13h30**	**12h15 - 13h30**	**12h15 - 13h30**	
PM	**15h30 - 16h30** Registration **16h30 - 18h30** Opening Session	**13h30 - 14h00** Small group discussion of Plenary 1 **14h00 - 15h00** Questions for Ball & Bass **15h30 - 16h30** Anniversary Session 1 **16h30 - 17h30** New PhDs	**13h30 - 14h30** Plenary 2: Borwein **14h30 - 15h00** Small group discussion of Plenary 2	**13h30 - 14h30** Questions for Borwein **14h30 - 15h30** Topic Groups **16h00 - 17h00** Anniversary Session 2 **17h00 - 18h00** Annual General Meeting	
Supper	**18h30 - 20h00**	**18h00** President's Dinner (Fort Henry)	**17h00** Boat Trip to Gananoque Theatre	**18h00 - 19h30**	
Evening	**20h00 - 21h00** Plenary 1: Ball & Bass **21h00** Dessert & Social			**20h00 - 22h00** Displays, Ad Hocs, & Dessert	

Introduction

Malgorzata Dubiel - President, CMESG/GCEDM
Simon Fraser University

It is my great pleasure to write an introduction to the *CMESG/GCEDM Proceedings* from the 2002 meeting, held at Queen's University in Kingston, Ontario.

A necessary part of the introduction to the *CMESG/GCEDM Proceedings* is an attempt to explain to readers, some of whom may be newcomers to our organization, that the volume in their hands cannot possibly convey the spirit of the meeting it reports on. It can merely describe the content of activities without giving much of the flavour of the process.

To understand this, one needs to understand the uniqueness of both our organization and our annual meetings.

CMESG is an organization unlike other professional organizations. One belongs to it not because of who one is professionally, but because of one's interests. And that is why our members come from mathematics departments and faculties of education, from universities and colleges, and from schools and other educational institutions. What unites them is their interest in mathematics and how it is taught at every level, and their desire to make teaching more exciting, more relevant, and more meaningful.

The 2002 meeting was a special one: This was our 25th meeting which was held at the very place where our first three meetings happened. Since such an anniversary is an excellent opportunity for both celebrations and reflections, the meeting included, in addition to all the regular components, several special sessions which followed the central theme: Lessons from the Past, Questions for the Future.

The opening session included a panel which brought back three keynote speakers from the first CMESG/GCEDM in 1997: John Coleman, Claude Gaulin, and Tom Kieren. All three were asked to reflect on the topics of their 1977 talks from today's point of view.

Tom Kieren, whose 1977 talk was titled "The state of research in mathematics education", spoke first. He reflected on some of the work done during the part 25 years in math education in Canada, on mathematical learning, on the development of mathematical knowing and understanding. And he also recalled the words of David Wheeler, the "spiritual father" of our organization: "Could we, as CMESG, publish something others will be interested in?"

Claude Gaulin, the only one to have attended all our 25 meetings, spoke in 1977 about innovations in teacher education programmes. In 2002, he remembered how he spoke then about teacher education in Quebec, where in-service education was (and still is) very strong and world renowned. He said that we need a broad picture on what is happening in teacher education—both in-service and pre-service—across Canada, and to reflect on what is needed now.

John Coleman, the one mathematician in the panel, spoke in 1977 on the objectives of math education. He said that the objectives are basically the same now as they were then. Possibly the most important one is to have students learn the joy of "mathematicizing". He cited a statement from the book "Enjoying Mathematics" by Keith Devlin, which was recently reviewed in *Mathematics Monthly*, to the effect that 85% of elementary school teachers in North America have math anxiety. This is something we need to change. Students pick up emotions quickly and they often "inherit" their attitudes towards mathematics from their teachers.

Three special Anniversary Sessions continued the theme Lessons from the Past, Questions for the Future throughout the conference.

The first session featured two mathematicians, Frédéric Gourdeau and Eric Muller, describing what our organization means to them and how it has influenced their understandings of the nature of mathematics, its teaching, and learning. Frédéric also shared with us some of his frustrations on being a mathematician in a group of math educators, trying to understand their view of the world and being bewildered by their language.

In the second session, Carolyn Kieran gave us a historical perspective on the emergence and growth of a community of mathematics education researchers in Canada.

In the third, Brent Davis and Roberta Mura attacked the topic of Lessons from the Past, Questions for the Future from the math education point of view. Or was it more a philosophy of math education perspective? Questions from the past and questions about the past: Can we learn from the past? Can we change the past? Can we change the future by changing the past? These, and many others questions asked there, will stay with us to reflect on for the next 25 years.

The most festive event of the anniversary celebrations was the Presidents' Dinner, held at the historic Fort Henry, with toasts to all the CMESG/GCDEM presidents and treasurers.

But of course all the exciting anniversary events were only additions to the traditional components of our meetings.

Working Groups form the core of each CMESG meeting. Participants choose one of several possible topics, and, for three days, become members of a community which meets three hours a day to exchange ideas and knowledge—and, through discussions which often continue beyond the allotted time, create fresh knowledge and insights. Throughout the three days, the group becomes much more than a sum of its parts, often in ways totally unexpected to its leaders. The leaders, after working for months prior to the meeting, may see their carefully prepared plan ignored or put aside by the group, and a completely new picture emerging in its stead.

Two plenary talks are traditionally part of the conference, at least one of which is given by a speaker invited from outside Canada who brings a non-Canadian perspective. And, traditionally, one of the talks is given by a mathematician, the other, by a math educator. This year, our non-Canadian talk turned out to be a double pleasure—two speakers instead of one, speaking of their collaboration and presenting both a mathematician's and a math educator's perspective on their topic.

All plenary speakers participate in the whole meeting; some of them afterwards become part of the Group. And, in the spirit of CMESG meetings, a plenary talk is not just a talk, but a mere beginning: it is followed by discussions in small groups, in which questions are prepared for the speaker. After the small group discussions, in a renewed plenary session, the speaker fields the questions generated by the groups.

Topic Groups and Ad Hoc presentations provide more possibilities for exchange of ideas and reflections. Shorter in duration than the Working Groups, Topic Groups are sessions where individual members present work in progress and often find inspiration and new insight from their colleagues' comments.

Ad hoc sessions are opportunities to share ideas, which are often not even "half-baked"—sometimes born during the very meeting at which they are presented.

A traditional part of each meeting is the recognition of new PhDs. Those who completed their dissertations in the last year are invited to speak on their work. This gives the group a wonderful opportunity to see the future of mathematics education in Canada.

The 2002 meeting in Kingston was a memorable one, both due to all the anniversary events, and to the great program of the conference itself. We all took back with us wonderful memories of the hospitality of Queen's University and of a great, thought-provoking program. Our warmest thanks to the local organizers, Peter Taylor and Bill Higginson, and to David Reid, our conference coordinator, for giving us a meeting which managed to preserve all the traditions of the CMESG meetings and to combine them effortlessly with all the retrospective and anniversary events.

Plenary Lectures

Conférences plénières

Toward a Practice-Based Theory of Mathematical Knowledge for Teaching[1]

Deborah Loewenberg Ball and Hyman Bass[2]
University of Michigan

Mathematics professor: *The situation is terrible: Only one of the students in my mathematics content course for teachers can correctly divide .0045789 by 3.45.*

Fifth grader: *Ms. Ball, I can't remember how to divide decimals. There's something my stepfather showed me about getting rid of the decimal point, but I can't remember what he said and, besides, I don't think that would work.*

With all the talk of teachers' weak mathematical knowledge, we begin with a reminder that the problem on the table is the quality of mathematics teaching and learning, not—in itself—the quality of teachers' knowledge. We seek in the end to improve *students*' learning of mathematics, not just produce *teachers* who know more mathematics.

Why, then, talk about teacher knowledge here? We focus on teacher knowledge based on the working assumption that how well teachers know their subjects affects how well they can teach. In other words, the goal of improving students' learning depends on improving teachers' knowledge. This premise—widely shared as it may be, however—is not well supported empirically. We begin with a brief glimpse of the territory in which the problem on which we are working fits. Our purpose is to set the context for our proposal for reframing the problem.

The Problem: What Mathematics Do Teachers Need to Know to Teach Effectively?

The earliest attempts to investigate the relationship between teachers' mathematics knowledge and their students' achievement met with results that surprised many people. Perhaps the best known among these is Begle's (1979) analysis of the relationship between the number of courses teachers had taken past calculus and student performance. He found that taking advanced mathematics courses[3] produced positive main effects on students' achievement in only 10% of the cases, and, perhaps more unsettling, negative main effects in 8%. That taking courses could be negatively associated with teacher effects is interesting because the negative main effects are not easily explained by the criticism that advanced mathematics courses are not relevant to teaching, or that course-taking is a poor proxy for teachers' actual mathematical knowledge. Such claims support finding *no* effects, but not *negative* effects.

So why might these variables be associated with negative effects? One explanation might rest with the compression of knowledge that accompanies increasingly advanced mathematical work, a compression that may interfere with the unpacking of content that teachers need to do (Ball & Bass, 2000a). Another explanation might be that more coursework in mathematics is accompanied by more experience with conventional approaches to teaching mathematics. Such experience may impress teachers with pedagogical images and habits that do not contribute to their effectiveness with young students (Ball, 1988).

Observational studies of beginning and experienced teachers reveal that teachers' understanding of and agility with the mathematical content does affect the quality of their teaching. For example, Eisenhart, Borko, Underhill, Brown, Jones, and Agard (1993) describe the case of a middle school student teacher, Ms. Daniels, who was asked by a child to explain why the invert-and-multiply algorithm for dividing fractions works. Ms. Daniels tried to create a word problem for three-quarters divided by one-half by saying that three quarters of a wall was unpainted. However, there was only enough paint to cover half of the unpainted area. As she drew a rectangle to represent the wall and began to illustrate the problem, she realized that something was not right. She aborted the problem and her explanation in favor of telling the children to "just use our rule for right now" (p. 198).

Despite having taken two years of calculus, a course in proof, a course in modern algebra, and four computer science courses, Ms. Daniels was unable to provide a correct representation for division of fractions or to explain why the invert-and-multiply algorithm works. In fact, she represented multiplication, rather than division, of fractions.

Many other studies reveal the difficulties teachers face when they are uncertain or unfamiliar with the content. In 1996, the National Commission on Teaching and America's Future (NCTAF) released its report which proposed a series of strong recommendations for improving the nation's schools that consisted of "a blueprint for recruiting, preparing, and supporting excellent teachers in all of America's schools" (p. vi). Asserting that what teachers know and can do is the most important influence on what students learn, the report argues that teachers' knowledge affects students' opportunities to learn and learning. Teachers must know the content "thoroughly" in order to be able to present it clearly, to make the ideas accessible to a wide variety of students, and to engage students in challenging work.

The report's authors cite studies that show that teacher knowledge makes a substantial contribution to student achievement. They argue that "differences in teacher qualifications accounted for more than 90% of the variation in student achievement in reading and mathematics" (Armour-Thomas, Clay, et al., 1989, cited in National Commission on Teaching and America's Future, 1996, p. 8). Still, what constitutes necessary knowledge for teaching remains elusive.

An important contribution to the question of what it means to know content for teaching has been the concept of "pedagogical content knowledge" (Grossman, 1990; Shulman, 1986, 1987; Wilson, Shulman, & Richert, 1987). Pedagogical content knowledge, as Shulman and his colleagues conceived it, identifies the special kind of teacher knowledge that links content and pedagogy. In addition to general pedagogical knowledge and knowledge of the content, teachers need to know things like what topics children find interesting or difficult and the representations most useful for teaching a specific content idea. Pedagogical content knowledge is a unique kind of knowledge that intertwines content with aspects of teaching and learning.

The introduction of the notion of pedagogical content knowledge has brought to the fore questions about the content and nature of teachers' subject matter understanding in ways that the previous focus on teachers' course-taking did not. It suggests that even expert personal knowledge of mathematics often may be inadequate for teaching. Knowing mathematics for teaching requires a transcendence of the tacit understanding that characterizes much personal knowledge (Polanyi, 1958). It also requires a unique understanding that intertwines aspects of teaching and learning with content.

In 1999, Liping Ma's book, *Knowing and Teaching Elementary Mathematics* attracted still broader interest in this issue. In her study, Ma compared Chinese and U.S. elementary teachers' mathematical knowledge. Producing a portrait of dramatic differences between the two groups, Ma used her data to develop a notion of "profound understanding of fundamental mathematics", an argument for a kind of connected, curricularly-structured, and longitudinally coherent knowledge of core mathematical ideas.

What is revealed by the work in the years since Begle's (1979) famous analysis? Although his work failed to show expected connections between teachers' level of mathematics and their students' learning, it seems clearer now that mathematical knowledge for teaching has features that are rooted in the mathematical demands of teaching itself. These are not easily detected by how much mathematics someone has studied. We are poised to make new gains on an old and continuing question: What do teachers need to know to teach mathematics well? But we are poised to make those gains by approaching the question in new ways.

Reframing the Problem: What Mathematical Work Do Teachers Have to *Do* to Teach Effectively?

The substantial efforts to trace the effects of teacher knowledge on student learning, and the problem of what constitutes important knowledge for teaching, led our research group[4] to the idea of working bottom up, beginning with practice. We were struck with the fact that the nature of the knowledge required for teaching is underspecified. On one hand, what teachers need to know seems obvious: They need to know *mathematics*. Who can imagine teachers being able to explain how to find equivalent fractions, answer student questions about primes or factors, or represent place value, without understanding the mathematical content? On the other hand, less obvious is what "understanding mathematical content" *for teaching* entails: *How* do teachers need to know such mathematics? What else do teachers need to know of and about mathematics? And how and where might teachers use such mathematical knowledge in practice?

Hence, instead of investigating what teachers need to know by looking at what they need to teach, or by examining the curricula they use, we decided to focus on their work. What do teachers *do*, and how does what they do demand mathematical reasoning, insight, understanding, and skill? We began to try to unearth the ways in which mathematics is entailed by its regular day-to-day, moment-to-moment demands. These analyses help to support the development of a *practice-based theory of mathematical knowledge for teaching*. We see this approach as a kind of "job analysis", similar to analyses done of other mathematically intensive occupations, from nursing to engineering and physics (Hoyles, Noss, & Pozzi, 2001; Noss, Healy, & Hoyles, 1997), to carpentry and waiting tables. In this case, we ask:

- *What* mathematical knowledge is entailed by the work of teaching mathematics?
- *Where* and *how* is mathematical knowledge used in teaching mathematics? How is mathematical knowledge intertwined with other knowledge and sensibilities in the course of that work?

How We Do Our Work

Central to our work is a large longitudinal NSF-funded database, documenting an entire year of the mathematics teaching in a third grade public school classroom during 1989–90.[5] The records collected across that year include videotapes and audiotapes of the classroom lessons, transcripts, copies of students' written class work, homework, and quizzes, as well as the teacher's plans, notes, and reflections. By analyzing these detailed records of practice, we seek to develop a theory of mathematical knowledge as it is entailed by and used in teaching. We look not only at specific episodes but also consider instruction over time, examining the work of developing both mathematics and students across the school year. What sort of larger picture of a mathematical topic and its associated practices is needed for teaching over time? How do students' ideas and practices develop and what does this imply about the mathematical work of teachers?

A database of the scale and completeness of this archive affords a kind of surrogate for the replicable experiment. More precisely, the close study of small segments of the data supports the making of provisional hypotheses (about teacher actions, about student thinking, about the pedagogical dynamics), and even theoretical constructs. These hypotheses or constructs can then be "tested" and, in principle, refuted, using other data with this archive itself. We can inspect what happened days (or weeks) later, or earlier, or look at a student's notebook, or at the teacher's journal for evidence that confirms or challenges an idea. Further, when theoretical ideas emerge from observations of patterns across the data, we can use them as a lens for viewing other records, of other teachers' practices, and either reinforce or modify or reject our theoretical ideas in line with their adaptability to the new data.

Structured data like those collected in this archive can constitute a kind of public "text" for the study of teaching and learning by a community of researchers. This would permit the discussion of theoretical ideas to be grounded in a publicly shared body of data, inherently connected to actual practice. As norms for such discourse are developed, so also would the expansion of such data sets to support such scholarly communication be encouraged. In our experience, disciplined inquiry focused on such a practice-based "text" tends to dissipate ideologically based disputes, and to assure that theoretical constructs remain connected to practice.

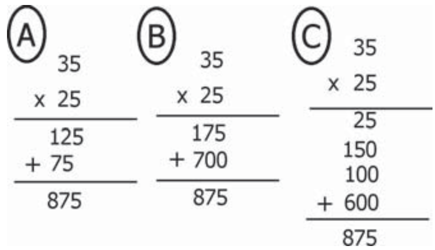
Even with such records of practice in which much is available to be seen, casual observation will no more produce insight about teaching and learning than unsophisticated reading of a good mathematics text will produce mathematical insight. Teaching and learning are complex and dynamic phenomena in which, even with the best of records, much remains hidden and needing interpretation and analysis. Our approach to this work has been to mobilize an interdisciplinary group representing expertise in teaching practice, in disciplinary mathematics, in cognitive and social psychology, and in educational research. Over time we have collectively crafted well-honed skills for sensitive observation of records (particularly video) of teaching practice. One of our research aims is to articulate some of the demands, skills, and norms that this entails; in short, a kind of *methodology of interdisciplinary observation of teaching*.

Our work uses methods of mathematical and pedagogical analysis developed in previous research (see, for example, Ball & Bass, 2000a, b; 2003). Using a framework for examining practice, we focus on mathematics as it emerges within the core task domains of teachers' work. Examples of this work include representing and making mathematical ideas available to students; attending to, interpreting, and handling students' oral and written productions; giving and evaluating mathematical explanations and justifications; and establishing and managing the discourse and collectivity of the class for mathematics learning. As we analyze particular segments of teaching, we seek to identify the mathematical resources used and needed by the teacher. For example, when a student offers an unfamiliar solution, we will look for signs of whether and how the teacher understands the solution, and what he or she did, and what the mathematical moves and decisions are. Our coding scheme includes both mathematical content (topics, procedures, and the like) and practices (mathematical processes and skills, such as investigating equivalence, reconciling discrepancies, verifying solutions, proving claims). The goal of the analysis is twofold: First, to examine how and where mathematical issues arise in teaching, and how that impacts the course of the students' and teacher's work together; and second, to understand in more detail, and in new ways, what elements of mathematical content and practice are used—or might be used—and in what ways in teaching.

What Mathematical Problems Do Teachers Have to Solve?

This approach has led us to a new perspective on the work of mathematics teaching. We see many things teachers do when teaching mathematics that teachers of any subject must do—keep the classroom orderly, keep track of students' progress, communicate with parents, and build relationships with students. Teachers select and modify instructional tasks, make up quizzes, manage discussions, interpret and use curriculum materials, pose questions, evaluate student answers, and decide what to take up and what to leave. At first, these may sound like generic pedagogical tasks. Closer examination, however, reveals that doing them requires substantial mathematical knowledge and reasoning. In some cases, the work requires teachers to think carefully about a particular mathematical idea together with something about learners or learning. In other cases, the work involves teachers in a kind of mathematical reasoning, unencumbered by considerations of students, but applied in a pedagogical context. Our analyses have helped us to see that teaching is a form of mathematical work. Teaching involves a steady stream of mathematical problems that teachers must solve.

Let us consider an example. Teachers often encounter students using methods and solutions different from the ones with which they are familiar. This can arise for a variety of reasons, but when teachers see methods they have not seen before, they must be able to ask and answer—for themselves—a crucial *mathematical* question: What, if any, is the method, and will it work for all cases? No pedagogical decision can be made prior to asking and answering this question. Consider, for example, three alternative methods for multiplying 35 × 25:



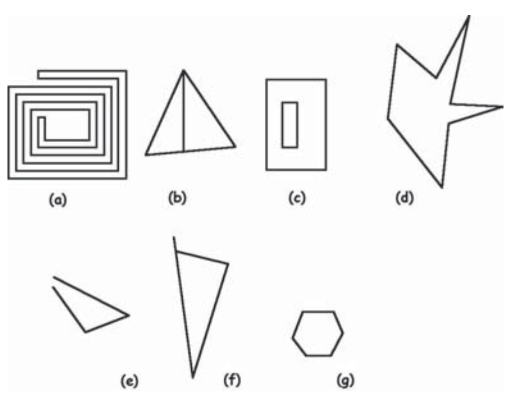
A teacher must be able to ask what is going on in each of these approaches, and to know which of these is a method that works for multiplying any two whole numbers. These are quintessential *mathematical*—<u>not</u> *pedagogical*—questions. Knowing to ask and how to answer such mathematical questions is essential to being able to make wise judgments in teaching. For instance, a decision about whether or not to examine such alternative methods with the students depends on first sizing up the mathematical issues involved in the particular approach, and whether they afford possibilities for worthwhile mathematics learning for these students at this point in time.

Being able to sort out the three examples above requires more of teachers than simply being able to multiply 35 × 25 themselves. Suppose, for example, a teacher knew the method used in (B). If a student produced this solution, the teacher would have little difficulty recognizing it, and could feel confident that the student was using a reliable and generalizable method. This knowledge would not, however, help that same teacher uncover what is going on in (A) or (C).

Take solution (A) for instance. Where do the numbers 125 and 75 come from? And how does 125 + 75 = 875? Sorting this out requires insight into place value (that 75 represents 750, for example) and commutativity (that 25 × 35 is equivalent to 35 × 25), just as solution (C) makes use of distributivity (that 35 × 25 = (30 × 20) + (5 × 20) + (30 × 5) + (5 × 5). Even once the solution methods are clarified, establishing whether or not each of these generalizes still requires justification.

Significant to this example is that a teacher's own ability to solve a mathematical problem of multiplication (35 × 25) is not sufficient to solve the mathematical problem of teaching—to inspect alternative methods, examine their mathematical structure and principles, and to judge whether or not they can be generalized.

Let us consider a second example. This example again helps to make visible the mathematical demands of simple, everyday tasks of teaching. Different from the first, however, it reveals that the mathematical demands are not always so closely aligned to the content outlines of the curriculum (in the example above, multiplication). Suppose that, in studying polygons, students produce or encounter some unusual figures and ask whether any of them is a polygon.



This is an natural mathematical question. Knowing how to answer it involves mathematical knowledge, skill, and appreciation. An essential mathematical move at this point is to consider the definition: What makes a figure a polygon? A teacher should know to consult the textbook's definition, but may well find an inadequate definition, such as this one, found in a current textbook:

> *A closed flat two-dimensional shape whose sides are formed by line segments.*

Knowing that it is inadequate requires appreciating what a mathematical definition needs to do. This one, for example, does not rule out (b) or (c) or (f), none of which is a polygon. But if the textbook definition is unusable, then teachers must know more than a formally correct mathematical definition, such as:

> *A simple closed plane curve formed by straight line segments.*

Teaching involves selecting definitions that are mathematically appropriate and also usable by students at a particular level. For example, fifth graders studying polygons would not know definitions for "simple" or "curve", and therefore would not be able to use this definition to sort out the aberrant figures from those we would call polygons.

To determine a mathematically appropriate and usable definition for "polygon", a teacher might try to develop a suitable definition, better than those found in the available textbooks. Consider this effort:

> *A sequence of three or more line segments in the plane, each one ending where the next one begins, and the last one ending where the first one begins. Except for these endpoints, shared only by two neighboring segments, the line segments have no other points in common.*

This definition, unlike the previous one in the textbook, is mathematically acceptable, as it does properly eliminate (b), (c), and (f), as well as (e). But a teacher would still need to consider whether or not her students can use it. Definitions must be based on elements that are themselves already defined and understood. Do these students already have defined knowledge of terms such as "line segments", "endpoints", and "plane", and do they know what "neighboring" and "in common" mean? In place of "neighboring", would either "adjacent" or "consecutive" be preferable? Knowing definitions for teaching, therefore, requires being able to understand and work with them sensibly, treating them in a way that is consistent with the centrality of definitions in doing and knowing mathematics. Knowing how definitions function, and what they are supposed to do, together with also knowing a well-accepted definition in the discipline, would equip a teacher for the task of developing a definition that has mathematical integrity and is also comprehensible to students. A definition of a mathematical object is useless, no matter how mathematically refined or elegant, if it includes terms that are beyond the prospective user's knowledge.

Teaching requires, then, a special sort of sensitivity to the need for precision in mathematics. Precision requires that language and ideas be meticulously specified so that mathematical problem solving is not unnecessarily impeded by ambiguities of meaning and interpretation. But the need for precision is relative to context and use. For example, a rigorous and precise definition for odd numbers as those numbers of the form $(2k + 1)$, or of even numbers as multiples of two, would not be precise for first graders first encountering the notion of "even number". Because they cannot decode the meaning of $(2k + 1)$ and do not have a definition of "multiple", the elements used to create a precise definition remain obscure and unusable to six-year-olds. Needed for teaching are definitions that are both correct and useful. Knowing what definitions are supposed to do, and how to make or select definitions that are appropriately and usefully precise for students at a certain point, demands a flexible and serious understanding of mathematical language and what it means for something to be *precise*.

Taken together, these two examples show that knowing mathematics in and for teaching includes both elements of mathematics as found *in* the student curriculum—that is, standard computational algorithms, multiplication, and polygons—as well as aspects of knowing and doing mathematics that are less visible in the textbook's table of contents—

sensitivity to definitions or inspecting the generality of a method, for example. These examples also provide a glimpse of how centrally mathematical reasoning and problem solving figure in the work of teaching.

Examples of Mathematical Problems of Teaching

To illustrate ways in which solving mathematical problems is a recurrent part of the work of teaching, we turn next to some examples. Each of our examples was chosen to show different aspects of the mathematical work of teaching, and to develop the portrait of the mathematics that teaching entails, and the ways in which mathematics is used to solve problems of teaching mathematics.

1. *Choosing a task to assess student understanding: Decimals*

   One thing that teachers do is monitor whether or not students are learning. To do that, on an informal basis, they pose questions and tasks that can provide indicators of whether or not students are "getting it".

   Suppose you wanted to find out if your students could put decimal numbers in order. Which of the following lists of numbers would give you best evidence of students' understanding?

a)	.5	7	.01	11.4
b)	.60	2.53	3.14	.45
c)	.6	4.25	.565	2.5

   Obviously, any of these lists of numbers can be ordered. One possible decision, then, is that the string makes no difference—that a correct ordering of any of the lists is as good as any other.

   However, a closer look reveals differences among the lists. It is possible to order (a) and (b) correctly without paying any attention to the decimal point at all. Students who merely looked at the numbers, with no concern for decimal notation, would still put the numbers into the correct order. List (c), however, requires attention to the decimal places: If a student ignored the decimal point, and interpreted the list as a set of whole numbers, he would order the numbers as follows:

   .6 2.5 4.25 .565

   instead of:

   .565 .6 2.5 4.25

   So what sort of mathematical reasoning by the teacher is involved? More than being able to put the numbers in the correct order, required here is an analysis of what there is to understand about order, a central mathematical notion, when it is applied to decimals. And it also requires thinking about how ordering decimals is different from ordering whole numbers. For example, when ordering whole numbers, the number of digits is always associated with the size of the number: Numbers with more digits are larger than numbers with fewer. Not so with decimals. 135 is larger than 9, but .135 is not larger than 9. This mathematical perspective is one that matters for teaching, for, as students learn, their number universe expands, from whole numbers to rationals and integers. Hence, teaching requires considering how students' understanding must correspondingly expand and change.

2. *Interpreting and evaluating students' non-standard mathematical ideas: Subtraction algorithms*

   Teachers regularly encounter approaches and methods with which they are not familiar. Sometimes students invent alternative methods and bring them to their teachers. In other cases, students have been taught different methods.

   Suppose you had students who showed you these methods for multi-digit subtraction. First, you would need to figure out what is going on, and whether it makes sense mathematically. Second, you would want to know whether either of these methods works in general.

$$\begin{array}{r} 307 \\ -\ 168 \\ \hline 2\ \text{-}6\ \text{-}1 \\ \hline 1\ 3\ 9 \end{array} \qquad\qquad \begin{array}{r} \overset{29}{\cancel{30}}{}^{1}7 \\ -\ 168 \\ \hline 139 \end{array}$$

The first method uses integers to avoid the standard, error-prone, method of regrouping. It surely works, for it reduces the algorithm to a simple procedure that relies on the composition of numbers, and does not require "borrowing". The second regroups 307 by regarding it, cleverly, as 30 tens plus 7 ones, to 29 tens and 17 ones. Asking mathematical questions, a teacher might ask himself: Even if the methods work, what would either one look like with a 10-digit number? Do both work as "nicely" with any numbers? Skills and habits for analyzing and evaluating the mathematical features and validity of alternative methods play an important role in this example. Note, once again, that this is different from merely being able to subtract 307 – 168 oneself.

3. *Making and evaluating explanations: Multiplication*

Independent of any particular pedagogical approach, teachers are frequently engaged in the work of mathematical explanation. Teachers explain mathematics; they also judge the adequacy of explanations—in textbooks, from their students, or in mathematics resource books for teachers.

Take a very basic example. In multiplying decimals, say 1.3 × 2.7, one algorithm involves carrying out the multiplication much as if the problem were to multiply the whole numbers 13 and 27. One multiplies, ignoring the decimal points.

$$\begin{array}{r} 1.3 \\ \times\ \ 2.7 \\ \hline 91 \\ 26\ \ \\ \hline 351 \end{array}$$

Then, because the numbers are decimals, the algorithm counts over two places from the right, yielding a product of 3.51.

But suppose one wants to explain why this execution of the algorithm is wrong:

$$\begin{array}{r} 1.3 \\ \times\ \ 2.7 \\ \hline 91 \\ 26 \\ \hline 11.7 \end{array}$$

and to explain why the standard algorithm works? In this typical instance, a student has not "moved over" the 26 on the second line, and has, in addition, simply placed the decimal point in the position consistent with the original problem.

Is it sufficient to explain by saying that the 26 has to be moved over to line up with the 6 under the 9? And to count the decimal places and insert the decimal point two places from the right?

These are not adequate mathematical explanations. Teaching involves explaining why the 26 should be slid over so that the 6 is under the 9: this involves knowing what the 26 actually represents. In whole number multiplication, if this were 13 × 27, then the 26 on the second line would represent the product of 13 and 20, or 260. In this case, the 26 represents the product of 1.3 and 2—260 tenths, or 2.6. Developing sound explanations that justify the steps of the algorithm, and explain their meaning, involves knowing much more about the algorithm than simply being able to perform it. It also involves sensitivity to what constitutes an explanation in mathematics.

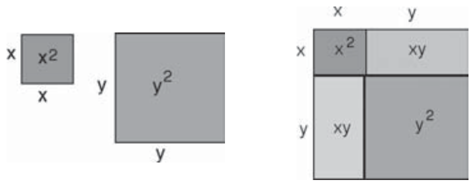
What Does Examining the Work of Teaching Imply About Knowing Mathematics for Teaching?

Standing back from our investigation thus far, we offer three observations. First, our examination of mathematics teaching shows that teaching can be seen as involving substantial mathematical work. Looking in this way can illuminate the mathematics that teachers have to do in the course of their work. Each of these involves mathematical problem solving:

- Design mathematically accurate explanations that are comprehensible and useful for students
- Use mathematically appropriate and comprehensible definitions;
- Represent ideas carefully, mapping between a physical or graphical model, the symbolic notation, and the operation or process;
- Interpret and make mathematical and pedagogical judgments about students' questions, solutions, problems, and insights (both predictable and unusual);
- Be able to respond productively to students' mathematical questions and curiosities;
- Make judgments about the mathematical quality of instructional materials and modify as necessary;
- Be able to pose good mathematical questions and problems that are productive for students' learning;
- Assess students' mathematics learning and take next steps.

Second, looking at teaching as mathematical work highlights some essential features of knowing mathematics *for teaching*. One such feature is that mathematical knowledge needs to be *unpacked*. This may be a distinctive feature of knowledge for teaching. Consider, in contrast, that a powerful characteristic of mathematics is its capacity to *compress* information into abstract and highly usable forms. When ideas are represented in compressed symbolic form, their structure becomes evident, and new ideas and actions are possible because of the simplification afforded by the compression and abstraction. Mathematicians rely on this compression in their work. However, teachers work with mathematics as it is being learned, which requires a kind of decompression, or "unpacking", of ideas. Consider the learning of fractions. When children learn about fractions they do not begin with the notion of a real number, nor even a rational number. They begin by encountering quantities that are parts of wholes, and by seeking to represent and then operate with those quantities.[6] They also encounter other situations that call for fractional notation: distances or points on the number line between the familiar whole numbers, the result of dividing quantities that do not come out "evenly" (e.g., 13 ÷ 4, and later 3 ÷ 4). Across different mathematical and everyday contexts, children work with the elements that come together to compose quantities represented conveniently with fraction notation. Meanwhile, their experiences with the expansion of place value notation to decimals develops another territory that they will later join with fractions to constitute an emergent concept of rational numbers. Teachers would not be able to manage the development of children's understanding with only a compressed conception of real numbers, or formal definition of a rational number. So, although such a conception has high utility for the work of mathematics, it is inadequate for the work of teaching mathematics.

Another important aspect of knowledge for teaching is its connectedness, both across mathematical domains at a given level, and across time as mathematical ideas develop and extend. Teaching requires teachers to help students connect ideas they are learning—geometry to arithmetic, for example. In learning to multiply, students often use grouping: 35 × 25 could be represented with 35 groups of 25 objects. But, for example, to show that 35 × 25 = 25 × 35, or that multiplication is commutative, grouping is not illuminating. More useful is being able to represent 35 × 25 as a rectangular area, with lengths of 25 and 35 and an area of 875 square units. This representation makes it possible to prove commutativity, simply by rotating the rectangle, showing a × b = b × a. Or, later, helping students understand the meaning of $x^2 + y^2$ and how it is different from $(x + y)^2$, it is useful to be able to connect the algebraic notions to a geometric representation:



Using these two diagrams helps to show that $x^2 + y^2$ is not the same as the $x^2 + 2xy + y^2$ produced by multiplying $(x + y)^2$.

Teaching involves making connections across mathematical domains, helping students build links and coherence in their knowledge. This can also involve seeing themes. For example, the regrouping of numbers that is part of the standard multi-digit subtraction algorithm is not unlike the renaming of fractions into equivalent forms. In each case, numbers are written in equivalent forms useful to the mathematical procedure at hand. To add two fractions with unlike denominators, it is useful to be able to rewrite them so that they have the same denominator. In subtraction, to subtract 82 – 38, it is useful to be able to rewrite 82 as "7 12" (7 tens and 12 ones)—also an equivalent form. Seeing this connection is useful in helping students appreciate that, to be strategic and clever in mathematics, quantities can be written in equivalent, useful forms.

Teaching also requires teachers to anticipate how mathematical ideas change and grow. Teachers need to have their eye on students' "mathematical horizons" even as they unpack the details of the ideas in focus at the moment (Ball, 1993). For example, second grade teachers may need to be aware of the fact that saying, "You can't subtract a larger number from a smaller one", is to say something that, although pragmatic when teaching whole number subtraction, is soon to be false. Are there mathematically honest things to say instead that more properly anticipate the expansion to integers, and the accompanying changes in what is true or permissible?[7]

One final observation about what we are finding by examining teaching as mathematical work: In our analyses, we discover that the critical mathematical issues at play in the lesson are not merely those of the curricular topic at hand. For example, in a lesson on subtraction with regrouping, we saw the students grappling with three different representations of subtraction and struggling with whether these were all valid, and, if so, whether and how they represented the *same* mathematical operation. They were examining correspondences among representations, investigating whether or not they were equivalent. Although the content was subtraction, the mathematical entailments of the lesson included notions of equivalence and mapping. In other instances, we have seen students struggling over language, where terms were incompletely or inconsistently defined, and we have seen discussions which run aground because mathematical reasoning is limited by a lack of established knowledge foundational to the point at hand. These lessons brought to the surface important aspects of mathematical reasoning, notation, use of terms and representation. Entailed for the teacher would be both the particular mathematical ideas under discussion as well as these other elements of knowing, learning, and doing mathematics. We have seen many moments where the teachers' attentions to one of these aspects of *mathematical practice* is crucial to the navigation of the lesson, and we have also seen opportunities missed because of teachers' lack of mathematical sensibilities and knowledge of fundamental mathematical practices.

Attending to *mathematical practices as a component of mathematical knowledge* makes sense. As children—or mathematicians, for that matter—do and learn mathematics, they are engaged in using and doing mathematics, as are their teachers. They are representing ideas, developing and using definitions, interpreting and introducing notation, figuring out whether a solution is valid, and noticing patterns. They are engaged in mathematical *practices* as they engage in learning mathematics. For example we often see students whose limited ability to interpret and use symbolic notation, or other forms of representation impedes their work and their learning. Similarly, being able to inspect, investigate, and determine whether two solutions, two representations, or two definitions are similar, or equivalent is fundamental to many arenas of school mathematics. Students and teachers are constantly engaged in situations in which mathematical practices are salient. Yet, to date, studies of mathematical knowledge for teaching have barely probed the surface of what of mathematical practices teachers would need to know and how they would use such knowledge.

Conclusion: Learning Mathematics for Teaching

What we know about teachers' mathematical knowledge, learning mathematics for teaching, and the demands of teaching mathematics suggests the need to reframe the problem of preparing teachers to know mathematics for teaching. First, although many U.S. teachers lack adequate mathematical knowledge, most know some mathematics—especially some basic mathematics. Identifying what teachers know well and what they know less well is an important question for leveraging resources wisely toward the improvement of teachers' opportunities to learn mathematics. What many teachers lack is mathematical knowledge that is useful to and usable for teaching. Of course, some teachers do learn some mathematics in this way from their teaching, from using curriculum materials thoughtfully and by analyzing student work. However, many do not. Inadequate opportunities exist for teachers to learn mathematics in ways that prepare them for the work, and few curriculum materials effectively realize their potential to provide mathematical guidance and learning opportunities for teachers. Also important to realize is that professional mathematicians may often not know mathematics in these ways, either. This is not surprising, for the mathematics they use and the uses to which they put it are different from the mathematical work of teaching children mathematics. They, too, in helping teachers, will have mathematics to learn, and new problems to learn to solve, even as they also contribute resources. This summary suggests that reframing the problem and working on it productively is both promising and challenging.

Our analysis suggests that teachers' opportunities to learn mathematics should include experiences in unpacking familiar mathematical ideas, procedures, and principles. But, as the polygon example shows, learning mathematics for teaching must also afford opportunities to consider other aspects of proficiency with mathematics—such as understanding the role of definitions and choosing and using them skillfully, knowing what constitutes an adequate explanation or justification, and using representations with care. Knowing mathematics for teaching often entails making sense of methods and solutions different from one's own, and so learning to size up other methods, determine their adequacy, and compare them, is an essential mathematical skill for teaching, and opportunities to engage in such analytic and comparative work is likely to be useful for teachers. As we examine the work of teaching, we are struck repeatedly with how much mathematical problem solving is involved. It is mathematical problem solving both like and unlike the problem solving done by mathematicians or others who use mathematics in their work. Practice in solving the mathematical problems they will face in their work would help teachers learn to use mathematics in the ways they will do so in practice, and is likely also to strengthen and deepen their understanding of the ideas. For example, a group of teachers could analyze the three multiplication solutions presented here, determine their validity and generality, map them carefully onto one another. They could also represent them in a common representational context, such as a grid diagram or an area representation of the multiplication of 35×25 (see Ball, 2003).

Seeing teaching as mathematically-intensive work, involving significant and challenging mathematical reasoning and problem solving, can offer a perspective on the mathematical education of teachers, both preservice and across their careers. It opens the door to making professional education of teachers of mathematics both more intellectually and mathematically challenging, and, at the same time, more deeply useful and practical.

Notes

1. This work has been supported by grants from the National Science Foundation (REC # 0126237) and the Spencer Foundation (MG #199800202).
2. The authors acknowledge Heather Hill for her contributions to the ideas discussed in this paper.
3. An "advanced course" was defined as a course past the calculus sequence.

4. Members of the Mathematics Teaching and Learning to Teach Project include Mark Hoover, Jennifer Lewis, Ed Wall, Rhonda Cohen, Laurie Sleep, and Andreas Stylianides.
5. These data were collected under a 1989 National Science Foundation grant to Ball and Magdalene Lampert, then at Michigan State University.
6. Understanding the development of ideas was implied by Dewey in his distinction between the psychological and the logical aspects of subject matter in *The Child and the Curriculum* (1902). By "psychological", he did not mean the way in which a particular idea might be learned, but the epistemological composition of its growth.
7. A group of prospective teachers suggested saying, "We can't subtract larger numbers from smaller ones using the numbers we have right now".

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The Experimental Mathematician: The Pleasure of Discovery and the Role of Proof*

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FIGURE 1.
Einstein's savage certainty

Abstract

The emergence of powerful mathematical computing environments, the growing availability of correspondingly powerful (multi-processor) computers, and the pervasive presence of the internet allow for mathematicians, students, and teachers to proceed heuristically and 'quasiinductively'. We may increasingly use symbolic and numeric computation visualization tools, simulation, and data mining.

Many of the benefits of computation are accessible through low-end 'electronic blackboard' versions of experimental mathematics.[1] This permits livelier classes, more realistic examples, and more collaborative learning. Moreover, the distinction between computing (HPC) and communicating (HPN) is increasingly moot.

- The unique features of our discipline make this both more problematic and more challenging. For example, there is still no truly satisfactory way of displaying mathematical notation on the web; and we care more about the reliability of our literature than does any other science. The traditional role of proof in mathematics is arguably under siege.
- Limned by examples, I intend to pose questions such as follow. And I offer some personal conclusions.

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* Editors' note: The text and images of this paper are drawn from the transparencies used during Jonathan Borwein's presentation. The original transparencies and other resources for the presentation are lodged at: www.cecm.sfu.ca/personal/jborwein/cmesg25.html. The corresponding paper is Borwein (2000), and a forthcoming book is Bailey, Bowein, & Devlin (in preparation).

Questions

- What constitutes secure mathematical knowledge?
- When is computation convincing? Are humans less fallible?
- What tools are available? What methodologies?
- What about the 'law of the small numbers'?
- How is mathematics actually done? How should it be?
- Who cares for certainty? What is the role of proof?

Favourite Examples

Many of my favourite more sophisticated examples originate in the boundary between mathematical physics and number theory and involve the z-function,

$$\zeta(n) = \sum_{k=1}^{\infty} \frac{1}{k^n},$$

and its friends.[2] They rely on the use of *Integer Relations Algorithms*—recently ranked among the 'top ten' algorithms of the century.[3]

Briggs

> ... where almost one quarter hour was spent, each beholding the other with admiration before one word was spoken: at last Mr. Briggs began "My Lord, I have undertaken this long journey purposely to see your person, and to know by what wit or ingenuity you first came to think of this most excellent help unto Astronomy, viz. the Logarithms: but my Lord, being by you found out, I wonder nobody else found it out before, when now being known it appears so easy".[4]

Introduction

Ten years ago I was offered the signal opportunity to found the Centre for Experimental and Constructive Mathematics (CECM) at Simon Fraser. On our website (www.cecm.sfu.ca) I wrote:

> At CECM we are interested in developing methods for exploiting mathematical computation as a tool in the development of mathematical intuition, in hypotheses building, in the generation of symbolically assisted proofs, and in the construction of a flexible computer environment in which researchers and research students can undertake such research. That is, in doing 'Experimental Mathematics'.

CECM

The decision to build CECM was based on:

(i) more than a decade's personal experience, largely since the advent of the personal computer, of the value of computing as an adjunct to mathematical insight and correctness;

(ii) on a growing conviction that the future of mathematics would rely much more on collaboration and intelligent computation;

(iii) that such developments needed to be enshrined in, and were equally valuable for, mathematical education; and

(iv) that experimental mathematics is *fun*.

Ten years later, my colleagues and I are even more convinced of the value of our venture—and the 'mathematical universe is unfolding' much as we anticipated. Our efforts and philosophy are described in some detail in the forthcoming book and in the survey articles.[5] Ten years ago the term 'experimental mathematics' was often treated as an oxymoron. Now there is a highly visible and high quality journal of the same name.

Fifteen years ago, most self-respecting research pure mathematicians would not admit to using computers as an adjunct to research. Now they will talk about the topic whether or not they have any expertise.

The centrality of information technology to our era and the growing need for concrete implementable answers suggests why we have attached the word 'Constructive' to CECM.

Plus ça change

While some things have happened much more slowly than we guessed (e.g., good character recognition (OCR) for mathematics, any substantial impact on classroom parole), others have happened much more rapidly (e.g., the explosion of the world wide web,[6] the quality of graphics and animations, the speed and power of computers).

Crudely, the tools with broad societal or economic value arrive rapidly, those that are interesting primarily in our niche do not.

Research mathematicians for the most part neither think deeply about nor are terribly concerned with either pedagogy or the philosophy of mathematics.

The Aesthetic Impulse

Nonetheless, aesthetic and philosophical notions have always permeated (pure and applied) mathematics. And the top researchers have always been driven by an aesthetic imperative:

> We all believe that mathematics is an art. The author of a book, the lecturer in a classroom tries to convey the structural beauty of mathematics to his readers, to his listeners. In this attempt, he must always fail. Mathematics is logical to be sure, each conclusion is drawn from previously derived statements. Yet the whole of it, the real piece of art, is not linear; worse than that, its perception should be instantaneous. We have all experienced on some rare occasions the feeling of elation in realizing that we have enabled our listeners to see at a moment's glance the whole architecture and all its ramifications. (Emil Artin, 1898–1962)[7]

Aesthetics and Utility

Elsewhere, I have similarly argued for aesthetics before utility.[8] The opportunities to tie research and teaching to aesthetics are almost boundless—at all levels of the curriculum.[9] This is in part due to the increasing power and sophistication of visualization, geometry, algebra and other mathematical software.

That said, in my online lectures and resources and in many of the references one will find numerous examples of the utility of experimental mathematics—such as gravitational boosting.

My Present Aim

In this setting, my primary concern is to explore the relationship between proof (*deduction*) and experiment (*induction*). I shall borrow shamelessly from my earlier writings.

There is a disconcerting pressure at all levels of the curriculum to derogate the role of proof. This is in part motivated by the aridity of some traditional teaching (e.g., Euclid), by the alternatives offered by good software, by the difficulty of teaching and learning the tools of the traditional trade, and perhaps by laziness. My own attitude is perhaps best summed up by a cartoon in a book on learning to program in APL (a very high level language). The blurb above it reads:

> *Remember ten minutes of computation is worth ten hours of thought.*

The blurb below it reads:

Remember ten minutes of thought is worth ten hours of computation.

Just as 'the unlived life is not much worth examining' (Charles Krauthammer et al.), proof and rigour should be in the service of things worth proving.

And equally foolish, but pervasive, is encouraging students to 'discover' fatuous generalizations of uninteresting facts.

Gauss, Hadamard, & Hardy

Three of my personal mathematical heroes, very different men from different times, all testify interestingly on these points and on the nature of mathematics.

Gauss

Carl Friedrich Gauss (1777–1855) once wrote, "I have the result, but I do not yet know how to get it."[10]

One of Gauss's greatest discoveries, in 1799, was the link between the *lemniscate sine* function and the *arithmetic-geometric mean* iteration. This was based on a purely computational observation. The young Gauss wrote in his diary that the result "will surely open up a whole new field of analysis".

He was right, as it pried open the whole vista of nineteenth century elliptic and modular function theory. Gauss's specific discovery, based on tables of integrals provided by Stirling (1692–1770), was that the reciprocal of the integral

$$\frac{2}{\pi}\int_0^1 \frac{dt}{\sqrt{1-t^4}}$$

agreed numerically with the limit of the rapidly convergent iteration given by $a_0 := 1.$, $b_0 := \sqrt{2}$ and computing

$$a_{n+1} \;:=\; \frac{a_n + b_n}{2}, \qquad b_{n+1} \;:=\; \sqrt{a_n b_n}$$

The sequences a_n, b_n have a common limit 1.1981402347355922074....

Which object, the integral or the iteration, is more familiar, which is more elegant—then and now?

... Criteria Change

'Closed forms' have yielded centre stage to 'recursion', much as biological and computational metaphors (even 'biology envy') have replaced Newtonian mental images with Richard Dawkin's 'blind watchmaker'. This experience of 'having the result' is reflective of much research mathematics. Proof and rigour play the role described next by Hadamard.

Likewise, the back-handed complement given by Briggs to Napier underscores that is often harder to discover than to explain or digest the new discovery.

> Hardy asked, 'What's your father doing these days. How about that esthetic measure of his?' I replied that my father's book was out. He said, 'Good, now he can get back to real mathematics'. (Garret Birkhoff)

Hadamard

A constructivist, experimental, and aesthetic driven rationale for mathematics could hardly do better than to start with an insight from Jacques Hadamard:

> The object of mathematical rigor is to sanction and legitimize the conquests of intuition, and there was never any other object for it.[11]

Hadamard (1865–1963) was perhaps the greatest mathematician to think deeply and seriously about cognition in mathematics.[12]

Hadamard is quoted as saying "... in arithmetic, until the seventh grade, I was last or nearly last" which should give encouragement to many young students. He was both the author of *The psychology of invention in the mathematical field* (1945), a book still worth close inspection, and co-prover of the *Prime Number Theorem* (1896):

The number of primes less than n tends to ∞ as does $\frac{n}{\log n}$.

This was one of the culminating results of 19th century mathematics and one that relied on much preliminary computation and experimentation.

In Part Because …

One rationale for experimental mathematics and for heuristic computations is that one generally does not know during the course of research how it will pan out. Nonetheless, one must frequently prove all the pieces along the way as assurance that the project remains on course. The methods of experimental mathematics, alluded to below, allow one to maintain the necessary level of assurance without nailing down all the lemmas. At the end of the day, one can decide if the result merits proof. It may not it may not be the answer one sought, or it just may not be interesting enough.

Hardy's Apology

Correspondingly, G.H. Hardy (1877–1947), the leading British analyst of the first half of the twentieth century, was also a stylish author who wrote compellingly in defense of pure mathematics. In his apologia, *A Mathematician's Apology*,[13] Hardy writes, "All physicists and a good many quite respectable mathematicians are contemptuous about proof". The *Apology* is a spirited defense of beauty over utility: "Beauty is the first test. There is no permanent place in the world for ugly mathematics".

That said, his comment that "Real mathematics … is almost wholly 'useless'" has been overplayed and is now to my mind very dated, given the importance of cryptography and other pieces of algebra and number theory devolving from very pure study. But he does acknowledge that, "If the theory of numbers could be employed for any practical and obviously honourable purpose ...", even Gauss would be persuaded.

The existence of Amazon or Google means that I can be less than thorough with my bibliographic details without derailing anyone who wishes to find the source.

A Striking Example

Hardy, on page 15 of his tribute to Ramanujan entitled *Ramanujan, Twelve Lectures*, gives the so-called 'Skewes number' as a "*striking example of a false conjecture*". The integral

$$\operatorname{li} x = \int_0^x \frac{dt}{\log t}$$

is a very good approximation to $\pi(x)$, the number of primes not exceeding x. Thus, li 10^8 = 5,761,455 while $\pi\,(10^8)$ = 5,762,209. It was conjectured that

$$\operatorname{li} x < \pi(x)$$

holds for all x and indeed it so for many x. Skewes in 1933 showed the first explicit *crossing* at $10^{10^{10^{34}}}$. This is now reduced to a tiny number, a mere 10^{1167}, still vastly beyond direct computational reach or even insight.

The Limits of Reason

Such examples show forcibly the limits on experimentation, at least of a naïve variety. Many will be familiar with the 'Law of large numbers' in statistics. Here we see what some number theorists call the 'Law of small numbers': *all small numbers are special*, many are primes and direct experience is a poor guide.

And sadly or happily, depending on one's attitude, even 10^{1166} may be a small number. In more generality one never knows when the initial cases of a seemingly rock solid pattern are misleading. Consider the classic sequence counting the maximal number of regions obtained by joining *n* points around a circle by straight lines:

$$1, 2, 4, 8, 16, 31, 57, \ldots$$

(Entry A000127 in *Sloane's Encyclopedia*.)

My Own Methodology

As a computational and experimental pure mathematician my main goal is *insight*. Insight demands speed and increasingly parallelism.[14] Extraordinary speed and enough space are prerequisite for rapid verification and for validation and falsification ('proofs and refutations'). One cannot have an 'aha' when the 'a' and 'ha' come minutes or hours apart.

What is 'easy' changes as computers and mathematical software grow more powerful. We see an exciting merging of disciplines, levels, and collaborators. We are more and more able: to marry theory and practice, history and philosophy, proofs and experiments; to match elegance and balance to utility and economy; and to inform all mathematical modalities computationally (analytic, algebraic, geometric, and topological).

This has led us to articulate an *Experimental* Math*odology*, as a philosophy[15] and in practice[16], based on:

(i) meshing computation and mathematics (intuition is acquired not natural);

(ii) visualization (even three is a lot of dimensions). Nowadays we can exploit pictures, animations, *emmersive reality*, sounds and other haptic stimuli; and

(iii) 'caging' and 'monster-barring' (Imre Lakatos's terms for how one rules out exceptions and refines hypotheses).

Two particularly useful components are:

- *graphic checks*: comparing $2\sqrt{y} - y$ and $\sqrt{y}\ln(y)$, $0 < y < 1$ pictorially is a much more rapid way to divine which is larger that traditional analytic methods.
- *randomized checks*: of equations, linear algebra, or primality can provide enormously secure knowledge or counter-examples when deterministic methods are doomed.

All of which is relevant at every level of learning and research. My own works depend heavily on:

(i) *High Precision* (computation of object(s) for subsequent examination);

(ii) *Pattern Recognition of Real Numbers* (e.g., using CECM's Inverse Calculator[17] and 'RevEng'), or *Sequences* (e.g., using Salvy & Zimmermann's 'gfun' or Sloane and Plouffe's *Online Encyclopedia*); and

(iii) Extensive use of *Integer Relation Methods*[18]: *PSLQ* & *LLL* and *FFT*.

Integer Relation methods are an integral part of a wonderful test bed for experimental mathematics. Ruling out things ('exclusion bounds') is, as always in science, often more useful than finding things.

To make more sense of all this it is helpful to discuss the nature of experiment. Peter Medawar[19] usefully distinguishes four forms of scientific experiment:

1. The *Kantian* example: generating "the classical non-Euclidean geometries (hyperbolic, elliptic) by replacing Euclid's axiom of parallels (or something equivalent to it) with alternative forms".
2. The *Baconian* experiment is a contrived as opposed to a natural happening, it "is the consequence of 'trying things out' or even of merely messing about".

3. *Aristotelian* demonstrations: "apply electrodes to a frog's sciatic nerve, and lo, the leg kicks; always precede the presentation of the dog's dinner with the ringing of a bell, and lo, the bell alone will soon make the dog dribble".
4. The most important is *Galilean*: "a critical experiment—one that discriminates between possibilities and, in doing so, either gives us confidence in the view we are taking or makes us think it in need of correction".

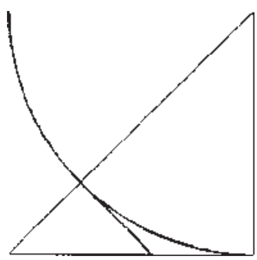
The first three forms of experiment are common in mathematics, the fourth (Galilean) is not. The Galilean Experiment is also the only one of the four forms that has the promise of making Experimental Mathematics a serious replicable scientific enterprise. I'll illustrate this point with some examples.

Two Things About √2

Remarkably one can still find new insights in the oldest areas:

Irrationality. We present graphically, Tom Apostol's lovely new geometric proof[20] of the irrationality of √2.

Proof. To say √2 is rational is to draw a right-angled isosceles triangle with integer sides. Consider the smallest right-angled isosceles triangle with integer sides—that is with shortest hypotenuse. Circumscribe a circle of radius the vertical side and construct the tangent on the hypotenuse, as in the picture.



The *smaller* right-angled isoceles triangle again has integer sides.

This can be beautifully illustrated in a dynamic geometry package such as *Geometer's Sketchpad* or *Cinderella*. We can continue to draw smaller and smaller integer-sided similar triangles until the area drops below $1/2$.

But I give it here to emphasize the ineffably human component of the best proofs, and to suggest the role of the visual.

Rationality. √2 also makes things rational:

$$\left(\sqrt{2}^{\sqrt{2}}\right)^{\sqrt{2}} = \sqrt{2}^{(\sqrt{2}\cdot\sqrt{2})} = \sqrt{2}^{2} = 2.$$

Hence by the principle of the excluded middle:

$$\text{either } \sqrt{2}^{\sqrt{2}} \in \mathbb{Q} \text{ or } \sqrt{2}^{\sqrt{2}} \notin \mathbb{Q}.$$

In either case we can deduce that there are irrational numbers α and β with α^β rational. But how do we know which ones?

This is not an adequate proof for an Intuitionist or a Constructivist.

We may build a whole mathematical philosophy project around this.

DISCOVER versus VERIFICATION

Compare the assertion that

$$\alpha := \sqrt{2} \text{ and } \beta := 2\ln_2(3) \text{ yield } \alpha^\beta = 3$$

... as *Maple* confirms. This illustrates nicely that verification is often easier than discovery. Similarly, the fact multiplication is easier than factorization is at the base of secure encryption schemes for e-commerce.

There are eight possible (ir)rational triples:

$$\alpha^{\beta} = \gamma\,,$$

and finding examples of all cases is now a fine student project.

Integrals & Products

Even *Maple* or *Mathematica* 'knows' $\pi \neq {}^{22}/_{7}$ since

$$0 < \int_0^1 \frac{(1-x)^4 x^4}{1+x^2}\,dx = \frac{22}{7} - \pi,$$

though it would be prudent to ask 'why' it can perform the evaluation and 'whether' to trust it.

In this case, computing $\int_0^t \dots$ provides reassurance. In contrast, *Maple* struggles with the following sophomore's dream:

$$\int_0^1 \frac{1}{x^x}\,dx = \sum_{n=1}^{\infty} \frac{1}{n^n}.$$

Students asked to confirm this, typically mistake numerical validation for symbolic proof:

$$1.291285997 = 1.291285997$$

Similarly

$$\prod_{n=2}^{\infty} \frac{n^3-1}{n^3+1} = \frac{2}{3} \tag{1}$$

is rational, while the seemingly simpler ($n = 2$) case

$$\prod_{n=2}^{\infty} \frac{n^2-1}{n^2+1} = \frac{\pi}{\sinh(\pi)} \tag{2}$$

is irrational, indeed transcendental.

Our Inverse Symbolic Calculator can identify the right-hand side of (2) from its numeric value 0.272029054..., and *Maple* can 'do' both products. But the student learns little or nothing from this unless the software can also recreate the steps of a validation. For example, (1) is a lovely telescoping product (or a 'bunch' of Γ-functions).

That said in each case computing adds reality, making concrete the abstract, and making some hard things simple. This is strikingly the case in *Pascal's Triangle*, which affords an emphatic example where deep fractal structure is exhibited in the elementary binomial coefficients.[21]

David Berlinski[22] writes:

> The computer has in turn changed the very nature of mathematical experience, suggesting for the first time that mathematics, like physics, may yet become an empirical discipline, a place where things are discovered because they are seen.

A sentiment I agree with, unlike others of his, in his *A Tour of the Calculus*.

Partitions & Patterns

The number of additive partitions of n, $p(n)$, is *generated* by

$$1 + \sum_{n\geq 1} p(n) q^n = \frac{1}{\prod_{n\geq 1}(1-q^n)}. \tag{3}$$

Thus, $p(5) = 7$ since

$$5 = 4 + 1 = 3 + 2 = 3 + 1 + 1 = 2 + 2 + 1 = 2 + 1 + 1 + 1 = 1 + 1 + 1 + 1 + 1.$$

Developing (3) is a nice introduction to enumeration via generating functions.

Additive partitions are harder to handle than multiplicative factorizations, but they can be introduced in the elementary school curriculum with questions like:

How many 'trains' of a given length can be built with Cuisenaire rods?

A modern computationally driven question is

How hard is $p(n)$ to compute?

In 1900, the father of combinatorics, Major Percy MacMahon (1854–1929), took months to compute $p(200)$ recursively via (3). By 2000, Maple would produce $p(200)$ in seconds simply by computing the 200th term of the series. A few years earlier, it required one to be careful to compute the series for $\Pi_{n\geq 1}(1 - q^n)$ first and then to compute the series for the reciprocal of that series!

This seemingly baroque event is occasioned by *Euler's pentagonal number theorem*

$$\boxed{\prod_{n\geq 1}(1-q^n) = \sum_{n=-\infty}^{\infty}(-1)^n q^{(3n+1)n/2}.}$$

The reason is that, if one takes the series for (3) directly, the software has to deal with 200 terms on the bottom. But if one takes the series for $\Pi_{n\geq 1}(1 - q^n)$, the software has only to handle the 23 non-zero terms in series in the pentagonal number theorem. This ex post facto algorithmic analysis can be used to facilitate independent student discovery of the pentagonal number theorem, and like results.

Ramanujan used MacMahon's table of $p(n)$ to intuit remarkable and deep congruences such as

$$p(5n + 4) \equiv 0 \mod 5$$
$$p(7n + 5) \equiv 0 \mod 7$$

and

$$p(11n + 6) \equiv 0 \mod 11,$$

... from data like

$$\begin{aligned} P(q) = 1 &+ q + 2q^2 + 3q^3 + 5q^4 + 7q^5 + 11q^6 \\ &+ 15q^7 + 22q^8 + 30q^9 + 42q^{10} + 56q^{11} \\ &+ 77q^{12} + 101q^{13} + 135q^{14} + 176q^{15} \\ &+ 231q^{16} + 297q^{17} + 385q^{18} + 490q^{19} \\ &+ 627q^{20}b + 792q^{21} + 1002q^{22} + \cdots \end{aligned}$$

If introspection fails, we can find the *pentagonal numbers* occurring above in Sloane and Plouffe's on-line *Encyclopedia of Integer Sequences*.[23] Here we see a very fine example of *Mathematics*: *the science of patterns* as is the title of Keith Devlin's book. And much more may similarly be done.

Changing Questions

The difficulty of estimating the size of $p(n)$ analytically—so as to avoid enormous computational effort—led to some marvelous mathematical advances by researchers including Hardy and Ramanujan, and Rademacher.

The corresponding ease of computation may now act as a retardant to mathematical insight. New mathematics is discovered only when prevailing tools run totally out of steam. This raises another caveat against mindless computing: Will a student or researcher discover structure when it is easy to compute without needing to think about it? Today, she may thoughtlessly compute $p(500)$ which a generation ago took much, much pain and insight.

Ramanujan typically saw results not proofs and sometimes went badly wrong for that reason.

So will we all.

Thus, we are brought full face to the challenge—such software should be used, but algorithms must be taught and an appropriate appreciation for and facility with proof developed.

High Precision Fraud

Below '$[x]$' denotes the integer part of x. Consider:

$$\sum_{n=1}^{\infty} \frac{[n \tanh(\pi)]}{10^n} \stackrel{?}{=} \frac{1}{81}$$

is valid to 268 places; while

$$\sum_{n=1}^{\infty} \frac{[n \tanh(\frac{\pi}{2})]}{10^n} \stackrel{?}{=} \frac{1}{81}$$

is valid to 12 places. Both are actually transcendental numbers.

Correspondingly the simple continued fractions for $\tanh(\pi)$ and $\tanh(\pi/2)$ are respectively.

$$[0, 1, 267, 4, 14, 1, 2, 1, 2, 2, 1, 2, 3, 8, 3, 1, \ldots]$$

and

$$[0, 1, 11, 14, 4, 1, 1, 1, 3, 1, 295, 4, 4, 1, 5, 17, 7, \ldots].$$

This is, as they say, no coincidence!

While the reasons are too advanced to explain here, it is easy to conduct experiments to discover what happens when $\tanh(\pi)$ is replaced by another irrational number, say $\log(2)$. It also affords a great example of fundamental objects that are hard to compute by hand (high precision sums or continued fractions) but easy even on a small computer or calculator. Indeed, I would claim that continued fractions fell out of the undergraduate curriculum precisely because they are too hard to work with by hand.

And, of course the main message, is again that computation without insight is mind numbing and destroys learning.

'Pentium Farming' for Bits of π

Bailey, Borwein, & Plouffe (1996) discovered a series for π (and corresponding ones for some other *polylogarithmic constants*) which somewhat disconcertingly allows one to compute hexadecimal digits of π without computing prior digits. The algorithm needs very little memory and no multiple precision. The running time grows only slightly faster than linearly in the order of the digit being computed.

Until then it was broadly considered impossible to compute digits of such a number without computing most of the preceding ones. The key, found as described above, is:

$$\pi = \sum_{k=0}^{\infty} \left(\frac{1}{16}\right)^k \left(\frac{4}{8k+1} - \frac{2}{8k+4} - \frac{1}{8k+5} - \frac{1}{8k+6}\right)$$

Knowing an algorithm would follow they spent several months hunting by computer using integer relation methods[24] for such a formula. Once found, it is easy to prove in *Mathematica*, in *Maple* or by hand—and provides a very nice calculus exercise. This was a most successful case of

REVERSE MATHEMATICAL ENGINEERING

This is entirely practicable. God reaches her hand deep into π: in September 1997, Fabrice Bellard (INRIA) used a variant of this formula to compute 152 binary digits of π, starting at the trillionth position (10^{12})—which took 12 days on 20 work-stations working in parallel over the Internet.

In August 1998, Colin Percival (SFU, age 17) similarly made a naturally or "embarrassingly parallel" computation of the *five trillionth bit* (on 25 machines about 10 times the speed of Bellard's). In *hexedecimal notation*, he got

$$\underline{0}7E45733CC790B5B5979.$$

The corresponding binary digits of π, starting at the *40 trillionth* place, are

$$\underline{0}0000111110011111.$$

By September 2000, the *quadrillionth bit* had been found to be '0' (using 250 cpu years on 1734 machines from 56 countries).

Starting at the 999,999,999,999,997th bit of π, one has:

$$111\underline{0}001100010000101101011000001 10.$$

A Concrete Synopsis

I illustrate some of the mathematical challenges with a specific problem.[25]

10832. Donald E. Knuth, *Stanford University*. Evaluate

$$\sum_{k=1}^{\infty}\left(\frac{k^k}{k!\,e^k}-\frac{1}{\sqrt{2\pi k}}\right).$$

1. A very rapid *Maple* computation yielded $^{-}0.08406950872765600$ ... as the first 16 digits of the sum.
2. The Inverse Symbolic Calculator has a 'smart lookup' feature[26] that replied that this was probably—$\frac{2}{3}-\zeta(\frac{1}{2})/\sqrt{2\pi}$.
3. Ample experimental confirmation was provided by checking this to 50 digits. Thus within minutes we knew the answer.
4. As to why, a clue was provided by the surprising speed with which *Maple* computed the slowly convergent infinite sum. The package clearly knew something the user did not. Peering under the covers revealed that it was using the *LambertW* function, *W*, which is the inverse of $w = z\exp(z)$.[27]
5. The presence of $\zeta(^1/_2)$ and standard Euler-MacLaurin techniques, using Stirling's formula (as might be anticipated from the question), led to

   $$\sum_{k=1}^{\infty}\left(\frac{1}{\sqrt{2\pi k}}-\frac{1}{\sqrt{2}}\frac{(\frac{1}{2})_{k-1}}{(k-1)!}\right)=\frac{\zeta(\frac{1}{2})}{\sqrt{2\pi}}, \tag{4}$$

   where the binomial coefficients in (4) are those of $\frac{1}{\sqrt{2-2z}}$.

   Now (4) is a formula *Maple* can 'prove'.
6. It remains to show

   $$\sum_{k=1}^{\infty}\left(\frac{k^k}{k!\;e^k}-\frac{1}{\sqrt{2}}\frac{(\frac{1}{2})_{k-1}}{(k-1)!}\right)=-\frac{2}{3}. \tag{5}$$
7. Guided by the presence of *W*, and its series

   $$W(z)=\sum_{k=1}^{\infty}\frac{(-k)^{k-1}\,z^k}{k!},$$

an appeal to Abel's limit theorem lets one deduce the need to evaluate

(6) $$\lim_{z\to 1}\left(\frac{d}{dz}W(-\frac{z}{e})+\frac{1}{\sqrt{2-2z}}\right)=\frac{2}{3}.$$

Again *Maple* happily does know (6).

8. Of course, this all took a *fair* amount of human mediation and insight.

TRUTH versus PROOF

By some accounts Percival's web-computation of π is one of the largest computations ever done. It certainly shows the possibility to use inductive engineering-like methods in mathematics, if one keeps one's eye on the ball.

To assure accuracy the algorithm can be run twice starting at different points—say, starting at 40 trillion minus 10. The overlapping digits will differ if any error has been made. If 20 hex-digits agree we can argue heuristically that the probability of error is roughly 1 part in 10^{25}. While this is not a proof of correctness, it is certainly much less likely to be wrong than any really complicated piece of human mathematics.

Fermat's Margins

For example, perhaps 20 people alive can, given enough time, digest alt of Andrew Wiles' extraordinarily sophisticated proof of *Fermat's Last Theorem* and it relies on a century-long program. If there is even a 1% chance that each has overlooked the same subtle error[28]—probably in prior work not explicitly in Wiles' corrected version—then, clearly, many computation-based ventures are much more secure.

This would seem to be a good place to address another common misconception:

No amount of simple-minded case checking constitutes a proof.

Four Colours Suffice

The 1976-7 'proof' of the

> Four Colour Theorem. *Every planar map can be coloured with four colours so adjoining countries are never the same colour*

was a proof because prior mathematical analysis had reduced the problem to showing that a large but finite number of bad configurations could be ruled out.

The proof was viewed as somewhat flawed because the case analysis was inelegant, complicated, and originally incomplete. In the last few years, the computation has been redone after a more satisfactory analysis.[29]

Though many mathematicians still yearn for a simple proof in both cases, there is no particular reason to think that all elegant true conjectures have accessible proofs. Nor indeed, given Goedel's work, need they have proofs at all.

> The message is that mathematics is quasi-empirical, that mathematics is not the same as physics, not an empirical science, but I think it's more akin to an empirical science than mathematicians would like to admit. (Greg Chaitin, 2000)

Kuhn & Planck

Much of what I have described in detail or in passing involves changing set modes of thinking. Many profound thinkers view such changes as difficult:

> The issue of paradigm choice can never be unequivocally settled by logic and experiment alone. … in these matters neither proof nor error is at issue. The transfer of allegiance from paradigm to paradigm is a conversion experience that cannot be forced. (Thomas Kuhn)[30]

... and

> a new scientific truth does not triumph by convincing its opponents and making them see the light, but rather because its opponents die and a new generation grows up that's familiar with it. (Albert Einstein quoting Max Planck)[31]

However hard such paradigm shifts and whatever the outcome of these discourses, mathematics is and will remain a uniquely human undertaking.

Hersh's Humanist Philosophy

Indeed Reuben Hersh's arguments for a humanist philosophy of mathematics, as paraphrased below, become more convincing in our setting:

1. *Mathematics is human*. It is part of and fits into human culture. It does not match Frege's concept of an abstract, timeless, tenseless, objective reality.
2. *Mathematical knowledge is fallible*. As in science, mathematics can advance by making mistakes and then correcting or even re-correcting them. The "fallibilism" of mathematics is brilliantly argued in Lakatos's *Proofs and Refutations*.
3. *There are different versions of proof or rigor*. Standards of rigor can vary depending on time, place, and other things. The use of computers in formal proofs, exemplified by the computer-assisted proof of the *Four Colour Theorem* in 1977, is just one example of an emerging nontraditional standard of rigor.
4. *Empirical evidence, numerical experimentation and probabilistic proof all can help us decide what to believe in mathematics*. Aristotelian logic isn't necessarily always the best way of deciding.
5. *Mathematical objects are a special variety of a social-cultural-historical object*. Contrary to the assertions of certain post-modern detractors, mathematics cannot be dismissed as merely a new form of literature or religion. Nevertheless, many mathematical objects can be seen as shared ideas, like *Moby Dick* in literature, or the Immaculate Conception in religion.[32]

To this I would add that for me now mathematics is not ultimately about proof but about secure mathematical knowledge.

Riemann

Georg Friedrich Bernhard Riemann (1826–1866) was one of the most influential thinkers of the past 200 years. Yet he proved very few theorems, and many of the proofs were flawed.

But his conceptual contributions, such as through Riemannian geometry and the Riemann zeta function, and to elliptic and Abelian function theory, were epochal.

In Conclusion

The experimental method is an addition not a substitute for proof, and its careful use is an example of Hersh's nontraditional standard of rigor. The recognition that 'quasi-intuitive' methods may be used to gain mathematical insight can dramatically assist in the learning and discovery of mathematics. Aesthetic and intuitive impulses are shot through our subject, and honest mathematicians will acknowledge their role. But a student who never masters proof will not be able to profitably take advantage of these tools.

Final Observations

As we have already seen, the stark contrast between the deductive and the inductive has always been exaggerated. Herbert A. Simon wrote:

> This skyhook-skyscraper construction of science from the roof down to the yet unconstructed foundations was possible because the behaviour of the system at each level depended only on a very approximate, simplified, abstracted characterization at the level beneath.[33]

Russell

> "… the chief reason in favour of any theory on the principles of mathematics must always be inductive, i.e., it must lie in the fact that the theory in question allows us to deduce ordinary mathematics. In mathematics, the greatest degree of self-evidence is usually not to be found quite at the beginning, but at some later point; hence the early deductions, until they reach this point, give reason rather for believing the premises because true consequences follow from them, than for believing the consequences because they follow from the premises." Contemporary preferences for deductive formalisms frequently blind us to this important fact, which is no less true today than it was in 1910."
>
> "This is lucky, else the safety of bridges and airplanes might depend on the correctness of the 'Eightfold Way' of looking at elementary particles."

It is precisely this '*post hoc ergo propter hoc*' part of theory-building that Russell so accurately typifies that makes him an articulate if surprising advocate of my own views.

In Summary

- Good software packages can make difficult concepts accessible (e.g., *Mathematica* and *SketchPad*) and radically assist mathematical discovery. Nonetheless, introspection is here to stay.
- "We are Pleistocene People" (Kieran Egan). Our minds can subitize, but were not made for modern mathematics. We need all the help we can get.
- While proofs are often out of reach to students or indeed lie beyond present mathematics, understanding, even certainty, is not.
- "It is more important to have beauty in one's equations than to have them fit experiment" (Paul Dirac).
- And surely: "You can't go home again" (Thomas Wolfe).

Notes

1. See Borwein, Borwein, Girgensohn, & Parnes (1996).
2. See Borwein & Bradley (1997).
3. See Dongarra & Sullivan (2000); Borwein & Borwein (2000). See also Borwein & Corless (1999); Bailey & Borwein (2000); and www.cecm.sfu.ca/projects/IntegerRelations/
4. Henry Briggs is describing his first meeting in 1617 with Napier whom he had traveled from London to Edinburgh to meet. Quoted from Turnbull (1929).
5. See Borwein, Borwein, Girgensohn, & Parnes (1996); Borwein & Corless(1999); Bailey & Borwein (2000); and Borwein & Borwein (2001). More technical accounts of some of our tools and successes are detailed in Borwein & Bradley (1997).
6. Our web site now averages well over a million accesses a month.
7. Quoted by Ram Murty (2000).
8. Borwein (in press).
9. An excellent middle school illustration is described in Sinclair (2001).
10. See Asimov & Shulman (1988).
11. In E. Borel (1928), "Leçons sur la théorie des functions", Polya (1981).
12. Other than Poincaré?
13. *A Mathematician's Apology* is one of Amazon's best sellers.
14. See Borwein & Borwein (2001).
15. See Borwein, Borwein, Girgensohn, & Parnes (1996).
16. See Borwein & Corless(1999).
17. ISC space limits have changed from 10Mb being a constraint in 1985 to 10Gb being 'easily available' today.
18. Described as one of the top ten "Algorithm's for the Ages". See Dongarra & Sullivan (2000).
19. In P. Medawar (1979).
20. *MAA Monthly*, November 2000, 241–242.

21. See wwu.cecm.afu.ca/interfaces/
22. Berlinski (1995).
23. At www.research.att.com/personal/njas/sequences /eisonline.html.
24. See Borwein & Corless (1999); Bailey & Borwein(2000); Dongarra & Sullivan (2000).
25. Proposed in the *American Mathematical Monthly* (November, 2000). Described in Borwein & Borwein (2001).
26. Alternatively, a sufficiently robust integer relation finder could be used.
27. A search for 'Lambert W function' on MathSciNet provided nine references—all since 1997, when the function appears named for the first time in *Maple* and *Mathematica*.
28. And they may be psychologically predisposed so to do!
29. This is beautifully described at www. math.gatech.edu/personal/thomas/FC/fourcolor.html.
30. In Regis (1986).
31. From Major (1998).
32. From "Fresh Breezes in the Philosophy of Mathematics'. *American Mathematical Monthly*, August–September 1995, 589–594.
33. Simon (1996), page 16. More than fifty years ago Bertrand Russell made the same point about the architecture of mathematics. See the "Preface" to *Principia Mathematica*.

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[Note that those entries with a CECM index number are available at http://wwv.cecm.sfu.ca/preprints/.]

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Working Groups

Groupes de travail

Report of Working Group A

Mathematics and the Arts

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Introduction

As we began to make plans for this working group, we were oriented by a shared concern: For the most part, we felt, when the terms 'art' and 'mathematics' are mentioned in the same sentence, attentions are prompted either to matters of elegance in argumentation or to such mathematically inspired or generated artifacts as Escher prints and fractal images.

To avoid this sort of tendency, we set out to frame our discussions by borrowing from Hans-Georg Gadamer (1990) who distinguishes the work of art from other cultural forms. Gadamer suggests that art, to be art, must fulfill two complementary functions. First, the work of art must *represent*—that is, it must call to mind something familiar, the viewer must be able to identify (with) it in some way. But mere representation is inadequate for an artifact to do the work of art. It must also *present*—that is, it must point to new perceptual and interpretive possibilities.

So oriented, our original intentions for the working group were:

- to investigate places where mathematics and the arts can co-emerge, informed by the intertwining histories of the arts with mathematics;
- to operate on several distinct levels simultaneously. (On the cultural level we asked, What do the arts and mathematics do? On the level of the classroom we asked, How might the experience of mathematics learning be artful? On the level of the teacher we asked, How might art help us reframe teaching?)

We also attempted to articulate the project proscriptively. In particular, we sought to avoid the temptations:

- to engage in the debate of whether or not mathematics is an art;
- to scavenge for 'artsy' activities (this was not a quest for activities and applications);
- to focus exclusively on the arts as our starting places in discussions of the relationships between art and mathematics.

Part of the frame of this working group was a compilation of activities drawn from the domains of music, literature, dance, and visual arts—other forms of cultural expression through which particular aspects of experience are either foregrounded or pushed into the shadows.

Day 1

Activity: Translating Rhythms

Our first day's activity had been designed originally as a way to introduce one of the functions of algebra—the compact, concise, and interpretable representation of some aspect of a real situation—to students beginning algebra. The activity had previously been used in a workshop setting with several groups of high school students, student teachers, and secondary school mathematics teachers. It involved the ideas of recognition, representation, and translation of a pattern among a number of different media, some of which are embodied and which offer the potential of artistic engagement.

Working in groups, participants began with a short excerpt of poetry and were asked to find the rhythmic pattern in it. The poems, by authors like e.e. cummings, Edgar Allen Poe and Christina Rossetti, had been chosen for their strong, quite regular rhythms. Participants were instructed, first, to read the poem as a group and work together to get a feel for its rhythm, and then to "translate" the poem's rhythm into movement and sound. Simple musical and rhythm instruments were available for use. Participants were also encouraged to "do" the rhythm of the poem using their voice without saying the words of the poem, and to move to the rhythm of the poem, using clapping, stamping, movement across the floor, body shapes, and so on. After a period of work, each group showed their sound and movement representations of their poem's rhythmic patterns to the whole group. (See Figure 1.)

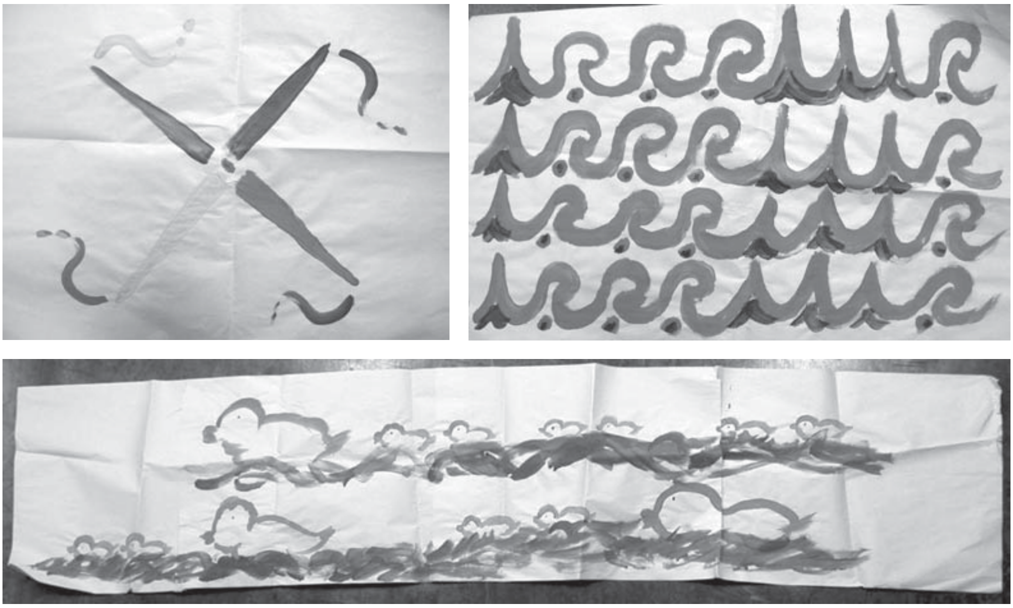

FIGURE 1. Some "translations" of the rhythmic patterns of poetry

The second phase of the activity involved a translation of the poem's rhythmic patterns to a kind of self-made algebra. The groups were asked to work together to find a way to write down the rhythm of their poem symbolically, working through several drafts to improve their written representations. They were told that a "better" symbolic representation was one that was:

1) *more compact,*
2) *a truer representation of the underlying rhythm of the poem,*
3) *more generalized, and*
4) *easier for other people to read and understand.*

These instructions were meant to reflect mathematical values for a good algebraic representation of a physical phenomenon. (See Figure 2.)

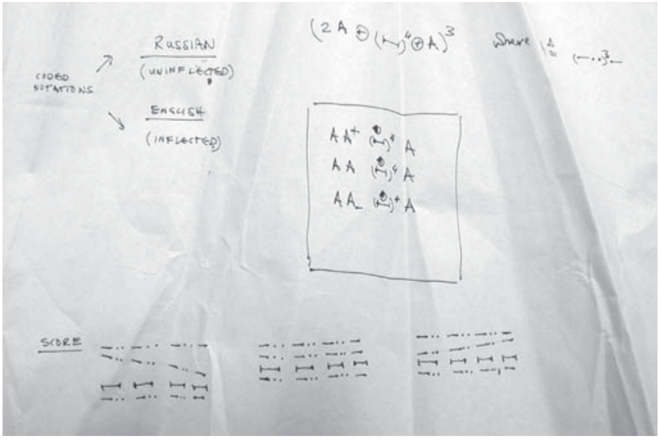

FIGURE 2. A symbolic representation of the rhythmic patterns in a poem

In the final phase of this activity, each small group received another group's best symbolic representation of a poem's rhythmic pattern. First of all, groups were asked to work together to interpret the rhythmic pattern represented by the symbols they had been given, using clapping, voice sounds, and so on, to re-embody the rhythm and to come to an agreed-upon interpretation of the symbols. Then they were instructed:

a) to use paints and other art materials to express the rhythmic pattern visually, so that other people could "feel" it, and
b) to write a new piece of verse using this rhythmic pattern.

At the end of the activity, the paintings, new verses and symbolic notations, and the original poetry excerpts were displayed in a "gallery showing".

Commentary

Most of the participants thoroughly enjoyed the activity; there was considerable excitement and energy in the room, and several attractive paintings were generated. However, during the discussion, many expressed a general unrest about the mathematical relevance of the activity. Were we doing mathematics? Was this an activity that could be used in the classroom? Were we just having fun? And though they didn't feel like we were doing much mathematics, there was a sense in which we were acting in a mathematical way, rather than an artistic one. For example, some felt uncomfortable about translating the poem into a rhythm and then into the notation. How could we ignore the meaning of the poem? How could we ignore previous interpretations of the poem? These translations did nothing but lose meaning, rather than gain any, which did not seem appropriate. There was also a sense in which the translations were over-analytic, rather than holistic, again not appropriate for the kinds of artistic activities in which we were engaged.

Many participants saw mathematics being most appropriate and present only when it came to creating the symbolic notation, which was used to mediate between the poem as experienced and realizations of its rhythmic content in paintings and new poems. Perhaps this translation was the most abstract, or simply felt more familiar. However, even the clapping of the poem can be seen as mathematical since it involves identifying, abstracting and representing the rhythmic structure of the poem—the very same processes used to describe rational numbers, say. While rational numbers are mathematical objects in the way a poem is not, the rainbow—which has been repeatedly mathematized—is no more a mathematical object than a poem.

In a more general way however, participants thought that this activity revealed a fundamental similarity between mathematics and the arts. Recalling Gadamer's description (noted above), mathematics like art can both provide a way to *represent* (call to mind that which has been experienced) and has a potential to *present* (point to something other than the immediate experience.

Day 2

Activity: Twelve-Tone Composition

The second day's activity involved composing 12-tone music according to some of the permutational rules derived by Schoenberg, Webern, and other modern classical composers in the early 20th century. These composers tried to break away from the traditional Western scale and structures of tonality and melody to create a new atonal music that could express the culture of the new century.

To do this, they abandoned traditional musical scales and worked with the twelve semitones of an octave on the piano keyboard, representing them as the numbers from zero to eleven. A twelve-element tone row was created using each of these numbers once in the sequence. Using permutational rules and conventions, the original tone row would be altered to create new rows of notes and chords, which the composer would combine to create a new composition.

We set up the room for the working group with a number of programmable electronic keyboards. Working in groups, and following instructions on a worksheet, participants established tone rows and permuted them using the operations of transposition, inversion, retrogression and compositions of these operations, and created hexachords and other chord patterns from their permuted strings of tones. By the end of the session, each group had performed and recorded their 12-tone atonal composition in concert for the working group.

Commentary

Again, there was much excitement and engagement during this activity. However, the presence of mathematics was much stronger in this activity. The rules of permutation were written and presented using mathematical language and ideas. In addition, the musical notes with which we composed are much like mathematical symbols. One group even attended to the ideas of modular arithmetic and group theory that were lurking about.

Many participants noted that we were using mathematics to create music; in fact, we were using mathematics to place constraints that would give rise to combinations and sequences of notes that would be difficult to invent otherwise. The work of Oulipo (a French experimental literature group) uses a similar strategy by imposing mathematical rules on writing; for example, the rule '$n + 7$' replaces every word by the one that is seven places further down in the dictionary.

The use of mathematical rules to generate music elicited many interesting reactions. Some participants empathized with how students feel when they have to use algorithms. There was a sense of following the (arbitrary) rules without really knowing where they were leading. Moreover, some believed that the sound and expressive quality of the music was constrained by adherence to these rules. In a way, some participants felt that we doing neither mathematics nor music.

In fact, we might have been doing neither. We were reminded throughout this hybrid activity that the cultural project of mathematics and the cultural project of music are not the same. But our attentions were also drawn to some deep resonances between these two realms of activity. For instance, melodies of sorts did arise through applications of some quite formal algorithms. Perhaps, we discussed, the difficulty encountered here has to do with the tension that arises between the opposing questions of "Why are we analyzing this?", as commonly met in studies of musical and other artistic forms, and "Why aren't we analyzing this?", more often met in mathematical contexts. As Jeanette Winterson (1997) expresses the point, "Art takes time. To spend an hour looking at a painting for an hour is difficult" (p. 5). There is a tension between the desire to move to an explicit formulation and the desire to linger in the complexity and potential ambiguity of the tacit and unarticulated. And accompanying this tension, there is often a forgetting that the original purpose of analysis was to deepen one's sense of the whole.

Day 3

First Activity: Fractal Cards

We began the third session by constructing 'fractal cards'—simple paper forms that are generated through recursive applications of simple cut-and-fold rules. Figure 3, for example, shows a card after three iterations of a single-cut + single-fold algorithm.

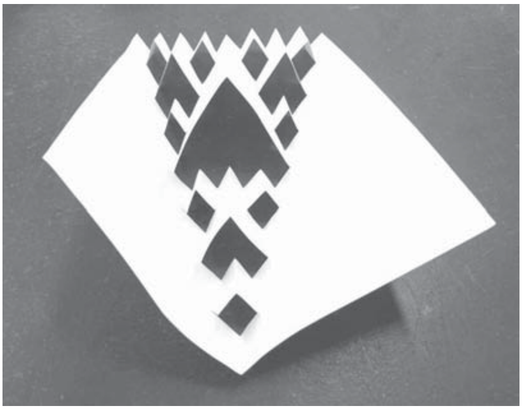

FIGURE 3. A fractal card

Departing somewhat from the structure of the two previous days, where emphases were on engagements with the activity, the main intention in the fractal cards activity was to frame a discussion of issues around the uses of these sorts of tasks within a mathematics class. Brent opened the discussion by mentioning his own reasons for using fractal cards and similar activities in his own teaching. In terms of formal mathematics, for instance, the activity can be used to address a range of curriculum topics, from simple number patterns though sequences and series.

The activity is also useful for pointing beyond traditional curriculum content—including, perhaps most obviously, recursion, scale independence, self-similarity and other topics from fractal geometry. In these regards, it was agreed that the fractal cards activity offers a useful example of the art-full potential of mathematics, in that it clearly fulfills the representational aspect of the work of art while it presents possibilities to open perception to the not-yet-noticed. For instance, as more and varied cards are generated, participants typically begin to notice some visual similarities to objects of the natural world. Trees, seashells, rock formations, and other natural forms might begin to be seen anew.

On this count, Brent mentioned that he usually 'book-ends' the fractal cards activity with a "look for other geometric forms" activity. Participants are first asked to make note of geometric shapes that they notice around them—a task that has consistently and reliably given rise to lengthy lists of Euclidean forms. That list is pushed aside during the fractal cards activity, after which participants are invited to repeat the exercise of looking for geometric shapes. Invariably, there is a certain level of surprise as they 'see' plants, clouds, natural surfaces, and others that they simply had not noticed the first time around.

Commentary

As in all the activities undertaken in this working group, there was a strong element of "art phobia" that accompanied this one. George Gadanidis commented that school art can be like school math—a matter of reproducing what the teacher does. Florence Glanfield said that she was initially afraid to try to make fractal cards because it was like "following the teacher's art" (and perhaps not being very good at it). She worried that, in both math and art, the subject may be seen as a completed object, with no room for creation, and that many children may be "lost" early on for this reason.

On the other hand, George said that creating fractal cards was like "holding infinity in your hand". It was noted that there was a delight and mystery in transforming the familiar and undifferentiated surface of a sheet of cardboard into a representation of infinity. Jon Borwein talked about the pleasure of presenting new ideas in mathematics, and showing the "livingness" of the culture. Brent suggested that activities like this one could be "smuggled in to interrupt the habits of the curriculum"—as curriculum is often perceived mainly in terms of representing established ideas, and rarely in terms of opportunity to present new ideas. He took the fractal card activity as an example of "teaching artfully" by opening up (here, literally) unseen possibilities. The folded paper of the fractal card is in fact an embodiment of the idea "multiply"—as in "multi-ply", or many-layered. Folded and cut paper gives a bodily experience of what multiplication feels like.

There was a discussion, led by Paul Betts, of ways that the intentions of the fractal card activity or any other artful activity could be undermined in classroom use—by 'worksheeting' the activity, so that it changes from a living, interrupting, surprising exploration to a dull, predictable one fully subsumed in the grinding routine of classroom days. Paul was concerned that, by worksheeting an activity, "infinite possibilities are destroyed in favor of one possibility, which one person chooses for everyone else—an example of the danger of privileged mathematics and a questioning of who decides what mathematics should be learned". For most participants in the working group, it seemed that simply writing out and photocopying the steps to any activity along with, perhaps, some questions, reduced its effect to one of prespecified, precooked mush.

There was general agreement, however, that such qualities as aesthetic appeal and the potential for surprise (as embodied in activities like fractal cards) might help to minimize the tendencies to reduce these sorts of activities to step-following routines.

Second Activity: Colour Calculator

For our final activity, we moved to the computer lab to work with a computer-based tool that enables a synaesthetic, sometimes surprising link between the visual and numerical. The goal was to illustrate the way in which this tool, the colour calculator, provides an aesthetically-rich environment for mathematical exploration—not so much by depending on artistic artifacts or materials, but by exploiting the vivid patterns available in the structure of numbers. Of the four activities, this one was the most squarely rooted in mathematics.

The colour calculator is a regular internet-based (hydra.educ.queensu.ca/maths/) calculator that provides numerical results, but that also offers its results in a colour-coded table. Conventional operations are provided; the division operation allows rational numbers while the square root operator allows irrational numbers. Each digit of the result corresponds to one of ten distinctly coloured swatches in the table.

The calculator operates at a maximum precision of 100 decimals digits, and thus each result is simultaneously represented by a (long) decimal string and an array or matrix of colour swatches. It is possible to change the dimension, or the width, of colour table. Thus, of particular interest in the colour calculator are the pattern-rich real numbers because they can be seen and understood as patterns of colour.

This graphical representation of number calls attention to and facilitates the perception of important classes of real numbers—terminating, periodic, eventually periodic and non-periodic—and some of their properties. In this following screenshot, the operation $^1/_7$ has been typed into the calculator. Using the button that controls the width of the table of colours, the different table dimensions have been selected, resulting in different colour patterns that highlight interesting aspects of the number's period.

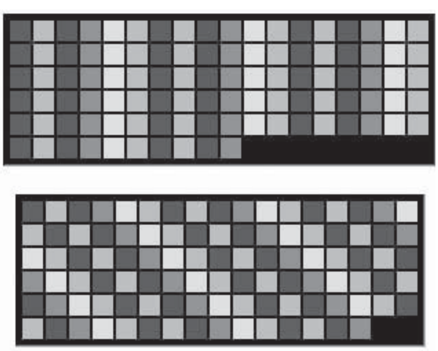

FIGURE 4.
Different representations of $^1/_7$

Participants were invited to "play around" with the colour calculator. Some noted the two-way movement between the numerical and the visual: one can begin with a numerical calculation and generate a visual representation, yet one can also begin with the visual (stripes, diagonals, solid tables, checkerboards, etc.) to probe the numerical. Especially in the latter case, the particular, personal attractions and emphases of the various participants shaped the distinct explorations they undertook and problems they formulated. The visual accessibility of the patterns made possible the kind of qualitative unity necessary for shaping the conjectures, ideas and abstractions that initiated their mathematical inquiry.

Commentary

There was a fascination in playing with the patterns generated by the colour calculator that made it hard to tear people away from their computer screens. Grace Orzech wrote: "This activity demonstrated to me how mathematical and sensorial exploration can merge. … Manipulation of the colour pattern raises questions about decimal expansions".

At the same time, the nature of individual work stations made it difficult to get participants to share their ideas about this activity. Each person was in contact with a computer, not with other participants. The mathematical context of this activity was very comfortable and familiar for all the participants; everyone could remember or predict the period of various rational numbers. The sense of surprise that young students might feel when working on the colour calculator was not available to most members of this group.

Nonetheless, there were moments of discovery of interesting patterns in the colour arrays—for example, when Susan Gerofsky found doublings in the squares of colour representing one and eight in the fraction $^{29}/_{17}$ and a cycling pattern of digits in its multiples.

Jon Borwein called up his own website's "π page" (www.cecm.sfu.ca/~jborwein/Pi_Talk.html) to show us a "colour calculator" representation of 1000 digits of π. After working with the regularities of rational numbers, members of the group expressed a fascination with the non-regularity of the irrationals, and saw this graphic representation as a convincing demonstration of the non-repeating nature of π.

Themes

A large part of the working group was spent in activity mode. While we also attempted to initiate discussion about the experiences provided by those activities, we were not always successful in explicitly articulating some of the conceptual frameworks that guided our planning. At times, there did not seem to be enough common ground in order to probe or reveal some of the underlying processes and values operative in mathematics and in the arts—whether common or opposing. Nevertheless, we were able to identify four issues that emerged, however briefly, in both our planning and in the group's discussions.

Theme 1: Synaesthesia

The theme of synaesthesia was central to many of the ideas explored in this workshop. "Synaesthesia" refers to the crossing of sensory boundaries in perception—for example, tasting colours, or hearing shapes, or perceiving musical sounds as textures.

Although synaesthesia may be categorized as merely a psychological anomaly, it is actually much more central to our everyday cultural practices than may be obvious at first. For example, the reading of written text is entirely synaesthetic; it is simply a learned process of "hearing shapes", by assigning phonetic sounds arbitrarily to the shapes of letters or characters, an assignment that becomes so automatic in readers that the letter "s" seems to hiss, the letter "p" to pop with a small explosion of air, and so on.

Similarly, the ability of trained musicians to sight-read music involves a training, which eventually becomes automatic, in "hearing shapes", so that a pattern of dots and lines may be heard directly as the opening bars of a symphony or as a popular song, complete with varied instrumentation.

Synaesthesia is a kind of translation—a translation among the senses, rather than between languages. The difficulties inherent in linguistic translation also apply to synaesthetic translation, as there is no one-to-one correspondence between the worlds of one language, or one sense, and another. Thus "translation is treason" in that there must be different features stressed and ignored (or, more strongly, appropriated and discarded) when a phenomenon is taken from one language, or one part of the sensorium, to another. We are familiar with this in considering the translation of sensory perceptions to mathematized realms of statistical data or algebraic models; it is also pertinent in synaesthetic translations.

Many of the activities introduced in this working group were at their core synaesthetic translations. The colour calculator makes a one-to-one translation between the numbers from zero to nine and a palette of ten colours. The patterns that appear to emerge with colour are patterns that were there all along, but in a perhaps less-accessible medium.

Similarly, as Susan comments, "I remember once surprising a psychological researcher by my ability to remember long strings of random numbers. The trick was that I 'heard' the numbers as the corresponding notes on a piano keyboard, and remembered the pattern of the melody instead of the strings of numbers themselves".

In translating numbers to colours, certain features of the numbers are necessarily lost—for example, we are no longer aware whether the numbers are even, odd or prime. What is stressed is the regularity of repeated patterns, their period, and perhaps the aesthetics of the arbitrarily chosen colours (their "beauty" or "boringness"). Relationships between the periods of different rational numbers can be found, and the colourful patterns created can be manipulated by changing the array size to create columns and diagonals of the same colour. Meanings may be added as a result of the translation to colour, and arbitrary features of the colouring may affect decisions about the aesthetics. For example, the combination of red and purple could be seen to be "more beautiful" than yellow and purple and thus more interesting, although the numbers they have replaced might not support such an interpretation.

Our first activity was designed as a series of synaesthetic translations, moving from written to spoken poetry, to music, movement, abstract symbolic notation, then to painting and the writing of a new poem. Part of the question addressed by this activity was whether *anything* would remain invariant throughout this series of translations. (Susan notes that, having led this workshop several times with a wide variety of participants, she has found that it *is* possible to abstract and re-present an invariant rhythm through these many sensory media.)

For many of the working group participants, the idea of stressing and ignoring in the context of the arts, rather than in mathematics or physics, seemed almost sacrilegious. They considered artistic objects to be inviolably holistic, despite the fact that artists, of necessity, engage in analysis, revision, and experimentation in their creation of new works. As mathematics educators, many members of our group felt it was wrong to read a poem just for its rhythm, or to create a piece of music simply by translating permutations on a string of numbers. In a sense, the media of translation themselves became an obstruction because of qualities attributed to them as artistic media. It was sometimes difficult to engage mathematics educators in artistic experimentation because it they did not consider it a realm in which they could play.

Theme 2: Aesthesia and Anaesthesia

Throughout the three days of the working group, we played with cognates of the word "aesthetic"—including synaesthetic, kinesthetic, and even anaesthetic—in the relationship between mathematics and the arts.

Formally, *aesthetics* refers to the branch of philosophy that is concerned with matters of beauty and taste. More generally, the term is often used in reference to the principles and properties of art.

Much has been said on matter of the aesthetics of (and in) mathematics—perhaps most commonly articulated in relation to the role of elegance, as, for example, a motivator for research and a property that can be used to distinguish good from not-so-good math. Indeed, some mathematicians think of themselves as artists and aestheticians of pure form.

Such discussions, however, often gloss over the sensorial meaning of the Greek term *aesthesis* (feeling or sensation) from which aesthetic derives. This original meaning, in fact, seems to be better preserved in its cognates *synaesthetic*, *kinesthetic*, and *anaesthetic*. Indeed, when this cluster of terms is pressed against an instance of "traditional mathematics instruction", one cannot help but be struck my the tremendous anaesthetic qualities of student experience—held in place and held apart by rigid desks, presented with abstractions that are stripped of all contact with the worlds of experience from which they arose. So presented, numbers numb.

The contribution of a more art-oriented sensibility, then, goes deeper that the explicit interconnections that might be articulated between, for example, rhythms and arithmetic.[1] The issue here is more than a matter of complementarity of cultural forms; it is a matter of the bodily origins of meaning, a recollection that mathematical understanding arises in one's situated engagements with the world. Art reminds us of our sensate beings, just as mathematics often prompts us to forget.

Theme 3: Abstraction

The process of abstracting—that is, of selecting certain features of a conception on which to focus while letting others drop into the background—pervades in mathematics. However, although abstraction is sometimes more naturally associated with mathematics, the working group activities revealed the way in which it also pervades in the arts. This should not be surprising since abstract thinking, which even young children use masterfully, is a fundamental, human sense-making strategy. Of interest though, are the particular practices of abstracting used in different disciplines. When we clapped our poems, we were stressing the rhythm of the poem while ignoring its content—the words. Even when we added the sounds of other instruments in acting out our poems, we were in some sense richly abstracting the rhythm of the poem, emphasizing and elaborating on one of its sensory modes—without necessarily peeling away toward some fundamental representation of the rhythm. Our purpose was to create an entertaining musical performance.

In moving from rhythm to notation however, we noticed how the mathematical values of terseness and conciseness influenced the next process of abstraction. Many groups took particular pleasure in creating a symbolic notation as compact as possible, one that in most cases lost all connection to the rhythm or meaning of the poem, and that retained only a symbolic relationship to it. How few symbols could be used? How elegant could the symbolic representation be? What other structures could be well exposed or expressed by using symbols? The notation frequently emphasized some of the other structures of the poem not specifically isolated by the clapping; for example, the repetition of verses could be encoded and compacted into a single symbolic instruction. The rhythm may even have been ignored while other structures were emphasized. In mathematics, the point is often to abstract away from all the sensory connections, and to identify the fundamental structures of phenomena by peeling rather than elaborating.

We note that mathematics and the arts both rely on abstraction, but that the values and purposes of abstracting are often different. According to Dissanayake (1995), elaborating is a core human behaviour, one that accounts for the artistic activities of humans in all cultures, which are often so time- and resource-consuming that they must serve a biological purpose. In reflecting on the nature of mathematical abstraction, and the notorious difficulties that abstraction presents for students, fractal cards stand out as one example of an activity in which sensory engagement can be coupled to formal abstraction.

Theme 4: The Role of Affect

Emotional excess, particularly the ebullient, creative, vivacious kind, forms part of our image of artists. Of course, the image of the manic, violent, and ego-centered artist exists as well. In both cases, emotion is commonly seen as being a central force in artistic endeavour. Superman may be an artist, but Clark Kent—in his passive, mechanical mode of nine-to-fiving—certainly is not. On the other hand, the mathematician is primarily asocial, isolated, and probably a little nerdy. These stereotypes have pat implications: the artist had better be emotionally engaged, while the mathematician had better remain cool, calm, and rational. Or perhaps even: the artist had better keep reason at bay while the mathematician had better leave earthly passions and desires at the office doorstep. Most know that these stereotypes and their implications are exaggerations and over-generalisations. Many concede that artists and mathematicians are both reasonable as well as emotional beings. However, while the role of emotion in artistic creativity might seem obvious, it may be more difficult to see how the mathematician need rely on affective responses in the course of mathematical inquiry.

The first activity highlighted the emotional involvement needed to read poems, clap them out, and perform them—involvement that was difficult for some to summon due to anxieties and insecurities. But what kind of emotional involvement did the colour calculator activity demand? Perhaps feeling surprised by a certain pattern, or curious about a certain fraction, or attracted to a certain asymmetry, or repulsed by a perceived dullness. In fact, all these emotions can colour the various features that will be noticed: the stripes of $^1/_7$ with grid width 14 may evoke sadness in one person whose grandfather wore a striped shirt while the sunny look of $^{67}/_{99}$ may evoke peace and warmth in another. These responses will guide the decisions that will be made during exploration: How can I avoid stripes? How can I get a sunnier fraction? While these examples may seem somewhat removed from the usual objects of higher mathematics, they may be illustrative of the important role of emotions in the process of mathematical inquiry, a process involving choices that cannot be based on logic alone (there is no logical reason to be more interested in stripes than in the colour of the sun).

Concluding Remarks

One of the questions that we've avoided in this report—and that we sought to avoid in the context of the working group—is the all-too-familiar, Is mathematics an art or a science?

For us, this question only makes sense if one ignores some key common factors of human experience—in particular, the complex, recursive processes that enable the emergence of personal and collective knowledge.

This is not to suggest that there is something to be gained by collapsing such domains of human activity as art and mathematics, or by attempting to place one in the service of the other. Rather, the suggestion is that, for us as educators, there is an obligation to attend not just to the products of human inquiry, but to the sights, sounds, and other aesthetic experiences that have prompted the human mind to knit primal sensations into abstract understandings.

As a working group, we came to no consensus as to how this might be done—merely to an agreement that it is something we must strive to do.

Note

1. We note that "rhythm" and "arithmetic" derive from the same root. For that matter, "art" and "arithmetic" also have common origins.

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Report of Working Group B

Philosophy for Children on Mathematics

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Participants

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Ann Kajander	Richard Pallascio	Dave Wagner

Introduction

The purpose of this working group was to consider a particular approach, called the *Philosophy for Children on Mathematics* (P4C/M), designed to engage children in a community of inquiry so that they might have the opportunity to consider philosophical questions that arise in mathematics. CMESG members from the Pacific Coast to the Atlantic and as far away as England found themselves drawn to this session for a variety of reasons: the age group of children that the approach is intended for; concerns with how to draw children into questioning, how to engage them in discussions; how to encourage students to think about what they are doing in the mathematics classroom; how to introduce pre-service teachers to the philosophy of mathematics; and to think about the kinds of answers that answer 'why' to mathematics questions. In our report we introduce the method that has been developed to create communities of inquiry for philosophizing on mathematics and the activities and discussions the working group participants engaged in as they themselves experienced the method for engaging children in philosophy.

Philosophy for Children on Mathematics (P4C/M)

Begun in the 1970s by the philosophers Matthew Lipman and Anne Margaret Sharp (Montclair University in New Jersey), the *Philosophy for Children* (P4C approach has been adapted to mathematics by a research team at CIRADE (Centre interdisciplinaire de recherche sur l'aprentissage et le développement en éducation) which includes a philosopher of education (Marie-France Daniel of the University of Montreal), specialists in mathematics teaching (Louise Lafortune of UQTR and Richard Pallascio of UQAM), and a resource person in philosophy (Pierre Sykes).

Lipman was a disciple of Dewey and interested in democratic education. He wrote novels with philosophical issues embedded and used those novels with children as prompts to foster philosophical discussions. However the CIRADE group was unable to identify novels that would foster philosophical questions more specific to mathematics. So they took up the challenge to write a novel that could be used in P4C/M. Their strategy was to write novels with characters about the same age as those who would be reading the books. The characters have reflections and preoccupations; but rather than finding answers they find questions and paradoxes. Those novels then are the prompt for pupils' questions and discussions within a "democratic community of inquiry". The CIRADE group has been working with school teachers and pupils to develop these communities of inquiry and have created a website, "L'Agora de Pythagore", http://Euler.Cyberscol.qc.ca/Pythagore/, which includes a forum for students from different classes to respond to one another's questions.

Like P4C, P4C/M's goal is to bring children to reflect on the nature of mathematics and on what they are doing when they are doing mathematics. Specifically, the intents of the program are to allow students to:

- philosophize together on mathematical concepts, notions and problems;
- adapt themselves, these concepts and notions;
- transpose mathematical ideas in life contexts;
- become aware of myths and prejudices about mathematics and its learning;
- lessen fears and negative attitudes with respect to mathematics;
- develop an interest and a self-confidence towards mathematics;
- develop autonomous, critical and caring thinking;
- favour cooperation between pupils.

In order to facilitate these intents the P4C/M involves a particular approach for developing a community of inquiry and for carrying out philosophical reflection and developing awareness of the nature of mathematics and mathematical thinking. The process involves the following stages:

- Group Reading
  The process begins by a shared oral reading from a mathematical novel. (See Appendix A for an example.)
- Questioning
  Following the reading students are invited to note some questions and then share their questions with the class. The teacher makes note of the questions on the board so that they are available for all the students to consider.
- Selection of a Question
  One question is selected from the many questions posed by the students. Each student is given the opportunity to argue for his or her question to be selected as the question to be considered for philosophical discussion. Once the arguments have been heard then the students vote on one question to consider.
- Individual Reflection Prior to the Community of Inquiry
  Students are asked to reflect on the questions that the class has chosen to discuss prior to the group dialogue.
- The Community of Philosophical Inquiry
  In order to foster students' thinking around the selected question, the teacher prepares a mathematics activity related to the question to be considered. As students carry out this activity, questions are raised and considered. After working through the mathematical activity students come together in a large group and discuss the philosophical question. The community of inquiry is one of philosophical dialogue and based on Socratic pedagogy. For more details about the community of inquiry see Lafortune, Daniel, Pallascio, and Sykes (1996).
- Group Review of Thinking Abilities
  Once the question has been discussed the process is not over. Rather, students are asked to consider their own thinking and the kinds of questions that they engaged in. Led by the teacher students consider both the higher level and the lower level thinking skills that were part of the process of the community of inquiry (see Appendix B).
- Personal Synthesis of Learning
  At the end of a community of inquiry, students are invited to do a personal synthesis, either of the learning they have acquired during the discussion that took place with the other pupils—or even to discern the ideas that fuel their thinking outside of the philosophical community of inquiry.

The working group participants were introduced to the model of the P4C/M (described above) by means of a video. In this video we followed two groups of students through the process. From the shared reading a number of questions were raised and one group of students settled on the question, "Why were the first philosophers and mathematicians interested in the construction of geometric figures using only a compass and an ungraduated ruler?" and another group, "What is the difference between geometry and mathematics?"

Viewing the video raised a number of questions for us concerning the intents and the process. The questions that were raised in the working group might be categorized as: those that are related to the students thinking, initiation into philosophical discussion and metacognition; those that question the process itself; and those that ask about the role of the teacher and teacher education. For example, some of the questions we asked included: *How is the P4C/M related to thinking maths and thinking science out of England that are based on a Piagetian model? What role does the mathematical activity play in fostering philosophical discussions? More specifically, what role does building a geometrical model have in answering philosophical questions? What role does the teacher play in this community and can it be democratic if she is shaping the questions and discussions? Is this approach being taught to student teachers?*

By viewing the video, working group members had the opportunity to see the process in action. In particular, we observed students in the shared reading, question posing, question selection, and then in mathematical activity. In the lesson on the tape, students were building representations of various geometrical forms. We wondered, what is the role of building geometric models to answer philosophical questions? Such mathematical activity is a significant part of the community of inquiry since it provides the occasion for thinking about the question at hand through discussion and by encountering obstacles, paradoxes and further questions. (See Appendix C for an example of a mathematical activity that is used to consider a philosophical-mathematical question.) A working group member pointed out how when doing physical activities they do not often work out right—so then there arises a need to do mathematics to explain why the thing that did not work is what we say it is.

Observing the highly structured nature of the process leading to a community of inquiry led us to ask about the search for criteria and justifications for ideas. We wondered—*Through the process are we trying to get students to form mathematics or the right mathematics? Are we asking them to form a community of inquiry in 'the' right way? Or can the particular community that emerges take on its own set of criteria? If students are left to decide the criteria for the community of inquiry how much does the teacher influence that? If we provide the labels for thinking then are we saying then we are allowing them to form a community of inquiry? What is the agenda and how does it play out? How does the task the instructor brings to the group direct the inquiry? What happens when students do something in a particular form? What are the implications of having a method taught?*

Richard was able to respond to many of our questions. He pointed out that the P4C/M method is very specific and is intended to initiate students into democratic processes. Students have been observed to ask why we are doing that problem. These students are thinking critically. They ask the hard /good questions. The philosophical dialogue and metacognitive work has consequences in other aspects of their lives.

It was also noted how the mathematics and philosophy that are being promoted by this method are connected back to the Greeks, and we were reminded that democracy and proof arose with the Greeks at about the same time. However there are differences that arise in the community of inquiry. For one thing, the historians don't have any evidence of doubt with the Greeks. Seeing this method being played out by the children, and the presence of doubt in their form of questioning, we think that this may speak to the way humanity progressed.

These are some questions discussed on the "Agora de Pythagore" by a class of 10- to 12-year-olds:

- Is it possible to construct a perfect cube?
- Are the mathematics discovered or invented?
- Are the mathematics useful to construct buildings?

- Might scientific or mathematic discoveries conduct humanity to its destruction?
- Does chance exist?
- Is the Universe boundless?
- Can we construct straight lines without a ruler?
- What is the shape of a star?
- What is a mathematical thought?
- Why do we find some mathematics in Nature?
- Are some numbers more useful than others?
- Where is the beauty in mathematics?
- Why are there so many rules in mathematics?

There is another dimension that is facilitated by this process; that is coming to an understanding that mathematics has a story. I am a human being; I can do mathematics; I can create mathematics, not just discover. Students begin to understand the importance of their contributions. When writing in their forum (internet discussion group) the students are writing for someone outside of their immediate community. The context of each has particulars that the other does not—and, hence, that students do not have as part of their experience. The students need common experiences (or problems) to create community.

Of interest throughout our introduction to the P4C/M the working group members was the role of the teacher. *What is the extent of her or his participation? Does she or he mediate? Does she or he vet questions? To what extent does her or his agenda get played out in the discussion? Is it truly a community of inquiry if all along the teacher's plan was to have a discussion about the nature of reality and the students have that discussion?* These questions and others around the process are reflective of the concerns the members of the working group had about the process to create a community of inquiry. Richard reminded us that this process was based on the work of Dewey towards educating for democracy and the Socratic method.

Some discussion around the possibility of introducing philosophy for children to our pre-service teachers led to Richard noting that people are skeptical. It is only when they do this with students that they realize that children will do this when invited to do so. Another member of the working group did have some success with his undergraduate students when he invited them to discuss the existence of a hypercube; this was very rich, he claimed.

Participating in a Community of Inquiry on a Philosophical-Mathematical Question

In the second—or "bridging"—session the working group members had the opportunity to experience all of the stages of a reflective activity on mathematics:

- collective reading of an episode of a philosophical novel on mathematics. (*The Mathematical Adventures of Michelle and Damian*. An excerpt can be found in Appendix A);
- questioning;
- selection of a question;
- mathematical activity;
- community of inquiry;
- review of the abilities used to think.

(We would encourage you, the reader, to read the excerpt provided in the appendices before reading the account of our experiences with a community of inquiry.)

As required by the method for a community of inquiry, we began the day with a shared reading from the novel. Then we each posed questions that were raised from the reading. (It is interesting to note how the energy level in the room went up and down as people posed questions and considered the questions of others.) Here are the questions we posed. Keeping with the method, each question has the questioner's name associated with it.

- *What do 'winners' and 'losers' mean? And does attitude influence this labeling. (Gary)*
- *Is there a finish line? Can anyone know all mathematics and be an authority? (Cathy)*
- *Is there a relationship between geometry and art? (Tom O)*
- *Is knowledge fixed? Is intelligence fixed? Are abilities fixed? (Joan)*

- *The boy talks about feeling like he failed the test, but how does he know that he failed the test? Is it possible in mathematics to know for yourself that are correct without outside verification? (David R)*
- *What kinds of mental images are mathematical? What kinds of mathematical images are connected to what we perceive as mathematics? (Ann)*
- *Are we predestined to be good at mathematics? Do visual images in mathematics have to have a physical connection? (Geoff)*
- *Can we use mathematical activity to process emotions? (David W)*
- *In mathematics do all people start with a big idea and have small ideas come from it? (Sue)*
- *Are mathematical ideas waiting to be brought into focus? (Lynda)*
- *What role does "looks like" play in mathematics? (Tom)*
- *Can mathematics be based on predictions? (Elaine)*

Once each of us had the opportunity to raise a question we were called on to defend it. This part of the process requires that individuals make arguments as to the importance or significance of their question for consideration by the community of inquiry. This stage of the process raised the most discussion about the method itself. People wondered what implications such a strategy could have on the students themselves. Richard responded to these questions.

What happens to the questions that are not used for the community inquiry? What can the teacher do with the questions? In this method it is important when students come up with questions to keep them, to validate them. Questions are displayed (on the board) and shared. This method teaches students to create better questions and at the same time they are learning principles of democracy by this. *What do we do with all the possibilities that emerge?* It sometimes happens that a group of questions are put together into types and responded to because they belong to the group of questions that is selected. Often however, questions that were not selected for group attention are addressed in responding to the ones that were posed for the group discussion. Questions can be left for the moment and yet still be present for the asking at later moments. *What are the criteria for arguing a particular questions? Must they come from the text?* Children argue based on their preoccupations. But they develop an understanding of good reasons as they engage in the process.

After responding to a number of initial concerns raised by this process. We had questions about the process for ourselves. *How close do we stay to the text? Can I argue against other questions rather than defend mine? Can we group questions together and form a new question?* Again Richard directed our discussion. Here are three different ways of philosophizing. This points to the need to set up the criteria first. The practicing of politics and the process of saying why our questions are really good is probably really important. If we could speak in support of our questions then we get to better understand what they are about and if they are worth pursuing. However, only a few of the members defended the questions they themselves posed. A few people suggested why we might consider someone else's question. Here is a flavor of that conversation.

> *"Now I am really interested in mathematics waiting to be fertilized," said Tom O in reference to Lynda's question of bringing mathematics into focus.*
>
> *"I picked my question on processing emotions because it is new to me and, I suspect, new to everyone," Dave W added. "Since we are trying to simulate the experience of the children philosophizing about mathematics, we should talk about a philosophical question that is new to us, just as the questions would be to the children in Richard's research." This seemed to have some impact on other members of the group.*
>
> *"I don't think I would be impacted by a discussion on something that I have heard before," Geoff pointed out as he contemplated the questions. "The eighth question is a one that I haven't heard or thought of before; hence I would be interested in it."*
>
> *"I am also interested in question 8 because we make the assumption that we can get rid of our emotions through art, etc.", Sue added.*
>
> *David R pointed out that, "All of these questions are about knowing things. Therefore we should begin with a question of knowing things. …"*

When no further reasons were forthcoming from members of the working group, Richard called the vote. But, again, the participants seemed bothered by this process and had questions and concerns. A very intense conversation broke out about whether this voting process was democratic and, even if democratic, was it potentially emotionally harmful. In particular, there was concern around the notion of public voting for a question. Someone wondered if this might not be harmful and work against the development of a positive attitude to mathematics. Another person wondered if this might not silence some students. *Voting on the questions? Does this hurt people?* In Richard's experience working with students he has found that the students play the rules and easily agree that choices other than their own are sometimes good to discuss. Often, they can transfer their ideas into the retained question. It is exactly what we do in our own dialog!

The question on voting was not resolved but the working group members did vote and chose to consider the question, *"Can we use mathematical activity to process emotions?"*

The next part of the method involves breaking at this point, leaving the students with the question and giving the teacher a chance to select and arrange a mathematical activity that has the potential to foster reflections around the question of inquiry. In a normal classroom situation the next lesson might be a week away however, in the working group the break was a mere 20 minutes, just enough time for coffee, When the working group members returned, they were asked to break into small groups and consider the following.

> Definition: A line is the shortest path between two points.
> Task: On a sphere, what kind of "polygons" can we find?

This prompt proved to be a very provocative one for the small groups. Not only was it difficult for Richard to break into this group's activity to call them back to the large group but they wanted to share and consider the mathematics the other groups had come up with.

One group found a one-sided polygon and a two-sided polygon. Take the equator, for example. It determines a one-sided polygon. Another group suggested that they could get an infinite number of polygons by dispersing points around the sphere. By example, we can construct a triangle with three right angles.

Although we found the question very interesting, we returned to the philosophical question we posed earlier. *Can we work through emotions with mathematics?* We wondered, *what does mathematics provoke for the mathematician or the doer? What does it provoke for the reader? What does it provoke for the student?*

Art is an expression is mathematics? The writer tries to make us cry. The artist wants us to be stunned. With mathematics do we want to put others on an emotional roller coaster? Can someone read the emotions from my mathematics? Do I attribute my emotional response to the artist? The emotion is engendered by mathematics. But there is the other question: Can we express emotions through mathematics?

These observations and questions led us to think about if there is a particular set of emotions associated with mathematics. *What emotions does mathematics bring? Puzzlement, for example—do you think puzzlement is domain-specific to mathematics? Is math a domain that brings out a different set of responses. Are challenge and frustration the same as puzzlement? When puzzled, I want to continue. When frustrated, I don't want to continue. When challenged, I must continue. There is a demand from the outside. Imperative, I have to do it. In a mathematical activity when we find something for the first time, we are jolted by it. I think there can be a tremendous amount of satisfaction in returning to a problem. For the teachers we present mathematics as something that should be taught because of the joy it brings. Coleman said to mathematize is to be joyous.*

For some people their personal experiences spoke to the questions of the emotional nature of mathematics. *There is a therapeutic value of mathematics. Can I immerse myself in mathematics? I choose problems and they depend on my mood. The way I go in a particular mathematics territory is based on the way I feel. There is a story about a mathematician who was saved from mathematics. A mathematical problem bothered him and his intrigue with it prevented him from committing suicide. Painting and mathematics distracts. And this is an important part of the sadness. Mathematics can help the child. It can be liberating. Mathematics is self-checking.*

And for others they couldn't help but think about their students' emotional responses to mathematics. *They questioned what we, as teachers, should be attempting to achieve with our* *teaching. With math we try to provoke positive emotions. When you share a bad model. There is a big element of doubt. But as teachers we want positive emotions with math.*

This led the group to consider *if there is an empirical way to study this question of emotions in mathematics? For example, divided-page exercises. Give them a problem and on one side they do the problem and the other side they write thoughts and ideas. Those teachers were using the exercises to help them work on their emotions. And we used math to help them process their emotions.*

Before we broke from our second day of deliberations, we asked participants who had reflections to jot them down and pass them on to us. These questions and reflections would provide the starting point for our last day of the working group.

Reflections on a Method for Creating a Community of Philosophical Inquiry

The third session was an opportunity to reflect on our experiences with the process and to further consider the questions that we raised about philosophizing in mathematics. There were some questions left from the previous session that we began with.

- What is the role of reasoning by analogy in mathematics? In philosophy?
- What makes an analogy a mathematical analogy?
- What makes an argument "good enough" in a process of inquiry? In a process of mathematical inquiry? In a process of philosophical inquiry?

These questions led us to consider the criteria of acceptance within a community of inquiry. How do we know if a question is a good question? Is there something about the question or can we only know in the student's response? We also wondered how does a teacher facilitate and guide the discussion? Our last day was devoted to responding to such questions about P4C/M.

Once again we were reminded of the connection to Greek philosophizing; this process uses the Socratic method and mathematical activities, selected by the teacher, to prompt the questioning and the discussion. Students learn to participate in philosophizing through participating in the community of inquiry. When the students discuss the question there are two factors: experience and argument. In the beginning the teacher is showing what is valued but as students engage in the process they learn what makes a good question and a good discussion. The teacher is also responsible for shaping the process through her selection of activities. As it is with any activity in the mathematics classroom, the activity is facilitated through a suitable set of questions. In these ways the teacher facilitates and shapes the discussion. The shape is philosophy. Students are not going to solve a particular question; but learn how to discuss.

Another question that was raised concerned the relationship between the issues raised (deliberately) in the novel and those questions that the students ask. *What is that relationship? And what did you put in the novel and why?* Richard explained that the writers began with myths about mathematics and learning mathematics and inserted them into the novel. This was deliberate to prompt philosophical questions.

Fascinated with the method, a number of the working group members were interested in identifying novels appropriate for older children. A number of suggestions arose in the conversation. Lakatos' *Proof and Refutations;* Abbott's *Flatland;* Garader's *Sophie's World*; Carroll's *Alice in Wonderland*; Litman's *The Discovery of Harry Stottlemeier;* Fadiman's *Fantasia Mathematica* and *The Mathematical Magpie.* Gary pointed out that the students' own lives present a source of stories. Dave W suggested student responses in mathematics could act as prompts for philosophical discussions.

Again we found ourselves turning to the central concern of the working group, students philosophizing. It is apt that we ended our three days of discussion around the question of why is it important to develop reflective thinking on mathematics for the student. Here is a collage of the ideas we expressed around these questions.

Metacognitive skills help students to connect concepts. We begin to create a story of mathematics. This is important because we need to participate in the language game. This kind of activity changes the nature of the mathematics children learn. Reflexive thinking changes (builds) founda-

tions, changes people, changes math. In some way you are forcing yourself to think about understanding in a very deep way. It changes what we mean by development. Reflexive thinking is important because you can't make sense without it.

Math as a human activity that is dynamic. Often we get so bogged down in application; here is a look at people's journeys. "The unexamined life is not worth living." Unexamined mathematics is not worth living. Maybe it is because we do not examine mathematics that we do not have hope, joy, love.

Krummeuher (2000) writes of the necessity of intellectual social activity to construct mathematics culture. We created a community of inquiry in front of participants and we talked about the question why 5/0 gives infinity. Today, I explored 1 and 0 with you!

Mathematics is a domain of human activities that is guided by a set of guidelines that is intended to acculturate children into our way of thinking. At the beginning of this session, I thought there wasn't enough time to engage in this process but that was when I was thinking about just me, and I thought about the fact that there are 15 of us and that gives the question more time.

This process helps make a distinction between doing mathematics and a study of mathematics. We can study mathematics as a mode of thinking. These are fundamental things we should be doing in schools. We can learn the mathematics we need for work on the job site but there is no other place where we can learn what mathematics does or its nature.

So then, we ended our session believing in the value of such a process and asking ourselves how can we make this kind of thinking happen in our classrooms. The participants were left with the questions:

1) Why is it important for the pupils to develop reflexive thinking of maths?
2) ... which environment to do that?
3) ... conditions for teachers?

David R —

1) Mathematics is a domain of human activity that involves certain unique values (e.g., In judging an argument). Reflection on mathematics allows for the application of those values by students and the guiding of the community of inquiry toward those values by the teacher (e.g., By questioning). This is an element in the agenda of mathematics educators to acculturate children into our way of thinking.
2) RP describes one. Vicki Zack has described another. The common feature is perhaps that students' thinking is heard, reheard, and the mode of expression is a focus of the teacher's attention, as much or more than the validity of what is being said. In other words the teacher is correcting the students' criteria for judging, not the judgements themselves.
3) See #2.

Tom K —

In what ways are these "ways of thinking" important for knowing maths?

In some sense most ideas/questions/concepts in mathematics are open for treatment as philosophical. But I think they are rarely treated as such in most mathematics classrooms at any level. Thus a focus on these "ways of thinking" vis-à-vis mathematical ideas allows the children, youth, or adults to be embodied in and related to the historical and contemporary communities of mathematical practice in a needed different dimensions. (To see this practically look at Brown's ad hoc question.)

Second such ways of thinking as enacted by individuals with others seem to me fully co-implicated in mathematics knowing itself. Even if one is engaged in a technical task the nature of this task is raised by the use of some reflexive thinking (e.g., differences, values, etc.).

Third (like my first point in some ways), such thinking in/about mathematics features mathematics as a human science. It prompts us to ask the kind of "emotion" related questions we did.

Fourth, using such thinking deliberately situates mathematics knowing in a collective setting with all of its attendant issues such as human interaction. Thus it is *done really* to all mathematics as contributing to central life skills for all.

Lynda —

Why is it important to develop reflexive thinking on math?

- To introduce students to the process of evaluating different solutions—what makes them different? What makes one better (mathematically)?
- To make explicit the process of constructing an argument, proof or narrative about a solution path.
- To nurture logical thinking, responsible citizens.
- To make "history" come alive—scratch a theory and you'll find a biography.
- To help students to see value of different mathematical representations and to understand why some communicate certain ideas better in certain circumstances for different purposes.

Which environment does that?

- Trusting
- One in which students encounter rich questions regularly
- One in which fertile mathematics resources abound—on walls, on computers, on bookshelves

Conditions for teachers

- Knowledgeable teacher (in the Shulman sense)
- Invitational teacher
- Reflective teacher

Rina —

Why is it important to develop students' reflexive thinking in math?

Reflection helps construct new knowledge and consolidate learned concepts. It does so by enabling the exploration and discovery of connections (with previously learned concepts and ideas), and relationships (among various mathematical ideas, topics and structures).

Reflection is also an essential component in problem solving—especially when solving non-standard problems. Helps explore various possibilities for solving the problem, an evaluate various solution attempts or parts thereof.

Reflection helps in change of attitudes & values.

"The unexamined life is not worth living."

Reflexive thinking changes math. Ontario curriculum documents forbid teachers from stating their own opinions/position.

Elaine —

Why might it be important to develop reflexive thinking on maths for the pupils?

- To engage in meta-cognitive thinking
- To develop critical citizens
- To develop people who take opportunities to step outside & consider consequences.
- To ask structural questions
- To engage students with different interests.

Anonymous —

- Attitudes
  - Aware of others' thinking and feelings in math
  - Expand their own definition of math
  - Positive attitude → success
  - Open to different solutions and opinions
- Problem solvers
  - Increase ability to solve???
  - Make connections → leads to sense making
- Thinking
  - Skills → higher order development

- Environment
  - Supportive
    - good, rich questions
    - valuable tasks to create common experiences
  - Social
    - physically encourages discussion
  - Tolerant
    - language used
    - body language
  - Time
    - to contemplate
    - work in small groups, whole group, individual

Dave W —

Why is it important to think reflectively on mathematics?

Socrates is reported to have said that the unexamined life is not worth living. Before relating this assertion to mathematical activity, I first ask why examining life might make it worth living, or at least make it better.

When I examine my activity I gain insight into my particular purposes and desires and become attentive to the possibility of alternative purposes and desires. Purpose and desire are intimately related to hope, joy, and love (and to despair, apathy, and hate). Self-examination opens up possibilities for being an agent of hope, joy and love.

Self-examination also increases my awareness of the particular standards and the values through which I evaluate my choices, and allows me to consider my standards and values in relationship to alternative systems. An awareness of the diversity and interconnectivity of value systems is necessary for peace and for mercy.

Furthermore, self-examination exposes the interconnectivity of authority, experience and reasoning. Awareness of these interrelationships is necessary for healthy faith and healthy skepticism.

Now, moving to mathematics, I ask myself the question, "Why is unexamined mathematical activity not worth doing?" (I don't mean "examined" in the external assessment sense.)

First, I think about desire and its connection to hope, joy and love (emotions?). Perhaps our traditional avoidance of thinking about the mathematics we do is related to our unfamiliarity with emotion in mathematics. Following Skovsmose's assertion that the way we do mathematics in school formats the way we solve problems in our world, we might hope that philosophizing about mathematics will format our society to legitimize emotion in responses to our world's problems.

With regard to standards and values I think of the work of Candia Morgan who exposes the apparent arbitrariness of what is valued in school mathematical writing. She suggests that mathematics teachers discuss with their students the values that underlie the assessment of their mathematical writing. What does such heightened awareness do? Does it relate to peace? to justice? When we expose the problems associated with justice, people tend to feel violated or guilty. What would happen in mathematics classrooms? Or, is there an irrefutable standard in mathematics?

There has been much research of proof in mathematics. Proof, it seems to me, has much to do with authority, experience and reasoning. As mathematics students or other mathematicians become more aware of these three factors in their proving, perhaps they will become more sympathetic to the tension between faith and doubt and more aware of the human face of mathematics.

Ann —

As a follow up to the working group, I decided to experiment with extending the P4C ideas to the secondary audience. To do so, I wrote the following story for my grade 12 advanced mathematics class to perform as a play. Readers should note that the play has been designed to appeal to this audience!

We attempted to do a version of the P4C idea in a 70-minute class. First I explained the idea, and then four volunteers performed the play unrehearsed, which was surprisingly successful. Then we brainstormed for questions, first in pairs, and then shared. One young man was sure his question was "no good" and refused to share it. I finally convinced him to tell me privately.

I assured him it was a great question, and it turned out to be the one chosen. It was "What happens to an infinitely big cube? Is it still a cube or does it change shape—can there even be an infinitely big cube?"

After going through the voting process to choose this question, debate continued for the rest of the class, about 35 minutes. The conversation still continued when the bell rang! Some students did not actively participate in the shared discussion, but I noticed them talking about the subject with their neighbors.

All in all it was a great experience, and the level of discussion was wonderful. I will certainly do it again. Here's the play:

Variables in the Alphabet Soup

A short play with 4 characters for (unrehearsed) classroom production, to stimulate formation of mathematical philosophical questions for classroom discussion.

Implied metaphors and inspirations:

- Elizabeth Drew's four "types" of gifted individuals;
- The story *Pearls in the Alphabet Soup* by Sandy Shiner, featuring the four Drews "types";
- Individuals in my grade 12 class.

Characters: Bruce, Vicky, Tom, Karen

Scene: sitting around the cafeteria lunch table. Karen is sitting at the table working hard on her math homework with a calculator. Vicky is eating soup. Tom is reading a race car magazine.

Bruce: [saunters in and slams his books and lunch bag on the table]
Karen: [punching numbers in on her calculator] Thanks jerk! You just messed up my numbers.
Bruce: I just had enough stupid numbers in math class with the "Battle ax". Hey don't you ever quit working?
Karen: We've got a test next period. I've got to be ready.
Tom: The test is on mathematical proof. What's a calculator got to do with that?
Vicky: You and your stupid questions! I'm not going anyway, I'm just going to fail.
Karen: How can you say that? You know you want to do well.
Vicky: Yeah but it's easy for you. I'm not good at math. [eats her soup]
Bruce: What kind of stupid soup is that?
Vicky: Alphabet. Wanna word?
Bruce: Cute.
Tom: Why is it always alphabet soup? Why not number soup?
Karen: Numbers, letters, what's the difference?
Bruce: At least letters mean something….
Karen: Don't numbers mean something?
Tom: Tell you what Vicky. Put something on that spoon.
Vicky: What?
Tom: Put one character from the soup on your spoon.
Vicky: [gives Tom a weird look but carefully puts something on her spoon.]
Tom: OK so here's the deal. If it's a number, you go to math class. If it's a letter, you skip.
Vicky: Great. It's an "X".
Bruce: Ha! You fail. [laughs]
Karen: An "X" can *mean* a number.
Bruce: Yeah. It can mean 'wrong' too.
Karen: No, like I mean a variable.
Bruce: Just what I need, math homework I can eat … barf!
Tom: Cool, Karen. Like, the "X" can be any number! So Vicky I take that back, you gotta go to class.
Vicky: [rolls her eyes] Oh brother.

Tom: [getting excited] Yeah, like, just think how many numbers there could be. Maybe even infinite numbers of them.
Bruce: Yeah, and they all fit in Vicky's bowl. [shakes his head]
Vicky: Are you guys on drugs? That's the stupidest thing I've ever heard.
Karen: At least in math you can always tell what's going on. There's always a right answer. You can always find out what "X" is.
Tom: You think so? But what about all those proofs?
Vicky: Yeah, if I could just remember them all…
Tom: You don't have to remember them. Just figure them out.
Vicky: I'll never figure math out, I'm not like you. It's just something you have to memorize. C'mon, let's get this stupid test over with.
[all get up and leave.]

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Appendix A • *The Mathematical Adventures of Michelle and Damian* (Daniel et al., 1996)

Chapter One

Michelle arrives home from school. Kicks off her shoes and goes into the house letting the screen door slam behind her. She goes straight to her room and as usual dumps her backpack in a corner and throws herself onto her bed. Ahhh! How good it feels!

Michelle likes her room. It's small, but comfortable, with a square floor.

"Oh! One could say it's almost a cube! Mrs. Toyama told us about cubes this morning, in geometry class. What exactly did she say?" Michelle frowns, tying to recall.

Slowly the words of her teacher, Mrs. Toyama, come back to her. Things always happen like that in Michelle's head. At first her thoughts form a kind of large dense cloud. Then, one by one, her ideas emerge from the depths of the cloud. It is only then that she can grasp and inspect them.

While continuing to think reflectively, Michelle lets her eyes wander around the room.

She wonders: "Can a room really be a cube or does it only look like a cube? Mrs. Toyama told us, I remember now, that on earth it was not possible to have an absolutely perfect cube. That's astonishing!"

Michelle tries to reflect further on this question, but she is tired. She gets bogged down in her ideas, gets impatient, and finally gives up.

"Tomorrow I am going to ask Mrs. Toyama to explain this. After all, she is the teacher! She must surely know all about geometry."

Michelle's thoughts take wing, freed from their mathematical problem. She starts to dream about Marco. She would so like him to be her boyfriend.

"Ahhh! Marco, what a special boy! He is so different from the others. And I think he's really handsome, even if the other girls in the class don't! And he's intelligent as well. If only he could show a little more interest in me."

Michelle allows herself to be engulfed by her daydreams. Everything is peaceful and pleasant in her room. Then suddenly, she sees a large orange sphere pass just over her head, like a demolition ball shattering walls. Her heart beats frantically, she opens her eyes and recognizes her twin brother who has come into her room and has hurled his stupid basket ball at the wall. What a pain!

• • •

"I have a problem, Michelle."

"Oh yes? Really, me too. You're my problem!"

In fact, Michelle adores her twin brother. They often play together. And they rarely squabble. Michelle very much admires Damian and believes he will eventually become a famous artist. Only she has never got around to telling him that. On the contrary, she always replies to him grumpily.

But Damian knows his sister and he acts as if he hasn't heard anything.

"No, please, listen to me. I have a serious problem. I think I've failed my math test again this afternoon.

"Why do you say that, Damian?"

"What a question! Because that's what I think, see!" replies Damian a bit surprised.

Michelle continues:

"You said: 'I think I've failed my math test again.' What made you say that? Your fear of failing? Or the predictions you spend your time making on the basis of bad experiences last year? Or what else?"

Damian responds honestly:

"I don't really know. It's just an impression I have."

"But have you, at least, good reasons for believing this?" Michelle demands. "Just because you failed some tests last year it doesn't mean that you will fail them all this year."

"I know, but I so hate math!" Damian replies stubbornly.

"Damian, you keep saying this: 'I hate math, students who don't like math are generally losers, so I am going to fail my tests.' With an attitude like that, it is not surprising that you suffer setbacks, you defeatist!"

"I'm not a defeatist but I find that math is good for nothing except provoking stress. And then, it's boring, difficult and so requires lots more work outside class time. Me, I prefer to play basketball with the boys in the class. Or to draw, alone in my room. I am very good at drawing."

"There I agree with you Damian. When it comes to drawing you are really brilliant."

After a moment's silence, Damian adds, eyes full of tears:

"So there, that's what I'm going to do anyway. I am going to draw frustrations in my room. That will do me good."

Appendix B • Thinking Skills Occasioned by P4C/M

Lower-order thinking skills

- Statement of opinion
- Explanation
- Description
- Simple definition
- Observation
- Precision
- Example
- Simple Question

High-order thinking skills

- Formulation of a hypothesis
- Doubt
- Comparison
- Categorization
- Justification
- Criticism
- Counter-example
- Nuance
- Contradiction
- Use of criteria
- Concrete syllogism
- Search for meaning quest

Appendix C • CUBE

A. *Actvity*

- Make up various drawings that could represent cubes.

B. *Discussion plan*

- Are the drawings you just made, during activity A, cubes or do they just look like cubes? Explain.
- What are the differences and the similarities between a cube and the geometric shape that represents it?
- Establish a parallel between a cube and some other elements. For example, take the word *tree*. Is the word itself a tree or just a concept that encompasses every type of trees that exists on the planet?
- What is the difference between the word *tree* and a real tree?
- Now take the name *Mathew*. Is the name Mathew a particular boy or simply a name given to some boys?
- Are there differences between pupils that are named Mathew or are they all identical?
- Now suppose you write a 4 on a sheet of paper and that you loose this sheet of paper. Do you think you have lost the number 4 forever or just the copy you had made of it?
- Can you draw a parallel between the geometric shape we call *cube*, the word *tree*, the name *Mathew* and the number *4*?
- Could it be possible that shapes, names and ideas exist in a perfect state in our minds only whereas their concrete representation is only approximate? Explain.

- Can you answer the question Matilda asks herself, "What is the difference between *being* a cube and *looking like* a cube?"
- Is a cube a square? Explain the similarities and the differences.
- What number does a cube make you think of? 1, 2, 3, 4, 5 or 6? Explain why.
- How many square faces can you find on a cube's surface?
- How many summits can you find on a cube's surface?
- How many edges can you find on a cube's surface?

Does the amount of faces, summits and edges vary according to the size of the cube? Depending on whether we refer to a small cube or a large cube? Why?

C. Mathematical activity

Give two examples of a development plan from which a cube can be built. Give two more examples of a development plan from which a cube cannot be built.

- Which of these development plans looks most like a cube? Why?

The pupils, working in teams, must find a development plan that is different from the previous examples and which will look like a cube when it is folded. Six square pieces of cardboard can be distributed to each team in order to help them imagine their own development plan.

- Can each development plan form a cube? Why?
- What are the characteristics of the development plans which can form a cube?
- What are the characteristics of a cube?
- Is the development plan a cube or does it just look like a cube?

The Arithmetic/Algebra Interface: Implications for Primary and Secondary Mathematics

Articulation arithmétique/algèbre: implications pour l'enseignement des mathématiques au primaire et au secondaire

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Les membres de notre groupe de travail proviennent de contextes différents, autant sur le plan géographique que sur le plan professionnel. Le groupe était en effet composé d'intervenants provenant de l'Ontario, du Québec, du Manitoba, de la Colombie-Britannique, de l'Alberta et des États-Unis, et donc de lieux où les programmes d'études, en ce qui concerne l'enseignement de l'arithmétique et de l'algèbre notamment, n'apparaissent pas nécessairement au même moment ni sous les mêmes formes. Sur le plan professionnel, ils ou elles interviennent comme enseignants ou enseignantes au niveau primaire ou secondaire, en formation initiale ou continue des enseignants au primaire et au secondaire, d'autres ont une expérience aux études avancées, et plusieurs des participants et participantes ont également une expérience d'intervention auprès d'adultes. Les contextes dans lesquels la question de « l'articulation arithmétique-algèbre » se pose sont donc multiples : ils touchent aux élèves de l'école secondaire appelés à faire cette transition (*middle school*), à la formation des enseignants, dans laquelle les étudiants sont appelés à se prononcer sur la compréhension éventuelle d'élèves, et sur leurs différentes stratégies de résolution face à des problèmes, au travail avec des adultes qui ont reçu un enseignement de l'algèbre mais ne l'ont pas réellement appris, et qui rencontrent des difficultés dans ce domaine. Cette variété de contextes, de positionnements vis à vis la thématique abordée dans le groupe de travail a contribué, comme nous le verrons dans ce compte rendu, à enrichir la discussion.

The First Shared Experience

Two shared experiences on the second day produced rich questions and discussions, rich perhaps because of the diversity of perspectives in the group. The first shared experience involved working on the following set of "algebraic problems" arithmetically and considering the differences between the arithmetic and the algebraic reasoning involved.

1) Three tennis rackets and four badminton rackets cost $184. What is the price of a badminton racket if it costs $3 less than a tennis racket?

   Trois raquettes de tennis et quatre raquettes de badminton coûtent 184$. Quel est le prix d'une raquette de badminton si celle-ci coûte 3$ de moins qu'une raquette de tennis?

2) Luc has $3.50 less than Michel. Luc doubles his money. Meanwhile Michel increases his by 1/7. Now Luc has 40 cents less than Michel. How much did each have originally?

   Luc a 3.50$ de moins que Michel. Luc double son montant d'argent. Pendant ce temps, Michel augmente le sien de 1/7. Maintenant Luc a 0.40$ de moins que Michel. Quel montant chacun avait-il au départ ?

3) The dance troupe Petitpas is giving its annual recital tonight. Tickets were all sold ahead of time and the caretaker must now organize the hall. If he places 8 chairs in a row, 3 spectators will not have a chair. If he puts 9 in a row, there will be 27 empty chairs. How many people are expected to attend the recital?

   La troupe de danse Petitpas donne son spectacle annuel ce soir. Les billets ont tous été vendus à l'avance et le concierge doit maintenant organiser la salle. S'il place 8 chaises par rangée, 3 spectateurs n'auront pas de chaises. S'il en met 9 par rangée, il restera 27 chaises disponibles. Combien de personnes attend-on à ce spectacle ?

4) By increasing his speed to 5 km/hr, a cyclist saves 37 minutes and 30 seconds. By diminishing his speed by 5 km/hr, he loses 50 minutes. What is his speed and the length of the track?

   En augmentant sa vitesse de 5 km/h, un cycliste gagne 37 minutes et 30 secondes. En diminuant sa vitesse de 5 km/h, il perd 50 minutes. Quelle est sa vitesse et la longueur du parcours ?

5) A man takes five and a half hours to hike 32 km. He starts by walking on flat terrain and then climbs a slope at 4 km/hr. He turns around at the top and returns on the same path to his starting point. We know he walked on the flat terrain for 4 hours (2 going and 2 returning) and that it took him twice as long to climb the slope as to descend it. Find the length of the flat part of his hike.

   Un homme met 5 heures et demie pour faire un trajet de 32 km. Il commence par marcher sur un terrain plat puis il monte une pente à la vitesse de 4 km/h. Il fait alors demi-tour et retourne au point de départ par le même chemin qu'à l'aller. Nous savons qu'il a marché pendant 4 heures (2 à l'aller et 2 au retour) sur le terrain plat et que la montée de la pente lui prend le double du temps que la descente. Calculer la longueur de la partie plate du trajet.

6) It takes a man five and a half hours to complete a certain hike. He starts by walking on flat terrain at a speed of 6 km/hr and then climbs a slope at 4 km/hr. He turns around at the top and returns on the same path to his starting point. We know that he descended the slope at 8 km/hr and that the length of the slope is 2/7 of the total distance walked. Find the total distance he walked.

   Il faut à un homme 5 heures 30 pour faire un certain trajet. Il commence par marcher sur une partie plate à la vitesse de 6 km/h et continue en montant une pente à la vitesse de 4 km/h. Il fait alors demi-tour et arrive au point de départ en faisant le même parcours qu'à l'aller. Nous savons que la vitesse de descente de la pente est de 8 km/h et que la longueur de la pente est les 2/7 du parcours total. Calculer la longueur du parcours.

Solving the problem arithmetically was not necessarily an easy task for participants. Some of the problems appear to be quite complex. The hope was that it would contribute to our understanding of one facet of the interface between arithmetic and algebra in a particular context, that of problem solving. The two first problems were discussed at length.

Solutions to the First Problem

For the first problem, the rackets problem, the group proposed a variety of solutions and some of these were the basis of considerable subsequent discussion.

Solution A: This trial procedure consisted of choosing a certain given number for the price of a tennis racket, then finding the corresponding price for the badminton racket and then the total amount for 3 tennis rackets and 4 badminton rackets. A new number was tried until the correct total price was reached.

13 + 13 + 13
10 + 10 + 10 + 10

26 + 26 + 26
23 + 23 + 23 + 23

28 + 28 + 28
25 + 25 + 25 + 25

Solution B: The total number of rackets, 7, was divided into the total price, $184, in order to get a ballpark number for the price of a racket. Then a trial and adjustment procedure was undertaken.

184 ÷ 7 = 26 …	First trial: 20	20	20			
		17	17	17	17	total: 128
Then adjustment to a lesser amount:		16	16	16		
		13	13	13	13	total: 93
A new adjustment, raising the price …						

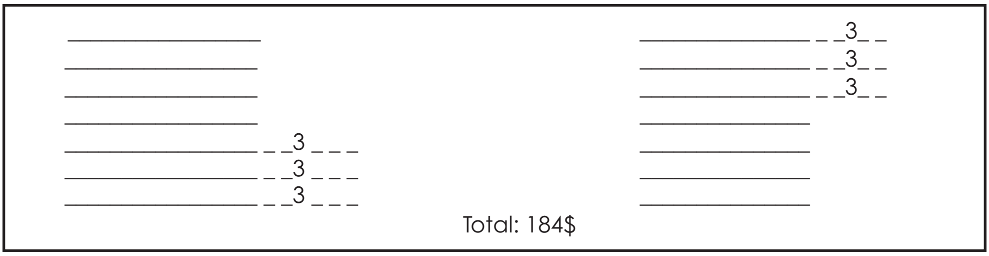
Solution C: This solution began with the fact that 4 of the rackets together cost $12 less. This was subtracted from $184 to get $172 (to have 7 rackets of the same price) and the latter amount was divided by 7 to get 24 and 4/7. The price of the more expensive racket was then fixed at three dollars more, 27 and 4/7. The price of the badminton racket was multiplied by 4 and that of the tennis racket by 3 with the result not coming out to $184.

Solution C led to some discussion on the difficulty of controlling the relationship "three dollars less than" and two corrected solutions (C1 & C2) by the group.

Solution C1: If they all cost the price of a tennis racket, then the bill would be $184 + $12. Dividing $196 by 7 gave the price of the tennis racket, $28 and thus $25 for the badminton racket.

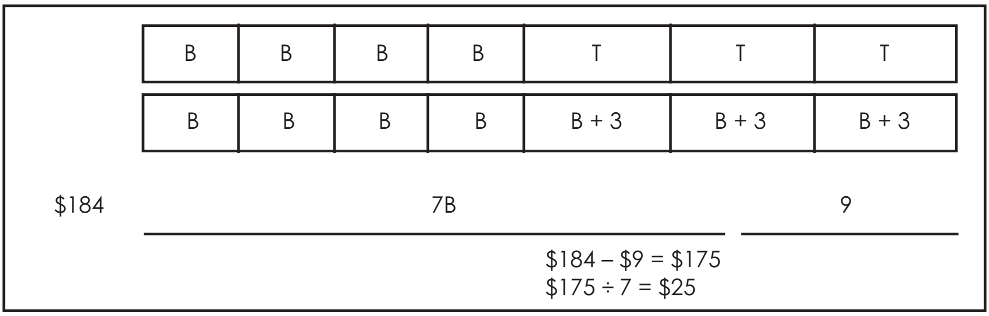
Solution C2: This solution began with the fact that a tennis racket costs $3 more than a badminton racket and so the three tennis rackets would cost $9 more altogether. If they all cost the price of a badminton racket, then the bill would be $184 – $9. Dividing this by 7 gave the price of a badminton racket.

It was noted that it is difficult to decide whether to add or subtract—controlling the relationship "$3 less than" and its influence on the total is difficult—and that sometimes a drawing is helpful in making the decision. Nadine offered her drawing of the problem situation and reminded us that at the turn of the 20th century this type of drawing could be found in the arithmetic problem solving sections of textbooks.



The solution that drew the most attention in the discussion later was the following:

Solution D:



Solutions to the Second Problem

The second problem (Luc and Michel problem) also led to a variety of solutions, the most discussed of which were the two following solutions—reproduced here with an attempt to reflect the way the solvers explained them.

Solution A:

- Michel augmente son montant d'argent de 1/7, on va donc choisir au départ un nombre divisible par 7, disons 7$. Luc a alors 3,50$. Il double son montant d'argent, il a donc maintenant 7$, et Michel 8$ (7$ et 1/7 de 7$). La différence entre leurs deux montants est de 1$, ce qui ne convient pas puisque la différence devrait être de 0,40$.
- On va donc diminuer le montant de Michel choisi au départ pour pouvoir avoir une différence moindre à la fin. Prenons 6,30$ (nombre aussi divisible par 7). Luc a alors 2, 80$ (3,50$ de moins que Michel). Luc double son montant d'argent, il a maintenant 5,60$. Michel augmente son montant de 1/7, soit de 0,90$, il a donc maintenant 7,20$. La différence entre les deux montants est alors de 1,60$. La différence a augmenté et non diminué....
- Il faut donc que j'augmente le montant de Michel et non que je le diminue (plusieurs membres du groupe avaient fait cette erreur, mettant ici en évidence une des difficultés du problème, le contrôle ici de l'effet des transformations sur les grandeurs en présence). Prenons 7,70$ (nombre divisible par 7) pour le montant de Michel, Luc a alors 4,20$. Luc double son montant, il a maintenant 8,40$ et Michel a 8,80$ (7,70$ plus 1/7 de 7,70$). La différence est bien de 0,40$.

Luc		Michel
3,50		**7**
7		8
	différence = 1$	
2,80		**6, 30**
5,60		7, 20
	différence = 1,60$	
4,20		**7, 70**
8,40		8, 80

Solution B:

- La différence entre les montants d'argent de Luc et Michel au départ était de 3,50$.
- Si les deux avaient doublé leurs montants d'argent, l'écart entre ceux-ci aurait alors été de 7$. Cependant cet écart n'est réellement que de 0,40$. On a donc réussi à regagner 6,60$.
- Luc a effectivement doublé son montant d'argent, mais Michel n'a pas réellement doublé son montant d'argent, il a juste augmenté celui-ci de 1/7. Il lui aurait fallu 6/7 de plus pour effectivement doubler son montant initial. Si on rajoutait 6/7 de la part de Michel, on aurait donc regagné 6,60$.
- Les 6/7 (de son montant de départ) correspondent donc à 6,60$.
- Michel avait donc 7,70$. Et Luc avait 4,20$ (3,50$ de moins).

(Montant de Luc au départ) _ _ _ _ _ (Montant de Michel au départ) ________

3,50$ (écart)

(Montant de Luc doublé) _ _ _ _ _ _ _ _ _ _ (Montant de Michel doublé)____________

7$ (nouvel écart)

_ _

0,40$ (écart réel)

_ _ _ _ _ _ _ _ _

6,60$ (regagné sur la différence)

The Second Shared Experience

The second shared experience involved watching a short video extract in which two future teachers were discussing their solutions to the following problem:

Another version of the Luc and Michel problem (distinct from that solved by the group)

Luc a 3.50$ de moins que Michel. Luc double son montant d'argent et Michel augmente le sien de 1.10$. Maintenant Luc a 0.40$ de moins que Michel. Trouve les montants que Luc et Michel avaient au départ.

Luc has $3.50 less than Michel. Luc doubles his money and Michel increases his by $1.10. Now Luc has 40 cents less than Michel. Find the amounts that Luc and Michel had originally.

Excerpt from a dyadic interview

Éric (EC) «algebraic» problem solver, and Mireille (MI), «arithmetical» problem solver. (A partial translation of the verbatim[1] was provided by Nadine.)

Notes that Mireille made as she explained how she solved the problem	L 3,50 M ? ,40 ? 3,50 − ,40 ——— 3,10 + 1,10 ——— 4,20

MV: Okay. Luc has $3.50 less than Michel does (she writes down L,M and 3.50 as above). Now to start with, I suppose that ...

EC: *Michel has at least* $3.50.

MV: Well, let's say ... yeah, you could say that. Okay, Luc doubles his money. ... *Well, when you get down to it, I go about it more using the difference between the two.* I know that he, here there's 3.50 separating them. Uh, Luc doubles his money whereas Michel increases his money by $1.10. So I know that here there was an increase of $1.10. But I don't know the amount that they had (she writes down the two ?)

EC: Okay.

MV: What I do know is that there was a difference and that afterwards, I've got Luc who's now got 40 cents less than Michel (she writes down .40). So I know that the difference between these two (she draws an arrow between $3.50 and $ .40) is $3.10.

EC: $3.10 you say ...

MV: *A difference of $3.10*, and I know already that...$1.10, here there was an increase of $1.10. So normally that would give the amount ...

EC: *... that Michel had*

MV: *Here, that Luc had.*

Because time was short, we did not observe the second video clip in which the same two students worked on the cafe croissant problem (see the problem with contradictory éléments below). We did, however, discuss the problem (the purpose here was to focus on the control of the process of solving problems in arithmetic and algebra) and Nadine provided a verbal description of the student interchange.[2]

The cafe, croissant problem

Au restaurant, une tasse de café et trois croissants coûtent 2,70$. Deux tasses de café et deux croissants coûtent 3.00$. Trois tasses de café et un croissant coûtent 3,50$. Trouve le prix d'une tasse de café et d'un croissant.

At a restaurant, a coffee and three croissants cost $2.70. Two cups of coffee and two croissants cost $3.00. Three cups of coffee and one croissant cost $3.50. Find the price of one coffee and one croissant.

One of the students (EC) immediately attempted an algebraic approach, writing three equations with two unknowns, solving two of them and then replacing the numbers found in the other one. When he observed that when he put them in the third equation, it didn't give him the right answer, he checked his method... He tried again with two other equations...He attempted three other algebraic solutions to the same problem ....he never returned to an analysis of the proposed relationships in the problem. On the other hand, the other student (MI) worked on the basis of the relations in the situation: Here I've got one coffe and three croissants; here I've got two coffees and two croissants, I've got one coffee more and one croissant less, and it costs 30 cents more. Then here, I have the same thing, I've got one coffee more and one croissant less. That costs me 50 cents more! That doesn't work!

Une certaine expérience partagée a ainsi constitué le point de départ de la discussion subséquente du groupe de travail. Elle portait d'une part sur la résolution arithmétique de problèmes usuellement présentés en algèbre, et l'explicitation de diverses solutions par les participants, et d'autre part sur le visionnement d'un extrait de vidéo dans lequel deux étudiants en formation des maîtres confrontaient leurs solutions (arithmétique et algébrique) à un même problème.

Les diverses solutions proposées par le groupe à certains problèmes, les discussions qu'ont provoquées certaines solutions plus spécifiquement, les réflexions issues de notre observation du vidéo par ailleurs, ont permis d'ouvrir sur un certain nombre de discussions. Nous reprendrons maintenant quelques uns des points les plus importants.

Qu'est-ce que l'arithmétique? Qu'est-ce que l'algèbre?

Le travail sur les différentes tâches, notamment la résolution arithmétique de problèmes, nous a amené à discuter longuement la différence entre arithmétique et algèbre: en quoi peut-on dire que cette solution est arithmétique ou algébrique? Où s'arrête l'arithmétique? Où commence l'algèbre? Par exemple certaines solutions arithmétiques présentées au problème des raquettes ont suscité une interrogation par certains participants: en quoi pouvait-on dire que cette solution était arithmétique et non algébrique? Ainsi, si une ligne (voir solution C, dessin proposé par Nadine) ou une boîte (voir solution D) est utilisée pour représenter les grandeurs en présence et leurs relations, ceci n'est-il pas une certaine façon de représenter l'inconnue et ne peut-on dire dans ce cas que le processus de résolution est algébrique? Le fait que certains élèves qui n'ont jamais reçu d'enseignement de l'algèbre produisent de telles solutions, ou encore que l'on retrouve des illustrations semblables (solution C) dans de vieux manuels d'arithmétique, plaident toutefois en faveur de voir celles-ci comme des solutions arithmétiques. La question de savoir si une solution est arithmétique ou algébrique est vite apparue au groupe comme risquant de nous enfermer dans une discussion stérile, et celle-ci a été abandonnée au profit de l'intérêt qu'il pouvait y avoir à encourager certaines solutions particulières dans une perspective de transition à l'algèbre.

What is arithmetic, what is algebra?

Some of the arithmetic solutions produced by participants led to challenges by others as to whether or not they could also be classified as algebraic. For example, if a line or box is used to represent the unknown amount, is that just another way of representing the unknown and is the solution process essentially algebraic? The fact that some students who have never been exposed to algebra produce such solutions argued in favour of viewing these as arithmetic. Trying to reach a conclusion about whether such a solution was arithmetic or algebraic was eventually abandoned in favour of a discussion of the interest of encouraging this particular type of solution as a stepping-stone to algebra.

Potentiel de certaines solutions arithmétiques pour un passage à l'algèbre?

Le travail autour des solutions proposées par le groupe à quelques problèmes nous a amené à discuter très longuement du potentiel de certaines de ces solutions: en quoi ces solutions sont-elles porteuses de sens, riches pour un éventuel passage à l'algèbre? Comment favoriser la transition à partir de celles-ci à l'algèbre? Par exemple, les solutions arithmétiques mettent dans certains cas en évidence un contrôle très grand des relations en présence ou des transformations sur les grandeurs (voir les solutions C1 et C2, ou D au problème des raquettes, ou la solution B, au problème de Luc et Michel) ou s'appuient sur des propriétés des nombres (exemple de la solution A, au problème de Luc et Michel), elles constituent un atout important dans la mathématisation des problèmes en algèbre. Ces solutions rendent compte par ailleurs dans certains cas d'une notation ou représentation globale intéressante (exemple de la solution C, illustration proposée, ou de la notation utilisée en D au problème des raquettes). Cette dernière solution apparaissait même a priori pour plusieurs des participants très proche d'une résolution algébrique (faisant référence à une notation symbolique et semblant opérer sur ce symbolisme dans la notation même utilisée). Toutefois, la discussion a mis en évidence que la lettre ici ne joue pas vraiment le rôle d'inconnue. Elle sert juste à désigner les quantités en présence, c'est en quelque sorte une étiquette (badminton, raquette), ce qui constitue en fait un obstacle dans la résolution algébrique ultérieure (où la lettre représente un nombre et la symbolisation des relations exprime une relation entre des nombres qui n'est pas une traduction directe). La question de la transition à l'algèbre exige donc davantage. Elle peut sans doute s'appuyer avantageusement sur les habiletés développées en arithmétique et certaines de ces solutions, mais le passage n'est pas une simple transposition d'une procédure à l'autre.

Do certain arithmetic solutions lead more easily into algebraic solutions?

Some arithmetic solutions produced in the group seemed to be more meaningful and offer a greater potential for an eventual passage to algebra. The question arose as to how to move from these particular solutions into algebraic ones. For example, arithmetic solutions exhibiting mastery of the relationships in the problem or of transformations of quantities (see solutions C1, C2 and D to the rackets problem, or solution B to the Luc and Michel problem), demonstrate skills that are important in mathematizing problems in algebra. These solutions also show interesting notation or global representations of the problem (for example, the illustration used in solution C and the notation in D). For many of the participants, solution D appeared to be very close to an algebraic solution in that it involved symbolic notation and operated on that notation. However the discussion brought out the view that the letter here did not really play the role of an unknown. Rather, the letter designated the quantities present and acted as a label (which is considered to be an obstacle in later algebraic work). Thus the passage to algebra requires additional insights and skills. It can certainly build on arithmetic skills and solutions but the passage is not a simple transposition from one to the other.

Tension dans la transition arithmétique-algèbre entre contextualisation et décontextualisation

Un point important soulevé par le groupe et sur lequel nous nous sommes longtemps attardés est celui de la tension, dans la transition arithmétique-algèbre en résolution de problèmes, entre la nécessité de partir du contexte, pour construire notamment un sens à l'expression algébrique élaborée ou à toute autre représentation, et la nécessité de quitter le contexte pour aller plus loin dans la résolution. Dans nos solutions arithmétiques, nous nous appuyons en effet fortement sur le contexte, interprétant constamment les quantités et relations en présence pour pouvoir opérer. Chaque partie de la solution s'appuie sur le contexte, peut être vérifiée en regard du contexte. Nous reconnaissons que tel n'est pas le cas en algèbre, où le contexte sert seulement au début de la résolution du problème lors de la construction de l'équation ou des équations, et à la fin du processus dans l'interprétation de la solution trouvée. Plusieurs des participants du groupe pensent que cet abandon du contexte est un des gros obstacles dans la résolution de problèmes en algèbre. Le problème du café croissant, et sa résolution par les deux étudiants en formation à laquelle nous avons fait allusion précédemment, le montre bien et fournit plusieurs arguments en faveur du maintient d'un lien avec le contexte. Ceux qui en effet s'engagent dans une résolution algébrique semblent tourner en rond, essayant de résoudre à plusieurs reprises deux équations à deux inconnues puis de remplacer dans l'autre équation, sans nullement contrôler ce qui s'y passe. Ceux qui essaient de résoudre arithmétiquement le problème restent en contact avec le contexte, et rapidement réalisent que le problème ne fonctionne pas, qu'il n'y a aucune solution.

Nous nous sommes demandés s'il ne serait pas possible de prévoir des allers retours entre contexte et manipulations algébriques, au moins dans les premières étapes de l'apprentissage de l'algèbre. Bien que cette question soit restée ouverte, quelques oppositions à cette idée ont été exprimées. Tout d'abord, l'algèbre est un outil pour résoudre une classe plus générale de problèmes et sa puissance réside justement ici dans l'abandon du contexte, d'autre part, il est extrêmement difficile, voire impossible, de donner à chaque étape de la manipulation une signification dans le contexte.

Tension between keeping the context and working in abstract

In our arithmetic solutions we were constantly reading and interpreting the problem situation. Every line of the solution was or could be checked for sense in the context of the problem. We recognized that this is not the case in algebra where the context serves only at the beginning and end of the solution process. Several participants thought that this abandonment of context was one of the big obstacles to problem solving with algebra. We wondered whether it would be possible to move back and forth between the algebraic manipulations and the context, at least in the early stages of algebra. Although this question remains open, there was some opposition to the idea: firstly, algebra is a tool for solving general problems and its power lies in the abandonment of context and secondly, it is extremely difficult to do—perhaps more difficult than the manipulations themselves.

The café / croissant problem above provided some arguments in favour of maintaining a connection with context. Those who leaped into an algebraic solution ended up going around in circles trying to solve 3 equations in two unknowns. Those who looked at the problem arithmetically or stayed in touch with the context, quickly realized the impossibility of a solution.

Survalorisation de l'algèbre et dévalorisation de l'arithmétique en résolution de problèmes

Dans l'extrait vidéo que nous avons visionné, la difficulté du futur enseignant de mathématiques au secondaire (ER) à comprendre la solution arithmétique produite par l'autre étudiante (MI), et son absence de volonté apparente à vouloir comprendre celle-ci, ont questionné les participants. À l'opposé, bien que MI ait eu de la difficulté à suivre le raisonnement algébrique de son coéquipier, elle a fait l'effort de comprendre celui-ci et a été tout à fait capable à la fin d'expliquer et de refaire ce raisonnement. Derrière l'indifférence du solutionneur «algébrique» envers la solution arithmétique de l'autre (qu'il perçoit comme de la magie), il est possible d'y lire une certaine supériorité de l'algèbre sur l'arithmétique, ce que le groupe a nommé «une certaine arrogance de l'algèbre».

Venant de travailler nous-mêmes sur des solutions arithmétiques à des problèmes, nous étions naturellement impressionnés par la solution arithmétique de MI et par le raisonnement sous-jacent mis en jeu. D'où notre étonnement à voir l'inhabileté du solutionneur algébrique à apprécier, lui de son côté, cette solution arithmétique. Les conséquences d'une telle attitude selon nous dans la classe sont importantes. Elle questionne en effet la capacité du futur enseignant à comprendre les stratégies des élèves et amène à penser que lorsque l'algèbre est introduite, tout raisonnement arithmétique est de fait évacué. Ceci peut nous laisser penser que le raisonnement arithmétique est de fait, au moment de l'introduction à l'algèbre et après, négligé, voire même qu'il régresse. Ceux qui ne rentrent jamais dans l'algèbre courent ainsi le risque d'être laissés de côté avec aucun outil pour résoudre les problèmes, et aucune confiance dans leur capacité à résoudre des problèmes.

Du point de vue de la formation des maîtres, un travail important est à faire, en valorisant entre autres les enseignants qui veulent comprendre les stratégies premières des élèves.

To what extent is algebra over valued and arithmetic under valued in problem solving?

In the video extract, we were all slightly appalled by the future high school teacher's inability and unwillingness to understand the arithmetic solution produced by the future special education teacher. Although the latter found her partner's algebraic solution difficult to follow, she did make the effort to do so and in the end was able to follow it. Behind the indifference of the algebraic solver towards the arithmetic solution, we read a sense of superiority attributed by both students to the algebraic solution and coined the term "the arrogance of algebra". Having just worked on arithmetic solutions, we were naturally quite impressed with the arithmetic solution and the mathematical reasoning involved. It was worrisome to recognize in the algebraic solver the inability to appreciate an arithmetic solution. The consequences of this in the classroom assure that once algebra is introduced, all arithmetic reasoning is outlawed. Hence arithmetic reasoning atrophies and those who never quite "get" algebra are left with no tools and no confidence to solve mathematics problems.

In teacher training, it also seems important to value teachers who want to understand students' primitive strategies.

Devrions-nous introduire des problèmes «d'algèbre» en arithmétique?

Nous avons aussi discuté la pertinence qu'il pouvait y avoir à introduire des problèmes classiques d'algèbre, comme ceux que nous avions examinés, pas nécessairement les derniers considérés comme complexes pour les élèves, mais d'autres plus simples, avant toute introduction à l'algèbre. Beaucoup d'arguments en faveur d'une telle introduction ont été mis en évidence par le groupe:

- Il y a plusieurs stratégies de résolution possibles comme nous l'avons vu, dont le potentiel est riche pour le développement d'habiletés en résolution de problèmes: essais erreurs raisonnés s'appuyant sur certaines propriétés des nombres; fausse position: on fait semblant que...en se donnant un nombre et on réajuste; travail sur les relations et comparaison. ...
- Le recours à plusieurs méthodes de résolution fait partie du curriculum (est requis par celui-ci)
- Il semble toutefois important dans ce travail d'aller au delà de la simple procédure d'essais-erreurs pour forcer une réflexion sur les relations. L'arithmétique, si elle est un appui important pour le passage ultérieur à l'algèbre, doit être une arithmétique relationnelle.
- Les notations utilisées, la manière dont nous rendons compte de ces stratégies, dont nous les explicitons, est aussi un appui important pour le travail ultérieur en algèbre: notations séquentielles versus notations globales (rendant compte de l'ensemble des relations en présence), recours à des illustrations aidant à contrôler les relations, présence possible d'une riche variété de notations, représentations (diverses représentations explicitant l'ordre de grandeur des notations ou leurs relations, notations symboliques ...)
- Le travail sur différents types de problèmes est possible: travail sur des régularités (exemple trouver la somme des 45 premiers nombres entiers rapidement...), problèmes mettant en jeu des relations de comparaison, développant une flexibilité à jouer avec ces relations de comparaison (se les représenter, les formuler de différentes façons ...)
- Il est possible de discuter certains critères avec les élèves dans le retour sur les stratégies (clarté à des fins de communication de celles-ci à quelqu'un d'autre, efficacité, certaine stratégies sont-elles plus efficaces que d'autres?: qu'arrive t-il si l'on change certaines données du problème, la solution fonctionne t-elle encore? ...)

La question de savoir si nous devrions enseigner certaines de ces stratégies , si nous devrions parfois insister sur la mise en évidence de certaines stratégies plutôt que d'autres a été posée.

Should we introduce "algebra" problems in arithmetic?

A number of arguments were made for introducing classic algebra problems such as those examined in the working group before any introduction of algebra.

- As we saw, a number of solution strategies emerge that are potentially rich for developing problem solving abilities: trial and error strategies grounded in number sense, trial and adjustment, work on relationships and comparisons. (It is important though to go beyond trial and error strategies and move to reflection on the relationships between the quantities in the problem.)
- The use of a variety of solution strategies is required by the curriculum.
- The notation used (sequential as opposed to global notation, recourse to illustrations, multiple notations and representations) and the ensuing discussions are important for future work in algebra.
- Work on a wide variety of problems is possible: on patterns or regularities (for example, find the sum of the first 45 whole numbers rapidly), on comparisons (expressing and representing them in different ways), …
- In discussions of strategies with the students, criteria can be established for clarity in presentation, efficiency in solutions (Are some solutions more elegant than others? What happens if we change some of the givens in the problem?)

The question arose as to whether or not we should teach certain of these strategies.

Devrions-nous continuer le travail en arithmétique après l'introduction de l'algèbre?

Notre travail sur les problèmes nous a amené à la conclusion que l'arithmétique ne devrait pas être mise de côté dans le travail en résolution de problèmes, après que l'algèbre ait été introduit. Nous avons trouvé quelques unes des solutions arithmétiques que nous avons partagées riches sur le plan du raisonnement mis en jeu. Toutefois, encourager des solutions arithmétiques et continuer le développement de l'arithmétique chez les étudiants tout au long de l'école secondaire ne fait pas vraiment partie des expériences des participants. Le curriculum arrête en effet l'enseignement de l'arithmétique en général lorsque l'algèbre est introduit.

Dans le programme récemment introduit en France, on réintroduit cependant l'arithmétique dans les dernières années du secondaire et même au niveau postsecondaire. Il reste à voir ce qu'on appelle ici arithmétique. Il y a un intérêt à considérer les raisons qui ont conduit les responsables de ce curriculum à vouloir y réintroduire l'arithmétique, ce dernier contribuant selon eux au développement d'une certaine rationalité mathématique. Les participants du groupe ont mentionné l'intérêt que pourrait avoir un tel travail en arithmétique, articulé par exemple autour de la théorie des nombres, pour le développement du concept de variable. La réflexion est ici à poursuivre.

What would be the benefits of continuing the study of arithmetic throughout secondary school?

Our experiences in the group work led us to the conclusion that arithmetic should not be set aside in problem solving work after algebra is introduced. We found some of the arithmetic solutions to the shared problems both simple and mathematically exciting. However, encouraging arithmetic solutions and continuing the development of students' arithmetic throughout high school had not been part of the experience or expectations of any of the participants. Nadine spoke about the new programs in France where arithmetic has been re-introduced in the last years of secondary school. There was considerable interest in that curriculum and conjectures that students would be much better prepared for tertiary mathematics particularly in the area of number theory.

One learning shared by all was that the meeting of arithmetic and algebra does not just occur in a year or two somewhere in middle school. It impacts on all of us wherever we are intervening in the school system.

Notes

1. Pour la recherche plus complète dont est tiré ce verbatim, voir S. Schmidt & N. Bednarz (2002), Arithmetical and algebraic types of reasoning used by preservice teachers in a problem-solving context, *Canadian Journal of Science, Mathematics and Technology Education, 2*(1), 67–91.
2. For more details, see Schmidt & Bednarz (2002).

Report of Working Group D

Mathematics, the Written and the Drawn

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Participants

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The written and the drawn—at first blush, two faces of mathematical communication on paper. I put pen to paper to express and perhaps communicate my ideas (in words, in drawings) across distance; more ambitiously, across time. But then I find myself putting pen to paper when doing mathematics by myself, in the moment, for myself. What is it I capture there, in word and image? And in what sense is it 'captured'? Words tell stories spreading across time; images paint pictures spreading across space. Somewhere uncomfortably between them falls mathematical symbolism, which we read in words; write in pictures; and imagine thereby to convey meanings that transcend both time and space. Much later, I attempt to communicate mathematics to you. What shall you make of what I have written; of what I've drawn?

In customary manner, our group's discussions were focused and informed by a series of group tasks spread over the three days, which is also how this report is structured, that is to say narratively rather than thematically. We can, as ever, only attempt to provide a brief taste of the tenor of engagement which occurred in this very coherent and stimulating group, illustrated / punctuated with comments from the participants (given in inverted commas and in italics) which were gathered towards the end of the third day in response to the question of what had stood out for them over the course of the group's working life together.

Day 1 • Saturday

Following a general introduction including working on some quotations from artists, a theme emerged which we explored more fully over the course of the sessions, namely that of the role of time in mathematics, not only in relation to images but also in the context of textual proofs. It seems to us an important and relatively unexamined notion with regard to mathematics. In order to approach this idea, the following task was offered to the group, who generally worked either alone or, predominantly, in pairs.

Task 1: A Euclidean cut proof

The order of the sentence statements in this proof have got scrambled and the first word(s) of each sentence cut off and placed in a pile. Can you discover the original, correct order to restore the proof? How did you work on this task?

Prime numbers are more than any assigned multitude of prime numbers.
[Euclid IX. Prop 20]

1. … it also measures EF.
2. … G is not the same with any of the numbers A, B, C.
3. … it be prime; then the prime numbers A, B, C, EF have been found which are more than A, B, C.
4. … it be measured by the prime number G.
5. … G is not the same with any one of the numbers A, B, C.
6. … the prime numbers A, B, C, G have been found which are more than the assigned multitude of A, B, C.
7. … if possible, let it be so.
8. … the least number measured by A, B, C be taken, and let it be DE. Let the unit DF be added to DE.
9. … EF not be prime; therefore it is measured by some prime number.
10. … G, being a number, will measure the remainder, the unit DF; which is absurd.
11. … by hypothesis it is prime.
12. … A, B, C measure DE; therefore G also will measure DE.
13. … EF is either prime or not.
14. … A, B, C be the assigned prime numbers. I say that there are more prime numbers than A, B, C.

Choose beginning words from the following list:

Then, First, Let, For, I say that, Now, Next, But, Therefore, And.

For us, one question which this task gave access to was the notion of a 'coding time' of a proof, that is where is the temporal centre of gravity of a proof by contradiction (as this is) and it was observed how many of the words used as logical 'glue', marking the structure of relatedness among consecutive and adjacent sentences, have both a temporal and logical sense in English.

"I have never done something like this before. This was wearing a real student hat rather than a pretend one. But my greatest appreciation was the pedagogic fallout when my attention was drawn to the word 'assigned' in the proof."

"The struggle with the Euclidean proof. What do you do to begin the task? Try to make sense of the assertion, identify 'troubling' words. Still not sure, but try to connect the opening words at the bottom of the page to some of the fourteen statements—a few seem to fit. Next highlight all the statements with a G or a DE in them and begin to see an emerging structure. Still muddy water. Take a stab maybe statement 14 is first followed by 8. We are triggered to think about the word 'absurd'. We began to appreciate how certain words are clues to structure and to appreciate the meaning of the word' measure' in a Euclidean sense and how it contrasts with our present meaning. Finally, we start to appreciate how pictures/diagrams are critical to understanding words like 'measure'."

"I was struck by the fact that even though I knew the way the proof goes, it still took a bit of effort to reconstruct, partly because the language was unfamiliar. The fact that Euclid proves the general case by a generic example, using just three primes, led me to think I could be a bit loser in my own presentation of proofs. A generic case that captures the essence of the proof ought to be enough."

Task 2: Drawing with words

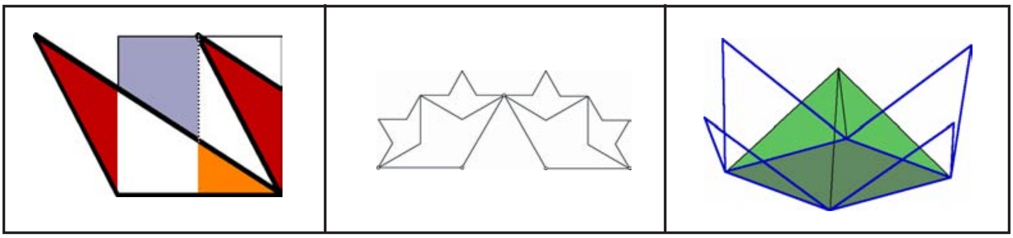

FIGURE 1.

Participants were paired and invited to sit back-to-back and to describe a static image (some of those we used are given above) provided to their partner who had to recreate it with pencil and paper from the words alone. We were endeavouring to get away from the oppositional words *versus* images to examining the differing strengths of both and to work on discovering the challenges transmuting one into the other. This sequence requires first images to be translated into spoken words by the 'encoder' and then those spoken words translated back into an image by the 'decoder'. Another way of describing this is to talk in terms of description and transcription.

Participants found parts of this task challenging. We discussed the question of a 'grammar of images' and how speech is linear in time whereas a diagram is not. What is involved in 'reading' a diagram? There is choice about where to begin and whether a global image is described, whether there are structured elements to clue into and draw on in the description or whether it is seen as an exercise in drawing with the speaker remotely directing the pen of the drawer. Words can both trigger and substitute for images. Images prompt words and reciprocally.

"The A-B back-to-back was illuminating. It reveals the illusion we have as instructors that what we describe is clear to our students—even though they don't have our background for interpretation."

"Word and image are two seemingly interesting dichotomous objects. However, as with all other dichotomies, they leave unsaid more than they suggest. But the process of generating images or generating words or of linking images and words involves a dynamic to-ing and fro-ing, often between words and images (or images and images, etc.). This dynamic process which is at the heart of our thinking is not captured by the dichotomy."

"In communicating images to others, it can be very helpful to others to use words in several different ways: globally, procedurally (how to go about constructing the image using a certain aspect as a starting point, etc.). there again, images and words work together in a dynamic way."

"The task of rearranging the steps of the prime number theorem was probably more difficult than that of communicating an image, but was unseen to our partner because there was less of the visual to draw upon. The productive dynamic between word and image was, thus, much less present."

Day 2 • Sunday

Image Archaeology using Dynamic Geometry Software

Day 2 was given over to explorations of the particular form of imagery associated with educational mathematics software such as *The Geometer's Sketchpad* or *Cabri*. Images in these environments are (usually) mathematical figures and diagrams that evolve and deform continuously in response to manipulation by the computer mouse, but that across such deformation maintain defining properties and essential invariances. Others (e.g., Scher, 2000) have written about how compelling these images are—at once familiar from our imaginations, and yet now, embodied and palpable, and therefore, more certain. In our day's experiments, we used *Sketchpad* to investigate how much of an author's story we can reconstruct from Dynamic Geometry artifacts.

In the first task, pairs of participants worked together to construct as many rhombi as they could within the Dynamic Geometry environment. While perhaps intended more to provide an orientation to the software tools than a task in itself, of course in discussion even this chore became fertile. Since each Dynamic Geometry figure has an unlimited number of appearances and configurations, participants had to come to a stronger definitions of "different rhombi" than governs variation in size and angles, in location and orientation. And once converging to a notion of "different *construction*" rather than simple "different *appearance*"—even to different construction even when appearance was unchanged—we still needed strategies for generating different constructions, given that many of us were far from our days of compass and straightedge facility.

This first task became more difficult and thought provoking at its mid-point, when each pair of participants had a virtual desktop collection of their various "successfully different" rhombi, as well as several partial rhombi or constructions abandoned somewhere along the path toward possibly becoming rhombi. At this point, opposing pairs switched computers, and the task became to examine the unfamiliar work now before us, and to reconstruct—for both the successful rhombi and the failed ones—what definitions of, or insights into, the nature of the rhombus had illuminated each of our now absent authors' construction attempts.

Here obviously we were both directed and constrained by the modes of inquiry facilitated by the software itself. Over the course of the activity, several clear approaches emerged. Through Sketchpad dragging, we could probe the dynamic behavior of the rhombus, almost as if it were a living thing. This takes the authors' work and invests it with a certain premise of purpose, and of achievement of that purpose. Through examining the Script View of a rhombus, we could read a propositional description of the salient mathematics of the construction—but (as with many a published paper in mathematics) in a format and a presentation that obscures much of the process through which its results were achieved.

Finally, by stepping forward and backward through the unlimited Undo and Redo "history" that the program stores of a user's work, we could follow and re-enact a step-by-step transcript of the absent authors' work, as codified or digitized by the software. This view far more than the other two was likely to reveal the distribution of the authors' efforts across the totality of their work product, as well as the inevitable tangents, digressions and dead-ends hidden from the final work—but at the same time, in its microscopic detail and absent any illuminating commentary, this form of inquiry lead to a vision of a process far more homogenous in texture than either dragging or the Script View, which were more likely to indicate what was significant or important in a construction even as they cast into shadow processes by which such attributes came to be.

A second activity revisited these themes from a perspective possibly closer to a typical student's. Here we provided Sketchpad microworlds (called "Drawing Worlds"—see Jackiw, 1997) in which the traditional compass and straightedge tools of the program had been replaced by "broken" equivalents, but in which the manner or effect of "brokenness" was systematic and amenable to mathematical characterization. (For example, the figure below shows an example straight line and circle drawn by the broken straightedge and compass tools of one of these microworlds.) Our charge was to use these broken tools to devise and execute strategies that explained how they had been broken, and—perhaps—what steps we might take to fix them. (In our example, the tools are incapable of rendering outside of some upper-right quadrant, and reflect everything into that quadrant. In this sense, they are "absolute value" tools—they'll work fine, provided we limit ourselves to the upper-right quadrant!)

While the mathematical means by which these suspect tools had been broken were rarely more sophisticated or involved than the mathematics of our previous rhombi constructions, here they were intentionally hidden. In our initial pursuits, participants relied heavily on the mathematical language of composed transformations and of complex conformal mappings to describe the pictures drawn by broken tools. In many cases, these vocabularies eventually seemed to be overkill, imposing an assumption of a certain level of complexity on problems usually involving (though not transparently!) simpler geometric operations.

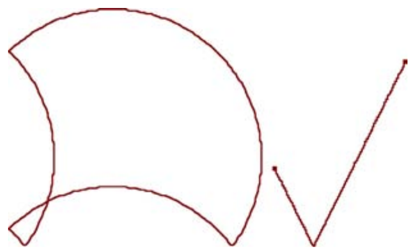

FIGURE 2. The Absolute Value Drawing World

"On day 2, my eyes were opened to powerful new ways to look at GSP can be employed. Pictures were constructed then deconstructed. A grid is such a powerful image to attest to in understanding the essence of unknown geometric transformations."

"I want to experience my activity in many ways—'I think by moving the mouse'. I was struck by how different representations form and illuminate. Language is linear, happening in time. Deliberately choosing to use/not use particular ways of representing a situation (e.g., algebraic, geometric) can create new connections, possibilities, insights."

"In dynamic geometry environments, as one is controlling the mouse, one is constantly making conjectures and testing them, conjectures which are an amalgam of interior words, thoughts, images."

"In the rhombus activity, getting into the geometry and having time to play was important; then, being able to play detective and look into others' constructions using 'undo' presented a new perspective. It brought to mind an idea about how to delve into the mind of a sketch creator (if one of our roles as teachers is to unpack the mathematics, then this is a useful vehicle). When you're faced with a picture, you can analyse it and derive meaning from it by focusing on various points, colours, etc., but you can't undo to see how it was made. In some cases, you probably don't care whether the artist drew the mast first or last, but in a geometry diagram, the unpacking using undo can reveal relationships that promote understanding." (For more on this point, see Nunokawa, 1994.)

Day 3 • Monday

Task 1: Monstrous visions

In this brief task, the group "confronted" a symbolic expression of no small visual complexity.

$$\frac{\sin\left[t\sqrt{x^2+y^2}-m\right]}{\sqrt{x^2+y^2}}+\frac{\sin\left[t\sqrt{(x-x_a)^2+(y-y_a)^2}-n\right]}{\sqrt{(x-x_a)^2+(y-y_a)^2}}$$

FIGURE 3. A Symbolic Monster

The goal of the exercise—which requires some preliminary group preparation—is to attend particularly to how, as individuals, our eyes track in their first seconds of vision over the expression's two-dimensional notational layout; to how we begin, visually, to harvest evidence of sense from the symbolism before (or as a very initial part of) developing a more thorough understanding of the expression's formal or propositional meaning. Where the broad purpose of the task was to add an encounter with symbolic notation, which sits so tenuously on the interface between the written and the drawn, to the group's catalogue of representational experiments, the task more directly seeks to experiment with Walter Whiteley's (2002) observation that, as a mathematician, he sees fundamentally differently from his students; that not only does (in this case, mathematical) experience shape sense perception at the most basic and immediate level (as Hoffman, 1998, develops at length), but that—in the case of the mathematician—this cognitive conditioning affects the apprehension of symbolic landscapes as well as iconographic or pictorial ones.

In your first moments inspecting the expression above, how do your eyes move, and what do you see? Raw alphabetic cacophony? A surface? A sine function? A sum? Second, how do these differences play out categorically? Whiteley's formulation distinguishes between the vision of "the mathematician" and "the student", but these are more evocative than rigid taxa. Does the experiment reveal any useful axes of distinction across the somewhat homogenous, somewhat diverse population of a CMESG working group?

Each of the "first visions" mentioned above (cacophony, surface, sine, sum) came from participants responding in the group, and each clearly construes the expression from a distinct perspective. To see it as a surface, for example, is to take a poet's or modernist painter's view: the choice suggests not only the strong preference for a palpable geometric dimensionality over the two wooden understudies named x and y, but also indicates a willingness to forgo all the potential meanings and still shadowy impact of the many symbols *beyond* x and y, to transcend that promised specificity in the leap to a geometric "sense" of the expression.

By contrast, seeing the expression as a sum suggests a grammarian's more clinical eye, pruning back all of the visual flow—of subscripts and radicals and brackets—to its barest symbolic skeleton parsed by order of operation (not that of left-to-right processing of written English). But many in the group found themselves drawn first to the **sin()** elements in the expression, over the square roots and squarings, the additions and divisions, the x and y and m and n. Is this the readerly eye's conditioned attention to, and preference for, letter strings it can concatenate into recognizable words? Or do such responses stem more from the mathematical familiarity of the syntax of sine functions than from the lexical weight of three-letter sequences (s, i, n) over solitaires (x, a, m)?

In trigonometry and elsewhere in school mathematics, sine ratios inexorably summon ideas of right triangles and unit circles, but sine functions invoke contexts in which **sin** often binds its neighboring coefficients and arguments into ideas of amplitude, frequency and phase. When we confront the monstrous expression above, does the associative strength of such templates offer hope that we will eventually squeeze all the rogue symbolic ink beyond **sin** into these supporting roles, these adjectival positions that only qualify or specialize the sine?

If the most immediate result of the experiment was to document the variety of things we think we see when we look at the same set of mathematical symbols, perhaps the least anticipated result was the degree to which our vision comes fully vested with our opinion, taste and prejudice. The moment the expression in question was revealed on the board, more than several members of the group issued loud groans—as if the act of beholding the expression above is itself somehow unpleasant. And—after a moment's silent feasting on the expression—once participants began describing the details on which their eyes first focused and the perceptions those first sights invoked, other participants in the (usually very collegial) group spoke out several times to object, to argue with or deny their colleagues' offered interpretations. This sort of response and debate would be fully expected if we were expected to "solve" the expression or "explain its meaning", but both before first revealing it and again throughout the discussion, the group clearly understood the task as "to say what you see", not "to say what it means". Thus we neither liked *what* we saw (the groans); nor *how* others saw (the arguments, the rejections). Disturbing.

To turn from the group's particular experience in response to this task to its wider concerns—the interplay of mathematical media and mathematical meaning—it is worth noting the provenance of the particular expression used in this task. This particular symbolic fragment is an end product of a recreational mathematics idyll in which one of us

(Jackiw) attempted to model smooth water surfaces disturbed by scattered pebbles—to build a Zen water fountain with dynamic graphing software (*Geometer's Sketchpad*). This activity began in the plane with a simple wave model graphed as $y = \sin(x)$. Adding a phase term, t, that changed over time causes the waves to "travel" with time, and dividing the result by x diminishes a wave's amplitude the further it travels from the origin (or "splash"). At this point, one has a reasonable model of a two-dimensional wave, except for a fracture in the water surface where x changes from positive to negative. Replacing x in the expression with its absolute value—in other words, using distance from the origin rather than signed distance—results in a symmetric and unfractured waveform, symbolically given as

$$y = \sin(|x| + t) \,/\, |x|$$

FIGURE 4. 2-D Portent of the Monster

To translate this idea to three dimensions simply involves replacing distance to the origin *along the number line* (the absolute value of x) distance to the origin *in the plane* (the square root of $x^2 + y^2$), so that the wave radiates across the plane in concentric circles from the origin. We are now quite close the "monstrous symbolism" of the task, which sums two such expressions to represent two pebbles splashing (one at the origin, the other shifted from the origin at some imagined point a), two sets of waves radiating and colliding. Where the symbolism seems monstrous, the graph of the surface $f(x, y)$ defined by it, delights.

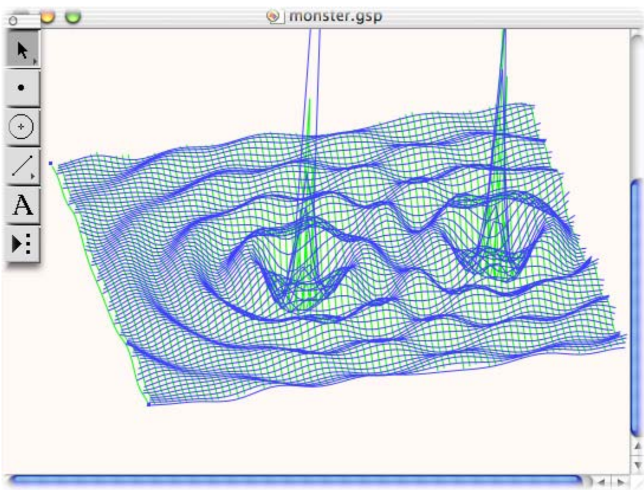

FIGURE 5. Two Waves Colliding in a Pool

The purpose of recounting this derivation in such detail is not to suggest that the anecdote retroactively arbitrates the correctness of the various interpretations that came up during group activity, but rather to inspect some of the group's developments from the perspective of one (possible) more fully-realized interpretation of what the symbolism represents. Where the group spread across (and even argued among) alternate visions of sum, sine and surface, we see in the anecdote an interpretation that reflects all three visions, with no one in dominance. (As the title of the previous illustration is "Two waves colliding in a pool", the interpretations—of sum, sine, and surface—correspond to alternating stress on the words "Two", "Waves", and "Pool".)

Second, we note that once we have access to a substantiating derivation, meaning begins to accrue to various parts of the expression as the geometric backstore shines light on individual symbolic ornaments: in the radical expressions we now see distances; in the primary addends, waves. And yet, while we can now relate the mathematical meaning of symbolic details to a larger narrative purpose ("the divisions decrease wave amplitude as the traveled distance increases"), these mathematical insights really obtain only in the particular. The overall expression grows out of these sensible significant components (additions, ratios, the sine function) eventually to become the monster, which then acts only as a vague placeholder for the ensemble of constituent narratives and meanings of its part. (This middle-level structuring of component parts evokes the geometric equivalent in the back-to-back drawing task from Day 1.)

The visualization of its plotted geometry—"Two waves colliding in a pool"—by contrast, seems tremendously detailed and precise, and has followed a reverse trajectory, in which the rather unevocative geometric components of the anecdote's first visualization (a few significant points on the plane, a pair of axes, a graph of the canonical sinusoidal curve) become, over the course of their elaboration, melded into a harmonious, organic and sculptural whole. This all contributes to a rather paradoxical inversion of our normal sense of the things, where the symbolic is the most exalted mathematical language, where we assume that symbolic declarations and propositions must invest mathematics with greater specificity, more precisely delineated details and more transcendent truths than can be found in modest imagery.

Task 2: Working on videotape and closing discussion

The concluding task and discussion drew on two extracts from videotapes: the first is an Open University tape of Dave Hewitt teaching school algebra, entitled *Working Mathematically on Symbolism in Key Stage 3*. In particular, the extract worked on order-of-operations embodied in space and sound (and is discussed further in Chapter 5 of Pimm, 1995). The second tape showed examples of the work of British artist Patrick Hughes, specifically his three dimensional sculpture-paintings which play significantly with viewer perspective and evoke the spirit of Magritte and his tradition of philosophical painting.

"A common theme for me was the unpredictability of communication from person to person, medium to medium and form to form (words to images, etc.) By unpredictability, I don't only mean loss of information or mis-transmissions, but also the possibility of evoking new insights, images and structures. In the final videotape session, it was the issue of maintaining the element of time while switching from verbal to written notation."

Themes:

Foregrounding circumstances where (as teachers) we are likely to incorrectly assume the "transparency of translation" (e.g., A to B drawing; Monstrous Symbolism)

Fluidity of Representations. Kinematic Figures; Enriched Images (Sketchpad). Symbolism as words vs. symbolism as image (Monstrous Symbolism). Dave Hewitt's sonic imagery.

What is the "grammar of images?" How to become more aware of it in geometry?

Reflections on the dimension of time and how we generally lose sight of its role in teaching mathematics—how Sketchpad and the operation of "undo" incorporates this dimension in meaningful ways, including providing insight into the work of the student. How this also ties into Ball and Bass' plenary presentation on how teachers (need to) can learn to understand (rather than feel insecure about or threatened by) students' ways of knowing (and doing) mathematics.

How writing and drawing have no distinct boundaries between them. For example, an equation is an image or a drawing in many ways, depending on who beholds it.

The role that time can play: seeing a particular geometric image does not provide a starting point for the way it was constructed through time.

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Des cours de mathématiques pour les futurs (et actuels) maîtres au secondaire

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Mise en situation

L'objectif de cet atelier de travail concerne les cours offerts par les départements de mathématiques des universités dans la formation des enseignants et des enseignantes au secondaire.

Questions initiales

1. *Objectifs.* Quelles sont nos attentes quant aux cours que nous dispensons aux étudiants en enseignement (et aux autres étudiants) ?
2. *Contenu et pédagogie*. Quelles sont les implications quant au style et aux approches à privilégier pour l'enseignement de ces cours ?
3. *Ressources*. Comment peut-on s'offrir de tels cours ? Comment s'assurer du meilleur recrutement ? Est-ce que ces cours pourraient être profitables à d'autres étudiants que nous avons déjà ? Cette clientèle hors enseignement est-elle moins spéciale et plus polyvalente que la clientèle en enseignement ?
4. *Liens avec les sciences de l'éducation*. Que peut-on dire à ce sujet ?
5. *Collaboration*. Y a-t-il un appui ou une opposition de la part de nos collègues mathématiciens ? Comment pouvons-nous changer les attitudes, et encourager un recrutement de professeurs pouvant travailler avec la clientèle en enseignement ?

Discussions

Types et caractéristiques souhaitables des cours des programmes de mathématiques pour les futurs (et actuels) enseignants au secondaire

1. Fondements pédagogiques.

Un tour de table (brainstorming / remue-méninges) nous a permis de faire ressortir certaines attentes. Idéalement, ces cours devraient présenter une « vision globale » intégrant diverses composantes telles que : la métamathématique et la méta-connaissance (incluant l'histoire et l'évolution des concepts mathématiques), la résolution de problèmes, la modélisation, les différents modes de représentation ; une réflexion sur les processus inhérents à l'activité mathématique ; des apprentissages individuels, des lectures et des explorations ; le développement de la confiance en soi en tant qu'apprenant, et aussi en tant que participant à des activités mathématiques ; l'habileté pour tous à communiquer (oral, écrit, écoute, lecture), et l'instauration d'une communauté mathématique ayant des groupes de travail et des liens avec d'autres communautés ; des projets. De plus, ces cours devraient permettre de faire preuve d'initiatives (ne pas se satisfaire d'une seule façon de faire un problème, mettre en évidence les avantages et les désavantages d'une méthode, encourager les discussions avec les autres) ; d'analyser les problèmes (examiner des cas simples, et d'autres plus complexes, tenir compte des expériences des étudiants) ; de respecter l'hétérogénéité des groupes d'apprenants ; de mettre l'accent sur la compréhension réelle des problèmes sans surcharger le contenu.

Il y avait un assentiment reconnaissant le bien-fondé des caractéristiques mentionnées, lesquelles seraient valables pour tout cours de mathématiques s'adressant aux futurs mathématiciens, de même qu'aux utilisateurs des mathématiques de d'autres disciplines (voir les commentaires de Morris Orzech).

2. Le contenu.

Nous n'avons pas discuté longtemps sur ce point car plusieurs éléments avaient déjà été signalés (voir 1), mais les participants semblaient d'accord à accepter les notions suivantes dont certaines étaient jugées essentielles : géométrie, théorie des nombres, modélisation, histoire des mathématiques, résolution de problèmes et démarche exploratoire, mathématiques discrètes, statistique et analyse de données, preuves mathématiques, séminaires sur les mathématiques fondamentales, technologie (intégrée aux cours et réflexion sur son apport).

3. Difficultés.

En se basant sur le vécu existant dans nos départements respectifs, plusieurs personnes entrevoyaient des difficultés éventuelles à l'implantation de nouveaux cours tels que suggérés précédemment. Les discussions ont principalement porté sur les points suivants : a) incidence monétaire - l'ajout de cours « spécialisés » pour les futurs enseignants peut engendrer des coûts supplémentaires alors que la situation financière de nos universités exige plutôt une réduction des dépenses reliées au fonctionnement ; b) ressources - la disponibilité de personnels qualifiés et intéressés à se porter responsables de cours préconisant une approche basée sur l'exploration, la découverte et l'utilisation de matériel didactique plutôt qu'une approche magistrale, est problématique pour la plupart des départements ; c) implications académiques - il y a le danger de ne pas couvrir entièrement le contenu d'un cours qui est préalable à un autre cours ayant des objectifs différents de formation (mathématiques vs enseignement des mathématiques) parce que généralement les cours à la formation des maîtres nécessitent plus de temps pour les discussions et les réflexions des notions étudiées ; de plus, pour le personnel enseignant, ces cours demandent habituellement un temps de préparation plus long, et une disponibilité accrue auprès des étudiants ; plus souvent qu'autrement, le nombre d'étudiants par groupe est souvent contingenté et le support d'auxiliaires d'enseignement n'est pas toujours assuré ; les difficultés relatives aux évaluations risquent d'être amplifiées ; il y a un réel danger de « mise à l'écart » à l'intérieur du département du professeur impliqué dans ce type de cours ; le manque de coordination dans la gestion des cours offerts entre les différents programmes peut amener d'autres problèmes.

4. Ressources et supports.

Afin de pouvoir créer ce type d'activités, il est nécessaire d'avoir accès à plus de ressources financières de la part des organismes subventionnaires et professionnels : NSERC, MAA, GCEDM, centres de recherche en mathématiques, etc. Un inventaire du matériel disponible sur le web et des volumes répondant aux conditions recherchées, réduirait sensiblement la préparation de ces activités (voir les commentaires de Peter Taylor). D'autres moyens favoriseraient grandement cette option : le travail en équipe multidisciplinaire, le perfectionnement des personnes intéressées (projet NEXT, MAA, …), une amélioration des communications avec les collègues des sciences de l'éducation (exemples : séminaires conjoints, planification commune), une meilleure coordination entre les divers ordres d'enseignement, la création de programmes de mathématiques répondant à des besoins diversifiés (mathématiques, enseignement, formation continue), l'opportunité pour les futurs enseignants de vivre des expériences d'enseignement et d'apprentissage par des stages. [Remarque : au Québec, nos programmes de formation des maîtres répondent à ce souhait car des stages totalisant plus de 700 heures dans le milieu scolaire sont obligatoires].

5. Formation continue.

Selon les commentaires entendus, l'incitation à poursuivre des études en prenant des « cours réguliers » de mathématiques n'a pas les effets escomptés dans la pratique des enseignants en exercice. Il en est de même pour les cours en éducation. Quel que soit le modèle choisi de formation continue, il devrait être un projet à long terme et inclure le « mentoring » et la participation des universitaires dans le milieu scolaire. Que penser d'un diplôme supplémentaire au baccalauréat ? Il devrait être à l'image d'une comète (un gros noyau brillant et une longue traînée) : deux semaines de travail intensif, suivies d'une journée à toutes les trois ou quatre semaines pour des discussions. Nous croyons que ce projet est également souhaitable pour les futurs enseignants.

6. Quelques personnes-ressources.

Plusieurs collègues du GCEDM ont produit et / ou travaillent avec du matériel et des activités qui semblent répondre aux caractéristiques et aux objectifs souhaités. Les membres suivants tiennent à être nommés à cet égard : Deborah Ball, Hyman Bass, Peter Brouwer, Sandy Dawson, Claude Gaulin, John Grant McLoughlin, Bernard Hodgson, Richard Hoshino, Miroslav Lovric, Morris Orzech, Medhat Rahim, Jean Springer, Peter Taylor, Walter Whiteley.

Commentaires individuels

Morris Orzech

J'aimerais nuancer les propos entendus concernant l'idée « …*de mettre l'accent sur la compréhension réelle des problèmes sans surcharger le contenu* » et « …*le bien-fondé des éléments mentionnés, lesquels seraient valables pour tout cours de mathématiques s'adressant aux futurs mathématiciens, de même qu'aux utilisateurs des mathématiques de d'autres disciplines* ». Ce que signifie « compréhension » et « surcharger le contenu » peut varier en fonction de l'auditoire visé. Dans certaines universités (ex. Queen's), les étudiants inscrits au programme de formation des maîtres ont des cours communs avec les étudiants de mathématiques. Dans certains cours, le nombre d'étudiants inscrits à un programme (mathématiques ou enseignement) peut dépasser le nombre d'étudiants inscrits dans l'autre programme, ce qui a pour conséquence que le cours donné semble moins approprié pour un groupe d'étudiants par rapport à l'autre. C'est un problème, et il ne faut pas le sous-estimer. Peut-être pourrions-nous offrir des cours dans lesquels les étudiants puissent avoir des projets différents, et leur laisser croire à l'égalité des chances de succès dans l'atteinte des objectifs visés, mais cette éventualité semble inacceptable. Si nous avons un cours qui fait partie du programme de mathématiques (baccalauréat), et que la clientèle est surtout composée de futurs enseignants, il faudrait accepter (et défendre ce point de vue, le cas échéant) qu'il y ait moins de contenu, ou moins d'approfondissement des notions étudiées, de telle façon que cela puisse être un compromis satisfaisant pour les étudiants de mathématiques.

Walter Whiteley

Une question qui a été soulevée dans le groupe de travail mais pour laquelle nous n'avons pas formulé de réponses à ce que je me souvienne, était de savoir s'il fallait impliquer les futurs enseignants dans les organisations professionnelles pendant qu'ils sont encore étudiants. Est-ce que l'introduction aux activités d'associations professionnelles (publications, congrès, colloques, sessions d'étude, ateliers) devrait commencer lors de la formations initiale ? Si oui, comment devrions-nous procéder ?

Peter Taylor

Dans un certains sens, il y a toujours un dilemme à produire un cours qui soit riche sur le plan pédagogique et sur le plan du contenu, car il faut adopter deux approches conflictuelles. Je crois que cela provient en partie de notre conception à l'effet que les cours de mathématiques doivent demeurer conformes à la nature même du sujet qui est formel, et de ce fait, ils doivent être organisés selon un ordre logique et hiérarchique, être rigoureux et explicites. L'approche que j'expérimente est l'utilisation d'une collection de problèmes à la fois mathématiquement riches et concrets, et j'essaie de porter l'attention uniquement sur ces problèmes, en espérant couvrir les thèmes mathématiques au programme, ou à tout le moins de rendre les étudiants suffisamment autonomes pour qu'ils puissent être en mesure d'étudier par eux-mêmes le contenu du programme. Toutefois, cette approche n'est pas simple ; notre conscience professionnelle et notre perception de ce que sont les mathématiques font en sorte que nous sommes toujours rappelés à l'ordre afin de remplir toutes les « ouvertures » (et comme le dit si bien Leonard Cohen, *quand ce sont les ouvertures qui laissent pénétrer la lumière*). Évidemment, le grand défi c'est aussi de trouver la collection de problèmes qui soient riches et valables. C'est la principale tâche du développement de programme.

Il existe, selon moi, une importante question non encore résolue. Le problème des ressources et de la faisabilité des cours que nous aimerions offrir, dépend en partie de l'obligation que nous nous imposons à donner des cours différents pour les étudiants qui poursuivront des études de cycles supérieurs en mathématiques, et ceux qui deviendront des enseignants. Même si cette exigence est sûrement valable pour les cours de dernières années (du baccalauréat en mathématiques), je ne suis pas certain qu'il doive en être forcément ainsi dans les premières années comme nous sommes portés à le croire. Je sais que Morris (Orzech) a discuté de ce point. Il est probable que les cours destinés spécifiquement pour les futurs maîtres ne soient pas adéquats pour les étudiants en mathématiques. Mais je crois qu'avec une peu d'imagination et des activités « qui sortent de l'ordinaire », il serait possible d'offrir des cours satisfaisant les deux groupes, et bien les servir ! Je crois que la faiblesse de l'argumentation habituelle vient du fait que nous oublions ce qui se passe à l'extérieur de la classe. Il est certainement vrai que des étudiants appartenant à différents programmes s'attendent à recevoir des cours différents durant leurs études. Quand j'étais étudiant, il n'y avait qu'un seul ensemble de cours (programme). Plusieurs de mes collègues sont devenus enseignants. D'autres sont devenus des gens d'affaire ou des professionnels (avocats, etc.). Quelques-uns, comme moi, avons poursuivis jusqu'au doctorat (Ph.D. dans différents domaines), et un ou deux en mathématiques. Je sais que ce que j'étudiais le soir était différent (je ne dis pas « meilleur ») de ce qu'ils étudiaient. Je n'avais pas besoin d'un contenu très développé — il y avait toujours du contenu à explorer même dans la petite bibliothèque de mathématiques au dernier étage du pavillon Carruthers ! Du contenu, il n'en manquait jamais. Tout ce que j'avais besoin, c'était d'un professeur qui avait une ouverture d'esprit et qui avait accès à une banque de problèmes passionnants (et j'en avais). Aujourd'hui, les bibliothèques sont beaucoup mieux garnies qu'elles l'étaient dans les années 60. Donc ça devrait être plausible de faire autrement. Quoi qu'il en soit, je ne suis pas encore entièrement convaincu de la possibilité d'opérer ainsi.

Report of Working Group E

Types and Characteristics Desired of Courses in Mathematics Programs for Future (and In-Service) Teachers

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Pedagogical Thrust of Such Courses

They should offer:

- the 'Big Picture', with meta-mathematics and metacognition (include history of mathematics, the evolution of ideas), problem solving, modeling, multiple representations;
- reflections on the processes of doing mathematics;
- revealing (not hiding) the processes;
- independent learning? reading, exploration;
- confidence as a learner and as a doer of mathematics;
- communication of mathematics: oral, writing, listening, reading, by all participants;
- a community of mathematics, including group work, connections of community to explorations;
- projects;
- never stop at one way to do a problem: highlight advantages/disadvantages, encourage curiosity about alternatives of others;
- layered problems: consider simpler versions, extensions, different levels of student experience;
- respect for the diversity of students, of learners;
- include both simplicity of problem and rich problems;
- work for deep understanding of material, without overcrowding content.

There was a recognition that, implicitly, we believe these are good characteristics of any mathematics course, including mathematics courses for future graduate students, and future users of mathematics in other disciplines. (But see comments below of Morris Orzech.)

Content of Such Courses

Such courses should cover or address:

- modeling;
- history of mathematics;
- problem solving and inquiry;
- capstone style seminars;
- discrete mathematics;
- number theory;
- geometry;
- statistics and data exploration;
- proof;
- technology (integrated and with reflections on when it assists).

Very little time was spent on this, as there was general agreement on most of these topics as desirable, and few of these topics as essential.

Obstacles to Offering Such Courses

The following were identified as obstacles (or potential obstacles) to offering such courses:

- money for what will appear to be 'extra courses';
- available math faculty with such experience and interest to be course directors;
- lack of these kinds of experience for mathematics faculty, and fear of the unknown;
- courses with too much material;
- prerequisite structures when current courses is prerequisite to following course with different objectives;
- time for this teaching, reflecting our sense that these characteristics are more time consuming for the instructor, per hour of scheduled student contact;
- class sizes and lack of other supports (TAs, etc.);
- constraints on assessment which lead to assessment at odds with the larger objectives;
- coordination among math instructors;
- isolation of individual instructors with these objectives, and sense that when instructors shift the entire structure of the course is up for grabs (that might not be such a problem).

Resources and Support to Overcome Obstacles

- more funding sources (e.g., push for change in NSERC)
- MAA/CMS/CMESG/Mathematics Research Centers as sources of support;
- accessible modules, units, web resources to reduce preparation of specific pieces;
- texts designed for courses with these characteristics;
- alternate resources (physical materials, activities, …);
- course design which builds the characteristics in (see comments of Peter Taylor below);
- team teaching;
- community of support (beyond the Department);
- professional development for current and future instructors of such courses (Project NEXT, MAA, …);
- mathematics/mathematics education connections and liaisons (e.g., joint seminars, shared planning);
- secondary/post-secondary connections and liaisons;
- joint mathematics/mathematics education post-baccalaureate programs;
- opportunities within Math Departments for pre-service students to have teaching and learning related experiences.

Features of In-Service Courses

- Warning that simply taking more 'regular math courses' has no measurable impact in classroom of in-service teachers;
- Warning that simply taking more 'regular education courses' has no measurable impact on classrooms of in-service teachers;
- Features should include long-term work (e.g., mentoring, integration with time of the instructors in the classrooms of the teachers);
- Post-baccalaureate? Comet model (intensive head plus a long tail), which consists of two weeks of intensive work plus one day every 3 or 4 weeks. We considered this topic with an implicit understanding that many features of this model would also be desirable for pre-service teachers.

Sources of Examples and Existing Resources

CMESG is rich in people who have produced or who work with rich materials, activities, resources, etc. which seem support the characteristics or features listed above. The following members of the group wanted to be mentioned in this regard: Deborah Ball, Hyman Bass, Peter Brouwer, Sandy Dawson, Claude Gaulin, Bernard Hodgson, Richard Hoshino, Miroslav Lovric, John Grant McLoughlin, Morris Orzech, Medhat Rahim, Jean Springer, Peter Taylor, and Walter Whiteley.

Individual Comments

Morris Orzech

I would caution against facile acceptance of the idea that "these characteristics [suitable for a course for prospective teachers] are good characteristics of any mathematics course, including ... for future graduate students, and future users of mathematics". A characteristic that needs nuance is that the course "work for deep understanding of material, without overcrowding content". What "understanding" and "overcrowding content" mean should vary with the target audience. In some universities (e.g., Queen's), pre-service secondary math teachers will take math courses with prospective math graduate students. In some courses one group will predominate (in size) over the other, and in each situation the resulting course is likely to feel less appropriate to one group than to the other. We should not try to "paper over" this. Perhaps we can learn to offer courses in which students can do quite different things and still be deemed to have achieved equal success, but this goal seems elusive. If we have a course that is accepted as part of a math major degree, and pre-service teachers are the main intended participants in the course, we should be ready to accept (and argue for if necessary) less content, or less understanding, of the type that would be suitable for prospective graduate students.

Walter Whiteley

One question that was raised in the Working Group, but for which we did not formulate an answer that I remember, was whether (and if so how) to involve pre-service teachers with professional organizations while they are still students. Should exposure to professional publications, workshops, and meetings begin in pre-service days? If so, how do we go about this?

Peter Taylor

In some sense there is always the problem that to make a course both pedagogically rich as well as content rich is to adopt two conflicting objectives. Partly this is due, I believe, to the notion that mathematics courses must remain true to the nature of the subject as it is formally recorded—that they should be organized in a logical hierarchical fashion and that they should be comprehensive and have all the details. They way around this that I am experimenting with is to fasten attention of a collection of rich concrete examples and focus on these and only on these, trusting them to carry the subject, at least to carry it forward enough for the student that she or he is enabled (and inspired) to continue independently to develop the subject. By the way, it's not easy to do this; our mathematical conscience and our historical notion of what the given subject "is" seem always to call us, siren like, to fill in all the cracks (when, as Leonard Cohen reminds us, it is the cracks that let the light in). Of course a big challenge is also to find the right collection of rich examples. That's the main task of curriculum development.

There is still a big question which is unresolved in my mind. The problem of resources and feasibility for the courses we want to be able to offer hinges in part on the assumption that we need different courses for math students who are going on to do a Ph.D. and those who are going to become teachers. Now this is certainly true for 4th-year courses (though teachers do not take many of those anyway) and some third year courses, but I am not convinced that it is as true as is commonly believed in first and second year, and often in third year. I know that Morris has argued this point above. It's probably true that courses which are designed solely with teachers in mind would not be "enough" or "right" or "adequate" for the others. But I believe that with some imagination and "thinking out of the box" courses could be designed to serve both parties—and serve them well! I think the flaw in the usual argument is that we forget about what happens outside of class. It's certainly true that different students would wind up doing different things in their study time. When I was an undergraduate there was only one set of courses. Many of my fellow students became teachers. Others became business people of different kinds, or professionals, lawyers, etc. A few pursued, like me, a Ph.D. (in various subjects!), and one or two of those were in math. I know that what I did in the evenings was different (I am not saying "better"!) from what they did. I did not need a lot of content—holy cow there was content to burn even in the small math library on the top floor of Carruthers Hall! Content was never in short supply. All I needed was a teacher who had some vision and access to problems of different kinds (and I had that). And resources are much more plentiful today than they were in the 1960s. So it should be possible. Anyway, I'm not yet convinced.

Topic Sessions

Sessions thématiques

High School Mathematics Teachers' Perspectives of Mathematical Word Problems

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Introduction

Citation of word problems on Babylonian clay tablets provides evidence of their ancient origins. They also have a long history of use in the teaching and learning of mathematics. But this use has earned them a negative reputation, particularly from the perspective of learners. This likely played a significant role in influencing researchers to focus their investigations on the relationship between word problems and learner as a way of improving the teaching and learning of word problems.

Studies on word problems have traditionally focused on the learner to study cognitive and affective factors that aid or hinder his/her performance as a problem solver. For example, studies on arithmetic word problems (Carey, 1991; Cummings, 1991; Debout, 1990; Fuson & Willis, 1989; Lewis & Mayer, 1987; Reed, 1999; Sowder, 1988; Verschaffel et al, 2000) have looked at the mathematical and linguistic structure of these problems in relation to the children's performance; factors that affect the difficulty of the problem for children; strategies and methods children use; the errors children make in their solutions; and children's suspension of sense making in doing word problems. A similar situation has existed for studies on high school algebraic word problems where, for example, the focus has been on students' errors and methods in the translation process (Clement, 1982; Crowley et al., 1994; Kaput & Sims-Knight, 1983; MacGregor & Stacey, 1993; Reed, 1999; Wollman, 1983).

In recent years, there has been increased focus on researching the mathematics teacher. This shift has accompanied recommendations to reform the teaching and learning of mathematics (e.g., NCTM 1989, 1991) and the assumption that teachers are a significant factor in bringing about this reform. In particular, teacher thinking is being viewed as an important factor in determining how mathematics is taught (e.g., Chapman, 1997; Ernest, 1989; Fennema and Nelson, 1997; Lloyd and Wilson, 1998; Thompson, 1992). This makes knowledge of mathematics teacher thinking about word problems an important component of understanding word problem instruction and ways to change or enhance it through teacher development programs. This paper focuses on teachers' perspectives of word problems based on the findings of a study of mathematics teacher thinking in the teaching of word problems. A brief discussion of how word problems are defined in the literature is followed by a brief description of the research method of this study, the findings in terms of ways in which the participants conceptualized word problems and implications of these findings.

What Are Word Problems?

Given their long history, it would seem that word problems should be easily recognizable and easily defined. Historically, in a word problem, a task or situation is presented in words, and a question is asked which sets out the goal that the solver has to attain. As Gerofsky (2000) concluded, "The form of mathematical word problems appears nearly unchanged throughout its long history" (p. 132). But the definition of word problems is not always clear-cut. For example, some consider word problems as including problems normally in symbolic mode expressed in words (e.g., What is 5 from 6? What is the difference between 8 and 5?) Others consider them to be only those that are "story problems". This is reflected in the different ways word problems are described in the literature, although the latter case tends to be the preferred view of word problems.

Leacock (1910), in his discussion of "the human elements in mathematical word problems", described word problems as "short stories of adventure and industry with the end omitted" (p. 118). Verschaffel et al. (2000) offered the following:

> Word problems can be defined as verbal descriptions of problem situations wherein one or more questions are raised the answer to which can be obtained by the application of mathematical operations to numerical data available in the problem statement. (p. ix)

Gerofsky (2000) in her study of the nature of word problems from the perspective of genre theory concluded:

> Literary analysis of word problems suggests that they are *like* religious or philosophical parables in their non-deictic, "glancing" referential relationship to our experienced lives, and in the fact that the concrete images they invoke are interchangeable with other images without changing the essential nature of the word problem or parable. (p. 131)

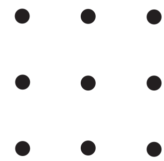
Word problems have also been described in terms of the structural components that characterize them. Verschaffel et al. (2000, pp. x–xi) summarized these components as:

> Mathematical structure, i.e., the nature of the given and unknown quantities involved in the problem, as well as the kind of mathematical operation(s) by which the unknown quantities can be derived from the givens.
>
> Semantic structure, i.e., the way in which an interpretation of the text points to particular mathematical relationships …
>
> Context, i.e., what the problem is about …
>
> Format, i.e., how the problem is formulated and presented, involving such factors as the placement of the question, the complexity of the lexical and grammatical structures, the presence of superfluous information, etc.

When word problems are viewed as genuine problems, the are also associated with the position and state of student required to solve them, e.g., the student wants something but does not know immediately how to get it. The problem still has a situation and goal, but the problem solver has a desire to attain the goal. In recent years, this added feature has also influenced the range of word problems in the mathematics curriculum. This range covers problems of two general categories, one consisting of routine, closed, algorithmic, or translation problems and the other consisting of non-routine, open, non-algorithmic, real/genuine, or process problems. Charles and Lester (1982, pp. 6–7) provided the following examples of these word problems, which they considered to be typical in mathematics education. They labelled these as simple translation, complex translation, process, applied, and puzzle problems, respectively.

> Jenny has 7 tropical fish in her aquarium. Tommy has 4 tropical fish in his aquarium. How many more fish does Jenny have than Tommy?
>
> Ping-Pong balls come in packs of 3. A carton holds 24 packs. Mr. Collins, the owner of a sporting goods store, ordered 1800 Ping-Pong balls. How many cartons did Mr. Collins order?
>
> A chess club held a tournament for its 15 members. If every member played one game against each other member, how many games were played?
>
> How much paper of all kinds does your school use in a month?
>
> Draw 4 straight line segments to pass through all 9 dots in Figure 1. Each segment must be connected to an endpoint of at least one other line segment.

FIGURE 1



Research Method

The primary objective of the larger project from which this paper is based was to understand the teaching of word problems from the perspective of the teacher. A summary of the research method of this project follows. Twenty-two participants were involved as the research subjects, consisting of experienced and preservice mathematics teachers at the elementary, junior high, senior high, and college levels. The main sources of data were interviews and classroom observations. The interviews were open-ended and dealt with the participants' thinking in three contexts: past, present, and future. The past dealt with their past experiences with word problems as both students and teachers focusing on teacher and student presage characteristics, task features, and contextual conditions The present dealt with their current practice with particular emphasis on classroom processes, planning, and intentions. The future dealt with expectations, such as possible changes due to personal or external factors. Classroom observations focused on the participants' actual instructional behaviours during lessons involving word problems. Special attention was given to what the teachers and students did during instruction and how their actions interacted. Complete units involving word problems were observed. The data were thoroughly reviewed by the researcher and two research assistants working independently to identify attributes (e.g., recurring conceptions/beliefs and intentions) of the participants' thinking and actions that were characteristic of their perspective of word problems and the teaching of word problems. These attributes were grouped into themes. Both attributes and themes were validated by comparison of findings by the three reviewers and triangulation of findings from interviews and classroom observations. The only aspect of the findings reported here deals with the high school teachers' perspectives of word problems.

The Nature of Word Problems: High School Teachers' Perspective

While the elementary teachers of the early grades insisted that they did not teach word problems, one of the college teachers explained that there is nothing like word problems, and one Grade 6 teacher always talked about *worded problems*, the high school teachers had no conflict with the term *word problems*. They resonated with it as a natural part of their mathematics vocabulary and curriculum. However, there was not always consistency among them in terms of how they viewed word problems. Thus a range of ways of characterizing word problems emerged from their thinking. These ways embodied some of the features of word problems described in the literature as previously discussed, in particular, the structural features and whether non-routine or routine. The high school teachers' perspectives of word problem are summarized here to reflect the five ways of making sense of word problems in their teaching: word problem as problem, word problem as object, word problem as tool, word problem as experience, and word problem as text.

A) *Word Problem as problem*

All of the teachers considered word problems to be real problems for students depending on particular circumstances associated with the student and the teacher. There were three of these circumstances that were dominant in their thinking, collectively.

(i) *Relationship between student and problem*

A word problem is a problem depending on the relationship between the student and it. As one teacher explained,

> All word problems are real problems if students have not encountered them before. … I don't think there's anything in the problem that makes it necessarily routine or non-routine. … No problem is routine if you've never seen it before.

Other ways in which the teachers described the relationship are:

> Students don't have a predetermined solution process.
>
> It's a problem you want to have the answer to, that is something that is needed, is practical, is worthwhile, that has some kind of relevance.
>
> It's like anything else that you don't know what the outcome will be and you're kind of game for anything else, so you just take your chances and you try and use the tools that are available to you, see what happens.

(ii) *Nature of problem/solution*

The nature of the problem/solution influences the relationship between student and problem and consequently whether or not it is a problem. This was viewed in terms of two situations. First, there are "problems for which students must deduce a structure to determine a solution." The teachers referred to these as traditional word problems, e.g., "you can type the problem" based on the structure. Second, there are "problems for which students must impose a structure on problem to create a solution". In this case, "you cannot type the problem, categorize it so that you can read it and do it". The teachers referred to these as interesting, intriguing, challenging. They also described them as:

> The ones where they have to bring quite a few different tools to solve them … and think on a lot of levels and have to bring a lot of things into play.
>
> They allow you to think and come up with a solution that may use different areas, techniques that you know about but combine it in different ways.
>
> [It is] one that is interesting, one that makes you see things differently, takes you down different paths.
>
> It doesn't require only one specific method, … [It] initiates discussion … promotes dialog.

(iii) *Teaching approach / teacher intent*

Finally, a word problem is a problem depending on when and how it is introduced to students by the teacher, i.e., it is dependent on the teacher's intent and teaching approach. For example, a teacher could take a potentially routine word problem and problematize it by presenting it before the routine approach is taught. As one teacher explained,

> If they [traditional word problems] are given to students at the right stages as something beyond their level of experience at this time… [they] could be used to practice their problem solving skills.

B) *Word Problem as object*

Word problem as object refers to the aspects of the teachers' perspectives that deal with the structural components of a word problem when they are viewed in terms of universal properties independent of the student. This view includes that a word problem has pre-existing or pre-determined mathematics and semiotic structures/contexts. For example, it has/is a:

> Concept taught, e.g., equations;
>
> Type of problem, e.g., coin, age, distance, number;
>
> Hidden math concept, …they're hiding the concept in the form of a word problem;

> Written statement in which mathematics will emerge;
>
> Clear language, written, clear in terms of what it wants.

As objects these properties should be transparent, i.e., the problem:

> Must have clear language, no extraneous information, clear about what want, not ambiguous.

C) Word Problem as tool

Word problem as tool refers to the aspects of the teachers' perspectives that deal with the relationship of word problems to the mathematics curriculum and the real world. In this context, there are two levels of word problems. Level 1 consists of situations that are generally tailor-made to illustrate the application of a mathematical concept or skill. As the teachers explained, these problems are:

> A means to apply concept or practice a skill they have seen most recently in class.
>
> Instances of applications of mathematical concepts … the whole point of them is to give the students experience in practicing that concept, similar application of that particular concept. … They're a way to make sense of the concepts in a context.
>
> They can be used to demonstrate or as an example, or simple application of a formula that students have been working with. … It's a simple way to introduce them to the application or is of this particular formula or rule that they may have derived … they will help them understand how that rule is employed in that particular kind of problems. Their role is basically to teach, help students make sense of mathematical concepts. … Help them understand what they mean, how they can be used and maybe give them idea of how their uses can be extended to more meaningful context in their own lives.
>
> It's a tool that they now have available, they can take to other problems hopefully real world problems eventually, other situations.
>
> They don't necessarily develop students' mathematical power but it's the first step.
>
> They aren't all that important, so if you have to cut corners some place and you don't have a lot of time, … they can be dismissed.
>
> I would include them to limited degree yes, because I want the students to be aware of them and to recognize them as routine problems and to know they exist, so even though they're not, not all that important for me, they are something that the students I feel need to be able to solve.
>
> I think the role has been to have kids take all of the content that they've seen in a chapter, unit and apply it to some situation, the belief being that if children are required to do that and they can do it
>
> They're extra, they're not necessary, they're trivial and they do little, most of the time I think to enhance a topic.

The level 2 word problems were viewed as providing a means of dealing with new situations and of fostering mathematical thinking, for example:

> They are intended to get at some in-depth thinking … serve to enhance and understand on a topic or depth of thought.
>
> They are [used] to get the kids to handle a new situation where it does not seem like anything that we've done before.
>
> They're another way of asking you how to do math and they're designed to encourage thinking, analytical skills, logical skills, reading skills. They can be applications of mathematical concepts, and therefore they can be used to reinforce math skills learned to develop understanding, and word problems have infinite variations. There's infiniteness to them. They're open-ended. They just keep going.

D) *Word Problem as experience*

Some of the teachers viewed word problem as experience. This refers to the phenomenological relationship between a word problem and a student, i.e., the components of a word problem are viewed in relation to the student as a lived experience and linked to intention/interest/value. Thus the meaning of the problem is personally determined and justified, i.e., it is dependent on the student and not the author of the problem. In order for students to accomplish this in the context of a positive experience, the word problems should:

> capture their attention;
>
> invite them, intrigue them and prod them to want to solve it;
>
> [be] a discussion or conversation in which something is unknown about the world of mathematics in which you are trying to invite the students to become a participant;
>
> [be] the students' story;
>
> help us experience the world;
>
> [be] a situation that has interest and appeal to student;
>
> have a context that's relevant to the students, because if they don't and if they're contrived in any way, they're not interesting to the students.

Without these characteristics, word problems become negative experiences. For example,

> You are fearful of those problems because you don't understand where they're coming from, you don't see they're connected to the mathematics you know … so it really becomes a problem because you can't make sense out of the wording.
>
> [For students] they're threatening. When they see word problems, they know that there's mathematics inherent in it but to them it seems hidden. It doesn't seem that it's something that invites them to apply their skills or to use their knowledge of something, but more threatens them.

E) *Word Problem as text*

Some of the teachers viewed word problems as text, i.e., conveyor of knowledge. As one teacher explained:

> Somebody needed to transfer information [about a specific mathematical skill/concept] to somebody else. So the best way to do it, other then just symbolically, was to write a sentence or a small scenario or a small story in which the information that they wanted to share was there. Then it became part of mathematical teaching …. In the same way that somebody originally might have wanted to share this question and they had to explain this question to somebody else so they could write it down for someone else to refer to later. Then we decided this would be a good way for us to share our request for students to make sense of mathematics by putting it into this same kind of a scenario.

Other teachers noted that they are:

> a way to share mathematical experience with another;
>
> stories from which you can extract mathematics;
>
> an opportunity for us to share mathematical experience

All of the experienced teachers viewed word problems as problem, object, and tool. The preservice teachers viewed them as object, problem (only in terms of the nature of problem/solution), and tool, but with less depth than the experienced teachers. The teachers who held the perspective of word problem as problem, object, and tool used a more traditional teaching approach, i.e., a show and tell approach. Those whose perspective included word problem as experience and text were more student-centred and inquiry oriented in their teaching. Their range of views of word problems seemed to allow them to select, modify, interpret, and understand word problems with the flexibility necessary to facilitate student understanding of word problems in a meaningful and realistic way. Thus they were more flexible in their teaching of word problems and more successful in motivating students to do word problems and helping them to understand how to solve word problems.

Implications

The findings suggest a possible range of ways of thinking about word problems that teachers could hold and that there seems to be an important relationship between the teachers' perspectives of word problems and their teaching. This has implications for teacher development in order to improve the teaching of problem solving. This could be done on two levels. First, although these ways are not intended to state how things should be but how they are and could be, they could form a basis for helping teachers to broaden the scope of their perspectives of word problems in a similar way. Second, and more importantly, they offer a structure, something against which other teachers could examine their own perspectives and assumptions, either through reaction against or resonance with what is offered, to understand their thinking and the relationship to their teaching.

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What are Critical Online Experiences for Mathematics Teachers and Students?

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Despite major differences in course delivery and focus, I suggest that critical experiences, whether they are online or face-to-face, or whether they are designed for teachers or students of mathematics, are essentially similar in one important aspect, namely, in their focus on good mathematics. I conclude that the core component of critical mathematics experience is a good mathematics 'story'—a mathematics experience that is worth living, in Dewey's (1938) sense. Teachers and students are aesthetically drawn to a good mathematics story.

What are critical experiences?

Critical experiences may be defined as those experiences of epiphany that cause us to reflect on our knowledge and beliefs and to see mathematics and mathematics teaching in new light. When such moments of epiphany occur for teachers, mathematics education artifacts—such as curriculum documents, classroom experiences, ideas from professional development workshops, journal articles, and so forth—can be thought of as inkblots where the image appears to shift and something new is seen, something that was not apparent before. As one teacher in one of our studies commented "I feel like [this experience] has cleaned my spectacles". Similar findings are reported by McGowen & Davis (2001b) where teachers noted that course experiences "opened [my] eyes to a new outlook on mathematics" (p. 444).

Components of critical experiences

We all have had such "critical" experiences. The challenge is to look back and identify what are the key components of critical experiences. Hawkins (2000) talks about scientists who in their childhood and youth experienced and enjoyed "investigative curiosity" that

> prepared them for the routines and rigors of didactic coursework that, for most other students, has simply turned off the light. But as successful students in formal coursework, most scientists think that the way they learned was the right way, forgetting the earlier, more self-directed work, or play, that first got them into good subject matter on their own. (p. 203)

It seems that although we have all lived through critical learning experiences it is not a simple task to identify what it is that made such experiences critical. Research on mathematics teacher development (Cohen & Ball, 1990; Gadanidis, Hoogland, & Hill, 2002a; McGowen & Davis, 2001a, 2001b; Stipek, Givvin, Salmon, & MacGyvers, 2001) indicates that the following are integral components of critical experiences for mathematics teachers:

- Teachers confront their beliefs about mathematics.
- Teachers have aesthetic experiences with mathematics.
- Teachers engage in practical inquiry.
- Teachers consider pedagogical implications in the context of relevant mathematics education literature.

Aesthetic aspects of critical experiences

In collaborative research with my colleague Cornelia Hoogland, a poet and language arts educator, and my research assistant Bonilyn Hill, we have concentrated on aesthetic aspects of critical experiences in online mathematics education courses for inservice and preservice teachers—courses that I have designed and taught. Contrary to the usual alignment of "critical" with "rigorously intellectual", participants in our studies demonstrate that critical engagement occurs within an aesthetic context (Gadanidis, Hoogland, & Hill, 2002a, 2002b). This was true for teachers with diverse mathematical backgrounds and attitudes that ranged between positive and negative. It is interesting to note that although aesthetic qualities of critical experiences for mathematics teachers are not explicitly identified in related studies, aesthetic qualities are in some cases implied. For example, McGowen & Davis (2001b) make use of phrases such as "we focused on a […] beautiful experience in establishing connections" (p. 439) and "the atmosphere […] was electric" (p. 440) to describe aspects critical experiences for mathematics teachers.

The aesthetic is a historically and biologically verifiable human predisposition (Dissanakye, 1992; Boyd, 2001), a means by which humans make sense of the world (Egan, 1997; Lakoff & Nunez, 2000) and an element of pedagogy and constructed environment (Eisner, 1985, 1988; Greene, 1995). Within this vision of the aesthetic, people are predisposed to their senses of rhythm and fit, balance, motion, and symmetry. Dissanakye (1992) talks about an "aesthetic sensibility" that "acts as one of our primary meaning-making capacities in all domains" (p. 25). Greene (1995) has characterized the aesthetic as an alertness, a "wide-awakeness". Greene says educators can learn from artists, whose way of comprehending experience is through perception, imagination, feeling and form. Aesthetic engagement has to do with being open to one's own sense of curiosity and intuition. As a researcher I value the open-ended nature of aesthetic questions (particularly important at the beginning of inquiry), as well as the choices I can make about following their hunches and intuitions.

Gilbert Labelle (2000), professor of mathematics at the Université du Québec à Montréal, says "I like mathematics because it is beautiful, full of surprises, and gives me complete freedom of thought." Feelings of surprise and beauty are emotional. Doing mathematics is emotional. Students also express these attributes. "Math is just another way of both creating meaning and describing it. It's lovely. I'm lousy at it, but I love feeling my brain tumble over as it understands something for the first time". The feeling of connectedness to stimuli that math provides this professor and student should not be mistaken for sentiment or mere personal expression. Lakoff and Johnson (1999, p. 176) state that emotion is "inextricably linked to perception and cognition" and "is better understood as the tension or excitement level produced by the interaction of brain processes of perception, expectation, memory and so forth".

The development of an online course

In this section I discuss my development of an online teacher education course [Online Course], and I make explicit references to the four critical experience components identified earlier. I also note similarities between the experiences provided in the Online Course and the critical experiences provided in face-to-face professional development sessions I had previously designed and implemented. Although this section deals with teacher education experiences, the discussion of aesthetic experiences with mathematics applies equally to students of mathematics. Also, the discussion dealing with beliefs about mathematics learning also has implications for students, as they also have developed beliefs about what it means to learn and do mathematics that influence their engagement in mathematics activity.

Teachers confront their beliefs about mathematics.

Prior to developing the Online Course I was a mathematics consultant for a large school district, dealing mostly with elementary mathematics program design and teacher development. This experience reinforced research findings that many elementary teachers view mathematics as procedures to be learned for getting right answers (McGowen & Davis, 2001a, 2001b; Stipek et al., 2001). One of the goals of district-wide professional development was to help teachers become aware of these beliefs and to examine them critically. Bringing teachers face-to-face with their unexamined beliefs about mathematics involved more than simply telling or showing teachers what mathematics is really like or how it may be different from their personal beliefs. Teachers were provided with opportunities to personally experience aesthetically-rich mathematical contexts where good mathematics stories unfolded and were experienced, which were different than the teachers' historical experiences with mathematics or the experiences they may have been providing for their own students. Likewise, teachers in the Online Course were provided with similarly mathematics experiences. Such experiences created a reflective context for examining personal beliefs in both the face-to-face and the online environments.

Teachers have aesthetic experiences with mathematics.

Experiencing good mathematics stories does not have to involve complex mathematics, especially for elementary teachers. Some of the mathematics experiences in the Online Course involved teachers in mentally solving arithmetic problems such as 16 × 24 and 156 + 78 + 9. These activities were chosen based on their positive effect in previous face-to-face workshops conducted for elementary teachers and parents. In such workshops, typically half the people in each group were asked to solve a problem like 16 × 24 or 156 + 78 + 9 in their heads and half the people to use pencil and paper. After a few minutes, people shared and explained the methods they used in their groups. Then the discussion was opened up and people shared and explained other methods. It quickly became apparent that the people who used paper and pencil methods had little to say. One reason for this was that most people used the same procedure. Another reason was that although they were able to describe the procedure they followed, they often were not able to explain why. Some people reverted to statements like "this is how it works—it's just a rule". On the other hand, people who solved the problem in their heads shared a variety of methods and they understood what they were doing and why they were doing it. They displayed pride in their individual approaches to problem solving. There was an excitement about mathematical thinking in the room, with people eager to share their personal methods and quick to express surprise and praise for unique methods that others shared. A palpable energy was created in this exchange.

The experience of mentally solving 16 × 24 or 156 + 78 + 9 is aesthetically rich in that the mental processes involved do not demand rule-based procedures. How people solve 16 × 24 depends greatly on how they personally interpret the problem. For example, some people may multiply 16 and 25 and then subtract the extra 16. Others may deconstruct the problem as 10 × 24 + 6 × 24. Many other solutions processes are possible—even ones that use algebraic structures like (20 – 4)(20 + 4). Given such problems, people are eager to share their solutions, they express interest and sometimes surprise in the solutions of others, and are motivated to try to come up with different solution processes. Open-ended inquiry, interest, surprise, and motivation are characteristics of an aesthetic approach.

Teachers in the Online Course noticed that their mental solution processes were "different than when I did it with paper and pencil because I solved my problem by starting with the bigger numbers first (left to right, not right to left!)". Such experiences appear to have helped teachers move towards questioning traditional views of mathematics and developing a deeper understanding of what constitutes mathematical activity and mathematical understanding in the context of addition and multiplication. "To me, the implications are that doing arithmetic mentally requires real understanding. The traditional way (on paper, doing the "ones" first) is more of a procedure to be memorized that requires little understanding".

Teachers engage in practical inquiry.

In face-to-face professional development, practical inquiry was facilitated through a double-session structure. Between sessions teachers tried out new ideas in their classrooms and shared their experiences and reflections in the second session. Teachers were encouraged to bring to the second session samples of student work. Many of the insights that teachers gained and shared arose from observations of students doing mathematics and thinking mathematically in the context of the new ideas that teachers tried in their classrooms. The Online Course involved teachers in practical inquiry in that teachers were asked to explore the thinking of others, including the thinking of their own students, in mentally solving problems like 16×24 and $156 + 78 + 9$. They shared and reflected on these observations in online discussions. Many of the teachers tried the problems with their students and discovered that they too used a variety of methods, and usually not standard paper and pencil procedures. This helped teachers realize that their mathematical thinking as adults was similar to that of their students and different from the standard paper and pencil procedures. Teachers were impressed by the creativity of student answers and questioned their reliance on paper and pencil procedures.

Asking teachers in the Online Course to mentally solve problems like 16×24 and $156 + 78 + 9$ and to share their solution processes also offered opportunity for practical inquiry into the nature of mathematics and doing mathematics. This set the context for discussions of related pedagogy. However, one would expect that practical inquiry would also involve experimenting with teaching practice, which was not a requirement of the Online Course. Unlike the face-to-face professional development described above, the Online Course did not explicitly ask teachers to experiment with new teaching ideas in their own classrooms. This is something that will be reconsidered when redrafting the Online Course.

Teachers consider pedagogical implications in the context of relevant mathematics education literature.

In face-to-face professional development sessions, ideas from mathematics education literature were shared and discussed. The Online Course gave teachers the opportunity to read such literature. Two articles about children inventing personal algorithms for arithmetic operations (Burns, 1994; Kamii et al., 1993) provided a context for teacher reflections on their thinking when mentally solving problems like 16×24 and $156 + 78 + 9$ and for considering pedagogical implications. Questions directed teacher attention to pedagogical issues. In contrast to the face-to-face professional development sessions where ideas verbalized may be forgotten, an advantage of the Online Course was the 'permanent' record of discussions. Many teachers revisited past discussions and created scrapbooks of 'good ideas' by copying sections of online transcripts in word processing documents. As was the case in the study by McGowen & Davis (2001b), teachers made important connections between their experiences and ideas in the articles they read.

> I do agree with Kamii and Burns' points of view. I think that by having the student discover a successful method they will be more likely to internalize and understand the concept. In coming up with their own methods they are doing the thinking the way their mind works. We can see [in our discussion] that everyone processes things differently.

Online mathematical romance

When mathematicians describe mathematics it is not uncommon that they talk about the pleasure of doing mathematics and the beauty that mathematics helps them see and create. For example, Karen Amanda Yeats (1999, p. 6), a mathematics student at the University of Waterloo, in Ontario, said "I like math because it's beautiful, and because working with it is fun. I considered going into music composition, and I really feel the aesthetics of the two subjects are very closely related. The search for elegance". Mathematicians' love of mathematics seems to be shared by young children. Sinclair (2000, p. 4) states that "I have found that most children love to do mathematics". In my own experiences of working with children in the primary grades, and with my own children at a younger age, I have found that they have a natural affinity towards mathematics. They love big numbers. They are fascinated by the idea of numbers less than zero, and many other mathematical concepts that we take for granted. Yet, somewhere between the curious mathematical minds of young children and the elegant mathematical minds of mathematicians, many students lose the joy of mathematics. Moving through the grades, "mathematics goes further and further away from the things they know and enjoy. As they start experiencing disinterest and continued failure, they become afraid of the subject" (Sinclair, 2000, p. 4).

Aesthetic mathematics experiences may be seen as romantic mathematics stories. Such positive experiences offer students (and teachers) opportunities to fall in love with mathematics or to continue and extend a romantic relationship with the subject. For young children, we need to provide mathematics experiences that fuel a continuing romance with mathematics. For many older children and adults, the challenge is to somehow set aside negative experiences with mathematics and to rekindle a lost romance. One way to identify the characteristics of romantic mathematics experience is to look at the type of mathematics that adults identify with various views of mathematics. In one of our research projects involving teachers taking a fully online mathematics teacher education course, we examined teachers' responses when asked to share their views of mathematics and to explain some of the reasons or sources for their views. We found it interesting that teachers responded by telling us stories—rather than make a list of characteristics.

Teachers' stories expressed one of three distinct aesthetic orientations towards mathematics. A positive orientation was manifested by statements such as "I LOVE math. I always have". Note the aesthetic nature of the teacher's statement—the verbal expression of delight and the use of capital letters to convey her emphasis. If this had been a face-to-face dialogue the teacher might have smiled or raised her voice pleasurably. Another teacher related the following story:

> I also acquired my love of mathematics from my parents. As a child, our cottage was 3.5 hours away from our house and my mom and dad kept me and my three sisters busy for the trip by playing lots of games—counting, logic puzzles and also doing a lot of singing. At the cottage we played lots of different card games—Fish, Crazy Eights, 31, Solitaire, Euchre, Cribbage—to name a few. I have taught my Grades 3/4 class the game Digits Place and they really love it. I gave them a homework assignment to teach the game to 2 people that night. I had an excellent response from the parents. All the parents really enjoyed playing it with their children. They also have asked for more "homework" they can do together as a family.

This teacher created continuity between her childhood experience and her teaching through mathematical play. One could imagine the teacher vicariously experiencing her remembered childhood pleasure through her students. Through games she helps create community among children and parents, and parents and school, and children and mathematics. Games offer students the opportunity to do mathematics in their heads where they are not bound by the rules of paper-and-pencil procedures—in fact, very few children rely on such rules when operating on numbers mentally (Kamii et al., 1993). It has been our experience working in elementary classrooms that when mentally adding numbers like 19 and 16 children will transform question to 19 + 1 + 15 or 14 + 16 + 5 or 10 + 10 + 9 + 6, and so forth. Such mental activity allows students to construct personal and powerful ways of understanding and to take pleasure in doing mathematics. In a classroom context, such activity makes for a good mathematics story. The mental methods students use are understandable to them and to their peers and they are personal. This stands in sharp contrast to the standard paper and pencil algorithm for adding number which is typically not understood, learned through rote and which is impersonal—it works the same way regardless of what the question is or what the student knows and understands. Given opportunities to share mental methods for adding numbers, students will be eager to explain and they will express surprise and delight at unique or interesting ways that other students use. They will be motivated to seek different methods.

Most teachers in the online course expressed a negative predisposition towards mathematics, as manifested in statements like "I grew up with very negative feelings towards math." In addition, some teachers expressed a passive attitude towards the subject. There wasn't a feeling of dislike or fear but neither was there a feeling of excitement or enthusiasm. As one teacher commented, "I did not hate math in school but I didn't love it either. I went through the motions". It was interesting to observe that although negative and passive stories about mathematics were primarily school based, positive stories of mathematics were all family based. As one teacher related, "I, like many of you, had many problem-solving car trips. I still get excited when I see a license plate that I can make ten with (using any means)". This is not to say that all home experiences of math are positive and school experiences are negative, rather, that in our study mathematical activity in the home was remembered fondly and its pleasurable affects continued to be felt in adult life. Although negative and passive stories focused on learning procedures and getting answers (or the difficulty in getting correct answers), positive stories focused on problem-solving processes. For example, note the aesthetic qualities of the earlier open-ended mathematical problem of "making ten" using "any means" which contrasts sharply with the more traditional approach that insists that one finds *the answer,* to, say, 5 + 5. The person "making ten" has the opportunity to use her imagination and to find personal, creative ways of looking at mathematically combined digits on license plates.

It is not uncommon for elementary teachers to have negative aesthetic associations with mathematics. Many openly and sometimes proudly admit that they do not like mathematics or that they do not feel confident mathematically. Changing teachers' perceptions of mathematics is an important first step towards improving classroom practice. McGowen & Davis (2001a) suggest that we need an "antidote" to teachers' conceptions of mathematics as learning procedures and getting right answers. Findings show that such conceptions are consistently associated with observed practice of teachers (McGowen & Davis, 2001a, 2001b; Stipek et al., 2001) and that teachers who hold such conceptions of mathematics have lower teacher self-confidence and enjoy mathematics less than teachers who hold inquiry-oriented conceptions (Stipek et al., 2001).

In another of our research projects we are looking at pre-service elementary teachers' reactions, in an online component of their teacher education program, to interviews with mathematicians expressing affection towards mathematics (Labelle, 2000; Sinclair, 2000; Yeats, 1999). It is interesting that almost all of the sixty pre-service teachers in the study—most of which entered the faculty of education experience with very negative attitudes towards mathematics—expressed positive aesthetic reactions to the mathematicians' views of mathematics and they shared personal examples of mathematics experiences that they found aesthetically pleasing. Listed below are excerpts from two pre-service teachers' reactions to the mathematician's views. Similar reactions were expressed by many of the other pre-service teachers in the study.

> My initial response to the question regarding the beauty of mathematics was one of disbelief! I honestly never considered such an adjective as applicable to the subject of math. Yet with a little introspection I can remember …

> After reading the interviews with the mathematicians, particularly Nathalie Sinclair's, I felt a creeping desire to tackle math again. The language that she used to describe her love of math was inspirational, as well as, poetic, appealing to my senses, and the possibility of actually loving math again. ... Maybe it is possible, after all, to enjoy math again. We'll see …

We believe that teachers naturally want to improve the mathematics stories they possess, live and help create in their classrooms. They, like their students, are aesthetically drawn to a good story and they naturally want to find ways of incorporating it in their lives by retelling it and reliving it and by improving the mathematics story they have internalized—the story they tell to themselves.

Mathematics as story

Human cognition is story-based. We think in terms of stories, we understand the world in terms of stories that we have already understood, we learn by living and accommodating new stories and we define ourselves through the stories we tell ourselves (Schank, 1990, pp. 218–219). Our lives make sense when shaped into narrative form (MacIntyre, 1984, p. 39). Story is a human symbol system used to comprehend events and entertain questions, and represent those events and questions in multi-modal ways. Multi-modal refers to story's ability to hold the complexity of experience in a single work, and to present sensory, emotional, and conceptual information simultaneously through image and metaphor. A story also has its own internal system of validation in that it is not a story unless it coheres, and complies with its own internal logic. In this way story forces people to concern themselves with the logic of their stories, with making sense of their own experience. Thus story provides its own justification, or *proves* itself.

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New PhD Reports

Présentations de thèses de doctorat

Socioeconomic Gradients in Mathematics Achievement: Findings for Canada from the Third International Mathematics and Science Study

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Introduction

A major objective of mathematics education in Canada is to provide opportunities for all students to successfully learn mathematics. Educational systems in Canada, however, have not achieved this goal as many students with low socioeconomic status, females, and minority students fail to achieve an adequate knowledge of mathematics. Much of the discussion regarding this lack of achievement concerns classroom resources and practices, school policies within educational systems, and the specific domain of mathematics achievement considered. My dissertation employs the concept of *socioeconomic gradient* to conceptualize a successful mathematics education in terms of the level of mathematics achievement and how equitably achievement is distributed within schools in the system. A gradient refers to a gap in schooling outcomes between minority and majority groups, or between males and females. The term "socioeconomic gradient" refers to the relationship between individuals' school achievement and their socioeconomic status (SES). SES describes a person's access to and control over wealth, prestige, and power. It is typically measured through factors such as income, the prestige of a person's occupation, and his or her level of education (see White, 1982). The steepness of socioeconomic gradient indicates the extent of social equity since it highlights the gap in school achievement between advantaged and disadvantaged groups. Shallow gradients indicate schooling outcomes distributed equitably among children with varying SES, while steep gradients demonstrate less equitable distribution. The socioeconomic gradient approach allows for an understanding of the processes associated with the variation in achievement levels and the variation in socioeconomic gradients.

The dissertation employs multilevel models and the 1995 Canadian data from the Third International Mathematics and Science Study (TIMSS) to address three main research issues: 1) the extent to which differences in mathematics achievement is attributable to gender, family background, classrooms, and the province where a student attends school; 2) whether the variation in achievement is specific to a mathematics domain; and 3) whether the variation among six provinces (Newfoundland, New Brunswick, Ontario, Alberta, British Columbia, and Quebec) in the levels of their mathematics achievement is associated with various aspects of school policy and practices. By addressing these questions, the dissertation attempts to identify factors that characterize effective school systems; that is, those with high achievement levels and shallow socioeconomic gradients. My thesis is that, understanding the processes that allow disadvantaged students to successfully learn mathematics is fundamental to achieving effective mathematics education for all Canadian students.

Theoretical Background

One can view mathematics learning from two complementary perspectives: the "individualist" perspective, and the socio-cultural perspective. The individualist perspective explains learning as an acquisition of knowledge through processes that occur within the individual. Individuals "actively construct their mathematical way of knowing" (Cobb, 1994, p. 13). The socio-cultural perspective envisions knowledge as being distributed, and that learning occurs through interaction with and participation in socio-cultural practices. From the individualist perspective, the process within the individual is more important, while from the socio-cultural perspective, the activities within learning environments are more important.

The main argument from the individualist perspective is that, during the formative years children acquire a network of ideas, and the connections among these ideas allow them to make sense of new information (see Schoenfed, Smith, & Arcavi, 1993; Siegler & Klahr, 1982). The conceptual organization of this new information occurs within the individual. When children encounter new information, they try to associate it with their existing knowledge and personal experiences (Mayer, 1992) in an attempt to construct a meaningful link between familiar and unfamiliar information. Sometimes, the new information is elaborated and integrated into an existing schema within an individual. Other times, the existing schema is adapted to fit the new information. Learning, therefore, involves processes of individual knowledge construction.

The major weakness in the individualist position is the less emphasis placed on how the context of the past and immediate social and cultural experiences and the role this context plays in the construction of mathematical knowledge. The socio-cultural perspective picks up this weakness and situates mathematics learning in a social and cultural context. The theoretical position of this perspective is motivated largely through the work of Vygotsky (Nunes, 1992), who argues that, in general, learning occurs when an individual internalizes a social experience through interacting with a peer or adult (Vygotsky, 1988). The process of learning occurs through cognitive processes that originate and form through social interaction. Vygotsky (1978) stresses the importance of social interaction with more experienced others through the concept of the "zone of proximal development" (ZPD) and the role of culturally developed instruments as psychological tools for thinking. The ZPD is defined as the distance between a child's independent problem-solving ability and his or her potential for success through collaboration with others. Leot'ev (1981) supports Vygotsky's view but stresses the importance of engagement in activity. He maintains that learning occurs through interaction and participation in activity. Other researchers emphasize the importance of locating learning in the co-participation in cultural practices (Lave & Wenger, 1991; Rogoff, 1990). In this model, the students' social engagements through interaction with more experienced others, and through participation in cultural activities are the driving forces for learning. From this perspective, the variation in the processes that allow students to interact with peers and teachers and fully participate in mathematics communal practices is the source of the variation in students' mathematics learning. In this sense, the learning environments are more important than an individual's cognitive processes.

A number of mathematics educators now consider individuals' cognitive process and their learning environments as equally important for understanding students' mathematics learning (see NCTM, 2000). The belief is that the two perspectives are reflexively related such that one does not exist without the other (Cobb, 1998). That is, cognitive processes within individuals and the context of students' active participation in classroom mathematical practices are both important for understanding students' success in mathematics learning. This is consistent with a view of a mathematics classroom as a community with norms and practices, and also, as "a collection of individuals who mutually adapt to each other" (Cobb, 1998, p. 1-44). This view suggests that the norms and practices of a mathematics classroom as well as the characteristics of individuals within the classroom are equally important in determining how students come to understand mathematics. Put in other words, individuals as well as classrooms are important unit of analysis in research that seeks to understand the processes for a successful mathematics education for all. Multiple unit of analysis, however, requires complex research techniques. The research technique employed in this dissertation involved complex statistical procedures for multilevel models.

Multilevel Models

Multilevel procedures allow researchers to estimate models with nested data sets so that multiple units of analysis are possible. The data set for this study involve students nested within classrooms in Canada. The multilevel statistical procedures for this study entail regression analyses within classrooms with estimates of intercepts and regression coefficients (gradients), and the variation of these estimates within and among classrooms. The analyses also employ multivariate, multilevel statistical procedures so that a multilevel analysis of the six domains of mathematics are carried out simultaneously. (These procedures are discussed in Chapter 3 of the dissertation). The analysis entailed: 1) tracing the sources of variation in students' mathematics achievement levels; 2) identifying classroom and school factors associated with this variation; and 3) describing the sources of differences among six Canadian provinces in their mathematics achievement levels.

The application of multilevel models in research is becoming increasingly popular, especially in medicine, economics, and education (Goldstein, 1995, 1997). Multilevel statistical models, and the research and practical issues they address, are highly relevant in many areas of mathematics education (see Frempong & Willms, 2002; Willms & Jacobson, 1990). Such an important and emerging research tool has yet to make its debut in the mainstream of mathematics education research. The application of this important statistical tool is significant in this dissertation as it attempts to demonstrate the usefulness of this tool in a research that extends our understanding the processes that provide opportunities for all students to successfully learn mathematics.

TIMSS

The Third International Mathematics and Science Study (TIMSS) provided data source for my analysis. TIMSS is a study of classrooms across Canada and around the world involving about 41 countries, which makes it the largest and most comprehensive comparative project to assess students' school outcomes in mathematics. The International Association for the Evaluation of Educational Achievement (IEA) coordinated TIMSS from Canada and the United States.

The main objective of TIMSS was to provide data on the teaching and learning of mathematics and science in elementary, lower and upper secondary schools around the world with the hope that analyses of these data would inform teachers, educators, and policy makers about the classroom processes associated with students' mathematics and science outcomes. The framework of the study presumes that certain processes linked to curriculum and instruction have a direct relationship with the students' achievement and their attitude toward these subjects.

TIMSS targeted three populations: population 1—students in adjacent grades containing a majority of 9-year-olds (Grades 3 and 4 in most countries); population 2—students in adjacent grades containing a majority of 13-year-olds (Grades 7 and 8 in most countries); population 3—students in their final year of secondary schooling (Grade 12 in most countries). This research study utilized the Canadian population 2 data describing the mathematics achievement levels of 13-year-old students in Canada. In Canada, these students are in Grades 7 and 8 (Secondaire I and II in Quebec). Both grades are part of the secondary school system in all provinces except British Columbia, where Grade 7 is part of the elementary program (Taylor, 1997).

The TIMSS Canada population 2 data were collected from a random sample of Canadian schools and classrooms. The random sampling and selection were carried out by Statistics Canada and data were collected in the spring of 1995. Over 16 000 students and their teachers and principals participated in the population 2 component of the study in Canada. Students wrote achievement tests that included both multiple-choice and constructed-response items which covered a broad range of concepts in mathematics. The students also responded to questionnaires about their backgrounds, their attitudes towards mathematics, and instructional practices within their classrooms. Principals completed a school questionnaire describing school inputs and processes, and teachers responded to questionnaires about classroom processes and curriculum coverage.

An important feature of TIMSS Canada is that five provinces—British Columbia (BC), Alberta (AB), Ontario (ON), New Brunswick (NB), and Newfoundland (NF) over-sampled their population such that sample sizes are sufficiently large to allow for inter-provincial comparisons. A sixth "province"—a collective group representing "Other French"—was created by isolating the students who wrote the TIMSS test in French. One will expect the majority of students in the "Other French" to come from Quebec, because there are comparatively few Francophone students who wrote the TIMSS tests in French from provinces such as Saskatchewan, Manitoba, and Nova Scotia whose students' population is comprised of Anglophones and Francophones.

Major Findings

Seven major findings emerged from the analysis:

1. Students within mathematics classrooms vary in their achievement levels according to their gender and family backgrounds. The achievement levels of females and low SES students are particularly lower in Proportionality, Measurement, and Fractions than other domains of mathematics.
2. Socioeconomic gradients vary significantly among classrooms, and there is some evidence that gradients decrease with increasing classroom mathematics achievement levels.
3. A more equitable distribution of achievement within mathematics classrooms was related to teachers avoiding practices which involve small grouping, where mathematics teachers are specialized, and in schools where pupil-teacher ratio is low.
4. Excellence in mathematics is possible in classrooms where a teacher's instructional practice is less traditional, where calculators are used regularly but computers are not used, where teachers regularly assign homework, where there are fewer disciplinary problems, where teachers specialize in mathematics instruction, where pupil-teacher ration is low, and where remedial students are not removed from regular mathematics classrooms.
5. The average socioeconomic status of a classroom has an effect on student achievement over and above the effects associated with a child's own family background.
6. There are large and statistically significant differences among the Canadian provinces, both in their levels of academic achievement in mathematics and in their SES gradients.
7. Some of the differences among the six provinces in their levels of academic achievement in mathematics are attributable to provincial differences in schooling processes.

Discussion of Findings

The relatively low achievement of females and low SES students in Proportionality, Measurement, and Fractions than other domains of mathematics could be attributed to a number of factors, including the possibility that teachers present concepts in these domains in ways that do not allow these students to utilize their knowledge from one domain to understand another. Further analyses revealed that, in general, within a mathematics classroom, students' achievement levels were not stable across the six domains of mathematics; that is, students with high scores in one domain did not necessarily have high scores in other domains. These findings indicated that a student's success or failure in mathematics learning is domain-specific and is also related to the backgrounds of students. More research is needed to determine whether the inconsistency in students' achievement across domains of mathematics is related to instructional practices. The process of learning mathematics involves building on prior knowledge and experiences so that one expects teachers to present mathematical concepts in ways that allow their students to connect ideas in mathematics.

As discussed earlier, socioeconomic gradient provides a measure of how well a classroom has achieved an equitable distribution of mathematics achievement along socioeconomic lines. Steep gradients indicate large disparities between advantaged and disadvantaged students within a classroom, whereas shallow gradients indicate a more equitable distribution of mathematics achievement. Excellence in mathematics achievement for all requires that one think about both equity and excellence together; that is, gradients need to be considered alongside levels of achievement. From this view, one would expect any schooling system, including those in Canada, to strive to achieve equity through educational policy and reform initiatives that are likely to bolster the achievement levels of less advantaged students up to those of advantaged students. School policies and practices that achieve equity but result in lower achievement levels for certain groups are undesirable. Also undesirable are schooling practices that result in disadvantaged students being disadvantaged in schools and classrooms. There are examples of schools that are successful in achieving both excellence and equity (e.g., see Lee & Bryk, 1989). The findings in this study indicate that gradients do vary significantly among classrooms, and that there is a modest negative relationship between excellence and equity; that is, mathematics achievement is equitably distributed in classrooms with high achievement levels. Thus, there are successful classrooms in Canada, but the most successful classrooms tend to be those where students from disadvantaged socioeconomic backgrounds excel in mathematics.

The finding pertaining to small grouping is not consistent with the expected theory at least with respect to the notion that such a practice would provide weak students lacking certain mathematics skills with the opportunity to learn them from their more advanced peers. The theory holds that interaction among students within small groups through discussion, debating, and expressing ideas creates the opportunity for multiple acceptable solutions to mathematics problems. The belief is that, through these interactions, students would experience cognitive conflicts, evaluate their reasoning, and enrich their understanding about mathematical concepts. However, as Springer, Stanne, and Donovan (1999) have noted, without the appropriate structures to make each member of a small group accountable for learning, the expected benefits of small groupings may not be realized, since the interaction would be in most instances merely sharing answers instead of ideas. A number of studies indicate that effective interactions characterized by high-level deliberations about issues that enhance conceptual understanding occur when teachers clearly define issues, give specific guidelines, and define roles for members in a group (see Johnson & Johnson, 1994). TIMSS data did not include variables describing structure and dynamics within the small groups and, therefore, the motivation of low SES students to interact with other students in their small grouping could not be evaluated. Further detailed study on the effect of small groupings on students' mathematics learning is needed.

The association of low pupil-teacher ratio and teacher specialization with equitable distribution of mathematics achievement makes sense as one would expect knowledgeable mathematics teachers to be deeply committed to the teaching of mathematics and could more easily keep up to date with latest curriculum developments and innovations in mathematics teaching. And in schools with low pupil-teacher ratio, one would expect small class sizes that will allow these teachers to utilize all resources and strategies at their disposal to ensure that all students excel in learning mathematics.

The finding regarding traditional instructional practice was expected as it is consistent with the contemporary views of mathematics educators that such an instructional practice makes students less active in classroom mathematics learning. The recommended instructional practice is students' active interactions and participation in mathematics, and problem-solving involving real-life experiences.

The relatively high achievement levels of classrooms in schools where remedial students are not removed from regular classrooms is also important. In a large number of schools within the provinces weak students are offered remedial classes. In some of these schools, however, students in remedial classes are removed from the regular classes. This is a form of tracking. The major motivation for this type of grouping in Canadian Grade 7 and Grade 8 classrooms is not known. In the United States, however, research indicates that this form of tracking is designed to ensure homogeneity of students in terms of their academic ability (see Mevarech & Kramarski, 1997). The belief is that teachers would be more efficient in teaching students with similar ability levels and, consequently, produce high achievement levels. This study contradicts this belief. The study indicated that it is possible to achieve excellence in mathematics even with weak students in regular mathematics classrooms. Removing the academically weak students from regular classrooms is inconsistent with socio-cultural learning theory as the practice denies these students the opportunity to learn from their more able counterparts.

The finding that students, irrespective of their SES backgrounds, are likely to score higher in mathematics if they are in classrooms with high mean SES is consistent with the findings of a number of studies pertaining to contextual effects associated with one's peer group, and it demonstrates the importance of peer interaction of talented and motivated students in classroom mathematics learning. The finding also calls for caution in the way students are distributed in schools as this could severely hamper the successful mathematics learning of students from disadvantaged home backgrounds. The term "double jeopardy" is used in this study to indicate that a child from a family with poor socioeconomic status has an even worse chance of success in a school where the majority of students are also from families with low socioeconomic status. A number of researchers have noted that when students are segregated through residential segregation, private schooling, or choice arrangements within the public sector, advantaged students benefit slightly, but disadvantaged students do considerably worse. And if the desire of a schooling system is to ensure quality mathematics outcomes for all students, then policies that tend to segregate students according to ability or socioeconomic status should be viewed with caution.

The analyses revealed that the six provinces can be clustered into three groups: Newfoundland, New Brunswick, and Ontario, with achievement levels which were below the national average; Alberta, and B.C., with achievement levels above the national average; and Quebec, with achievement levels well above the national average. The provinces also varied in their SES gradients. The SES gradients were relatively shallow in Quebec but steep in Newfoundland and New Brunswick. The SES gradients for Ontario, Alberta, and B.C. were close to the national average. The findings indicated that the provinces of Quebec, Alberta, and B.C. with high mathematics achievement levels tended to have shallow gradients, whereas the other provinces, Newfoundland and New Brunswick, with lower mathematics achievement levels tended to have steep gradients. This finding indicates that mathematics achievement is equitably distributed in provinces with high achievement levels. In other words, some provinces excel in mathematics, but the most successful provinces are those where disadvantaged students from low SES backgrounds excel in mathematics. This is quite evident in the way provincial achievement levels are distributed along socioeconomic lines. The variation is wider at the lower SES levels than at the higher SES levels suggesting that provinces with high average mathematics achievement levels tend to do so by raising the achievement levels of their low SES students. The distribution also indicated that students from low SES families are likely to have high achievement levels if they attended schools in Quebec and low achievement levels if they attended schools in Newfoundland, New Brunswick, or Ontario. For students from high SES families, attending a school in Quebec may give them a slight advantage but would not matter in any of the other five provinces. The study also provided evidence that classroom mathematics achievement differences and gradients are linked to differences in the provinces, indicating that a student in a classroom might successfully learn mathematics, while another student of similar family background may not be as successful, simply because of the province in which the student received mathematics instruction. Quebec, Alberta, and B.C. have relatively high proportions of their classrooms that are successful in achieving excellence and equity. The provincial achievement levels and their SES gradients were stable across the six domains of mathematics. This means that a province with a high achievement level and shallow SES gradients in one domain also tends to have high achievement levels and shallow gradients in the other domains of mathematics.

The analyses demonstrated that students' background characteristics could not account for all the differences among the six provinces in their achievement levels. However, the inclusion of variables describing mathematics instructional practices and other school processes explained some of the differences in achievement levels between Quebec and the other provinces, and all the variation in achievement levels among the other five provinces. There are no major differences between Quebec and the other provinces in their classroom instructional practices. Quebec differs from the other provinces in its low pupil-teacher ratio, its specialized mathematics teachers, and in the small proportion of schools where students are removed from regular mathematics classrooms. Incidentally, these variables are also associated with excellence and equity within mathematics classrooms so that one can be confident in attributing some of the differences between Quebec and the other provinces to differences in these school processes.

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Light and Shadows (in) Knowing Mathematics and Science

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> "I'm flustered!" she exclaimed as she reluctantly handed in her test on mathematical functions. "Yes," I said reassuring myself, "a test is certainly not the time you want to be encountering a problem-type for the first time. My tests are designed to see what you have been doing all along and whether you have practiced enough." Disgruntled, the student left the classroom to commiserate with her classmates just outside the door, presumably measuring their heart rates before sprinting off to their next exercise in futility.
>
> For the first time I can remember, doubts began to creep into my mathematical mind (and heart) as I gathered the tests together and left the classroom. Normally, at about this time, I like to pat myself on the back for a test well made—a test that reflects the undeniable challenging nature of mathematics; a test that separates the 'haves' from the 'have-nots', if you know what I mean. I am being facetious of course, since *I* do not even know what that means, except for believing that it symbolizes unmistakable mathematical arrogance. Just because I survived my school days on the privileged side of this binary opposition, does not mean it is the only way to 'know' math. I started wondering how I could continue to comfortably contradict myself, as I *preached* math for all and *practiced* math for the few. (Nolan, 2001, p. 26)

The above passage is one of several author reflections in my recent doctoral dissertation entitled *Shadowed by light, knowing by heart: Preservice teachers' images of knowing (in) math and science*. Best described as a feminist critical narrative, this dissertation explores women preservice teachers' experiences of learning math and science. From a personal perspective, the study emerged out of my own experiences of learning math and science, and from common encounters with expressions of the shadows in what it means to know. In other words, my research really began as an exploration into why so many people think of (and describe themselves as being) "good at" or "not good at" math and science. If I have learned anything during my research over the past few years, it is that everyone has a story (or two) about their school math and science experiences. When asked what my research is about, I usually fumble my way through various descriptors—math, science, knowing, experiences, gender, answers, 'good at', etc., etc. Once I have put together a coherent sentence from within my state of research immersion, I always eagerly anticipate the person's response. And there is always a response. The following story relates one such unexpected daily expression of what it means to know (in) math.

Bakery Math

My request to mix cookies, obtaining equal numbers of each kind,
was fairly straightforward, especially with the combination of 5 kinds
and 40 cookies in total.

The task of calculating the cost was not as enthusiastically embraced,
however. The dozen and half-dozen quoted prices lent themselves more
readily to customers abiding by multiples of 6.

It is never my intention to draw attention to a person's mathematical
(dis)interests. When I volunteered a method for figuring out the cost, I felt
the need to pardon my interference. "Don't mind me,
I am a math teacher, and I'm always problem solving."

The cashier looked surprised. "A math teacher?" Before I could nod and
shrug my shoulders in casual agreement, she added,
"I've never been very good at math."

"To this day", she reminisced, "I can still remember sitting at the kitchen
table, doing one math problem after another.
I guess that's the only way to learn it—
to do them *over and over* again.
But I was never very good at it."

(Nolan, 2001, p. 24)

Stories like this made me ponder the spaces between the dichotomies of "good at it" and "not good at it"; between knowing/NOT knowing, light/dark, presence/absence. Western society uses dichotomies frequently to describe (defend? protect?) our knowledge. Exploring the presence/absence dichotomy as related to what it means to know became critical to me. I realized the importance of exploring how preservice teachers' images of knowing highlight many such presences and absences, clearly portraying reasons for such dichotomous thinking. For instance, preservice teachers expressed a real and present pressure to be "all knowing" in mathematics and science or to fake it. My participants frequently informed me that they felt pressure form students, parents, and administrators to know all the answers. They expressed considerable fear and anxiety about not having answers for anyone and everyone about anything and everything. Also, my conversations with the preservice teachers drew attention to the fact that there is inadequate space between the dichotomies of knowing and not knowing for a partial, situated knowing. With/in an educational system that describes knowing in terms of grades, these preservice teachers judged their knowledge according to the marks they received. A high mark meant they knew the subject and were good at it; a low mark (as they often received in math and science) told them exactly the opposite.

Things started to make sense for me (and, at the same time, become less sensible) as I blended these ideas with the light/dark dichotomy. In my dissertation, I use the metaphor of light to explore the to-and-fro movement between light and dark in the form of shadows. In this paper, I present a few such shadows as shaped by the voices of my participants.

Shadowed by Discourses of Ignorance

Michael (1996) states that "… ignorance cannot be treated as simple deficit; it entails active construction" (p. 111). In describing three different discourses of ignorance, Michael draws attention to the fact that we actively formulate a relationship with/to science. In my research, this relationship was frequently characterized by Michael's discourse of ignorance related to mental constitution. My participants often found themselves stating up front, in any conversation about science and math, that they were unsure and that they were just not scientifically or mathematically minded.

> Right away, when that Tangram was in front of me, I said to somebody 'I can't do these'. I didn't know that yet! I just say it because it's easier … you have a safety net. Then every-

> body knows you can't do it. And if you're the only one at the table who can't do it then it doesn't matter, you know? (Ursula, in Nolan, 2001, p. 156)

My participants often expressed a relation of dependence on those who know math and science and they were quick to evaluate their own knowledge (in its partial and often informal form) as being of little value. This is one critical shadow of exploration between light and dark, between knowing and not knowing.

Shadowed by Gendered Ideology

There were many articulated and unarticulated classroom messages that my research participants referred to in our conversations, directing one's attention to shadows created by a form of gendered ideology at work to exclude.

> I don't know if this has anything to do with your research but I loved our female math teacher … but I always felt like she was teaching to the male students in the class. Her favourites were always the males. And a lot of other people noticed that too. It was mostly with help—if something wasn't explained as well as it could be, she would offer them more help than she'd offer me. That was basically the big thing that I noticed. (Elsie, in Nolan, 2001, p. 129)

> With math and science … I feel like it's partially to do with being a girl. If you're a girl and you say 'oh, I can't do this'—if all the girls say 'oh, I can't do this'— it's like 'ok, well, you don't really need it anyway'. But if a guy were to say it, [the teacher would say] 'well, try harder, work it out, come over and I'll help you'. With a girl it's 'oh well, you did your best, you just can't do it'. They never said it directly, but if the girls said they didn't understand, the teachers would say 'oh well'. But with the guys, they'd spend a lot more time helping them to get it. (Helen, in Nolan, 2001, p. 128)

Such classroom articulations (and unarticulations), as vividly remembered by my participants, feed into the hegemony of who is supposed to succeed at math and science and, alternatively, whose mathematics and science success surprises us as educators. As Lewis (1993) states, "…discourse is an action taken upon the world. These actions may be transformative or they may be tenaciously preservative of the status quo; whatever the case, discourse is socially negotiated through power" (p. 114). I believe that gendered messages like those of Elsie and Helen, still perpetuated in many math and science classrooms, are preservative of status quo in their creation of unexplored shadows between who is supposed to know and not know.

Shadowed by Answers

For my research participants, ambiguity in/and mathematics represented the unimaginable. Their experiences of learning mathematics taught them that mathematics is very close-ended and answers (always present) are simply either right or wrong. In addition, since there was no ambiguity as to right and wrong, they also perceived no ambiguity then as to who can and cannot do mathematics. According to their experiences, without answers in mathematics, there was no knowledge to speak of.

> That's what all those tests were—'you know what I want you to know, or you don't'. And it was an 'x' or a check mark. There was no discussion. It was just a right or wrong thing. (Ursula, in Nolan, 2001, p.51)

> It's an uncomfortable feeling if there isn't an answer or you don't know it or you can't find it. That concept of 'if you don't have an answer then you don't have knowledge'. Yeah, they go hand-in-hand. (Maude, in Nolan, 2001, p. 119)

Shadowed by Questions

My participants often spoke of a desire to know and understand *why* in their experiences of learning math and science. Instead of being deconstructed, such questions were often met by manifestations of power in the classroom. A teacher's response of 'you just do; don't ask why' was frequently misinterpreted by my participants as a weakness in their own knowledge and in their abilities to learn mathematics and science.

> We were having a discussion and many of us were saying that we did not like math class in high school because we would always want to know why the teachers were doing something, and they would just say 'that's just the way it is.' We discussed how frustrating this was, and still is. I have always had this problem, in every single math class I have ever taken. (Helen, in Nolan, 2001, p. 122)

Do shadow spaces exist between questions and answers? There is a sense of power associated with being the one who has the answers. Unfortunately, this often means there is a sense of powerlessness associated with not having answers. A view of the teacher as being all-knowledgeable leaves no room for realistic levels of ambiguity and uncertainty. This was never more clearly demonstrated than in a recent teaching experience of mine. I recently accepted a position as instructor for a class of students enrolled in general equivalency courses in high school mathematics and science. My role was to assist students as they worked in their textbooks on their respective subjects. As I sat at the teacher's desk, reflecting on my own research and the spaces between knowing / not knowing, presence / absence, and light / dark, I was constantly reminded of how people really do not want to reside in the in-between ambiguous spaces between questions and answers.

The students.
Nine in number.

One graphs systems of equations on a graphics calculator and wonders why he wasted all that time and graph paper in school.

Another ponders why $(^{-}6)^2$ is not 12.

A man draws a blank as he flips through the pages of the biology chapter, seeking the right words to place in the questions' blank spaces.

Hands on chin, she says it's been almost a year since she's done this math, but it'll come.

From under the baseball cap, his head never leaves kinematics and I wonder why *he* has no questions for me.

The other kinematics student needs to know if she's supposed to just READ about ticker tapes, since she certainly can't be expected to DO the experiment, right here, right now.

The man with the cap and plaid shirt, breathing heavily as he returns from writing a math test, is clinging to a possible 50 as he says 'some of the stuff … just gone.'

The most advanced student has circular function angst over the inverse of inverse sine functions, as if he'll ever use them.

And the man at the back table calls me over every so often to interpret what they (the nebulous textbook) want for an answer, as his just aren't matching those in the back.

And it's a room full of answer-want
it boosts my self-confidence to give them what they want
it pumps me up to be able to take away some of their frustration
And then I return to the pages of my own research
where I am trying to understand
why answers are important
and what it all means
to have knowledge
why it matters
if I'm tested
on what
I know
(not)

(Nolan, 2001, p. 17)

This paper has briefly outlined a few of the questions that I grappled with in my dissertation. The format of my dissertation reflects the multi-layered nature of such questions, especially in connecting what it means to know in math and science to what it means to know in educational research. The research text is presented through a kaleidoscopic writing format, highlighting, and acknowledging how thinking and learning are not linear processes. It is my belief that the integrity of a critical research project, seeking to illuminate shadows and ambiguities in knowing, would be entirely compromised through a traditional linear dissertation text. Instead, the text embodies to-and-fro movement between theory and practice, between author and participant experience, between knowing and not knowing. There is no final dissertation chapter outlining future directions, results, conclusions, or answers for this research project. Instead, spaces are created for pondering critical questions in education without presuming that, at the end of two hundred pages, answers will be provided. After all is said and done, however, the question remains: do we still have knowledge if we don't have answers?

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Supporting Student Efforts to Learn with Understanding: An Investigation of the Use of *JavaSketchpad* Sketches in the Secondary Geometry Classroom

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The choice of this topic for my doctoral study stemmed from my ongoing interest in using computers to display mathematical ideas. From the start I was captivated by the dynamic computer images produced by *Cabri Géomètre* (Baulac, Bellemain, & Laborde, 1992), and *The Geometer's Sketchpad* (Jackiw, 1991). Later, after experimenting with *JavaSketchpad* (Jackiw, 1998) I began to study the use of pre-constructed dynamic sketches. I considered that they could have some potential advantages since: they can be accessed from school or home; they can be used by those who do not know software construction techniques; they eliminate the need to spend class time on constructing; and they provide an opportunity for students to manipulate representations of objects, and to thereby develop "the ability to take apart in the mind, [and] see the individual elements" (Goldenberg, Cuoco, & Mark, 1998)

Since there had been no systematic analysis of how students use web-based dynamic sketches, I decided to investigate the benefits and limitations of using such sketches in the secondary program.

JavaSketchpad

JavaSketchpad converts sketches constructed with *The Geometer's Sketchpad* to HTML format. The resulting images can be viewed and manipulated through any Java-compatible web browser.

Like sketches pre-constructed with Sketchpad or Cabri, *JavaSketches* can be dragged to enable reasoning about invariant properties and to provide evidence about the validity of conjectures. Pre-set relationships, such as measurements and ratios, change to match the change that has occurred as a consequence of dragging. *JavaSketchpad* supports action buttons to hide or show additional details, to move points, and to animate objects; however, elements cannot be deleted and items cannot be constructed.

Background

Those who have examined the dynamic geometry environment have found evidence that the act of constructing contributes to the growth of mathematical understanding. Hoyles and Noss (1994) report that when an improperly constructed figure falls apart under dragging, the student is forced to notice relationships among the geometric objects. Hadas and Hershkowitz (1999) point out that the experience of not being able to construct an object that seems intuitively possible to make, is a powerful incentive for students to investigate geometric ideas.

While not minimizing the importance of constructions, some researchers believe that pre-constructed sketches can also play an important role in the development of mathematical understanding. Whiteley (1999; personal communication, 2000) contends that pre-constructed dynamic geometry diagrams are valuable as learning tools because they provide the opportunity for students to examine the connections between geometric objects—a necessary stage before students can effectively carry out many constructions.

However, pre-constructed sketches are visual, dynamic objects and research shows that visual reasoning presents challenges. Eisenberg and Dreyfus (1991) report that visual thinking requires more cognitive effort. Mathematical pictures and diagrams, in particular, are difficult to interpret because they contain a great deal of information, represented in a concise but "nonsequential" format (Goldenberg, Cuoco, & Mark, 1998). Interpreting a dynamic diagram is even more complex. In a recent study of *change blindness,* Rensink (2000) found that we are only able to grab four to six visual objects at once and that focused attention is needed to notice change when objects are moving.

Extensive studies of *Cabri*, *The Geometer's Sketchpad*, and *Geometry Inventor* (Logal, 1994) have led to an understanding of how students use dynamic geometry software. For example, we know that a geometry problem cannot be solved simply by perceiving the images on a Cabri screen, even if these are animated. The student must bring some explicit mathematical knowledge to the process (Arzarello, Micheletti, Olivero, Robutti, Paola, & Gallino, 1998). The present study concerns dynamic geometry, but focuses specifically on how students reason with pre-constructed, web-based sketches.

The Study

I chose a case study approach for the research and used multiple sources of information—observation field notes, videotaping/audiotaping of selected student pairs, a student questionnaire, and interviews with the teachers. I analysed the raw data by coding, developing categories, describing relationships, and applying simple statistical tests where appropriate.

The Participants

Three mathematics classes from two different secondary schools participated in this study. The 69 students were enrolled in the Ontario Grade 12 advanced mathematics program (replaced in 2002), which covered topics in algebra, geometry, analytic geometry, and trigonometry (Curriculum guideline, 1985). The study focused on congruence and parallelism, the first section in the geometry unit. Although the students had done introductory work on deductive geometry related to congruence and parallelism in Grade 10 and on similarity in Grade 11, none had worked with dynamic geometry software.

Students worked in pairs and in each class, several pairs were studied in more depth by audiotaping or videotaping their activities.

The Sketches and Labsheets

Using *JavaSketchpad*, I prepared four web-based dynamic sketches for students to explore, and one sketch for a group discussion. Labsheets that accompanied the sketches provided directions, questions and space for work.

Problems chosen as the basis for the web-based sketches were similar in difficulty to those in the student text, *Mathematics: Principles and Process, Book 2* (Ebos, Tuck, & Schofield, 1986) and related to triangles and quadrilaterals.

Each sketch was created to address particular student difficulties in deductive geometry. The study teachers reported that students often have trouble selecting information from a given diagram—specifically: a) noticing relevant details, b) focusing on the whole shape, c) picking out smaller triangles within a larger diagram, d) mentally separating overlapping shapes, and e) understanding that rotated or reflected copies of a shape are congruent to the original.

Each of the sketches supported the possibility of arriving at a solution from a transformation perspective as well as from a straightforward application of congruency theorems. The intention was to allow students to use symmetry considerations, a) to visually confirm or negate conjectures, and b) to develop a new perspective on geometric relationships.

Example

Day 1, task 2 (see Figure 1) was designed to address student difficulties with overlapping triangles and selection of triangles. This task also aimed to create uncertainty, by including pairs of triangles that could not be proven congruent with the given information. This is not usually done in textbook problems.

To help students notice details:

- the four pairs of congruent triangles were shaded in four different colours;
- given equal angles were shaded red;
- information could be toggled off and on to allow details to stand out;
- triangle pairs could be separated;
- measurements for the given angles and lengths were displayed;
- measurements updated as the sketch was dragged.

To help students pick out a shape within the larger diagram:

- overlapping figures could be separated;
- colour was added to emphasise the shapes;
- colour was used to overlay angles and sides within the shape.

Labsheet questions were designed to focus students on exploring, noticing, interpreting, deducing, and extending, as shown in Table 1.

In this task students were asked to prove that BA = BC. To do this, students needed to deduce at least one piece of information from the given information before proceeding to use ASA (angle, side, angle) or other congruency theorem, (i.e., all options involved at least two steps). All chosen pairs of triangles were reflections, and congruency could also be established or not established by considering what would happen if one member of the pair was reflected.

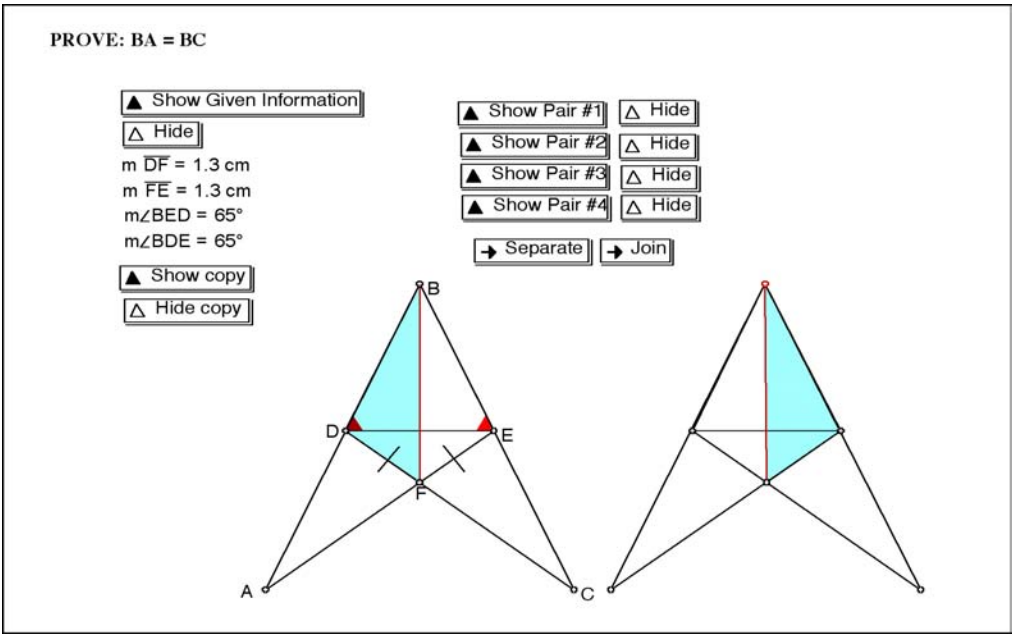

FIGURE 1. Day 1, task 2 – *Javasketch* 3 – View on selecting: "Show Given Information," "Show Copy," and "Show Pair #1".

TABLE 1. Purpose of labsheet questions, Day 1, task 2.

Focus	Question or Direction
Explore	Drag each red point.
Notice	Observe the measurements.
Interpret	Write two additional facts that you know and explain why they are true.
Deduce	If you proved the pair congruent, how would this help you prove BA= BC?
Extend	What is an alternative explanation for the congruency of triangle ABC and triangle FCB?

Findings

Two major and distinct themes emerged from the research data: how the students used the sketches, and how students interacted with elements of the learning environment.

Use of Sketches

In examining the raw data I focused on how students used colour, motion, and action buttons, and analysed their actions and comments with respect to the levels of geometric thinking in the van Hiele model (van Hiele, 1986). Hoffer (1981) defines these levels as: 1) recognition, 2) analysis, 3) ordering, 4) deduction, and 5) rigor. I expected the study students to be at level 4; however, many students seemed to be working at a level 2, still unsure of basic terminology, and properties of triangles and quadrilaterals.

Although students initially dragged points, most used the sketches in static form. This practice indicated that students were not able to examine relationships between different instances of a figure, characteristic of level 3. In some cases, avoiding dragging led to erroneous deductions because a sketch had been dragged to show a special case.

The observation that many students appeared to be working at a low level might be explained by de Villiers' contention that students from a higher geometric thinking level often work in "lower" levels (de Villiers, personal correspondence, 2001). Another possibility is that the dynamic environment has its own set of levels that students pass through in developing geometric thinking skills. The use of dragging to examine relationships between instances of a figure could be associated with an as yet unidentified level.

Several examples in the study showed that the use of the software helped some students move to a higher level of geometric thinking. In an early task Tara and Mary worked at listing details or "trolling for triangles" (Whiteley, personal correspondence, 2000):

Tara: *So in this case do we have side, angle, side? Cause they share the same side, right?*
Mary: *Yeah. So can you say side, angle, side?*
Tara: *Yeah. ... Put triangle BD ... is that BDF?*
Mary: *Yeah.*
Tara: *Yeah, triangle BDF ... no ... what about ... all it does is side, side, where's the angle? We don't know an angle for sure.*
...
Tara: *Well this one you can figure out.*
Mary: *I know ... but how? Just put ... isn't it side, angle, side? ... Yeah, just put side, angle, side.*

Later, they were able to consider motion as a method of exploration even though the sketch was not sufficiently flexible.

Tara: *Cause that means that's the uh—the median? The centre point I mean.*
Mary: *Yeah.*
Tara: *I dunno, is it? Can you turn it around? Probably not. You can't turn it around.*

In another example, Doug and Sal made a conclusion based on how a figure "looks", a characteristic of level 1 geometric thinking.

> Doug: *Angle E is 90 degrees. Well, I'm thinking this is—because it* looks *like it, right?*

Later, they were focused on justifying their reasoning:

> Doug: *But that's an angle within the triangle. We want the whole triangle. What does that little angle prove? It just proves that on the little triangle they're equal but we want the big triangle. You get what I mean?*

Results showed that pre-constructed sketches share important characteristics with mathematical illustrations or pictures. They present information all at once, in non-sequential format, to a user who does not have the constructor's knowledge of the relationships between and among objects. But since they are also dynamic they require students to draw information from a changing model. In order to interpret the information in the sketch students must be able to notice detail, see and use change, and apply (mathematical) knowledge.

Colour, markings, measurements, and built-in motions helped study students to notice details. However, only a few students were able to use change to explore. Paul and Sue were two students who made a conjecture and discovered their error by dragging the sketch.

> Sue: *Ok, if DB right bisects AC then the parallelogram will become a square.*
> Paul: *Two diagonals bisect each other at right angles then the parallelogram becomes a square.* [After dragging the sketch they were rather surprised.]
> Sue: *Obviously it's not a square now ... it's a parallelogram.*
> Paul: *Still a parallelogram—so we were wrong.*
> Sue: *So we have to erase all of it.*
> Paul: *Let's take a look if the sides are all equal.*

The need for students to bring mathematical knowledge to the exploration of a dynamic geometry sketch, documented by Cabri researchers such as Laborde (1998), was very apparent in the study. In this example, Pat and Dave's lack of precision in geometric language hampered their progress:

> Dave: *G is the midpoint.*
> Pat: *No H is the midpoint. G is the line in the middle.*
> Dave: *I'm going to write, 'H is the line that crosses the midpoint'.*
> Pat: *Is H the line or the point?*
> Dave: *H is the line that crosses the midpoint.*
> Pat: *H is a point, not a line.*

This pair's poor grasp of geometric concepts meant that they progressed very slowly through the task questions. Nevertheless, the pre-constructed sketches did help them correct misconceptions as shown in the following exchange:

> Pat: *No, it says rotate. ….*
> Dave: *How do you rotate it? You can't unless it's round. You can only rotate it. ...*
> Pat: *Oh, it rotates on one point. …*
> Dave: *Yeah so—so it stays in one point.*
> Pat: *It goes in a circle. It goes around the midpoint.*

Throughout, the onscreen image was a central feature of the learning situations. Each sketch acted as a shared space for partners to explore and communicate. Students pointed to the sketches, traced onscreen figures, encouraged one another to drag or click, and gazed intently at the images as if trying to soak up the details.

Interactions in the Learning Environment

During the tasks the students interacted with their teacher, their peers, and the task materials. Data suggested that many of the interactions that led to profitable discussions were initiated by questions and peer responses that paralleled the interventions of a good teacher identified by Towers (1999).

In her study, Towers researched intervention styles and strategies that teachers use and identified those that support the growth of mathematical understanding. Towers found that the most effective interventions were the *shepherding* style in which the teacher uses "subtle nudging, coaxing, and prompting", the *inviting* strategy, in which the teacher suggests a "new and potentially fruitful avenue of exploration", and *the rug-pulling* strategy in which the teacher deliberately introduces an idea that "confuses and forces the student to reassess what he or she is doing" (Towers, 1999, pp. 201–202). The importance of rug-pulling is also supported by dynamic geometry research. Hadas, Hershkowitz, and Schwarz (2000) found that uncertainty spurs student need to explain "why" something is true.

In some of the peer interventions during the study we can see evidence of *shepherding*. For example, here Paul tries to help Sue focus on the whole:

Sue: *Okay go ahead, separate them. Oh, it's different. Okay, we know that… Okay just a minute. If these two angles ...*

Paul: *You're working inside the thing again. Just look at ... see the red part? Stare at the red parts. Blur out the black parts. No looking at the black parts ... look at the triangle.*

Questions that encouraged the most significant discussions and actions during the study were those that used the inviting or rug-pulling format. For example, in response to: "How can the information provided by these images be used to explain why DABC is congruent to DFCB?" Pat and Dave had the following conversation.

Pat: *It breaks it apart so it's easy to picture it.*
Dave: *They match. It matched it with the other one.*
Pat: *It shows us that they're congruent. …*
Dave: *The mirror effect shows that ABC's just like FCB.*

This question, more open ended than the typical "prove that triangle ABC is congruent to triangle DEF", drew many students into animated discussions.

There were also several examples of "rug-pulling" questions. In Day 1, task 2, pairs of triangles were included that did not have three pieces of information available to prove them congruent. This confused students because their experience of geometry problems was limited to situations that could be solved. Students reacted to this surprising situation in several ways. Some spent a great deal of time—certain that they must be missing something. Others made up information. After I led one pair through an organized checklist, they realized they did not have enough information, but they were not confident enough to abandon the search until I arrived. This episode demonstrates the importance of teachers circulating throughout sessions and also of gathering students together to discuss ideas and conclusions.

The evidence that questions and directions "intervened" in the learning environment in such a significant way led me to examine their effect more closely. I categorized the intervention interplay between the labsheet questions and the *JavaSketches* as shown in Table 2. This table indicates the deep connection between questions/prompts and sketch affordances. For example, if a question invites exploration, the sketch must provide alternate

TABLE 2. Intervention Interplay

Labsheet	JavaSketch
Focuses attention	Draws attention
Prompts action	Provides affordances (i.e., tools with which to act)
Invites exploration	Provides alternate paths
Introduces uncertainty	Supports experimentation
Checks understanding	Provides a shared image

paths for students to investigate. Otherwise, the act of exploring is trivialised. And when rug-pulling is used, pre-constructed sketches must include appropriate experimental tools to help students overcome confusion.

Conclusion

The results of the study indicated that tasks centred on pre-constructed web-based sketches provide effective interaction opportunities for student pairs, and intervene in the learning situation in ways that parallel those of a good teacher. To support student efforts sketch and labsheet must complement one another in the roles of focusing attention—drawing attention, prompting action—providing affordances, inviting exploration—providing alternate paths, introducing uncertainty—supporting experimentation, checking for understanding—providing a shared image. When materials are developed with attention to these parameters, such tasks can provide valuable opportunities for student learning.

Several directions for future research are suggested by the study results. In particular, study students were not automatically able to make use of motion in explorations of dynamic diagrams. This behaviour suggests two questions for future investigation. First: Is the van Hiele model (or any other traditional model) appropriate for describing the development of geometric thinking skills when motion is involved? Second: How can we teach students to explore dynamic diagrams?

The study also showed that the interplay between what we ask and what tools we provide must be considered when designing mathematical tasks that include pre-constructed sketches. More work is needed to determine how to develop this interplay.

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Ad Hoc Sessions

Séances ad hoc

Conceptualizing Limit in Calculus (Why is it still a problem?)

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This ad hoc session was a discussion of students' conceptions of limits related to the focus of my research. The intention was to invite feedback from attendees based on their experience.

The two foundational notions for calculus are function and limit. The idea of function preceded the arrival of the calculus (although in a more restricted form than is used today). However, it was not until 200 years later that mathematicians agreed on a definition of the limit, which might suggest that this is not a readily intuitive idea. Based on my own teaching experience and the research literature, both notions are perennially problematic for learners. Research studies repeatedly find that learners have incomplete or inaccurate conceptualizations of the limit at least during learning, and likely even ongoing (Ferrini-Mundy & Graham, 1991; Sierpinska, 1994; Szydlik, 2000; Tall, 1991; Williams, 2001), and a whole calculus reform movement has emerged. So why is the limit concept still a problem?

Cartwright (1970) reports on the variety of ways mathematicians and engineers conceptualize problems: "pure" mathematicians who think in applications; "applied" mathematicians who think in mathematical abstractions; engineers who think in terms of the problem at hand. She also says that it is not clear how these conceptualizations arise, nor whether there is any relation between them. If professionals display such conceptual diversity, it is possible that individually and collectively learners also have diverse conceptualizations, and perhaps more tenuous as well, suggesting that there might be value in probing learners' conceptualizations from different points of view.

My goal is to pursue this and investigate conceptualizations across contexts, such as geometric, graphical, analytical, and others, and also across approaches, such as responding to situations, discovery exercises, constructions and counterexamples to force focusing on properties and definitions, and to look for themes and patterns. The discussion and feedback did provide suggestions for further ways to probe students' conceptualizations, as well as contacts for communication as the research proceeds.

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Amplifying Mathematical Intelligence Using Web-Based Interactive Activities?[1]

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When I was going to university I took a year off and worked with my uncle as a carpenter. One morning I arrived at his house and he was under the hood of his station wagon pounding on the engine with his hammer. 'What's the matter', I asked. 'The car won't start again', he said. 'So', I asked, 'what exactly is the connection between the car not starting and you pounding it with a hammer?' He gave it one last bang and slammed the hood down. 'I have no idea', he said, 'but I hope it hurt!' We both started laughing.

As more and more of us are connected to the internet, web-based activities have the potential of changing how and what mathematics is learned. But like my uncle's hammer, web-based activities can sometimes *help* kids understand by exploring math relationships and making new connections, they can sometimes *make no difference* by providing only entertainment, and they can sometimes *hurt* by reinforcing misconceptions or wasting valuable learning time.

In the early 1990s I had the opportunity to observe, over long periods of time, Grade 10 students working with Geometer's Sketchpad. More recently, I have been part of a team designing and delivering inservice on IBM's Teaching and Learning with Computers Software, for all grade 1-4 teachers in an Ontario school district. I have also had the opportunity to observe teachers using the IBM software with their students.

It seems to me that computer-based resources have a lot of potential in affecting how and what is learned in math classrooms. At the same time, my experience cautions me that computer technology is a tool whose effect is greatly determined by the users (and here I have both teachers and students in mind).

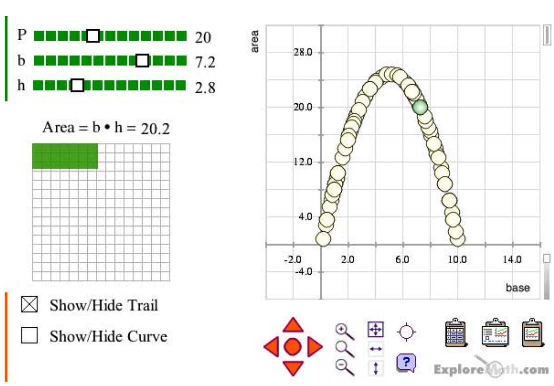
The question that I am now considering is this: To what extent can web-based activities act as intelligence amplifiers, enabling students to think at higher mathematical levels?

Discussed below is the potential of two web-based math activities on the topic of area and perimeter. Both activities are freely available to teachers and students at www.ExploreMath.com.

Maximizing Area

Suppose you have 20 meters of fencing for making a rectangular dog pen. What dimensions would you make the pen so that its area is as large as possible?

You can explore this problem using the activity on the right.



- Set the perimeter, or P, slider to 20.
- Move the base, or b, slider. Notice how the diagram of the rectangle changes.
- Also, notice how the value of the area is changing, and how the point represent-

ing the area is moving on the graph. What dimensions result in the largest area?

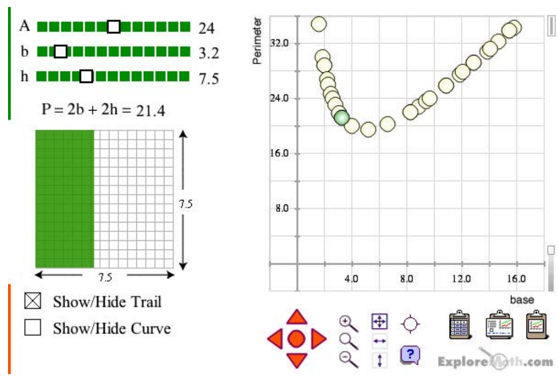
- Explore this problem for different perimeters. Make a conjecture about the general solution to the fencing problem.

Minimizing Perimeter

Suppose you need to make a rectangular dog pen with an area of 24 square meters. What dimensions would you make the pen so that its perimeter (or fence) is as small as possible?

You can explore this problem using the activity on the right.

- Set the area, or A, slider to 24.
- Move the base, or b, slider. Notice how the diagram of the rectangle changes.
- Also, notice how the value of the perimeter changes, and how the point representing the perimeter moves on the graph.



- What dimensions result in the smallest perimeter?
- Explore this problem for different areas. Make a conjecture about the general solution to the fencing problem.

At their most elementary level, these explorations help students understand the relationship between area and perimeter. Students can also explore the functions that generate the area and perimeter graphs and discover that the former is a parabola and the latter is a hyperbola. What a great way to see conic sections in action.

Conclusion

It is not uncommon to see area and perimeter taught as separate topics that are 'mastered' in isolation from one another. The web-based area and perimeter activities described above have the potential of increasing the mathematical focus on the relationships between area and perimeter. They also are tools for students to think with when exploring such relationships.

Two questions worth researching with respect to the area and perimeter activities are:

- How will teachers use them in their teaching?
- How will students use them to think mathematically?

Note

1. This report was inadvertently omitted from the *Proceedings of the 2000 CMESG/GCEDM Annual Meeting*.

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www.ExploreMath.com

Grades 5-6 Teachers' Algebra Teaching Beliefs and Practices[1]

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Algebra is a language for representing and exploring mathematical relationships. Current curricular views of algebra emphasize multiple representations of relationships between quantities and a focus of student attention on the mathematical analysis of change in these relationships (NCTM, 2000, p. 37; Ontario Ministry of Education, 1997, p. 52). The relationships component of such a definition of algebra is what gives purpose and meaning to the language aspect of algebra. Without a focus on relationships, the language of algebra loses its richness and withers to a set of grammatical rules and structures.

The teachers in the study suggested that there is not a lot of algebra in Grades 5-6. They talked about patterning, looking at a set of numbers and seeing "how you go from one number to the other". They also discussed solving simple equations through trial and error. One of the Grade 6 teachers shared an experience of trying to teach students how to solve simple equations more formally: "I was teaching Grade 6, a sharp bunch. There was hardly anything in the textbook on algebra. … So I went from a math book I had from Grade 7 and 8 and I took the simplest pages … $a - 7 = 10$ and I tried to teach them $a - 7 + 7 = 10 + 7$ and what you do to one side you do to the other side, the way I was taught algebra".

It is interesting that the word 'modeling', or other phrases with similar meanings found in the Ontario curriculum descriptions of algebra (Ontario Ministry of Education, 1997), were not used by the teachers when discussing their conceptions of algebra. And there was not a sense that the teachers viewed the exploration of mathematical relationships and the analysis of change as integral components of algebra, as is the case in the Ontario curriculum document. What is also interesting is that the teachers' focus on the formal solution of equations is not part of the Ontario curriculum in Grades 5-6, nor is it an explicit expectation in the Grades 7-8 curriculum (Ontario Ministry of Education, 1997, p. 60). The teachers' view of algebra matches the typical view of algebra that emerges from research, where the teaching of algebra is instrumental rather than relational, with a dominance of symbolic algebra over other representations (Borba & Confrey, 1996; Kieran, 1992; Kieran & Sfard, 1999).

In the study, teachers used the web-based Maximize Area activity, shown in Figure 1, whose focus is on algebraic relationships in the context of area and perimeter. When using the Maximize Area activity in their teaching of mathematics, teachers tended to place a greater focus on algebraic relationships. The activity may have acted as a pedagogical model for teachers' classroom practice, shifting the teaching focus from the learning of isolated area and perimeter concepts to algebraic relationships among these concepts.

When asked if there was algebra in the student activities relating to the Maximize Area activity, the teachers talked about the area equation, and the substitution of values in the equation. "A mystery, an unknown, that's what I see. That's algebra, right?" Another teacher described algebra as "numbers and unknown letters, solving by doing the same to both sides". When asked if the relationships between area and perimeter that were explored in the Maximize Area activity were part of algebra, teachers said they didn't think so. One teacher said that some algebra was involved in creating a table of values, as substitution in an equation was used to find values. The teachers recognized the increased focus on relationships in the Maximize Area activity, compared to their regular teaching. However, they did not associate these relationships with algebra, even though that connection is in the Ontario mathematics curriculum document. They saw the relationships simply as area and perimeter relationships.

One may wonder whether this makes a difference in teaching mathematics—that is, as long as teachers focus on algebraic relationships when they teach about area and perimeter, does it matter that they do not see it as algebra? This diminished view of algebra makes a difference for at least two reasons. First, algebra is typically a strand of mathematics in curriculum documents. The focus on relationships integrates and relates the mathematics strands that students study. Isolating algebra to a symbolic and manipulative category reduces its richness as a topic of study. Second, many teachers view area and perimeter as an elementary school topic of study and algebra as a secondary school topic of study. As elementary teachers naturally try to prepare their students for success in secondary school mathematics, the richness of their view of algebra will likely make a difference in what algebra they teach and how they teach it.

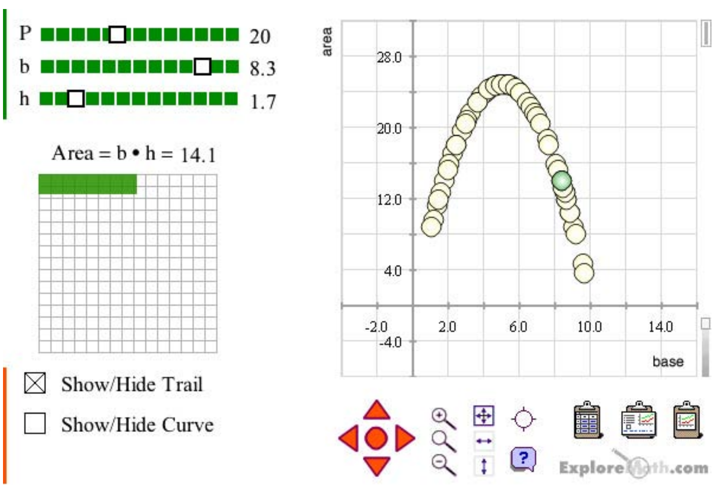

FIGURE 1. Maximize Area activity showing area diagram and graph.

Note

1. This report was inadvertently omitted from the *Proceedings of the 2001 CMESG/GCEDM Annual Meeting*.

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Researching the Effect of Interactive Applets in Mathematics Teaching

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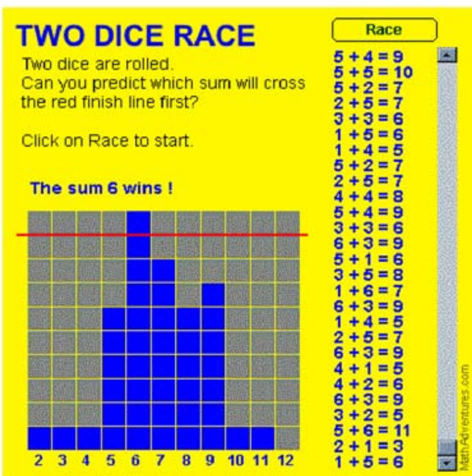
In my recent research of teachers' use of interactive applets in their teaching there was some evidence that certain applets may play the role of pedagogical models (Gadanidis 2001). In 2002–2003, I am conducting two follow-up studies to investigate the effect of the availability of interactive applets on (a) the lesson planning of pre-service mathematics teachers and (b) the mathematical thinking of students. Both studies will use structured, task-based interviews (Goldin 2000) where participants will be presented with two tasks, one of which will have an interactive applet available. For example, consider the probability applet shown on the below (MathAdventures.com 2002).



In the case of pre-service mathematics teachers planning a lesson on probability at the Grades 7-8 level, the following research questions will be considered:

· Are there differences between teachers planning such a lesson with and without the interactive applet available?

· More specifically, does the availability of an interactive applets that may be used to explore mathematical relationships affect the potential student mathematical thinking and performance that is facilitated in the lessons designed by teachers?

In the case of mathematics students performing a probability investigation, the following research questions will be considered:

· Are there differences between students performing the task with and without the interactive applet available?

· More specifically, does the availability of interactive applet affect student mathematical performance?

Levels of student performance will be analyzed based on the four-level scheme described below.

Level	Emphasis
1	Recalling mathematical facts and simple skills.
2	Applying mathematical procedures to solve routine problems.
3	Understanding and explaining mathematical relationships.
4	Extending understanding to new contexts or more general cases.

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Creativity and the Psychology of Mathematical Invention

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> *The genesis of mathematical creation is a problem which should intensely interest the psychologist. It is the activity in which the human mind seems to take least from the outside world, in which it acts or seems to act only of itself and on itself, so that in studying the procedure of geometric thought we may hope to reach what is most essential in man's mind.*
> – Henri Poincaré (1908/1956, p. 2041)

In the beginning of the last century an inquiry into the working methods of mathematicians was published in the journal *L'Enseignement Mathématique*. The inquiry, created by psychologists Claparede and Flournoy, appeared in two parts—1902 volume IV and 1904 volume VI—and was made up of 30 questions probing the creative process of mathematicians. In 1908 there was a lecture delivered by Henri Poincaré to the Psychological Society in Paris entitled Mathematical Creation. Although Poincaré was aware of the aforementioned questionnaire the publication of the results came after he had already laid the groundwork for his presentation. Poincaré's lecture and subsequent essay contains within it autobiographical fragments that tell an insightful story of moments of illumination.

These two events conspired to create perhaps the most famous of treatments on the subject of mathematical invention. Jaques Hadamard was simultaneously inspired by Poincaré's lecture and annoyed by Claparede and Flournoy's inquiry. Although he saw great potential in the published survey, Hadamard saw some shortcomings in it. According to Hadamard the survey failed in that there was a lack of prominence on the part of the respondents.

"Who can be considered a mathematician, especially a mathematician whose creative processes are worthy of interest?" Hadamard made note of the fact that, of all the respondents to the questionnaire, not one was noteworthy. So, he chose to revisit the survey in his own way. He reformulated the questionnaire devised by Claparede and Flournoy, made additions, and used it to survey personal friends—mathematicians whose prominence was beyond reproach. In 1943 he gave a series of lectures on mathematical invention at the Ecole Libre des Hautes Etudes in New York City. These talks were subsequently published as *The Psychology of Mathematical Invention in the Mathematical Field* (Princeton University Press).

Intrigued and inspired by Hadamard's work I sought to recreate a portion of his survey and solicit responses from more contemporary but equally prominent mathematicians—Field's medal winners, members of prestigious academies, and so on. Results from this study were presented in the form of a series of mathematicians' quotes.

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Logical Reasoning and Mathematical Games

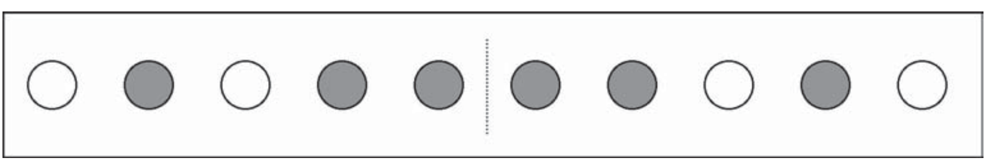
Ralph Mason, *University of Manitoba*
Janelle McFeetors, *University of Manitoba*

This presentation outlined the results of a teaching experiment in the Logical Reasoning unit of grade 11 academic mathematics. The students explored the structure of mathematical games. They formed, tested, and communicated conditional statements about effective strategies to create an experience base with logical reasoning.

Mathematical games invite logical reasoning, because analysis of patterns in the play can determine who wins. Eight games were chosen, each with a limited number of moves to be considered, but still requiring some sophistication in analysis to yield an overall strategy. One game was used by the class as a whole, to establish the expectations for the inquiry to come. Then pairs of students each pioneered a mathematical game, searching for winning strategies.

As an example, consider the game *One or Two*, played with ten markers in a row (see picture below). Two players in alternating turns take either one marker or two adjacent markers. Whoever gets the last marker wins. However, at the beginning of the game, the first player is restricted to taking only one marker. The explorers tried to find, state, and justify winning moves (and make sense of the restriction on the first move). Here are two very different statements of strategy from two pioneers of *One or Two*.

> *Just remember that a combination of isolated pieces and adjacent groups of 2 pieces are very powerful.*
>
> *The second player should keep the board symmetrical. However, if the first player ruins the symmetry by moving two adjacent pieces in the middle, then there is no advantage to having the second move.*



The structure of the classroom project provided students with differentiated and authentic audiences for statements of logical reasoning. Although all members of the class could appreciate general statements of strategy about a particular game, students who had pioneered the same game demanded more precise justifications of specific strategies. After noticing which aspects of their game would yield analyses that led to winning strategies, all students succeeded in formulating, testing, and amending logical claims. All students succeeded in expressing conjectures and conclusions, using if-then logical statements to describe their game-playing strategies. A significant feature of their mathematical communication was the naming of game properties and key situations.

One limitation of the use of mathematical games for mathematical reasoning emerged in the project. Rather than leading students to construct complete conditional chains of logic, the games context provided a strong temptation to justify the value of a strategy empirically by just playing the game again. Although a mathematically rigorous justification of a strategy would deal with every possible option of play, in the games context even a critical student audience tended only to demand that the strategically reasonable options of play be addressed.

Overall, the project has reinforced its central principle for effectively engaging students in logical reasoning. Students need to experience a full sequence of exploration and analysis in a bounded context before learning to make and defend formal and absolute statements.

Collective Mathematical Thinking

Immaculate Namukasa
University of Alberta

Social and cultural accounts of learning raise awareness towards how socio-cultural aspects constrain learning. In addition recent studies on cognition draw from socio-psychology, anthropology, and ecology to construe learning as not strictly individual-based. System theorists observe that human social systems as well as animal communities demonstrate "cognitive" behavior (Davis & Sumara, 2000). What possibilities are created for learning when the classroom is construed as a complex body capable of cognition?

From complexity theory the classroom may be viewed as a network of individual students acting and interacting to produce emergent behavior that lie on the scale above the individual students (Waldrop, 1992). This complexity perspective seems to offer ways of explaining classroom collective behavior, which Simmt and I observed in a yearlong junior high classroom research. The metaphor of classroom-as-organism helped us to understand many learning events that occurred in the classroom as adaptative behavior of the collective structure. We observed the social organizational level not only for interactive behavior (such as classroom practice and norms) but also for structural behavior of the social body as a learning structure.

For instance, in a lesson on transformational geometry the question about objects with the most lines of symmetry, to use Rotman's (1993, p. 39) phrase, "intrusively pushed itself to the forefront" of the classroom consciousness. The teacher had not, at least not in detail, anticipated the mathematical inquiry that arose. She had prepared her introductory lesson to explore objects with 1 up to 4 symmetry lines. But after she together with the students explored the square as an object with 4 lines of symmetry some students, spontaneously, sought to find objects with 8, 16, many, lots, and then infinite lines of symmetry. In the discussion that arose, both in the whole classroom and in small groups, students made conjectures about: (a) the circle as an object with the most lines of symmetry; (b) the relation between lines in a circle and planes of symmetry in a sphere; (c) what it empirically meant for a circle to have infinitely many lines of symmetry. The joint project evidenced in this lesson appeared to have enabled majority students to energetically engage in genuine inquiry about reflectional symmetry.

Collective learning characterizes many classrooms especially those in potentially rich learning settings. By focusing on the structural behavior of the classroom as a social system we might be able to observe behavior that at most times goes undistinguished when we solely focus on individual students as the only acting and interacting bodies in the classroom. Moreover it appears that what unfolds in the collective—rather than what happens in the individuals' heads—might be what a teacher can appropriately influence. Complexity theory's approach to social behavior as behavior above the level of individuals, yet co-arising with individual behavior might offer insight for teaching.

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Proofs and Refutations on the Web: Mathematics Environments for Grades 7 & 8

Geoffrey Roulet
Queen's University

In *Proofs and Refutations*, Imre Lakatos (1976) presents a story of mathematical knowledge growth that is dramatically different from the picture painted by the formalism found in mathematics texts and research journals. In place of an assumptions-proof-conclusions straight line path, Lakatos presents a much less organized process involving cycles of: conjecture, attempts at proof, production of counterexamples, and conjecture refinement. This image suggests that mathematical invention is a struggle for all, and that full participation in the subject is not reserved for brilliant minds that somehow just see the problem solutions. Lakatos' story can provide students with both an invitation to and a general heuristic for mathematical exploration.

Lakatos' work is not appropriate reading for students below the senior secondary school years, but it is still possible to present his message to younger pupils. Students can experience the *Proofs and Refutations* process when working in a mathematical environment; an amalgam of a potentially stimulating problem, tools to support exploration, and social and physical structures that encourage collaboration. A mathematical environment begins with a context from which questions naturally arise. The problem itself and the materials and tools available to the class are designed to suggest potentially valuable methods for exploration. Students in small groups and seated in arrangements that encourage conversation, test out tentative ideas and construct arguments in a collaborative setting before presenting their conjectures to the whole class. Class activity moves back and forth between small group exploration and whole class reporting and consolidation. Such environments have been developed and tested by myself and others at a variety of grade levels and experience has shown that students' work does follow a pattern similar to that described by Lakatos, except that student groups often do not feel a pressing need to develop "proofs" that go beyond simple arguments based upon multiple examples.

With support from the Imperial Oil Charitable Foundation at Queen's University MSTE Group project has constructed a website that will provide mathematical environments on a wider geographically distributed level. In *math-towers.ca* we have developed a structure, the tower, that will support visiting pupils' collaborative exploration of multi-layered problems. On the ground flour of a tower, visiting groups are presented with a problem context and an initial question to explore. Each group is assigned a laboratory room where they find Java applet tools to support exploration and areas for recording their emerging ideas. Once a group feels that they have something to say about the question, they may publish their results on a scroll nailed to their laboratory door. They are now permitted to wander the tower's halls, reading and commenting upon the published work of other groups. After a period of laboratory work and sharing with others, a group may visit the "Trial Chamber" where they can try out their understanding. Here, using their conjectures, the pupils make predictions concerning a sequence of situations. If the match is good a door opens and the group is invited to climb the stairs and work on the next level of the problem. Eventually the students emerge onto the tower's ramparts to be greeted by a congratulatory banner.

We have programmed our site so that we can use the tower structure to support a variety of explorations by inserting appropriate problem descriptions, laboratory tools, and trial chamber tasks. There is also a research support facility that captures all student work and allows researcher paced playback.

Math-towers.ca should be ready for its first student visitors in January 2003.

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Being in a Mathematical Place: Immersion in Mathematical Investigation

David Wagner
University of Alberta

Two grade 10 classes were immersed in mathematical investigation projects for the first time. I used selections of their written work and transcripted dialogue to explore their experience.[1] The investigation projects were modelled after the more formalized investigations that have been common in the United Kingdom.

I think of a mathematical experience as a multidimensional place encompassing the geographical place as well as three other elements for which we use metaphors of place—the topic of discourse, the place in time and relational positioning. My observation of the grade 10 students and teachers immersed in a mathematical place new to them was reminiscent of my experiences of immersion in a culture new to me. Thus, my interpretation of their experiences tended toward comparison between immersions in these two kinds of places.

In describing ways of being in a mathematical place, I distinguish between immersion and tourism, and between different roles filled by guides. A mathematical tourist avoids emotional involvement in problems and is uninterested in the complexity of the landscape. By contrast, signs of mathematical immersion include captivation, creativity and the uncovering of interconnectedness.

First, I consider an example of captivation. In one set of classroom transcripts, a normally lazy student is so engaged in his investigation that he vehemently resists his peers' efforts to stop. In a subsequent interview, he called his investigation "a break" from the tedium of usual mathematics classes, though he admitted that he normally did as little work as possible in most classes. We might say "he had a problem", but it is more descriptive to say "the problem had him". He was captivated by the problems in the mathematical landscape and he could not leave. I still wonder what made the problem real for him. He thought it was the deceptive simplicity of its posing.

A second example demonstrates how creativity is dependent on the absence of an available solution. In this second set of classroom transcripts, ownership of a problem shifts according to established relations of authority. A student is distracted from her unique, viable approach to the investigation by her teacher's voice of authority. With this shift, her pronouns shift from "I" to "we" and finally to "you" when her mathematics is taken away by her teacher's presupposition that there would be one right way to approach this problem—his way. Her creative idea was lost in this episode.

The third example points to the need for teachers to direct students' attention to the interconnectedness of their mathematical explorations. One student group's written work displayed two good approaches to the same problem—one verbal (with no diagrams) and one diagrammatic (with virtually no verbal clues). There was no explicit connection made between the different approaches. If the teacher were to have been attentive to the interconnectedness of these two approaches to the same problem, he could have directed his students to become more aware of the interconnectedness that they uncovered.

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[1] For a more in-depth interpretation of this research see Wagner, David (2002), *Being in a Mathematical Place: Brief Immersions in Pure Mathematics Investigation*, unpublished masters thesis, University of Alberta.

Special Anniversary Sessions

Séances spéciales d’anniversaire

Reflections on 25 years with CMESG

Eric Muller
Brock University

Abstract

Not all reflections are real. Frédéric Gourdeau and I will try to make ours as true to our dual reality allows: the reality of a mathematician, and the reality of a teacher. As mathematicians we travel regions of mathematics taking photographs of interesting events, unforgettable characters, exotic birds, beautiful flowers, striking monuments, unusual landscapes. ... For the photo album, we select some of these photographs, throwing away those that are technically imperfect or those that do not capture the moment we remember. As mathematics teachers we open up this album and attempt to recreate the moment that made that picture so memorable. We can hear the sigh—oh no, not another family photograph! CMESG, through its members and its meetings, has and continues to have an important influence in the development of our understanding of the nature of mathematics and of its teaching and learning at the post secondary level.

This is an unusual paper. One is rarely invited to reminisce in public and in print about influences of decisions and direction taken in one's professional life. CMESG is clearly a very different Group and this is not an ordinary meeting, it is its 25th anniversary. This paper demonstrates one of the many non-conventional approaches taken by CMESG. In these writings you will find reflections on a personal journey, a journey that was not part of my planning as a mathematician in a university setting. It started with an invitation to attend a meeting of a group of individuals who were exploring ways to bring Canadian mathematics educators together. This group grew into the now well-established CMESG, a Group that was to substantially influence my teaching and my views on the learning of mathematics at the post secondary level. Some 25 years ago, as a research mathematician I was immersed in the subject as I always have been. At that time in preparation for a course, I took snapshots of many different mathematical objects often repeating the activity on the same object but focusing on different aspects. With my professional eyes I would select the best snapshots for my class, and using a particular theme I then placed these in a sequence for my presentation. Like most mathematicians the sequence was logically organized starting with definition or theorem, followed by explanations, implications, abstractions, and rounded off with well-chosen examples.

It was my interactions with CMESG members and my participation in the Group's annual conferences that I began to question the information provider model, the only model I had experienced in my university education. For this presentation I will highlight a few of the many undergraduate mathematics education issues that I have explored—they are not raised in any particular order. For long standing CMESG members these are not new issues, I have the excuse that I can bring them up again as this session is focused on reflections of the impact of 25 years of CMESG on a mathematician.

Individual mathematical concept development—the teaching sequence vs. the learning sequence

The teaching sequence used by many mathematicians is not the sequence that we most commonly use for learning about the world around us. The natural learning sequence starts with an encounter with a new object or action. As active learners, we then look for other examples of this object or action, and within these examples, search for common properties. Other objects or actions arise and we decide whether they belong to the set under consideration. It is only after extensive experience, that we develop our definition. In most cases mathematicians also see the definition or theorem as a culminating activity; the definition is the activity that determines boundaries for the object or action. Our definition is a function of our state of knowledge. It is alive; we can amend it as our knowledge grows and as we come across new experiences that challenge where we have placed the boundaries. I leave with you two questions. When are students mature enough to reverse the more natural sequence of learning? And, as mathematicians, do we provide for this change in approach?

When I show a mathematical object, one of my snapshots, it seems natural to assume that my students and I are seeing the same thing. Although the two-dimensional photograph is the same, my experience of this object is very different. For me it is a completely unsatisfactory two-dimensional projection of an n-dimensional dynamic world of space, time, emotion, sound, heat, wind, etc., etc. What an incredibly difficult task I face as a teacher to reconstruct the n-dimensional environment from its two dimensional representation. This can be especially challenging if I have discarded other snapshots of the same situation taken from different vantage points.

As teachers we have the possibility of introducing environments in which the students take their own snapshots. Examples of these are the Lénárt Sphere[1] or the computer environment "Journey through Calculus" developed by one of my colleagues Bill Ralph.[2] When all my students and I have worked through an activity in one of these environments we have a very real common experience. Everyone is present when the snapshot is taken! Building on this common experience we are able to look for explanations, implications, generalizations, abstractions that lead to the formulation of a definition. From a mathematician's point of view, these environments do bring in new challenges and concerns. The most important one for me is the need to assess whether the learner has made the transition from the environment to the mathematics. Learners may function very well within a given environment but what mathematics are they learning? To explore this further John Mason and I led a Working Group at the 2001 CMESG meeting. It had the title "Where is the Mathematics",[3] and had for initial goal to address the questions "How does mathematics emerge from playing games, from using structural apparatus and from mathematical instruments? How can this be planned for, enhanced, and exploited?". Whether these environments provide more than a common experience is still to be explored, whether they engage more undergraduate students into mathematics (not only into the activity) than the students who would have been engaged anyway by a more traditional teaching approach still needs to be documented.

Course and program development—the sequencing theme

For a presentation, many different themes can be used to sequence the snapshots. Some themes are easier for the audience to follow than others. For example a sequence on flowers is more easily discernible than a time sequence. In most situations the time sequence is only evident to the presenter who actually lived through the experiences in that particular time order. Thus a sequence that may be eminently logical to the presenter, may appear quite haphazard to the audience. Most mathematical themes are only evident to the mathematician who has an overview of the subject. It is no wonder that for many people mathematics is a game of the mathematical gods, only they are privy to how the presentation fits together and where it is all leading to. Most mathematicians provide the destination by first providing the definition or theorem and then developing its content. Unfortunately, as was discussed in the previous section, and for many learners the definition or theorem is not comprehensible and is not seen as a destination.

When sequencing mathematical topics in courses and in programs, mathematicians build mainly on a technical hierarchy of mathematics. In this technical hierarchy it is, for example, necessary to understand the structure of a function before applying the rules of differentiation. Mathematics also possesses a conceptual hierarchy. This hierarchy is less well understood and therefore plays a less important role in the sequencing of topics in courses and programs. In the conceptual hierarchy it is, for example, possible to develop an understanding of slope without understanding the structure of functions. How are these two hierarchies related in the development of our understanding of mathematics and its applications? This question may not have been very important fifty years ago, however, I believe that it is now central to undergraduate mathematics education, because learners have access to computer technology that possesses all the technical capabilities required of the first two years of university mathematics. In their exploration of the implications of of Symbolic Mathematical Systems in Mathematics Education, Hodgson and Muller[4] provide the analogy of the development in transportation. Each of the bicycle, car and airplane provide different opportunities. Insisting that one has to go from Toronto to San Antonio by bicycle will enthuse a rather small audience! The question that needs to be addressed in mathematics education, and more specifically at the undergraduate level is, what would sequencing within mathematics courses and within mathematics programs look like if it was based on a conceptual hierarchy, and the requirements of the technical hierarchy were left to technology? In other words, is it possible for learners to access mathematical concepts which are higher on the conceptual hierarchy scale if they can by-pass some of the technical hierarchy now readily available in computer software? Resolving this question has important implications on the flexibility of learner access to mathematics. I explored this in a presentation to the Third Southern Hemisphere Conference on Undergraduate Mathematics Teaching in South Africa.[5] University departments of mathematics have made some progress in providing more flexibility of learner access in their service courses. However little has been done for mathematics majors. Even less attention has been paid to students planning to become mathematics teachers; future elementary mathematics teachers are not even on the radar screen of many university mathematics departments. Why is it that special mathematics courses are introduced for engineers, computer scientists, and others, while the group most able to ensure the future well being of mathematics education, is seen as having no specialized needs or is disregarded completely if the focus is mathematics education at the elementary level? University mathematics departments need to reflect on the role that technology can play in facilitating student action in mathematics and on the way it can provide flexibility of student access to mathematics.

Awareness of the teaching environment, student expectations, assessment, etc.

Although I am not spending anytime discussing these issues I still see them as very important issues that mathematics departments need to address. Without CMESG, I would not have spent the time and effort to reflect on many issues which impact the teaching and learning of mathematics but which have little to do with the subject itself. I find it interesting that those students, who provide additional information in course evaluations, write mainly about issues that pertain to their expectations, their experience of the teaching and learning environment and their evaluation of the assessment process. All these situations are very closely tied to things that the department can affect, namely, mathematics program of study, use of technology, resources, faculty attitudes, etc. There are many others over which the department has no control. Departments and the universities should not underestimate the tremendous amount of energy and goodwill that is required to make sustainable changes in teaching environments, program reviews, etc. It is crucial for mathematics departments to provide more than lip service to faculty who spend more than the minimum time and effort to such mathematics education activities.

Making long lasting changes in mathematics education at the university level

Making mathematics education changes in departments of mathematics is a real challenge. Mathematicians have graduated through a system that prizes individualism and specialization, usually in a narrow field of mathematics. Discussion, let alone collaboration, in areas of teaching and learning mathematics are rather rare. The consequence is that many learner centred developments in a course or program are lost when a faculty moves to another course or another set of responsibilities. I truly believe that significant program changes will only be sustained if there is a group of faculty 'champions' to support the change. One needs to realize that there is very little mathematics education memory in Departments of Mathematics and that in general program reviews are curriculum based with little time and effort devoted to issues of pedagogy and cognition.

In my many years at Brock University I have paid special attention to developing a mathematics education memory, I have focused my efforts on changes that would impact more than my individual classroom. Such lasting changes take a lot of effort, consume much time, and require the goodwill of many colleagues. Looking back over my publications I note that one of the first papers[6] that explored an aspect of mathematics education was co-authored by five members of our Department. In retrospect this was our first effort at using technology in service courses. It has taken twenty years for the Department to respond in a universal and significant way to technology. The change was in line with what John Conway[7] wrote in 1997: *"We have to embrace technology, I don't mean just tolerate it; embrace it and celebrate it.... The professional mathematics community must adapt and learn how to best incorporate technology into instruction. With the existence of powerful, inexpensive computers, I see mathematics departments rethinking their entire curriculum.... Otherwise we are out of business".* The significant response occurred in 1999 when our Department of Mathematics completely reviewed its programs, basically starting from scratch and paying attention to many of the issues I have raised today. What was significant about this review was that issues of pedagogy and learning environments were raised and discussed for every course and for every program. We are much further ahead than many mathematics departments but we still have a long way to go! Those of you who are interested in what mathematics first year students are able to do when they are let loose with powerful mathematics technology can look at the site which is under development

http://www.brocku.ca/mathematics/mica02/

The first year MICA course[8] used Java in 2002 and is moving to VisualBasic.net in 2003.

In closing this brief reflection on CMESG's impact on my university teaching career I wish to touch on the future of this Group.

Looking at the future for CMESG

In my presentation I spent some time exploring the future of CMESG. Let me point to a few items which are, in my view, very important priorities. At the top of the list I place the CMESG meetings and their proceedings, and I follow this with the arms length support of the journal *for the learning of mathematics*[9] started by David Wheeler and now edited by David Pimm. The meetings provide a friendly environment where mathematics educators, from across Canada, are able to meet and discuss issues and research in mathematics education. The journal provides international visibility for the Group. I believe that CMESG would benefit from an increased membership of mathematicians from Faculties of Mathematics and Sciences. Generating additional members from this group should improve with time as these Faculties provide more emphasis and support for those who are interested in addressing issues of teaching and learning.

My final comments concern ICMI and the question, should there be a Canadian ICMI sub-commission? If the answer is yes, then what role could CMESG play in it? It is clear to me that such a sub-commission should not duplicate what is already being done. What would be of benefit to mathematics education in Canada? Neither the Canadian Mathematical

Society nor CMESG have been successful at bridging the gap between mathematicians in universities (in Mathematics Departments and in Faculties of Education) and mathematicians in schools and other post secondary institutions. Could a Canadian ICMI sub-commission engender better collaboration across institutions and levels?

There is no better way for me to end these reflections than with a question. CMESG is built on trust and collaboration where questions are the norm and where answers are seen as stepping stones to new questions.

Acknowledgement

Frédéric Gourdeau and I would like to thank the program organizers for inviting us to speak at this very special meeting celebrating the 25th anniversary of CMESG held at Queen's University in Kingston, the venue where the founding meeting of this organization was held.

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Réflexions d'un mathématicien sur le GCEDM

Frédéric Gourdeau
Université Laval

En premier lieu, je tiens à remercier les organisateurs de m'avoir demandé de contribuer à la rencontre anniversaire avec mon collègue Eric Muller. Je suis honoré que l'on m'ait demandé de partager la tribune avec Eric Muller et c'est avec grand plaisir que j'ai accepté cette invitation.

As you can immediately notice, this paper will hop from French to English, from English to French. This is a true reflection of my presentation at Kingston. Pourquoi avoir choisi de faire une présentation en utilisant deux langues ? En partie parce que je ne connais pas de solution parfaite au dilemme que pose la nature bilingue de notre groupe, mais surtout parce que le respect que je ressens lorsque je suis aux rencontres du GCEDM me permet de faire un tel choix. I hope that those of you who read mainly English will have a chance to practice their reading skills in French. Les francophones auront au moins une partie du texte dans leur langue.

Bien que je ne sois pas un membre du groupe depuis aussi longtemps qu'Eric, j'ai déjà dans mon album de famille du GCEDM plusieurs photographies que j'aimerais vous présenter. Puisque ma présentation se veut avant tout un témoignage personnel, il importe sans doute de préciser que je suis revenu aux mathématiques après un passage de quelques années en coopération internationale, années pendant lesquelles j'ai appris énormément des gens que j'ai côtoyés. Je retrouve au sein du GCEDM une passion et un engagement humain qui me rappellent ceux que je retrouvais alors dans mon travail. Loin d'être anecdotique, cet aspect du groupe me paraît fondamental et il en sera donc largement question dans la suite du texte.

Getting to know CMESG/GCEDM

My first photograph features an important member of this group who played a key role in my development as a mathematics educator: Bernard Hodgson.

Bernard est à l'origine de ma participation au GCEDM tout comme il est à l'origine de ma présence actuelle en mathématiques. Alors que je n'apprêtais à être engagé par l'Université Laval, Bernard m'entretenait déjà des différentes organisations nationales et internationales oeuvrant en enseignement des mathématiques. Que de sigles à connaître : GDM, AMQ, GRMS, SMC, NCTM, ICMI, … Je me souviens qu'un groupe se dégageait du tableau qu'il brossait pour moi : un groupe qui avait été plus important que les autres dans son cheminement en enseignement des mathématiques.

Ce groupe, évidemment, est le Groupe canadien d'études en didactique des mathématiques. Les rencontres annuelles du groupe étaient, selon Bernard, des moments à ne pas manquer. De ses propos se dégageait une photographie complexe.

Here I must briefly interrupt my discussion and point out the mathematically rich concept that underlies it. My first photograph about this group, call it A, contains one man, Bernard Hodgson, who is a member of the group. In this photograph, this man is talking about another photograph of the group, call it B: thus B is a part of A. Now Bernard is himself part of that second photograph. Thus, it can be argued that A is a part of B, which is a part of A, which is a part of … and it goes on. Thus I claim that there is something fractal about this discussion, through self-similarity. (Fans of Harry Potter may also note that the man is talking in the photograph.) Moral: perhaps this talk doesn't have depth, but it is more than one-dimensional.

Let me now continue on the main theme. The portrait of CMESG that Bernard was describing to me was that of the group that had the most influence on him with regards to his views and understanding of mathematical education. This was largely due to the informal atmosphere in which the group operated, only contrasted by the commitment that its members had to mathematical education. He was convinced that I would like the group and mentioned the names of many people, important members of the group, like Sandy Dawson, Claude Gaulin, Bill Higginson, Carolyn Kieran, Tom Kieren, Eric Muller, David Wheeler, and others. But who were they?

Halifax, 1996, et l'importance des groupes de travail

Cela m'amène à ma seconde photographie, prise à Halifax en 1996. Je me rends à ma première rencontre annuelle du groupe, ne sachant trop ce que ce groupe mythique me réserve. Quels gourous m'y attendent ? Seront-ils vêtus de longues toges blanches ? Comble de malheur, Bernard Hodgson ne peut m'y accompagner. Y aurait-il un piège ? Heureusement, l'accueil chaleureux de Mary Crowley et Yvonne Pothier me permet de me détendre un peu.

Le premier groupe de travail auquel je prends part est dirigé (est-ce le bon mot ?) par Bill Byers et Harvey Gerber et porte sur les preuves en enseignement des mathématiques. L'éclair frappe le (pas très) jeune et (pas tout à fait) innocent mathématicien que je suis : Je me souviens de questions percutantes (pour moi) posées par Joel Hillel, notamment, mais aussi par d'autres.

We wonder if the teaching of proof was not largely a ritual. Are most students simply going through what the teacher, as a student, has gone through, even if in fact the vast majority of students don't get the point of some / most proofs in some courses (like first year calculus)? I have to confess that questions like this one asked by someone else would probably not have mattered as much to me. But it mattered to me enormously when Joel asked some of these questions because of how he was talking about it and who he was: he had been giving courses similar to those I had as a student, and similar to some that I had started giving as a lecturer. I had not given a lot of thought to the reasons why we taught a course in a specific way, with detailed proofs or not, as it may be. My focus had been to understand the mathematics and the proofs as a student, and to try to help students understand them as a lecturer.

This, for me, is an example of one of the greatest features of CMESG: the group brings together mathematicians who are serious about mathematical education and provides a context in which discussion will occur. There may well be other groups that do this, but CMESG does it extremely well.

Par la suite, le groupe de travail est divisé, ce qui permet aux membres davantage intéressés par la formation des enseignants de se réunir. Je suis alors avec Nadine Bednarz, Sophie René de Cotret et Linda Gattuso notamment, et j'écoute, j'apprends. Nous discutons de la base des arguments invoqués lors de preuves, de l'axiomatique. Que peut-on prendre pour acquis pour donner une preuve, quels sont les axiomes de base ? Ces questions m'apparaissent alors fondamentales.

Ce genre de questions ainsi que les différents angles que l'on peut prendre pour aborder une question dans le cadre d'un groupe de travail illustrent bien l'importance d'avoir du temps. Du temps pour explorer plusieurs aspects d'une problématique, mais aussi du temps pour écouter et réfléchir. De plus, la composition du groupe de travail telle que je l'ai vécue à Halifax permet aussi au mathématicien (masculin, dans ce cas) d'apprendre au contact des didacticiennes (dans ce cas, le féminin est de rigueur), de s'enrichir de perspectives et d’analyses différentes. Il s’agit là encore de traits particuliers du GCEDM qui, s’ils ne lui sont pas uniques, en sont cependant des caractéristiques essentielles.

Conférence plénière

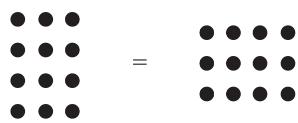
La rencontre de Halifax compte aussi une conférence plénière de David Henderson. David brosse un tableau de l’enseignement qu’il a reçu et de l’importance accordée au formalisme dans sa formation. Cette importance, il la questionne de manière fort bien appuyée. Il nous parle de son cheminement personnel et mathématique. Il nous entretient de la géométrie sphérique, absente de sa formation comme étudiant. Cette absence, il l’attribue en partie au fait que la géométrie sphérique ne soit pas axiomatisée alors que les mathématiques modernes et le formalisme sont la norme.

Parmi les idées les plus importantes émises par David Henderson, je citerai la suivante (parue aussi dans les actes de cette rencontre) :

Alive mathematical reasoning includes living proofs,
that is, convincing communications that answers—Why?

On ne parle ici ni d’enlever les preuves, ni de les conserver comme rite de passage.

As an example of this definition of proof, David contrasted the classical proof using double induction that multiplication is commutative (for natural numbers) with the following:



Comme autre exemple, David présente le cas d’une étudiante qui, dans l’un de ses cours, insistait qu’elle avait une preuve de l’égalité des angles opposés par le sommet à l’aide de deux rotations de 180 degrés. David relate que cela lui a pris beaucoup de temps pour finalement accepter qu’effectivement, il s’agit bien d’une preuve.

Ici, axiomatique et formalisme ne sont pas les critères de validité d’une preuve.

La conférence plénière de David Henderson est remarquable, mais elle n’est pas unique. Plusieurs présentations faites au GCEDM m’ont permis de mieux articuler ma vision des mathématiques et de leur enseignement.

These types of presentations, and the possibility to discuss them with others, allow me, as a mathematician, to think about mathematics in a way that I can not usually do. It permits me to redefine what I mean by mathematics, to acquire a broader and fuller understanding of the essence of mathematics.

Being part of a community: my first pizza run

Sur cette photographie, il n’y a que des gens. David Wheeler, Sandy Dawson, Bill Higginson, Tom Kieren et Marty Hoffman notamment, avec lesquels je fais ma première *pizza run*, à Halifax. Ils discutent des débuts du groupe, des rencontres passées. Je me sens privilégié. J’apprends les motivations de ces gens qui ont mis le groupe sur pied, qui en sont des membres fidèles. Je vois et je sens la passion qui les anime. Je commence à faire partie du groupe.

De retour en classe

Je quitte ma première rencontre annuelle avec questions et remises en question. Que garder du formalisme ? La nature des mathématiques et le besoin d’une axiomatique sont remis en cause. Qu’est-ce qui peut être considéré comme étant une preuve exactement ? Que faire comme enseignant ? Quelles sont les bases communes entre moi et mes étudiants, entre ces étudiants et leurs élèves ? Que peut-on prendre pour acquis dans un cours, dans le développement d'un thème ?

De retour en classe, je cherche à développer des activités qui permettent aux futurs enseignants au secondaire d'acquérir une vision plus riche des mathématiques. Il y aura des lectures de textes, la rédaction d'un essai, ainsi que davantage d'activités exploratoires, de simulations. Cet enseignement se situe dans le cadre d'un cours pour les futurs enseignants conçu par Bernard Hodgson et Charles Cassidy et qui porte déjà en lui ces questions. Le cycle commence pour moi, comme il a commencé pour eux avec le GCEDM.

La toile mathématique s'enrichit

Les rencontres qui suivent sont pour moi l'occasion de poursuivre les réflexions amorcées et d'en entamer de nouvelles. Ainsi je commence à mieux mesurer l'immensité de mon ignorance face à l'histoire et à l'algèbre grâce à ma participation au groupe de travail de John Mason et Louis Charbonneau à Lakehead, au groupe thématique de Israel Kleiner la même année (Abstract algebra: a problem-centred and historically focused approach) et au groupe de travail de Louis Charbonneau et Luiz Radford lors de la rencontre de Montréal (Histoire et enseignement des mathématiques).

And what can be said about mathematics and society? As a person, I have always needed to reconcile mathematics with the world we live in. CMESG provides me with opportunities, once again. I can hear conferences from, and speak with, people like Ubiratan D'Ambrosio, Jill Adler, Ole Skovsmose or Sandy Dawson. It is wonderful for a mathematician, like me, to be able to reflect about different aspects of the relationship between mathematics and society, and opportunities to do so are not often seen in mathematics research conferences. These encounters provide me with other ways to see the role of mathematics in the community, for instance by linking it with democracy, culture, or development (a loaded word which I deliberately choose to use).

This multitude of possibilities enables me to constantly reshape my vision of mathematics and to come in contact with different and challenging new perspectives and with new areas of investigation as a mathematical educator.

The darker side: mathematics education and jargon

Not everything is easy for a mathematician at CMESG, and I must mention one sore point: it is sometimes difficult for the mathematician to feel part of the conversation. The first time it happened to me, everyone was talking about some turtle named "logo" which had (or was it that it could have, or should have) revolutionized the teaching of mathematics. I later realized my ignorance about logo also had a lot to do with my age. But it certainly was not the last time I was to feel out of place.

I later heard of embodiment, which even comes in threes, and of enactivism, which still eludes me. And here I cannot blame language and pretend that it is simply because I am not a native English speaker. It is the same in French!

Je me rappelle mon embarras lorsque l'une des chercheures émérites du groupe me parlait de problèmes de nature épistémologique. J'éprouve d'ailleurs toujours un certain malaise à utiliser ce mot, tout comme je ne peux parler de la théorie didactique des situations de manière naturelle.

However, we mathematicians can learn! There are other terms which I can now use as if they were natural to me. For instance, I used only to do maths but now I can sometimes say that I mathematize! I can even describe a problem as a rich learning situation without blinking. To sum it up, CMESG educates the mathematician: we learn lots of new words, and, sometimes, their meaning too.

The lighter side: fun maths

Ici, des pizzas et des bouts de papier, des jeux et des questions. Étonnamment peut-être, mon travail de mathématicien me donne rarement l'occasion de faire des maths amusantes et légères, de me poser des questions gratuites, de m'amuser mathématiquement. Encore une fois, c'est sans doute davantage le climat qui règne lors des rencontres du groupe qui permet de poser de jolis problèmes et de s'embarquer dans des aventures mathématiques pour le plaisir.

Le plaisir que j'éprouve à m'engager dans des activités mathématiques avec d'autres me fait découvrir de nouvelles manières de faire des mathématiques et me donne des idées pour mon enseignement. Je découvre ou redécouvre la beauté de certaines idées mathématiques, inconnues de moi ou oubliées depuis longtemps.

Conclusion

Les rencontres que j'ai eues l'occasion de faire grâce au GCEDM, qu'elles soient intellectuelles ou personnelles, continuent de nourrir mon enseignement et mes réflexions. En cela, elles influencent la formation de centaines de futurs enseignants de mathématiques au secondaire. Je crois que cette influence est extrêmement positive.

Lors de la rencontre de Kingston, je n'avais pas à tirer davantage de conclusions puisque Eric Muller le faisait pour moi, puisant dans son expérience et ses réflexions basées sur 25 années avec le groupe. Je suis confiant que la version écrite de sa contribution jouera ce rôle à nouveau.

A Historical Perspective on Mathematics Education Research in Canada: The Emergence of a Community[1]

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This paper describes the Canadian mathematics education research community—from its preemergence in the early 1920s up until 1995. Because data collection ended in 1995, the more recent scholars of the community are not included in this story. A future update will hopefully rectify this situation. In preparing the chapter on which this paper is based, I first asked myself, "What defines a research community? What are its characteristics? How does a research community develop? Are there events one can point to that could be said to have contributed to its emergence?" I finally decided to use Etienne Wenger's (1998) concept of *communities of practice* as a unifying thread to describe those happenings that I felt were pertinent to the formation of the Canadian community of mathematics education researchers. In my analysis of the events that fostered the emergence of our community, I weave together the concurrent growth of research communities both abroad and within the provinces and the roles that these played. The milestones I present are intended to illustrate the spirit of its overall evolution. In taking a particular focus, I have regrettably missed certain individuals or events that merit inclusion; for this, I offer my sincerest apologies.

1. What is a Community of Practice?

According to Wenger:

> We all belong to communities of practice. At home, at work, at school, in our hobbies—we belong to several communities of practice at any given time. And the communities of practice to which we belong change over the course of our lives. In fact, communities of practice are everywhere. ... In laboratories, scientists correspond with colleagues, near and far, in order to advance their inquiries. Across a worldwide web of computers, people congregate in virtual spaces and develop shared ways of pursuing their common interests. ... We can all construct a fairly good picture of the communities of practice we belong to now, those we belonged to in the past, and those we would like to belong to in the future. We also have a fairly good idea of who belongs to our communities of practice and why, even though membership is rarely made explicit on a roster or a checklist of qualifying criteria. Furthermore, we can probably distinguish a few communities of practice in which we are core members from a larger number of communities in which we have a more peripheral kind of membership. (pp. 6–7)

However, not all groupings and associations are communities of practice. The focus is on practice. For example, the neighborhood in which one lives may be called a community, but it is not a community of practice. More precisely, according to Wenger, communities of practice have the following three dimensions that associate practice and community, all of which are required for communities of practice: mutual engagement, joint enterprise, and shared repertoire.

Mutual Engagement

Mutual engagement involves, according to Wenger, "taking part in meaningful activities and interactions, in the production of sharable artifacts, in community-building conversations, and in the negotiation of new situations" (p. 184). He adds:

> [communities of practice] come together, they develop, they evolve, they disperse. ... Thus, unlike more formal types of organizational structures, it is not so clear where they begin and end. They do not have launching and dismissal dates. In this sense, a community of practice is a different kind of entity than, say, a task force or a team. ... A community of practice takes a while to come into being. (p. 96)

Joint Enterprise

Joint enterprise concerns the way in which members do what they do. Wenger emphasized that "because members produce a practice to deal with what they understand to be their enterprise, their practice as it unfolds belongs to their community in a fundamental sense" (p. 80). Practice is, in fact, shaped by the community's way of responding to conditions, resources, and demands.

Shared Repertoire

Wenger described the shared repertoire of a community of practice as follows:

> The repertoire of a community of practice includes routines, words, tools, ... concepts that the community has produced or adopted in the course of its existence, and which have become part of its practice. The repertoire combines both reificative and participative aspects [*reificative*: documents, instruments, forms, etc.; *participative*: acting, interacting, mutuality, etc.]. It includes the discourse by which members create meaningful statements about the world, as well as the styles by which they express their forms of membership and their identities as members. (p. 83)

The above description would appear to focus more on the repertoire of an existing community. But, as Wenger pointed out: "A community of practice need not be reified as such to be a community: it enters into the experience of participants through their very engagement" (p. 84). Indeed, the reificative aspects of a community are of two types. There are those that are the products of the enterprise, such as—for a community of researchers—research reports, publications, and so on. There are also those that are more related to the processes engaged in by the community. The latter might include the taking on of a more formal, organizational structure, but this is not essential for a community to exist. In either case, as Wenger emphasized: "Reification is not a mere articulation of something that already exists. ... [It involves] not merely giving expression to existing meanings, but in fact creating the conditions for new meanings" (p. 68).

As reificative aspects can yield evidence related to a community's coming together, emerging, and developing, I first present two rather broad examples of reificative aspects of the shared repertoire: doctoral dissertation production and government-funded research (note that master's theses would have been included as an example in this category, were it not for the challenge of obtaining reliable information regarding their production across the country over the past century). The overview that is presented in the following section not only signals the growth that occurred during approximately three-quarters of a century but also helps situate the later discussion of the communities of practice that emerged at both the local and national levels and the interactions between them.

2. An Overview of the Growth of a Community

When Jeremy Kilpatrick (1992) wrote a history of research in mathematics education in the 1992 *Handbook of Research on Mathematics Teaching and Learning*, he argued that disciplined inquiry into the teaching and learning of mathematics in the United States and elsewhere in the world had its beginnings in the universities. Therefore, signs of the beginnings of a Canadian community of mathematics education researchers were sought in the universities.

The Early Canadian Research Related to Mathematics Education

The first doctorate from a Canadian university for research that was related to school mathematics was awarded in 1924, a Doctor of Pedagogy from the University of Toronto (U of T) (*Dissertation Abstracts* 1861–1996). The dissertation had the title *Practice in Arithmetic or the Arithmetic Scale for Ontario Public Schools*. It was followed by three more in 1929, 1943, and 1945 at the same university. These first dissertations centered on surveys of the teaching of arithmetic, the development of arithmetic evaluation instruments, and the diagnosis and remediation of arithmetic learning problems. The subject matter of these dissertations suggests that, as early as 1920 at U of T, there were individuals, perhaps even a group, whose main research interest was mathematics education.

In fact, there was very definitely an interest in school arithmetic at U of T, and this preoccupation preceded by several years the 1924 dissertation just mentioned. It seems that in the late 1880s John Dewey had been contacted by James McLellan, who was Director of Normal Schools for Ontario and a professor of pedagogy at U of T (see Dykhuizen 1973, p. 60), to write a psychological introduction to a book that McLellan was authoring on educational theory and practice. The outcome of that collaborative effort was published in 1889 (McLellan 1889), but more interestingly it led to a second book on the study and teaching of arithmetic, *The Psychology of Number and its Applications to Methods of Teaching Arithmetic* by McLellan and Dewey in 1895. However, McLellan passed away in 1907, at the age of 75, and thus had no direct role in the supervision of the first dissertations.

The doctoral research related to school mathematics that was carried out at U of T from the 1920s to the 1940s was succeeded by similar work in the late 1940s and early 1950s at Université de Montréal (one dissertation in 1947) and Université Laval (one dissertation in 1951). In 1948, U of T added another to its set of dissertations related to school mathematics. Thus, the total number of dissertations completed at Canadian universities on research related to the mathematics curriculum or the teaching and learning of school mathematics during the 1924–1951 period was seven. Even though a national community of practice was still far from being a reality, it was clear that small groups had begun to be involved in mathematics education research in Ontario and Québec by the middle of the twentieth century. But progress during these years was very slow. According to the U.S. scholar Ellen Lagemann (1997), mathematics education as a research discipline in its own right did not exist in many countries prior to the late 1950s and early 1960s, at which time educational research began to be more discipline-based (see Kilpatrick 1992 for an extensive discussion of this process both in the United States and abroad). Nevertheless, the topics of interest in these early Canadian dissertations related to mathematics education reflected themes that were equally of interest south of the border.

It took the post-World War II population boom to provide a jump-start to the mathematics education research enterprise in Canada. The population, which in the 1920s stood at 9 million, rose to 14 million in 1950 and then 18 million in 1960, registering in that latter decade the highest percent of increase since the years 1900–1910. To accommodate the growing numbers of students in the 1960s, the already existing universities had to expand, and new ones were created. The number of graduate programs increased, too, which meant that more research would now be done than ever before.

A Period of Growth for the Universities

Of the several universities created in Canada in the post-World War II years, the two that were formed in Montréal—Concordia University in 1964, and Université du Québec à Montréal (UQAM) in 1969—made an innovative decision regarding the intersection of mathematics and education. Scholars who were interested in the teaching and learning of school mathematics were affiliated with mathematics departments rather than education departments. This was a period of intense educational reform in the province of Québec.

The traditional eight-year classical colleges of the French-language educational system of the province were disbanded in the 1960s and replaced by high schools, *CEGEPs* (from Collèges d'enseignement général et professionnel [in English, Colleges of General and Professional Training]; i.e., colleges that dispensed both preuniversity and technical or vocational courses), and universities. Those who had taught at the upper levels in these classical colleges were integrated into the new CEGEPs and universities of the province. The same reform that closed the classical colleges also brought an end to the French- and English-language normal schools and transferred instructors and students alike to both the new and existing universities. Similar events with respect to the transfer of normal school teachers to the universities occurred in other provinces as well, but at different times; for example, in Alberta, this changeover took place in 1945.

It might have been expected that once the teacher-training faculty had been incorporated into universities across the nation, they would soon enough get involved in research. But it took time because many of the freshly appointed education professors had to work at obtaining doctoral degrees themselves and learning about the research process. At some universities across the nation, a tension arose between the role of teacher trainer and that of researcher, and was not resolved for several decades in many education faculties. As of the early 1990s, there continued to be faculties of education in several universities where the emphasis was clearly on teacher training. Research was simply not part of the culture of these faculties of education, as was indicated by the lack of graduate programs with a research component. At such universities, students might have been able to obtain a master's degree, but had to go elsewhere if they wished to continue on to the doctoral level in education. In 1990, there were merely seventeen Canadian universities where it was possible to obtain a doctoral degree involving research related to mathematics education (see Kieran and Dawson 1992).

Growth in Dissertation Research Related to Mathematics Education

The years from 1955 to about 1969 were years of continuing gradual growth in Canada for dissertation-based research related to school mathematics. During this period of population increase, of reform in the educational systems of various provinces, and of the beginnings of mathematics education as an identifiable field of research study in many countries of the world, the number of school-mathematics-related dissertations in Canadian universities showed a modest increase. As well, the production moved beyond the universities of Ontario and Québec. The Universities of Alberta and British Columbia had also begun to develop research groups interested in school mathematics.

But the period of most intense growth in dissertation research related to mathematics education in Canadian universities occurred from the late 1960s onward. See table 1 for the number of dissertations related to school mathematics for which doctoral degrees were awarded in Canadian universities from 1924 up to 1995. The data from table 1 are re-presented in graphical form in figure 1 so as to see at a glance the periods that were peaks with respect to Canadian math education dissertation production. Note the rise in the mid- to late 1970s (of the 43 doctorates awarded during the years 1974–1979 for research related to mathematics education, 17 were from the U of A and the remainder from six other universities across the country). This rise during the 1970s was followed by a period of slower growth, until the 1990s when the sharpest increase took place—resulting in the highest peak in 1994 with twenty-one dissertations.

Over the 72-year period from 1924 through 1995, the lone entrant of the early years—U of T—was joined during the latter part of this period by several other universities. Nevertheless, the majority of the 80 doctorates (63 of them, or 79 percent) for research related to mathematics education that were awarded from 1990 to 1995 came from 6 universities (U of T, including OISE; U of A; Université Laval; Université de Montréal; UQAM; and UBC). The remaining 17 doctorates awarded during this period were from 11 other universities across the country. By the end of the 1990s, there had been a 60 percent increase in the number of universities offering doctoral programs in education over that of 1990. It had become possible to earn a doctorate for research in mathematics education in all provinces of the country except for New Brunswick, Prince Edward Island, and Newfoundland (see table 2 for province-by-province totals of dissertations produced during the 1990–1995 period for research related to mathematics education; data reflect the recency of the doctoral programs in Saskatchewan, Manitoba, and Nova Scotia). The increase in mathematics education dissertation production over the 72-year period suggested the presence of communities of mathematics education researchers at certain universities and, along with other events to be discussed, reflected as well the growth in the Canadian mathematics education research community at large.

TABLE 1. The yearly number of Canadian-university doctoral dissertations related to mathematics education for which a degree was awarded during the period 1924–1995 (A skipped year indicates that no dissertations were produced.)

1924	1	1951	1	1964	1	1971	5	1977	4	1983	1	1989	8
1929	1	1955	3	1966	1	1972	3	1978	6	1984	4	1990	10
1943	1	1956	1	1967	3	1973	4	1979	9	1985	5	1991	7
1945	1	1957	1	1968	1	1974	8	1980	6	1986	4	1992	18
1947	1	1959	3	1969	3	1975	8	1981	6	1987	8	1993	13
1948	1	1962	1	1970	3	1976	8	1982	4	1988	3	1994	21
												1995	11
												Total:	212

Note: This compilation is extracted from the following data bases: *Bibliothèque Nationale du Canada* 1947–1981; *Bibliothèque Nationale du Canada* 1981–1984; *Canadian Education Index* 1976–1996; *Dissertation Abstracts* 1861–1996; and specialized university listings such as that of *Université du Québec à Montréal* 1996–1997. For work carried out during the early part of the period and for which the data bases provided no descriptors or abstracts, only the title of the dissertation could be used for deciding whether the content was related to school mathematics.

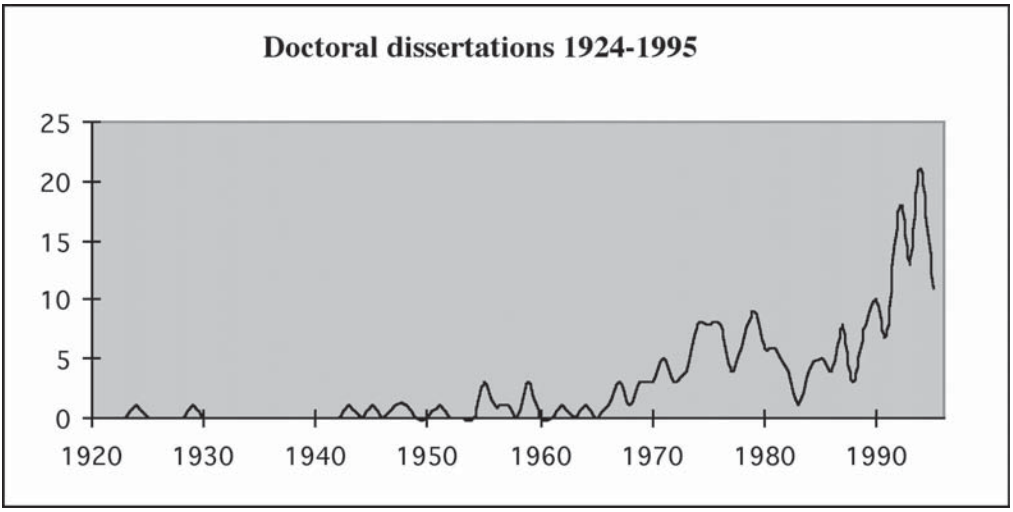

FIGURE 1. A graphical representation of the growth in the number of doctoral dissertations related to mathematics education that were produced in Canada between 1924 and 1995

TABLE 2. Province-by-province distribution (in a west-to-east order) of mathematics-education-related dissertations for which a doctorate was awarded during the 1990–1995 period in Canada

Province	Number of dissertations
British Columbia (BC)	10
Alberta (AB)	13
Saskatchewan (SK)	2
Manitoba (MB)	1
Ontario (ON)	23
Québec (QC)	30
New Brunswick (NB)	0
Nova Scotia (NS)	1
Prince Edward Island (PE)	0
Newfoundland (NF)	0

Federal Government Funding of Mathematics Education Research

The continued increase in doctoral dissertation research in mathematics education was but one of the global indicators of the growth of the Canadian community. Another indicator, one that is also linked with reificative aspects of the shared repertoire, was government-funded research. Nearly 50 years elapsed between the production of the first dissertation related to school mathematics in Canada and the initial awarding of federal funds in 1970 for mathematics education research carried out by university faculty (or independent scholars associated with a university). However, as table 3 illustrates, the 1970s were not especially productive for federally funded research in mathematics education. It was not until 1983 that such projects became more significant in number. (Note that federal funding for mathematics education research was under the control of the Canada Council, which was set up in 1957, and then under the Social Sciences and Humanities Research Council which replaced it in 1978.)

The low figures in table 3 for certain provinces reflect the fact that universities in some of those provinces had not, as of 1990, developed doctoral programs where one could obtain a degree for research related to mathematics education. Thus, academics in those universities had not, in general, applied for federal research funds. In contrast, the high figures for the province of Québec reflect the emergence of communities of researchers in the 1970s and 1980s who were strongly encouraged and supported at both the university and provincial government levels (I will say more about this in a later section when I treat the communities of practice in various provinces).

By graphically overlaying the data on doctoral dissertation production with those on federally-funded research projects (see figure 2), one obtains an overview that suggests three phases of growth over the years 1924–1995. The years up to approximately 1967 can be considered the years of *preemergence* of the community—the number of doctoral dissertations had not increased dramatically and no research projects related to mathematics education had yet been funded. The years from 1967 to approximately 1983 can be considered the years of *emergence*—there was significant growth with respect to doctoral dissertation production and mathematics education research by university faculty had begun to be funded, even if somewhat sporadically. The years from 1983 onward can be considered the years of *continued development* that followed the middle phase of emergence—doctoral dissertation production had gone on to reach new highs, after a brief slowdown period, and federally funded research had come into its own. In fact, during the third phase, both dissertation production and federally funded research greatly increased together—a sign that the community had already emerged.

TABLE 3. Number of new research projects in mathematics education in Canada funded by the Canada Council (1957–1977) and the Social Sciences and Humanities Research Council (1978–1995)

	Province (arranged in a west to east order)										
Year	BC	AB	SK	MN	ON	QC	NB	NS	PE	NF	**Total # projects funded**
Up to 1970	...	...	...	...	...	...	...	...	...	...	**0**
1970	...	...	...	...	...	1	...	...	...	...	**1**
1971	...	...	...	...	...	1	...	...	...	...	**1**
1972	...	...	...	...	...	...	...	...	...	...	**0**
1973	...	1	....	...	...	1	...	...	...	...	**2**
1974	...	...	...	...	...	1	...	...	...	...	**1**
1975	...	...	...	...	...	...	...	...	...	...	**0**
1976	...	...	...	...	...	...	...	...	...	...	**0**
1977	...	...	...	...	...	...	...	...	...	...	**0**
1978	...	1	...	...	...	...	...	...	...	...	**1**
1979	...	...	...	...	...	...	...	...	...	...	**0**
1980	...	...	...	...	...	...	...	...	...	...	**0**
1981	...	...	...	...	...	...	...	...	...	...	**0**
1982	...	...	...	...	...	...	...	...	...	...	**0**
1983	1	...	...	...	1	2	...	...	...	...	**4**
1984	...	...	...	...	...	1	...	1	...	...	**2**
1985	...	1	...	...	1	2	...	...	...	...	**4**
1986	1	...	...	...	...	3	...	...	...	...	**4**
1987	...	...	...	...	2	5	...	...	...	...	**7**
1988	1	1	...	...	2	2	...	...	...	...	**6**
1989	...	...	1	...	...	6	...	...	...	...	**7**
1990	1	1	...	...	...	4	...	...	...	...	**6**
1991	1	1	...	...	...	1	...	...	...	...	**3**
1992	...	...	...	...	1	2	...	...	...	...	**3**
1993	1	1	...	...	1	4	...	...	...	...	**7**
1994	...	1	...	...	...	5	...	...	...	...	**6**
1995	...	...	...	...	3	3	...	...	...	...	**6**
Totals/ province	6	8	1	0	11	44	0	1	0	0	**71**

Note: The first year in which a given project was funded is the year used, as is the province of the principal investigator. Sources: *Annual Reports* of Canada Council (1958–1978) and *Annual Reports* of Social Sciences and Humanities Research Council of Canada (1978–1996).

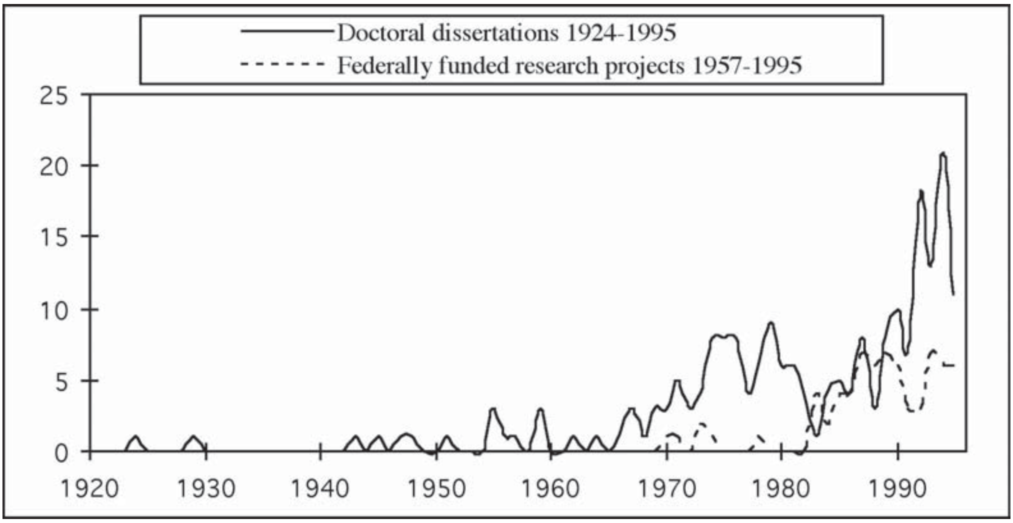

FIGURE 2. The number of doctoral dissertations related to mathematics education during the period 1924–1995 compared with the number of federally funded research projects related to mathematics education (Note that the federal funding agency was not established until 1957.)

3. International Influences and Interactions Related to the Emerging Canadian Community of Mathematics Education Researchers

Few countries develop their research communities in isolation. The influences of, and interactions with, mathematics education researchers in other countries, along with research-related events that took place outside Canada, all served to shape the Canadian community. In this part of the paper, the focus switches to the activities of individual Canadians and the roles they played both at home and abroad, particularly during the 1960s and 1970s—activities that had an effect on the evolution of the Canadian community of mathematics education researchers. As will be seen, the impact of several of these individuals was felt in both the local and the national communities of practice that they worked to develop.

Early Interactions with the United States Mathematics Education Research Community

The United States was a source of influential ideas with respect to the growth of research in mathematics education in Canada not only during the initial three decades of the 1924–1995 period, when the first doctoral dissertations were produced at U of T, but also in the decades that followed, especially from the mid-1950s through the 1960s. Douglas Crawford in the 1970 *History of Mathematics Education in the United States and Canada* has described some of the interactions that occurred between Canadians and Americans during the 1950s and early 1960s. But the most obvious source of influence of the U.S. mathematics education research scene on the developing Canadian community during these years was its journals. The U.S. research journals played an important dual role, especially in the late 1960s and 1970s, in that they not only enabled the growing number of Canadian mathematics education researchers to read about the kinds of research that were being conducted in the United States but also published the work of members of the newly emerging Canadian research community. These journals thus provide a window on the research activities of many Canadians during that period.

For example, in 1969, the *Review of Educational Research* (*RER*) published a special issue on mathematics education research. In that special issue was a paper by Tom Kieren on "Activity Learning," which contained several references to the mathematics education research activities being engaged in by fellow Canadians. The *Journal for Research in Mathematics Education (JRME),* which published its first issue in January 1970, also featured the work of Canadian researchers during its early years, for example, Tom Kieren, Daiyo Sawada, Walter Szetela, W. George Cathcart, David Robitaille, Frank Riggs, Doyal Nelson, Lars Jansson, Jim Sherrill, and D. Kaufmann. Equally important to note is that these early contributors to *RER* and *JRME* were primarily from Canada's western provinces—in particular, from U of A and UBC—reflecting the ties that had developed between the English-speaking mathematics educators of western Canada and their National Council of Teachers of Mathematics (NCTM) counterparts in the United States. This western Canada-United States connection parallels the pull felt by many of the French-speaking researchers of the country, along with some of their anglophone colleagues, toward Europe (e.g., to the International Commission on Mathematical Instruction and its quadrennial International Congresses on Mathematical Education, the soon-to-be-formed International Group for the Psychology of Mathematics Education, and the Commission Internationale pour l'Étude et l'Amélioration de l'Enseignement des Mathématiques).

International Interactions

Developments in mathematics education research at an international level in the late 1960s and 1970s attracted the attention of Canadian researchers. Canadians played a role in the growth of international associations of researchers, while at the same time coming into contact with fellow researchers from other parts of Canada and thereby forging the relationships needed for a Canadian community of practice.

The early International Congresses on Mathematical Education in the 1960s and 1970s

The first International Congress on Mathematical Education (ICME), which was held in Lyon, France, in August 1969, drew twenty-three Canadians—sixteen from Québec, six from Ontario, and one from Manitoba. It exposed them to the work and ideas of several invited speakers, including the research-related plenary presentations of Ed Begle from the United States, Efraim Fischbein from Israel, and Zoltan Dienes from Canada (see ICME-1 1969). Dienes was a researcher, born in Hungary, who had spent several years in Sherbrooke, Québec, directing the research center he founded there in the 1960s.

The next ICME, held three years later in Exeter, England, attracted even more Canadians than the previous one had; this time, fifty-two Canadians attended—twenty-three from Ontario, fourteen from Québec, seven from British Columbia, three from Nova Scotia, three from Alberta, one from New Brunswick, and one from Manitoba. The program allotted more time to research than ICME-1 had, with two of the thirty-nine working groups targeted explicitly towards discussion of research on learning and teaching (ICME-2 1972), one chaired by Efraim Fischbein on the psychology of learning mathematics and another chaired by Bent Christiansen on research in the teaching of mathematics. At this ICME, among the Canadians attending were Marshall Bye, Douglas Crawford, Claude Gaulin, Bill Higginson, Claude Janvier, Raynald Lacasse, and Richard Pallascio.

The increasing worldwide interest in mathematics education, and in mathematics education research in particular, was reflected in Canadian participation at the third ICME in Karlsruhe, Germany, in 1976. Fifty-one Canadians attended: twenty-nine from Québec, thirteen from Ontario, five from Alberta, one from Manitoba, one from British Columbia, one from Saskatchewan, and one from Prince Edward Island. Participants had the opportunity to congregate with, and talk to, many active mathematics education researchers from around the world, and to hear about current international research activities, such as those reported by Heinrich Bauersfeld and by Jeremy Kilpatrick. But, by far, the most important research-related event that occurred at ICME-3 was the formation of the International Group for the Psychology of Mathematics Education (originally IGPME, then changed to PME), a group that was to become the largest association of mathematics education researchers in the world. Not only did Canadians play an important role in the genesis of this international group, but their participation in its creation served also to increase the interest in the research enterprise at home and to contribute a particular identity to the emerging Canadian community.

The formation of the International Group for the Psychology of Mathematics Education

At the founding meeting of PME in Karlsruhe, 100 or so persons were present, including six Canadians: Claude Dubé, Nicolas Herscovics, Joel Hillel, Claude Janvier, Dieter Lunkenbein, and David Wheeler—all of whom were from Québec. The Canadians provided a considerable amount of leadership and support in the setting up of PME. The contributions of Nicolas Herscovics to the founding of PME were remembered by Efraim Fischbein at the opening session of a much later PME conference, in 1995, when he delivered a eulogy in memory of Herscovics, who had passed away the previous year:

> Nicolas was instrumental in setting up a Committee, in suggesting the election of a president and in contributing to the project of organizing, as quickly as possible, the first international conference of the organization. The activity of Nicolas was a decisive factor in the creation of the new body. Nicolas understood from the beginning that one had to create an organizational body which would facilitate interaction in that area [the psychology of mathematics education], would promote common research efforts, would contribute to new ideas, new research methods, and would confer on mathematics education, a theoretical and investigative dimension which it was lacking before. (Fischbein 1995, p. 1)

Canadians continued to play a major role in PME and remained active through the late 1990s. They assisted in the direction of PME in the following capacities: Eight served on the international committee (Jacques Bergeron, Claude Gaulin, Gila Hanna, Nicolas Herscovics, Claude Janvier, Carolyn Kieran, Gerald Noelting, and Vicki Zack); one was elected to the presidency for a three-year term from 1992 to 1995 (Carolyn Kieran); and three Montréalers (Jacques Bergeron, Nicolas Herscovics, and Carolyn Kieran), supported by many others from the Québec community of mathematics education researchers, hosted the eleventh annual PME conference in 1987. By the end of 1995, PME had 684 members, of whom 26 were Canadian—from British Columbia on the west coast to Newfoundland on the east coast.

Since PME's beginnings, Canadians have both drawn on the research that was presented by international colleagues at the annual conferences and also contributed their own work. One example that suggests how the research of Canadians may have been an influence on the larger PME community is reflected in the book published by PME in 1992, *Mathematics and Cognition* (Nesher and Kilpatrick 1992). The major authors in that book were all either Canadian, French, or Israeli, with the Canadian chapters focusing especially on the learning of early arithmetic and algebra. Additionally, Canadians served as contributing authors to the chapters on the epistemology and psychology of mathematics education, language and mathematics, and advanced mathematical thinking.

The role of the Commission Internationale pour l'Étude et l'Amélioration de l'Enseignement des Mathématiques in the development of French-speaking Canadian mathematics education researchers

An important international association involving French-speaking Canadians during the early years of their developing practice as mathematics education researchers was the Commission Internationale pour l'Étude et l'Amélioration de l'Enseignement des Mathématiques (CIEAEM), which had been set up in 1950. According to Claude Janvier (personal communication, September 1997), "Québecers were not researchers in the late 1960s and early 1970s; they were university teachers looking for the best curricula and the best approaches for teaching that curricula." Janvier emphasized that those were the years when the "new math" movement was being experienced. Teachers had to be retrained. In their search for answers to questions regarding new approaches to teaching mathematics, Québec mathematics educators came across examples of European research. So they decided to attend CIEAEM conferences where much of this research was being generated and discussed. The style of research appealed to Québec participants: critical questioning of what was being promoted in many countries under the name of new math. According to Janvier, the Québecers were also attracted to the voice being given in CIEAEM conference discussions to teachers and their experience.

4. Local Communities of Practice

The images presented thus far suggest that communities of mathematics education researchers had been established here and there throughout the country. Indeed, local communities of mathematics education researchers had either emerged or were in the process of emerging during the 1960s and 1970s in several provinces. Not all of these communities were at the same point of development at the same time. Nor were they all of the same size or strength. But there was no question as to their existence. Space constraints do not allow me to cover all of the provinces, such as for example, Manitoba and the work of Lars Jansson, or Nova Scotia where Yvonne Pothier and Mary Crowley were instrumental in developing a local community of practice, or the efforts of Lionel Pereira-Mendoza in Newfoundland, or the more recent work of Vi Maeers in Saskatchewan.

As will be seen, the local communities were the roots of the national community, a community whose period of emergence included a reificative act of a more formal nature in 1978. However, these communities were more than roots. They continued to develop, in interaction with the national community, during the 1980s through the 1990s. As will be seen from the examples of local communities of practice to be presented, it was not always the case that communities of a provincial nature emerged. In some provinces, structures—either of a formal or informal nature—had simply not been put into place that would encourage the development of a single provincial community. In contrast, in other provinces, the local communities of practice had become quite interconnected.

The Communities of Practice in Alberta

From about 1965, the University of Alberta (U of A) was an active center of mathematics education research, with the work of Doyal Nelson and Sol Sigurdson. Tom Kieren's arrival at U of A in 1967 signaled a bustling period devoted to theorizing in this local community of mathematics education research. In the 1970s, the research by Kieren, his colleagues, and their many graduate students focused on such themes as models of the use of concrete materials, rational number construct theory, rational number mechanisms, and neo-Piagetianism. The group's strong interest during the 1970s in constructivism developed further when, in the 1980s, Maturana and Varela's theories were brought into play.

Another local community of mathematics education researchers in the province of Alberta was situated at the University of Calgary where Bruce Harrison became a faculty member after being the first graduate of the new mathematics education doctoral program at U of A in 1968. One of Sigurdson's students, Harrison had focused his doctoral research on elaborating and testing the work of Richard Skemp on reflective thinking. This community continued its evolution throughout the decades that followed, a period that included the arrival of Olive Chapman during the 1990s. In another area of Alberta, at the University of Lethbridge, the mathematics education group benefited from interactions during the 1970s with Sigurdson from U of A on "direct meaning" teaching and curriculum.

The Alberta community of mathematics education researchers, from its early days in the 1960s and 1970s up to the late 1990s, was characterized by a strong theoretical leaning toward a nonrepresentationist, constructivist, enactivist perspective. It could be said that theory building was a main feature of their joint enterprise. And this interest was reflected in the dissertation topics of several of the mathematics education doctoral students trained in Alberta, including the early graduates such as Bruce Harrison, Sandy Dawson, and Bill Higginson. Tom Kieren advised many of the U of A students—in fact, he guided to completion more mathematics education Ph.D. students than any other Canadian academic during the last three decades of the twentieth century. As the Ph.D. graduates from U of A took up posts at other Canadian universities, characteristics of that community of practice took root at these other sites. And because many of these researchers continued to interact with fellow Canadians in the development of a joint enterprise, the Canadian community of mathematics education researchers reflected some of the features of the extended Alberta community.

The Communities of Practice in British Columbia

In the 1960s, concomitant with the arrival of the new math era, the mathematics education department of the University of British Columbia (UBC) mushroomed to become what was then the largest in Canada. As the new math period drew to a close, David Robitaille assumed the headship of the department, and he and Jim Sherrill in particular redirected the department to contemporary concerns.

Shortly after his arrival, David Robitaille was asked to take over the clinic for school children having trouble with mathematics, a project with which the department had been involved for some time. The diagnostic research focus gradually evolved during the late 1970s and 1980s into more qualitative work on children's misconceptions and on constructivist approaches to learning. Examples include studies on problem solving and metacognition (e.g., the work of Walter Szetela, Jim Sherrill, and Tom Schroeder), and decimal-fraction learning within a Piagetian perspective (the research of Doug Owens). As well, there was a growing interest in large-scale evaluation. This interest was exemplified in the role that UBC mathematics education researchers played in both the second and third international mathematics studies, in particular, as the international coordinating center (directed by David Robitaille) for the entire TIMSS study from its outset and until the end of 1993.

The story of the development of the UBC community of mathematics education researchers illustrates the multifaceted nature of the joint enterprise of that community, which had produced a shared repertoire covering many different themes of research, from diagnostic work to problem solving to large-scale evaluation. This diversification continued through the 1990s when new faculty were hired: Susan Pirie, with her interest in models of students' understanding; Ann Anderson, who focused on mathematics learning in young children; and Cynthia Nicol, whose research centered on the education of mathematics teachers.

During these same years, another local community of mathematics education researchers developed in British Columbia—at Simon Fraser University (SFU). The SFU community was characterized by a consistent theme that ran through its approach to mathematics education research over the years—that of studying the implementation of innovative teaching practices that arose from an investigation of the nature of mathematics. That theme was originated by John Trivett, who joined the SFU faculty of education in 1967; was strengthened in the early 1970s when Sandy Dawson was hired; and became fully developed with the addition of Tom O'Shea in the early 1980s. The theme arose out of, and was based on, a close collaboration between the mathematics department at SFU (Len Berggren, Harvey Gerber, and others) and the above mathematics educators of the faculty of education.

The same theme that characterized the mathematics education research at SFU from the 1960s through the 1980s was reinforced and elaborated when Rina Zazkis joined the faculty in the early 1990s. When Rina arrived, not only did she work closely with members of the mathematics department, continuing the tradition established by her colleagues, but she also taught courses in that department, in particular the mathematics-for-teachers course that the mathematics department had offered for a number of years. Rina also used this opportunity to begin an investigation of the understandings that preservice teachers have of number theoretic concepts. The close connection between mathematicians and mathematics educators, that was one of the distinguishing features of the local community of mathematics education researchers at SFU, will be seen to be a characteristic as well of the national community of mathematics education researchers from the late 1970s onward.

The Communities of Practice in Ontario

In 1965, U of T's departments of Graduate Studies and Research were transferred to the newly created Ontario Institute for Studies in Education (OISE), an autonomous institution with an affiliation agreement with U of T. Despite OISE's newly acquired prominence on the graduate education scene of Ontario in the 1960s and the past history of U of T in doctoral research related to mathematics education prior to the 1960s, events in the 1960s and 1970s seemed to work against the growth of a unified community of mathematics education researchers centered at OISE. The new math movement brought several actors onto the Ontario mathematics education stage, but they belonged to different communities, each with its own forms of engagement, enterprise, and shared repertoire. In fact, these communities held quite opposite views of what was important in mathematics education and of what it might mean to do research in mathematics education.

In 1985, Gila Hanna joined the OISE faculty, after having been a research associate there since 1978. Her main research interests focused on gender studies and the role of proof in mathematics. She also advised most of the doctoral candidates in mathematics education in Ontario from 1985 through the late 1990s. These years, which were important ones in the growth of the mathematics education research community at OISE, signaled a period of intense activity that was due in no small measure to her leadership. In 1999, just a year before her retirement from OISE-U of T, Gila established a new bilingual Canadian journal, which she would coedit with two colleagues. These years also witnessed important work being carried out by the educational psychologists at OISE, for example, Robbie Case's research on rational number and Rina Cohen's on the learning of mathematics in Logo environments.

While a local community of practice in Ontario was developing at OISE in the late 1980s, other communities of mathematics education researchers that were emerging in Ontario included one at the University of Western Ontario, where Doug Edge, Eric Wood, Barry Onslow, and Allan Pitman were active, and another at the University of Windsor, which centered on the work of Erika Kuendiger and her colleagues and students.

The last local community of practice in Ontario to be discussed, but by no means the smallest or the most recent, is the community of mathematics education researchers that emerged at Queen's University in Kingston during the 1970s. A faculty of education had been established there in the late 1960s, during the period when teachers colleges and normal schools in Ontario had begun to be affiliated with the universities. The corps of active researchers in mathematics education at Queen's included Hugh Allen, Douglas Crawford, and Bill Higginson. Under Bill's leadership, the group established Queen's as an important center of mathematics education and mathematics education research, in particular in the area of technology applied to the teaching and learning of school mathematics.

A significant feature of this local community of mathematics education researchers was the involvement of some of the faculty from the mathematics department. Working relationships were developed between the mathematics educators and mathematicians such as John Coleman. Peter Taylor was another Queen's mathematician who actively collaborated with the mathematics educators. Taylor, known for his research on the teaching of calculus, was honored in the 1990s for his contributions to mathematics education by a 3M Teaching Fellowship awarded by the Society for Teaching and Learning in Higher Education.

Other Ontario mathematicians who conducted research in mathematics education included Pat Rogers, then of York University, whose research focused on increasing the participation of women in university mathematics courses, research for which she, too, was awarded a 3M Fellowship in the 1990s. Similarly, Eric Muller from Brock University and Ed Barbeau from U of T (as well as Bernard Hodgson from Québec) were recipients of the Adrien Pouliot award for sustained contributions to mathematics education in Canada, an award given each year since 1995 by the Canadian Mathematical Society. Other awardees included the group from the University of Waterloo for their work on Canadian mathematics competitions at the secondary school level.

The productive interactions between mathematicians and mathematics educators that had been fostered at Queen's from the late 1970s through the 1990s were reflected in other such collaborative work occurring in the province. For example, in the 1990s, Eric Muller and Ed Barbeau, along with Gila Hanna and Bill Higginson, served on the Forum of the Fields Institute that played a major role in the revision of the Ontario secondary school mathematics courses. Interactions such as these between mathematics educators and mathematicians set in place the mechanisms for the creation of a center for mathematics education at the Fields Institute.

The Communities of Practice in Québec

The 1960s were exciting years in Québec. Zoltan Dienes, a recent arrival to Canada, had just set up his research institute, Centre de Recherche en Psycho-Mathématiques, at Université de Sherbrooke. His research center attracted visitors from all around the world, thereby exposing Québecers to international mathematics education research. Other signs of research activity in Québec in the 1960s and 1970s included the research centers such as Institut National de la Recherche Scientifique. However, an event that was among the most significant with respect to the emergence of a community of mathematics education researchers in Quebec was the creation of a government funding agency that paralleled the federal SSHRC. In 1970, the Programme de Formation des Chercheurs et d'Actions Concertées (FCAC) was set up by the newly formed ministry of education of Québec. One of the key elements of FCAC funding, as well as that of the Fonds pour la Formation de Chercheurs et l'Aide à la Recherche (FCAR) that replaced it in 1984, was the encouragement of teams of researchers, including teams drawn from various universities. Table 4 gives the number of FCAC/FCAR-funded projects in mathematics education research between 1972 and 1995.

In comparison with the data presented earlier in table 3, which showed that mathematics education research was hardly present on the federal funding scene prior to 1983, the data of table 4 reveal that mathematics education research was in fact being funded in Québec during the years 1972–1983 and that 1981 was an especially productive year for the emerging community. Thus, as of 1970, Québec researchers had access to two governmental funding agencies in contrast with researchers from the rest of the country who could only submit research proposals to the federal funding agency SSHRC.

The first recipient of an FCAC grant for mathematics education research, in 1972, was Claude Gaulin (with Hector Gravel as co-investigator). Gaulin was one of Québec's pioneers in mathematics education research, having carried out studies on the teaching of fractions from 1966 to 1971 with colleagues from Collège Ste-Marie, a college that was incorporated into Université du Québec à Montréal (UQAM) in 1969. Another pioneer from the 1970s was Dieter Lunkenbein from Université de Sherbrooke, who conducted research on the teaching of geometry in the early grades. And at Université Laval was a mathematician named Fernand Lemay, whose theoretical reflections were a great influence on the conceptual and epistemological thinking of some of the Québec mathematics education researchers of that time.

An additional development of significance for the growing Quebec community of mathematics education researchers during these years was related to a series of retraining courses for mathematics teachers provided by the ministry of education from 1965 to 1970. These courses evolved into the highly successful in-service distance education program for the

TABLE 4. Number of new research projects in mathematics education in Québec funded by FCAC (1970–84) and FCAR (1984–95)

1972	1	1976	2	1980	0	1984	3	1988	5	1992	1
1973	0	1977	1	1981	9	1985	4	1989	2	1993	1
1974	0	1978	2	1982	3	1986	1	1990	1	1994	2
1975	0	1979	4	1983	4	1987	2	1991	2	1995	2

Note: The first year in which a given project was awarded is the year used. Sources: *Subventions accordées* of FCAC (1971–73), *Crédits alloués* of FCAC (1973–77), *Répertoire des subventions allouées* of FCAC (1977–79), *Crédits alloués: équipes et séminaires* of FCAC (1979–84), *Rapports annuels* of FCAR (1984–91), and *Répertoire des subventions octroyées: soutien aux équipes de recherche* of FCAR (1990–96).

retraining of mathematics teachers throughout the province (known as the PERMAMA program), a program that not only involved many of the province's mathematics educators but also served as a basis for elaborating some of their research orientations. Another important event of the 1970s was the formation of the Groupe des Didacticiens en Mathématiques (GDM), an association of Québec mathematics educators interested in research.

During the late 1970s, as well, FCAC became more structured and began to give both more and larger grants. The excitement that had been generated with the creation of the international PME group at Karlsruhe in 1976 led to the formation of new teams of funded researchers in Québec, for example, the group of Joel Hillel and David Wheeler, followed by the collaboration of David Wheeler and Lesley Lee, at Concordia University, and the team situated at U de M of Nicolas Herscovics from Concordia University and Jacques Bergeron from U de M. The latter researchers' work on the learning and teaching of early number in the late 1970s and 1980s attracted several doctoral students to the team (e.g., Jean Dionne, Nicole Nantais, Bernard Héraud), who in turn became advisors to later graduate students of mathematics education at other universities in Québec. The work of Roberta Mura at Université Laval in the 1980s on women in mathematics brought certain socio-cultural issues to the fore in Québec mathematics education research. And Anna Sierpinska's arrival at Concordia in 1990 from her native Poland injected new dimensions into the research being carried out on understanding and epistemological obstacles.

Another event that aided the growth of the Québec mathematics education research community was the creation of CIRADE in 1980, a center whose initial roots were located in the Centre de Recherche en Didactique that was set up when UQAM was established in 1969. Mathematics education research flourished there during the 1980s and 1990s. Several international seminars and colloquia were held; these involved not only the members of CIRADE and their international visitors but also many other mathematics education researchers of Québec. Such colloquia focused, for example, on representation and the teaching and learning of mathematics organized by Claude Janvier, on epistemological obstacles and sociocognitive conflict, and on approaches to algebra—perspectives for research and teaching, organized by Nadine Bednarz, Carolyn Kieran, and Lesley Lee.

The 1980s were also the years in Québec when the potential of the computer programming language, Logo, as a mathematical exploration tool, sparked the interest of many researchers. Just about all Québec universities and colleges had their Logo groups of mathematics education researchers in the 1980s, such as the UQAM collaboration of Benoît Côté, Hélène Kayler, and Tamara Lemerise, as well as the inter-university team of Joel Hillel, Stanley Erlwanger, and Carolyn Kieran. During that same decade, the 17 full-time faculty members of the mathematics education section of the UQAM Mathematics Department made that group the largest ever contingent of mathematics education researchers across the country.

5. A Major Reificative Event for the Canadian Community of Practice: The Formation of the Canadian Mathematics Education Study Group / Groupe Canadien d'Étude en Didactique des Mathématiques

Let us return for a moment to the late 1960s and early 1970s. Local communities of practice had been evolving in various provinces since then. Members from some of these communities had come together at the early ICMEs, where connections among them had been formed. Then, in 1976, the third ICME had led to the establishment of the international PME group of researchers. The momentum created by these events beyond Canada's borders sparked not only an increased interest in mathematics education research at home but also a need for a structure that would permit members of the various local communities of Canada to get together. An occasion would soon present itself, even if it was planned with a somewhat different purpose in mind.

In 1977, John Coleman, Bill Higginson, and David Wheeler invited thirty mathematicians and mathematics educators from across Canada to join them at a mathematics education conference at Queen's University, Kingston, Ontario (sponsored by the Science Council of Canada) to discuss the general theme "Educating Teachers of Mathematics: The Universities' Responsibility." The conference had been convened primarily as part of the follow-up to the Science Council's Background Study No. 37 (Beltzner, Coleman, and Edwards 1976) to consider the place and responsibility of Canadian universities in the education of teachers of mathematics. Wheeler (1992) wrote that "one purpose of the conference was served by the mere fact of bringing participants together and the consequent pooling of ideas and information by those who have overlapping interests but seldom meet" (p. 2). Despite the intended "teacher education" agenda of the meeting, Coleman wrote in a letter accompanying the proceedings of the 1977 conference, "The meeting was noteworthy for the fact that, as far as we were aware, there had never been a comparable gathering of [Canadian] university staff whose prime concern was *research* in mathematics education" (Coleman, Higginson, and Wheeler 1978, p. 1 of accompanying letter, emphasis added). Wheeler (1992) pointed out that "the encounter generated a demand from many of the participants for further opportunities to meet and talk" (p. 1). The Science Council supported a second invitational meeting in June 1978 at which the decision was taken to establish a continuing group to be called the Canadian Mathematics Education Study Group (CMESG)/ Groupe Canadien d'Étude en Didactique des Mathématiques (GCEDM). This reificative act was a very important one for the emerging Canadian community of mathematics education researchers. The formation of this group would further enable the connectedness that had already been developing among individual researchers across the country, as well as extend the joint enterprise over a broader base. At the close of the 1978 meeting, the participants voted for an acting executive committee; a formal constitution was approved at the 1979 meeting; and the first elections under the terms of the constitution took place in 1980.

Tom Kieren, in his plenary address at the 1977 meeting entitled "Mathematics Education Research in Canada: A Prospective View," emphasized the "need for *much more* interrelated mathematics education research to tackle the problems [of mathematics education]" and suggested that "perhaps our small numbers in Canada and our personal interrelationships will allow us to engage in such interrelated research" (Kieren 1978, p. 19). He then offered a few recommendations to effect the cooperation needed in Canadian mathematics education research, among which was the regular meeting of groups of researchers and teachers to discuss problems of mathematics education in Canada.

CMESG/GCEDM tried to find some balance between focusing on teacher training and on research. Wheeler (1992), in a historical retrospective of CMESG/GCEDM written in 1992, described the concerns of CMESG/GCEDM as follows: "The two main interests of CMESG/GCEDM have been teacher education and mathematics education research, with subsidiary interests in the teaching of mathematics at the undergraduate level and in what might be called the psycho-philosophical facets of mathematics education (mathematization, imagery, the connection between mathematics and language, for instance)" (p. 5). However, because many Canadian mathematics education researchers were also responsible for the training of mathematics teachers and did in fact focus their research on teacher training, the two main spheres of interest were intertwined.

CMESG/GCEDM has attempted to provide a forum where research could be discussed—and even where new research partnerships could be formed—as well as set up an encouraging atmosphere where novice researchers could find out how to begin. For individuals coming from universities or provinces where no local community of mathematics education research practice had yet emerged, this latter provision was extremely important. Through its activities, CMESG/GCEDM gave some mathematics educators a taste for research. Wheeler (1992) wrote that CMESG/GCEDM "has shown them that their puzzlement about some aspects of mathematics is shared by many mathematicians; and it has shown some mathematicians that learning can be studied and that teaching might be made into something more than flying by the seat of the pants" (p. 8). The fact that the study group included among its active members both mathematicians and mathematics educators gave a particular flavor to the nature of the research enterprise as engaged in by its participants. One of the aspects of this particularity was a fairly wide vision of what it means to do research in mathematics education, as suggested by the following: "The Study Group takes as its essential position that the teaching of mathematics and all the human activities that are connected to it can, and should, be *studied*, whether the study has the form of an individual's reflections, the reasoned argument of professional colleagues, or the more formal questioning of empirical or scholarly research" (Wheeler 1992, p. 8).

From the beginning, the format of the four-day CMESG/GCEDM meetings fostered a unique form of mutual engagement of its participants. Three half-days were spent within one of the working groups. Designed to be the core activity of the meetings, these working groups were based on themes related to research, teacher development, and mathematical topics. During the 1990s, a novel feature was added to the annual meeting programs: the reporting by new mathematics education doctoral graduates of their dissertation research. This feature became a standard component of the program and had the effect of encouraging younger mathematics education scholars to join the community. But it succeeded in doing more than that. It made provision for the community of practice that came together at CMESG/GCEDM meetings to be a community of learners in which new practices and new identities were formed for both the existing members and the new members. Wenger (1998) argued that "engagement is not just a matter of activity, but of community building and ... emergent knowledgeability" (p. 237) and that "practice must be understood as a learning process, ... learning by which newcomers can join the community and thus further its practice" (p. 49); "from this perspective, communities of practice can be thought of as shared histories of learning" (p. 86). From its beginnings in 1978, CMESG/GCEDM succeeded in creating both the accumulation of a history of shared experiences and the development of interpersonal relationships—processes that, according to Wenger, are characteristically entailed in the work of engagement of a community.

In describing the dimension of a community of practice that is the shared repertoire, Wenger emphasized the ways of doing and talking about things, as well as the reified written forms of its work. CMESG/GCEDM remained rather steady in size—about sixty people attending the annual meetings, with a core of regulars present every year—so the participative aspects of the community stayed quite constant over time. The only written trace of the annual meetings is the proceedings, but these do not always manage to convey the spirit of the annual get-togethers. One has to look further to obtain a sense of the reified repertoire of this national community, for example, to the publications of its members or to the journal *For the Learning of Mathematics* (*FLM*). This journal, which was established by David Wheeler in 1980, often published the texts of various contributions made at the annual CMESG/GCEDM meetings. When David retired in the mid-1990s, the administration of the journal was handed over to CMESG/GCEDM. David (1997) emphasized, however, that the journal would not become the source of "Canadian news and views," but would continue to retain its international character. Nevertheless, the "Canadianness" of the journal was articulated by Bill Higginson in a special 1997 "retirement of the founding editor" issue of *FLM*:

> Let me point to two other aspects of FLM that have loomed large for this reader. The first is the extent to which it has been for me a quintessentially Canadian publication in the best possible sense of that term. Partly that has been because of geography. The journal was born in the bilingual richness of Montréal (subliminally, I suspect that the real meaning of FLM is *Front for the Liberation of Mathematics*) and then, like many other institutions and individuals, succumbed to the siren call of mellower British Columbia. More importantly, however, are its close links with one of David Wheeler's other legacies, the Canadian Mathematics Education Study Group (Groupe Canadien d'Étude en Didactique des Mathématiques) the small but vital organization which he was instrumental in creating in the late 1970s. ... The other unique feature of FLM for me has been the extent to which it exemplifies what I would like mathematics education to be. I have always found the inside front cover proclamation of the journal's aims ("... to stimulate reflection on and study of ...") to be a succinct and graceful statement. (Higginson 1997, p. 18)

Because of David Wheeler's influence on the Canadian community of mathematics education researchers from the 1970s to his passing in 2000, the above remarks of Higginson can be said to be related not merely to *FLM*; they relate as well to the spirit and to the "ways of doing and talking about things" of a national community whose emergence and development were stimulated by the founding of CMESG/GCEDM.

The national community of mathematics education researchers and CMESG/GCEDM were not one and the same, even if it was difficult at times to disentangle them. But CMESG/GCEDM encouraged community building and it was this community building that was so vital to the growth of a national community of mathematics education researchers. This is not to say that, when CMESG/GCEDM meetings were over, members did not return to their local communities and work at the continued development of these communities. But they also participated in a national community by, for example, collaborating in joint research teams with other Canadians, consulting on their research projects, coadvising doctoral students from other universities across the country, and organizing research colloquia and conferences with fellow Canadian researchers.

The annual meetings of CMESG/GCEDM continued to contribute to the emergence and later development of the national community of practice; however, there were additional events that played a role as well. One of these was the formation of the North American chapter of the International Group for the Psychology of Mathematics Education (PME-NA), in which Canadians participated both as founders and regular contributing members. Another was the preparation for and participation in ICME-7, held in Québec City in 1992, an event that entailed the involvement in one form or another of all Canadian mathematics education researchers.

6. The Canadian Mathematics Education Research Community at the End of the Twentieth Century

The main focus of this paper has been a description of the events related to the emergence of the Canadian community of mathematics education researchers, an emergence which could be said to have occurred in the block of years from the mid-1960s to the mid-1980s. The discussion of those events also touched upon the period of preemergence prior to the mid-1960s, as well as the years of continued development from the mid-1980s onward. What remains is to take a final look at the community in the 1990s.

In 1993, Roberta Mura of Université Laval conducted a survey of all mathematics educators who were faculty members of Canadian universities in order to learn more about the community that they constituted. Mura (1998) stated that "since the vast majority of universities do not have mathematics education departments, 'mathematics educator' is a label that individual members of various departments may or may not choose to apply to themselves" (p. 106). She therefore sent questionnaires to all those whose names appeared in the CMESG/GCEDM mailing list or in the CMESG/GCEDM research monograph produced by Kieran and Dawson in 1992, as well as to any other university-based Canadian mathematics educators known to these recipients. Of the 158 questionnaires sent out, 106 were returned; of these, 63 were retained as they were considered to belong to the target population. To be retained, one had to have answered positively to both of the following questions: (a) Do you hold a tenured or tenure-track position at a Canadian university? and (b) Is mathematics education your primary field of research and teaching? (Mura estimated that the total number of Canadians satisfying these two conditions was about 100, coming from approximately twenty-eight universities across the country.)

Mura reported that 44 of the 63 were men (70 percent). The mean age of the 63 respondents was fifty years, with a range from thirty to sixty-four. Forty-one of the respondents (65 percent) spoke English at work and 22 (35 percent), French. Of the 63 who did acknowledge mathematics education as their primary field, 47 (75 percent) worked in education departments, 13 (21 percent) in mathematics departments, and three had joint appointments. Eleven of the 13 employed in mathematics departments worked at two Québec universities, Concordia University and Université du Québec à Montréal, where mathematics education was a section of the mathematics departments. Concerning their education, 56 of the 63 respondents (89 percent) held doctoral degrees—46 in education, eight in mathematics, and two in psychology. For 57 percent of the survey participants, their highest degree was from a Canadian university, while for 33 percent it was from a U.S. university (the remaining 10 percent were from various other countries). Regarding the supervision of doctoral students, 29 percent had directed the research of at least one doctoral student.

Mura asked, "How do you define mathematics education?" In responding to this open-ended question, many referred to the goals of their field—some in theoretical terms, others in practical terms. Twenty-two respondents identified the aim of mathematics education in a way classified as "analyzing, understanding and explaining the phenomena of the teaching and learning of mathematics" (Mura 1998, p. 110). Twenty-one respondents assigned to mathematics education the goal of improving the teaching of mathematics and the facilitation of its learning. But Mura pointed out that these two identified goals are not mutually exclusive:

> In fact, four respondents integrated elements of both tendencies in their definitions of mathematics education. Contrary to what one might expect, even withdrawing these four individuals, the group who expressed a theoretical orientation and the group who expressed a practical orientation do not differ substantially from each other in their involvement in research as measured by the number of publications and communications, the number of theses supervised and manuscripts reviewed, membership in editorial boards, participation in joint research projects, co-authored publications and exchange of information with colleagues in Canada and abroad. (p. 110)

Despite some intersection of goals, the main characteristic of the community as uncovered by Mura was its diversity: *"Le portrait dessiné par les résultats de l'enquête est celui d'une communauté professionnelle diversifiée"* ["The portrait drawn by the survey's results is that of a diversified professional community"] (Mura 1994, p. 112). This diversity was based partially on the fact that the Canadian community consisted of both anglophones and francophones, each group having a different history that was clearly related to the school systems in which many of them taught before becoming university academics. But diversity also existed within the strictly anglophone communities of practice where various perspectives on what was important in mathematics education existed. Another facet of the community's heterogeneity was related to the fact that it included persons who were trained as mathematicians but who considered mathematics education to be their primary field of research and teaching. Many of these individuals tended to focus their research on the learning of mathematics by undergraduate students (e.g., Muller 1991; Taylor 1985). Consequently, they often reported their research at meetings of mathematicians, such as the Canadian Mathematical Society or the Mathematical Association of America. These researchers also published their work more often in the journals and monographs of those mathematical societies than in the usual mathematics education research periodicals. Wenger (1998) argued that it is the community that creates its own practice. In this regard, the community of mathematics education researchers that was created in Canada was one whose practice was markedly characterized by diversity.

However, the multifaceted nature of the Canadian community was not attributable solely to linguistic factors or to the discipline of initial training. There were also differences among Canadian mathematics education researchers that were related to the theoretical tools they used for framing research questions and for analyzing data. Some of these differences were evident from the 1980s in, for example, the theoretical perspectives held by U of A researchers. But, the variety of theoretical perspectives increased even more across the Canadian mathematics education landscape during the 1990s when a widespread shift toward theorizing occurred. One of the indicators of this shift was the mix of espoused theoretical positions that were discussed within the 1994 CMESG/GCEDM working group on "Theories and Theorizing in Mathematics Education" (led by Tom Kieren and Olive

Chapman). Much of the previous research of the Canadian community had tended by and large to be Piagetian in spirit, often focusing on constructivism, cognitive conflict, and epistemological obstacles, and usually paying less attention to the role of cultural and social factors. But in the 1990s, theoretical frameworks broadened considerably to include, for example, Vygotsky's socio-cultural psychology, Brousseau's theory of didactical situations, and the interactionist perspective of Bauersfeld—a shift that could also been seen on the international scene.

As the 1990s ended, there could not be said to be one perspective that characterized the Canadian community of mathematics education researchers. Even though it was possible to speak in the 1990s of a French or Italian or German school of thought in mathematics education research, there was no such single view in the Canadian community. The Canadian community of mathematics education researchers was basically quite eclectic with respect to theories and theorizing. However, theoretical sameness, even though it may exist in a community of research practice, is not required, for, as Wenger (1998) argued, "If what makes a community of practice is mutual engagement, then it is a kind of community that does not entail homogeneity; indeed, what makes engagement in practice possible and productive is as much a matter of diversity as it is a matter of homogeneity."

Note

1. This paper is an abridged version of a chapter, "The Twentieth-Century Emergence of the Canadian Mathematics Education Research Community", to appear in *A History of School Mathematics*, edited by George Stanic and Jeremy Kilpatrick, which will be published by the National Council of Teachers of Mathematics in 2003.

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Lessons from the Past, Questions for the Future

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I graduated from the Grande Prairie Composite High School in 1977, twenty-five years ago. In fact, the Class of '77 held its 25th anniversary reunion the very same weekend as this 25th anniversary meeting.

When I decided to skip that event and to attend this one, I found myself feeling both a little guilty and a little, well, 'Who cares?' Being the self-analytic sort, I wondered about those conflicting emotions. As near as I can tell, the tension seems to be rooted in mathematics—or, at least, in numbers.

Twenty-five. That's obviously important. But why? It seems to be because most of us have 10 digits on our upper limbs … and when you collect together as many people as you have fingers, altogether you have 100 digits … and when you let the earth revolve around the sun as many times as 10 people have digits on their upper limbs, you have a century … which is a foundational notion in our construals of history, but an experientially inaccessible stretch of time, unless you invoke the cultural habit of dividing in half and then dividing in half again to arrive at the magic that is a comprehensible quarter century.

So, what we have here in the *need* to celebrate 25th anniversaries is a collision of biological, mechanical, and cultural phenomena, all pressed through the ringer of 'number' to generate something that is simultaneously arbitrary and significant.

It strikes me that one of the really important transitions in mathematics education research over the most recent passage of that arbitrary and significant span of time has been the emergent realization that matters of mathematics and mathematics learning arise in the collisions of the mechanical, the biological, and the cultural. This is no small change. As I aim to develop, mathematics education has tended to rely too exclusively on mathematics—or, perhaps more appropriately, on prevailing beliefs about the nature of mathematics—to orient and inform its activities.

Of course, in broader academic terms, that insight isn't really so recent. Almost exactly a century ago, such matters were taken up by researchers and theorists who helped to trigger a new transdisciplinarity in academia. Among the emergent discourses of the time were three that I'm going to draw on today: pragmatism, phenomenology, and psychoanalysis. (In other words, I've decided to speak to the last century, not just to the last 25 years, and I look across all academic domains, not just mathematics education.)

I frame my references to these fields with a bit of unpopular wisdom:

> Arguments against new ideas generally pass through three distinct stages, from 'It's not true', to 'Well, it may be true, but it's not important', to 'It's true and it's important, but it's not new—we knew it all along'. (Barrow, 1995, p. 1)

When Dewey, Husserl, and Freud, along with their colleagues, were working out the details of pragmatist, phenomenological, and psychoanalytic theories a century ago, it is certainly the case that most around them regarded their insights as falling into the *'It's not true'* category.

That changed through the 20th century, as many their ideas gained a gradual acceptance, inside and outside of academia. That is, some core insights of pragmatism, phenomenology, and psychoanalysis slipped from *'It's not true'* into *'Well, it may be true, but it's not important'*.

This was particularly the case over the last 25 years in mathematics education research—as pragmatist insights were imported in crates labeled "constructivism", phenomenological insights were squeezed in the door labeled "critical theory", and psychoanalytic insights came to be represented, somewhat appropriately, in the commonsense and largely unconscious discourses of teachers and researchers.

I'm going to try to connect insights from these fields to developments in mathematics education research over the past quarter-century, hoping to foreground some important lessons and some pressing questions. In the process, I also want to point to what I worry might be a slippage of insights from the category of '*Well, it may be true, but it's not important*' into the category of *'It's true and it's important, but it's not new—we knew it all along'.* I think that we need to work very hard to maintain sharp edges on these ideas so that we can do as much innovative work as possible before they're dulled and included among the blunt instruments wielded to maintain an existing order.

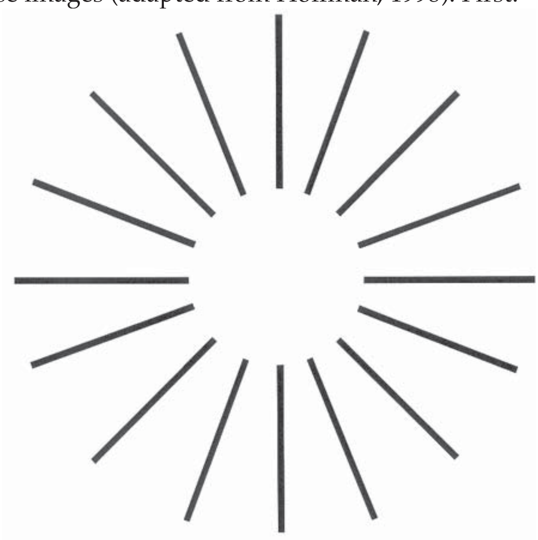
Anyway, back to the fact that I'm missing the 25th anniversary reunion of the Class of '77 of the Grande Prairie Composite High School, you will note that I have chosen a G-word (for Grande) and a P-word (for Prairie) in each of my four subtitles.

1. Lessons and Questions from Geography and Pragmatism

The first broad set of issues that I want to address is the geography of mathematics education research—its borders, topography, climate, and so on.

Before getting too mired in these notions of geography and borders, I should mention that I don't intend them literally. These strike me as ideas that reflect the very tendency that they name: the human habits of imposing boundaries and exaggerating edges, even when no edges exist. Two quick exercises in perception can help to demonstrate this point.

Consider these images (adapted from Hoffman, 1998). First:



Most people see a completed circle here. Usually, not only does the observer project a circle, but one that is brighter than (and perhaps even floating above) the background.

Second:

Where these gray blocks meet one another, it appears that they lighten up slightly. In fact, in terms of ink density, each box is uniformly shaded. What's more is that the difference in tone between adjacent blocks is not actually as great as it might appear. If you lay a pencil across one of the borders, you'll see the adjacent blocks are almost the same shade.

The point? Our perceptual systems impose borders that aren't there and amplify edges that are—which, as has been explained to me, makes perfect sense. Boundaries are the most useful information in the environment, so one might expect vision to be oriented to fishing out such details and exaggerating them.

We humans extend these tendencies from the realm of the *perceptual* into the realm of the *conceptual*. Humans are forever imagining and exaggerating differences. It's fundamental to our self-definition, to our collective identifications, to our having a world.

These proclivities are certainly represented in the manner in which the territory of mathematics education is defined. At the moment, for instance, its terrain is criss-crossed by borders that are used to distinguish between

- back-to-basics & problem-solving / posing;
- fragmentation / reduction & holism;
- procedural understanding & conceptual understanding;
- mental discipline & physical experience;
- teacher-centeredness & learner-centeredness;
- individual interests & societal interests;
- empowerment & subjugation;
- nature & nurture;
- theory & practice.

It would be easy to stretch out this list. But I'll stop here, and invoke the help of John Dewey who wrote in 1910:

> intellectual progress usually occurs through sheer abandonment of questions together with both of the alternatives they assume—an abandonment that results from their decreasing vitality and a change of urgent interest. We do not solve them, we get over them. (p. 19)

In terms of the *internal* geography of mathematics education, one of the strong lessons of the past 25 years is that the map suggested by these dyads is a poor one, and that we have a whole host of imagined borders and exaggerated edges that we need to get over.

I should mention, by the way, that my point is not that we shouldn't be drawing borders. On the contrary, we have to, in terms of both physiological tendency and interpretive necessity. The point is that we should be careful about how we draw them, always willing to erase them and draw new ones.

Perhaps the most important influence in prompting the recent move to this more mindful attitude was pragmatism, a North American philosophical movement that is most often identified with John Dewey.

Pragmatism can't be readily summarized. But some of its key insights can be succinctly represented:

- *contingency and adequacy of truth*: Pragmatically speaking, *truth is what works*—what is adequate to the situation. Truth is not about a perfect correspondence with an existing reality, but about a good-enough system of interpretation to maintain one's viability.

  This point should call to mind radical and social constructivist sensibilities—and quite appropriately so. There is abundant evidence that both Piaget and Vygotsky were strongly influenced by pragmatist philosophy.

- *evolutionary unfolding of knowledge*: Dewey made this point rather strongly a century ago. The idea is that many dynamic forms acquire their 'shapes' by continuously adapting themselves to immediate circumstances. By 1900, Darwinian processes had already been applied to such phenomena as personal understanding, collective knowledge, culture, and social institutions—in addition to biological species.

- *co-emergent characters of agent and setting*: Of course, when one part of a system changes, it often triggers changes in other parts of the system. Darwin suspected as much, and the idea of co-emergence was further developed by the pragmatists as a core principle in the emergence of understanding, knowledge, culture, and so on.

An important lesson of pragmatism, incorporated into the field of mathematics education over the past 25 years, is that the geography of the field is entirely problematic—both internally (in terms of the issues that tend to predominate) and externally (in terms of the manner in which the field is rendered distinct from other domains).

If I were to attempt to sum up these issues into lessons from the past and questions for the future, I would suggest something along the lines of:

A Geographical and Pragmatist Lesson:
(To borrow from Gregory Bateson, who borrowed from Alfred Korzybski:) The map is not the territory … but once the territory is mapped, it changes things.
Geographical and Pragmatist Questions:
1) Which of the current tensions are worth worrying about, and which do we need to get over? 2) How do we fit the largely accidental characters of mathematical knowing and mathematics knowledge into the deliberate structures of school mathematics?

2. Lessons and Questions from Geology and Phenomenology

When studying "earth science" in Grade 8, I was shaken by the realization that the solid ground beneath us is, in fact, not at all solid.

I had a good science teacher that year in Mr. Lakusta. It was a time of tremendous oil exploration in the area of northern Alberta that I lived, and Mr. Lakusta was in the habit of collecting samples of rock strata cut out by oil drills. He used them to illustrate that the ground we walked on was not bedrock, as I had surmised from my daily watching of the Flintstones, but many, many layers of sediment in varying states of metamorphosis.

This lesson in geology is an image that might be used to describe the work of phenomenology—which asks questions about what's buried beneath the literalized, common sense conceptions of events and experience. It answers questions about the nature of knowledge in terms of layer upon sedimented layer of literalized metaphor, as opposed to the common assumption of a bedrock of logic.

It's very difficult to understand what phenomenology is all about in this era of empiricism and rationalism. Both empiricism and rationalism are rooted in a faith in formal logic. Although they're often discussed as opposites, they share the same core belief that, in order to understand a phenomenon, you have to be able to break it down into its fundamental pieces.

Phenomenology doesn't deny that this reductionist attitude is a powerful one, especially when applied to mechanical events. However, phenomenology suggests that the rationalist-empiricist attitude tends to embody at least one fundamental oversimplification: It usually ignores the role of language.

Much of our collective knowledge is woven into and through our habits of speech, deposited there by humans thousands of years ago and thousands of kilometers away. Because of the temporal, physical, and cultural distances, we tend to lose track of what our ancestors were pointing to when a new term was invented or a different metaphor was used.

We are born into a languaged world. Our predecessors did most of the hard conceptual work of dividing and interpreting the world for us, and part of the generous inheritance they left us was the fact that we get a great deal of information and insight for free, just by learning to speak. However, we usually use language as though it were a transparent medium, as though words corresponded with objects and events in the world—as though language attached us to the bedrock of reality.

Language doesn't do that. Language consists of sedimented layers of metaphors, analogies, and other figurative forms. Phenomenology has been instrumental in triggering an explosion of research into this matter over the past 25 years. There's too much to mention, but one notable effort was recently published by George Lakoff and Rafael Núñez (2001) under the very confident title, *Where mathematics comes from.*

Lakoff and Núñez work from the premise the there are two main categories of knowledge: the primitive knowing that arises from the fact that we have bodies that move through the world, and the associative elaborations by which that primitive knowledge is refined and knitted into more sophisticated understandings.

Their departure from the last few centuries of philosophy is around the assertion that we humans are not all that logical, we're mainly analogical. Now, that idea has been pretty much embraced by cognitive science. But use of the idea has tended to be restricted to discussions of language development. In particular, it hasn't much been taken up in discussions of mathematical understanding—in large part because it has been assumed that mathematical understanding is a matter of logical thought, not analogical thought.

Lakoff and Núñez attempt something remarkable. They begin by tracing metaphors that underpin counting and adding—things like "adding as collecting sets of objects together" and "counting as walking forward or climbing upward"—and then they move through layers and layers and layers metaphors in an attempt to demonstrate a figurative—not logical—grounding of Euler's equation, $e^{\pi i} + 1 = 0$.

I'm not sure they succeed in that goal. Recently published reviews of the book from prominent mathematicians seem to suggest that they've fallen short. However, they do manage to demonstrate the significance of metaphor in both the cultural development of mathematics and in the emergence of personal understanding. On the latter, there is a compelling argument to be made that, like language, mathematics is *NOT* constructed on the solid bedrock of logic. It is mired in layers and layers of literalized metaphor (Rorty, 1989), which in turn rest on such rudimentary bodily experiences and reaching, walking, and sticking things in our mouths.

Lest you think these claims are off the wall, I would urge you to undertake an exercise in geology some time. Make a list of mathematical terms—basic ones—and then go to an etymological dictionary to dig into their pasts. Here are a few examples out of geometry:

Notion	Root	Original Meaning
Angle	Latin, *angulus*	ankle
Basic	Greek, *bainein*	step
Line	Latin, *linum*	linen thread
Normal	Latin, *norma*	carpenter's square
Parallel	Greek, *parallelos*	alongside (one another)
Plane	Latin, *planum*	roam about
Point	Latin, *pungere*	fist
Rule	Latin, *regula*	wooden ruler
Standard	Latin, *stare*	stand, upright
Straight	German, *streccan*	stretch

These are all body parts, bodily activities, or physical artifacts. The abstract concepts represented in the left hand column, that is, are metaphoric elaborations of embodied knowledge. Some very formal, crisp ideas derive from some very informal, fuzzy bodily experiences. This appears to be true on the levels of both personal knowing and collective knowledge.

Now, it you buy all that, there are huge implications for the teaching of mathematics—and especially in the early years of schooling when the first layers of interpretation are established. In particular, it would suggest that we should be spending a lot more time on educating people's intuitions than on insisting on demonstrations of formal knowledge.

A caveat of this point is that our language and our mathematics are not benign or inert. They are technologies that we use to manipulate the world. This is one of the profound realizations of phenomenology. Knowledge systems are technologies, wielded to transform the world. This realization also has important implications for discussions of mathematics learning. Arguments on the role of more recent electronic technologies have taken up a lot of air time recently, and those arguments have tended to be framed by troublesome assumptions about the natures of both technology and human thought. We would do well to recall around such matters that Plato, more than 2000 years ago, lamented the advent of writing. He was worried that reliance on the technology of the written word would dull memory. And, in fact, it has been demonstrated to do just that. What Plato couldn't know was the tremendous informational advantage that writing gives us by allowing us to offload so many details, freeing up consciousness for other worries.

One of the points that contemporary debates over the use of technology tend to ignore is the manner in which available technologies transform intellectual possibilities. Calculators and computers are seen as things imposed on a divinely wrought curriculum, and that tendency has left us with some laughably outdated curriculum topics and teaching methods. Given that leaders in electronic technologies are confidently predicting direct interfaces between the brain and the internet in the not-too-distant future, I would dare to suggest that a rethinking of our curricula is on the verge of becoming critical. Technologies aren't add-ons. They affect the geological substrate of our activity.

A Geological and Phenomenological Lesson:
We're not logical creatures in the main. We are principally analogical, and our perceptions-and-conceptions are framed by sedimented layers of interpretive habit.
Geological and Phenomenological Questions:
1) How are bodily experience, language, and mathematics entwined? *2)* Can we conceive of a mathematics pedagogy that is mindful of its past and responsive to its present, rather than being obsessed by an imagined future?

3. Lessons and Questions from Geometry and Psychoanalysis

Another 19th century thinker who has had a profound influence on how we think about matters of thought and learning is Sigmund Freud, through his psychoanalytic theory—in particular, his development of 'the unconscious' as the unseen part of the iceberg of personal knowing. Freud argued the case that most of what we know, we don't know that we know. We just know it. Or, more precisely, we just do it.

Unfortunately, Freud coupled the unconscious to the assumption that humans are naturally aggressive, greedy, selfish, duplicitous, sex-crazed, and cruel, with only a veneer of social responsibility. The past century of research into "human nature" has revealed that Freud was quite right about the influence of nonconscious awareness. It has also demonstrated that he was quite incorrect in his belief that we are essentially evil creatures. Human nature, it seems, is as plastic and as situationally specific as most things human.

On the matter of the nonconscious, research into perception has revealed that while our bodies are fitted with something in the order of 10 million sensory receptors (when you add together light sensitive cells in the retina, tastebuds on the tongue, nerve endings in the skin, and so on), the consciousness of a typical human can accommodate in the order of 10 discernments each second. We are immersed in a sea of sensorial possibility, but are capable of being consciously aware of only the tiniest droplet at any given moment. (See Norretranders, 1998, for an elaborated discussion of the research and its implications.)

But the fact that we're not consciously aware of the sea does not mean that it doesn't profoundly influence what we do. On the contrary, we are constitutionally coupled to the worlds around us—shaped by and shaping, dancing with, conversing with—in the most elegant of ways … ways to which we are almost always totally oblivious. But, in terms of consciousness, we tend to operate only in the veneer of worldly events.

Let me develop an example of that veneer: the contribution of mathematical knowledge to the shape of contemporary teaching. The point that I aim to develop here is that *explicit beliefs about the teaching of mathematics have tended to be framed by formal mathematical notions that have been incorporated into implicit beliefs about what thought is and how learning happens.*

Mathematics teaching, that is, has been mathematized. So has, for that matter, just about every aspect of our modern world. And Western culture has been very selective in its choice of mathematical truths to embody in its mathematics teaching. They're drawn mainly from Euclidean geometry.

Let's look at the word *line* and some of its Euclidean relatives.

Modern term	Derivation	Some current usages and associated terms
right	> Latin *rectus*, straight	right angle, righteous, right handed, right of way, right/wrong, human rights
rect-	> Latin *rectus*, straight	rectangle, correct, direct, rectify, rector, erect
regular	> Latin *regula*, wooden straightedge	regulation, regulate, irregular
rule	> Latin *regula*, wooden straightedge	ruler, rule of law, rule out, rule of thumb, broken rule
line	> Latin *linum*, flax thread	linear, time line, line of text, line of argument, linear relation, sight line, linear causality, toe the line
ortho-	> Greek, *orthos*, straight	orthodox/unorthodox, orthodontics, orthogonal, orthopedic
straight	> German *streccan*, stretch	straight up, go straight, straight answer, straight talk, straight and narrow, straight-laced

This table is hardly exhaustive. I *strictly limited* myself in this *project* to words that are *explicitly aligned* with *prominent* Euclidean notions. In *truth*, I could easily *justify* an *extended list* that includes many, many other *ordinary* terms whose ancient roots and contemporary associations have to do with lines and linearities—including each of the words italicized in this paragraph.

My point here is not that the emergence of this particular web of associations represents some sort of error or conspiracy. It is, rather, that these terms are pervasive. They are present in English-speakers' communications and infuse habits of interpretation—and, in the process, they do a particular sort of work. They help to project and maintain a 'right' and 'correct' sense of how things are. For instance, as I hope is evident in the third column of the table, beneath the literal surface of these terms is a mesh of rightness and wrongness, or correctness and falsehood, of straightness and queerness. In English, straight lines are knitted into how we think about good and evil, truth and deception, morality and deviance.

If you think I'm exaggerating, then consider the connotations of words that mean, literally, not straight: *twisted*, *bent*, *kinky*, *crooked*, *warped*, *deviant*, and the like.

Straight-ness or right-ness is not just a pervasive part of the collective conceptual world, it is built into our physical worlds. Just look around. Or think about the rectangular timetable pasted on the corner of the rectangular tabletop in the rectangular classroom at the end of the rectangular hallway in the rectangular school on the rectangular block in the rectangular city that you might locate on a map by using a rectangular grid that looks a lot like the rectangular timetable pasted on the corner of the rectangular tabletop. Whereas you would be hard-pressed to find many reasonably close approximations of rectangles in what tends to be called the natural world, the modern Western world is utterly rectangulated—conceptually and physically.

This matter comes into even more dramatic relief through a similar analysis of origins, meanings, and current associations of terms linked to another key Euclidean form: the 90˚, or *right* angle.

Modern term	Derivation	Some current usages and associated terms
Standard	> Latin *stare*, stand (i.e., make a right angle to a planar surface)	standardized tests, standard form, raising standards, standard time, standard units, standard deviation, standard of living
Normal	> Latin *norma*, carpenter's square	normal curve, normalize, normative, normalize, normalcy, normal fork, normal child
Perpendicular	> Latin *pendere*, to hang	dependence, expense, independent, pendulum, suspend, suspence

I won't drag you through a history of the emergence of modern conceptions of normality (but will suggest the histories provided in Davis, Sumara, & Luce-Kapler, 2000 and Foucault, 1990). I will however flag the fact that, via probability and statistics, mathematics and mathematized sensibilities have played a central role in the invention and imposition of the normal child, standardized curriculum, age appropriatism and the like.

The point here is that, by simple virtue of the fact that I'm a citizen of a mathematized society, I can't help but be sucked into linearized and normalized patterns of association. And it can demand a great deal of effort to get out of the ruts carved by a common knowledge.

A Geometric and Psychoanalytic Lesson:

We aren't very aware of what we do. Beneath the veneer of straightforwardness and normality, things are actually quite twisted and very knotty.

Geometric and Psychoanalytic Questions:

1) What sorts of no-longer-conscious mathematized assumptions are used to give shape to whatever it is we're doing when we claim to be teaching mathematics?

2) Where might we look for and how might we incorporate a more appropriate set of forms and images into our thinking about mathematics, learning, and teaching?

4. Lessons and Questions from Gaia and Plectere

It would be quite wrong of me to end on that note, especially because mathematicians have done a wonderful job of developing alternative geometries that present us different lenses on the world. In these few closing comments, I want to shift away from lessons learned and questions to ask, to remark on where I think we might look for insight in response to some pressing issues.

There's too much to say in this regard, but I would point to a favorite set of images out fractal geometry—which might be characterized as a mathematics of twistedness and knottyness.

Some have gone so far as to dub fractal geometry "the mathematics of nature" or "the geometry of the surprise"—and for good reason. Both the recursive generation of fractal forms and their consequent qualities of scale independence and occasional self-similarity do prompt attentions to a great many natural processes and natural forms.

In response to the realization of an over-Euclideanized school mathematics, I believe that fractal geometry offers us a much better (but by no means ideal) means to image or map. One might, for example, draw an analogy between the nested, emergent qualities of a fern frond and the nested, emergent qualities or aspects of human activity and the domains that have been devoted to study of those aspects.

(Images are from Davis et al., 2000, pp. 72–73.) One might also highlight the different images for movement, away from a straight line into a predictable future, and toward a spray of emergent possibility, an ever-expanding realm of the possible.

It's interesting to note that this geometry has arisen in a historical moment of increased interest in co-implication, dynamic evolution, complex emergence, and the like. For me, this is more than coincidence. I think it flags the profound manner in which our mathematical insights are entwined with our cultural sensibilities—and, with that flag, the tremendous moral and ethical implications of mathematics research and mathematics teaching. These activities don't occur in vacuums. They are neither inert nor benign.

One of the major triggers of this sort of 'systems thinking' was James Lovelock's Gaia Hypothesis—the idea that the biosphere might be understood through the metaphor of an organism rather than the pervasive metaphor of a machine.

These sorts of sensibilities have coalesced into a couple academic discourses—namely ecology and complexity science. These fields are interested in complex co-emergences. What's more is that they're begun to be interested in precisely the same thing as formal education: the pragmatics of complex transformation. A few ecologically minded educational researchers—although not so many in mathematics education—have developed the idea that we might reframe our efforts in terms of complex participation at various levels of physical, biological, and cultural organization.

The term *complexity* derives from the Latin *plectere*, to braid—as echoed in modern terms implicate, explicate, complicate, implicit, explicit, complicit, perplex, duplicity, and plexus. Complexity science seems to have come of age in recent years, as it's shifted from an emphasis on description of such complex phenomena as heart function, the self-organization of neurons in the embryo's brain, social structures, cultural evolution, and so on, toward deliberate efforts to affect such structures. Complexivists, for example, have done some important work in identifying conditions that are necessary for complex emergence—conditions that, I might note, are met in this sort of meeting, but that are usually absent in mathematics classrooms.

A Gaian and Plecterean Lesson:
Complexity and complicity arise in nested layers of co-implication.

With this lesson in mind, I would like to close by (re)citing Carolyn Kieran's 2002 (re)citation of Tom Kieren's 1977 plenary address, in which she/he/they remarked on the

> need for *much more* interrelated mathematics education research to tackle the problems [of mathematics education] … perhaps our small numbers in Canada and our personal interrelationships will allow us to engage in such interrelated research. (Kieran, p. 180)

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Lessons from the past, questions for the future: méditation sur thème imposé

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Introduction

Perchè — quando si è sbagliato — si dice « un'altra volta saprò come fare », quando si dovrebbe dire: « un'altra volta so già come farò? »

– Cesare Pavese, *Il mestiere di vivere*[1]

Ma première réaction au titre proposé a été : « Lessons from the past? We don't learn any! » Nous n'apprenons rien de nos expériences passées, encore moins de celles des autres, nous répétons toujours les mêmes erreurs. (Je ne suis pas d'un naturel optimiste, Pavese ne l'était pas non plus.)

Après réflexion, pourtant, j'ai dû admettre que cette conclusion ne pouvait être tout à fait acceptable, puisqu'elle était elle-même le fruit d'expériences passées et constituait donc bel et bien une leçon du passé, fût-elle la seule!

« Apprendre une leçon » peut signifier simplement « savoir que les choses sont ainsi » et pas nécessairement « modifier son comportement ». Il fallait distinguer les deux sens de cette expression.

L'échec de ma première leçon m'en suggérait immédiatement une seconde — c'est-à-dire une première, puisque la précédente n'en était plus une : méfions-nous des déclarations trop catégoriques! Ce n'est pas une leçon spécialement agréable; il est bien moins fatigant de penser et de s'exprimer sans nuances. Les démagogues le savent bien : la nuance confond le public et entrave l'action. Toutefois, justement, nous avons affaire à l'éducation et non à la démagogie. Si ma leçon permet de distinguer les deux, c'est bon signe.

J'ai bien senti le danger que cette nouvelle leçon ne se retourne contre elle-même, comme la première : n'était-elle pas justement une déclaration un peu trop nette? Sans doute que si, mais j'ai voulu résister à la tentation de m'enliser dans ce terrain.

On ne peut donc pas nier l'existence de leçons du passé. Reprenons alors avec plus de calme et un brin d'optimisme.

Introduction, prise deux

If I have seen farther than others, it is because I have stood on the shoulders of giants.

– Isaac Newton

Que nous enseigne le passé? D'abord, qu'il est passé. Nous pouvons le regretter ou nous en réjouir, c'est selon. Toutefois, dans bien des cas, avec le passage du temps, nos jugements sur les événements et sur les courants de pensée se modifient, des certitudes se lézardent et s'effritent, des théories sur l'éducation sont supplantées par d'autres, parfois davantage à cause d'un changement de valeurs que de nouvelles découvertes. Il est prudent alors de ne pas trop s'attacher aux idées du présent, de s'abstenir d'opinions trop tranchées et de croyances trop fermes, de ne pas se lier, corps, cœur et âme, à une conception particulière de l'enseignement des mathématiques.

La vision cumulative de la science évoquée par la citation de Newton n'est pas au goût du jour. On a critiqué la vision absolutiste des mathématiques; la même critique doit s'adresser à une vision absolutiste de la philosophie et de la didactique des mathématiques. Si les mathématiques ne peuvent livrer la vérité et ne peuvent aspirer à la permanence, la philosophie et la didactique des mathématiques le peuvent encore moins (y compris lorsqu'elles rejettent une vision absolutiste..., mais résistons à la tentation de nous enliser!). Ce qui, aujourd'hui, nous apparaît comme la vérité et le bon sens en matière d'enseignement pourra fort bien être perçu comme erroné et ridicule dans quelques années. Je vous propose une expérience : songez aux théories passées de mode (positivisme, modernisme, behaviorisme, structuralisme, etc.) et au jugement que vous portez sur elles. Essayez ensuite de vous placer d'un point de vue futur et de regarder de ce point de vue les théories en vogue actuellement, essayez d'imaginer ce que l'on en dira dans 10 ans, dans 30 ans ou dans un siècle... Toutes les générations ont cru être sur la bonne piste et ont estimé que leurs prédécesseurs s'étaient fourvoyés. Pouvons-nous penser sérieusement que nous sommes l'exception, que *nous* ne faisons pas fausse route, que notre piste est réellement la bonne? La contemplation du sort des théories passées nous incite à la prudence, à l'humilité et au non-attachement (avec modération!). Voilà donc ma leçon du passé : il nous faut regarder nos croyances avec du recul et éviter de les prendre trop au sérieux, de nous prendre trop au sérieux.

D'une part, donc, il semble difficile, en didactique, de se tenir « debout sur les épaules de géants » et d'imaginer les générations futures debout sur les nôtres. D'autre part, comment y renoncer? Quelle serait la valeur d'une discipline où nous rejetterions systématiquement la vision de nos prédécesseurs pour voir ensuite la nôtre mise de coté à son tour par nos successeurs? Ne finirions-nous pas par tourner en rond? N'est-ce pas, du moins un peu, ce que nous faisons? Peut-être, en didactique — et plus généralement en éducation — avons-nous un peu trop soif de changement et souvent tendance à procéder par réaction, à nous définir par opposition à une théorie précédente, ou concurrente, que nous érigeons en rivale et que nous « démonisons » pour nous justifier de la condamner en bloc. Cette attitude est au cœur du drame cyclique de la réforme des programmes, dont le scénario pourrait se résumer comme suit :

1. On juge la situation catastrophique : on prétend qu'au sortir de l'école les élèves ne savent pas grand-chose et comprennent encore moins, que leurs connaissances sont désuètes, inutiles et inapplicables, que l'école ne fournit de préparation convenable ni à la vie ni aux études supérieures. On crie au scandale. Il est toujours possible de dresser un constat d'échec, peu importe la situation : tout est dans la façon de s'y prendre pour observer et mesurer le phénomène;
2. On cherche un coupable. On impute le désastre au curriculum (on pourrait aussi mettre en cause la compétence du personnel enseignant, ce qui renverrait aux faiblesses des programmes de formation des maîtres);
3. On produit un nouveau curriculum en réaction au précédent : on le définit par contraste, on raisonne par dichotomies et on propose des ruptures radicales. On fait table rase. On risque alors de donner aux nouvelles idées, bonnes en soi, une application d'une étendue exagérée, qui les pervertit et les voue à l'échec à leur tour. Comme on dirait en didactique, on les pousse au-delà de leur « domaine de validité »;
4. On implante le nouveau curriculum, souvent rapidement, et on l'évalue, fréquemment de façon prématurée, sans tenir compte du temps nécessaire à un changement en profondeur. (La patience est une autre leçon qu'il nous faudrait apprendre du passé.) Il s'en suit un nouveau constat d'échec et une reprise du cycle[2].

Peut-être qu'une certaine humilité et un peu de détachement à l'égard de nos propres idées nous prédisposeraient à percevoir dans les théories concurrentes autre chose que des défauts et à profiter de tout élément utile qu'elles pourraient contenir; cela nous aiderait également à ne pas vouloir pousser nos idées au-delà de leurs limites et à prévenir ainsi de nouveaux dégâts ou, du moins, à les circonscrire.

L'enseignement, nous disait David Wheeler il y a quatre ans, est une sorte de bricolage, ce n'est pas une science, car cela suppose un consensus sur des théories de base qui est loin d'être atteint, si jamais il devait l'être[3]. J'avais lu une mise en garde similaire dans un petit livre que m'avait offert Fernand Lemay, un de mes collègues qui se sont chargés de ma formation en didactique. Dans cet ouvrage[4], Krishnamurti soutenait qu'aucune méthode ni système ne peut fournir la bonne sorte d'éducation (p. 23). Il recommandait de ne pas penser selon des principes et de ne pas suivre de méthode, car, selon lui, cela conduit à accorder plus d'importance à la méthode qu'à la réalité des élèves (p. 25).

Dans ce qui suit, je développerai cette méditation sur les vicissitudes de l'enseignement des mathématiques à la lumière de la leçon que j'ai dégagée du passé, en prenant comme point de départ mon histoire personnelle.

L'enseignement des mathématiques : les maths modernes, la réforme des programmes et la tradition

Le traité prend les mathématiques à leur début, et donne des démonstrations complètes. Sa lecture ne suppose donc, en principe, aucune connaissance mathématique particulière, mais seulement une certaine habitude du raisonnement mathématique et un certain pouvoir d'abstraction.

– N. Bourbaki, *Éléments de mathématique, Mode d'emploi de ce traité*

Mon tout premier contact avec la didactique des mathématiques, à part mes expériences comme élève, a eu lieu dans le contexte de la réforme des maths modernes, vers la fin des années 60. J'étudiais les mathématiques à l'université, en Italie, et l'on m'avait invitée à donner quelques heures de cours dans le contexte d'une activité de perfectionnement d'enseignantes et d'enseignants. Je devais leur parler d'ensembles, de relations d'équivalence et d'autres notions de ce genre.

J'étais pleine d'enthousiasme pour ces idées que je venais de découvrir, j'étais heureuse de les partager, et le projet de les introduire à l'école me paraissait bon, car il me semblait répondre à un besoin réel. Je me souvenais, par exemple, d'avoir été frustrée, au secondaire, par l'absence d'une définition du mot « fonction ». Il y avait les polynômes, les fonctions trigonométriques, les logarithmes et les exponentielles. En existait-il d'autres? On me disait que oui. Cependant, qu'est-ce que c'était une fonction au juste? Mystère! C'est seulement à l'université que l'on m'avait enfin révélé qu'une fonction était une correspondance univoque entre deux ensembles. (À l'apogée des maths modernes, on envisagera d'enseigner cela au préscolaire!) Cette définition m'avait libérée d'un long malaise. Un peu plus tard j'ai pris connaissance d'une variante de cette définition, à savoir qu'une fonction est un sous-ensemble du produit de deux ensembles respectant certaines conditions, et cette autre formulation m'avait plu également, non seulement parce qu'elle faisait ressortir le lien avec l'idée familière de graphique, mais aussi parce qu'elle livrait d'emblée la fonction tout entière et qu'elle ne contenait aucune suggestion de mouvement (comme ce va-et-vient entre les deux ensembles suggéré par le mot « correspondance » dans la première formulation). Je trouvais cela satisfaisant et apaisant.

En fait, la teneur en maths modernes de mon éducation a été très faible : nulle à l'école, relativement modeste à l'université. Si j'en ai appris un peu plus, c'est en raison de mon initiative personnelle de tenter de lire, je dis bien « tenter », ce fameux traité qui ne supposait, en principe, aucune connaissance mathématique particulière. Cela a été frustrant, bien sûr, mais je n'ai pas été rebutée par la chose comme d'autres qui en ont été gavés en bas âge. Au contraire, je me souviens de moments de réel plaisir, comme lorsque j'ai lu qu'un couple (a,b) pouvait se définir comme l'ensemble $\{a,\{a,b\}\}$. Jusque-là, la notion de couple m'avait agacée, puisque je ne voyais pas comment distinguer (a,b) de (b,a) sans importer en mathématique des notions physiques telles que « droite » et « gauche » ou « avant » et « après » (j'ai toujours aimé mes mathématiques très pures)[5].

Aujourd'hui, il est rare que l'on ne qualifie pas de « faillite » le mouvement des maths modernes et le discours structuraliste qui le sous-tendait. Les exposés systématiques, genre « axiomes, définitions, théorèmes, démonstrations », qu'il s'agisse des *Éléments* d'Euclide ou de ceux de Bourbaki, sont donnés comme exemple d'une mauvaise approche didactique. Pourtant, la motivation, du moins la motivation initiale, derrière ces traités était d'ordre didactique[6]. Leur but était de présenter les concepts et les résultats de base d'une discipline, de façon cohérente, claire et ordonnée, avec un degré optimal (pas nécessairement maximal) de généralité.

N'est-ce pas là des intentions louables? Où est l'erreur? Je ne m'y attarderai pas longtemps, car les critiques sont bien connues : les maths modernes ont donné lieu à de nombreux excès et dérives, mais, même sans cela, cette approche, qui aspirait pourtant à introduire à l'école des « vraies » mathématiques, des mathématiques alignées sur celles qui se pratiquaient à l'université, avait le défaut d'offrir aux élèves un savoir achevé, sans leur montrer par quels chemins on en arrivait à s'intéresser à telle question, à la circonscrire au moyen de tels concepts, définis de telle manière. Elle occultait les raisons du choix des définitions et des axiomes, qui pouvaient paraître alors purement gratuits. Les problèmes auxquels répondaient les théorèmes, les cas particuliers à partir desquels on avait bâti abstractions et généralisations, les stratégies qui avaient permis d'obtenir les résultats, bref tout le processus de création, demeuraient cachés.

Il fallait donc (re)donner aux élèves la chance de se familiariser avec ce processus, de faire appel à leur intuition, d'élaborer les concepts de façon graduelle et de tenter de résoudre des problèmes. Il fallait leur permettre de passer par toutes les étapes qu'une présentation axiomatique les forçait à sauter. Cependant, cet excellent programme risque lui aussi de provoquer des effets pervers, notamment en conduisant à dévaloriser, voire éliminer, l'étape finale de systématisation et d'organisation du savoir.

L'indignation contre l'erreur des maths modernes et l'engouement pour la résolution de problèmes ont entraîné une certaine indifférence, presque de la méfiance, à l'égard de tout ce qui est abstraction, théorie mathématique, système ordonné de résultats. Si avant on négligeait l'activité mathématique, le processus, maintenant on risque d'en oublier le produit. Voilà donc ma première question pour l'avenir : comment trouver et maintenir un juste équilibre entre les deux[7]? Dans la conjoncture actuelle, le défi me semble être d'éviter que les concepts demeurent au stade d'intuitions, emprisonnés dans des représentations qui ne devaient jouer qu'un rôle d'échafaudage[8], et d'éviter que les résultats (les solutions des problèmes) s'accumulent sans que l'on se soucie de les organiser en structures.

« Le mode d'exposition suivi est axiomatique et abstrait; il procède le plus souvent du général au particulier. »[9] Aujourd'hui, cette façon de procéder et l'idée même d'« exposition » sont frappées d'anathème. L'orientation actuelle en didactique veut que l'on procède du particulier au général. Cela paraît avalisé autant par le bon sens que par la recherche. J'ai tout de même à ce sujet des préoccupations de deux ordres : 1) que la nouvelle façon de procéder soit appliquée correctement, que l'on prenne réellement le temps, que l'on fasse vraiment l'effort, de se rendre au « général », que l'on ne se perde pas dans la multitude des « particuliers »; 2) que l'on n'érige pas en dogme une façon unique de procéder. Même si pour l'instant elle semble être la meilleure, elle ne l'est sans doute pas pour tout le monde et en toute occasion.

L'histoire devrait nous inciter à faire preuve de retenue. Nous reconnaissons aisément l'intérêt d'un principe de précaution en repensant à l'aventure des maths modernes et en nous disant : « Ils auraient dû... », mais le même principe vaut tout autant pour les tendances dominantes actuelles! Il est facile, maintenant, d'exhiber des horreurs tirées des manuels scolaires d'époques révolues[10], mais mettons-nous à la place de nos collègues du futur et essayons de regarder le matériel didactique contemporain à travers leurs yeux. Peut-être pourrons-nous déjà entrevoir ce qui leur paraîtra risible. (Je pense, par exemple, à certaines « situations-problèmes significatives »...) Si nous avons de la difficulté à apprendre des leçons du passé, peut-être que voyager mentalement dans l'avenir pour regarder en arrière vers le présent pourra nous aider, pour ainsi dire, à apprendre des leçons du futur!

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J'ai contrasté l'esprit des nouveaux programmes et celui des maths modernes, alors qu'il est plus courant de l'opposer à celui de l'enseignement dit « traditionnel ». Les maths modernes sont un épisode bien délimité dans l'histoire de la didactique, un adversaire déjà vaincu. L'ennemi par rapport auquel se définissent les nouvelles orientations est l'enseignement « traditionnel ». On le dépeint comme la transmission de faits et de techniques au niveau intellectuel le plus bas qui soit. Sans doute cela a-t-il existé, et malheureusement cela existe-t-il encore, mais supposer une uniformité dans la tradition plurimillénaire de l'enseignement des mathématiques est une simplification inacceptable. En fait, cette tradition contient *déjà* les principales idées qui animent la réforme actuelle.

Il y a un siècle, par exemple, Mary Boole, épouse de George Boole, prônait et pratiquait déjà une approche par découverte[11] :

> For mathematical purposes, all influence from without, which induces the pupils to admit a principle as valid before his own unbiased reason recognises its truth, come under the same condemnation (p. 9).
>
> Qualities of a teacher [...] Great reserve on the part of the teacher in even stating to pupils the special conclusions to which he has been led, lest he should arrest the normal exercise of their investigating faculties (p. **1**).
>
> [The teacher's] object should be to efface himself, his books, and his systems; to draw aside a curtain from between the child and the process of discovery, and to leave the young soul alone with pure Truth (p. 14)

L'auteure de ces propos n'était pas une visionnaire isolée. Au contraire, Boole fait allusion à des théories éducatives de son époque plus radicales que les siennes, condamnant *tout* apprentissage mécanique, théories dont elle se démarque en adoptant une position plus modérée, en reconnaissant, à côté de moments privilégiés d'apprentissage pleinement conscient, l'utilité de périodes d'entraînement (p. 15).

Le caractère récurrent des préoccupations à propos de l'enseignement des mathématiques ressort bien d'un autre passage du même texte. Boole y rapporte l'indignation d'un professeur devant l'incapacité des étudiants à se servir de leurs connaissances mathématiques dans leurs études de génie ou de physique. Cela, poursuit-elle, a rallumé l'intérêt pour une question qui avait été négligée pendant une ou deux générations, mais qui avait retenu l'attention de savants *60 ans plus tôt*, à savoir (p. 20) : « What are the conditions which favour a vital knowledge of mathematics? ». L'idée de connaissances vivantes me semble très proche du discours contemporain sur la compétence à se servir des mathématiques dans des contextes variés.

Un autre aspect des nouveaux programmes que l'on peut retracer dans la tradition concerne justement l'accent mis sur l'utilité des mathématiques. La part importante du « temps d'antenne » réservé aux mathématiques à l'école se justifie par le fait qu'elles « sont partout » et qu'elles sont devenues indispensables à la vie en société. Axer l'enseignement des mathématiques sur leur utilité n'était certainement pas le souci des promoteurs des maths modernes, mais ce n'est pas non plus une idée nouvelle. À ce propos, la tradition oscille entre deux pôles. D'une part, l'étude des mathématiques est conçue comme une poursuite intellectuelle gratuite, « pour l'honneur de l'esprit humain », comme l'écrivait Jacobi et le répétait Dieudonné[12]. Une anecdote célèbre illustre bien cette vision. On raconte qu'un élève, après avoir appris un théorème, a demandé à Euclide à quoi cela lui servirait. Euclide aurait ordonné alors à son esclave de donner une pièce de monnaie au garçon, puisque ce dernier avait besoin de tirer un avantage de ce qu'il apprenait[13]. L'objet de l'histoire est de promouvoir une attitude désintéressée et idéaliste envers le savoir. On peut y voir aussi, et dénoncer, une attitude arrogante et élitiste d'hommes privilégiés[14], mais il reste que tout le monde est en droit d'aspirer à une part de loisir à consacrer, éventuellement, à la spéculation gratuite.

D'autre part, la tradition comprend aussi, et depuis longtemps, une vision plus utilitaire des mathématiques. Sans parler des mathématiques babyloniennes qui s'exprimaient essentiellement par des problèmes de nature économique, même dans la Grèce classique, il semble que l'éducation mathématique des jeunes, jusqu'à 14 ans, avait une orientation surtout pratique[15]. Par la suite, pour les élèves de 14 à 18 ans, l'inclusion dans le curriculum de matières plus abstraites, comme l'astronomie et la géométrie, et les dangers d'en pousser l'étude trop loin faisaient l'objet de discussions. Platon, qui souhaitait enrichir le contenu mathématique des programmes d'études, citait la meilleure éducation des enfants égyptiens, ce qui n'est pas sans rappeler le rôle des comparaisons internationales dans les débats contemporains...

Qu'est-ce donc que la tradition? Un héritage culturel précieux à chérir ou une tyrannie étouffante contre laquelle on doit se révolter si l'on veut progresser?

Pour améliorer quoi que ce soit, nos conditions de vie ou l'enseignement des mathématiques, il faut innover (l'inverse n'est pas vrai), et pour innover, il faut s'écarter de la tradition, c'est évident. D'où la connotation négative du terme « traditionnel[16] ». Cependant, une tradition aussi ancienne, riche et variée que celle de l'enseignement des mathématiques, une tradition qui contient tout et son contraire, n'est sans doute pas à mettre au rancart en bloc! Et, ne l'oublions pas, même ce qui nous semble dépassé pourra être revalorisé plus tard. Pensons au domaine artistique, où les styles sont constamment réévalués.

Il en va ainsi, en éducation, des valeurs et des méthodes d'enseignement. Il ne faudrait alors peut-être pas oublier entièrement certaines valeurs démodées, telles la clarté et la vérité. La clarté était jadis parmi les qualités considérées comme les plus désirables pour l'enseignement. Quant à la vérité, elle a été, traditionnellement, la qualité idéale de la connaissance attendue par les élèves et dispensée par les maîtres. C'était ce que j'attendais et recherchais quand j'étais élève. Cela n'exclut pas l'esprit critique; au contraire, celui-ci doit être bien éveillé pour tester la vérité des connaissances proposées. Si l'école renonce à dispenser la vérité, elle laissera un vide, un désir insatisfait qu'il faudra combler par d'autres moyens. D'ailleurs, que cherchons-nous lorsque nous faisons de la recherche, si ce ne sont des connaissances vraies? Bien sûr, dans le domaine intellectuel, la vérité n'est pas absolue, elle varie selon les points de vue, mais cette affirmation aussi est une vérité. L'existence de plusieurs niveaux de vérité et son caractère relatif n'impliquent pas que ce concept soit dépourvu d'intérêt.

Il ne faudrait pas écarter non plus la possibilité de façons d'apprendre autres que celles qui tiennent actuellement la vedette, soit la résolution de problèmes, la recherche personnelle ou collective, l'exploration, la découverte et la discussion. Il n'est pas impossible d'apprendre aussi par l'écoute, l'observation, l'imitation, la lecture, l'entraînement, la pratique, voire la mémorisation[17]. N'avons-nous pas appris nous-mêmes par un mélange de ces approches? Pourquoi rejeter ce qui a fonctionné pour nous? Personnellement, je crois que j'ai beaucoup appris des livres. À l'occasion, même de livres qui se situaient au-delà de ma « zone de développement proximale ». Je ne comprenais pas, mais je voulais comprendre, je me disais qu'un jour je comprendrais. Ces livres constituaient pour moi un but, un horizon vers lequel marcher.

En somme, pour ce qui est de la tradition, il me semble qu'un regard sur l'histoire, même un regard sur quelques fragments seulement, nous incite à faire preuve de prudence, d'une part, afin de ne pas créer une image stéréotypée de ce qu'est la tradition de l'enseignement des mathématiques et, d'autre part, afin de ne pas rejeter entièrement et définitivement des éléments de cette tradition qui pourraient s'avérer encore profitables.

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Revenons maintenant à la réforme actuelle des programmes. J'ai parlé de l'importance de rechercher un équilibre entre processus et produit, entre créativité et systématisation du savoir. Une autre problématique où prudence et modération me semblent nécessaires concerne l'utilisation des mathématiques. La question que je voudrais poser à l'avenir à ce sujet est la suivante : comment éviter l'abus des mathématiques, la mathématisation à outrance, à tort et à travers? Y a-t-il moyen de contrer cette tendance par l'éducation? La nouvelle insistance sur les situations-problèmes favorisera-t-elle une attitude critique à ce propos ou, au contraire, contribuera-t-elle à empirer la situation?

Je ne veux pas parler ici de la fabrication d'armes, du clonage de monstres ou de la survente de billets d'avion. Une éducation critique peut sensibiliser au rôle des mathématiques dans tout cela, mais je doute que la didactique offre des moyens d'endiguer le mal. Le problème que je veux soulever est moins grave, mais quand même irritant et davantage de notre ressort. Il s'agit de l'emploi de formules, d'images ou de termes mathématiques à mauvais escient, là où ils n'apportent rien à la compréhension d'une situation et deviennent même une source de confusion. Cette pratique revient, encore une fois, à pousser trop loin une bonne idée et nuit au projet de montrer aux élèves l'utilité des mathématiques et de leur apprendre à s'en servir.

La prolifération de schémas inspirés des maths modernes en constitue un exemple anodin, mais typique. Désormais, il ne reste plus rien, à l'école primaire (et bien peu au secondaire), de ces notions de théorie des ensembles qui ont été la marque de commerce des maths modernes et la cible des railleries de leurs détracteurs[18]. Entre-temps, par contre, certains éléments du langage graphique qui les accompagnait, comme les diagrammes de Venn, sont passés, dénués de leur sens, dans l'usage courant. Ironiquement, on en trouve de nombreux exemples parmi les illustrations du nouveau programme pour le préscolaire et le primaire au Québec[19], programme qui, justement, a évacué du contenu d'étude les dernières traces de ce langage! Loin d'éclairer quoi que ce soit, la plupart de ces ovales et de ces flèches jouent, au mieux, un rôle purement décoratif. Souvent ils trahissent et encouragent un flou intellectuel qui se traduit par des schémas dans lesquels des flèches colorées remplacent des connexions logiques que l'on aurait du mal à expliciter.

Pourquoi s'inquiéter de ces pratiques maintenant, à l'heure de l'implantation d'un curriculum qui met l'accent justement sur l'utilisation des mathématiques? Celui-ci ne devrait-il pas éduquer à en faire un usage judicieux? En principe, oui. Cependant, les contextes qui se prêtent à des activités adaptées aux élèves et dans lesquels les mathématiques jouent un rôle véritablement significatif ne sont pas si faciles à trouver. Je crains — à tort, je l'espère — que devant cette pénurie on ne se rabatte sur des situations artificielles où l'on plaque des éléments mathématiques sans trop se soucier de la pertinence de l'opération, comme l'on met des schémas inspirés des diagrammes de Venn en guise d'illustration d'un texte.

Ne nous faisons pas d'illusions, l'introduction à l'école des mathématiques appliquées, pour ne parler que de cet aspect de la réforme actuelle, demande au personnel enseignant un effort majeur de perfectionnement, un effort comparable à celui qui a été exigé à l'époque par les maths modernes. Je me souviens encore très bien du sentiment d'incompétence que j'ai éprouvé au début de ma carrière lorsque j'enseignais la géométrie projective et que des étudiants d'architecture m'ont posé une question pratique, portant sur la couverture d'un toit. J'ai oublié la question, mais je me rappelle que je me suis sentie paralysée, entièrement dépourvue de moyens pour aborder une question de ce genre. Je crois que beaucoup d'enseignants et d'enseignantes n'ont pas plus de préparation à cet égard, aujourd'hui, que je n'en avais alors, après quatre ans de spécialisation en mathématiques, ou que leurs collègues d'antan n'en avaient sur la théorie des ensembles, l'algèbre linéaire et l'algèbre abstraite.

C'est le moment de tirer profit des leçons de l'histoire. Le perfectionnement est indispensable, mais il n'est pas suffisant pour le déploiement optimal d'un curriculum radicalement nouveau. S'il est vrai, comme l'a rappelé Vicki Zack dans un commentaire écrit remis à la fin de la séance, que l'on peut apprendre, changer et se développer tout au long de sa vie, je pense que ce ne seront pas tous les enseignants et les enseignantes qui le feront, et qu'il est parfois difficile de se débarrasser d'habitudes acquises dans sa jeunesse.

Ce n'est que lorsqu'une génération entière d'enseignantes et d'enseignants aura reçu une formation *initiale* en accord avec le nouveau curriculum que celui-ci pourra donner sa pleine mesure. D'ici là, il faudra traverser quelques décennies de transition pendant lesquelles la plupart des maîtres n'en sauront pas beaucoup plus que ce qui se trouve dans les manuels de leurs élèves et garderont une vision des mathématiques et de leur enseignement plus ou moins décalée par rapport aux nouvelles orientations. Dans le passé, nous n'avons pas eu la patience d'attendre une si longue période. L'aurons-nous cette fois-ci? Choisirons-nous d'endurer les ratés inévitables et de les corriger graduellement ou en ferons-nous un argument pour condamner la réforme et changer de cap une fois de plus?

Conclusion

> *A popular misconception is that we can't change the past—*
> *everyone is constantly changing their own past, recalling it,*
> *revising it. What really happened? A meaningless question. But*
> *one I keep trying to answer, knowing there is no answer.*
>
> – Margaret Laurence, *The Diviners*[20]

Tirer des leçons de l'histoire est une démarche éminemment subjective, qui consiste à interpréter des souvenirs. Encore faut-il que ces derniers soient disponibles. Malheureusement, la mémoire humaine est tout sauf fiable et les documents se révèlent souvent ambigus ou incomplets, certains ayant été perdus ou détruits, d'autres étant devenus indéchiffrables, sans compter toutes les pensées et tous les événements qui n'ont pas été enregistrés et dont la mémoire ne s'est pas transmise. Nous ne saurons jamais ce qui s'est passé dans toutes les classes de mathématiques et quels en ont été les effets. Les souvenirs que nous interprétons posent eux-mêmes problème. L'incertitude n'est pas l'apanage de l'avenir.

La prudence est donc souhaitable, aussi, lorsque nous reconstruisons le passé. Difficile de savoir ce qui est arrivé « en réalité », si tant est que la question ait un sens. Nous suivons notre tendance à créer des récits cohérents, et cela nous permet d'y lire des leçons. Je vous en ai proposé une, à vous maintenant de fouiller dans vos souvenirs pour en trouver d'autres.

Notes

1. Cesare Pavese, *Il mestiere di vivere (Diario 1935-1950)*, Torino, Giulio Einaudi editore, 1964, p. 99. Il s'agit de l'entrée datée du 25 avril 1938 : « Pourquoi — quand on s'est trompé — dit-on « une autre fois, je saurai comment faire », quand on devrait dire : « une autre fois, je sais déjà comment je ferai »? » Cesare Pavese, *Le métier de vivre*, traduit de l'italien par Michel Arnaud, Paris, Gallimard, 1987, p. 120.
2. Parfois, comme l'a rappelé Lesley Lee lors de la période de discussion, ce scénario n'est pas suivi et on lance une réforme sans fournir de justification. Une culture qui valorise le changement et la nouveauté fait que cette pratique rencontre peu d'opposition.
3. David Wheeler, « The commonsense of teaching », dans Yvonne M. Pothier (dir.), *Proceedings of the Annual Meeting of the Canadian Mathematics Education Study Group, University of British Columbia, May 29–June 2, 1998*, Mount Saint Vincent University Press, 1998, p. 98.
4. J. Krishnamurti, *Education & the Significance of Life*, San Francisco, Harper & Row, 1981/1953, p. 125.
5. Voilà une autre leçon du passé, du moins de mon passé : l'apprentissage qui fait le plus plaisir est habituellement celui qui résout un malaise préexistant, ou satisfait une curiosité préexistante.
6. Frédéric Patras, *La pensée mathématique contemporaine*, Paris, Presses universitaires de France, 2001, p. 16; Maurice Mashaal, « Bourbaki. Une société secrète de mathématiciens. Un groupe se forme », *Pour la science*, fév.-mai 2000, p. 6–9; *Le matin des mathématiciens, entretiens sur l'histoire des mathématiques présentés par* Émile Noël, Paris, Éditions Belin-Radio France, 1985, p. 43.
7. D'après Frederick Leung, une plus grande attention accordée au produit, plutôt qu'au

processus, est une des caractéristiques qui distinguent l'enseignement des mathématiques en Asie de l'Est (les pays de culture confucianiste) de celui qui se pratique en Occident (les pays anglo-saxons). Cette plus grande attention au produit, ainsi que tous les autres traits qui, selon Leung, définissent l'identité asiatique dans l'enseignement des mathématiques, me semble pourtant se retrouver aussi, avec autant de relief, dans la tradition « occidentale » : Frederick K.S. Leung, « In search of an East Asian identity in mathematics education », *Educational Studies in Mathematics*, vol. 47, nº 1, 2001, p. 35–51. Par ailleurs, j'ai appris de mon collègue Christian Laville que, dans l'enseignement de l'histoire, il existe une tension similaire entre curriculums modernes centrés sur la pensée historique (le processus) et curriculums anciens centrés sur le récit historique (le produit).

8. Roberta Mura, « L'épaisseur d'un décimètre carré », *La Revue canadienne de l'enseignement des sciences, des mathématiques et des technologies*, vol. 1, nº 3, 2001, p. 291–303.
9. N. Bourbaki, Éléments de mathématique, Mode d'emploi de ce traité.
10. Voir, par exemple, Maurice Mashaal, « Bourbaki. Une société secrète de mathématiciens. Les « maths modernes » à l'école », *Pour la science*, fév.-mai 2000, p. 83.
11. D.G. Tahta, *A Boolean Anthology. Selected Writings of Mary Boole on Mathematical Education*, Derby, U.K., The Association of Teachers of Mathematics, 1972. Merci à David Pimm d'avoir cité ce texte dans un de ses articles et de m'avoir ainsi permis de le découvrir.
12. C.G.J. Jacobi, *Gesammelte Werke*, vol. 1, Berlin, 1881, cité par Frédéric Patras, *op. cit.*, p. 4, note 4. Dieudonné a intitulé un de ses ouvrages *Pour l'honneur de l'esprit humain* : Jean-Alexandre Dieudonné, *Pour l'honneur de l'esprit humain : les mathématiques aujourd'hui*, Paris, Hachette, 1987.
13. Thomas L. Heath, *Greek Mathematics*, New York, Dover Publications, 1963, p. 10.
14. On répète souvent et avec un peu trop de désinvolture que, si les mathématiques telles que nous les connaissons ont été façonnées dans la Grèce classique, c'est parce que les penseurs grecs étaient « des hommes libres » : Denis Guedj, *Le théorème du perroquet*, Paris, Éditions du Seuil, 1998, p. 179, cité par Bernard Hodgson, « Pourquoi enseigner les mathématiques à tous ? », dans Elaine Simmt, Brent Davis et John Grant McLoughlin (dir.), *Proceedings of the Annual Meeting of the Canadian Mathematics Education Study Group, Université du Québec à Montréal, May 26–30, 2000*, p. 164. Lorsqu'on tient des propos de ce genre, il faudrait prendre soin de ne pas contribuer à perpétuer une image idéalisée de cette société antique et rappeler que ces « *hommes* libres » devaient leur liberté non pas à une supposée démocratie, mais au travail des femmes et des esclaves.
15. Thomas L. Heath, *op. cit.*, p. 7–10.
16. Dans d'autres contextes, pourtant — dans l'artisanat par exemple —, ce terme n'a pas la même connotation.
17. Selon Leung (voir la note 7), le rôle accordé à la mémorisation, même avant qu'une pleine compréhension soit atteinte, est le deuxième aspect qui caractérise l'enseignement des mathématiques en Asie de l'Est par rapport à ce qui se fait en Occident. Le troisième élément est la vision de l'étude comme un travail sérieux, difficile et pénible. Leung contraste cela et la recherche, en Occident, d'une manière d'apprendre qui soit agréable, voire amusante. Encore une fois, je noterais que les caractéristiques que Leung attribue à la vision occidentale de l'enseignement des mathématiques n'en résument pas toute la tradition, celle-ci comprenant également des courants qui accordent autant d'importance aux aspects que cet auteur considère comme typiques de l'approche asiatique. Il suffit de penser, par exemple, à la célèbre réponse de Ménechme à Alexandre (ou d'Euclide à Ptolomée) à savoir qu'il n'existe pas de chemin royal en géométrie : Thomas L. Heath, *op. cit.*, p. 158.
18. Voir par exemple la caricature intitulée « Intersection d'un bébé et d'une pomme de terre », dans Didier Nordon, *Les mathématiques pures n'existent pas!*, Paris, Actes Sud, 1981, p. 6.
19. Ministère de l'Éducation du Québec, *Programme de formation de l'école québécoise. Éducation préscolaire. Enseignement primaire*, 2001. Voir tous les schémas, notamment ceux des pages 8, 43, 99, 125, 197, 253 et 257.
20. Margaret Laurence, *The Diviners*, Toronto, McClelland & Stewart, 1988/1974, p. 70.

Appendices

Appendices

APPENDIX A

Working Groups at Each Annual Meeting

1977 *Queen's University, Kingston, Ontario*
- Teacher education programmes
- Undergraduate mathematics programmes and prospective teachers
- Research and mathematics education
- Learning and teaching mathematics

1978 *Queen's University, Kingston, Ontario*
- Mathematics courses for prospective elementary teachers
- Mathematization
- Research in mathematics education

1979 *Queen's University, Kingston, Ontario*
- Ratio and proportion: a study of a mathematical concept
- Minicalculators in the mathematics classroom
- Is there a mathematical method?
- Topics suitable for mathematics courses for elementary teachers

1980 *Université Laval, Québec, Québec*
- The teaching of calculus and analysis
- Applications of mathematics for high school students
- Geometry in the elementary and junior high school curriculum
- The diagnosis and remediation of common mathematical errors

1981 *University of Alberta, Edmonton, Alberta*
- Research and the classroom
- Computer education for teachers
- Issues in the teaching of calculus
- Revitalising mathematics in teacher education courses

1982 *Queen's University, Kingston, Ontario*
- The influence of computer science on undergraduate mathematics education
- Applications of research in mathematics education to teacher training programmes
- Problem solving in the curriculum

1983 *University of British Columbia, Vancouver, British Columbia*
- Developing statistical thinking
- Training in diagnosis and remediation of teachers
- Mathematics and language
- The influence of computer science on the mathematics curriculum

1984 *University of Waterloo, Waterloo, Ontario*
- Logo and the mathematics curriculum
- The impact of research and technology on school algebra
- Epistemology and mathematics
- Visual thinking in mathematics

1985 *Université Laval, Québec, Québec*
- Lessons from research about students' errors
- Logo activities for the high school
- Impact of symbolic manipulation software on the teaching of calculus

1986 *Memorial University of Newfoundland, St. John's, Newfoundland*
- The role of feelings in mathematics
- The problem of rigour in mathematics teaching
- Microcomputers in teacher education
- The role of microcomputers in developing statistical thinking

1987 *Queen's University, Kingston, Ontario*
- Methods courses for secondary teacher education
- The problem of formal reasoning in undergraduate programmes
- Small group work in the mathematics classroom

1988 *University of Manitoba, Winnipeg, Manitoba*
- Teacher education: what could it be?
- Natural learning and mathematics
- Using software for geometrical investigations
- A study of the remedial teaching of mathematics

1989 *Brock University, St. Catharines, Ontario*
- Using computers to investigate work with teachers
- Computers in the undergraduate mathematics curriculum
- Natural language and mathematical language
- Research strategies for pupils' conceptions in mathematics

1990 *Simon Fraser University, Vancouver, British Columbia*
- Reading and writing in the mathematics classroom
- The NCTM "Standards" and Canadian reality
- Explanatory models of children's mathematics
- Chaos and fractal geometry for high school students

1991 *University of New Brunswick, Fredericton, New Brunswick*
- Fractal geometry in the curriculum
- Socio-cultural aspects of mathematics
- Technology and understanding mathematics
- Constructivism: implications for teacher education in mathematics

1992 *ICME–7, Université Laval, Québec, Québec*

1993 *York University, Toronto, Ontario*
- Research in undergraduate teaching and learning of mathematics
- New ideas in assessment
- Computers in the classroom: mathematical and social implications
- Gender and mathematics
- Training pre-service teachers for creating mathematical communities in the classroom

1994 *University of Regina, Regina, Saskatchewan*
- Theories of mathematics education
- Pre-service mathematics teachers as purposeful learners: issues of enculturation
- Popularizing mathematics

1995 *University of Western Ontario, London, Ontario*
- Autonomy and authority in the design and conduct of learning activity
- Expanding the conversation: trying to talk about what our theories don't talk about
- Factors affecting the transition from high school to university mathematics
- Geometric proofs and knowledge without axioms

1996 *Mount Saint Vincent University, Halifax, Nova Scotia*
- Teacher education: challenges, opportunities and innovations
- Formation à l'enseignement des mathématiques au secondaire: nouvelles perspectives et défis
- What is dynamic algebra?
- The role of proof in post-secondary education

1997 *Lakehead University, Thunder Bay, Ontario*
- Awareness and expression of generality in teaching mathematics
- Communicating mathematics
- The crisis in school mathematics content

1998 *University of British Columbia, Vancouver, British Columbia*
- Assessing mathematical thinking
- From theory to observational data (and back again)
- Bringing Ethnomathematics into the classroom in a meaningful way
- Mathematical software for the undergraduate curriculum

1999 *Brock University, St. Catharines, Ontario*
- Information technology and mathematics education: What's out there and how can we use it?
- Applied mathematics in the secondary school curriculum
- Elementary mathematics
- Teaching practices and teacher education

2000 *Université du Québec à Montréal, Montréal, Québec*
- Des cours de mathématiques pour les futurs enseignants et enseignantes du primaire/Mathematics courses for prospective elementary teachers
- Crafting an algebraic mind: Intersections from history and the contemporary mathematics classroom
- Mathematics education et didactique des mathématiques : y a-t-il une raison pour vivre des vies séparées?/Mathematics education et didactique des mathématiques: Is there a reason for living separate lives?
- Teachers, technologies, and productive pedagogy

2001 *University of Alberta, Edmonton, Alberta*
- Considering how linear algebra is taught and learned
- Children's proving
- Inservice mathematics teacher education
- Where is the mathematics?

APPENDIX B

Plenary Lectures at Each Annual Meeting

1977	A.J. COLEMAN	The objectives of mathematics education
	C. GAULIN	Innovations in teacher education programmes
	T.E. KIEREN	The state of research in mathematics education
1978	G.R. RISING	The mathematician's contribution to curriculum development
	A.I. WEINZWEIG	The mathematician's contribution to pedagogy
1979	J. AGASSI	The Lakatosian revolution*
	J.A. EASLEY	Formal and informal research methods and the cultural status of school mathematics*
1980	C. GATTEGNO	Reflections on forty years of thinking about the teaching of mathematics
	D. HAWKINS	Understanding understanding mathematics
1981	K. IVERSON	Mathematics and computers
	J. KILPATRICK	The reasonable effectiveness of research in mathematics education*
1982	P.J. DAVIS	Towards a philosophy of computation*
	G. VERGNAUD	Cognitive and developmental psychology and research in mathematics education*
1983	S.I. BROWN	The nature of problem generation and the mathematics curriculum
	P.J. HILTON	The nature of mathematics today and implications for mathematics teaching*
1984	A.J. BISHOP	The social construction of meaning: A significant development for mathematics education?*
	L. HENKIN	Linguistic aspects of mathematics and mathematics instruction
1985	H. BAUERSFELD	Contributions to a fundamental theory of mathematics learning and teaching
	H.O. POLLAK	On the relation between the applications of mathematics and the teaching of mathematics
1986	R. FINNEY	Professional applications of undergraduate mathematics
	A.H. SCHOENFELD	Confessions of an accidental theorist*
1987	P. NESHER	Formulating instructional theory: the role of students' misconceptions*
	H.S. WILF	The calculator with a college education
1988	C. KEITEL	Mathematics education and technology*
	L.A. STEEN	All one system

1989	N. BALACHEFF	Teaching mathematical proof: The relevance and complexity of a social approach
	D. SCHATTSNEIDER	Geometry is alive and well
1990	U. D'AMBROSIO	Values in mathematics education*
	A. SIERPINSKA	On understanding mathematics
1991	J .J. KAPUT	Mathematics and technology: Multiple visions of multiple futures
	C. LABORDE	Approches théoriques et méthodologiques des recherches françaises en didactique des mathématiques
1992	ICME-7	
1993	G.G. JOSEPH	What is a square root? A study of geometrical representation in different mathematical traditions
	J CONFREY	Forging a revised theory of intellectual development: Piaget, Vygotsky and beyond*
1994	A. SFARD	Understanding = Doing + Seeing ?
	K. DEVLIN	Mathematics for the twenty-first century
1995	M. ARTIGUE	The role of epistemological analysis in a didactic approach to the phenomenon of mathematics learning and teaching
	K. MILLETT	Teaching and making certain it counts
1996	C. HOYLES	Beyond the classroom: The curriculum as a key factor in students' approaches to proof*
	D. HENDERSON	Alive mathematical reasoning
1997	R. BORASSI	What does it really mean to teach mathematics through inquiry?
	P. TAYLOR	The high school math curriculum
	T. KIEREN	Triple embodiment: Studies of mathematical understanding-in-inter-action in my work and in the work of CMESG/GCEDM
1998	J. MASON	Structure of attention in teaching mathematics
	K. HEINRICH	Communicating mathematics or mathematics storytelling
1999	J. BORWEIN	The impact of technology on the doing of mathematics
	W. WHITELEY	The decline and rise of geometry in 20th century North America
	W. LANGFORD	Industrial mathematics for the 21st century
	J. ADLER	Learning to understand mathematics teacher development and change: Researching resource availability and use in the context of formalised INSET in South Africa
	B. BARTON	An archaeology of mathematical concepts: Sifting languages for mathematical meanings
2000	G. LABELLE	Manipulating combinatorial structures
	M. BARTOLINI BUSSI	The theoretical dimension of mathematics: A challenge for didacticians
2001	O. SKOVSMOSE	Mathematics in action: A challenge for social theorising
	C. ROUSSEAU	Mathematics, a living discipline within science and technology

NOTE

*These lectures, some in a revised form, were subsequently published in the journal *For the Learning of Mathematics.*

APPENDIX C

Proceedings of Annual Meetings

Past proceedings of CMESG/GCEDM annual meetings have been deposited in the ERIC documentation system with call numbers as follows:

Proceedings of the 1980 Annual Meeting	ED 204120
Proceedings of the 1981 Annual Meeting	ED 234988
Proceedings of the 1982 Annual Meeting	ED 234989
Proceedings of the 1983 Annual Meeting	ED 243653
Proceedings of the 1984 Annual Meeting	ED 257640
Proceedings of the 1985 Annual Meeting	ED 277573
Proceedings of the 1986 Annual Meeting	ED 297966
Proceedings of the 1987 Annual Meeting	ED 295842
Proceedings of the 1988 Annual Meeting	ED 306259
Proceedings of the 1989 Annual Meeting	ED 319606
Proceedings of the 1990 Annual Meeting	ED 344746
Proceedings of the 1991 Annual Meeting	ED 350161
Proceedings of the 1993 Annual Meeting	ED 407243
Proceedings of the 1994 Annual Meeting	ED 407242
Proceedings of the 1995 Annual Meeting	ED 407241
Proceedings of the 1996 Annual Meeting	ED 425054
Proceedings of the 1997 Annual Meeting	ED 423116
Proceedings of the 1998 Annual Meeting	ED 431624
Proceedings of the 1999 Annual Meeting	ED 445894
Proceedings of the 2000 Annual Meeting	not available
Proceedings of the 2001 Annual Meeting	not available

NOTES

1. There was no Annual Meeting in 1992 because Canada hosted the Seventh International Conference on Mathematical Education that year.
2. *Proceedings of the 2002 Annual Meeting* have been submitted to ERIC.