CANADIAN MATHEMATICS EDUCATION STUDY GROUP

GROUPE CANADIEN D'ÉTUDE EN DIDACTIQUE DES MATHÉMATIQUES

PROCEEDINGS / ACTES
2003 ANNUAL MEETING

Acadia University
May 30 – June 3, 2003

EDITED BY:
Elaine Simmt, *University of Alberta*
Brent Davis, *University of Alberta*

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27th Annual Meeting
Acadia University
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Contents / Table des matières

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Schedule

	Vendredi/ Friday May 30 mai	**Samedi/ Saturday May 31 mai**	**Dimanche/ Sunday June 01 juin**	**Lundi/ Monday June 02 juin**	**Mardi/ Tuesday June 03 juin**
Matin/ Morning		**09h00–12h15** Groupes de travail / Working Groups			**9h00–11h00** Table rond/Panel Discussion
					11h30–12h30 Séance de cloture/ Closing Session
Aprés-midi/ After-noon		**12h15–13h45** Dîner / Lunch			
		13h45–14h30 Petits groupes/ Small Groups	**13h45–14h45** Conférence 2/ Plenary 2: Sierpinska	**13h45–14h30** Petits groupes/ Small Groups	
		14h30–15h30 Discussion de/of Archibald	**14h45–15h45** Groupes thématiques/ Topic Groups	**14h30–15h30** Discussion de/of Sierpinska	
	15h30–16h30 Registration/ Inscription				
	16h30–17h30 Ouverture/ Opening Session	**16h00–17h00** Groupes thématiques/ Topic Groups	**16h15–17h15** New PhDs/Thèses	**16h00–17h30** AGA / AGM	
	17h30–19h00 Souper/Supper	**17h00–18h30** Teacher Workshops ou/or Grand Pré	Grand Pré	**17h30–18h30** Ad Hocs	
Soir/ Evening	**19h00–20h00** Conférence 1/ Plenary 1: Archibald	**18h30–20h00** Souper/Supper			
	20h00 Dessert & Social				

Introduction

Malgorzata Dubiel - President, CMESG/GCEDM
Simon Fraser University

It is my great pleasure to write an introduction to the *CMESG/GCEDM Proceedings* from the 2003 meeting, held at Acadia University in Wolfville, Nova Scotia.

Alas, this volume cannot possibly convey the spirit of the meeting it reports on. It can merely describe the content of activities without giving much of the flavour of the process. Therefore, this introduction will attempt to tell you a little about our organization, and about our meetings.

CMESG/GCDEM is unlike other professional organizations: One belongs to it not because of who one is professionally, but because of one's interests. And that is why our members come from mathematics departments and faculties of education, from universities and colleges, and from schools and other educational institutions. What unites them is an interest in mathematics and how it is taught at every level, and a desire to make teaching more exciting, more relevant, and more meaningful.

Our meetings are unique, too. One does not simply attend a CMESG meeting the way one attends other professional meetings, by coming to listen to a few chosen talks. You are immediately part of it; you live and breathe it. And, even though the format, embedded in tradition, does not change much from year to year, there are always new and sometimes unexpected additions, to make each meeting special. The Acadia meeting was no exception!

Working Groups form the core of each CMESG meeting. Participants choose one of several possible topics and, for three days, become members of a community that meets three hours every day to exchange ideas and knowledge and, through discussions that often continue beyond the allotted time, create fresh knowledge and insights. Throughout the three days, the group becomes much more than a sum of its parts—often in ways totally unexpected to its leaders. The leaders, after working for months prior to the meeting, may see their carefully prepared plan ignored or put aside by the group, and a completely new picture emerging in its stead.

The topics of the working groups are chosen and the leaders invited long in advance of the publication of the meeting announcement, to allow the leaders time to prepare, and the participants time to reflect and choose the group they wish to join, which is not an easy task. In Acadia, however, in addition to the three "official" working groups announced in the program, the executive approved a last-minute request from a group of "rebels" to create a fourth one: an "ad-hoc" working group on the Canadian school mathematics curriculum. The idea for this group originated at the Canadian Forum on School Mathematics, which took place two weeks earlier in Montreal. CMESG was a partner in the Forum, and we decided that such a group will be very timely, because of the importance of the Forum and its possible impact on the school mathematics in Canada. This group brought us the document "A Mathematics Curriculum Manifesto".

Two plenary talks are always part of the conference. Traditionally, one of the talks is given by a mathematician, and the other by a mathematics educator. This reflects both the membership and the focus of our organization. In the spirit of CMESG meetings, a plenary talk is not just a talk, but a mere beginning. It is followed by discussions in small groups, which prepare questions for the speaker. After the small group discussions, in a renewed plenary session, the speaker fields the collectively generated questions. The plenary speakers participate in the whole meeting; some of them afterwards become members and strong supporters of our group.

Topic Groups and Ad Hoc presentations provide further possibilities for the exchange of ideas and reflections. Shorter in duration than the Working Groups, Topic Groups are sessions in which individual members present works in progress and often find inspiration and new insight from their colleagues' comments. One of the Acadia Topic Groups consisted of a report and a discussion on the Canadian Forum on School Mathematics. But by far the most memorable Topic Groups this year were the two presentations given by Zoltan Dienes, who joined the meeting to share some of his theoretical insights and teaching activities.

Ad hoc sessions are opportunities to share ideas, which are often not even "half-baked"—sometimes born during the very meeting at which they are presented.

A traditional part of each meeting is the recognition of new PhD's. Those who completed their dissertations in the last year are invited to speak on their work. This gives the group a wonderful opportunity to see the future of mathematics education in Canada.

The meeting ended with a panel on recent curriculum changes across Canada, with members representing the four major regions of the country: Western Provinces, Ontario, Quebec, and Maritime Provinces.

Local organizer, David Reid, realized that that the meeting will provide him with access to his colleagues' knowledge, experience and enthusiasm, and recruited several conference participants to take part in the workshop for local teachers, which accompanied the conference.

Such a rich program did not provide for much free time. But the participants were presented with several exciting possibilities to fill the free time they had. Those who came early to Wolfville were invited to admire the beauty of Acadia on a hike to Cape Split. An excursion to Grand Pré National Historic Site provided an opportunity to explore Acadian history. The evenings were spent on counting chimney swifts, and late night pizza runs, which have been a tradition at our meetings for many years. And, as the closest pizza places were located just opposite a great Irish pub, Acadia's pizza runs somehow managed to end there, as beer seemed to go quite well with math puzzles.

The 2003 meeting in Wolfville was a memorable meeting, in large part thanks to the local organizers: David Reid and his team. Thanks, David, for the great job!

Special Announcement:
Zoltan Dienes, Extraordinary Member of CMESG/GCEDM

At the 2003 CMESG annual meeting Dr. Zoltan Dienes joined us to share some of his theoretical insights and teaching activities. In recognition of his contributions to mathematics education in Canada and world-wide, CMESG took this opportunity to welcome Zoltan as an Extraordinary Member.

Zoltan Dienes grew up in Hungary, France and England, and lived and worked all over the world during his long career in mathematics education, making him a true citizen of the world. We can claim him as a member of the Canadian mathematics education community, however, in light of two important periods of residence here.

After finishing his PhD in mathematics at University College, London, in 1939, Zoltan taught mathematics in schools and then at universities in England. In the 1950s he began more systematic study of the psychology of learning mathematics. It was during this period that he developed the Multi-base blocks that he is perhaps most famous for, as well as publishing ground breaking books like *Building up Mathematics.*

In 1961 he moved to Australia to work at the University of Adelaide, and began to travel widely, working in New Guinea, Hawaii, Hungary, Italy, Geneva (with Piaget), Boston (with Bruner), San Francisco, and Minneapolis, as well as being a founding member of the International Study Group for Mathematics Learning. It was during these travels that he first visited Canada, travelling to Sherbrooke, where he found a warm welcome for his ideas about teaching, although the snow and –25˚ temperatures gave him a welcome of a different kind.

In 1966 he was appointed director of a new research centre at the Université de Sherbrooke, where he spent the next nine years (aside from continuing to travel all over the world). Mathematics educators come from around the world to work with him at Sherbrooke. He describes this time as the zenith of his working life.

For three years in the 1970s he worked at Brandon University (though he was based in Winnipeg) on projects to provide better opportunities for Aboriginal students to learn mathematics. He then left Canada to pursue projects in Italy and England. In 1986, twenty years after he first moved to Canada, he returned to live in Wolfville, Nova Scotia. From Wolfville he continues to explore new ways to teach mathematics, and maintains contacts with international colleagues through the Internet.

Through all these travels he was accompanied by Tessa, his devoted wife, and various subsets of his five children. Tessa and Zoltan recently celebrated their sixty-fifth wedding anniversary.

If you are interested in learning more about Zoltan's ideas and tracing further his contributions to mathematics education here and abroad, your university library will have copies of his fundamental theoretic work. You may also be interested in his *Memoirs of a Maverick Mathematician* recently published by Upfront Publishing, and other works on his website, www.zoltandienes.com.

Plenary Lectures

Conférences plénières

Using History of Mathematics in the Classroom: Prospects and Problems

Tom Archibald
Acadia University

Introduction

Historical examples have been proposed for use in the mathematics classroom to achieve a variety of aims. In primary and secondary school, the aim may be to give a human context to an otherwise arid subject or to demonstrate cultural relevance of specific mathematical achievements to a particular group. In the upper years of high school or in university, historical studies are being used to provide insight into the origin of a particular set of mathematical ideas or to give insight into why certain results are thought of as important. These and other possible uses are well represented in a growing literature on the uses of history in the classroom. The recent establishment and rapid growth of the History of Mathematics Special Interest Group of the Mathematical Association of America suggests that, on this continent at least, history is booming as a field of interest among mathematics educators.

In this lecture I propose to give an overview of these recent developments and then to discuss some of the problems associated with bringing history into the mathematics classroom. Practical issues include but are not limited to: getting bogged down in historical detail at the expense of actually covering the topic the teacher of professor hoped to cover; combining the stultifying boredom of history with the difficulty of mathematics to make the subject even more repellent; and grossly oversimplifying history in order to make some pedagogical point, so that students leave with a manufactured history that is actually incorrect. A more general class of problems arises from the fact that mathematics instructors rarely have much background in the study of history.

This will be a personal view from an historian of mathematics who happens to work in a mathematics department and has a lot of experience teaching undergraduates, many bound for teaching careers. The main aim is to help provoke a bit of discussion on the subject. My main exposure to K–12 teaching is that of everyone else (I experienced it as a student), together with experience working on textbook materials for these levels, again over 20 years ago. This involved a certain amount of classroom visiting, meeting with teachers and curriculum people, and so forth, and also the design and execution of teachers' manuals for math texts at all levels. Anyway, despite my amateur status in pedagogical matters, I'm sure our discussions will illuminate various controversial points and bring out different points of view.

I'll begin with a discussion of various aspects of how history is being put into the classroom at different levels, including a discussion of some resource materials. I note that when I say "the classroom", I am referring to any level whatsoever, but I'll try to make this more precise in what follows when necessary. I then propose to spend some time discussing various practical and theoretical problems with the introduction of history into the mathematics classroom. Finally, I'll conclude with some general observations and suggestions. I should say that I am starting from the viewpoint, perhaps not universally shared, that the historical development of mathematics is uniquely and centrally important to human history and to the configuration of the present world, and that a great deal more attention ought to be paid to mathematics in the educational system from beginning to end. This would require a rather complete reform of prevailing cultural values in this country, which I don't think will happen, but in any case I think we can try to move in the right direction.

History in the Mathematics Classroom

Let's begin with a quiz. The questions involve the history of mathematics (broadly understood to include some applications).

1. (a) How did the number system we use originate? What are its advantages over other systems?
   (b) Why are there 360 degrees in a complete revolution?
2. Why did arithmetic come to be taught in the schools? Geometry?
3. Why does the week have 7 days? Why are they named in the order that we give them?
4. (a) Who discovered Pythagoras' theorem? (Hint: in answering this question, you need to analyze what "discover", "theorem", and "Pythagoras" mean.)
   (b) What is the relation between the square of a number and a geometric square?
5. (a) Who were the first Europeans to use reasonably advanced mathematics on what is now Canadian territory?
   (b) What kind of mathematics was known in the Canadian region before the European arrival?
6. Why weren't there more women who did famous things in mathematics (and science) before 1950 (or your favourite date)?
7. State one connection between Shakespeare and mathematics.
8. What are the main arguments against the existence of negative numbers?
9. Give some examples of how mathematics and society have interacted. Has the interaction in your examples been good or bad for society, in your view?
10. State one mathematical discovery or invention of the past 50 years other than the Fermat-Wiles theorem.
11. Find the square root of 152399025. Don't use a calculator.

Not all of these questions have known answers, and some of the answers are debated or very complicated. The point of the list is to suggest that students planning to teach mathematics should have some idea of how to find out about the topics in question. You'll notice that most of them are not the kinds of things that are usually dealt with in mathematics classes. I hope you'll agree that most of them are reasonably important, though some, while interesting, are a bit trivial-pursuitish.

I wanted to mention, then, at the outset, that the history of mathematics is a way for teachers to address the important question: Why should everyone be interested in mathematics? It can also be very helpful in teaching specific mathematical topics and skills, and this is the main thrust of most work in the area of the relation between history of mathematics and pedagogy. As we all know, in fact, there is considerable interest internationally in attempting to use history in the mathematics classroom. There is, for example, an association, the International Study Group on the Relations Between History and Pedagogy of Mathematics (HPM), which is an affiliate of ICMI. This organization has a new website, http://www.mathedu-jp.org/hpm/index.htm, which includes recent issues of its newsletter, which I know many of you are familiar with. Luis Radford is on the advisory board of the association. This group has existed since 1972.

Since 1984, there have been regular discussions at the ICME international meetings on history, many of which have resulted in volumes of essays. One example, *Using History to Teach Mathematics*, edited by Victor Katz of the MAA, came out of the 1996 meeting in Braga, Portugal. The recent book by Jan van Maanen and the late John Fauvel (2000), *History in*

Mathematics Education, is a survey of the state of the art, based on a study in which the editors sought responses to a number of questions. We'll look at some of these later. This work contains an annotated bibliography of 300 items, which obviously I can't duplicate here. I'll note that the paperback edition of this book is a mere US$65.00.

The issue of giving future or present educators some training in the history of mathematics that is specifically oriented toward pedagogy has also been taken up in a variety of ways. For example, textbooks on the history of mathematics such as those by Victor Katz, David Burton, or Jeff Suzuki all emphasize (if not exclusively) topics which late elementary or secondary teachers may well encounter and provide exercises which have specific pedagogical content. An example from Katz (2000):

> Compare the various notations for unknowns used by the mathematicians discussed in the text [from the 15th and 16th centuries]. Write a brief essay on the importance of a good notation for increasing a student's understanding of algebra. (p. 381)

Such texts usually aim at the future secondary teacher. We'll come back to this issue.

In addition, for a number of years (six?) the NSF in the US supported a summer school, organized through the MAA by Fred Rickey and Victor Katz, on the use of the history of mathematics in teaching. This course was intended to give a more professional kind of historical training to college or university teachers with an interest in the history of mathematics who were (in most cases) already teaching such courses.

All of these points, then, signal a lively level of professional interest in putting history into the mathematics classroom. There are a number of reasons cited as to why one would want to do this, and I'll give a few of these now, though the list is once again far from exhaustive and the points overlap.

- *Motivation for learning specific mathematical topics*. Historical anecdotes add colour and depth to what might seem rather arid. A nice example from undergraduate mathematics (though it could be done in high school) is the cubic formula. With the story of rivalry, betrayal, and public competitions it becomes a good deal more than just a calculation.
- *Illumination of mathematical concepts*. To take another example from algebra, we could consider the geometrical proof of the quadratic formula due to Al-Khwarizmi. It can be a great moment to have students realize that a geometrical square and an algebraic square are somehow related, and that completing the algebraic square and completing the geometrical square can be strictly related to one another.
- *As a superior vehicle for the introduction of certain topics*. It's all very well to teach computer science students what bases other than decimal mean, since they have to work with binary, logic circuits, and so on in other courses. However, most students find this artificial in the extreme (and hard). Introducing the idea of base-60, discussing hours, minutes, seconds, and the 360-degree circle ties the topic to previously studied work and to daily life.
- *Providing a cultural context for mathematics*. Why do we work with fractions? This can be nicely discussed (for example) in the context of the Egyptian Rhind papyrus, where the proper allocation of food to slaves depends on the ability to work with fractions. This feature has been particularly employed by those involved in minority education programs in the US, and is perhaps one of the most controversial aspects of the use of history. Here one could mention Paulus Gerdes' book, *Geometry from Africa*, which proposes such topics as "Geometrical ideas in crafts and possibilities for their educational exploration." I'll return later to the controversial aspects here.
- *Providing meaning for mathematical topics*. One of the biggest single problems in mathematics education at any level is trying to get the student (and, rather too often, the teacher) to understand that mathematical operations and manipulations have meaning. A very large percentage of students entering university consider mathematics as a set of actions performed by a pencil or calculator according to rules made up by the teacher or the textbook author. Historical, contextual examples like those mentioned already can complement the problem-solving component of curriculum.

- *Providing historical role models*. One can debate how profound this is, but I think it is beyond argument that Einstein's low algebra marks, the death in a duel of Galois at 21, the examples of Sophie Germain, Kovalevskaya, Benjamin Banneker have importance for certain students. We may also mention the important motivating role of unsolved problems (like, for a long time, the Fermat problem) or unsolvable problems like angle trisection.
- *Helping students realize that mathematics is a part of human activity, a critically important one for all kinds of reasons*.

I already mentioned that these points overlap considerably, and aren't exhaustive. Doubtless you can think of others, or think of other ways of describing the function of the various examples I cited. Proponents of history point to such things as advantages, and I think it is hard to argue that these are not virtuous taken in themselves.

Yet behind every brilliant educational idea lurks a multitude of problems, and I'd like to go on to discuss some of those at this point.

Historical Examples in the Classroom: Problems

There are various classes of problems and controversies associated with the use of history in the mathematics classroom. Let's begin with the controversies. Some people would hold that spending much time on history in the mathematics classroom is a bad idea. The arguments here take several forms.

First, one can simply argue that there is no time. If we wish to cover the required curriculum, goes the argument, and ensure that students have the necessary level of skill to progress, we don't have time for giving history lessons. There is an easy reply here, which is that the topics have to be introduced and taught somehow, and that some motivation has to be provided. For the historically committed and historically knowledgeable teacher, this argument will not hold much water. More broadly, though, if the curriculum were to come to require a certain amount of history to be covered, this does entail practical problems of teacher training which I'll return to below.

On a somewhat deeper level, I'll cite a sentence from a recent email I received from a teacher of Native Canadians (Campbell Ross), who is interested in trying to incorporate more narrative into the teaching of mathematics in order to make it relevant to his students. As he puts it,

> The competing view appears to be that, beyond systems for counting, mathematics is an abstract, formal, non-culturally contextualized, 'sui generis' field of knowledge based on rules of logic/reason; consequently, the teacher of whatever student should concentrate on building up the student's internalization of the self-contained world of mathematics.

As an historian, of course, that isn't my view of mathematics. There is no doubt, though, that it is one which is widespread among mathematicians, and that some very serious and dedicated educators hold a similar view. I suppose that the response here is to state that, used effectively, historical examples will contribute to precisely this improved internalization of concepts. It's hard for me to argue that they will contribute to the impression that mathematics is somehow self-contained, or even that abstractness and formality are essential features of mathematical activity, at least in the way these terms seem to be used in this citation. It is quite true that there are historical writers who hold this view, however. The late Andre Weil famously couched his analysis of the work of Euclid in a discussion of semigroups, for example; and it is not especially unusual to find historical writers saying something like the following: "You can see here that C. Neumann is using a version of the contraction mapping theorem of Banach" some 50 years before the birth of the latter. This type of discussion is not without interest, but it is not history in the way I understand the term.

I mentioned earlier that the use of history to give cultural context to mathematics had controversial elements. Here one could mention the comments by Martin Gardner in his review, in 1998, of *Multicultural and Gender Equity in the Mathematics Classroom: The Gift of Diversity* (1997 Yearbook). Gardner classes this work as "fuzzy", and complains about various aspects of the presentation. Historical examples as such are not decried, but the idea that one should teach different mathematics to different groups, that mathematics has a special culturally specific content, is targeted by Gardner with his usual wit and verve. I think mathematics educators should be concerned if time is spent examining patterns of woven baskets and using that to satisfy a geometry requirement instead of studying Euclidean geometry and proof. And of course, instruction time is precious.

A key concern here is the general question of curriculum change. University mathematics professors (and those in disciplines that use mathematics) are constantly being made aware that curriculum change takes place behind their backs; and they are rarely happy with the results. A move to a more "culturally relevant" curriculum for certain groups may ultimately disadvantage those groups in a competitive setting. So if history is coming into the curriculum, many in that group would like to see it in an ancillary role, as a support to the teaching of traditional topics.

This is perhaps the place to mention that I myself use historical examples rather infrequently in my mathematics teaching. At least, I think it is infrequent—a recent non-historian visitor to one of my classes felt it was full of historical remarks. The main exceptions are that I use Leibniz' proof of the product rule (as well as the more standard calculus textbook proof); and that I like to talk about Cauchy's definition of the sum of a series as contrasted with Euler's when introducing Taylor series and so on. But apart from those points, outside of my history courses I tend to avoid historical approaches. Why?

At the university level, historical treatments of mathematics are rarely simpler than a present-day treatment, and frequently they are far worse. Try learning about conic sections from Apollonius, or calculus from Newton, or Riemann surfaces from Riemann! To be sure, some works may have conceptual advantages—I don't think there is any better place to learn about elliptic functions, theta-functions, and modular functions than Jacobi's *Fundamenta Nova*, though I think more recent books like McKean and Moll's *Elliptic Curves*, itself very historical, should be read along with it. But students who will be taking subsequent courses need to know standard methods, and exposing those methods takes all the time available.

So this brings me to a variety of practical, mostly not very ideological issues in presenting historical examples as part of the regular curriculum, at whatever level.

1. *Getting bogged down in historical detail at the expense of actually covering the topic the teacher or professor hoped to cover.*
   This can happen at any level. For example, figurate numbers are great fun and there is much one can do in discerning formulas for generating the next one, for sums of such numbers, and so on, just by working from the pictures. A detour via the Pythagoreans might be interesting up to a point, but it probably obscures the objective. Worse is the situation where the historical subject is really rather complicated, such as the invention of the geometric representation of complex numbers. This seems innocent enough, but the history is rather complicated (one needs to know spherical trigonometry) and somewhat controversial, and most of the writings until Gauss are also pretty tangled.

2. *Combining the stultifying boredom of history with the difficulty of mathematics to make the subject even more repellent*.
   I think the description of this suffices. However one should note that most students don't know any history at all. One is therefore faced with giving rather long explanations, for example, about the French revolution or Hitler and the Holocaust. These examples are really worth doing (in fact, I think mathematics class is very good place to touch on both these subjects, at the appropriate level), but time is lost. The elementary teacher has the glorious possibility of integrating some of these, and in fact the war in Iraq presents a real opportunity to do something about Mesopotamia, mathematics, writing, astronomy, astrology … and one could even involve the *Star Trek* episode about Gilgamesh and Enki-lil.

3. *Grossly oversimplifying or altering history in order to make some point, pedagogical or otherwise, so that students leave with a manufactured history that is actually incorrect. This happens for ideological reasons as well. This points out the need for very careful choice of historical reference material*.
   Changing history in order to suit your argument is unfortunately pretty common in mathematics textbooks. It's also done for dramatic effect or for ideological reasons. I don't want to suggest that there is usually malicious intent. I'll give two examples. One is the persistent statement that Newton is the inventor of limits, and that he used them in his *Principia* to derive differential equations for the laws of motion. This statement is inexact. It is trivially inexact in that Newton never wrote the laws of motion as differential equations. This was first done explicitly as far as I know by Euler in 1750, 23 years after Newton's death. The relationship between limits and the calculus in Newton's work is also in fact pretty complicated. One might note that it isn't especially harmful if students walk around with such imprecisions in their heads. It isn't any more important to most people than the relative position of the pancreas and the spleen.

   Another type of example comes up in connection with "culturally-based" histories. Here I would mention the writings on Vedic mathematics. This rather large body of literature suggests that the central ideas of the mathematics we call Greek in fact date back to Vedic writings from 3200 BCE or earlier. Now, I don't have the linguistic expertise to evaluate these arguments fully. Nor do I believe that the Greeks invented everything traditionally ascribed to them in the 19th century, such as mathematical proof, etc. Nor, furthermore, do I really believe that our mathematics is the direct result of Greek mathematics (and here I can follow the literature quite well, though I'm not an expert). However, I don't find the claims for Vedic mathematics especially plausible, and at best I think we can regard them as controversial (for one thing, they depend on accepting specific arguments for dating texts made in the late nineteenth century—I'm automatically suspicious when I see the claim that sine tables were compiled by Aryabhatta, who was teaching mathematics at the age of 23 in 742 BCE, when there is no detailed reference to the argument that puts him at that time. Our problem here is twofold: for one thing, these writings are taken by many as completely true (and they are used currently by Hindu nationalists as part of a claim of cultural superiority). For another, faced with an ardent claim of absolute truth, most teachers, and most people of any kind, don't have the means to evaluate the claim.

   One of the most annoying classes of example of false history comes in the fabrication that is deliberate. Here I will mention the work of E.T. Bell. Amazingly, Bell's work was reprinted recently by the MAA without comment. There is a lot of literature of the past 50 years that has been devoted to disproving some of Bell's statements, yet these errors still propagate.

   This brings us to our final problematic point. This is one that something can be done about. It is:

4. *The fact that mathematics instructors rarely have much background in the study of history*.
   Mathematicians rarely study history formally, and many mathematics teachers likewise don't have much historical background. Despite the existence of mathematicians who have written truly interesting historical works, often such ventures are tendentious. (An amazing example is the work "Life, Art and Mysticism" by the topologist L. J. E. Brouwer, which begins with the creation of Holland and asserts, among other things, that at one time humans and animals existed in happy peace.) The critical judgement of sources, the interpretation of competing claims, the avoidance of projecting the present onto the past, and the various fashions that lurk among professional historians, are not things they are trained in. Happily, the best textbooks in the history of mathematics and the materials on HPM are generally quite sensitive to these issues. In North America and internationally, a history of mathematics course is a more and more common part of the curriculum, and in a number of states such courses are required for future mathematics teachers. Thus, provided the professors

of these courses use good materials and realize that they have something to learn, we can hope that this problem is being addressed to some extent. I can't resist stating the preference that far more institutions should hire professional historians of mathematics, not only for this reason.

What is to be done?

Fauvel and van Maanen, in the study leading to their fat book, formulated a number of questions relevant to the use of history in teaching.

1. How does the educational level of the learner bear upon the role of history of mathematics?
2. At what level does history of mathematics as a taught subject become relevant?
3. What are the particular functions of a history of mathematics course or component for teachers?
4. What is the relation between historians of mathematics and those whose main concern is in using history of mathematics in mathematics education?
5. Should different parts of the curriculum involve history of mathematics in a different way?
6. Does the experience of learning and teaching mathematics in different parts of the world, or cultural groups in local contexts, make different demands on the history of mathematics?
7. What role can history of mathematics play in supporting special educational needs?
8. What are the relations between the role or roles we attribute to history and the ways of introducing or using it in education?
9. What are the consequences for classroom organization and practice?
10. How can history of mathematics be useful for the mathematics education researcher?
11. What are the national experiences of incorporating history of mathematics in national curriculum documents and central political guidance?
12. What work has been done on the area in the past?

Noting once again that the essays in their book address these questions, I would like to give my own views on the answers to some of these. I intend these as discussion points.

1. I would like to advocate the position that the history of mathematics should be integrated into instruction from very early on. I don't see any harm, for example, in telling first grade pupils that the symbols they are learning came from India and showing symbols used in the old days elsewhere depending on the ethnic composition of the class. Of course, specific materials are needed for a large-scale integration. I expect some issues of this kind will be addressed in the workshop led by Louis Charbonneau and Irene Percival (see pp. 39–54 in this volume).

2. & 3. History of mathematics should be taught to future mathematics educators, either before they are admitted or (ideally, I think) during their program. If one has a class composed entirely of future teachers, one can reasonably ask for students to complete exercises with a specifically pedagogical focus (lesson plans, etc.) as in the example from Katz that I cited at the beginning of the talk. The people who teach such courses should ideally have some continuing education along the lines of the NSF Summer Institute I mentioned, with participation of both historians and educators.

   This is perhaps a moment to dwell a little on the nature of the materials that one would use in such a course. Katz's very fine survey textbook has a lot of very positive features. It is historically accurate, with excellent references to recent literature, good problems, including quite a few problems and discussion questions that are specifically teaching-related. It is probably the broadest culturally as well, achieving very broad geographic and ethnic coverage in a very reasonable way. It has a small drawback, which is that students really find it hard to read. So: it's a great resource book. A more readable competitor is the recent book of Berlinghoff and Gouvea called *Math through the ages: a gentle introduction for teachers and others*. This is certainly an easier read mathematically and on all grounds, is pretty scrupulous historically, and for a lot of people would be a good starting point.

6. I think it is certainly the case that cultural context plays an important role in student motivation, and I think historical or culturally specific materials have a role to play here. I don't think this should cut into the traditional topics for those who are bound for university science programs, which implies that mathematics will require additional time in the curriculum. In other words, one of the reasons for the failures in teaching the traditional curriculum is the failure to take enough time to supply sufficient context. Here again, very significant teaching materials and in-service resources are needed.

9. What are the consequences for classroom organisation and practice? This I really should leave to the experts, but to sound a sixties note which still echoes, it seems to me that teaching mathematics in a historically and culturally informed way offers many opportunities for integration of teaching among now-different subjects. In present modes of organization, this is very difficult in high school because of subject specialization. I repeat that additional time is needed for successful mathematics instruction.

Concluding Remarks

In the best of all possible worlds, then, students would understand mathematics as a broad cultural phenomenon of general relevance while attaining skill levels that will enable them to succeed in careers of their choice. Doubtless this is absurdly utopian. I believe, however, that efforts in this direction employing the history of mathematics will have substantial benefits. While the overall goal of a mathematically well-educated public might remain elusive, it doesn't mean the effort should not be made. In the words of Robert Browning:

> Ah, but a man's reach should exceed his grasp,
> Or what's a Heaven for?
> (Andrea del Sarto, lines 97–98)

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Research in Mathematics Education
Through a Keyhole

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Introduction

When Caroline Lajoie invited me to give a talk at the 2003 CMESG conference, I asked her if the program committee had any expectations with regard to the theme of the talk. Here are excerpts from the response of one member of the committee with whom the others have basically agreed.

> I would love it if she were to talk about the privileging and under-privileging of different forms of research in math ed.... I'd love to hear her talk on that kind of meta-level, of what we need math ed research to try to be. I want to be stirred up, angered, forced to recognize how partial my [or rather] the standard outlook is.... Invite Anna to tell us if there's anything resembling a big picture out there, for math ed research. Or is it all just a tug-of-war over curriculum between the Good Guys and the Bad Guys, with the good-hearted but inneffectual math education researchers spiraling around the power brokers, like the electrons in an atom of uranium?

These words are an expression of a dream, among mathematics educators, for a common direction that would unite teachers, policy makers, and researchers, with researchers collaborating to accumulate knowledge and thus contribute to our understanding and improvement of school practice. But this dream becomes less and less realistic in view of the tendency of mathematics education to multiply and diversify its epistemological foundations, theoretical approaches, methodologies, and research questions.

> In the past, mathematics educators drew largely from psychological frameworks and theories, but contemporary researchers are increasingly demonstrating the insights that may be learned from additional frameworks. Central among the frameworks now drawn upon are sociology ..., sociocultural theory ..., politics ..., mathematics ..., philosophy ..., history ..., anthropology .... But scholarship is anything but simple, and whilst we may agree that breadth of thinking is critical to the evolution of ideas, and that different frameworks should be considered and employed, we must also be wary that mathematics education is a relatively new and young field and that *too much breadth will cause a scattering of focus and preclude opportunities for consolidation and identity*. (Boaler, 2002; the emphasis is mine).

The "breadth and diversity" is growing not only through importing theories from neighbouring domains, but also as a result of the productivity of mathematics educators themselves.

> [I]t has become the norm rather than the exception for researchers to propose their own conceptual framework rather than adopting or refining an existing one in an explicit and disciplined way. This prolific theorising may be represented as the sign of a young and healthy scientific discipline. But it may also mean that theories are not sufficiently examined, tested, refined, and expanded. A theory may be used mainly by its creators and their students rather than by a large number of independent and experienced researchers. It may be used for only one particular type of research study, of population, of methodology, or of context. Equally, essentially the same issues and research questions may be being

> described and analyzing by a multiplying array of parallel theories. One concern may be that such proliferation of theories, influences and frameworks may lead to mathematics education becoming a tower of Babylon, where many strive, with excellent intentions, to provide light for their colleagues, but few listen, read and take into account their' colleagues' ideas and work. (Reflections on *Educational Studies in Mathematics, ESM* 50.3)

Theoretical frameworks enrich the vocabulary of mathematics education research and practice. But a term, which has a precise technical meaning within its home theory, functions as a mere metaphor outside of it, its sense stretched at will by the users. This *abus de langage* does not contribute to developing the theory, but rather leads to transforming the theory into a fashionable discourse that may influence how people talk about their practices, but helps neither in understanding practice nor in predicting the consequences of interventions into practice.

One response to these concerns and challenges has been the engagement of some mathematics educators in "meta-studies" of mathematics education research such as attempts at outlining a definition of the domain (e.g., most papers collected in Sierpinska & Kilpatrick, 1998), more or less comprehensive syntheses of research approaches and theories (e.g., Sierpinska, 1995; 1996; 2002; Niss, 1999), comparison of theories (e.g., Boero et al., 2002), and analyses of articles published in leading journals and conference proceedings over the past 10 or more years (e.g., Boaler, 2002; Lerman et al., 2002; Hanna & Sidoli, 2002).

My previous meta-studies focused mostly on theoretical frameworks and referred to a personally biased selection of papers. For this conference, I decided to take a more empirical approach and have a good look at a more or less "random sample" of papers, in the sense that they were not especially selected to represent this or that issue, theme, theoretical perspective, mathematical topic, etc.

I took all 55 research reports included in Volume 4 of *PME 26 Proceedings* (Norwich 2002). Editors of the proceedings put the reports in alphabetical order with respect to the presenting authors' names; therefore there was no bias with regard to topics or research issues. At least from this point of view, the choice of the texts was random. But it is a very small sample, compared to the number of papers that are published worldwide. Moreover, the papers were written for a PME conference, which is a very special kind of conference, with its own interests and biases. Choosing papers published in a variety of journals or presented at different conferences could produce very different results. This is why the most I could obtain from the study of the 55 PME research reports is a limited vision of mathematics education research, "through a keyhole", and not the "big picture" of where mathematics education is going, hoped for by the program committee.

1. Analysis of Research Reports Published in Vol. 4 of *PME 26 Proceedings*

In reading the reports I had several questions in mind: What was the research problem? What were the results of the research? What were the tools of the research? In particular, were mathematical tasks used as tools in the research? How were they presented? Was their choice justified and discussed? Questions about mathematical tasks reflect my own particular bias as a reader of mathematics education papers, researcher, and practitioner. I consider the design, analysis and empirical testing of mathematical tasks, whether for the purposes of research or teaching, as one of the most important responsibilities of mathematics education.

A natural question would be, *In what kind of theory was the research grounded?* But, having looked at this question in my previous meta-studies, I will not devote much attention to it in this paper.

I will refer to the papers in the 4th volume of the *PME 26 Proceedings* by their ordinal number of appearance in this volume, from *1* to *55*, and not by their authors' names. This is not to be interpreted as a sign of lack of respect for the authors' work but my strategy of reinforcing the idea that the choice of texts has been rather arbitrary. I wish to focus the readers' attention on the texts and their contents and not on the persons who wrote them. From the point of view of the features I am looking at and their distribution within the sample, these papers could have been written by another set of persons. However, I could not completely conceal the source of my data from the reader because I wanted to obtain falsifiable results and not only some kind of prophesies. By reading the reports for themselves, readers have the possibility to verify (and therefore contest) my statements about them. The reports are listed in an appendix to this paper.

2. Research Questions

2.1. Overview

There were five categories of research questions according to whether they focused on the teaching of mathematics (24 reports, ~44%), the learning of mathematics (24), research methodologies (5, ~9%), mathematics teacher education (1, ~2%), and methods and instruments of assessment (1) (see Figure 1).

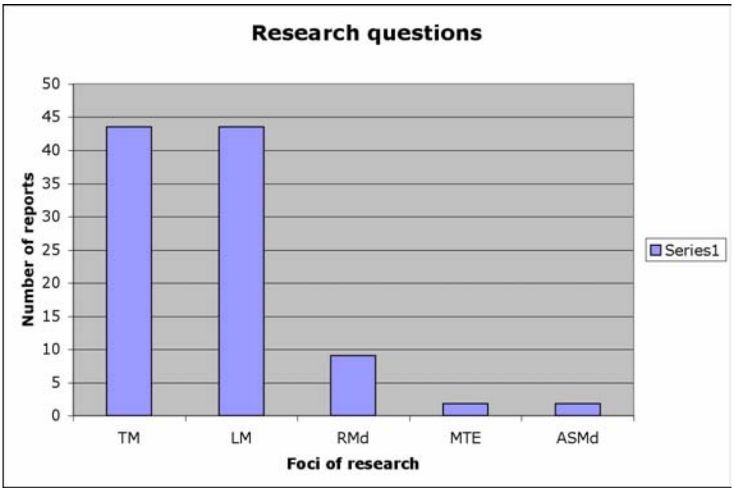

FIGURE 1. Foci of research questions (TM:= teaching mathematics; LM:=learning mathematics; RMd:= research methods; MTE:= mathematics teacher education; ASMd:= assessment methods)

Let us compare this distribution of foci of research with the one obtained by Lerman et al. (2002), in relation, for example, to those of special issues of *Educational Studies of Mathematics* between 1990 and 2002:

> We can see a strong focus on content across the years, from Aspects of Proof in 1993 to Infinity in 2001, as well as focus on pupils' learning and on teaching.... [W]e can notice an absence of engagement of the mathematics education research community with other agencies, in terms of policy or politics. There is no discussion of curriculum, and there is [only] one issue on assessment. Mathematical pedagogy seems to be the main interest of the research community .... (Lerman et al., 2002)

Four out of the 17 special issues were devoted to specific mathematical domains or processes (proof, proof in dynamic geometry environments, statistics and infinity); more than a half of the special issues were devoted to learning and teaching of mathematics in general (9 out of 17). The ESM special issues and our sample share the focus on learning and teaching, especially the development of reasoning and proving (often in a computer environment), little interest in assessment, and lack of interest in discussing the curriculum and general policy/political issues.

Concerning the mathematics involved in the research of our sample, there was a large variety of content and process areas, with fractions, simple equations, and reasoning/proof leading the list with at least five reports devoted to each. These are classical domains, but the idea and process of defining in mathematics is a relatively new domain, which seems to attract more and more attention (3 reports).

2.2. Questions about the Teaching of Mathematics

There are, in principle, four approaches to questions about teaching a subject:

· Descriptive: How is the subject actually being taught?
· Analytic: What are the factors that influence teaching practices?
· Experimental: What happens if the subject is taught differently?
· Prescriptive: How to best teach the subject?

2.2.1. Actual teaching practices

Five papers (*18; 21; 32; 43; 52*) offered descriptions of teaching practices, often obtained within the frame of larger projects aimed at identifying effective teaching practices.

The authors of (*18*) looked at the process of acquiring a proficiency in the management of a whole-class discussion by a teacher. Those of (*43*) analyzed one teacher's practices in attending to and interpreting students' interventions; they described the development of the teachers' ability to hear the students' voices without "over-hearing" or "under-hearing" them.

Authors of (*32*) used data obtained in a large-scale numeracy project in Australia, related to the teaching and learning of measurement, to observe that,

> [Contrary to the belief that] effective learning is the result of curriculum policy statements, resource provision, school leadership, or other factors amenable to central policy decision ... there are marked differences in achievement between classes, irrespective of geographic or socio-economic variables, and ... these differences are attributable to the teachers. (32)

The research led the authors to conclude, "Effective teachers seemed to be able to articulate focused, developmentally appropriate and engaging activities for their students, and engage them actively in interrogating these experiences". These teachers' behavior was not "tested" in the project against a control group; rather, the observation of this behavior and its effectiveness was a spontaneous outcome of the research.

The aim of (*21*) was to build a model of teachers' view of effective use of computer-based tools in mathematics teaching. This was an "insider's" model of effective technology based pedagogy. On the other hand, (*52*) proposed an "outsider's" view of such pedagogy by describing and discussing the teacher and students' behaviors in a few classes on differential equations where the Maple computer algebra system was used.

2.2.2. Factors that influence teaching practices

Several papers looked at factors that influence teachers' practices, for example, teachers' views of mathematics (*8*); teachers' views about pedagogical constraints (*27*); teachers' beliefs about gender differences in mathematics (*29*); teachers' knowledge of mathematics (*23*); official assessment (*25*). The last research (*25*) aimed at verifying if richer official assessment methods indeed lead teachers to focus more on deeper conceptual understanding in their teaching practice. Factors of change as perceived by teachers themselves were identified in (*33*).

2.2.3. Teaching experiments: What happens if the subject is taught differently?

Teaching experiments can be done on a *small scale* of one classroom or even a few students, but they can also be done on a *large scale*, involving many schools, teachers and students. Moreover, a teaching experiment may consist in trying out a small sequence of activities on a chosen topic, or a complete program of teaching.

2.2.3.1. Accounts of small scale teaching experiments

Ten papers referred to experimenting, or, more precisely, trying, on a small scale, a preconceived teaching approach. Four papers looked at the potential of a piece of technology for the learning of mathematics. Among these four, two papers reported experiments with the dynamic geometry software *Cabri-geometry II*, which exploited the features of this software (such as dragging) to promote theoretical thinking and deductive reasoning

(*2; 28*). The other two papers reported teaching experiments with different kinds of software: logo (*22*); and a combination of walkie-talkies, cartographical software and the Global Positioning Device (*54*). Report (*22*) also aimed at developing students' theoretical thinking and, particularly, their proving competence in mathematics. Research in (54) was concerned with modeling and developing spatial thinking (2nd grade students were engaged in the task of drawing a map of the school surroundings).

Modeling was also at the center of interest in (36), where the so-called Cultural Conceptual Learning-Teaching Model (CCLT) was applied in designing activities for children. The activities involved making links between school and out-of school mathematics. The question was if the activities bring the expected better performance of children both in school mathematics and in their judgment of daily life situations.

Report (*4*) focused on ways of teaching the idea and process of defining in mathematics. The author was pondering the usefulness of one tentative situation for engaging students in the construction of a definition. The situation was meant to reveal students' conceptions of definition to the teacher and thus create conditions for their improvement.

Two papers referred to trying approaches to teaching algebraic thinking and skills. Authors of (*17*) were testing the usefulness and pitfalls of introducing equations by way of a geometric model of equations (product ax corresponds to the area of a rectangle a by x), as students go on to study more complex linear equations. Research reported in (*19*) tried an approach to teaching problem solving aimed at helping students move beyond the "arithmetic" unknown on to the "numerical" unknown and eventually to the "algebraic unknown".

Fractions were the mathematical concept targeted in the teaching approaches described in (*7*) and (*38*). Authors of (*7*) wanted to evaluate the potential of a teaching program for second grade children aimed at preparing them for understanding fractions later on in their schooling; the program used specially designed manipulative material and collective games in building an initial understanding of sharing out. In (*38*), the author was testing classroom activities (related to fractions) developed on the basis of a socio-constructivist model of teaching.

2.2.3.2. Evaluation of a teaching program on a larger scale

Report (*3*) referred to the same large-scale teaching experiment in K–3 in Australia ("Count Me Into Measurement" program) as (*32*) but aimed at its more comprehensive evaluation by participating teachers. Teachers were asked to formulate their opinions about what they liked and what they would rather change in the program and if they thought children made progress in their understanding of measurement.

Report (*49*) evaluated a 10-lesson unit for teaching the notions of chance and sampling variation in Grade 5. The unit was taught to 82 children from three schools in Tasmania (Australia). The evaluation was based on a comparison of results in pre- and post-tests. Authors also formulated hypotheses about the factors of improvement of students' conceptions with respect to (a) their performance before instruction, and (b) the performance of Grade 3 students in a similar teaching experiment.

2.2.4. Prescriptive analyses: How to best teach the subject?

Recommendations and ideas for teaching can be based on theoretical arguments only, or on empirical arguments, or—as is most often the case—on a mixture of the two. I counted one report as clearly belonging to this category (*20*), and I hesitated about another one (*48*).

The authors of (*20*) drew on their previous teaching experiments to outline a series of activities in the Cabri environments aimed at improving students' understanding and learning of ratio and proportion. Their justification of the activities was based on the results of their earlier teaching experiments and on "theoretical" arguments in the restricted sense of reference to several authorities in psychology (Piaget) and mathematics education. This paper was prescriptive with respect to teaching in a rather explicit way.

Report (*48*) advocated activities focused on the construction of examples of all kinds (examples of objects satisfying a definition; examples of application of a technique; examples of problems; examples of proofs, etc.), which the authors considered as a learning/teaching strategy in mathematics. The argument was theoretical but supported by reference to published results of empirical research of other authors. The main theoretical assumption was a definition of learning as "growth and adaptation ... of personal, situated, example spaces", and of the role of teaching as "provid[ing] situations in which this can take place". The paper offered ample justification of this definition and thus contributed to the theory of learning. But it also gave examples of useful and not so useful teaching situations from the point of view of this theory of learning. Thus the report could also count among papers focused on teaching. However, since the paper speaks more about learning than about teaching and does not describe aspects of a teaching situation other than formulating the task, I decided to count this report as belonging to those focused on the learning of mathematics.

2.3. Questions about the Learning of Mathematics

There were three kinds of issues about learning in the sample of articles:

- Theories: how people learn (a) mathematics in general, (b) specific mathematical concepts or processes;
- Descriptions: how and/or how well a particular person or group of persons learned a particular mathematical idea;
- Analyses: the social, cognitive and/or affective factors that influence/ interfere with the learning of (a) mathematics in general, (b) specific mathematical concepts or processes.

2.3.1. How people learn mathematics in general

Three papers were interested in developing theories or models of learning mathematics in general (*6; 34; 45; 48*). I have already described (*48*) above, so I will only speak of the remaining three.

The author of (*6*) looked at a number of theories about cognitive development and proposed a (selective) synthesis of these for the purposes of explaining and predicting processes of mathematics learning (calling it the theory of "fundamental cycles of cognitive growth"). One of the theories was the SOLO Model (Structure of the Observed Learning Outcome; Biggs & Collis, 1982).

Authors of (*34*) were interested in testing/illustrating the nested model of situated abstraction (the "RBC" model: recognizing—building-with—constructing) by applying it to describe the process of learning exponential functions by a pair of students. The authors were particularly concerned with observing the processes of construction and consolidation of mathematical knowledge in the students, and thus providing a more precise description of these processes in the model.

Report (*45*) aimed at characterizing flexible mathematical thinking based on the notion of flexible thought developed in neuropsychology, referring to the "ability to use long-term declarative knowledge in novel situations". The authors proposed the following characteristics of flexible mathematical thinking: ability to interpret solutions different from one's own; using an idea across contexts; using multiple representations and being comfortable with ambiguous notations; raising "what if" questions; moving easily between an association and its reverse. These characteristics were illustrated with examples of concrete students' behavior in novel mathematical situations.

2.3.2. Theories of learning specific mathematical concepts or processes

Four papers aimed at contributing to the development of theories of learning specific mathematical concepts or processes (*11; 14; 39; 41*).

Report (*11*) contained an analysis of the symbolic thinking involved in understanding and solving word problems using algebraic equations, which explained some students' reluctance to collect like terms in an equation representing the quantitative relations in the problem. The goal in (*14*) was to develop a consistent descriptive language for research on mathematical reasoning in young children; so far such language was mainly developed for adolescents and adults. Authors of (*39*) wanted to verify if the "illusion of linearity" misconception also holds in learning probability, as it holds in the domain of geometry, which they had studied before. The aim of (*41*) was to identify cognitive sources of students' difficulties in solving equations of the type $-x = a$, and, more generally, equations involving negative numbers.

2.3.3. Descriptions of learning a specific mathematical concept or process

As many as nine papers offered descriptions of individual processes of learning a piece of mathematics, with often a judgment of how well they have learned it (*1; 5; 9; 12; 26; 35; 37; 46; 55*). The samples of individuals ranged from 2 to 277.

Report (*1*) described two nine-year-old children constructing a schema related to the notion of equivalent fractions, in the context of a constructivist teaching experiment. The aim was to show that, contrary to what some researchers say, whole number mental schemas are not an obstacle to the development of schemas related to equivalence of fractions. Authors of (*9*) described how several 8–11 year-old children solved addition and subtraction problems, in terms of a theory of conceptual development in mathematics. The aim was to identify obstacles to learning elementary arithmetic in terms of the theory.

Report (*46*) looked at young children's solutions of sentences with unknowns, based on an a priori analysis of such tasks. Report (*55*) offered a description of 5th graders' mathematical argumentation in the context of a challenging task. Report (*37*) described secondary students' conceptions of algebraic inequalities.

Report (*5*) described how several 6–8 years old children expressed their ideas for randomness in a two-dimensional continuous space, through [software] tools for directing and redirecting the simulated movement of balls.

Authors of (*12*) identified personal definitions of the definite integral and ways of working with this concept in a sample of 41 English upper secondary school students.

Report (*26*) offered a description of four 12th grade students' conceptions of mathematical definition.

In (*35*) authors investigated 277 novice university students' understandings of a formal definition of an equivalence relation, especially their understanding of the quantifiers in this definition, by using the procedure-process-procept-concept theory of advanced mathematical thinking.

2.3.4. Identification and study of factors influencing/interfering in learning

Seven papers reported research aimed at identifying factors, which may influence the learning of mathematics (*10; 15; 16; 24; 40; 47; 51*). The factors identified and studied were:

- Social, such as the social background of the students and power relationships in the classroom, were found to influence the interpretation and solving of a word problem, which refers to contexts of students' out-of-school life (*10*);
- Environmental or contextual; report (*40*) asked, "what [features of a learning] environment assist students' conjecturing and proof?" An environment with only physical linkages was compared with an environment where both physical and Cabri models of linkages were used; report (*16*) compared the levels of students' interpretation of graphs of motion after activities in two kinds of environment: pencil and paper (control group) and activities with a motion sensor connected with a graphing calculator; in (*47*) the author was looking at the effect of students' exposure to different contexts (physical, geometric) in different classroom settings (standard, reflective plenaries) on students' understanding of vectors.
- Emotional, such as positive affect (*51*) and the implicit emotional memory system in an individual (*24*);

- Cognitive, such as mathematical background, the ability to distinguish between correct and incorrect proofs, and scientific reasoning skills (*15*).

2.4. Methodological Issues

Five papers were concerned with research methodologies (*13; 31; 44; 50; 53*).

The aim in (*13*) was to improve the way of using the SOLO model in identifying levels of students' responses and make it less arbitrary or prone to subjective interpretations. The research problem in (*31*) was to find a framework for studying why some students remain focused on a mathematical investigation and others abandon it. Report (*50*) tested the potential of videos with seemingly random experiments as a research tool in studies of students' conceptions of probability. Report (*53*) examined the usefulness of Stiegler's overlapping wave theory to map the distribution of Stages of Early Arithmetical Strategies in a large population of children.

In the essay paper (*44*), the author described the differences between a positivist and poststructuralist epistemology and questioned, in general terms, the methods and results of research in mathematics education.

2.5. Questions Related to Mathematics Teacher Education

In many papers teachers of mathematics featured as subjects of research, but problems specifically related to teacher education were tackled in only one paper. The paper was devoted to the development of tools for pre-service teacher education (*30*) and looked at the potential of activities based on watching and commenting upon videos of teacher-student interactions in pre-service teacher training.

2.6. Questions about Assessment Methods and Instruments

Problems related to the development of methods and instruments of assessment of students' mathematical competence were specifically addressed in (*42*). Authors critically analyzed tasks used in the 1995 TIMSS performance assessment for grade 8 students administered in the Netherlands, the testing circumstances and the marking protocol, and the students' responses.

3. Results of Research

3.1. Overview

Most papers reported on research in progress and their answers to the research questions were tentative. But researchers drew conclusions from their research, described the various tools of their research and these could also be considered "results". I counted 71 such results in the papers, 35 of which I called "conclusions", and 36—"products". Some conclusions confirmed previously obtained results or common knowledge (12 or ~17%); some appeared to be new findings (13 or ~18%), and some were refutations of previously obtained results or common knowledge (10 or ~14%). Some products were "practical" such as teaching proposals (12 or ~17%) and material products, e.g., software or videos (4 or ~6%). Other products (20 or ~28%) were theoretical contributions to: theories of teaching and/or learning (14 or ~19%); methodology of research (4 or ~6%); epistemology of mathematics (1); and philosophy of mathematics education (1). (See Figure 2 and Table 1.)

The results did not exactly match the research questions in terms of numbers of papers. For example, I listed 5 papers as concerned with methodology of research, but only 4 papers as containing a contribution to methodology of research as a "product". This is because (*44*) contained only a critique of epistemologies underlying certain research methodologies and described some aspects of a different epistemology (poststructuralism), but did not translate this epistemology into actual research actions, or a constructive and usable alternative methodology of research.

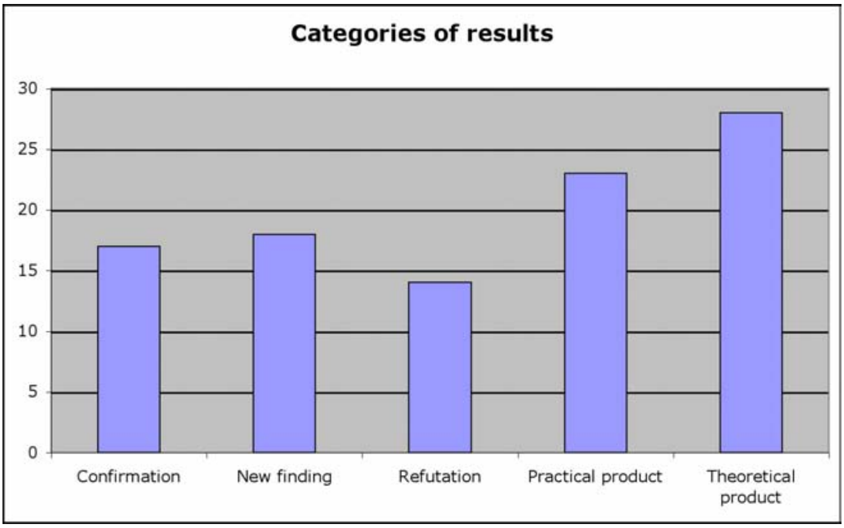

FIGURE 2. Categories of results

Type of result		List of papers in which the result appeared	No. of results in category
Conclusion	Confirmation	2; 4; 9; 10; 11; 15; 26; 32; 35; 37; 39; 51	12
	New finding	8; 16; 29; 30; 32; 35; 38; 40; 42; 46; 47; 53; 55	13
	Refutation	1; 5; 8; 12; 17; 23; 25; 33; 42; 52	10
Theoretical product	Contribution to the teaching/ learning of math	6; 8; 11; 14; 18; 21; 24; 27; 34; 41; 43; 45; 47; 48	14
	Methodological tool	13; 31; 51; 53	4
	Epistemological analysis	26	1
	Philosophy of math. education	44	1
Practical product	Material product	5; 30; 50; 54	4
	Teaching proposal	2; 3; 4; 7; 8; 19; 20; 22; 28; 36; 49; 54	12

TABLE 1. Types of results

3.2. Details and Comments on the Results

3.2.1 Confirmation

One sometimes wonders if large-scale studies that only confirm what we know from ordinary teaching experience are worth the effort. An example of such common sense knowledge is: if a student has the necessary declarative knowledge, good judgment about mathematical arguments, and a scientifically oriented mind then he or she is better prepared for proving in mathematics than if he or she has important gaps in declarative knowledge, cannot tell a correct proof from an incorrect one, has a non-scientifically oriented mind and cannot solve even simple word problems related to plausible situations. This sounds trivial. The following conclusions from a study involving 669 Grade 7 students do not sound as trivial, although they say just a little bit more than that:

> [M]athematical knowledge, the ability to evaluate correct and incorrect proofs, and scientific reasoning influence students' performance in proving.... Students have difficulties in identifying incorrect solutions as incorrect. It is significantly easier for them to classify correct solutions as correct.... [I]t is easier [for the students] to evaluate proofs than to formulate proofs by themselves..... [T]here are highly significant differences concerning the students' knowledge of proof methods and their evaluation of the correctness of proofs.... In the domain of scientific reasoning ... students are often guided by plausible arguments even if these arguments are not logically consistent.... If students could not solve a plausible task, they could not solve tasks presented in an unusual context either. On the other hand, if they could solve plausible tasks they had a high probability to solve tasks presented in an unusual context. (*15*)

Another large-scale study confirmed that students' learning depends on the qualifications of their teachers (*32*). The importance of the teacher's interventions was also a conclusion in a report from a classroom observation of Cabri-geometry activities aimed at the development of conjecturing and proving (*2*).

Research reported in the papers confirmed also massive students' difficulties with, and common misconceptions related to: abstract thinking (*35*), equations (*46*), inequalities (*37*), and probability (*39*). Small-scale studies also confirmed students' difficulties with mathematical thinking, its more abstract and formal aspects (*4; 9; 10; 11; 26*).

It is certainly important to take into account the relationship between affect and cognition, in teaching and research, but there was nothing new or surprising in the following conclusion:

> The characteristics of students undertaking task-centered learning can be seen in instances of learning through 'discovered complexity' ... which occurs during task completion where a group of [students] perceive intellectual and conceptual complexities not evident at the commencement of the task.... When a complexity is discovered, the group spontaneously formulates a question that leads to higher-level thinking ... [and gets into a state of] 'flow' [which is] an optimal learning condition that may occur when a person works just above their present skill level on a challenge almost out of reach. Individuals or groups in flow become so engrossed with the task at hand that they lose awareness of self, time and the world. (*51*)

3.2.2. Refutation

Results that put into question our previous beliefs or findings have a more revitalizing effect on a domain of research than those that merely confirm them. At least they stir some discussion and provoke an elaboration of our ways of thinking about a phenomenon.

In a study of 41 English high school students' definitions and images for the concept of definite integral, authors of (*12*) found that students who learn the concept in a conceptually and experientially oriented curriculum (involving classroom discussions of conceptual ideas, experiencing them in use and going back to refine the conceptual ideas) seem to have no less difficulty with the concept than those who learn it through a procedural approach. Also (*42*) reports a disappointment with a progressive curriculum (Realistic Mathematics Education in Dutch schools), which has not produced better performance in students, this time—in the expected practical, investigational skills using mathematizing.

Another hope crumbled under the evidence provided in (*25*). It is unfortunately not true that, "by including items that require students to solve more complex types of problems, teachers will be more likely to provide opportunities to do the same in class". The large-scale research in US involving 63 4th grade teachers has shown that,

> while teachers are aware of the test and have made some instructional changes in terms of specific teaching strategies, the changes that have been made tend to focus mostly on strategies and techniques such as use of small group instruction or manipulatives [and writing] rather than changes in, for example, the nature of discourse that is taking place in the classroom.... In almost two thirds of the lessons where manipulatives were used, they were used in a very procedural manner, where the teacher generally told the students what to do with the

> materials, and the students did it as best they could.... [C]lassroom discourse did not foster substantive conversations among students.... [Teachers] rarely insisted on [students finding and understanding multiple strategies for solving problems].... In almost 80% of all cases the teacher rarely asked students if their answers were reasonable. If a student gave an incorrect response, another student provided, or was asked to provide, a correct answer, but there was little discussion of an appropriate strategy to solve the problem. In an additional 15% of all cases, the teacher may have asked students if they checked whether their answers were reasonable, but did not promote discussion that emphasized conceptual understanding. (*25*)

This finding suggests that, if "teachers teach to the test", it is not because teachers will teach to any test, but because tests reflect the ways teachers most commonly teach. When a new type of assessment is introduced, after some years of its use, it is modified to better fit with the teachers' practice: tests are transformed to match teaching, not the other way round.

A successful program involving 80 pre-service teachers has been used as a counter-example in (*8*) against the findings of previous research that "mathematics education programmes appear to have only a limited effect on pre-service elementary teachers capacity and willingness to change their teaching".

In observations of small numbers of children or students working in specially designed environments, researchers found counter-examples to:

- Statements about concept formation such as, whole number multiplicative mental schemes function as obstacles to the construction of schemes related to fractions (*1*);
- Piagetian theory of the construction of the idea of randomness in children (*5*);
- The pedagogical principle that teaching should proceed from the concrete to the abstract (*17*);
- The belief that teaching and learning aided by a Computer Algebra Systems will necessarily support the development of more conceptual thinking because the system will take care of the procedural and computational processes (*52*).

Report (*33*) showed examples that teachers' perceived factors of professional development and change are different from those identified by researchers because (a) they are difficult to reveal in short-term studies, and (b) they are often not considered as important by researchers.

Based on interviews and observations of classes on the same topic (Thales theorem) of two teachers with the same mathematical background, the author of (*23*) concluded, not surprisingly, that the mathematical background does not determine the teaching style and approach, or even the perceived curricular objective of the unit. I was hesitating whether I should classify this result as a "confirmation" or a "refutation". The author seemed to view the result as a refutation, so I counted it in this category, although the statement is so obviously false that its rejection is not worthy of an argument.

3.2.3. New finding

Many findings, if not exactly confirming some well-known statements, were not very surprising. For example, the fact that, at the beginning of their studies, only 13% of a population of 80 pre-service teachers considered mathematics as "challenging problem solving" while about one third (31%) found mathematics "difficult and unpleasant" (*8*), is not a revelation for anyone who has ever worked with pre-service elementary teachers. Nor will it surprise anyone that mathematics teachers' beliefs about gender differences in mathematics are not much different from common stereotypes such as "girls avoid using intelligence" and "boys attract most of teacher attention" (*29*). Perhaps the result is a bit disappointing for those of us who hope that teachers "should know better".

The general trends in the distribution and changes with age of the most advanced addition and subtraction strategies that a child can use among 23 121 children between 4 and 9 years of age in Australia (*53*) are also not unexpected. But the paper also contains a description and analysis of the strategies and refers to a sets of tasks specially designed for revealing the strategies in children, and this is less trivial. Too bad, though, that there is so little information about the design of the tasks in the report.

An interesting observation was obtained as a by-product of a research on the use of videos in teacher education (*30*). In the frame of a "larger project aimed at helping prospective elementary teachers better understand the depth of knowledge necessary for teaching elementary school mathematics ... 4 prospective elementary school teachers, 4 experienced elementary teachers, and two mathematics educators (with doctorates)" were observed while watching a video of several teaching scenes between a teacher and one 5th grade girl on the subject of fractions (assessment of her knowledge of relative magnitude of fractions; procedural teaching followed by practice; and teaching with manipulatives and conversation), and interviewed afterwards. The surprising and rather shameful (for mathematics education researchers) observation was that, of the three groups, the mathematics education researchers revealed the lowest level of cognitive functioning (this was measured using a technological device) during the observation of the video. Teachers were the most active cognitively. The mathematics educators "made comments on the structure of the interview, on expectations, and on technological aspects of the videotape". Teachers commented on the child's conceptual problems with fractions and tried to understand the child's thinking; they commented also on the child's attitude. They made conjectures about the reasons behind the child's reluctance to make sense of the symbols of fractions and operations with them and blamed the kind of teaching she must have been exposed to. They tried to think about ways of teaching the child the concept ("something concrete to help her understand"). They were also more observant than mathematics educators about little details of the student's behavior (e.g., she never used the word "equals"). Mathematics teachers had expectations about the situation based on their own experience and had "a strong interest in mathematics learning, and thus found this video interesting, causing them to reflect on what they were seeing".

Some studies offered modest conclusions from *descriptions* of teaching/learning situations (*38; 46; 55*). For example, a detailed analysis of the author's own teaching experience (related to fractions) in (*38*) allowed him to highlight what worked and what did not and why. The report (*55*) provided evidence that in appropriate teaching/learning environments even quite young students (5th grade) are able to provide quite sophisticated mathematical arguments.

Identification of factors responsible for bringing about desired students' cognitive behaviors figured in conclusions of several papers, but the identified social, emotional and cognitive factors were not new, and the evidence supporting the impact of the environmental factors was not very strong. For example, in (*40*) researchers compared two learning environments for Grade 8 students aimed to engage them in conjecturing and proving: one using only mechanical linkages and the other both the mechanical linkages and their Cabri-geometry models. The report contained a description of the behaviors of two pairs of students each working in one of the two environments. Conjecturing and correct proofs were obtained more easily in students who worked both with the mechanical and the computer models. From this observation the authors concluded,

> It would seem, then, that the imagery ... of ... Cabri made a substantial contribution to the conjecturing process, but also challenged these students to produce an explanation. The features of dynamic geometry software—constructions based on Euclidean geometry, accurate measurements, the tracing of loci and the drag facility—make the software highly suitable for exploring the geometry of linkages. Rather than eliminating the need for proof, the convincing evidence and the unique opportunities for exploration and discovery that the software provided gave the students the confidence and desire to go ahead and prove their conjectures. However, the tactile experience and satisfaction of working with actual physical linkages may also represent a significant motivational aspect, at least for some students, and should not be overlooked. (*40*)

Thus, although the two learning situations were different in many ways, and only two pairs of students were observed, the authors pointed to the use of Cabri as the main factor of success of one of the pairs.

Similar criticism could be made with respect to research reported in (*16*) in which the cognitive behaviors with respect to graphs of motion in two groups of students were compared. In the experimental group the instruction used graphing calculators connected with a motion sensor and in the control group the motion sensor device was not used. The comparison was made based on students' ability to interpret the drawn graph of motion in the two groups. The experimental group's understanding was reported to have evolved towards a more mathematical and correct interpretation of the graph in terms of increasing and decreasing functions ("transformational reasoning"). The control group's understanding was characterized by a confusion of distance and velocity, an incorrect interpretation of the graph of a motion, and lack of transformational reasoning (only about 3–4 out of 16 students in the control group reached a correct interpretation of a graph of motion). The authors concluded that a learning environment in which students have an opportunity to go back and forth between a mathematical model of a motion and the physical experience of the motion is more likely to produce a better understanding of graphs than an environment which does not offer such opportunities.

Another comparison of learning environments was made in (*47*), which led the author to point to "reflective plenaries" classroom setting and "embodied" approach to the teaching of vectors as bringing about more success on a certain type of tasks involving vectors among students. It was unclear to me, in what way these contexts were different from "standard" approaches and how it would be possible to eliminate other aspects of the classroom situations (including teachers' actions and students' motivation and abilities) to conclude that these two factors were indeed essential in bringing about the differences in performance.

Other results which I counted as new findings were contained in (*32*)—description of effective teaching practices; (*35*)—description of students' misconceptions about equivalence relation; (*42*)—explanation of unexpectedly low results on certain TIMSS tasks in Dutch students.

3.2.4. Material products

The material products described or briefly mentioned were:

- A computer environment for the development of the idea of randomness in 6–8 year old children (*5*);
- Video recordings of interview and teaching situations for the purposes of pre-service teacher education (*30*);
- Video recordings of seemingly random experiments for the purposes of research on students' understandings of probability (*50*);
- Cartographical software for young (seven year old) students (*54*).

3.2.5. Methodological tools

Research focused on methodological questions produced concrete methodological tools in three papers (*13; 31; 53*). But also (*51*) described in detail a methodology of transcript analysis from group work, and this can be considered a (by-)product of the research. This methodology was used in the aim of identifying students' autonomy, spontaneity and creativity in the abstraction process, based on a synthesis of three perspectives: Cognitive (the RBC model of situated abstraction); Social (external vs internal motivation to go on with the task); and Affective (indicators of positive affect were assumed to be: eyes on the task; bodies leaning towards the task; unaware of the world around; participating in the interaction; students building on each others' comments ["latching"]; exclamations of pleasure).

3.2.6. Contributions to theory development

Researchers contributed to theories of learning and of teacher's professional knowledge, and to epistemology of mathematical concepts and processes.

Most contributions to theory came, of course, from research explicitly aiming at theory building, namely theory of learning mathematics in general (*6; 34; 45; 48*) and theory of learning specific concepts and processes (*11; 14; 41*). Contributions to theory came also from research focused on factors influencing teaching practices (*8; 27*) and learning (*24; 47*). However, the theoretical contribution of this research did not always consist in only the identification of these factors. The author of (*47*), for example, proposed a synthesis of two theories of learning (embodied character of knowledge and process-object encapsulation of actions). Contributions to theory of teaching were found also in papers focused on actual teaching practices (*18; 21; 43*); finding a language to analyze and structure teaching practices resulted in theory construction.

Results	Confir	Findi	Refut	Theo	Meth	Epist	Phil	Mater	T.prop
Questions Teaching practices	32	32	52	18; 21; 43					
Factors of teaching	8; 29	8; 23; 33; 25	8; 27						8
Teaching experiment	2; 4	38	17					54	2; 3; 4; 7; 19; 22; 28; 36; 40; 54
How to best teach									20
Theory Learn-general				6; 34; 45; 48					
Theory Learn-specific concepts	11; 39			11; 14; 41					
Description of learning	9; 26; 35; 37	35; 46; 55	1; 5; 12			26		5	
Factors of learning	10; 15; 51	16; 40; 47		24; 47	51				
Methodology		53			13; 31; 53		44	50	
MTE		30						30	
Assessment		42	42						

TABLE 2. Research questions and results

3.2.7. Teaching proposals

If a paper contained an account of a teaching experiment, this description counted as a "teaching proposal" (*2; 3; 4; 7; 8; 19; 22; 28; 36; 49; 54*). If a paper argued how a subject could be taught based on theoretical or empirical arguments, this also counted as a teaching proposal (*20*).

Table 2 contains an overview of the relationship between research questions and research results in the 55 papers in volume 4 of the *PME 26 Proceedings*.

If I may risk a conjecture, this table suggests that the most interesting results about teaching, i.e., refutations, are more likely to come from search for factors influencing teaching than from descriptions of teaching or teaching experiments. However, description of learning processes turned out to be quite fruitful in refutations. This could perhaps be explained by the fact that we have more developed theories about learning than about teaching mathematics and therefore our descriptions of learning are more analytical, while descriptions of teaching are more phenomenological. Search for factors influencing teaching practices brings in an analytic approach that sharpens our understanding of the processes and yields more interesting results.

4. Mathematical Tasks as Tools in Teaching and Research

The notion of "mathematical task" and the history of the term in mathematics education would require a separate dissertation. There have been debates related to its meaning, and this meaning has changed over the years and fashions in educational discourse. For some time it even seemed to be banned from researchers' vocabulary as reflecting a backward philosophy of mathematics and mathematics education. Its place was taken over by expressions such as "problem-situation", "activity", "didactical situation", "teaching / learning situation", etc. The term "task" evoked the dark ages of "task analysis" and "task variables" in a mathematics education without the cognitive subject, never mind the child, the person, the student, the learner, the teacher, the socio-cultural group, or the community of practice (see, e.g., Goldin & McClintock, 1979, which represents already a huge progress with respect to those dark ages by mentioning "context" and "situation" variables, but still gives no account of actual students' reactions to the tasks whose variables are analyzed).

But, no matter what emotive meanings we attach to mathematical tasks and how we want to call them, they remain the very fabric of mathematics teaching and learning. They are also a crucial component of many research projects. Mathematical tasks can be regarded as tools of research on a par with methodological tools such as statistics or coding schemes for qualitative data analysis. A teaching proposal contains tasks, and tasks are also needed to probe the effectiveness of a teaching approach. Studies in the psychology of mathematics learning require specially designed tasks. Asking students to tell us how they learn or understand a given concept may not give us much information about students' actual learning habits and understanding.

In this paper, I use the expression "mathematical task" in a broad sense to refer to any kind of mathematical problem, with clearly formulated assumptions and questions, known to be solvable in predictable time by students. A mathematical task may be open (e.g., not all information may be given; there may be many solutions or even interpretations), but it is not a research mathematics problem for which mathematics experts have not so far found a solution.

Different tasks are needed for different purposes. Students' response may be very sensitive to even small changes in a formulation of a task, or its mathematical, social, psychological, and didactic contexts. This is why I think it is so important to justify the choice of the mathematical tasks used in a research, not just in terms of the general goals and theoretical framework of the research, but in terms of the specific characteristics of the task. A task may be set in different contexts and formulated in different ways; it is important to be aware of the possible variants and reflect on the influence on the results of the research of the choice of one of these variants rather than another. This reflection makes explicit the boundaries of the generality of conclusions that can be drawn from the research.

In the 55 research reports I am looking at in this paper, 47 used one or more mathematical tasks, for various purposes. The 8 reports where tasks were at most alluded to or mentioned but not explicitly formulated as part of the research were *8; 21; 24; 25; 29; 31; 33; 44*. In (*24*), there appeared a task, namely the equation $4 + 2x - 6 + 5x = 95$, but only in a citation. It was given as an example of a piece of mathematics that would trigger the feeling of anxiety in a student. In (*25*) no specific task was formulated, but the paper contained a categorization of tasks used by teachers. Two axes of categorization were given. The key of the first was the level of conceptual thinking involved; thus tasks could be categorized into (a) memorization only, (b) doing procedures where the focus is on producing a correct answer, (c) doing procedures to develop a deeper understanding, (d) tasks requiring complex and non-algorithmic thinking. The key of the other categorization was the level of creativity required in the solution; tasks could thus be categorized into (a) practice tasks (imitate teacher's procedure) and (b) non-practice tasks (invent new solution method, analyze a mathematical situation, or generate a proof).

I will now look at the 47 reports where particular mathematical tasks were formulated and used in research

4.1 Presentation of Tasks in the Reports

4.1.1. Kinds of presentation of the tasks

I distinguished (and coded) the following kinds of presentation of the tasks, according to whether or not they were justified (and how task-specific was the justification) and problematized.

11TJ := The particular given task is justified, i.e., it is explained how the given task allows to reach pedagogical or research goals. The justification is based on explicit principles or theory and refers to the specific features of the task.

10TJ := A general rationale for the task is provided, i.e., it is explained what goals the task was expected to achieve, but it is not explained exactly which features of the particular task were essential in achieving these goals.

00TJ := The task is not justified.

1TP := The task is problematized, i.e., variations on the task are debated and there is a discussion of the effects of such variations on the learning or on the research results.

0TP := The task is not problematized.

Table 3 gives the frequencies of the 6 categories of presentations in the reports, in absolute values of number of papers and percentages relative to the 47 papers with tasks. Table 4 lists the reports corresponding to the categories of presentation.

Thus 33 (70%) of the 47 reports contained at least some justification of the tasks, but only 18 (38%) problematized the tasks. However, a complete, specific justification and problematization of tasks was found only in 14 (30%) of the reports.

	TP		
TJ	0	1	Totals
00	13 (28%)	1 (2%)	14 (30%)
10	10 (21%)	3 (6%)	13 (28%)
11	6 (13%)	14 (30%)	20 (43%)
Totals	29 (62%)	18 (38%)	47 (100%)

TABLE 3. Frequencies of different task presentations in all 47 reports with tasks

	TP	
TJ	0	1
00	*2, 6, 9, 13, 14, 18, 19, 27, 30, 43, 45, 51, 53*	*10*
10	*1, 7, 15, 16, 17, 20, 22, 34, 35, 41*	*3, 40, 42*
11	*11, 12, 26, 36, 49, 55*	*4, 5, 23, 28, 32, 37, 38, 39, 46, 47, 48, 50, 52, 54*

TABLE 4. Reports with task presentations of different categories

	TP		Number of papers
TJ	0	1	
00	*2; 19*	—	2
10	*7; 20; 22*	*3*	4
11	*36; 49*	*4; 28; 54*	5
Totals	7	4	11

TABLE 5. Frequencies of different task presentations in the 11 teaching proposals with a task

It would be reasonable to expect that the 11 (out of the 12) teaching proposals which contained a mathematical task (there was no particular mathematical task given in (*8*)) would be awarding more attention, than the papers in general, to justification and problematization of the tasks. However, while nine of these papers offered some kind of justification, only 3 both fully justified and problematized their tasks (*4; 28; 54*). In three of the proposals the justification was done in very general terms and tasks were not problematized (*7; 20; 22*). In two proposals the tasks were justified in specific terms but not problematized (*36; 49*). In total, an even smaller percentage of research proposals (36%) problematized the tasks than the entire sample of papers (38%). (See Table 5.)

Of course, the reader might say that it is not possible to fully justify the choice of a task in a short, 8-page report for a conference. This is true, but the fact that the authors, in making choices about what to write about in a report, massively decided to sacrifice exactly this justification, does say something about what we value and what we don't in the mathematics education research community.

4.1.2. Types of tasks

Inspired by a categorization of tasks proposed by Crespo (2003), I identified tasks used in the reports according to the following characteristics: routine vs. non-routine, contextualized vs. non-contextualized, exploratory (open) vs. non-exploratory, allowing for multiple vs. single interpretation, using a computer environment vs. another kind of environment. Each paper was coded with a string of letters each preceded by 0 or 1, representing the above characteristics: xR-xC-xE-xM-xK. For example, the string 0R-1C-1E-0M-1K represents a task that is non-routine, contextualized, exploratory, but not allowing for multiple interpretations of its formulation, and situated in a computer environment.

In the 47 reports with tasks, 40 (85%) used non-routine tasks; 31 (66%) referred to out-of school contexts; in 35 (75%) the tasks were exploratory; 30 (64%) allowed for multiple interpretations; and 29 (62%) were to be done in a computer environment.

4.2. Some Details about the Classification of Task Presentations

This section is meant to give an illustration of how the presentations of the tasks in the papers were categorized. It is impossible, in the frame of this paper, to justify why each presentation was classified as it was, because this would require devoting about half a page per each of the 47 papers, which contained a mathematical task.

4.2.1. Task justified in general terms and not problematized (10TJ-0TP)

Let me start by giving an example of a paper where the task was justified in general terms but not in terms of its specific features in relation to the goals of the task. In report (*1*) (classified as 10TJ-0TP-0R-1C-0E-1M-1K) the aim of the research was to "test the hypothesis" that "children could reorganize their whole number knowledge to build schemes for working with fractional quantities and numbers ... in meaningful ways". The aim of the particular teaching episode reported in the paper was to engage students in constructing the scheme of "commensurate fractions" or an early children's idea of what is known as equivalent fractions. The tasks used in the episode were about composition of fractions (e.g., children expressing $^1/_2$ of $^1/_3$ of a pizza in terms of a fraction of the pizza and explaining how they worked out the result). The tasks were about fractions and the study was about children's construction of fraction mental schemes, therefore the choice of the task is justified in a general sense. But little is said in the report about why the context of composition of fractions was considered propitious for the development of commensurate fractions scheme:

> In the teaching episode that took place in April in the second year, we introduced composition of fractions as a problem situation that might bring forth recursive partitioning and an awareness of the inverse relation between the resulting fraction and the original whole. The context was established using our computer environment TIMA: Sticks to represent

> pizzas that could be cut into so many slices.... The theme of baking pizzas continued in the next teaching episode.... Our goal was to provoke the children into thinking about different fractional names for quantities of pizza, based on a number of slices in a pizza.... In the next teaching episode, we decided to use a 24-part pizza stick.... (*1*)

Two children's responses to the tasks support the researchers' intuition that the context "might" be good for what they wanted them to achieve, but this does not justify why the researchers have chosen this context. Was it just a lucky guess or were there theoretical reasons that have guided the researchers?

In (*34*) (classified as 10TJ-0TP-0R-1C-1E-1M-1K) only a general justification of the tasks could be found. Underlying the report was a larger project of developing problem situations for high school whose aim was to foster a function approach to algebra. In particular, the activities aimed at the construction, by the students, of models of growth phenomena using algebraic variables. The "general justification" was:

> The problem situations, which are the milestones of the algebra course, were designed to give opportunities to students' construction of new knowledge structures concerning mathematical concepts (algebraic variables and models) and of various mathematical processes (hypothesizing, making generalizations, testing hypotheses, interpreting representational information, solving and justifying). (*34*)

The "problem situations" discussed in the report were concerned with exponential growth, but the situations were contextualized and did not contain any explicit algebraic formulae. Construction of an algebraic model was hoped to be the result of the students' explorations of the situation.

4.2.2. Task justified in general and specific terms and problematized (11TJ–1TP)

There were 14 reports within this category of presentation of mathematical tasks (*4, 5, 23, 28, 32, 37, 38, 39, 46, 47, 48, 50, 52, 54*). All tasks were non-routine; 6 were contextualized, 8 exploratory, 10 open to multiple interpretations, and 4 were situated in a computer environment. I will give a few examples to illustrate how the decision about classifying a presentation as "task justified in specific terms", "task problematized", "exploratory task", "task open to multiple interpretations", "contextualized task" was made.

In the study on students' understanding of inequalities described in (*37*) non-routine (for the participating student populations), but not situated tasks nor tasks open to various interpretations were used (11TJ-1TP-0R-0C-0E-0M-0K). There were three tasks:

> Task I: Examine the following claim: for any a in $\mathbf{R}$, $ax < 5 \Rightarrow x < 5/a$;
> Task II. Examine the following statement: for any $a \neq 0$ in $\mathbf{R}$ $ax < 5 \Rightarrow x < 5/a$;
> Task III. Solve the inequality $(a - 5)x > 2a - 1$ where x is a variable and a is a parameter. (*37*)

I classified these tasks as non-routine because, according to the authors, "parametric equations and inequalities are not commonly discussed in classes" in the countries of the participants of the study.

The focus of the study was "dividing an inequality by a non-necessarily positive factor". The choice of the tasks was justified as follows:

> Research findings indicate that when solving rational inequalities, students frequently multiplied both sides of the inequality by a negative number without changing the direction of the inequality.... It was also reported that students encounter difficulties when solving mathematical tasks, presented in a way different from the way they are used to.... Taking into account these data, we constructed tasks I, II and III.... The tasks were designed to provide information regarding students' distinction between the sufficiency of the '$a \neq 0$' condition given in Task I, and the insufficiency of the '$a \neq 0$' condition given in Task II, a condition that is sufficient in case of equations. Our aim was to see whether students regarded this limitation as sufficient for the inequality as well. While Tasks I and II asked the students to examine the claim regarding the equivalence of parametric inequalities, Task III dealt with the same issue in a different manner, asking the students to solve a similar given parametric inequality. (*37*)

This explanation was qualified as a "specific justification", because it referred to the specific features of the tasks and not only to general theories or principles. In their discussion, the authors problematized the choice of the tasks, showing how misleading it would have been if only Task I was given to the students. Here is the beginning of this detailed discussion:

> [E]xamination of ... solutions to Task I revealed almost no intuitive, erroneous ideas. Most students correctly rejected the statement.... [M]any students used zero as their counterexample.... Had we stopped here, we might have assumed that most students have a good formal understanding of such parametric inequalities. However, an examination of their responses to Tasks II and III revealed that this was not the case.... (*37*)

In (*39*) (11TJ-1TP-0R-0C-0E-0M-0K) authors referred to past research where they have tried "to identify influencing task variables" to explain students' tendency to "deal with each numerical relation as though it were linear" (the illusion of linearity). They looked at the phenomenon in geometry then and presently set up to look at it in the domain of probability. In particular they started looking at the scope of the linearity illusion in the binomial chance situation: will students consider the relation between the variables n, k, and p and the final probability P in this situation to be linear? [The relation is, theoretically, as follows: If in an experience two outcomes are possible, one (called "success") with probability p and the other (called "failure") with probability $1–p$, then the probability of obtaining k successes in n independent repetitions of the experiment is: $\mathrm{P}(\mathrm{S}_n = k) = \mathrm{C}(n,k)\, p^k\, (1–p)^{n-k}$]. More precisely, the questions were: Do secondary school students have a good qualitative insight (in terms of higher/smaller probability) into the effect of a variation of the different variables (n, k, p) that determine a binomial chance setting? To what extent these students have a tendency to quantify these qualitative insights as proportional relationships between n, k, and p on the one hand and P on the other? The authors discussed the following situation:

> The participants in a TV game can roll a fair die 12 times. If they obtain 6 at least 4 times, they win a car. On Christmas day, the game leader is in a generous mood and tells the participants that they are allowed to roll the die 24 instead of 12 times, so that their chance of winning the car is doubled. (*39*)

In the game leader's reasoning, a linear relationship between n and P was assumed. But, authors say, one can vary the situation to claim linear relationships between k and P or p and P. One can also vary two variables. This discussion led the authors to base their research on a set of variations on the above situation. All items in their tests were about rolling a fair die. Here is an example of a qualitative item where n was varied:

> I roll a fair die several times. The chance to have at least 2 times a 3 if I can roll 4 times is (a) larger than, (b) smaller than (c) equal to the chance to have at least 2 times a 3 if I roll 5 times. (*39*)

Quantitative items included:

> I roll a fair die several times. The chance to have at least 2 times a 6 if I can roll 12 times is 3 times as large as the chance to have at least 2 times a 6 if I can roll 4 times. True or not true? Why?
> I roll a fair die several times. The chance to have at least 2 times a 5 if I can roll 6 times is equal to the chance to have at least 1 time a 5 if I can roll 3 times. True or not true? Why? (*39*)

The set of tasks was justified by the need to systematically vary the variables n, k, and p.

My decision to classify the presentation of the tasks in (*39*) as "Task Problematized" (1TP) was based on the following statement in the conclusion to the report:

> Another remaining question, which we will address in our future research, is to what extent our findings are significantly affected by the way in which the test items were administered to the students. It could be argued that so many students fell into the linearity trap because they were seduced to do so, by confronting them with proportional statements with which they had to either agree or disagree. The tendency to reason linearly might considerably decrease when an open-answer format is used. (*39*)

In (*50*) (classified as 11TJ-1TP-0R-0C-0E-1M-0K), problematization of tasks used to study and teach probability was the very source of the authors' research:

> A major hindrance to probability research with children is, ironically, randomness itself. In other areas of mathematics the concrete experiences needed to support children's learning can be prepared and predetermined. However, when engaging children in probability experiments and games, the outcomes naturally remain unpredictable. With adults and adolescents this difficulty is overcome by providing hypothetical situations and lists of outcomes, but a less abstract approach is generally desirable for children. A surprisingly little used solution is to create videos of seemingly random experiments. This enables researchers and educators to present predetermined and consistent experiences to different groups of children at different times. (*50*)

Authors reported on two studies where they used videos of such seemingly random experiments; they varied the tasks the second time round and discussed the effect of these changes on the students' responses. I considered the discussion of the variants of the task as another expression of "Task Problematized" feature of the presentation.

In the above examples, tasks were non-routine, not contextualized and not exploratory. An example of a task which is non-routine, not contextualized but exploratory and inviting multiple interpretations in a computer environment could be found in (*28*) (11TJ-1TP-0R-0C-1E-1M-1K), whose aim was to design tasks "to study the relationship between students' actions [particularly dragging] and reasoning in Cabri geometry problems". One of the tasks, given to a 19 year old student was the following:

> Draw a quadrilateral ABCD. Construct the perpendicular bisectors of each side. Label the four points (say, M, N, P, Q) at the intersection of the perpendicular bisectors from adjacent sides. If you move ABCD what happens to the inner quadrilateral MNPQ? (a) Investigate the relationship between the internal angle at A and the internal angle at M. (b) For what types of external quadrilateral is MNPQ a parallelogram? (c) Find a quadrilateral ABCD that is similar of its inner quadrilateral MNPQ, i.e., an enlargement/reduction. (*28*)

Multiple interpretations of the situation are possible. No assumptions about the quadrilateral ABCD are made, therefore different students could start from a very different initial figure (someone could draw a non-convex quadrilateral) and this could lead to different paths of reasoning. Concave ABCD is implicitly ruled out by talking about "inner" and "external" quadrilaterals, but this occurs only in question c). Also, the mutual position of points A and M is subject to multiple interpretations, so students may be answering different questions (b).

A similar task was also used in (*2*) (classified as 0TJ-0TP-0R-0C-1E-1M-1K):

> You are given a quadrilateral ABDC. Construct the perpendicular bisectors of its sides: a of AB, b of BC, c of CD, d of DA. H is the intersection point of a and b, M of b and c, L of c and d, K of a and d. Investigate how HMLK changes in relation to ABCD. Prove your conjectures. (*2*)

In (*2*), the theory of instrumentation was invoked in the introduction but not used to justify the choice of the particular task. It would be interesting to analyze the differences between these two tasks. For example, the task in (*2*) still contained the possibility of multiple interpretations with respect to the quadrilateral ABCD but no longer the ambiguity of the positions of the vertices of the quadrilateral obtained from intersections of perpendicular bisectors.

In (*28*), the choice of the task was justified in general terms by saying: "the task was structured to build up a sequence of reasoning" and reference to some principles of design of tasks in the Cabri environment: "equal consideration [should be given] to soft construction methods" and "the focus of observation should be the interplay between the physical/visual aspects of different modes of dragging and the language of argument". The specific justification was not very elaborate, but it was there:

> This construction is also described in Arzarello (2000) with students exploring the degenerate case, but this version starts with angle relationships, which are familiar to UK students, and has structured goals that leave open further enquiry. (*28*)

The author problematized the tasks she used in her research by saying, in the conclusions,

> It may be helpful not just to exploit the potential of soft dragging methods in making sense of and establishing a context for a theorem, but also to include support for reorganising the problem and expressing the context as a construction, which can be dragged in a desired way. (*28*)

Authors of (*23*) (classified as 11TJ-1TP-0R-1C-0E-1M-0K) referred to tasks related to the Thales theorem used by two teachers with similar backgrounds in their classes. The justification and problematization of the tasks was done using the voices of these observed and interviewed teachers. In describing teachers' actions, the authors described in general terms the types of tasks the teachers used, without precise formulation, but they reported the teachers' concerns with the tasks and their problematization of these. One teacher used problems involving a real life situation (whence the "1C" in the above classification of the paper) asking students to identify a proportion and to find the fourth proportional value. He attributed his difficulty in reaching this goal in the lesson to the inadequacy of the task he had given the students: "with the first two problems, the pupils had difficulties with the proportion they had to identify"; "I realized that the formulation of the problems was too long compared to what pupils were used to". He found that the real life context was too complicated; "I overestimated their abstraction abilities", etc.

5. Conclusions

5.1. Theory: Breadth without Depth

In my analysis of the 55 research reports, I have not focused specifically on the underlying theoretical frameworks. But it was difficult not to notice that theory occupied a lot of space. A great multitude of theories were mentioned in the papers and contributions to theory constituted about 19% of the results. However, theories mentioned in the introductions or "theoretical frameworks" sections did not always play an essential role in the research. Of course, in principle, a consistent use of a theoretical framework to, say, analyze a teaching episode, does make a difference in what is seen and how (see, e.g., Steinbring, 1998; or Even & Schwartz, 2003). The problem was, however, that theories mentioned in the "theoretical framework" appeared to perhaps "inspire" the research but they were not necessarily consistently applied in the research. It would be interesting to conduct an empirical study to describe the role that theories, mentioned in the "theoretical framework" sections of mathematics education papers, indeed play in the research. Work in this direction has already begun (see Lerman et al., 2002), and I am curious to see further publication of its results.

5.2. Weakness of Results

For all the theory production in mathematics education, conclusions from research remain shaky or weak. Do we really need more empirical evidence for people's difficulties with abstract mathematical thinking? Perhaps we could now use what we know about the distribution of thinking styles among students in designing our curricula, textbooks, and teaching methods so that they reflect this very distribution rather than the distribution that the "mathematicians in us" dream about (as proposed by Amitsur in an interview, Sfard, 1998). Only a few results obtained in the 55 research reports could count as strong, that is, both interesting (surprising) and well justified or documented. Many results were teaching proposals whose value is mainly inspirational, for teaching and for research. But they often did not contain information that would allow the reader to judge what were the essential variables of the teaching situations. Moreover, few researchers took into account the constraints of the teacher's work and factored these constraints into their teaching proposals.

5.3. Scarcity of Studies on Assessment

We also need to be more constructive (rather than merely constructivist) in producing and putting to the test the results that bridge theory and practice. One area in dire need of this kind of constructive research is assessment. This activity is one of the main activities of a teacher, and it is also the least pleasant and the most difficult one. Without a serious consideration of the problem of assessment, most innovative approaches and teaching proposals never make it through the phase of initial experimentation by enthusiastic researchers and committed teachers. Yet, relatively to other issues, very little research has been devoted to assessment in mathematics education. Among the 55 reports, one research focused on methods of assessment and assessment tasks, and important issues related to assessment were raised in two more papers.

5.4. Call for a Come-back of Task Analysis

This brings me to the problem of task analysis in mathematics education.

People are doing interesting teaching experiments but they choose to say very little about the reasons underlying the choice of the tasks and what difference would it make if they changed the tasks in this or that respect. This puts into question the possibility of their reproduction in different teaching or research conditions, because it is not known what is arbitrary and what is essential, and in what way, in the tasks.

Tasks in the reports were mostly 'exploratory', or part of initial activities used to introduce a new idea. This is understandable from the point of view of research, but then this research is very far from the everyday concerns of the teachers. If we assume that knowing is based on a system of connections, then we must allow students the time to build and consolidate these connections. The reports are silent on the problem of the construction of systems of tasks which aim at consolidation of knowledge.

In general, the tasks were not the kind of tasks that would be used in day-to-day teaching or assessment. In research we need tasks capable of revealing students' most hidden conceptions and misconceptions; we do not merely test students' knowledge; *we put it to the test*, we probe it, we want to know all about it. In school assessment we use tasks that allow students to prove *that* they have a sufficient grasp of the basic required material and skills. We are not even interested in knowing too much about *what* each particular student knows and even less in *how* he or she knows it. Two aspects of teaching could explain this: time constraints (evaluation of several classes of students in our charge must be done in a timely fashion) and the fact that school is interested in proving its success and not its failure. Research, on the other hand, has a lot of time to analyze students' work and is interested in proving that "traditional" teaching (any current or ordinary teaching is deemed "traditional") is ineffective and therefore in need of reform. This justifies research. Each institution defends its reason of being.

The need for analyzing and problematizing tasks could be explained from the point of view the methodology of research, but also from the point of view of our relations with two professional groups, teachers and mathematicians. These relations are not always very good (Bartolini Bussi & Bazzini, 2003; Goldin, 2003) and there is a lot of misunderstanding about what research in mathematics education is actually doing. But mathematical problems, of which tasks for learners constitute a subset, are a common ground on which all these three groups meet.

Mathematics education could substantially contribute to the discussion, if only it took the problem of tasks and task variables seriously again, capitalizing on all that has been found in the last 20–30 years about students' learning and teaching practices.

5.5. Final remark

We need a lot more critical studies that would question and find weak points in our research methodologies, theories, as well as conclusions and other products of research. We need, indeed, to be "stirred up"—cognitively stimulated in mathematics education research. There should be more debate, more sharp criticism, more hard analytic thinking about the phenomena we study and about the validity of our claims about them. Otherwise, like in the mathematics education researchers in (30), the level of our cognitive functioning will fall behind that of pre- and in-service teachers, whom we are supposed to guide in their professional development.

I don't know if my talk at the CMESG 2003 meeting in Wolfville managed to "stir up" the audience as much as expected by the program committee. I hope to have at least demonstrated that there is still a lot to do in mathematics education research.

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Appendix: Authors and Titles of Reports Included in Vol. 4 of PME26, 2002 Proceedings

1. J. Olive, 'The construction of commensurate fractions'.
2. F. Olivera & O. Robutti, 'How much work does Cabri do for the students?'
3. L. Outhred, 'Teaching and learning about measurement: Responses to the "Count Me Into Measurement" Program'.
4. C. Ouvrier-Buffet, 'An activity for constructing a definition'.
5. E. Paparistodemou, R. Noss & D. Pratt, 'Explorations of random mixtures in a 2D space continuum'.
6. J.E. Pegg, 'Fundamental cycles of cognitive growth'.
7. P. Perera & M. Valdemoros, 'Manipulative help in verbal sharing out of continuous and discrete wholes problem solving'.
8. A. Pietilä, 'The role of mathematics experiences in forming pre-service elementary teachers' views of mathematics'.
9. D. Pitta-Pantazi, E. Gray & C. Christou, 'Mental representations in elementary arithmetic'.
10. N. Planas & M. Civil, 'The influence of social issues on the re-construction of mathematical norms'.
11. L. Radford, 'On heroes and the collapse of narratives: A contribution to the study of symbolic thinking'.
12. S. Rasslan & D. Tall, 'Definitions and images for the definite integral concept'.
13. C. Reading, 'Quantitative justification of SOLO cycles'.
14. D.A. Reid, 'Describing young children's deductive reasoning'.
15. K. Reiss, F. Hellmich & M. Reiss, ' Resoning and proof in geometry: Prerequisites of knowledge acquisition in secondary school students'.
16. O. Robutti & F. Ferrara, 'Approaching graphs with motion experiences'.
17. T. Rojano, E. Filloy-Yague & A. Solares, 'Cognitive tendencies and the interaction between semantics and algebraic syntax in the production of syntactic errors'.
18. S. Rota & R. Leikin, 'Development of mathematics teachers' proficiency in discussion orchestration'.
19. G. Rubio, 'Solution of word problems through a numerical approach. Evidences on the detachment of the arithmetical use of the unknown and the construction of its algebraic sense by pre-university students'.
20. E.F. Ruiz-Ledesma, J.L. Lupiañez Gomez, M. Valdemoros, 'Didactical reflections on proportionality in the Cabri environment based on a previous experience with basic education students'.
21. K. Ruthven & S. Hennessy, 'A practitioner model of the successful use of computer-based tools and resources to support mathematics teaching and learning'.
22. A.I. Sacristán & E. Sánchez, 'Processes of proof with the use of technology: Discovery, generalization and validation in a computer microworld'.
23. V. Sánchez & I. Escudero, 'Integration of domains of knowledge in mathematics teachers' practice'.
24. W. Schlöglmann, 'Affect and mathematics learning'.
25. R. Schörr, S. Bulgar & W.Y. Firestone, 'Testing and fourth grade teaching'.
26. K. Shir & O. Zaslavsky, 'Students' conceptions of an acceptable geometric definition'.
27. J. Skott, 'Belief research and the actual and virtual communities of a novice teacher's practice'.
28. C. Smith, 'Designing tasks to explore dragging within soft constructions using Cabri-géomètre'.
29. R. Soro, 'Teachers' beliefs about gender differences in mathematics: "Girls or boys?" scale'.
30. J.T. Sowder & R.A. Philipp, 'Use of eye-tracking technology to determine the best use of video with prospective and practising teachers'.
31. N. Stehliková, 'A case study of a university student's work analysed at three different levels'.
32. P. Sullivan & A. McDonough, 'Teachers differ in their effectiveness'.
33. P. Sztajn, 'Changes in mathematics teaching: Learning with an experienced elementary teacher'.
34. M. Tabach & R. Hershkowitz, 'Construction of knowledge and its consolidation: A case study from the early-algebra classroom'.
35. D. Tall & E.-T. Chin, 'University students' embodiment of quantifier'.
36. W. Tsai, 'Connecting children's in-school with out-school mathematics by using mathematical writings'.

37. P. Tsamir & L. Bazzini, 'Algorithmic models: Italian and Israeli students' solutions to algebraic inequalities'.
38. R. Tzur, From theory to practice: Explaining successful and unsuccessful teaching activities (case of fractions)'.
39. W. Van Dooren, D. De Bock, F. Depaepe, D. Janssens & L. Verschaffel, 'Secondary students' illusion of linearity: Expanding the evidence towards probabilistic reasoning'.
40. J. Vincent, H.L. Chic & B. McCrae, 'Mechanical linkages as bridges to deductive reasoning: a comparison of two environments'.
41. J. Vlassis, 'About the flexibility of the minus sign in solving equations'.
42. P. Vos & W. Kuiper, 'Exploring the potential of hands-on investigative tasks for curriculum evaluations'.
43. T. Wallach & R. Even, 'Teacher hearing students'.
44. M. Walshaw, 'Telling other peoples' stories: knowledge production and representation lessons'.
45. L.B. Warner, J. Coppoly jr. & G.E. Davis, 'Flexible mathematical thought'.
46. E. Warren, 'Unknowns, arithmetic to algebra: two exemplars'.
47. Anne Watson, 'Embodied action, effect and symbol in mathematical growth'.
48. Anne Watson & J. Mason, 'Extending example spaces as a learning/teaching strategy in mathematics'.
49. J.M. Watson & B.A. Kelly, 'Grade 5 students' appreciation of variation'.
50. J. Way & P. Ayres, 'The instability of students' probability notions'.
51. G. Williams, 'Associations between mathematically insightful collaborative behavior and positive affect'.
52. C. Winsløw, 'Semiotic analysis of Dreyfus' potential in first-year calculus'.
53. R. Wright & P. Gould, 'Mapping overlapping waves of strategies used with arithmetical problems'.
54. N. Yiannoutsou & C. Kynigos, 'Seven year olds negotiating spatial concepts and representations to construct a map'.
55. V. Zack, 'Learning from learners: robust counterarguments in fifth graders' talk about reasoning and proving'.

Working Groups

Groupes de travail

L'histoire des mathématiques en tant que levier pédagogique au primaire et au secondaire

The History of Mathematics as a Pedagogic Tool in Grades K–12

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Introduction

The sessions were organized around six questions concerning ways in which the historical dimension of mathematics could be used in elementary and/or secondary classrooms.

1. Why should (or should not) historical perspectives be included in the mathematics classroom?
2. How can anecdotes and primary sources be used, and what qualities should they should have to be of benefit in mathematics education?
3. What are the benefits of introducing historical instruments (measuring instruments at the secondary level, calculating devices at the elementary level)?
4. How can studying the evolution of a mathematical concept improve students' understanding (secondary level)?
5. What interdisciplinary connections are fostered by historical approaches to mathematics (elementary level)? How are the history of mathematics and history in general related in the classroom?
6. David Wheeler's question from CMESG 1981,"What should a teacher know of the history of mathematics?", was modified to become "What sort of courses on the history of mathematics could/should be offered by Mathematics or Education departments?"

Questions 1, 2, and 6 were discussed by the whole group together, but for the others we split into two sub-groups to discuss aspects of historical approaches particularly relevant to elementary or secondary teachers.

THEME 1: Pourquoi introduire l'histoire des mathématiques dans les classes ?

Les enseignants se plaignent souvent de ce que les programmes de mathématiques sont trop chargés et qu'il est dès lors difficile de passer à travers dans leur intégralité. Dans ce contexte, l'utilisation de l'histoire des mathématiques ne peut-elle pas apparaître comme un élément additionnel qui vient densifier encore davantage le contenu du programme ?

Cette question se pose d'autant plus au Québec où les nouveaux programmes des niveaux primaire et secondaire contiennent une composante appelée « Repères culturels » qui prescrit l'utilisation de l'histoire des mathématiques dans les cours. Il convient donc de s'interroger sur les raisons motivant l'intégration de l'histoire dans les classes de mathématiques. Le groupe de travail a identifié quelques objectifs pouvant justifier cet usage. La liste qui suit se veut une synthèse des discussions du groupe. Elle ne suit en rien l'ordre chronologique de nos propos.

1) *Lutter contre le mythe voulant que les mathématiques furent l'objet d'une création spontanée.*
Dans la population en général, les mathématiques sont perçues comme étant statiques et ayant de tout temps la forme que nous lui connaissons aujourd'hui. L'idée même que puissent s'ajouter aux mathématiques des éléments jusqu'ici inconnus semble souvent incongrue. Dans cette vision des mathématiques, il y a fort peu de place pour la créativité et l'évolution. Les mathématiques sont ce qu'elles sont. Dès lors, apprendre les mathématiques s'apparente à peu de chose près à apprendre un poème. La mémoire est mise à contribution beaucoup plus que la compréhension. L'histoire s'attaque à ce mythe en mettant en évidence des moments où les mathématiques prennent une forme différente de celle qu'elles ont aujourd'hui ou, à tout le moins, les mathématiciens du passé marquent des étapes, des changements, des nouveautés modifiant le monde mathématique de leur époque.

2) *Fournir un environnement culturellement riche.*
Le mythe mentionné en 1 camoufle la complexité du monde mathématique, c'est-à-dire ce monde comprenant la discipline, les mathématiciens et l'intégration de ceux-ci dans leur univers physique, social et intellectuel. En saisissant cette complexité, même sans vraiment la comprendre, l'élève est placé en position de voir les mathématiques comme une activité humaine. Les mathématiques prennent alors une teinte culturelle qui enrichit à la fois leur perception des mathématiques et leur vision de la culture.

3) *Donner un sens à la question « Pourquoi et comment le monde est-il devenu ce qu'il est aujourd'hui ? »*

   Tout en reprenant en partie le thème précédent, le pourquoi et le comment de notre monde permet de cibler des éléments plus précis participant à la construction d'une conception des mathématiques en tant qu'activité humaine. Nous avons retenu quatre éléments en relation avec la question.

   a) Prise de conscience par l'élève de la permanence du monde.

      L'univers temporal de l'élève se limite essentiellement à la durée de sa propre existence. Or, pour prendre conscience de la nature évolutive des mathématiques, ou de toute autre activité humaine, il importe que l'élève prenne d'abord conscience de la permanence de l'univers. Seulement alors pourra-t-il concevoir que ce monde antérieur a pu influencer le sein.

   b) Développement d'un sentiment d'humilité par rapport à l'état actuel du monde et à la façon dont nous faisons les choses.

      Vivons-nous dans le meilleur des mondes ? Notre monde est-il meilleur que ceux des hommes qui nous ont précédés ? La naïveté de ces questions découle de l'absence de vision historique. Les civilisations se sont développées, ont atteint un apogée puis ont connu un déclin. L'histoire nous montre les réalisations remarquables de nos ancêtres. Nous avons nos façons de faire. Nos ancêtres avaient les leurs. Par là, nous pouvons prendre conscience qu'il y a d'autres façons de faire que les nôtres. En appliquant ces considérations aux mathématiques, il s'ensuit que nos mathématiques ne constituent qu'une forme de mathématiques et qu'il y a d'autres façons de faire des mathématiques que celles qui nous sont maintenant familières.

Ainsi, la vision des mathématiques comme une discipline très dogmatique se trouve élimée.

c) Prise de conscience que les éléments clés pour évaluer ce qui importe le plus en mathématiques changent d'une époque à l'autre.

Dans une société donnée, à une époque donnée, le contexte social et historique influencent fortement la place qu'y occupent les mathématiques. Les mathématiques servent à résoudre des problèmes. Son efficacité et son importance seront donc jugées à l'aulne de l'importance pour cette société des problèmes ainsi résolus. Cela n'implique toutefois pas que ces problèmes soient pratiques. En effet, le contexte intellectuel dans lequel s'insèrent ces problèmes influence leur nature. Ainsi, d'une époque à l'autre, d'une civilisation à l'autre, la nature même des mathématiques change, étant parfois purement intellectuelle, parfois essentiellement pratique, parfois encore à quelque part entre les deux. Il n'existe donc pas de critères absolus pour juger de l'importance et même de l'efficacité des mathématiques.

d) Développement chez les élèves du sentiment qu'ils peuvent contribuer à l'évolution de la société et des mathématiques.

Les trois éléments, a, b, et c, que nous venons de mentionner devraient développer chez les élèves la perception que les mathématiques s'inscrivent, comme toute autre activité humaine, dans le flux des grands mouvements historiques. Les mathématiques évoluant, elles ont fait l'objet de discussions, de controverses, et même de contestations. Elles ne doivent donc pas être traitées comme un dogme. Dès lors, il est possible de ressentir à son égard des malaises, des frustrations, ou un enthousiasme qu'il est possible d'exprimer. Les mathématiques se discutent. Ses interactions avec la société ambiante aussi se discutent.

4) Accéder à une intuition de ce que sont les mathématiques, par exemple de la complexité des liens entre application et théorisation.

Tout ce que nous avons dit jusqu'à maintenant présente une image beaucoup plus complexe des mathématiques que ce qui est habituellement véhiculé. Néanmoins, même si complexité rime souvent avec difficulté, il n'en reste pas moins que la complexité manifeste la richesse d'un domaine. L'étude de l'histoire de certains concepts peut se révéler très riche à cet égard. Par exemple le concept de fonction qui, pourtant relativement récent dans l'histoire des mathématiques, occupe une place centrale aujourd'hui. De même, les numérations anciennes qui, par leurs différences avec notre système actuel, permettent de mieux le comprendre et d'en saisir le fonctionnement.

5) Voir qu'il est normal que la compréhension d'un concept mathématique exige beaucoup d'efforts.

L'évolution des mathématiques s'étale sur plusieurs milliers d'années. Dans cette seule constatation, on trouve une indication que la construction des concepts mathématiques implique une période de mûrissement. L'étude des travaux des grands mathématiciens montre que même eux ont investi beaucoup de temps et d'énergie pour résoudre les problèmes qui les ont rendus célèbres. Plusieurs problèmes, ou conjectures, ne sont toujours encore résolus, et cela parfois plusieurs siècles après avoir été énoncés. Il est dès lors normal qu'un élève rencontre des difficultés à saisir sans effort l'essence d'un concept. Les élèves devraient prendre conscience de cela.

6) Percevoir la différence entre un raisonnement mathématique et un raisonnement historique

Un raisonnement historique est global alors qu'un raisonnement mathématique est local. Il s'ensuit qu'une preuve historique se démarque d'une preuve mathématique Pourtant, toutes les deux veulent convaincre. Elles le font cependant de manières différentes. Derrière ces différences se cachent diverses notions de vérité. Manipuler ces deux types de preuves, et donc manipuler ces deux types de vérités et de raisonnements, vient enrichir la notion de preuve.

7) Intriguer et motiver les élèves avec des anecdotes.

   Écouter une anecdote plaît toujours. Une anecdote fait habituellement ressortir le côté humain des événements quotidiens. Elle comporte aussi une organisation des actions et des faits qui surprennent et intriguent. Elle met en valeur le caractère et le comportement des mathématiciens. Toutefois, même si les anecdotes permettent de mieux connaître un mathématicien, dévoilent-elles pour autant quelque chose sur les mathématiques elles-mêmes ? Rarement. Les anecdotes méritent certes une place dans une classe de mathématiques, mais il ne faudrait pas se limiter à cette seule présence de l'histoire dans les cours.

8) Établir des ponts avec les autres disciplines.

   Contrairement aux mathématiques scolaires, les mathématiques vivantes ne se sont pas développées en vase clos. Aussi, l'histoire se révèle un très bon prétexte pour établir des ponts avec les autres disciplines, idéalement par des activités interdisciplinaires impliquant par exemple l'histoire et la géographie, mais aussi la langue.

9) Révéler, pour l'enseignant, des difficultés possibles que peuvent rencontrer les élèves.

   Le principe voulant que l'ontogenèse reproduise la phylogenèse est aujourd'hui fortement contesté. Néanmoins, l'étude du développement historique d'un concept mathématique et des difficultés et obstacles alors rencontrés peut aider les enseignants à débusquer des difficultés que pourraient rencontrer les élèves et à mieux planifier de la sorte leur enseignement.

Tous ces éléments pointent positivement vers l'utilisation de l'histoire dans les classes de mathématiques. Néanmoins, comme le souligne Elaine, l'histoire peut aussi avoir des effets négatifs. Ainsi, les livres d'histoire destinés aux élèves ne disent pas tout et court-circuitent même parfois la vérité mathématique. Elaine donne l'exemple d'une enseignante du primaire qui croyait fermement que π valait $^{22}/_{7}$ après avoir lu des textes à saveur historique. Par ailleurs, certains enseignants ne croient pas que l'utilisation de l'histoire puisse se faire de façon adéquate, étant donné surtout le temps limité dévolu aux mathématiques.

THEME 2: Anecdotes and primary sources.

Two pages of information about Gauss taken from *Historical Connections in Mathematics* (Reimer & Reimer, 1992) provided a starting point for discussing the value of anecdotes from the history of mathematics. Participants first commented on the potential danger of focusing too much on great mathematicians. The material we studied included anecdotes about Gauss' ability with numbers at a very early age, and some members of the group felt that such stories might send students the message that if they do not demonstrate a remarkable talent at an early age they will not be able to become mathematicians or even be able to understand school mathematics. Louis noted that such stories could also give the impression that mathematicians were "weird" people, but that not all mathematicians presented the same risk: anecdotes of Euler often portray him as a "normal" family man, surrounded by his numerous children.

There was also concern that anecdotes focus more on the mathematicians than the mathematics, but the famous story about the ten-year old Gauss' addition of the first hundred numbers was produced as a counter-example. Elaine explained how she had used this anecdote at the end of the "handshakes" problem-solving activity, after first asking the students to find their own ways to add the numbers generated in this investigation. It was suggested that students who reinvent Gauss' method should be congratulated for "thinking like a mathematician", and Irene commented on her students' delight at being told that they had rediscovered the Egyptian method of multiplication.

Bernard defended the focus on the mathematicians themselves, suggesting that people were always interested in knowing about geniuses and compared the situation to music appreciation. He noted that when people have an interest in music it creates the desire to know more about the great composers and interpreters, but this knowledge does not generate any anxiety that they are unable to perform at the same level. However, in the case of music, the interest is often already present, whereas students' interest in mathematics is usually low, and he felt that anecdotes could be used to create this interest in the same way as the movie *Amadeus* created a general interest in Mozart's music.

David suggested an alternative approach to using history in the classroom. He explained that during the period in which Japan had no connection with the Western world, mathematics was a popular amusement for those who had leisure time. The *Ladies Diary*, an eighteenth century magazine, was also mentioned as example of mathematics specifically designed for recreational use. It was suggested that students should regard the readers of this magazine, and their Japanese counterparts, as role models, rather than professional mathematicians, and David compared mathematics to basketball: It is unlikely that students will become professional players, but they will still enjoy playing for fun.

Another anecdote which generated discussion was that of Gauss as a perfectionist, unwilling to publish his work until he was certain it was correct. Although the text presented this as a positive characteristic, we noted that this secretive practice was detrimental to the overall development of mathematics, as others spent time working on ideas which Gauss had already explored. It was suggested that a discussion about this story could help students see the importance of being open about their work in mathematics and encourage them to communicate their ideas to others and work collaboratively. However some participants wondered whether the teachers using this resource would have sufficient mathematical background to make this connection.

The discussion of primary sources focussed on material based on Robert Recorde's *Ground of Artes*. We looked at one page from an original 1552 text, and two pages from *Exploring Mathematics Through History* (Eagle, 1995), a modern classroom resource which includes activities based on Recorde's text. The excerpts of historical documents included in Eagle's book printed the words in modern fonts, but sometimes used the archaic spelling, as in the example shown here. David considered that this was an uncomfortable compromise, noting that the unusual spelling made it difficult to read. He suggested that it was better to have a copy of the original document with the modern spelling parallel to it, as shown in Appendix 1. This would convey a sense of "this is old", without requiring students to focus their attention on deciphering it. However, the enthusiasm with which some members of the working group started to decode the original text suggests that some students would find this a stimulating exercise: The modern version did not generate such an animated response.

> Numeration is that Arithmeticall skill, whereby we may duely value, expresse and reade any summe proposed or else in apte figures sette downe any Number.
> **Scholar.** Why? then me thinketh you put a difference betwene the value and the figure.
> **Master.** Yea so doe I: for the value is one thing, and the figures are another thing, and that commeth chiefly of the places wherein they be set. And first marke, that there are but ten figures that are used in Arithmetik: and of those ten, one doth signify nothing, which is made like an o, and is called a Cypher. The other nyne are called signifying figures, and be thus figured,
>
> 1 2 3 4 5 6 7 8 9
> and this is their value
> i. ii. iii. iiii. v. vi. vii. viii. ix.

We looked at the mathematical content of two short texts. In the first (shown here), Recorde explains the "ten figures that are used in Arithmetic" in terms of their "value" as expressed by Roman numerals. This text vividly portrays to students the relatively recent nature of the numerals they use everyday. However, the Scholar's comment about "a difference between the value and the figure" can serve as a pedagogical reminder about the significance of place-value. For the second excerpt, we had both the original source and a modern transcription (see Appendix 1). In this passage, Recorde explained the algorithm for multiplying two digits over 5. David noted that the use of alternative algorithms was good even without the historical context, as it shakes up students' notion that there is something essentially perfect about the standard algorithm they learn in school. There was also interest in the fact that Recorde provided an algorithm for calculating results which we expect children to memorize today. This text led us to consider which facts we should expect children to know and which should be figured out.

THEME 3.1: Des instruments pour la mesure de la longueur ou de la hauteur.

Pour cette partie de l'atelier, deux pages furent distribuées aux participants. Elles contenaient des illustrations d'instruments de mesure des longueurs et des hauteurs, illustrations tirées principalement du tome 2 du livre, maintenant classique, de D.E. Smith, *History of Mathematics*. (Voir l'Annexe 2.) Nous devions étudier chacun de ces instruments afin de comprendre, uniquement à partir de l'illustration, son usage et les mathématiques sous-jacentes à son utilisation. Le fait d'avoir plusieurs instruments sous les yeux visait à nous donner un aperçu à la fois de divers types de problèmes de mesure de longueur ou hauteur, mais aussi des diverses façons de résoudre ces problèmes. Les instruments illustrés sur les deux pages datent des XVI^e^ et XVII^e^ siècles. Seuls deux d'entre eux ne sont pas de cette période. Nous avons donc collectivement déterminé les mathématiques pertinentes à chaque instrument, et, par le fait même, discuter de celles-ci et de l'efficacité relative des instruments. Nous ne reprendrons pas ici ces discussions. Nous invitons plutôt le lecteur à entreprendre lui-même cet exercice, seul ou de préférences avec d'autres. Disons simplement que nous avons tous retiré de ce travail un plaisir profond.

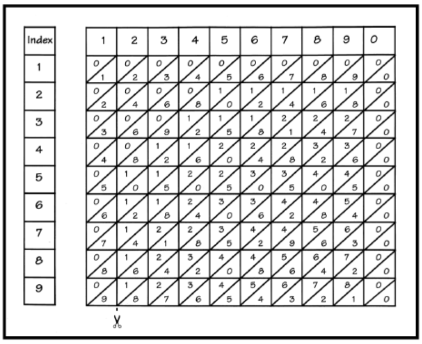
Après s'être familiarisé rapidement avec ces instruments, la discussion s'est portée sur l'utilisation que nous pouvions faire des instruments en classe. La question qui a retenu principalement notre attention a été formulé par Elaine : « Is it necessary, in a classroom, to present those instruments in an historical context? Is it not sufficient to simply present these instruments as instruments? » Nous ne sommes parvenues à aucune réponse évidente ou définitive. La plupart des collègues pensent que pour apprécier le contenu historique de ces illustrations, les élèves doivent savoir d'abord comment les lire et les interpréter. Par exemple, s'intéresser à la façon dont les personnes s'habillent, aux chapeaux qu'elles portent, au style même de l'illustration. Ce n'est qu'outillés de la sorte que les élèves pourront prendre conscience que tous les instruments, sauf deux, datent d'une même époque. Tom croit que le contenu historique de ces illustrations peut rebuter ses élèves. Ces derniers étant exposés uniquement à la vision américaine des choses accueilleront sans doute ces illustrations par un « What's that !!? » plein de mépris pour ce qui n'appartient pas à leur monde. Comment, dès lors, peut-on justifier à leurs yeux le temps et l'énergie nécessaire pour donner un sens à ces illustrations ? Nous sommes confrontés à nouveau au manque de motivation des élèves non seulement pour les mathématiques, mais aussi pour le monde et, plus important encore, pour ce qui fait que le monde est ce qu'il est aujourd'hui. Face à ce constat, plusieurs croient que les enseignants ne doivent pas abdiquer. Tout le contraire. Ils devraient plonger leurs élèves dans un flot d'informations culturelles et historiques, non pas pour que les élèves les apprennent, mais plutôt pour leur donner l'opportunité, dans un premier temps, de se rendre compte simplement de l'existence de ce genre d'information, et, dans un deuxième temps, d'établir des relations affectives et intellectuelles avec ces dernières. Comme Louis le suggérait, nous sommes un peu dans une situation similaire à celle d'un jeune enfant qui apprend à parler. Si nous lui adressons la parole dans une « langue de bébé », il utilisera un vocabulaire limité et des phrases aux structures déficientes. Par contre, s'il est plongé dans un milieu au langage riche et de haut niveau, il progressera au fur et à mesure de sa compréhension des mots et des structures pour, à long terme, atteindre le niveau linguistique de son environnement. Le plaisir et l'intérêt provoqués par une illustration datant de la Renaissance se comparent à la curiosité et à l'intérêt déclenchés par la copie d'un texte original auxquels nous faisions référence dans la section précédente. Dans les deux cas, il s'agit de plaisirs éduqués. Pour arriver là, les élèves ont-ils besoins d'une éducation formelle et explicite ?

Probablement pas, à la condition d'être plongés dans un environnement scolaire riche.

En terminant cette session, Bernard nous a suggéré de visiter le très beau site consacré aux instruments mathématiques : http://www.museo.unimo.it/theatrum/

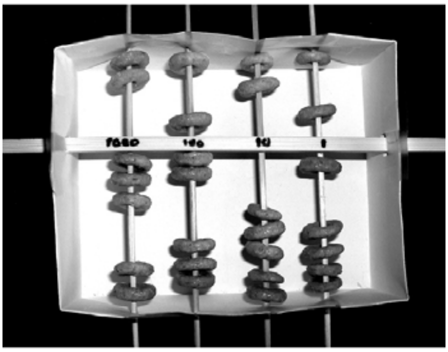
THEME 3.2: Calculating instruments and algorithms

Napier's Rods were a popular calculating instrument in the seventeenth century and are easy for students to investigate. Our session started by looking at a blackline master (Reimer & Reimer, 1992), which students can cut up to get their own set of Rods. However, even before we explored how the Rods could be used, there was considerable discussion about the pattern-solving activities that the uncut sheet provided. Individual strips were then handed out. Since several group members had not seen the Rods before, they played the role of students, which helped the rest of the group to realize what information had to be provided, and what students would be able to work out for themselves. During this exploration, the connection with the Lattice method of multiplication became apparent.



There was considerable discussion as to the purpose of introducing Napier's Rods. Some group members felt both the Rods and the Lattice method were interesting as alternative algorithms, but they did little to help children understand the regular method of multiplication. However, others felt that there was some conceptual work involved, in the sense that all the partial products are clear in these methods, whereas these are disturbed by the carrying necessary in the more concise method now used. We also noted that the Rods and Lattice method were historically significant because they were the first algorithms used which are similar to the present-day method: before these methods were introduced, multiplication had been performed by various doubling procedures. A different aspect of the Rods' historical importance is that their appearance in the seventeenth century provided a simple means of multiplication for those who had no knowledge of multiplication facts, a point which also makes them very popular with children in the intermediate school grades today. Although we agreed that Napier's Rods were a interesting calculating instrument to introduce to students, most participants felt that this work privileged the history rather than the mathematics.

We next considered the Chinese abacus, and agreed that students would enjoy making one for themselves. Irene demonstrated an origami construction for the box in which the rods are placed, and a discussion of the many mathematical questions which can be generated from such origami creations followed. Several group members were unfamiliar with the abacus, so we explored how to manipulate the beads to count, add and subtract. We then noted that abacus use highlights the regrouping process, and also encourages students to think of numbers as complements of five and ten, a crucial step in the efficient use of the abacus, but one which is also very important in the regular addition and subtraction techniques taught today. Bernard made the point that the Chinese abacus makes a very authentic introduction to alternative ways of calculating, as it is still in use today.



Irene then demonstrated an earlier Chinese calculating device, the Counting Board and Counting Rods, similar in appearance to a large chess board and toothpicks respectively. As with the abacus, we explored how numbers were represented, added and subtracted, but an additional feature of the Counting Rods was the two-colour system used: red for positive numbers and black for negative. This enabled the Chinese to calculate with signed numbers over two thousand years ago, and provides a useful discovery tool for students exploring the addition and subtraction of integers. A discussion of the written number system based upon the Counting Rods led us away from the topic of calculation, but produced some interesting ideas about the value of encouraging student awareness of both the number words and symbols used in countries around the world today.

David suggested that the traditional Chinese method of numeration, which uses a named place value system, provides a model which might help students understand place-value: thirty-five would be written as 3T5 rather than simply 35. The Base-20 origins of several number systems were discussed. Bernard mentioned the French use of "quatre-vingt", and David reminded us of Lincoln's use of the phrase "four score and seven years ago" in his Gettysburg address. After some discussion of the illogical number word order used at various points in several European languages, Hassan explained that the Arabic practice of reading from right to left avoided such difficulties: the number 234, which English speakers read as "two hundred thirty four", is read in Arabic as "4, 3 tens and 2 hundreds", thus eliminating the necessity of first knowing the place-value of the largest term. In the work of al-Khwarizmi and his successors, mathematical sentences were written in the same direction as the text, but in recent times, Moroccan educators have modified this practice. In an attempt to reconcile the Arabic direction of reading with the international mathematical symbolism, mathematical sentences within a literal text are written from right to left, whereas those which stand alone read from left to right.

The talk of number order led to comments about the various algorithms for addition and subtraction, and whether these operations should start with the digits on the right or the left. We noted that students who invented their own algorithms for working in different bases usually started working with the largest place-values, as is often done in mental and contextual mathematics. Elaine explained the base-complement method of subtraction, and other methods were also shared.

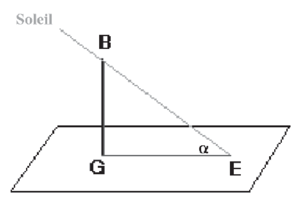
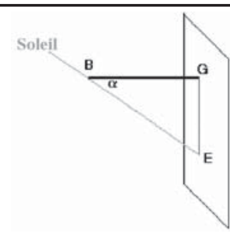
THEME 4: Évolution d'un domaine mathématique : la trigonométrie

L'histoire de la trigonométrie est généralement méconnue par les enseignants. Par ailleurs, elle constitue un thème intéressant. On y trouve, entrelacés, les nombres et la géométrie, l'astronomie et les instruments. De plus, la notion d'angle, et par ricochet la question de la mesure de ce dernier, y joue un rôle central. C'est pourquoi Louis avait décidé de proposer d'aborder ce thème. Selon lui, le meilleur livre contenant des informations sur ce sujet est celui de Victor Katz (1993).

Dans un premier temps, Louis a résumé l'histoire de la trigonométrie. Le mot trigonométrie apparaît pour la première fois en 1595 dans le titre du livre de Pitiscus (1561–1613) intitulé *Trigonometriae sive, de dimensione triangulis, Liber*. La date tardive de l'apparition d'un nom pour désigner ce domaine des mathématiques suggère qu'auparavant la trigonométrie n'était pas considérée comme un domaine indépendant. De fait, avant la Renaissance, on peut considérer qu'il y avait deux types de trigonométrie, la trigonométrie dans le cercle et la trigonométrie des ombres.

La trigonométrie dans le cercle constitue un sous-domaine de l'astronomie. Cette trigonométrie a été développée par les astronomes grecs Hipparque (190–120 avant notre ère) et Ptolémée (100–178 de notre ère). À l'aide de calculs trigonométriques, les modèles géométriques mis au point par ces deux astronomes peuvent décrire et prédire les mouvements apparents des corps célestes, en particulier des planètes. Ces calculs reposent principalement sur les tables de la longueur des cordes déterminées par les angles au centre inférieurs à 180° dans un cercle de rayon donné. Ces longueurs de cordes correspondent en gros à notre sinus. En fait, on a $sin(\alpha) = {}^{1}/_{2}\, corde(2\alpha)$. C'est pourquoi nous pourrions aussi appeler cette trigonométrie la trigonométrie du sinus. Dans le premier chapitre de l'*Almageste*, Ptolémée calcule une table de cordes pour les angles allant de $^1/_2$° à 180°, par sauts de $^1/_2$°, en utilisant des identités lui permettant, connaissant la corde d'un angle α et celle d'un angle β, de déterminer la corde de $180° - \alpha$, celle de $^{\alpha}/_2$ et celle de $\alpha - \beta$. Il emploie aussi l'équivalent de la loi des sinus et de la loi des cosinus. Ce furent les astronomes indiens du V^e^ siècle qui, dans leur souci de limiter la longueur des calculs, commencèrent à utiliser le sinus d'un angle à la place de la corde. En effet, ils avaient remarqué que le calcul de $^1/_2$ *corde*(2α) revenait très souvent dans les calculs astronomiques. Aussi, produirent-ils des tables correspondant à cette expression.

La trigonométrie des ombres, pour sa part, fait partie l'astronomie pratique, c'est-à-dire de cette partie de l'astronomie nécessaire à la détermination de la latitude d'une ville, de l'orientation d'une mosquée vers La Mecque, etc. Dans cette trigonométrie, l'outil principal est l'ombre, comme nous le voyons dans les définitions suivantes d'al-Biruni (973–1055).

GB est un bâton d'une longueur donnée GE est l'ombre directe (Cotangente) de l'angle α: BE est l'hypoténuse de l'ombre directe (cosécante) de l'angle α	
GB est un bâton d'une longueur donnée GE est l'ombre inverse (tangente) de l'angle α: BE est l'hypoténuse de l'ombre inverse (sécante) de l'angle α:	

Cette trigonométrie pourrait aussi porter le nom de trigonométrie de la tangente. De ces définitions, utilisant notre terminologie, découlent les identités suivantes:

- $\cot_{\mathcal{B}}^2(\alpha) + \mathcal{B}^2 = \mathcal{B}^2 \csc_B^2(\alpha)$, $\mathcal{B}$ = la longueur du bâton BG
- $\tan_{\mathcal{B}}^2(\alpha) + \mathcal{B}^2 = \sec_{\mathcal{B}}^2(\alpha)$,
- $\tan_{\mathcal{B}}(\alpha) = \sin_{\mathcal{B}}(\alpha)/\cos_{\mathcal{B}}(\alpha)$ (ici une propriété et non une définition).

La trigonométrie du triangle rectangle émergera de la trigonométrie des ombres. Les rapports trigonométriques pour leur part ne seront définis qu'au XVII^e^ siècle, lorsque les fractions décimales seront plus généralement utilisées. Auparavant, les lignes trigonométriques étaient simplement des longueurs de segments, dépendant d'un angle et de la longueur donnée d'un bâton dans le cadre de la trigonométrie des ombres, ou d'un angle et de la longueur donnée du rayon dans le cadre de la trigonométrie dans le cercle.

Nous pourrions élaborer aussi sur le choix des unités pour mesurer les angles et les rayons. Résumons cela en quelques points.

- Hipparque: le rayon est mesuré avec une unité de longueur égale à la longueur de l'arc de cercle déterminé par un angle d'une minute. Il y donc homogénéité entre la mesure des angles et la mesure linéaire du rayon. Le choix de la longueur du rayon R se réduit alors nécessairement à 3438 unités correspondant à un arc d'une minute, car on doit avoir $R = (360 \times 60)/2\pi$.
- Ptolémée: Il n'y a pas d'homogénéité entre la mesure des longueurs de segments et la mesure des angles. Le rayon du cercle est choisi égal à 60 unités pour faciliter les calculs alors que les angles sont mesurés en degrés.
- Regiomontanus (1436–1476): Comme chez Ptolémée, il n'y a pas d'homogénéité entre la mesure des longueurs de segment et la mesure des angles. La longueur du rayon du

cercle est choisi égale à 60×10^3 unités pour éviter les fractions alors que les angles sont mesurés en degrés. (Napier popularise à partir de 1619 les fractions décimales par l'usage de ses logarithmes.)

- Euler (1707–1783) ayant à sa disposition les fractions décimales choisit R = 1.
- James T. Thomson (1873) et Alexander J. Ellis (1874): les arcs déterminés par les angles au centre sont mesurés avec la même unité que le rayon. Il y donc homogénéité entre la mesure des angles, en radians, et la mesure linéaire. Le choix de la longueur de R = 1 impose alors que le cercle ait une circonférence de longueur 2π.

Ces quelques remarques sur la mesure des angles nous incitent à questionner l'importance d'introduire les radians uniquement au moment d'aborder les fonctions trigonométriques, comme on le fait au Québec.

Dans la discussion qui a suivi, très peu a été dit sur l'histoire de la trigonométrie et son utilisation dans l'enseignement ou même comme outil permettant de mieux analyser les difficultés rencontrées dans l'enseignement de la trigonométrie. Nous pouvons sans doute en conclure que pour avoir une telle discussion, il aurait fallu au préalable que chacun ait plus d'informations sur l'histoire de la trigonométrie. Normal… direz-vous. Par ailleurs, nous avons fait un tour de table sur la façon dont est abordée la trigonométrie dans les différentes provinces et aux États-Unis. En général, la trigonométrie débute en dixième année avec la trigonométrie du triangle rectangle suivie de la trigonométrie du cercle et de l'étude des fonctions trigonométriques pour les élèves de onzième année qui se destinent à l'université. Une difficulté se manifeste alors en dixième année, l'application des lois des sinus et des cosinus aux triangles contenant un angle obtus. Il va s'en dire que cela pose problème étant donné que, jusqu'alors, sinus et cosinus n'ont été définis que pour des angles contenus dans un triangle rectangle et donc de mesure inférieure à 90°.

THEME 5.1: Interdisciplinary Connections

This session focussed on ways in which the history of mathematics can provide a link between mathematics and other curriculum areas, and also between different strands of mathematics itself. We first considered the connection with language arts, and discussed the information given in Johnson (1994) concerning the linguistic origins of mathematical terms. Participants felt that such explanations could help students to recall the meaning of the words. We liked the imagery of some of the less familiar word origins, such as the Greek words "kolindros" (roller) for cylinder, and also noted that telling students the origin of such words gives them an indication of the importance of Greek mathematicians in the development of geometry. We then looked at the origin of some common mathematical symbols, information also given in the Johnson's book, and discussed the significance of Recorde's decision to represent "equals" by a pair of parallel lines "bicause noe 2 thynges can be moare equalle" (*The Whetstone of Witte*, 1557).

The most obvious cross-curriculum connection to be gained through the history of mathematics is of course that between mathematics and social studies. We discussed various ways of setting the historical scene for particular mathematical topics, such as stories, pictures and time-charts, and also noted the possibilities for making geographical connections, particularly in studying the evolution of mathematical ideas such as the Hindu-Arabic number system. Irene explained the programme of "Time-Travel Days" developed by an elementary teacher in British Columbia, in which Grade 3 children "travel" to ancient civilizations, not just to study their history or geography but also to investigate their mathematics. The historical context was often dramatized further, with children role-playing students of the period. The ancient school setting provided an authentic context in which to learn the appropriate numbers and algorithms.

The connections between mathematics and art are apparent in many aspects of geometry. We looked at two activities dealing with tessellations created by Islamic artists (Zaslavsky, 1994). These activity sheets included information on the historical background of this artwork, and we felt that this made the work more interesting than the standard textbook activity of simply tessellating an arbitrary shape.

The history of mathematics can also highlight connections between different branches of mathematics. At the elementary level, arithmetic and geometry can be linked through a study of Pythagorean figurate numbers, and we looked at some activities on this topic (Reimer & Reimer, 1992). An investigation of figurate numbers lends itself to the use of simple manipulatives (such as Bingo chips), and we felt that their use would not only keep the children more involved with the work, but would also provide an historically appropriate representation of the Pythagoreans playing with pebbles on the sand.

We ended the session by noting that even though such cross-curricular work is extremely valuable, it requires teachers to be aware of both the history of mathematics and more general history. Although there are now many resources available to provide this information, finding time to study them is not easy, and so only the most enthusiastic teachers will pursue this route.

THEME 5.2: Histoire des mathématiques et histoire générale

Quelles relations doit-il y avoir entre l'histoire des mathématiques que nous introduisons dans nos cours et l'histoire générale ? Cette question a fait tout de suite resurgir la question posée lors d'une autre session par Elaine « Is it necessary, in a classroom, to present those instruments in an historical context? Is it not sufficient to simply present the instruments as instruments? » Aujourd'hui, la technologie nous fournit des outils pour nous aider à comprendre, et souvent voir à l'aide de simulations, comment fonctionne un instrument. Toutefois, de telles simulations ne bousculent en rien la croyance des élèves voulant que le monde a toujours été ce qu'il est aujourd'hui. Tom croit qu'il faut provoquer les élèves en leur posant des questions comme « Quand la terre a-t-elle été mesurée pour la première fois ? » ou « Les miroirs d'Archimède utilisés lors du siège de Syracuse ont-ils vraiment pu mettre le feu aux galères romaines ? » Mais encore là, la réponse à de telles questions ne permet pas nécessairement aux élèves de développer une conception du passé et de l'écoulement du temps. Ces réponses doivent être placées dans une perspective historique. Dans ce dessein, Geoffrey suggère, comme premier moyen, l'usage d'une ligne du temps. Par celle-ci, il devient possible d'établir des liens avec d'autres domaines, comme les arts, la construction de machines, les religions, etc. Tom renchérit en rappelant que des livres d'histoire illustrés aident à donner une certaine représentation d'une époque et devraient donc être largement utilisés. Un exemple, le livre de Kathryn Lasky intitulé *The Librarian Who Measured the Earth* (1994) qui s'adresse à des enfants de 4 à 8 ans. Reste à savoir comment utiliser une ligne du temps ? Louis mentionne que les didacticiens de l'histoire recommandent fortement que la ligne du temps soit construite par les élèves eux-mêmes à partir d'un squelette de ligne du temps où l'on ne retrouve que deux ou trois dates repères, comme –1000, le début de notre ère, 1000 et 2000. Cette ligne devrait être visible en permanence dans la classe. Petit à petit, au fil des activités, les élèves viendront y placer des illustrations, des informations qui, à ce moment-là, seront pertinentes pour eux. Dans la classe de mathématiques, lors d'une activité impliquant par exemple Pythagore, les élèves placeront sur la ligne du temps le portrait de Pythagore et d'un temple de son époque. Tous s'entendent pour dire que les activités impliquant l'histoire des mathématiques devrait faire référence à l'histoire générale. Des références successives à cette dernière, par le biais de la ligne du temps, d'illustrations, de musique, enrichissent les composantes d'histoire des mathématiques et contribuent à humaniser la perception des mathématiques qu'ont les élèves. Nadine signale qu'on peut pousser encore plus loin ces liens en créant un coin d'histoire dans sa classe, coin décoré avec, par exemple, des rideaux aux motifs égyptiens, ou des morceaux de papyrus, ou des objets rappelant l'Égypte antique.

THEME 6: "What should a teacher know about the history of mathematics?"

Our final session started by reviewing the first paragraph of the paper with the above title which David Wheeler presented to CMESG in 1981. This listed five important aspects of mathematics which Wheeler considered every teacher should know, of which the historical dimension was merely one of several perspectives mentioned in his last point. Faced with this inventory, group members acknowledged that very few conference participants would claim to be competent in all these areas! However, the importance of historical and cultural approaches to mathematics was stressed, and we recalled the many points that were raised in our first session (see Theme 1). Eric also noted that a course on the history of mathematics appears to be a requirement for teacher education institutions to gain accreditation from NCATE (National Council for Accreditation of Teacher Education).

Elaine suggested that one role of the CMESG should be to make recommendations to Canadian universities about courses, and much of the session was spent discussing the type of course on the history of mathematics which could/should be offered. Participants considered that history of mathematics courses should be taken by all teachers to give them a broader education in mathematics, and also noted that such courses would be particularly useful if the historical material was closely related to concepts taught in the elementary and secondary school. Although the tongue-in-cheek suggestion that a history of mathematics course should be offered through the history department was met with laughter, France reported that a course offered at the University of Montreal, taught jointly by historians and mathematicians, was very well-received.

The relative importance of "history" and "mathematics" in a "history of mathematics" course was discussed. Eric felt strongly that such courses had to require students to actually do mathematics, rather than just read about mathematics and how other people had done it, and he suggested that the historical aspect of the course is to provide the environment in which the mathematics is presented. Others agreed, noting that much of the mathematics developed prior to calculus can indeed be handled by students with a good high school background. On the other hand, Elaine considered that courses dealing with metacognition or the metacontent of mathematics were very valuable, even though they would not include a high level of mathematical activity. Louis stressed the importance of the historical dimension, and made a case for invoking students' curiosity about the general path of evolution before specifically relating it to mathematics.

David raised the issue of a "mathematics appreciation" course, comparable to those in music or art departments. While several people liked this idea, the general consensus was that this was in a different category to history of mathematics courses, although Anna pointed out that financial restraints sometimes meant that the two approaches had to be taught together. However, as noted above, history of mathematics courses could well help students to appreciate mathematics by providing a context in which to make sense of the material.

Several members of the group had taught or knew of courses related to the history of mathematics. In some mathematics departments, students need to have studied mathematics at a high level (six mathematics courses showing a breath of knowledge as well as a high level of understanding), whereas others have less stringent requirements, and welcome students who are not mathematics specialists.

Seminar courses were suggested as a way of involving the students more than is the case in a traditional lecture course. By taking part in a sharing of information, rather than simply listening to the lecturer, the students might learn to become more critical of the material they find in books or on the Internet, and could also help them to become more aware of the work done by historians. Participants agreed that teachers using history in their mathematics classrooms at any level are often concerned that they will not be able to give a completely accurate picture. Indeed, many mathematics teachers like their subject specifically because of its structure and the sense of truth and security that it offers. We agreed that it was important to find ways to relieve this fear of inaccuracy, and most participants felt that it was worth providing information to stimulate children's interest, even if some of the stories were of doubtful authenticity. It was noted that if we want teachers to include historical material in their classrooms, we should model this behaviour in the courses we teach to them. The value of having posters in the room was also mentioned.

We discussed teachers' need to find resources for teaching historical perspectives. Several sources were suggested, but David pointed out that pre-service teachers are given so much information to absorb in a short time that it is hard for them to assimilate it all. The list which follows is a response to his suggestion that we compile a list of group members' favourite history of mathematics texts:

Dahan-Dalmedico, A. & Peiffer, J. (1986). *Une histoire des mathematiques.* Routes et dédales, Paris:Editions du Seuil (Coll. Points - Sciences).

Ifrah, G. (1994). Histoire universelle des chiffres. Paris: Editions Robert Laffont. [English text (1998) also recommended. *The universal history of numbers*. London: Harvill Press.]

Katz, V. J. (1993). *A history of mathematics: An introduction.* New York: Addison-Wesley.

Kline, M. (1972). *Mathematical thought from ancient to modern times*. Oxford University Press.

Mankiewicz, R. (2001). *The story of mathematics*. Cassell & Co. [French text (2000) also recommended. *L'histoire des mathématiques*. Paris: Seuil].

Seife, C. (2000). *Zero: the biography of a dangerous idea*. New York: Penguin Books.

Schmandt-Besserat, D. (1999). *The history of counting*. New York: Scholastic Inc. (recommended for elementary students and teachers)

Smith, D. E. (2001). *Number stories of long ago*. Reston, VA: National council of Teachers of Mathematics. (recommended for elementary students and teachers)

Smith, S. M. (1996). *Agnesi to Zeno: Over 100 vignettes from the history of math.* Berkeley, CA: Key Curriculum Press. (recommended for secondary school students and teachers)

Swetz, F.J. (Ed.) (1994). *From five fingers to infinity*. Peru, IL: Open Court Publishing Company.

Recommendation of the Working Group:

Universities should offer a course for pre-service teachers in which mathematics is presented in an historical context, showing the contributions made by many cultural groups.

Acknowledgements

The image of Napier's Rods on page 45 is used by permission of the AIMS Education Foundation (Fresno, CA; http://www.aimsedu.org; 1.888.733.2467). It was originally published in *Historical Connections in Mathematics, Vol. 1* (1992).

All other images are in the public domain.

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Eagle, M.R. (1995). *Exploring maths through history*. Cambridge, UK: Cambridge University Press.

Johnson, A. (1994). *Classic math: History topics for the classroom.* Palo Alto, CA: Dale Seymour Publications.

Katz, V. J. (1993). *A history of mathematics: An introduction.* New York: Addison-Wesley.

Lasky, K. (1994). *The librarian who measured the Earth*. Boston: Joy Street Books.

Recorde, R. (1552). *The ground of artes*, London: Reynold Wolfe.

Recorde, R. (1557). *The whetstone of witte*. London: I. Kyngston.

Reimer, L., & Reimer, W. (1992). *Historical connections in mathematics, Vol. 1.* Aurora, ON: Spectrum Educational Supplies Ltd.

Smith, D.E. (1925). *The history of mathematics, Vol. II.* Dover edition 1958.

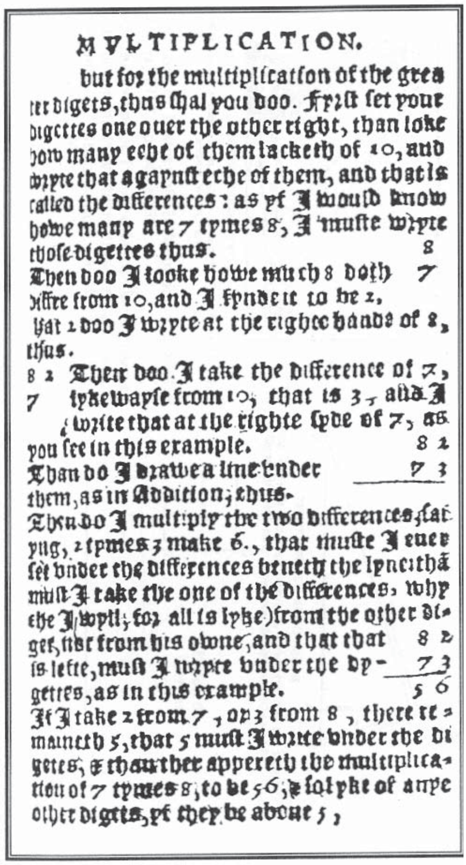
Wheeler, D. (1981). *What should a teacher know about the history of mathematics?* Paper presented at the Annual Meeting of the Canadian Mathematics Education Study Group, Edmonton, AB, June 5–9.

Zaslavsky, C. (1994). *Multicultural math: Hands-on activities from around the world.* New York: Scholastic Professional Books.

Annexe/Appendix 1

The images in these appendices are taken from D.E. Smith's (1925) *The history of mathematics, Vol. II* (republished by Dover in 1958). They are reprinted here with the permission of Dover.

Ground of Artes, 1552 edition



MVLTIPLICATION.

but for the multiplication of the greater digets, thus shal you doo. Fyrst set youre digettes one ouer the other right, than loke how many eche of them lacketh of 10, and wryte that agaynst eche of them, and that is called the differences: as yf I would know howe many are 7 tymes 8, I muste wryte those digettes thus.

8
7

Then doo I looke howe much 8 doth differ from 10, and I fynde it to be 2, that 2 doo I wryte at the righte hande of 8, thus.

8 2
7

Then doo I take the difference of 7, lykewayse from 10, that is 3, and I wryte that at the righte syde of 7, as you see in this example.

8 2
7 3

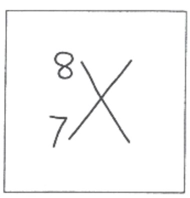
Than do I drawe a line under them, as in Addition, thus.

Than do I multiply the two differences, sayyng, 2 tymes 3 make 6., that muste I euer set under the differences beneth the lyne: than must I take the one of the differences, (whyche I wyll, for all is lyke) from the other diget, not from his owne, and that that is lefte, must I wryte under the dygettes, as in this example.

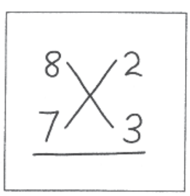
8 2
7 3
5 6

If I take 2 from 7, or 3 from 8, there remaineth 5, that 5 must I wryte under the digettes, & than ther appereth the multiplication of 7 tymes 8, to be 56, & so lyke of anye other digits, yf they be aboue 5,

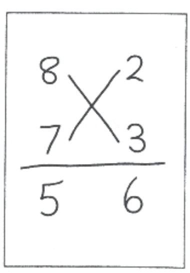
Transcription of different (unspecified) edition of the same text: Note the slight differences between the two editions (from Eagle, 1995).



But for the multiplication of the greater digits, thus shall you do. First set your digits one over the other digit, then from the uppermost downward, and from the nethermost upward, draw straight lines, so that they make a cross commonly called saint Andrews cross, as you see here: as if I would know how many are 7 times 8, I must write those digits thus.

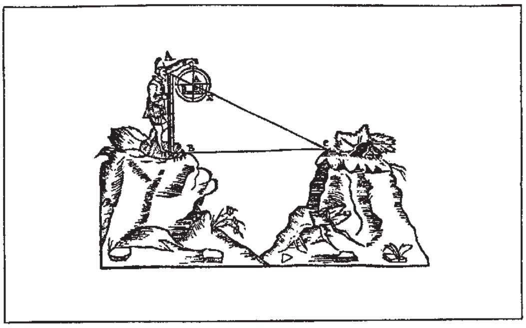


Then do I look how much 8 doth differ from 10, and I find it to be 2, that 2 do I write at the right hand of 8, at the end of the line, thus. After that I take the difference of 7 likewise from 10, that is 3, and I write that at the right side of 7, as you see in this example. Then do I draw a line under them, as in addition, thus.



Last of all I multiply the two differences as saying: 2 times 3 makes 6, that must I enter under the differences, beneath the line: then must I take the one of the differences (which I will, for all is like) from the other digit (not from his own) as the lines of the cross warn me, and that which is left must I write under the digits. As in this example. If I take 2 from 7, or 3 from 8, there remaineth 5: ye 5 must I write under ye digits: and then there appeared the multiplication of 7 times 8 to be 56. And so likewise of any other digits if they be above 5.

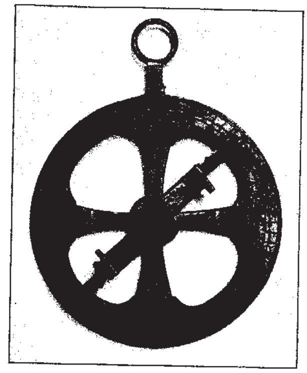
Annexe/Appendix 2



THE ASTROLABE IN SIMPLE MENSURATION

From Bartoli's *Del Modo di Misvrare*, Venice, 1589, showing simple work in a crude kind of trigonometry

(1)



(1) CHAMPLAIN'S ASTROLABE

Found near the Ottawa River about 1870. It was made in Paris in 1603. This is the type of astrolabe known as the planisphere. From the collection of Samuel V. Hoffman, New York

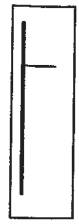
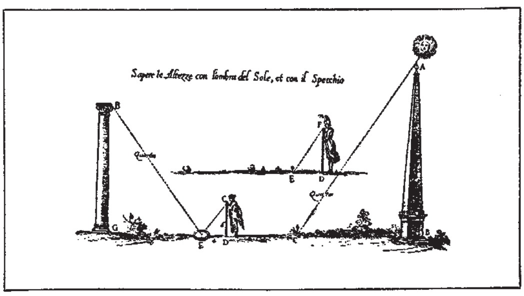
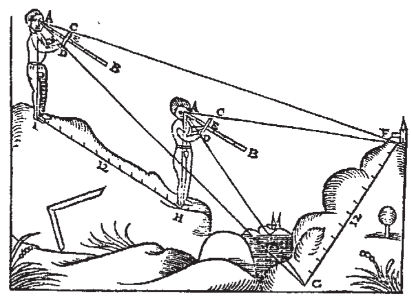


Fig. 1
Bâton de Gerbert
(2)



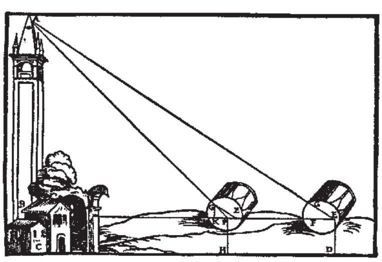
USE OF THE SHADOW AND THE MIRROR IN MEASURING HEIGHTS

(1) From Giovanni Pomodoro's *La Geometria Prattica*, Rome, 1624



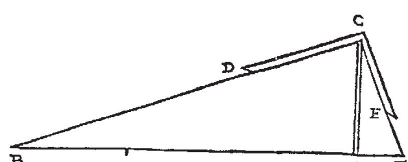
(1) THE BACULUM, OR CROSS-STAFF

From Oronce Fine's *De re & praxi geometrica*, Paris, 1556, showing the metho of measuring distances



(1) DRUMHEAD TRIGONOMETRY

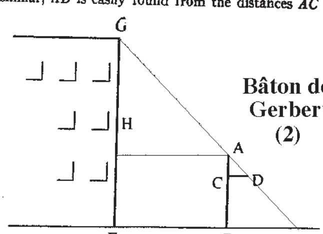
A common method of triangulating in the 16th century. From Belli's *Libro del Misvrar*, Venice, 1569



THE QUADRATUM GEOMETRICUM

From Oronce Fine's *De re & praxi geometrica*, Paris, 1556. The two triangles being similar, *AB* is easily found from the distances *AC* and *AF*

(1)

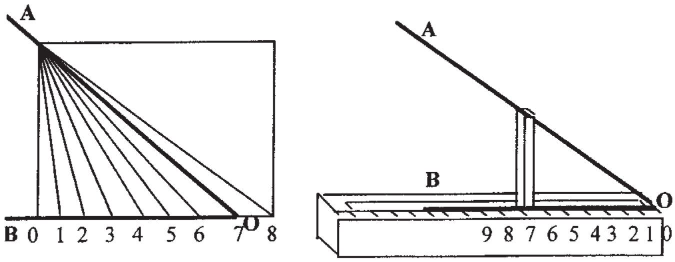


Bâton de Gerbert
(2)

(1) **D.E. Smith, History of Mathematics , t. 2, édition Dover, 1958, pp. 345 à 358.**

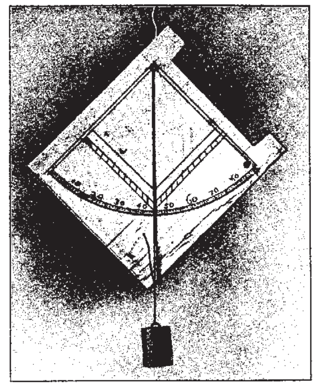
(2) **L. Charbonneau, Histoire des mathématiques et enseignement des mathématiques au primaire, Instantanés Mathématiques, vol. XXXIX, aut. 2002, pp. 21 à 36.**

Annexe/Appendix 2 (cont.)



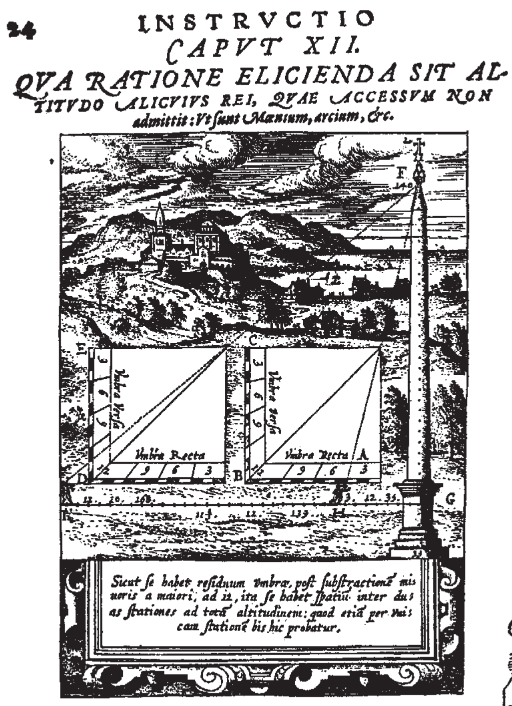
(3)

Figure 9.2: Instruments to measure sekts



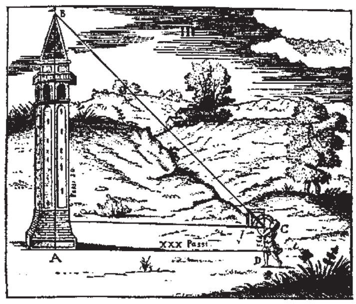
(1) SMALL IVORY QUADRANT

Italian work of the 18th century. If we sight through holes in the two projections on the upper right-hand edge, the angle of elevation is indicated by the plumb line and the arc. The original is 5 cm. by 5.9 cm. From the author's collection



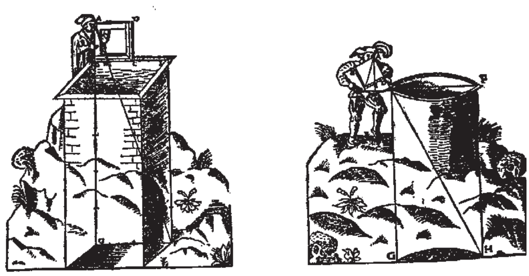
(1) EXPLANATION OF THE QUADRANT

From *De Quadrante Geometrico*, usually referred to Cornelius de Judeis, Nürnberg, 1594, but in fact written by Levinus Hulsius. Cornelius made the drawings with the help of Martin Geet



(1) USE OF THE QUADRANT

From Ottavio Fabri's *L' Uso della Squadra Mobile*, Trent, 1752



(1) THE QUADRANT

From the *Protomathesis* of Oronce Fine, Paris, 1530–1532

(3) H.N. Jahnke et al. The use of original sources in the mathematics classroom, dans J. Fauvel, J. van Maanen, History in Mathematics Education : the ICME Study, Kluwer, 2000, pp. 291 à 328, p. 301.

Teacher Research: An Empowering Practice?

Florence Glanfield, *University of Saskatchewan*
Louise Poirier, *Université de Montréal*
Vicki Zack, *St. George's Elementary School, Montreal*

Participants

Laurinda Brown	Alex Lawson	Louise Poirier
Rina Cohen	Ralph Mason	Laurent Theis
Doug Franks	Doug McDougall	Jo Towers
Claude Gaulin	Janelle McFeetors	Dave Wagner
Florence Glanfield	Cynthia Nicol	Elizabeth Wood
Heather Kelleher	Sara Perkins	Vicki Zack
Carolyn Kieran	Eileen Phillips	

Overview

The purpose of this working group was to consider teacher research. Some have spoken of the uniqueness, the insider status of the teacher-researcher, the requirement of spiralling self-reflection on action, and the intimate dialectical relationship of research to practice (Anderson & Herr, 1999, p. 12), and have stated that "insider, practitioner research has its own unique set of epistemological, methodological, political, and ethical dilemmas" (Anderson, 2002, p. 24). There are issues of ethics, of power, of who's in charge (i.e., whose agenda), as well as of the individual teacher's personal dilemmas and concerns. The three of us who were asked to be the leaders of the working group, Vicki, Florence, and Louise, have been involved in practitioner research for a number of years, as teacher-researchers (Vicki), and as academics working collaboratively with primary and secondary teachers and teachers of special-needs children, facilitating and supporting classroom research (Florence and Louise).

In opening the workshop we offered a number of aspects as possible foci for discussion. The first two were general. What is teacher research? How does one do it? We also drew upon Anderson (2002) and Anderson & Herr (1999) for other issues: Is practitioner research really research? Is practitioner research a separate epistemological entity? Why do practitioner research? Should all teachers do practitioner research? Should faculties of education prepare education practitioners to do education research?

In this report, we will summarise briefly below the questions which were pursued by the members of the working group over the three days, and their responses, and will end with selections of some of the reflective comments written by the participants on the last morning. The participants eloquently expressed in writing the issues of greatest import to them, and we have tried to include as complete as possible a selection of their writing as space allowed. We also acknowledge that we have missed other questions or specific comments that arose during the discussion. We apologize for those missed comments and questions.

During the session Vicki, Florence, and Louise touched upon their work and personal challenges. A number of the working group participants who are involved in researching in their classrooms gave short presentations during the first two morning sessions: Janelle

McFeetors, Jo Towers, Eileen Phillips, as well as Vicki Zack. Louise gave a presentation during the last morning session on her work with teachers of special needs children. Louise's summary of what she presented is part of the write-up which follows.

We will briefly note the thinking of the group around some of the questions and refer the reader to the more fully elaborated responses of some of the participants.

Is practitioner research a separate epistemological entity? We discussed teacher research, with an emphasis by most of the participants on the aspect of research with a small-*r* (see Jo Towers, Ralph Mason, Janelle McFeetors). The question is for us still an open one, that is, there was much more to be discussed in relation to this idea.

Should all teachers do practitioner research? "Yes," said most (see Alex Lawson, Jo Towers), thinking in the main in terms of teacher research as inquiry rather than as the more formal academic research. At least one person, Vicki Zack, said "no." Requiring that all teachers should do research implies that teaching alone is not enough, and perhaps reflects a lack of understanding of the demands and drain of teaching, let alone teaching and researching.

How does one "get" teachers to do research, teachers who do not have graduate degrees or ones who are not doing research as part of a degree program (Claude Gaulin)? Needed are some specifics, and suggestions for how to help teachers get started (Doug McDougall). How does one encourage teachers, support them (Carolyn Kieran)? How does the teacher start? (Here we are considering the teacher with no previous experience with research ideas.) What is needed is a reflective stance, stance of inquiry. One person (Dave Wagner) suggested that it need not start with a question, that indeed a question can be stultifying. Carolyn Kieran referred to Eileen Phillips mentioning a "theme of interest" and contended that what Eileen was manifesting, and what perhaps other preservice or inservice teachers might not feel, was the passion. What is needed, said Carolyn, is a trigger, un déclenchement.

Should faculties of education prepare education practitioners to do education research? Some thought yes. Jo Towers presented aspects of the preservice program at University of Calgary. All 400 students in the pre-service program engage in a major inquiry project in their last semester. The aim is to encourage teachers who will be inquirers. Jo added a caveat: When out in the schools, because the pre-service students asked many questions, at times they were seen as needy and not knowing. In some instances, the school system was not ready.

Other aspects which emerged:

What triggers the need to do teacher research? Some mentioned that the need might be triggered by a sense of wonder (Janelle McFeetors), a feeling—of elation, surprise, disappointment (Eileen Phillips). Doug McDougall contended that we need something more practical than that.

How is teacher research different from excellence in teaching? Carolyn Kieran posed this question. Eileen Phillips addressed Carolyn's question: "How does teacher research differ from good practice?" by noting that collecting data, framing, keeping detailed notes, heightened awareness, and committed looking were some of the elements which distinguished the research stance from a teaching stance. Eileen asks her students to make notes and help her collect data; it is important not to let the ideas/information slip away. The intent is slightly different if the information is aimed at parents and children, or at research. One participant mentioned that one needed to distinguish mere curiosity from research. David Wheeler's phrase that "research is disciplined curiosity" was cited.

Why do teacher research? How does one sustain it? Vicki spoke to the group about the gains as well as the constraints of being a teacher-researcher in the elementary classroom which she has done for the past twelve years; she stated that while researching from the inside has been generative and transformative, it has at the same time been very demanding of time and energy. Many of the working group participants expressed their belief in the generative benefits for the teacher of doing research in her/his classroom (see post-session reflections).

How does one do teacher research? How do you choose what to study? Bissex (1987) has said that it may not start with a hypothesis to test, but a wondering to pursue, a need to know or a need to change or a need to find out more. The commitment is intense. There is too much work involved in researching to work on something that is not a passion.

Participants spoke of *critical incident*s (Laurinda Brown), and of John Mason's *Discipline of Noticing*, and of the voice of the other inside you (Laurinda Brown). They spoke too of collecting data 'for the hunches you feel.' Vicki Zack cited from McCarthy (2000): "My [advice] ... to teacher researchers is, then, to respect your intuitions, take seriously the moments in your teaching that touch you, make you want to cry or sing or shout. Pay close attention to events you feel you need to understand or which you want to memorialize in your writing. If the moment "refuses to go away" ... I urge you to study it. Try to figure out why you were so affected, which of your assumptions, your closely held beliefs, were challenged by the incident" (Fishman & McCarthy, 2000, p. 270). Others stated that the teacher researcher must carefully watch and intentionally listen. She/he collects data—so that she/he has evidence to substantiate that 'you heard what you thought you heard.'

Is it necessary to communicate the research? Carolyn Kieran proposed that the research has to be shared and members of the group seemed unanimous in agreeing with this position. Indeed, Lawrence Stenhouse whose first-generation work in teacher research in the late 1970s gave rise to a movement, underlined the importance of written findings, documentation, peer criticism, and written reviews. Vicki Zack pointed out that those who follow Ann Berthoff's approach put a questioning approach at the heart of teacher research and do not of necessity require publication of findings.

Is it important that the research findings be novel? See Rina Cohen's reflections on this question.

What is the role of adminstrators? Eileen Phillips spoke of some of the ways in which administrators can nudge and support teachers who wish to do research. Doug McDougall shared his reflections on the importance of the involvement of administrators in the teacher-research project in which he worked with secondary school teacher-researchers.

A need for work in teacher research in mathematics. Is it different from work for example in language arts /reading/ writing? (See Cynthia Nicol, Dave Wagner.) Claude Gaulin noted that we need a reference list on teacher research in mathematics education.

Can you be a reseacher even if you do not know that you are one? The response was "No." An example given as an analogy was David Wheeler speaking about the weaving in a basket. It is only mathematics when you mathematise it, when you see the mathematics in it.

Research with teachers, research by teachers. We asked Louise Poirier to write a summary of her presentation to the group; her summary follows immediately below. Her theme was research with teachers; she presented an example of collaborative research, a project with teachers of special needs children.

For the past few years a group of researchers involved in research with teachers have been developing a model for collaborative research (see Desgagné, Bednarz, Couture, Poirier, & Lebuis, 2001). L'approche collaborative de recherche en éducation : un rapport nouveau à établir entre recherche et formation. *Revue des sciences de l'éducation*, XXVII(1), 33–64.). In our working group, this model was illustrated with a project involving teachers of mentally handicapped children.

The initial question came from the teachers: "Is it possible to push back the limits of mentally handicapped children in terms of learning the concept of number? How to do so?" When I first met the teachers, I had no previous knowledge about mental deficiency and I told them so. They had the knowledge and experience of teaching to these pupils. Some of these teachers had been working with them for more than 20 years. On the other hand, I had another kind of knowledge: mathematics, how to teach teachers, and mathematical development in children. So together, the teachers and myself, we would develop learning activities adapted to the needs of their pupils. In that way, collaborative research aims at a mediation between the practitioners' community and the researchers' community by bringing together the two communities. It assumes the co-construction of an object of knowledge. Collaborative research links research and continuous formation; it involves a joint reflection activity which is a research activity and a learning activity.

For example, this project had a professional development goal by raising questions concerning mathematics notions contained in the curriculum and concerning the proposed learning activities adapted to these children. It also had a research goal by analyzing teaching situations and their potential for conceptual development among these pupils (analysis of classroom videos and the teachers "journals"). For three years, we had monthly meetings where we would discuss learning activities, the students' reactions to these activities, we would analyze their difficulties, the ways they would solve a problem, we would talk about teaching strategies and classroom management. What was proposed was a regular, planned alternation of in-class experimentation of the approach with group reflection on this process. What came out of this project were quite unique and different teaching strategies because of the bringing together the two different perspectives and expertise. What came out of the project was a different outlook on the learning of maths of mentally handicapped children as two of the teachers interviewed at the end of the project told us:

H: ... this project helped me in understanding why I was teaching those concepts.
C: It gave us time to stop and think.
H: Yes, we were fed by the group because in the heat of the moment, in action, you do not have time. To work with you has persuaded me that this way of working is the way that reflects best these children's brains. These are children that are developing. One has to admit this because if you do not, you better work somewhere else.
C: What we knew intuitively, you gave us the theoretical background.

Participant Reflections

Laurinda Brown

"Special Needs children have minds too." (Patrick Keane)

A small group of 11-year-olds is separated out from their year group to be taught together for mathematics and English as a class of special needs students. Their expectation was that they would be controlled and given closed, practice tasks to learn to read and do arithmetic. Their teacher Patrick Keane, saw his role as questioning and supporting students to make their own decisions and to rely on themselves.

This image arose in my mind listening to the story of collaborative research and teachers seeing their students differently—having more faith in them. Part of research is becoming aware of connections, linking images and learning from them. It seems to be the inquiring stance of everyone in the process that is the key for me from these few hours of reflection and discussion. When the students in Patrick's class had questions arising they operated like any other person would. When teachers of other subjects in the school complained to him about his students he would ask what they were being given to do and say passionately in reply to their statements 'Special Needs children have minds too.' By this he meant that they were able to learn, to make connections out of their own need to know, to be in their world and to wonder. His role was to learn about his students, to support and challenge them.

As a teacher in the UK it is also possible to be positioned as if with no mind: to 'deliver' the Numeracy Framework as if that is an un-problematic thing to do ... but, as I have been strongly reminded these three days, teachers are professionals with autonomy within their classrooms. Questioning is the more natural stance, teachers have minds too and are individuals, not a different class of human being. Their focus of interest or theme of involvement emerges out of the creativity and the permission to reflect. How can teachers be supported in the knowing that they have minds in this process? Not everyone will want to do the rigorous studies of Eileen or Janelle but the inquiry is always present.

The teacher-educator or university lecturer is also part of this process of inquiry through teacher education programmes and in-service work and can take their own inquiring stance into their work with developing teachers. The image that is strongly with me as I finish this writing is of how this inquiring stance is infectious—inquiring teachers support inquiring students who develop a new relationship with mathematics that is not so hard edged and certain (Jo's, Janelle's, Louise's).

We all seem to value listening and hearing and the energy of interrogating our own practices. We all have minds too! I wrestle with how to encourage descent into habit without atrophy, keeping the learning anew, re-searching as a way of being for student teachers, teachers, children and myself. This is where I find the noticing paradigm to be so useful in linking the new with the previous history of all of my life allowing restructuring and transformation. Writing for me is part of this process of restructuring. Critical reading is another part. Collaborative working is also a factor. At various times these activities feel more or less like research projects.

Rina Cohen

The question that has occupied my mind the most during our meetings (and in-between meetings) is: Which type of teacher research write-ups, or publications, will be of value to others—both teachers and researchers—many years from now? What kind of teacher research publications (or products) will become "classics" for future generations?

A related question that came up was: "Should teacher research that does not include anything "new" be published? Could it be of any value to others, or to the field of education in general?

To answer the latter question we need to first clarify what we mean by "new." Of course, we can ask: *"New" for whom?* That's a key question. For those in the field who believe in 'formal' research methods, teacher narratives may not be recognized as findings of the research. Of course, such narratives are always unique and different from each other, so they are "new" in that sense. But one might ask: Does a particular teacher narrative (or research report) include any "new" ideas or conclusions that have never been published before?

The answer may depend on the type of teacher research report written. If indeed it is a very detailed narrative of the *process* that the individual teacher has gone through during the research project(s)—then in my mind there is bound to be in it some new perspectives, new ideas, experiences or ways of relating to experiences. There will always be at least some subtle differences among different teachers' reports.

If the type of research report is of a more "formal" nature, e.g., using the more academic style commonly used in research journals (e.g., including a section on 'findings' and then discussion, conclusions, etc.) then one may ask: does the paper include a "new" research finding? One that has never been published before?

I personally believe that in addition to the latter type of publications including some "new" findings, the teacher narratives describing and reflecting on the *process* of the teacher research in great detail are also invaluable resources for teachers *and* researchers. They have the potential to capture the subtle nuances of teaching/learning processes that may not be included in a 'summary of findings' section of a more formally written report.

To be of interest for the wider audience, such teacher narratives need to be well written and include enough details about the *context* of the research, and "thick" descriptions of the actual experiences plus researcher reflections to allow future readers to benefit from reading them.

Florence Glanfield

The image that continues to come forth for me is that of a quilt. A quilt where the patches are the 19 individuals in our working group, all satisfying the quality of "being human," all brought together by the possibility of the conversation about teacher research. Throughout the nine hours we were together, common threads of bringing the pieces together were illuminated. Some of the common threads I "noticed": the importance of engaging in professional conversations about our thinking and practice with colleagues who are working in different schools and universities; I continued to hear how the triggers for research became noticed through conversations with people; questioning how and what we've come to know; passion about mathematics education and "enlarging the sphere of the possible" for us as individuals, for children learning mathematics, and for teachers. Once again I am reminded of how we move back and forth between being "in community" and being "in ourselves." I was so appreciative that Janelle, Jo, Vicki, Eileen, and Louise shared their stories. The power of narrative led to wonderful in depth discussions about specific practice and related theory. What does this mean for me as a teacher? I've thought a lot about my pre-service teaching and am asking "in what way(s) is my teaching an inquiry model for my students?"

Carolyn Kieran

Louise ended her brief presentation on Day 3 by remarking that one of the highlights of her collaborative research project with the teachers of the mentally-handicapped children was the "bringing together of the two communities." This bringing together of the two communities represents one of the important ideas that emerged for me from this working group.

When the working group began, I had wondered how the university researcher might encourage teachers to become researchers in their own class. Thanks to the discussions that took place over the past three days, I now see that this perspective is ill-founded.

Teachers need, first, a support system in their own schools that encourages the asking of questions and the pursuit of a teaching practice involving inquiry and the sharing within the school community of the processes and products of these inquiries. But that is not enough. Teachers need other resources that they can draw upon to move forward in their own thinking. These other resources can come from books, journals, interactions with math education researchers from the university, and from feedback provided by audience participants when teacher researchers talk about and present their results.

Thus, both schools and universities have important roles to play in the support and development of teacher researchers. School principals, vice principals, and other teachers are the home community in which the teacher researcher is fostered, encouraged, and grows. The roots of this research may be found in prior pre-service courses, in post-graduate research work, or in the school community itself. But at a certain moment, the teacher researcher who has not already established a network of outside resources will likely need to draw upon other voices in order to further evolve. There the university mathematics education researcher may be able to help.

However, structures need to be put into place in order to allow the cross-fertilization of the two communities to occur. As teacher research is a process that needs time in order to grow and develop, the structures that are set up need to take the nature of these processes into account.

Alex Lawson

If we are looking in math ed at the end, that is, supporting instruction in the class that engenders greater understanding of and enjoyment of mathematics in children, then the question is: "Are there types of teacher research projects that can play a role in this? And, are they *necessary* to shifting math ed instruction in this way?"

If there are teacher research projects that support the development of instruction of this type are these the projects of a few? (We have always had outstanding special projects/schools/teachers or, are these projects a model open to many?)

If there *are* effective wide-scale projects, what do they look like? We have good pictures of effective small projects.

If teachers don't participate in some sort of critical community, including some type of professional reading, critique (i.e., teacher research), then I would suggest that we continue to promote teaching as craft rather than teaching as professional activity. Certainly within Ontario, the government emphasis, consciously or unconsciously, is the maintenance of teaching as craft, particularly at the elementary level, particularly in math.

Ralph Mason

I have chosen to put words to the questions that I am considering. Generally, I am coming to perceive teacher research as a powerful and empowering possibility when done in ways that take the processes and purposes of researching seriously. That means that it isn't something I believe could or should be done by large numbers of teachers. I am moving away from positions that want to dilute the concept of teacher research, and in so doing perhaps co-opt the processes of teachers inquiring professionally into their practices, their students' learning, and their contexts.

1. The necessities of teacher-as-researcher:
   a. Intention—her/his interest, desire to understand; to grow; to contribute beyond.
   b. Research capability —an extension (natural?) or teaching?; an intensification of an inquiry stance?; Learned by reading?—what?—how much? Is reading research a teachable skill?; Learned through apprenticeship?—mentoring—collaboration—coaching; learned through PD or MEd? Learned in B. Ed?
   c. Passion—confidence; desire for rigor; desire for the full process; community; collegiality; mutuality.
   d. Context, opportunity, time.
2. But which necessities are preconditions?
   Whichever, should we not foster the preconditions, rather than build on swampy ground? E.g., teaching inquiry as stance and a set of processes; building professional discourse communities.
3. When should teachers do research?
   Vicki as a completion of her teaching.
   Eileen as a transition from teaching.
   Janelle as a teacher who is also a student in a masters program.
   Evan as a very special preservice teacher candidate completing a required task.
4. Writing/presenting for others.
   Do *we* hunger for more research to read? Is writing for teachers, for teacher-researchers, for academics the goal? [It must be targeted to be focused.] Should what counts as writing from research stay the same, now that what counts as research has been stretched? Will we *teach* the writing of research? (See choices in 1b.)

5a. Why do we want more research? More research articles? I don't think we do. Do we want less teaching? I don't think we do. Why are we *not* proposing that we researchers *be teachers*?
If more teachers did more research as teachers, what would they do less of? Would it be good if they did less community-of-discourse building, fewer courses on learning of math, fewer inter-classroom visits, less participation on school improvement initiatives or self-selected reading? Or parenting, or gardening, or lunch time help for kids, or helping the kids leading the drama presentation, or challenging the curriculum or the school attendance policy. I don't think it would.

5b. Continuing the critical questioning: What will *we* do less of, to foster or enable more teachers to succeed as researchers? Will we do less research ourselves, to assist teachers to do more? I don't think we have the time or resources to do the research we now need to do.

6a. We teach math in grade 3 even if it seldom includes the formal deductive reasoning that some say makes math truly math. But if it leads to math later (if) then it's proto-math, maybe, and that's a good thing. Is there a need for proto-research? Maybe that's something we can add to the way that we think of teachers learning to do professional inquiry as an inherent part of their practices.

6b. Which of us will write accessible and invitational pieces for the teachers, to foster their passion, inform their processes, exemplify what they could write?

Doug McDougall

I was particularly impressed with the presentations and insights over the last three days. We explored the key areas of teacher as researcher: motivation, teacher knowledge, supports, developing good questions, research methodology, research "rigorness," and challenges. The examples were essentially from those with masters and/or doctorate degrees (or pursuing those degrees).

Our biggest challenge, however, is the practicing teacher without academic credit motivation. I believe that all teachers should be doing research. I had hoped that we would have explored ways to assist teachers in doing this research. The main focus of successful teacher research in this situation is the type and degree of administrative support. Eileen gave some good suggestions on how administrators can encourage teacher researchers to be able to conduct research. However, we need to be more insightful in what specific supports are needed.

One type of support is time. In a research project that I conducted this year, we paired secondary school teachers to encourage them to change their mathematics teaching practice. The principal provided time for the pairs to visit each other's classroom and to attend research meetings. This extra time allowed for an exchange of ideas and visitations that could not otherwise have occurred. At the end of the project, the teachers said that the single most important aspect of the project for them to feel successful and valued was the time given by the principal.

A second support is the creation of a professional learning library. Teachers need to read about new developments and successful practices in order to further their understanding of the phenomena under study. A principal can assist teachers by creating a professional library of books and journals. School boards can also be expected to have a professional library to assist teachers in getting appropriate articles and references.

Teacher research needs to be valued within the educational community. Principals can use staff meetings to introduce research on teaching and learning. They can also provide opportunities for teachers to present their findings and encourage feedback from other teachers and professionals. School boards can provide opportunities for research sharing amongst teachers.

A second major topic from the working groups was the role of preservice education programs in the development of teacher researchers. As a teacher educator, I believe that all students should be engaged in research. The methodology of creating a research project is essential for preservice teachers to understand the research literature and for their own personal improvement in teaching.

Teachers have many questions about teaching. They need to have methods to find answers to those questions. I believe that professional reading, research articles, other teachers, other professionals, and the community (as well as their students) provide an excellent source of data. In addition, however, they may need to gather their own data from students on their own practice in order to answer the questions. If student teachers have had experience in research, they would feel more comfortable and more confident to ask questions.

Additionally, student teachers/teachers would be more likely to encourage their own students to ask questions in class and to prepare a method of answering those questions. Teacher modeling is essential for exploration in mathematics. Teachers being classroom explorers and researchers provide a better and sound basis for the development of teacher experts!

Janelle McFeetors

One place to begin is to make a broad distinction between "outsider" researcher and teacher-research. At the heart of this distinction is the notion of *authority* and *intention*. For teacher-research, it is the teacher that directs the purposes and foci of the research that is being conducted in the classroom. If the teacher-research happens to be collaborative with outside researchers, it still is the teacher that directs the study. I'll say more about intentions in a moment.

I think we run into some trouble when we use the word 'research' to describe or label what teachers do when they engage systematically into phenomena (both teaching and learning) in their classroom. I think we need to move towards a conception of inquiry. And, yes, this is a part of what good teachers do in their classrooms. But this inquiry is guided by strong *intentions*. One intention (and must be, I think, even though it is strong language) is that the teacher is a *learner*. It is a learning journey (both personal and professional) that the teacher engages in as she inquires into her practice and her students' learning. Another intention is to improve teaching and learning in the classroom.

Teacher-inquiry differs significantly from "outsider" research, too, in that it is from the *inside* of classrooms. The teacher lives in pedagogical relationship with her students (van Manen, 1991) and that relationship affects what the teacher hears and sees in each student. She hears and sees more than any other could. What that means for research in the classroom is that as a teacher reads and interprets data, it is done at a more rich and complex level than any other researcher could do it. In other words, there are things about students, learners, individuals, learning, teaching, and classroom contexts that could not be *heard* and *understood* if it was told by another other but the teacher and students.

Of course, that draws in the idea of what is done with the data and interpretation afterwards—especially in terms of publication and sharing with others. I'm not sure. But it does also draw into the discussion the different elements of traditional research that would be required to form some sense of rigour that would constitute teacher-inquiry as research. We certainly discussed the role of questions in this process. And I believe that we (inquiry teachers) are constantly questioning the phenomena occurring in the classroom. So a sense of wonder is at the core of teacher-inquiry.

One thing that I am left wondering and struggling with is the role of reflection in this process. Is there a line to be drawn between teacher-reflection and teacher-inquiry? To be sure, reflection is a necessary component of inquiry—but is reflection sufficient for inquiry?

By the way, thank you so much for the opportunity to talk about my research. It always helps me to share and to consider the questions that are posed.

Cynthia Nicol

Looking back at my notes and thinking more about the teacher research session ... I think the session prompted me to think more about (at the time of the session and in the months following) these questions:

- The question of "What is TR (teacher research)" wasn't really a question for me during the session but it has in some form become an issue that I've re-visited. It is not so much a question of "what is" but more what counts as "good" TR. The issue also involves how I might help the preservice teachers I work with extend their views of what counts as research and what counts as good TR. In viewing TR projects that last year's students completed, this year's group was very critical about the very personal nature of the stories told and actions pursued. They claimed that this was not research. They objected to the use of the first-person in reporting the research and some claimed that many projects were "too self-reflective". These are preservice teachers in a problem-based learning cohort who were very experienced at reading educational research—although as I see now, few had opportunities to read and critique TR reports. Just as we would like our students to consider alternative ways of knowing, learning, and teaching mathematics we need also consider how we can as teacher educators help our students envision alternative ways of making sense of their teaching experiences and the contribution this can make to the educational community? How can we help preservice teachers participate in the learning to develop the "local knowledge of teaching, learning, and schooling" as Cochran-Smith and Lytle (1999) call it.
- What makes TR in mathematics education or self-study in mathematics teacher education different from TR in other areas, or is it different? Are preservice teachers more likely to have the conceptual frameworks needed to pursue TR in literary subject areas rather than mathematics? To what extent does the teacher's understanding of mathematics need to become a focus of teacher research in mathematics education? Interestingly much of

the well-published "first-person" research in mathematics education does not draw upon teacher research or action research as a framework (consider the work of Martin Simon, Deborah Ball, Ruth Heaton). What are the risks for teachers to lay out and make public their developing understandings of the mathematics they are teaching?

- During the session we talked about making TR public. Some mentioned that it was more challenging to publish pieces that were TR or self-study works. I think this brings into question the standards of quality that are used to judge TR and self-study work. Is the work compelling, significant, valuable (and valuable to whom)? I think Cochran-Smith and Lytle (1999) refer to the methods and knowledge critiques of TR that have been launched. But Feldman (2003) also brings up the issue of validity. He posits an existentialist orientation might help us report and study our teaching beyond solipsism to examining who we are as teachers or teacher educators. Changing teaching practice involves changing not only the practice but what it means to be a teacher. Feldman suggests that if teacher research or self-study research is to make knowledge claims and understanding that can be used and shared by others then the issue of validity needs to be addressed. Considering how to make TR more trustworthy Feldman argues that we need to make public and make explict the ways that we makes sense of or represent our research.
- I think Feldman is mainly referring to how to make self-study research trustworthy to academics and policy makers. But the question remains how we might make TR in mathematics education accessible, valuable, and trustworthy for teachers. How can teachers present their research to other educators? What kind of representations of research do teachers have the time and context to engage with/in?

These are some of the questions that arose for me as a result of the session. I think the presentations by those "doing" TR were very good and the article was likely a good anchor to focus discussion and initiate further debate. Perhaps some hands-on working with data, or viewing some video clips, or reading a TR study and considering how alternative ways of re-presenting the data might enhance the trustworthiness of it may have added an interactive component to the session beyond whole and small group discussions. I quite enjoyed the session and being part of the group.

Eileen Phillips

Many questions surfaced over the days and the ones that intrigued (and that continue to intrigue me) the most were/are:

- How are the activities associated with teacher-research different from, and similar to, exemplary teacher practice?
- What does teacher-research look like/sound like to an outside viewer and to the teacher-researcher? Can it be identified by an outside observer?
- Is teacher-research identifiable by the teacher-researcher as a unique set of feelings/actions within the act of teaching? How can I tell when I am engaged in research and when I am not—i.e., when I am 'just teaching'?
- Can one be an exemplary teacher without being a teacher-researcher?
- Why would someone want to do teacher-research?

For future sessions I would like to discuss the questions:

- What is possible for a teacher researcher using teacher-research methods that is not possible using other types of classroom-based research (such as outside observer/expert)?
- What claims can a teacher-researcher make that are unique to his/her situated context?
- What are the areas that teacher-research is not suited to address?

Reflections after the working group sessions:

It seems that teacher-research is still a young field—newer to many others than it is to me. Many of the beliefs that I have carved out for myself through practice, discussion and reading are not as generally agreed upon as I thought. For example, it seems that many people are still striving to define teacher-research as a series of actions without looking closely at the intentions behind the actions. This helped me to realize more fully that, for me, teacher-research is about intention and about attention.

Research actions within a classroom are not necessarily teacher-research. The teacher has to be willing to take the results of those actions and work with them to produce a "story" that helps illuminate something that has happened or that is happening. The teacher also needs to be willing to try to situate this story within the stories of others. How is what is being attended to in one classroom related to that which is being attended to in other classrooms? How is what I do one year related to what I do the next? If I work on the same problem year after year with my students, how does this become teacher-research? How will the knowledge about learning and about teaching—in one student, in one situation, in one classroom, etc.—become part of the knowledge of the greater whole?

In our group's discussion it became clear that we all felt that one condition of research was that it needed to be reported. We were very open to the methods of reporting that we agreed would fulfill this condition: talking to others, informal presentations, formal presentations, papers, reports, thesis dissertations, book chapters, journal articles, and so on. So, communication seemed to be an essential feature of teacher-research.

This being the case, where are the teacher-research journals? Where might a teacher-researcher seek publication? NCTM journals are one place and more general magazines like *Chatelaine* seem to be another. Provincial specialist associations also publish teacher-research. However, there does seem to be a need for more publication venues. The group briefly discussed this and I left full of the desire to see the start-up of a journal for this purpose—one that is broader than Mathematics in scope of topics.

Louise Poirier

What is teacher research?

1. Research on teachers?
2. Research by teachers?
3. Research with teachers?

For me, teacher research is not about research on teachers but research by teachers. Can I label "teacher research," collaborative research which is research with teachers? If by research, one means "inquiry" or questioning one's practice then I would say yes. If by research, one means "academic research" building body of knowledge, I am not sure of my answer. At first, the teachers I have worked with in the different projects were not seeing themselves as doing research. They had a conception of research as something being done by a "researcher" (someone somewhere in the university) who would sometimes come in to collect data and would never be heard from again. But as the projects would "grow" and develop, the teachers would talk about the research and them doing research. Their perspective on research was changing. If one has to produce "scientific knowledge" in order to be a researcher, then I would say that the teachers involved in the different collaborative research projects are not researchers. But if that body of knowledge is also professional knowledge about teaching then I would say yes. In the different projects, there were activities written down and shared, books were published. Teachers talking to other teachers.

Key issues in order to talk about teacher research:

- to have an initial question, idea, intuition;
- to have a community to rely on: literature, other teachers, professors;
- to have support to find space and time to reflect—support from the school, the school board, the ministry of education;
- to communicate the findings.

Jo Towers

I was initially captured by the posing of the statement: "All teachers should engage in research" or "All teachers should be researchers." I have kept coming back to this statement over the last three days, trying to decide whether I still agree with it, because my first reaction was to strongly agree with the statement. As we have progressed over the three days, we have begun to unpack the term "teacher- research" and I suppose I am now less inclined to agree with the initial statement if that statement implies a formal mode of data collection, data analysis and reporting. I think when I hear or read "All teachers should be researchers" I understand it as "all teachers should be inquirers," and on this the working group members seem to be in agreement. It is probably not a reasonable expectation that all teachers will be (formal) researchers—nor could any of us hope to keep up with the field of research thereby created. However, I do think it is reasonable to expect that all teachers inquire into practice, not simply enact it, but this needs to begin in initial teacher education. This is, of course, a core principle of the teacher education programme at the university in which I work.

The idea of teacher-research requiring neither a split in attention nor a conflict in intention (Wilson, 1995) also captured me (again—I had read Wilson's piece before, but it was nice to be reminded of this issue). I think we have been fortunate in this group in having several teacher-researchers who helped us to see how this is lived in a classroom. I found myself idly imagining if it might be possible to have someone study me studying my own practice, as I am sure it must be having some influence on my students. What I mean is, what difference does it make for students being in a teacher-researcher's classroom?

The notion of a community of (or perhaps community for?) teacher-researchers seemed to emerge as a significant idea for us over the three days.

Louise's first question (on her final overhead) seems significant and one that ought to be pursued: In what ways are practitioners essential to developing knowledge in education research?

Laurent Theis

Au cours des trois jours de colloque, j'ai appris à considérer la recherche par les enseignants comme une suite « logique » d'un bon enseignement. D'ailleurs, les limites entre le bon enseignement et la recherche semblaient être floues tout au long des discussions. Il semble alors que le degré de formalisation dépend de chaque enseignant.

Les questions que je me suis posé par rapport à la recherche par l'enseignant se rapportent surtout à la formation des enseignants. Comment peut-on amener nos étudiants vers une bonne pratique, et, éventuellement, vers le rôle d'un enseignant chercheur? Il me semble important dans ce contexte d'amener nos futurs enseignants à se questionner par rapport à leur pratique : apprentissages réalisés par leurs élèves, erreurs, stratégies d'enseignement, etc. Ce questionnement, qui fait d'abord partie d'une bonne pratique, peut alors mener, si elle est formalisé davantage vers une question de recherche.

Plusieurs éléments me paraissent importants dans ce contexte:

· Comme formateurs, il serait souhaitable que nous aidions nos étudiants à se questionner, et à se poser les « bonnes » questions.
· Nous devons essayer à les amener à s'intéresser à ce que font, apprennent ou pensent leurs élèves.
· Les futurs enseignants doivent acquérir des connaissances conceptuelles solides pour pouvoir questionner les démarches des enfants et leur pratique.

Je me demande alors si les travaux de fin d'études, dans lesquels on demande aux étudiants d'investiguer plus en profondeur une question de leur choix peut être un moyen intéressant pour former nos étudiants à une bonne pratique d'abord, et éventuellement au rôle d'un enseignant chercheur. Bien sûr, au niveau méthodologique, ce travail ne peut pas avoir la même rigueur qu'un travail de maîtrise, mais on peut quand même espérer que l'étudiant y apprend à poser une question, à l'analyser plus en profondeur, et à la communiquer de manière écrite.

Peut-être que ce travail n'amènera pas chacun des futurs enseignants vers une pratique comme enseignant chercheur, mais on peut quand même espérer qu'il leur permet d'évoluer vers une bonne pratique, et qu'il éveille chez certains futurs enseignants l'intérêt de formaliser davantage leur questionnement sous forme d'une recherche dans leur classe.

Dave Wagner

I came to this working group with an active interest in collaborative research and a passive interest in the benefits gained by teachers who research.

For me, at the heart of research is its role in coming-to-know. And I think of knowing as a dynamic thing; I am interested in knowing, the verb, more than knowledge, the noun. Within me I sense a tension between a more formal conception of research, which is closely connected to the scientific method (with a question, a method, communication for the purpose of inviting confirmation or reproduced results), and a less formal sense of research, which I think of as a way of looking again, re-search.

There is more than one way of coming-to-know well. We don't have to start with a question. In some areas, asking a question damages the resultant knowing, the understanding of how we live well. I assert that asking a question always changes what we see. There can be good in what we see whether we ask a question or not, whether we share results or not, whether we follow disciplined/traditional methods of looking or not.

Most of our discussion centred on the value of bringing traditional research (*à la* scientific method) into teacher practice. I am also interested in the potential value of bringing pedagogic intention into our big-*R* research considerations. When I work with a practitioner-collaborator, for example, I have access to the complexity of the classroom situation. This complexity makes analysis of classroom events/relationships much more difficult but much richer. I am interested in the richness that flows from such interpretation.

Also, when I as a researcher have the experience of immersion in a classroom working in close collaboration with a practitioner, I find myself asking different questions of other more traditional research reports that I read. I tend to notice the lack of due consideration of classroom complexities.

Another interest of mine is the potential for seeing students in the classroom as collaborators. What unique perspectives do they lend to my interpretations? I saw some of this valuing of student voice in Eileen's account of her research, for example.

Finally, I ask now how all this is unique in the study of mathematics teaching and learning—as opposed to the study of writing, or of collaborative learning models, for example? I don't have an answer to this question yet, not even a tentative answer. However, I believe that it is an important question for further consideration.

Vicki Zack

In preparing for the workshop I thought hard about my personal stance and positioning. I was delighted during the workshop to hear so many questions/ideas which resonated with my own experiences, questions, concerns.

1. Should all teachers engage in research?
2. What is necessary—the trigger—to get someone to take on the challenge (Carolyn)? Carolyn felt that what Eileen Phillips had was a "theme of interest"—a passion, and that possibly Claude's students did not have it. Jo (Towers) showed us how this interest or passion can be cultivated, and her example of one pre-service student finally feeling a connection with research (Deborah Ball's student Shea, with 6 being both an even and an odd number) and feeling part of a community of inquirers. I was struck too by Jo pointing out that in some school settings, because her graduates, now in-service teachers, are posing questions, they are at times seen as needy.

Questions that struck me:

1. What distinguishes teacher research from the work a good teacher does on an everyday basis? A number of people spoke to this question, Eileen most fully and eloquently on the third day.
2. Can you be a researcher even if you don't know you are one?
3. Interesting and of great importance to me: What distinguishes the teacher-researcher situation from the outsider researcher situation? Janelle's term: the "pedagogical relationship" (van Manen) was a term and idea I was seeking and now can use.
4. In what ways are practitioners (teachers, but also psychologists etc in schools) essential to developing knowledge in education research (Louise)?
5. In what ways do teachers need researchers [in order to perform their teaching practice]?
6. In what ways does collaborative research hold out potential for in-service education (Louise)?

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Images of Undergraduate Mathematics

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The theme that the working group was given to discuss, named "Images of Undergraduate Mathematics," is certainly very broad and far-reaching. Thinking about it, we published the following abstract:

> "I hate math!", "What is Fermat's Last Theorem about?", "I really liked your lecture on infinity.", "Fractals are cool but I hated those area and perimeter calculations.", "Do I have to teach that calculus course again?", "Do all mathematicians look like that guy in *Good Will Hunting*?", "I have always liked math, and was good at it.", "This textbook is useless.", "Why do we need all that geometric stuff?", "Who is Sophie Germain?", "Chinese students are expected to do well in mathematics.", "What is all this theory good for?", "The problem is that students don't learn that stuff in high school anymore.", "Why are you bothering me with questions, just give me the damn answer!"
>
> Images, opinions, and views of mathematics are uncountable ... so are emotions and stereotypes. The comments above come from students' comments in course evaluations and journal entries, faculty comments over coffee, and comments in the media. Can you tell which is which?
>
> This working group plans to look at this large and complex space of undergraduate mathematics—to discuss, investigate, and analyze, in an attempt to describe what it looks like. We will not restrict our attention to courses for mathematics majors only: "service" courses will also be considered and we will explore what each type of course can learn from the others.
>
> There are many approaches (historical, cultural, ethnomathematical, teaching/learning, epsilon-delta/no epsilon-delta, etc.), and viewpoints (undergraduate students, university lecturers, high school students and teachers, mathematics education researchers, media and popular culture, etc.) What attracts students to mathematics? What repels them? How can we keep interested students from "turning off" to mathematics? How can attract those able students who are already turned off? How to perceptions and misconceptions play into these issues?
>
> We'll be searching for interesting facts, fresh ideas, and creative insights. What can we learn? Can it help us appreciate mathematics more? Can it lead towards improvements in the way we teach and learn mathematics? What other questions can we possibly ask (and answer?)

Moreover, to provoke, motivate, and suggest possible directions of discussion, we produced a four-page handout (see Appendix).

Two-and-a-half days of (somewhat unfocussed) discussion resulted in the following list of statements and issues.

1. *Mathematics is difficult.* Instructors and teachers need to be honest and straightforward about it.
2. *Mathematics is useful.* Good reasons to learn mathematics include gaining valuable skills and increased chances of finding a good job.
3. *Mathematics is exciting.* This excitement, enthusiasm, motivation for mathematics needs to be 'transferred' to our students.
4. *Mathematics is not calculus.* However, at most universities calculus is the only course that a large majority of students take. Although calculus is essential to understanding certain areas of mathematics and applications, there are other mathematics disciplines that are of equal importance, but are not taught to non-mathematics majors. New courses at the entry-level are needed.
5. *Mathematics education is communication.* We need to facilitate communication between high school teachers, college/university mathematics instructors, and mathematics educators.
6. *List of knowledge and skills.* Is it possible, and on what level (local, provincial) to create a list of knowledge and skills that high school students possess (i.e., that are expected of them).
7. *Mathematics instruction needs small classes.*
8. *Mathematics education needs the support of NSERC, CMS, and others.*

Although we have touched upon numerous subjects, we will focus our presentation on the above statements.

Ad 1

Mathematics—on any level, from elementary school to graduate courses—is, for most of our students, a difficult subject to learn. They know it, and we know it. We should be honest about it, and tell our students that they are about to learn something difficult. With our support and help, and lots of work on their own, our students should be able to learn the material.

Issues to consider include:

- How to teach mathematics to the 'lower end'—i.e., to the students who are not adequately prepared for a university course in mathematics;
- 'Upper end' students usually get neglected; need to create challenging contexts for them;
- Drill has a place in learning mathematics; it can also serve as a motivational tool;
- Reading a mathematics textbook is not easy—we need to teach our students how to read mathematics.

Ad 2

We know that mathematics is useful. But what about our students? How can they conclude, reading a calendar description of a course (which is usually very short, and more often than not, quite vague), why it's important for them to take it? How will they benefit from it? How does the course fit, more globally, into their program? What skills are they going to gain, and why are those skills important and relevant to them?

Course syllabus is an ideal medium to address these questions. Of course, it cannot answer all students' questions and concerns, but could certainly be a good start.

Consider, as an example, the list below, given in the syllabus for the first-year science calculus course (Math 1A3) at McMaster University.[1] It is expected that the Math 1A3 course will:

- give you a detailed discussion of basic concepts of calculus of functions of one variable;
- give you some experience in relating mathematical results obtained using calculus to solu-

tions of problems in other disciplines and to “real-world” problems;
- give you experience in constructing and interpreting graphs of functions, so that you will be able to interpret pictorial data obtained from various sources (computers, reference manuals, instruments, various reports, etc.);
- give you experience in reading and writing mathematics, so that you will be able to communicate your mathematical and technical ideas to others and use various reference sources;
- teach you how to use computer software to enhance understanding of the material and to solve various problems.

This is just a start. In lectures, when the opportunity arises, students’ attention is brought to the above items. This way, they can see how the promises given in the syllabus ‘materialize’ in context of the course. The list of course objectives, that accompanies the above list gives further information on what will be happening in the course:

- to learn about basic concepts of calculus (function, limit, continuity, derivative, integral);
- to learn how to think logically (mathematically);
- to learn how to communicate mathematics ideas in writing;
- to learn about mathematics as a discipline (what is a definition? theorem? why do we need to prove statements in mathematics? why does mathematics insist on precision and clarity?).

Some universities use so-called ‘mission statements’ or ‘rationale for a course’ statements to precisely describe their courses in terms of knowledge an expectations. It was suggested that, in some format, these statements should enter university’s official document (‘course calendar’). Statements about the course should also include the following:

- ‘location’ of the course in terms of other courses (what are prerequisites); what courses are sequels to it; how does the course fit into a ‘general philosophy’ of a particular program;
- detailed list of material that students are expected to know or be familiar with;
- suggestions on how to review the background material, possibly with a good reference.

Given the reasons, and assuming that our students are convinced that learning mathematics is useful, how do they learn mathematics?

The above-mentioned syllabus document for the McMaster calculus course includes the following:

> Learning mathematics (physics, chemistry, philosophy, etc.) requires dedication, discipline, concentration, significant amount of your time, and hard work.
>
> To learn mathematics means to understand and to memorize.
>
> To understand something means to be able to correctly and effectively communicate it to somebody else, in writing and orally; to be able to answer questions about it, and to be able to relate it to known mathematics material. Understanding is a result of a thinking process. It is not a mere transfer from the one who understands (your lecturer) to the one who is supposed to understand (you).
>
> How do you make yourself understand math? Ask questions about the material and answer them (either by yourself, or with the help of your colleague, teaching assistant, or lecturer). Approach material from various perspectives, study solved problems and work on your own on problems and exercises. Make connections with previously taught material and apply what you just learnt to new situations.
>
> It is necessary to memorize certain mathematics facts, formulas, and algorithms. Memorizing is accomplished by exposure: by doing drill exercises, by using formulas and algorithms to solve exercises, by using mathematics facts in solving problems.
>
> The only way to master basic technical and computational skills is to solve a large number of exercises. You need to drill, i.e., solve (literally) hundreds of problems.
>
> It is not really possible to understand new mathematics unless one has mastered (to a certain extent) the required background material.

These are not statements to be taken for granted. Rather, they are supposed to start a discussion on the topic, motivate students to think about hows and whys of their learning.

Ad 3

For most of us, various aspects of doing mathematics are quite exciting. It could be working on a research problem, or trying to develop a new approach to teaching certain topic, or creating a good problem-solving set for our students (or all of the above). We need to 'transfer' this energy to our students. If we teach with enthusiasm and are excited about the material we are discussing, our students will be better motivated and will learn mathematics better. Although learning style is a matter of personality, there are certain attitudes that we all can 'learn' and 'act' when we lecture/teach. If we show interest in what we teach, so will our students.

Ad 4

At most universities calculus is the only course that most of the students take. It is the 'ultimate' mathematics service course, taught (quite often) to large audiences, made up of majors in almost every field: from political science and business, to business and health sciences. Often, calculus also serves as a 'gatekeeper'—a course through which all mathematics majors must (first) pass—and a 'filter' for other disciplines (e.g., engineering) looking for a quick enrolment-management tool.

Although calculus is essential to understanding certain areas of mathematics and its applications, there are other mathematics disciplines that are of equal importance, but are not taught to non-mathematics majors. New courses at the entry level are needed. For instance, students could profit from an entry-level course in linear algebra, probability, number theory, or discrete mathematics.

Why not have a course that discusses mathematics used in the human genome project? Or viruses? These could be very rich courses, spanning across areas as diverse as geometry, combinatorics, and probability.

Further suggestions for courses offered at entry-level:

- Inquiry-type course (e.g., McMaster has 'Inquiry in Mathematics' in its offering of first-year courses);
- Problem-based courses;
- Problem-solving courses;
- Foundations course (e.g., University of Waterloo);
- Geometry (e.g., SFU has a course on Euclidean geometry);
- Philosophy of math course (e.g., University of Glasgow);
- Interdisciplinary courses (e.g., Queen's University has a math and poetry course at the upper-year level; could there be an entry-level analogue?).

Among the issues to consider and useful information are the following:

- Courses developed for math teachers at Brock University became mainstream math courses;
- Some students are not allowed to take first year courses, even remedial, because they do not qualify. (Should we have courses that allow these students to take math at university level?);
- Danger of early streaming: students who change their minds are forced to retake courses
- High failure rates in first-year calculus courses;
- Review prerequisites scheme to broaden the base of students;
- Role of calculus as a filter/gatekeeper should be re-examined.

Ad 5

Communication amongst university/college mathematics instructors, mathematics educators in faculties of education, and high school teachers is very important. Opportunities abound for interaction: math contests in the schools, summer math camps, professional development workshops, and local/regional mathematics associations are some examples. We need to look for ways to foster dialogue and interaction. CMESG and CMS can play a lead role here in making the mathematics community aware of the possibilities that exist and providing easy, online access to lists of organizations and key contacts.

The aforementioned dialogue would be greatly facilitated if mathematics departments would consider hiring in the area of mathematics education, or making cross-appointments with faculties of education. Joint projects between mathematicians and mathematics educators are another possibility.

Ad 6

The question of identifying a list of essential mathematical skills that students should acquire in high school is a sensitive one. Any attempts in this direction will need to be phrased positively ('These are skills that will help you succeed.') rather than as an admonishment ('If you don't master this list of skills, you will struggle with university mathematics.') There is also the risk that such a list will be seen as "bashing" high school teachers.

One thing seems clear: the high school curriculum is over-packed. It is difficult for students (and teachers) to know what is essential when there are so many topics to be covered in so little time. The same is true for university mathematics courses. We need to take a critical look at the mathematics curriculum at all levels in an effort to provide our students with the time to explore, savour and become familiar with the truly essential ideas in mathematics. Can we teach fewer topics? What is essential and what is not?

This notion was the focus of Working Group D and we will allow them to elaborate on it.

Ad 7

Humanities departments have been more successful at arguing that small class size is integral to the way their subject is taught. The argument applies equally to mathematics but we have for too long been willing to accept large classes, especially at the introductory level.

Our best teachers are needed in first-year classes. This is the 'make or break' time for many students. A bad experience can cost us mathematics majors and ill-serve students in other disciplines.

We need to involve younger faculty too but tenure and promotion considerations often militate against their involvement in teaching issues and curriculum development. (If NSERC were to support research in mathematics education, some junior faculty would take advantage of it. It was noted in our working group that the situation is quite different in the U.S., where the NSF supports both 'traditional' mathematics research and research/projects of an educational nature; many highly funded NSF projects have focussed on the first-year experience in mathematics.)

Ad 8

Mathematics and research in mathematics education need full recognition by bodies/organizations that financially (and otherwise) support other areas of mathematics and science.

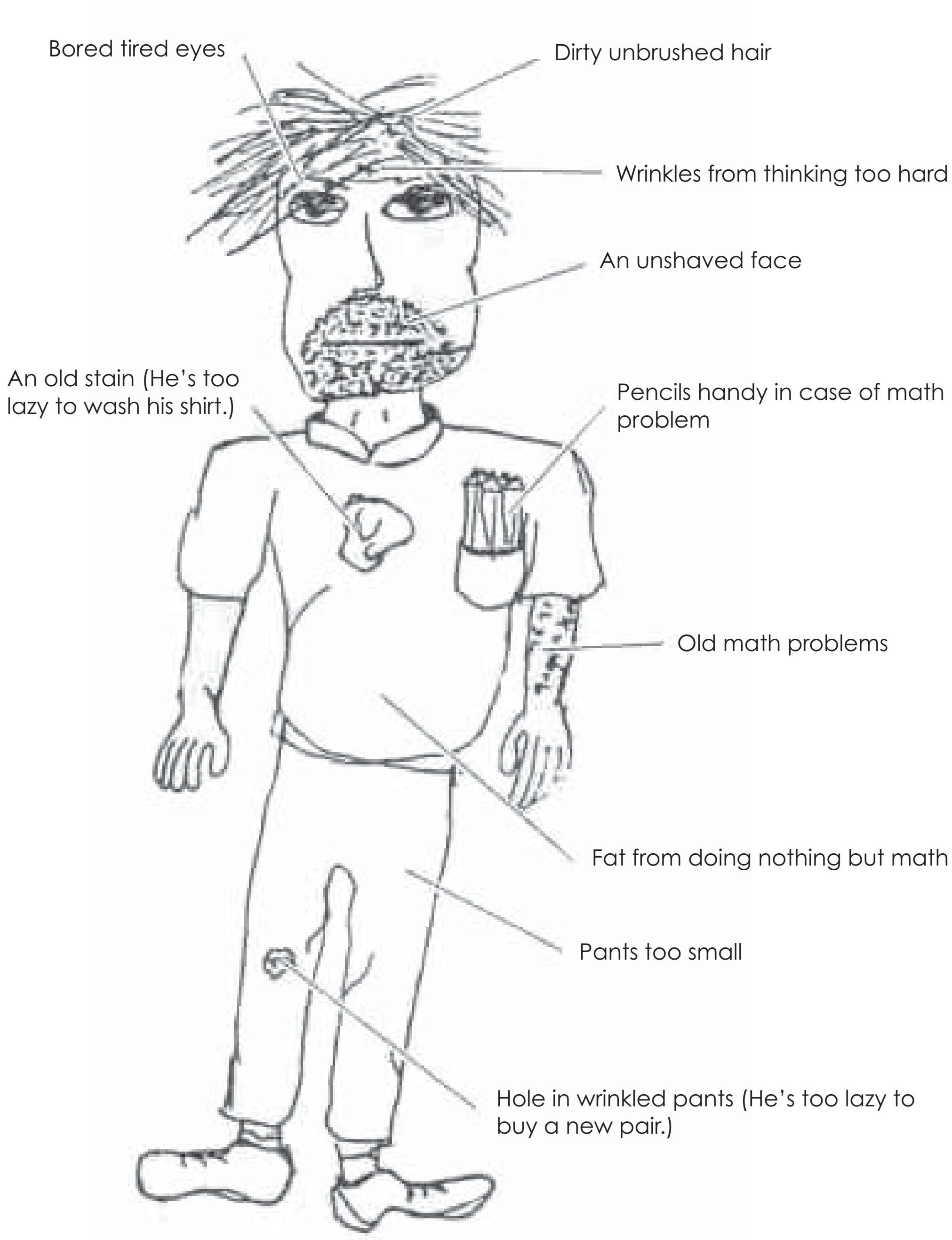
Levels of recognition include:

- Local: tenure based on strong teaching record and record of research in mathematics education ('scholarship of teaching');
- CMS needs to support research in mathematics education more strongly (supporting projects that promote mathematics is a very positive sign, but the support should not stop there); stronger ties between mathematics and mathematics education are needed;
- Funding agencies, such as NSERC, should support research in mathematics education.

Note

1. Parts are taken from a similar course taught at University of Waterloo that one of the authors (M. Lovric) taught some time ago.

Appendix


Bored tired eyes
Dirty unbrushed hair
Wrinkles from thinking too hard
An unshaved face
An old stain (He's too lazy to wash his shirt.)
Pencils handy in case of math problem
Old math problems
Fat from doing nothing but math
Pants too small
Hole in wrinkled pants (He's too lazy to buy a new pair.)

The Guild of Mathematicians believes that all things in the universe can be understood in simple, rational, mathematical terms. More importantly, they believe that understanding the mathematical terms that drive the universe gives a person power. The mathematicians have their origins in the more practical Engineers' Guild. During a routine lecture dealing with the forces on bridges, the Founder, Christo Meridian, had a revelation. With a few non-trivial alterations of fundamental equations, Meridian realized that he could greatly increase his understanding of forces. Extrapolating his process to the basic equations of time and space, he could begin manipulating the world around him.

The Mathematician's Guild has several functions in Bostonia. First, they run the three universities, educating students in accounting, navigation, physics, and of course, mathematics. They also manage the economy of Bostonia by studying the supply, demand, and price of goods and manipulating markets to keep the economy strong. They set standards of weights, measures, and purity of goods and precious metals. Finally, they perform esoteric research in using mathematical understanding to alter time and space.

Public Images Of Mathematics[1]

Definition of 'image of mathematics'

A review of past literature shows that there is not yet a consensus on the definition of 'image of mathematics'. This term has been used loosely and interchangeably with many other terms such as conceptions, views, attitudes and beliefs about mathematics. However, in this study, I choose to adopt both Thompson's (1996) and Rogers' (1992) suggestions and define the term image of mathematics held by a person as some kind of mental picture, or visual or other mental representation, originating from past experiences of mathematics, or from talk or other representations of mathematics, as well as the associated beliefs, attitudes and conceptions. As an image originates from past experiences, it can comprise both cognitive and affective dimensions. Cognitively, it relates to a person's knowledge, beliefs and other cognitive representations. Affectively, it is associated with emotions, feelings, and attitudes. Thus image of mathematics is conceptualised as a mental picture or view of mathematics, presumably derived as a result of social experiences, either through school, mass media, parents or peers. This is also understood broadly to include all visual or metaphorical images and associations, beliefs, attitudes and feelings related to mathematics and mathematics learning experiences.

Slightly more than half of the UK total sample reported a liking of mathematics but one third of them reported a disliking of mathematics. The percentage disliking mathematics is highest among the youths of age group 17–20 years (44.0%) and among the students of non-mathematics options (49.6%). These alarming negative attitudes towards mathematics among the youths and the non-mathematics students raise concern, because these groups represent the future workforce of the nation.

'... there is always enormous pleasure in manipulating numbers a kind of problem solving activity in its own right. So, if I have a mathematical problem that I basically, provided that I understand the rules, then I have quite a lot of pleasure in manipulating that out of whatever it might be. Although I am largely a verbal person, I am also have great fun in solving mathematical problems'

metaphors …
mathematics as a journey
mathematics as a skill
mathematics as a daily life experience
mathematics as a game or puzzle

myths …
mathematics is difficult
mathematics is only for the clever

factors of influence
mathematics learning experiences in school,
personalities and teaching styles of mathematics teachers
parental support and motivation (mostly father)
an individual's own personal interest in mathematics (whatever its source),
peer influence and support.

cultural differences

68% of the UK students and teachers listed inherited mathematical ability as the most important or second most important factor contributing to overall mathematical ability, whereas 79% of the Malaysian sample listed effort and perseverance as the most important or second most important contributing factor.

Many adults of most Anglo-American countries are not embarrassed to proclaim their ignorance or poor performance in mathematics, unlike on other subjects.

If the public image of mathematics is negative, then according to Howson and Kahane (1990), the image of mathematicians is even worse. They are regarded as "arrogant, elitist, middle class, eccentric, male social misfits. They lack social antennae, common sense, and a sense of humour" The most common public image of a mathematician has been furnished by a physicist (Einstein) rather than a mathematician.

Negative views about mathematics (and science) and mathematical myths have been claimed to be one of the contributing factors to some teething problems in mathematics education. These problems include: low performance in mathematics and adult numeracy, low enrolment of mathematics and science students, shortage of mathematics and science teachers in school.

Note

1. This material is drawn from Lim Chap Sam & Paul Ernest (1999), "Public Images of Mathematics" (published in *Philosophy of Mathematics Education Journal, volume* 11; 1999). The complete article is available at http://www.ex.ac.uk/~PErnest/pome11/art6htm. The portions included here are reproduced with permission.

Report of Working Group D

A Mathematics Curriculum Manifesto

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Background

Breaking with CMESG traditions and protocols, this working group was conceived, proposed, and approved during the week before the annual meeting actually began. We appreciate the cooperation of the CMESG in responding to this unusual initiative. The main impetus for this unorthodox action was a recommendation that was made at a forum on school mathematics that had been hosted by the Canadian Mathematics Society (CMS) in Montréal in mid-May. A purpose for the forum was to examine ways in which the CMS might participate meaningfully in efforts to reform school mathematics in Canada.

One prominent topic of discussion at the forum was the "overstuffed closet" of Canadian high school mathematics curriculum. It seemed that a near-universal theme in teachers' and researchers' responses to proposed pedagogical innovations was that an overprescribed curriculum, with long, detailed specific lists of topics, militates against any sort of meaningful change to their teaching. The perception is that an unwieldy syllabus compels lockstep, fragmented progress and does not allow time for, let alone encourage, lingering engagements. Coupled to this perception is a widespread belief that current curricula are 'held in place' by university-based mathematicians. All-too-often, the rationale for maintaining (or even adding to) a curriculum topic is that mastery of that topic is needed to ensure that university-bound students are prepared to survive undergraduate mathematics.

With these issues in mind, a proposal went forward at the CMS Forum that university-based mathematicians prepare and publish a statement on the sorts of competencies and other preparations that are understood as necessary for success in university mathematics courses. The goal of this proposal, and of our work, is not to have university mathematicians prescribe, or even describe the necessary mathematical curriculum or the pedagogy. The goal is that university mathematicians describe what mathematics is, and what background of experiences is appropriate preparation for doing mathematics.

We, as prospective working group leaders, felt that the CMESG conference could be a good place to continue discussion of the topic, given both the timing of the meeting (immediately after the CMS forum) and the mix of teachers, educational researchers, and mathematicians that is present at every CMESG gathering. The response within the working group, and from the CMESG executive, confirmed this judgment.

The Task

The May 2003 CMS Mathematics Education Forum was actually the second of three (the first was held in Quebec City in May, 1995), with the third tentatively scheduled for late spring, 2005. (See Dubiel, pp. 87–89 of this volume for further details.) The principal intentions for the second forum were to present participants with opportunities to identify important issues and to propose courses of action that would—hopefully—help to frame the third forum.

One of the working groups at the Montréal forum recommended that, by the 2005 gathering, there should be a clear statement from the CMS about what students need from school and university mathematical experiences, worded to lift this perceived burden and to open up more engaging possibilities for teaching and curriculum. It was agreed that such a statement would not consist of a detailed list of topics, but would address the broader goals of such a curriculum. The end-in-view was a curriculum that offered rich coherent mathematical experiences over many concepts, in contrast to what is often described in terms of (as) superficial and fragmented encounters with a great many disconnected topics.

As group leaders, we elected to frame the collective's efforts with the task of making recommendations to the CMS for the content and wording of a 'manifesto'. More specifically, we proposed that a goal of the working group would be to draft a statement that might be useful to members of the CMS as a starting place for discussions—to focus, to mine, to hone, to problematize, and to elaborate.

The Discussions

As is often the case, our discussions revolved in large part around the meanings of key terms. It quickly became apparent that at least part of our task involved the interrogation and redefinition of such seemingly transparent notions as 'curriculum' and 'mathematics'.

The word *curriculum*, in particular, seemed subject to two incompatible interpretations. On the one hand, curriculum was used in reference to formal 'must-do' lists of topics and expected levels of learner competence. On the other hand, curriculum was also used to refer to the obligation to engage in mathematically rich tasks that engender encounters with particular topics. The two conceptions are not necessarily incompatible: A well conceived list can help to frame rich engagements. However, an over-engineered list—one that reduces broad topics into micro-competencies and rigid sequences—can have quite the opposite effect as it compels a fragmented approach to instruction.

Unfortunately, it would seem that the 'must-do' conception of curriculum prevails, and it has contributed to a popular conception of mathematics as an amalgam of discrete, mechanical procedures that are so often dismissed as irrelevant in the oft-heard statement, "I was never very good at math". This proved to be a critical point in our discussions. For many, and perhaps most adult Canadians, 'mathematics' is understood in terms of their fragmented, algorithmic high school experiences with the subject matter. Our opinions on the nature of mathematics and mathematical engagement—as mathematicians, mathematics education researchers, and mathematics teachers—would seem to have little impact on popular and dominant beliefs about and attitudes toward the subject matter.

Given this backdrop of popular opinion, it seemed to us reasonable to conceive of our task in terms of reverse-engineering a curriculum from a conception of mathematics as more integrated and imaginative. The following were among the qualities that we hoped might be embodied in a mathematics curriculum:

- Mathematics arises when engaged in extended investigations and tasks.

· Students prepare to do mathematics by doing mathematics.
· A conception of curriculum as a prespecified, micro-detailed list of topics, algorithms and competencies prevents the extension of activities; a more flexible understanding of curriculum can promote the development of critical mathematical abilities.
· Mathematics is evolving with new experimental approaches, new topics, and new tools.

Such reframings of 'mathematics' and 'curriculum' prompted our discussions toward several other issues, including matters of prerequisites, uniformity of student experience, and the simultaneous need to rethink university-level mathematics. Regarding prerequisites—a notion that underpins contemporary linearized curricula, widespread 'readiness' testing, and endless debates over 'basics'—it was agreed that current emphases on preparing learners for future studies were at the expense of immediate and deep engagements with mathematics. An emphasis on prerequisites, for example, can support a conception of a singular route through mathematics and eclipse the important realization that mathematics has multiple pathways. It was also agreed that in many cases the claimed prerequisites were an illusion—material covered but not material learned and available as grounding for the next course. It was during discussion of this issue that we agreed that an important element in a CMS manifesto would be a statement to the effect that rich extended activities on broad topics can be expected to draw in most of the topics that might be identified in a curriculum as important—and that those topics that are not addressed in a particular activity can likely be set aside to emerge in a different activity or to recur when needed.

On the issue of uniformity of student experience, it was noted that major curriculum revision efforts over the past decade, provincial and interprovincial, have been in large part framed by a perceived need for learners to follow similar curricula. This 'need' is prompted by a recognition of high mobility of Canadians. The most common argument for a unified and highly regulated curriculum is that students, quite literally, should be on the same page so that they can move without interruption between schools and jurisdictions.

We agreed that the CMS manifesto would have to address this prominent concern. A consensus among us was that the argument of mobility, in and of itself, was just as applicable to the conception of mathematics curriculum that our group had begun to articulate: In brief, students whose mathematics experiences have been framed by flexible engagement with meaningful inquiries would be well prepared to a move among similar settings. In fact, it was argued, a revisioned curriculum might help to ease the problems that are often associated with students moving from one location to another. Learners who have been involved in activities that foreground interconnections of concepts, inventions of new possibilities, and so on might be expected to be able to adapt well to new environments.

All of this being said, an issue that university-based mathematicians must themselves address is the fact that many undergraduate courses can be described in very much the same terms that are criticized here: over-engineered, focused on mastery of disconnected topics, and so on. It was thus acknowledged that the manifesto would also have to include some sort of commitment to the transformation of university mathematics courses.

With these considerations foregrounded, the working group undertook the task of a draft manifesto on our final day together. The statement on page 83 is a synthesis of pieces that were developed in small and whole group discussion.

An important aspect of the words chosen for the statement, and the words omitted, were the tactical understanding of context and appropriate voice. In proposing this statement, we agreed that this proposal is a gamble on the deepest priorities of university mathematicians. Although most teachers have some concerns about the missing skills of their students, we concluded that a deeper discussion of the larger conceptual background which would prepare students to effectively learn and apply mathematics at the university level. There were some concerns about the expressions from university faculty in other disciplines, whose priorities do not center on the rich experiences of doing mathematics. The proposed statement would come from university mathematicians, and the discussion must be thoughtfully focused on the deeper issues, not fall into the trap of a quick survey of 'topics you want covered' as has sometimes happened in the past.

It was also recognized that this discussion is also linked to the discussions of Working Group C on the undergraduate curriculum and pedagogy. We foresee further discussions of these connections both within university departments, and at future CMESG conferences (see next steps below).

It is important to note that the text of the manifesto is written in the voice of the CMS. This mode of expression was adopted for purely pragmatic reasons. Framing the text in this way enabled us to speak more directly and to avoid endless qualifications. It is not in any way our intention to 'put words in the mouth' of the Canadian Mathematics Society—merely to offer, in as concise terms as we are able, suggestions that we hope are useful in their own discussions.

Next Steps

Despite having actually achieved the task that had been set, the group did not in any way see its work as completed. In addition to the circulation of a version of the above draft to members of the CMS for discussion, the following tasks presented themselves in our closing discussions:

- the collection of background research to support discussions within the CMS;
- participation with the CMS in the identification of commissions (likely CMS-based) for background reports, surveys, and so on, to be available as support of CMS statements;
- potentially, the preparation of a parallel document by CMESG on pedagogical issues to complement the mathematical issues emphasis of the CMS statement;
- potentially, a follow-up Working Group at the 2004 CMESG meeting, this one perhaps concerned with the topics of Working Groups C and D at the 2003 meeting;
- participation in the mathematics education sessions at the CMS meetings in December 2003 (Vancouver) and June 2004 (Halifax);
- engagement with other regional groups, including Fields (Mathematics Education Forum), PIMS (Changing the Culture, 2004), and AMQ;
- possible submission of discussion documents to regional newsletters, *CMS Notes*, and/or the CMESG newsletter;
- sharing (and perhaps collaboration) with other subject areas (including physics, biology, and statistics);
- preparation of a report as follow-up to the CMS Forum group.

At the time of this writing (November 2003), we have followed up on several of these steps. A draft version of the manifesto was presented for discussion at the June 2003 meeting of the CMS Education Committee and the draft Manifesto was a focus of discussion at the June meeting of the Mathematics Education Forum of the Fields Institute. It has also been circulated on some lists of teachers in Ontario, and to the person heading up the review of the K–12 Ontario Mathematics curriculum. It will now be the focus of discussion in a Mathematics Education Session at the CMS Winter Meeting in Vancouver, December 2003.

A Manifesto

(Drafted by a Canadian Mathematics Education Study Group Working group, for discussion with the Canadian Mathematics Society)

The CMS endorses the general aims of the current K–12 commonly found at the beginning of mathematics curriculum documents across Canada. However, we believe that the structure of these curricula is an obstacle to student learning of mathematics. Over-specified and fragmented lists of expectations misrepresent what mathematics is and militate against deep and authentic engagement with the subject—which, in turn, reduces recruitment and retention of people into the mathematical sciences.

The aim of this document is to describe the necessary preparation for the student who intends to study mathematics in university. We are aware that this statement also implies that change is necessary in undergraduate mathematics programs, including the mathematics programs offered to pre-service teachers.

The practice of mathematics is constantly evolving. Important new approaches include modeling, and numerical and symbolic work with computers. Student needs in such a changing environment cannot be met by adding more topics (or substituting new content for old) within an already overstuffed curriculum. They must be addressed in a more fundamental way.

We find that:

- students coming out of high school mathematics must be able to engage effectively with complex problems; they require the ability to 'think mathematically'—that is, to investigate the mathematics in a situation, to refine, to expand, and to generalize;
- students' mathematics concepts must be woven into a connected set of relationships;
- students must be able to independently encounter and make sense out of new mathematics.

These aims should have priority over any specific selection of content; and it is our judgment that it is impossible to achieve these objectives if teachers are required to cover each item on a curriculum list.

In support of our view, we point out that:

- the need for detailed lists of prerequisites in mathematics has been exaggerated. While there is some hierarchy of concepts, a more appropriate image of mathematics centers on the rich problems themselves with their relationships among concepts and that highlights both multiple entrance points into topics and multiple directions for expanding one's practice.
- a mathematical topic that appears isolated to the students and the teacher reveals a problem of placement and/or selection. Choose topics that offer opportunities to generalize and to connect.
- there are diverse modes of mathematical practice, ranging from established paths and practices of logical reasoning to modeling, investigation, and technology-supported experimentation.

Although a de-emphasis on checklists would result in variations between schools, we believe that the approach to mathematics described herein would not increase problems connected to student movement among schools and educational jurisdictions because it focuses on a central goal of mathematics education—namely, teaching students to think mathematically about a broad range of situations.

While we have not yet made explicit recommendations, we hope that, in the list of this statement, ministries and boards of education will re-examine the following:

- the structures of curriculum documents and the designs of resource materials;
- support for teachers' initial and on-going development of professional knowledge;
- assessment and reporting of students' abilities to engage with mathematically rich problems, to think mathematically, and to make sense of mathematics.

The CMS is committed to supporting teachers and curriculum developers in these difficult and important tasks.

Topic Sessions

Sessions thématiques

A Report on the Canadian School Mathematics Forum 2003
UQAM, May 16–18

Malgorzata Dubiel
Simon Fraser University

The Canadian Mathematical Society was a driving force behind the Canadian School Mathematics Forum, held at UQAM, May 16–18, 2003.

This was not the first forum on school mathematics organized by the CMS. The first took place May 7–9, 1995, in Quebec. In a revolutionary way, it brought together representatives of all those in Canada who are interested in school mathematics: mathematicians, math educators, teachers of mathematics from every level, representatives of school boards, ministries of education, industry, and parents. And even though no specific action followed, setting the stage for all those people to talk to each other, compare experiences, share ideas, and make connections, was perhaps the most important legacy of that forum. In British Columbia, it inspired the BC Miniforum for Education in Mathematics in December 1995, and later the annual PIMS Changing the Culture conference. In Ontario, its spirit is reflected in the Fields Institute Mathematics Education Forum.

The 2003 Montreal Forum, subtitled "Comparison of Experiences", was designed to be the first of two, with the second one slated to take place in Toronto in 2005. There were 148 delegates from across Canada, representing school and college teachers, university faculty in mathematics and mathematical education, and other stakeholder groups. They met for two and a half days at the UQAM campus. Pierre Reid, Minister of Education of Quebec, was the Honorary President of the Forum.

Five main themes guided the discussions:

I. Comparison of experiences;
II. Critical thinking;
III. Mathematics in the modern school: goals and challenges;
IV. Teachers' education and development;
V. A vision for the future.

Each theme had a plenary talk dedicated to it and a panel discussion probing the issues related to it. In addition, the participants were divided into sixteen working groups, which covered a wide spectrum of issues in mathematical education in Canada.

The main goal for this forum was to identify directions and themes to work on towards the next forum. At the post-forum wrap-up session, the scientific committee discussed the themes which had been suggested by the working groups. Below are those that were identified as most pressing:

The Canadian school mathematics curriculum

There was a strong perception that the present school mathematics curriculums across Canada—with the possible exception of that in Quebec—are overloaded and over-engineered. This situation is the result of pressures felt across Canada to include more and more topics and to cover them in less and less time. Some of the blame is placed on universities, and the rationale often given for maintaining current programs of study is: "The kids will need this if they're going to be successful at university math". But students' learning does not increase proportionally to the number of concepts covered—and so, the end product of such curricula is frequently a student with a toolbox full of disconnected knowledge, but not necessarily well-prepared for university courses.

This implies that, in order for the situation to change, a statement should be issued by the Canadian Mathematical Society on the preparation required by students who intend to study mathematics, engineering, computing science, or the "hard" sciences in university. The work on such a statement should be carried on in preparation to the next forum.

Elementary school teacher education

The importance of elementary mathematics education cannot be underestimated. Because students' attitudes to mathematics are formed during their elementary school years, we need to pay more attention to those who teach them during these years. In contrast to most countries in Europe and Asia, where math specialists are introduced early, across North America, elementary school teachers are required to teach all subjects. This creates a situation where too many of our elementary school teachers have little or no background in mathematics. Even worse, they are often math anxious, due to their own school experience with mathematics.

At present, mathematics requirements for elementary school teachers vary among provinces and among universities. The courses offered by universities sometimes fail to give this group adequate preparation. If we want this situation to change, we need to require that future elementary teachers must take both mathematics content and mathematics methods courses. There is some expert knowledge in mathematics that teachers need, and it is not always being taught in teachers' courses at present. Therefore, we have to review these courses, and ensure that they help students to understand and appreciate mathematics, and inspire them to be better teachers. The CMS should consider creating a task force to draft guidelines on what a good course for future elementary school teachers should look like.

Linking provincial associations of mathematics teachers

In Canada, education is a provincial responsibility. Because of this teachers' professional organizations are also provincial, and there is no national platform enabling them to meet and work together. This makes it very difficult for teachers to conduct any discussions on mathematics education in Canada as a whole. The only place Canadian mathematics teachers meet at present are NCTM annual and regional meetings in the US. However, NCTM is not a Canadian organization and does not address the needs of our teachers. The Montreal forum has given representatives of several provincial organizations an opportunity to meet on Canadian ground. CMS should consider playing a role in creating an umbrella organization to link the provincial organization together and providing web space to facilitate contacts.

Mathematics education for Aboriginal communities

Mathematical education for Aboriginal communities is quickly becoming a pressing issue across Canad— and especially in Saskatchewan, Manitoba, and Alberta. The number of young people in Aboriginal communities is growing much faster than in others. If those young people are to be successful members of their communities, their educational needs will have to be addressed in the context of their culture. The CMS is committed to working together with the Aboriginal committee on these issues.

Changing attitudes towards mathematics

Public perceptions of mathematics do have a considerable impact on mathematical education. These perceptions and attitudes should be changed towards more positive ones if we want to make any improvements in how mathematics is taught. We should put more effort and energy into this, by demonstrating the applications of mathematics, the problems it can solve, and its beauty.

Maintaining the momentum of the forum

The forum brought together many stakeholders, enabled them to share their experiences and discuss issues and problems. It generated a lot of enthusiasm for working together towards improving teaching mathematics in Canadian schools. To harvest this enthusiasm, we need to continue the work collectively in preparation to the next forum. We need also to maintain contact with ministries and the provincial associations.

CMESG/GCEDM was identified as an important partner in this work, since our organization brings together people interested in teaching mathematics from across Canada. Our meetings provide great opportunities for continuing the forum discussions, and our members have the expertise and knowledge relevant to all the topics identified at the forum.

For more information on the forum, see

http://www.cms.math.ca/Events/CSMF2003/

The Theory of the Six Stages of Learning with Integers[1]

Zoltan Dienes
Extraordinary Member of CMESG/GCEDM

Stage 1. Free interaction

In the case of the study of integers, this first stage will already have been experienced, as all that is necessary is to be aware that out of two kinds of objects sometimes there are more of one kind than of the other, sometimes less, and sometimes there are as many of one kind as of the other. Most people will have also experienced "counting" these differences and will be aware that there are two things that they look for in the differences:

(i) Out of which of the two kinds are there more and which less?
(ii) How many more or less ?

This is probably done by siblings arguing with each other about one of them having more or less candy than the other, or going to bed later or earlier than the other or through any such simple "more or less" situation which we all encounter very early in life.

Stage 2. Playing by the rules

This is the stage in which we discover some rules and learn to play by them. Alternatively we could also invent our own rules and play by them, as we shall see in the examples that follow. This stage could be called: learning to play a game with rules.

(i) The dance story

People coming to a certain place of entertainment have to obey the following rules:

(a) Everyone must enter or leave by going through the refreshment room.
(b) Anyone entering the refreshment room from the outside must choose a partner of the opposite sex, if there is any such person in the refreshment room. If there is no such person, he or she must wait until one turns up.
(c) You are only allowed to dance with a person of the opposite sex.
(d) Everyone must dance, while the music is playing, if there is a person of the opposite sex that they can dance with.

The dance hall leads off the refreshment room and is very big, and there are always lots of couples dancing, as dancing is that town's favorite pastime.

Now here are some questions:

Question 1: There are 3 girls and a boy in the refreshment room (1 boy and 1 girl are just going into the dance hall). Then 2 boys and 1 girl come in from the outside. When the dancers have gone into the dance hall, who is left waiting in the refreshment room?

Answer: One girl is left waiting. There were 2 less boys than girls at first, 1 more boy than girls came in, and we were left with 1 less boy than girls (zero boys being 1 less than 1 girl). More briefly described we could say:

2 less "added to" 1 more results in 1 less

Question 2: There are 5 boys and 2 girls in the refreshment room (of course 2 boys and 2 girls are about to go dancing). Suddenly 3 girls are called home. Who is left in the refreshment room.

Answer: Six boys. The 3 girls might all have been dancing in the dance hall. When they left, the 3 boys, their partners, joined the other three boys in the refreshment room, leaving 6 boys there.

The 2 girls who were about to enter the dance hall might have been two of the three who had to go, leaving 5 boys in the refreshment room. The third girl will have been in the dance hall, and when she left, her partner would have joined the 5 boys in the refreshment room, again making 6 boys.

3 more boys than girls   minus   3 less boys than girls

leave behind   6 more boys than girls

or   3 more   minus   3 less   leave   6 more

The manager of the place of entertainment has a large family, lots of boys and lots of girls. At times when few people are ordering in the refreshment room, he sends one of the children to knock on the window of the refreshment room. This means that everyone must leave. But it also means that

(i) if the "knocker" is a boy, then everyone must send back a person of the same sex,
(ii) if the "knocker" is a girl, then everyone must send back a person of the opposite sex.

If several people go and knock on the window at the same time, then each person leaving must send back as many persons as the number of people who came to the window, of the same sex for each boy at the window, and of the opposite sex for each girl at the window.

Question 3: There are 3 girls in the refreshment room. Then 2 girls from the manager's family come and knock on the window. When the "replacement customers" arrive, who will be in the refreshment room?

Answer: Six boys. This is because each girl who leaves must send back 2 boys. So the 3 girls together will send back 6 boys.

3 less boys than girls   window   2 less boys than girls

result in   6 more boys than girls

If we call "window-ing" "multiplying", then we have

3 less   times   2 less   equals   6 more

It is not hard to work out that by the "window rules" we shall have

more $\times$ more $=$ more,   more $\times$ less $=$ less

less $\times$ more $=$ less,   less $\times$ less $=$ more.

(ii) Walking East and walking West

Johnny is walking up and down the road to exercise himself. The road runs in an East-West direction. He walks a certain distance East, then turns round and walks back in a Westerly direction, then again turns East and so on, until he gets tired. At the end of his walk he stops and calls his father to pick him up in the car, as he is too tired to walk home!

He will either finish East of his starting point or West of his starting point, or maybe he will finish exactly at his starting point.

Question 1: Johnny walks 1 km East and then 3 km West and has a rest. He thinks he can do another walk and so now does 2 km East and 1 km West. Does he end up to the East or to the West of his starting point and how far?

Answer: He ends up 1 km to the West. We can write the two walks like this:

2 less East than West followed by 1 more East than West

is the same as 1 less East than West

or 2 less then 1 more is the same as 1 less

So you see that you can "add" Johnny's walks and always obtain another walk. But what about "subtracting" walks? You might just want to "subtract" a walk from one of Johnny's walks that he never actually made on that particular walk. Anyhow, what does it mean to "subtract a walk from another walk"?

We could ask what part of the walk Johnny had done without having done a certain portion of it? For example:

Question 2: 10 km East, 7 km West – 3 km East, 2 km West

would mean: What would Johnny have done if instead of 10 km East he had done 3 km less towards the East, and instead of 7 km West he would have done 2 km less towards the West? He would have done 7 km East and 5 km West.

Answer: 3 more – 1 more = 2 more.

But suppose we wanted to know what Johnny would have done had he not done 3 km East and 8 km West? He never did 8 km West, so it seems that the problem has no sense.

In order to make the problem "soluble", we can introduce the notion of

EQUIVALENT WALKS

and say that in this walking game, any walk can be replaced by any other EQUIVALENT WALK.

Two walks are equivalent if when they both start at the same spot, they also end at the same spot.

In our example we can replace Johnny's walk by 12 East, 9 West. We can now "subtract" 3 East, 8 West and end up with 9 East, 1 West. So this "subtraction" can be expressed as

3 more – 5 less = 8 more

Johnny and his sister Mary both go out walking. But this time they just go for short walks in their back yard. But before going out, they throw a die. Three of the faces of the die are painted red, the other three green. The red faces have the numbers 1, 2, and 3 written on them (one number on each face), and the same is true for the green faces.

They decide that for every meter that Johnny walks, Mary has to do as many meters as what was shown on the die when they had cast it. If it had come up green, then Mary would always go in the same direction as Johnny, but if it had come out red, Mary would always have to walk in the opposite direction to the one that Johnny was taking. They both walk on a little path that runs from East to West.

Question 3: Johnny walks 3 m East, 5 m West, and 6 m East. The die shows a red 2. What does Mary do?

Answer: Mary will do 6 m West, 10 m East, 12 m West, or

4 more East red 2 yields 8 less East

You could throw the die more than once and either carry out the "commands" one after the other, or "count" whether there will be more green than red or less green then red in your total of the die-numbers. You can then count the "more green than red" as "same direction commands" and the "less green than red" as "opposite direction commands".

If "more" means "more East than West" and "less" means "less East than West", and "more" means also "more green than red" and "less" means also "less green than red", we can easily verify that we have the rules:

More or less East than West steps	More or less green than red die numbers	More or less East than West steps
more	more	more
more	less	less
less	more	less
less	less	more

The "rules of the game" are now well defined and we can add, subtract and "multiply" walks.

(iii) The circles and squares rules

You must have a whole pile of counters available, some should be circular and some square shaped counters.

A circle and a square together is called a "zero pair".

Two piles of counters are EQUIVALENT to each other, if one can be obtained from the other by adding and/or removing some zero pairs.

Two piles of counters equivalent to each other can be exchanged with each other in what follows.

A pile in which there are no zero piles is said to be in standard form. A pile in standard form has only circles in it, or it has only squares in it.

Rules for adding: We add two piles by simply putting the two piles together to make a united pile.

Rules for subtracting: We subtract a pile from another pile by simply removing the pile to be "subtracted" from the pile from which it is to be "subtracted". If the pile "to be subtracted" cannot be found in the pile from which it is to be subtracted, we simply add enough zero pairs to the latter pile until the "subtraction" becomes possible.

Multiplication is done in the following way: Make a pile A. This is the "multiplicand pile". Then make a pile B. This is the "multiplier pile". Then make a third pile, the "product pile" or the pile C, in the following way: For every counter in pile A put as many counters of the same shape in pile C as there are circles in pile B. Also for every counter in pile A put as many counters of the opposite shape in pile C as there are squares in pile B.

We can refer to piles in which there are more circles than squares as "more piles" and piles in which there are less circles than squares as "less piles".

In what follows we shall refer to circles briefly by writing the letter C and to squares by writing the letter S.

Question 1: Do we get a "more pile" or a "less pile" if we add

CCCCCSSSS and CCSSSSSSS ?

Answer: The first pile is a "1 more C pile", the second pile is a "5 less C pile". Putting them together we get a "4 less C pile".

Question 2: From the pile CCCSSSSSS take away the pile CCCCCS. What kind of pile do you get?

Answer: We have to add two zero pairs to the first pile before we can remove a pile like the second pile from it. So the pile EQUIVALENT to the first pile that we have to consider is

CCC CS CS SSSSSSS

We can now remove CCCCCS and get SSSSSSSS. We have done the following:

(4 less) – (4 more) = (8 less)

Question 3: Into pile A (multiplicand pile) let us put S S S S C
Into pile B (multiplier pile) let us put S S C
The product pile will be:

CCCCS CCCCS SSSSC

which is a "3 more C pile", so we have the "multiplication":

(3 less) x (1 less) = (3 more)

We have now given three different activities for essentially the "same thing". In what way they are the "same thing" will be clearer in Stage 3.

Stage 3. The comparison stage

The three activities have been given precisely in order that only what is common to all three should be eventually retained. In order to see the common elements of the activities, we must see what corresponds to what as between activities. If this is not already quite obvious, here is a "dictionary", which "translates" one activity into one of the others.

Dance	Walks	Counters
boys girls	steps East steps West	circles squares
people present at the dance	set of steps East and/or West	pile of circles and squares
only boys in refreshment room	more East steps than West steps	more circles than squares
only girls in refreshment room	less East steps than West steps	less circles than squares
people arrive in refreshment room	do one walk and then do another	put two piles of counters together
people leave the place of entertainment	find the walk that would have been done without a part of the walk	From a pile remove a pile that is in it
n boys knock on the window	*n* more green die numbers than red die numbers	*n* circles more than squares in the multiplier pile
n girls knock on the window	*n* less green die numbers than red die numbers	*n* circles less than squares in the multiplier pile

In each activity we are concerned with the difference between the number of one kind of "element" and the number of another kind of "element".

In the dance these elements are boys and girls, in the walks they are displacement units East and West and green and red die numbers, in the case of the counters, they are circles and squares.

In each case we decide which of the two kinds we "count". In the dance activity, we "count" the boys, since we say "more boys than girls" and "less boys than girls" and not "less girls than boys" and "more girls than boys". It is immaterial which of the two kinds is chosen to be "counted", but a convention must be made, so that we know which are "more" situations and which are "less" situations. In the case of the walks we "count" the East displacements, in the case of the counters we "count" the circles as "elements of reference".

Stage 4. Representation

We have seen how the three activities are similar. It should therefore be possible to represent them all on the same figure. Below is such a suggested representation:

```
 -9 -8 -7 -6 -5 -4 -3 -2 -1  0 +1 +2 +3 +4 +5 +6 +7 +8 +9

9,0   8,1   7,2   6,3   5,4   4,5   3,6   2,7   1,8   0,9
   8,0   7,1   6,2   5,3   4,4   3,5   2,6   1,7   0,8
      7,0   6,1   5,2   4,3   3,4   2,5   1,6   0,7
         6,0   5,1   4,2   3,3   2,4   1,5   0,6
            5,0   4,1   3,2   2,3   1,4   0,5
               4,0   3,1   2,2   1,3   0,4
                  3,0   2,1   1,2   0,3
                     2,0   1,1   0,2
                        1,0   0,1
                           0,0
```

The number after the comma refers to the elements which we "count". For example (3,4) means that there is 1 more of the second kind than of the first kind.

The pairs with the same differences are always in the same column. At the head of each column a "more situation" is denoted with a plus sign and a "less situation" with a minus sign. The "as many as" situation is denoted by zero. In this way we obtain the NUMBER LINE, on which the "more" and the "less" situations are represented in order. Each "point" on the number line represents not just one situation, but an infinite number of situations. What is common to all situations in the same column is that they have the same "moreness" or "lessness" property.

It is easy to see that if we perform an addition in any of our three activities, we shall be starting at a certain "point" and move to the right if we "add" a "more situation" and move to the left if we "add" a "less situation".

It is also easy to see that if instead of "adding" we "subtract", then we move to the left if we subtract a "more situation" and we move to the right if we subtract a "less situation".

Multiplication can be seen to be a kind of "inflation" of the distance of our starting point from the point representing zero. Multiplying by a "more" we remain on the same side of zero as our starting point, multiplying by a "less" we also inflate but also change "sides".

If "less" "more" situations are regarded as "opposites" of each other, it is also easy to check that "subtracting" will yield the same result as "adding the opposite".

Clearly instead of "more" we could use its Latin equivalent, namely the word "plus". Also instead of "less" we could use the Latin word "minus", which means "less".

In the next stage we shall see how the conventional symbol system can be developed by taking the above properties of the number line as starting points. It might also be useful to play about with some less conventional ways of symbolizing, to stop ourselves from "cheating" by referring to various rote learned algorithms which we remember from childhood!

Stage 5. Symbolization

We can now "play about" on the number line and develop a "language" to record what we find.

We could use the conventional plus and minus signs for adding and subtracting respectively, and use the plus and minus signs as suffixes when we wish to say "more" or "less". For example to symbolize the adding of a "2 more" to a "3 less", we could write ...

$$2_+ + 3_- = 1_-$$

... where the equal sign means "comes to the same result as".

The above is a "mathematical sentence". The subject of the sentence is the left hand side, the predicate is the equal sign together with the right hand side. Or, if you prefer, you could think of the left hand side as one subject, of the right hand side as another subject, and of the equal sign as the predicate, which says something about the two subjects, namely that they are equal.

We can use the lower case letters $x, y, z, \ldots$ to denote "points on the number line yet to be chosen". If we write the same letter more than once in the same sentence, the same "point" must be picked each time that letter occurs. On the other hand, we are allowed to pick the same point for two different letters.

We must not forget that each "point" represents a whole lot of pairs whose elements differ from each other by the same amount. A whole class of such pairs having the same difference between its elements is called an INTEGER.

Let us do some investigations.

Choose two integers, namely two points on the number line. Call them x and y respectively. It is easy to check that if we start from the "point" x and "add to it the point y", we shall arrive at the same result as if we had taken the point y and then had "added to it the point x". We have to remember that "adding a plus point" means moving to the right, and "adding a minus point" means moving to the left.

If we are convinced of the truth of the above, we can write

$$x + y = y + x$$

We could use another method of writing the above. An upper case S could mean "The sum of" as long as there are two integers following whose sum we can be talking about. So the above finding could also be written as

$$\mathrm{S}\,x\,y = \mathrm{S}\,y\,x$$

If we take an "as many as" situation, represented by the "point" 0 and any other point x and "add them", we shall have the point x as a result. This we can write as

$$0 + x = x \quad \text{or} \quad \mathrm{S}\,0\,x = x$$

Now we can try something involving three "points" or integers. Let us call them x, y, and z respectively. Suppose we "add" x and y and to the "sum" we "add" the point z. Do we arrive at the same final point if to the point x we "add" the sum of the points y and z? Whatever points you choose for x, y, and z, you will find that the two ways of grouping and adding the integers involved always results in the same final integer. So we can write

$$(x + y) + z = x + (y + z) \quad \text{or} \quad \mathrm{S\,S}\,x\,y\,z = \mathrm{S}\,x\,\mathrm{S}\,y\,z$$

where the bracket convention is the usual one, namely that any operations within brackets must be carried out before any operations outside them are performed.

In the "subject" $\mathrm{S\,S}\,x\,y\,z$ it should be noted that the second S refers to adding x and y, while the first S refers to adding $\mathrm{S}\,x\,y$ and z. In the "subject" $\mathrm{S}\,x\,\mathrm{S}\,y\,z$ the second S tells us to add y and z, while the first S tells us to add x and $\mathrm{S}\,y\,z$. Note that in the second symbol system we have no need of brackets.

We can arrive at corresponding properties for our multiplication, by examining what happens on the number line. We can use upper case P for "the product of", and so "discover" the following "rules of the integer game", using a dot symbol for multiplying:

$$x \cdot y = y \cdot x, \quad 1 \cdot x = x, \quad (x \cdot y) \cdot z = x \cdot (y \cdot z)$$

or

$$\mathrm{P}\,x\,y = \mathrm{P}\,y\,x, \quad \mathrm{P}\,1\,x = x, \quad \mathrm{P}\,\mathrm{P}\,x\,y\,z = \mathrm{P}\,x\,\mathrm{P}\,y\,z$$

There is also an interesting property in our "integer game" which connects the two operations, namely that of "adding" and that of "multiplying". It is not hard to see that for any three integers x, y and z we have

$$(x + y) \cdot z = (x \cdot z) + (y \cdot z) \text{ and } x \cdot (y + z) = (x \cdot y) + (x \cdot z)$$

or

$$\mathrm{P}\,\mathrm{S}\,x\,y\,z = \mathrm{S}\,\mathrm{P}\,x\,z\,\mathrm{S}\,y\,z \text{ and } \mathrm{P}\,x\,\mathrm{S}\,y\,z = \mathrm{S}\,\mathrm{P}\,x\,y\,\mathrm{P}\,x\,z$$

Adding and multiplying are BINARY OPERATORS. By this is meant that we need two integers before we can either add them or multiply them (bis being Latin for "twice").

We shall need another OPERATOR to express things precisely. We need a "reverser", namely an operator that turns a "more" into a "less" and a "less" into a "more". For the moment let us stick to our S, P symbol system, and denote our REVERSING OPERATOR by means of an upper case N. This means that, for example

$$\mathrm{N}\,4_{-} = 4_{+} \quad \text{or} \quad \mathrm{N}\,3_{+} = 3_{-}$$

and so on. "Reversing" a point x will send us on the opposite side of zero to the one in which we are, but at the same "distance" from the zero. Clearly, if we reverse twice in succession, we find ourselves where we were before. So one obvious property of our N operator is

$$\mathrm{N}\,\mathrm{N}\,x = x \quad \text{for any integer } x$$

It is also quite easy to see that if we add an integer to its "opposite", we shall arrive at the zero point. So we can write

$$\mathrm{S}\,x\,\mathrm{N}\,x = 0 \quad \text{for any integer } x$$

What about reversing the sum of two integers? Do we get the same result if we add the reversed integers? This is also easy to check on a few examples, and so we can write

$$\mathrm{N}\,\mathrm{S}\,x\,y = \mathrm{S}\,\mathrm{N}\,x\,\mathrm{N}\,y \quad \text{for any two integers } x \text{ and } y$$

What happens if we reverse the product of two integers? Do we get the same result as if we take the product of the reversed integers? Unfortunately not so! But we do get the same if we multiply the first integer reversed by the second integer. This "discovery" can be written like this:

$$\mathrm{N}\,\mathrm{P}\,x\,y = \mathrm{P}\,\mathrm{N}\,x\,y \quad \text{for any two integers } x \text{ and } y$$

There are a number of other properties we can "discover", and then symbolize, using our new language. For example

$$\mathrm{P}\,2\,x = \mathrm{S}\,x\,x, \quad \mathrm{P}\,3\,x = \mathrm{S}\,\mathrm{S}\,x\,x\,x, \quad \mathrm{P}\,\mathrm{N}\,x\,\mathrm{N}\,y = \mathrm{P}\,x\,y$$

In fact there is clearly no end to the "description" of what we can do with our number line.

So there is a problem. How much of the above type of description is SUFFICIENT for describing EVERYTHING that can be done? This is a very vague question, so let me try to make it more precise:

Suppose we have written down a certain number of the properties of the number line. Then let us suppose that we have found some RULES through the application of which we can DERIVE some of the other properties that are seen to be true on the number line. How many properties and which ones do we have to write down, so that, using our RULES, we can arrive at ANY property that is verifiably true on the number line?

In the next section we shall make an attempt to give such a set of properties and such a set of rules.

The initial properties will be called AXIOMS. The CHAIN, leading from the AXIOMS to some other property, using the rules, will be called a PROOF. The property reached at the end of the chain, will be called a THEOREM.

We have seen that subtracting an integer comes to the same thing as adding its "opposite", so if "taking the opposite" is taken care of by means of the N operation, there is no need for any explicit "subtraction" in the system.

Stage 6. The S and P game. Formalization

Game 1 • The "well formed formulas" or W'FF's

A WW'F is defined as follows:

- (i) Any lower case letter by itself, or any numeral by itself is a W'FF.
- (ii) N followed by a W'FF is a W'FF.
- (iii) P followed by two W'FF's is a W'FF.
- (iv) S followed by two W'FF's is a W'FF.

Numerals 1,2,3, … will be written instead of $1_+, 2_+, 3_+$, … for brevity's sake.
Negative numerals will be written N1, N2, N3, N4, ...
Here are some W'FF's:

$$3,\ \mathrm{N}\,x,\ \mathrm{P}\,3\,4,\ \mathrm{S}\,4\,\mathrm{N}\,3,\ \mathrm{P}\,x\,\mathrm{S}\,y\,\mathrm{N}\,z,\ \mathrm{P}\,\mathrm{P}\,2\,3\,4,\ \mathrm{P}\,2\,\mathrm{P}\,3\,4$$

To play this game each player grabs a handful of pieces. Each player then tries to make a W'FF as long as he or she is able to with the pieces grabbed. The longest W'FF is the winner. Each player can challenge the W'FFness of the opponent's W'FF. A successful challenger then becomes the winner.

We shall assume that S 1 1 = 2, S 1 2 = 3, S 1 3 = 4, and in general S 1 $n = n + 1$.

Game 2 • Transforming a W'FF into another W'FF

Substitution rule: Any lower case letter in a W'FF an be replaced by any W'FF, as long as the same letter is always replaced by the same W'FF.

In what follows, x and y and z stand for any W'FF's. It is permitted to replace any combination on the left of one of the following "equations", with the combination that stands on the right of that "equation"or vice versa.

(1) $\mathrm{P}\,1\,x = x$,
(2) $\mathrm{P}\,x\,y = \mathrm{P}\,y\,x$
(3) $\mathrm{P}\,\mathrm{P}\,x\,y\,z = \mathrm{P}\,x\,\mathrm{P}\,y\,z$
(4) $\mathrm{S}\,0\,x = x$
(5) $\mathrm{S}\,x\,y = \mathrm{S}\,y\,x$
(6) $\mathrm{S}\,\mathrm{S}\,x\,y\,z = \mathrm{S}\,x\,\mathrm{S}\,y\,z$
(7) $\mathrm{P}\,\mathrm{S}\,x\,y\,z = \mathrm{S}\,\mathrm{P}\,x\,z\,\mathrm{P}\,y\,z$
(8) $\mathrm{P}\,x\,\mathrm{S}\,y\,z = \mathrm{S}\,\mathrm{P}\,x\,y\,\mathrm{P}\,x\,z$

We can write the equal sign between a W'FF and any other W'FF into which we have transformed it by the rules.

Here are some examples of transforming a W'FF into another W'FF:

Start with $\mathrm{P}\,\mathrm{S}\,x\,1\,x$ (obtained from the left of (7) by replacing y by 1 and z by x)

Now writing the right of (7) with these "values", we have

$$\mathrm{S}\,\mathrm{P}\,x\,x\,\mathrm{P}\,1\,x,$$

but by (1) $\mathrm{P}\,1\,x = x$, so we have

$$\mathrm{S}\,\mathrm{P}\,x\,x\,x.$$

So we have transformed $\mathrm{P}\,\mathrm{S}\,x\,1\,x$ into $\mathrm{S}\,\mathrm{P}\,x\,x\,x$. So we can write

$$\mathrm{P}\,\mathrm{S}\,x\,1\,x = \mathrm{S}\,\mathrm{P}\,x\,x\,x.$$

Let us connect S 2 2 to 4.

$$\mathrm{S}\,2\,2 = \mathrm{S}\,\mathrm{S}\,1\,1\,2 = \mathrm{S}\,1\,\mathrm{S}\,1\,2 = \mathrm{S}\,1\,3 = 4$$

For the second step we use rule (6). Here is a long chain connecting

$\mathrm{S\,S\,P}\,x\,x\,\mathrm{P}\,2\,x\,1$ to $\mathrm{P\,S}\,x\,1\,\mathrm{S}\,x\,1$

$\mathrm{S\,S\,P}\,x\,x\,\mathrm{P}\,2\,x\,1$, now use $2 = \mathrm{S}\,1\,1$ and we get

$\mathrm{S\,S\,P}\,x\,x\,\mathrm{P\,S}\,1\,1\,x\,1$, and by rule (7) we have

$\mathrm{S\,S\,P}\,x\,x\,\mathrm{S\,P}\,1\,x\,\mathrm{P}\,1\,x\,1$, now by rule (1) we have

$\mathrm{S\,S\,P}\,x\,x\,\mathrm{S}\,x\,x\,1$, and then using rule (6) we have

$\mathrm{S\,S\,S\,P}\,x\,x\,x\,x\,1$, now using rule (1) again we can write

$\mathrm{S\,S\,S\,P}\,x\,x\,\mathrm{P}\,x\,1\,x\,1$, and now by using rule (8) we have

$\mathrm{S\,S\,P}\,x\,\mathrm{S}\,x\,1\,x\,1$; we can use rule (5) and get

$\mathrm{S\,S}\,x\,\mathrm{P}\,x\,\mathrm{S}\,x\,1\,1$, then by rule (6) we get

$\mathrm{S}\,x\,\mathrm{S\,P}\,x\,\mathrm{S}\,x\,1\,1$, then again by (5)

$\mathrm{S}\,x\,\mathrm{S}\,1\,\mathrm{P}\,x\,\mathrm{S}\,x\,1$, then again by (6)

$\mathrm{S\,S}\,x\,1\,\mathrm{P}\,x\,\mathrm{S}\,x\,1$, then by (1)

$\mathrm{S\,P}\,1\,\mathrm{S}\,x\,1\,\mathrm{P}\,x\,\mathrm{S}\,x\,1$, then by (7)

$\mathrm{P\,S}\,1\,x\,\mathrm{S}\,x\,1$, and finally by (5) we have

$\mathrm{P\,S}\,x\,1\,\mathrm{S}\,x\,1$, so we have made the desired connection!

Let us see whether any of our rules can be "derived" from any of the others. For example, could we connect the left and the right hand sides of equation (8), by using the rules (1) to (7)?

Let us try and start with $\mathrm{S\,P}\,x\,y\,\mathrm{P}\,x\,z$.

Using rule (2) we can write

$\mathrm{S\,P}\,y\,x\,\mathrm{P}\,z\,x$, then by rule (7) we can write

$\mathrm{P\,S}\,y\,z\,x$, and again by rule (2) we can write

$\mathrm{P}\,x\,\mathrm{S}\,y\,z$, so we have made the connection.

Rule (8) is not INDEPENDENT of the other rules, as it can be DERIVED from them. Are there any other rules listed that are DEPENDENT or DERIVABLE from the others ?

Try to make other "connections". Some are easier than others. For example we can easily "connect"

$\mathrm{P}\,2\,x$ with $\mathrm{S}\,x\,x$

Start with $\mathrm{P}\,2\,x$, then use the fact that $2 = \mathrm{S}\,1\,1$ and write

$\mathrm{P\,S}\,1\,1\,x$ then by (7) we can write

$\mathrm{S\,P}\,1\,x\,\mathrm{P}\,1\,x$ and then by (1) we get

$\mathrm{S}\,x\,x$ and we have made the connection.

Game 3 • The rules for working with N.

Here are the "connecting rules" for using with W'FF's containing N:

(9) $\mathrm{S}\,x\,\mathrm{N}\,x = 0$
(10) $\mathrm{N\,N}\,x = x$
(11) $\mathrm{N\,S}\,x\,y = \mathrm{S\,N}\,x\,\mathrm{N}\,y$
(12) $\mathrm{N\,P}\,x\,y = \mathrm{P\,N}\,x\,y$

As an example let us try to connect $\mathrm{P\,N}\,x\,\mathrm{N}\,y$ to $\mathrm{P}\,x\,y$.

Start with $\mathrm{P\,N}\,x\,\mathrm{N}\,y$, then using rule (10)

$\mathrm{N\,N\,P\,N}\,x\,\mathrm{N}\,y$, then by rule (12)

$\mathrm{N\,P\,N\,N}\,x\,\mathrm{N}\,y$, then by (10) we have

$\mathrm{N\,P}\,x\,\mathrm{N}\,y$, then by rule (2) we can write

$\mathrm{N\,P\,N}\,y\,x$, then using rule (12) again we have

$\mathrm{P\,N\,N}\,y\,x$, and finally by rule (10) we can write

$\mathrm{P}\,y\,x$, but by rule (2) we can also write

$\mathrm{P}\,x\,y$

So we can write $\mathrm{P\,N}\,x\,\mathrm{N}\,y = \mathrm{P}\,x\,y$

It might be fun to start with $\mathrm{P\,S}\,x\,1\,\mathrm{S}\,x\,\mathrm{N}\,1$ and try to connect it to $\mathrm{S\,P}\,x\,x\,\mathrm{N}\,1$. How would you do it? And how do I know that it can be done?

Start with $\mathrm{P}\quad \mathrm{S}\,x\,1\quad \mathrm{S}\,x\,\mathrm{N}\,1$ by (7)

$\mathrm{S}\quad \mathrm{P}\quad \mathrm{S}\,x\,1\;x\quad \mathrm{P}\quad \mathrm{S}\,x\,1\quad \mathrm{N}\,1$ by (7) twice

$\mathrm{S}\quad \mathrm{S}\quad \mathrm{P}\,x\,x\quad \mathrm{P}\,1\,x\quad \mathrm{S}\quad \mathrm{P}\,x\,\mathrm{N}\,1\quad \mathrm{P}\,1\,\mathrm{N}\,1$ by (6)

$\mathrm{S}\quad \mathrm{S}\quad \mathrm{S}\quad \mathrm{P}\,x\,x\;\mathrm{P}\,1\,x\quad \mathrm{P}\,x\,\mathrm{N}\,1\quad \mathrm{P}\,1\,\mathrm{N}\,1$

Before going on it would be good to be convinced that $\mathrm{P}\,x\,\mathrm{N}\,1$ and $\mathrm{N}\,x$ can be connected;

$$\mathrm{P}\,x\,\mathrm{N}\,1 = \mathrm{P\,N}\,1\,x = \mathrm{N\,P}\,1\,x = \mathrm{N}\,x \qquad (13)$$

and putting $x = 1$ we also have

$$\mathrm{P}\,1\,\mathrm{N}\,1 = \mathrm{N}\,1 \qquad (14)$$

So we use (1) (13) and (14) and go on to

$\mathrm{S}\quad \mathrm{S}\quad \mathrm{S}\quad \mathrm{P}\,x\,x\quad x\quad \mathrm{N}\,x\quad \mathrm{N}\,1$ then by (5)

$\mathrm{S}\quad \mathrm{S}\quad \mathrm{S}\quad x\quad \mathrm{P}\,x\,x\quad \mathrm{N}\,x\quad \mathrm{N}\,1$ and by (5) again

$\mathrm{S}\quad \mathrm{S}\quad \mathrm{N}\,x\;\mathrm{S}\quad x\quad \mathrm{P}\,x\,x\quad \mathrm{N}\,1$ then by (6)

$\mathrm{S}\quad \mathrm{S}\quad \mathrm{S}\quad \mathrm{N}\,x\quad x\quad \mathrm{P}\,x\,x\quad \mathrm{N}\,1$ and by (5) again

$\mathrm{S}\quad \mathrm{S}\quad \mathrm{S}\quad x\quad \mathrm{N}\,x\quad \mathrm{P}\,x\,x\quad \mathrm{N}\,1$ so by (9)

$\mathrm{S}\quad \mathrm{S}\quad 0\quad \mathrm{P}\,x\,x\quad \mathrm{N}\,1$ and by (4)

$\mathrm{S}\quad \mathrm{P}\quad x\,x\quad \mathrm{N}\,1$

and so we connected $\mathrm{P\,S}\,x\,1\,\mathrm{S}\,x\,\mathrm{N}\,1$ to $\mathrm{S\,P}\,x\,x\,\mathrm{N}\,1$.

You have known all the time that S stands for "The sum of", that P stands for "The product of" and that N stands for "The negative of". But the "game" can be played without knowing this! But you must take care that S and P are followed by two well formed formulae and N by just one!

The "game" shows how complex the situation is if we wish to rely on absolutely formal methods. The traditional method of algebra teaching is neither intuitive nor rigorously formal, but falls between these two stools which are very far apart from each other. If we want to be intuitive, let us use all sorts of materials such as little square shapes, trays, cups and beans, colored counters or balance beams. If we wish to be "formal", let us be really formal, and become aware of every formal step that allows us to go from one expression to another, supposedly algebraically identical with each other.

It is really salutary for a teacher to go through step by step, with a mathematical tooth comb, that which is required to establish formally that

$$(x+1)^2 \quad = \quad x^2 + 2x + 1$$

or in our formal system that

$$\mathrm{P\,S}\,x\,1\,\mathrm{S}\,x\,1 \quad = \quad \mathrm{S\,S\,P}\,x\,x\,\mathrm{P}\,2\,x\,1$$

or that

$$x^2 - 1 \quad = \quad (x + 1)\,(x - 1) \quad \text{or}$$

$$\mathrm{S\,P}\,x\,x\,\mathrm{N}\,1 \quad = \quad \mathrm{P\,S}\,x\,1\,\mathrm{S}\,x\,\mathrm{N}\,1$$

The advantage of an unfamiliar system is that we cannot "slur" over anything. We must use the rules one by one in an orderly succession and so become aware of how much is taken for granted when the "usual" rules are applied and thought to be adequately "formal".

One advantage of the "S and P system" is that parentheses are not required. Another advantage is that it follows closely how the operations referred to would be expressed in the "vernacular". For example:

P 3 S 4, 5 "means" "Find the product of 3 by the sum of 4 and 5".

Once the fine detail of the "proofs" has been appreciated and truly understood, students can safely return to the conventional ways of expressing mathematical statements as they are usually found in text books.

Note

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Exploring the Fragility and Robustness of Students' Mathematical Knowing

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My ongoing research focuses on the relationship between teaching and learning, and specifically on the relationship between teachers' classroom interventions and the growth of students' mathematical understanding. As part of a broad study of the ethics of mathematics knowing in action that I've been conducting with Elaine Simmt and Lynn Gordon of the University of Alberta, I've been focusing recently on the role of the teacher, asking what constitutes an ethical intervention with respect to both children's learning and the discipline of mathematics.[1] This has pressed me to consider the ways in which teachers and students, and indeed students themselves, are placed in relationship with one another through their mathematical ideas.

Jo Boaler's recent work (2001, 2003), and particularly her recent *For the Learning of Mathematics* article on the relationships between knowledge, practice, and identity in mathematics classrooms (2002), has prompted me to think hard about the interconnections between the teacher-student relationship and mathematical ideas. In reading her 2002 piece a number of threads started to come together for me concerning my research and my practice as a teacher educator. Boaler talks about the ways in which different forms of teacher practice elicit different relationships between students and mathematics, practices which, as learning is a process of becoming, therefore transform (i.e., either reinforce or conflict with) students' emerging identities. This account felt very familiar to me. I work within a teacher education programme that is field-oriented, learner-focused, and inquiry-based; a programme that it is becoming clear is unique not in *how much* or *what* its graduates know, but in *how* they hold their knowledge—in their identities as beginning teachers. So, if, as my experience in our teacher preparation programme tells me and as Boaler claims, teaching practices can shape not only how much is learned but the *form* of the knowledge that is produced, then we are obligated to pay close attention to the teaching practices and to those forms of knowledge. In the domain of mathematics that means paying close attention to the emergence of mathematical ideas.

Robust And Fragile Images

I've been studying children's mathematical ideas and images for many years, and there are many different ways to approach such a study. Recently, though (and I'll come in a moment to how I came to be thinking in these terms) I've chosen to investigate the emergence of children's mathematical ideas by focusing on the fragility and robustness of children's ideas and images. My interest in students' images is prompted by the grounding of much of my analysis of students' mathematical thinking in Pirie and Kieren's work on the growth of mathematical understanding, wherein the concepts of Image Making and Image Having are central. Like Pirie and Kieren, then, I do not intend my use of the word image to be restricted to ideas that are, or can be, pictorially represented (see Pirie & Kieren, 1994).

My approach of considering students' robust and fragile mathematical images was, I think, prompted many years ago by my viewing of a British TV documentary from the eighties called "Twice Five Plus the Wings of a Bird". In one short piece a researcher is asking a child to count cookie monster cutouts on a board. There are nine cookie monster shapes arranged asymmetrically around the edge of the rectangular board, and the child repeatedly counts ten. I've seen this video clip many times, and every time I am captured by what I have labelled the "tyranny of ten". The number ten is reinforced so strongly in our culture that it seemed that it was impossible for this child to conceive of there not being ten. It seemed to me as though he figured that there couldn't possibly be just nine cookie monsters. Now, obviously there are strategies that he can be taught that will help him know when to stop counting, but it was the robustness of this child's image of ten that has stayed with me.

More recently, I have been observing what you might call the tyranny of fourteen. My two and a half year-old daughter loves to count. Like many children her age she can subitise about four items, count accurately eight or nine items, (observing one to one correspondence and usually pointing to each item with her finger), but when faced with more than nine items no matter how many there are she always counts fourteen, this being the highest number in her verbal counting string at the moment. For some reason, the one-to-one correspondence breaks down at nine and the string of counting words from ten to fourteen tumbles from her mouth as she randomly counts items.

Robust Images

Both of these are examples of what I am referring to as robust images. The idea of ten (cookie monsters) and the idea of a large collection of items being the largest number in the string (fourteen for my daughter) seem to be robust images that are somehow problematic. Not that I am concerned about the boy with his cookie monsters and my daughter's fourteen. I expect both of these problematic images to be resolved in time, but the idea of a *problematic* robust image interests me because intuitively robust images would seem to be the kind of image we would want to encourage children to form. When, though, might a robust image be problematic, and how are such problematic robust images created and how are they dispelled? More recently, then, I have begun to use the language of robust and fragile mathematical images as a framing for my tracing of students' growing mathematical understanding within my research, using some of the following questions to frame my exploration: When and how and why do particular images emerge? What, or who, occasions robust understandings? When are robust images problematic? What role do fragile images play in the growth of mathematical understanding? Does (or how does) a focus on fragility and robustness illuminate the relationship between teacher and student?

Problematic Robust Images

I'm going to offer here one brief example taken from an interview I conducted last year with three Grade 6 students. I asked them to find the area of a parallelogram, given only the diagram and a ruler. They created a solution that involved splitting the parallelogram into a rectangle and two right triangles, and then determined that one of the outer triangles would fit into the rectangle four times (see Figure 1).

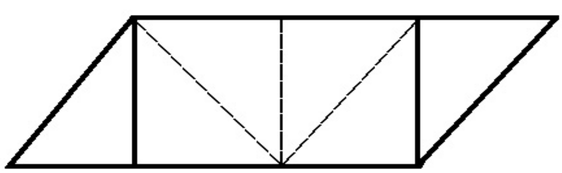

FIGURE 1

As the interviewer, I pressed them on this 'four-triangle' image to determine whether they believed it to be a general property of parallelograms. I first asked them whether they thought the property was "special about the way I drew" the original parallelogram, or whether it was true of "all parallelograms". Their initial reaction was to suggest that the property was generalisable, so I pressed further, offering what I intended to be a counter-example that I described as a "really slanted one" (drawing freehand, as I spoke) and asking if they thought the property was true of this figure too. The three students carefully studied my freehand drawing and annotated it by adding lines perpendicular to the base showing the inner rectangle and triangles (much like Figure 1), however as the diagram was drawn freehand it was not completely symmetrical, and this became a critical feature, allowing them to conclude that the 'four-triangle' image was not true of this (freehand, non-symmetrical) diagram (as the two outer triangles were not "equal"), but would still be true were the diagram to be a true parallelogram.

The students therefore rejected my 'counter-example' and insisted that the 'four-triangle' image is a general property of parallelograms. They seemed to have a very robust image for area of a parallelogram. It is interesting to note, however, that later in the interview the students did try to use this 'four-triangle' image to determine the area of a parallelogram they constructed, only to discover that it was inadequate. Hence, though their image seemed solid—robust—it was "fragile enough" for their own constructed counter-example (but not the counter-example I offered) to disrupt it. This raises interesting, though currently unresolved, questions for me concerning the role of teacher-generated examples (and by extension teacher generated counter-examples and proofs). How effective are teacher-generated counter-examples in disrupting students' problematic robust images?

Fragile Images

The vignettes in this section are taken from interviews I did with some Kindergarten students, conducted in their last week in Kindergarten. In class they had been working on what their teacher called "number stories" and I'll give just two brief examples of the kinds of tasks in which she had them engage. For the first task the students had handprint worksheets (several pairs of hands drawn on the paper). They also had four blocks, all the same colour. They were to gather the blocks in their hands, then split them so that some (or all) blocks were in one hand and some (or no) blocks were in the other. Then they were to draw the representation of the blocks in the handprints on the sheet. At the end of the class they were brought together to tell their number stories, of the form "I have x blocks in this hand and y blocks in this hand and that makes z blocks altogether".

In another task they worked with four pattern blocks. They had available two different colours, red pattern blocks and blue pattern blocks, and they were to make different combinations of red and blue blocks, always totalling four. So for example the number stories she intended them to create were something like "zero blue blocks and four red blocks makes four altogether". The children represented their number stories on paper with red and blue pattern block stickers and in whole group discussions the teacher tried to tease from them the verbal number story to go along with their picture, though many children preferred to describe their pictures as, say, flowers or animals, rather than use numbers in their descriptions. Hence, it was fairly characteristic of the group that they seem to resist using numbers in their stories. It was also characteristic of the teacher to try to draw the number story from the student. I heard her say many times variations of "How many here? And how many here? So how many altogether?" It should be noted that in the presentation of this work at the CMESG meeting in Wolfville, NS, members of the audience commented on the teacher's use of 'story' to describe the representations she was seeking, pointing out that it is, perhaps, not surprising that the children resisted the 'How many here? How many here? So, how many altogether?' form of the 'story', preferring instead more elaborate narratives. It was suggested, for instance, that the 'number story' narrative is stripped of many of the usual elements of a story, such as plot, character, and setting.

The following vignettes come from a task-based interview I did with two of the children from the class, Jason and Michael. I used a task similar to those they had been working on in class, but using a significantly larger number of blocks. There were nine red pattern blocks and the first task for the students was to count the blocks. Following this, I asked them to create number stories by repeatedly separating the blocks into two containers to explore different ways of making nine. The interview was conducted about two months after they started the number story work in class, though the teacher had repeated the tasks intermittently over the intervening time. I therefore expected them to be familiar with the *task*, even though nine blocks was probably pushing the boundary of competence for some of the students (but certainly not all).

Of course, although this is the first year of formal schooling for these students, and they are engaged in elementary mathematics, the capacities needed to be able to complete the task successfully are not trivial ones. Lakoff and Núñez (2000, pp. 51–52) offer the following list of cognitive capacities needed to count:

· *Grouping capacity:* To distinguish what we are counting, we have to be able to group discrete elements visually, mentally, or by touch.
· *Ordering capacity*: The objects to be counted typically do not come in any natural order in the world. They have to be ordered, that is, placed in a sequence.
· *Pairing capacity*: We need a cognitive mechanism that enables us to sequentially pair individual fingers [or counting words] with individual objects, following the sequence of objects in order [commonly known as one-to-one correspondence].
· *Memory capacity*: We need to keep track of which objects have been counted.
· *Exhaustion-detection capacity*: We need to be able to tell when there are "no more" objects left to be counted.
· *Cardinal-number assignment*: The last number in the count is an ordinal number, a number in a sequence. We need to be able to assign that ordinal number as the size—the cardinal number—of the group counted.
· *Independent-order capacity*: We need to realise that the cardinal number assigned to the counted group is independent of the order in which the elements have been counted. This capacity allows us to see that the result is always the same [the Piagetian concept of conservation of number].

As the following descriptions show, these capacities seemed to ebb and flow during the interview. This in itself has implications for the robustness (or rather fragility) of the images the two boys were able to develop. At the beginning of the interview the two boys confirmed with me that they recalled doing 'number stories' with their teacher. I offered them a pile of red blocks (nine) and asked them to tell me how many blocks there were. Jason seemed to have difficulty counting the haphazard grouping, so I assured him he could touch the blocks if he needed to. He began placing the blocks in a line, saying that he needed to put them in a line and that this helped him to count them. The boys agreed there were nine, then spontaneously re-counted (in a different order), getting nine again. Jason exclaimed "Still nine!", and seemed surprised, suggesting that perhaps for him the independent-order capacity is not yet stable. Next, we moved on to the 'number story' task. I asked the students to distribute the blocks into two containers, then tip out the containers into two piles of blocks and "tell the number story" as they had done in class. Though they had confirmed with me at the start of the interview that they remembered doing number stories in class, they seemed confused by instructions, so I helped to structure the task by asking them to tip out each container in turn, determine how many blocks were in each pile, and then figure out how many blocks there were altogether. On this occasion, Michael's pot contained five blocks and Jason's four, which they accurately counted, then agreed (after counting all the blocks) that there were nine altogether.

After walking them through the task in this way, I asked them to repeat it. This time, Michael's pot contained four blocks and Jason's five. I asked them if this was the same or different to last time and they said it was different. I asked if there were the same number of blocks altogether as last time, or was the total different, and they both agreed that it was different (despite counting all the blocks and getting nine again). Puzzled by their response, I wondered if it was a memory problem and they simply couldn't remember how many blocks they had had last time, and asked them how many there were last time. Jason could remember how many each pot contained last time (five in Michael's and four in his own), but thought that altogether there were six. Michael, however, insisted that last time it was "four to four", but couldn't remember how many altogether. This suggests that the memory capacity identified by Lakoff and Núñez was failing each of them.

We moved on. Next time, the pots contained two (Michael) and seven (Jason) blocks respectively. Jason's spontaneous response to my prompt of "So how many altogether?" was "Six!" I asked them how they could check if six was correct, and Michael suggested "Could we write a six?" Though clearly this was not a means by which the result could be verified, I acquiesced, and for a while they practised writing the number six. I then asked them to check their result by counting all the blocks, which they did, getting nine, and they then practised writing the numeral nine. In the next iteration of the task, the pots contained six (Michael) and three (Jason) blocks. They tipped out the blocks to count them; Jason correctly counting three and Michael first counting five then spontaneously re-counting to get six. This suggests that Michael may be beginning to pay closer attention to Lakoff and Núñez's pairing capacity (one-to-one correspondence). I asked how many blocks there were altogether and Jason immediately answered "Six". Michael counted each block and said "Nine". I decided not to pursue the discrepancy this time.

They repeated the task one more time, and I concluded the interview by asking them if they thought there were any more number stories they could do with the blocks. Jason thought he had one, and each boy grabbed a handful of blocks. Jason had six and Michael three, but when they counted them altogether they double-counted some blocks and reached a result of eleven. I showed some surprise and Michael remarked that it was "past ten". Jason said that they must have "counted one over". They recounted the blocks together aloud (pointing at, but not touching, individual blocks) and missed one, getting eight. Michael said "not eleven!" in a tone that suggested that he was now satisfied with the new answer. Jason, however, didn't seem satisfied (suggesting perhaps that his independent-order capacity was strengthening) and re-counted carefully, getting nine, commenting again on counting "one over". The following exchange occurred as I tried to determine just how robust or fragile were their images—their sense of 'nine' in this context:

Jo:	Why do you think it's nine? Why not eleven or eight, because you counted those as well?
Michael:	Because we got it wrong.
Jo:	How do you know?
Michael:	Because ten come after eleven.
Jo:	[*Surprised*] Ten comes after eleven?
Michael:	Ten first, then eleven.
Jo:	Oh. Ten first, then eleven. So how did you know that nine was the right number? [*Pause*] Why not eight? Because you counted eight.
Michael:	Because we keep forgetting.
Jason:	Hmmm. Erm, we forgot one of the pattern pieces.
Jo:	Oh. How do you know, though?
Jason:	And, and, er ...
Michael:	Stuck with math!
Jason:	That's what I said. That was my only guess ...
Jo:	Okay.
Jason:	... in my mind.

I discontinued this line of questioning and decided that this pair had probably had enough of the task for one day. During analysis of this data I was captured by what seemed to be the fragility of the images of counting and conservation of number for these students. On closer study though, I began to recognise the ebbing and flowing of particular capacities, suggesting that the images, though fragile at this stage, are contributing to the students' growing understanding of number. Returning for a moment to the list of cognitive capacities offered by Lakoff and Núñez, we can see evidence of many of these capacities stengthening and thereby contributing to the development of more robust images of number for these students. For example, some of the capacities seem solid for both boys, such as the cardinal-number assignment capacity. In each case of counting there seemed no doubt that each boy was sure that his final verbalised number represented the result of counting. Similarly, they seemed to be developing the capacity of exhaustion-detection, even though they still made mistakes in the execution of this skill. Both seemed to have some strength in the grouping and ordering capacities, with Jason perhaps lagging behind in his need to physically touch or move blocks to count them and in his need to place blocks in a particular order (a straight line) to count them, whereas for the most part Michael was able to count blocks without touching or ordering hem (though again, he sometimes made mistakes). Again, though both boys made mistakes in their execution of the pairing (one-to-one-correspondence) capacity, there is evidence that each of them understood the need to match one block with one counting word to count accurately.

It is with the two remaining capacities (memory and independent-order or conservation of number) that we observe the least confidence. It was obvious that these students' memories failed them on occasion and that there is evidence that their image of number does not yet reliably rest on conservation of number. This was echoed in my study of other children from this same Kindergarten class, including Sean, who, at one point in his interview (conducted with another student, Nina), was adamant that though he had counted eight blocks on the table and Nina nine, they could both be right. This was in stark contrast to Nina who was reluctant to even continue with the number story task after the first iteration because she was sure that there were nine blocks and would be nine each time because she had "already counted them" and it would be "the same as last time". The image that "it would be the same as last time" did not seem to be a robust one for either Michael or Jason.

Closing Remarks

To return to the questions that framed my exploration of these ideas, then, I have found this data helpful in beginning to illuminate the role of fragile and robust images in the growth of mathematical understanding. Careful study of videotaped episodes enables us to trace how particular images emerge, though the issue of who or what occasions robust (and fragile) understandings continues to puzzle. The data presented here provoke me to think deeply about the way in which students are placed in relationship with each other through their mathematical ideas. In this regard I should note the significance of the collective in Natalie, Stanley, and Thomas' generation of a very robust image for area of a parallelogram, in contrast to the absence of the sense of collective purpose (between students) evident in the tapes of the Kindergarten students. As Martin and I, based on an extensive study of the entire data corpus of Natalie, Stanley and Thomas' interactions, have posited elsewhere (Martin & Towers, 2003), these students' engagement with the mathematics exhibits striking features of a *collective* growth of mathematical understanding. This could not be said to be true of Michael and Jason's engagement with mathematics. In terms of occasioning robust mathematical understandings, then, the study of the collective seems to be offering a fruitful language and structure for understanding the processes at play. However, as Thomas, Mulligan, and Goldin (2002) claim in their analysis of children's imagery in relation to the structure of the number system, much of young children's imagery is transitory, suggesting that certain aspects of the child's structuring of the number system is fluid and constantly changing. Fragile images then, such as some of those I suggest are described in the vignettes with Michael and Jason, also seem to be important to examine, and I hope that others will join me in pursuing such studies.

Note

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The Role of Mathematics Contests

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Abstract

Mathematics contests are perceived by some to be written papers that will be used to identify award winners among a relatively elite group of students. While some validity can be granted to such a perception, the role of mathematics contests appears to be misunderstood, in general. The spirit of many existing initiatives places greater emphasis upon offering opportunities for collaboration, broadening the mathematical experiences of students, and developing mathematical abilities at a personal level. This topic group will focus attention on a range of national and regional initiatives, many of which offer models for mathematical community building.

The discussion will extend into consideration of potential benefits for professional development including opportunities for bridging across various levels. Extensive experience including involvement with problem writing teams, seminars with students, and participation in math leagues has shaped personal reflections on this subject. A critical examination of mathematics contests is encouraged as we consider the place of mathematics contests in mathematics education.

Reflections

Mathematics contests have provided me with a rich source of professional development as an educator and a learner of mathematics. As a high school student, it was experiences in junior and senior math leagues offered by the Metropolitan Toronto Separate School Board and Scarborough Board of Education, respectively, that figured prominently in my mathematical development. These experiences provided challenges and opportunities to dialogue about problems, to learn unfamiliar mathematical concepts such as modular arithmetic, and participate in genuine problem solving activities. Since that time I have been actively involved in problem solving at various levels including the development of problems suitable for various contests. Presently these involvements include service on the Gauss Contest Committee, ongoing preparation of the math league problems (along with colleagues Bruce Shawyer and Peter Booth at Memorial) for the provincial Newfoundland and Labrador High School Math League, and contributions to junior and senior high school contests in British Columbia. Clearly it is my perception that mathematics can be enhanced through such experiences. The value extends into three principal areas: the development of mathematical problem solving and a closer approximation of what it means to do mathematics; insight into mathematical learning and the experiences of others through extensive collaborations with teachers in different contexts, as well as through exchanges of diverse perceptions of common entities such as a particular curriculum or a specific problem; and finally, enrichment of one's own mathematical scope—particularly relevant on a personal level as an individual in a Faculty of Education who holds cross/adjunct appointments to Mathematics and Statistics Departments.

Surprising Results and Limitations

Problems intrigue me. It is so challenging to attempt to fairly assess problem solving. The curriculum here in Atlantic Canada promotes problem solving as a fundamental aspect of the mathematical experience at all levels. However, the assessments are unable to effectively test this core principle. It is indeed difficult to provide problems that are both accessible and nonroutine within a constrained framework such as an exam. In fact, mathematics contests may provide some insight into this challenge. Imagine the surprise of our Gauss Committee upon learning that only 18% of Grade 8 students in a school district (with over 1200 participants) correctly answered Question 4 on the 2003 contest. That is a far cry from the desired 80 to 90 percent for such an early question. The multiple-choice question required the participants to identify the "square of the square root of 17". Seemingly the language was too difficult. The committee learned a valuable lesson from this surprising result. If the square root symbol had appeared above a numerical representation of 17 squared, perhaps the correct answer would have been selected more often. Why share such a tale? The limitations of contest questions are real. It is difficult to fairly assess such knowledge. However, it is also valuable to gain such knowledge through a forum that does not carry high stakes or the same pressures to conform to ministry standards or public pressures for accountability. Contests offer an avenue through which mathematical ideas can be shared in a less threatening manner while providing opportunities to engage in problems that may be unfamiliar to all. Are contests fair? Likely not. ... NO, they are not. Some students receive extensive preparation, whereas, most participants simply write the contest with no prior sense of the experience. In fact, this latter point represents one of the dark sides of contests. Steps should be taken to ensure that students are aware that they are not expected to get a high score simply because they are used to such results. The interpretation and dissemination of raw scores must be handled judiciously, thus ensuring that the contest is less likely to be a negative experience.

Data from contest results can provide rich information that is helpful in attempts to identify weaknesses or strengths. In the aforementioned example, the language likely precipitated the problems though it is plausible that squares and square roots are also problematic concepts. An international journal, *Mathematics Competitions*, is a rich source of discussion on regional, national, and international contests. Extensive data from the responses to the 1997 Flanders Mathematical Olympiad provided the basis for a presentation at a Canadian Mathematics Society Education Session (1999) entitled *Popular Distractors: One Avenue into the Mathematical Thinking Underlying Errors in Math Contests.* The presentation focused on problems for which a distractor was more popular than the correct answer. One such example is drawn from the First Round of the Flanders Mathematical Olympiad 1996–1997. The question was answered correctly by only 12.96% of the participants. The incorrect responses of B and C accounted for over 74% of the responses. The question including choices is offered here:

> How many of the following four statements are true?
>
> Out of three successive odd numbers
>
> (1) exactly two are prime.
> (2) at least two are prime.
> (3) at least one is prime.
> (4) at least one is not prime.
>
> (A) 0 (B) 1 (C) 2 (D) 3 (E) 4

The small number of correct responses suggests a misconception or a subtle point of confusion with the topic at hand, namely, the interplay between odd and prime numbers. In fact, the error is likely due to overgeneralization. Consider an odd number ending in 5. This number must be divisible by 5 and hence, cannot be prime—unless of course it is 5. Further, we know that any set of three consecutive odd numbers must contain a multiple of 3. Again, this would seem to imply that such a number in each set must not be prime. In fact, this is true except for the case when 3 is one of those numbers. Apparently only a small percentage of participants correctly identified the particular details necessary to account for these instances. It is possible that some of these participants may have incorrectly answered that at least one of the numbers must be prime. In fact, the observation that $91 = 7 \times 13$ is needed to identify the smallest set of three successive odd numbers, namely, 91, 93, and 95, in which none of the numbers are prime. There are no true statements among those listed. Therefore, the correct response is (A) 0.

Another example from the Second Round of the Flanders Mathematics Olympiad (1995-1996) suggests a different error. The question features four polynomials and asks how many of them can be factored into a product of two real polynomials of strictly smaller degree. The correct answer of 3 (selected by 10%) is overshadowed by the response of 2 offered by 59% of participants. In my opinion, two of the expressions appeared easy to factor; however, the real factorization of the expression $x^4 + 16$ is not so evident. The low level of correct responses reflects the difficulty with one expression rather than an inability to factor. Such questions are unfair as a rule in that they seem to penalize those with a general grasp of material by providing one excessively difficult piece of content. This points to another limitation of contests. The style in which answers are presented needs to be aligned with the expectations for a strong performance. Some questions are entirely appropriate as multiple-choice questions, whereas others seem to require written solutions in order to justify their inclusion. The respondents also have preferences and it would be improper to generalize an ability to present strong written solutions from an ability to excel in multiple choice format contests.

Hidden Benefits

The value of mathematics contests is perhaps less evident to those who have not been involved in such initiatives. The idea of contests as sources of data and insight has been raised in the prior section. It should be noted that each contest usually holds its own surprises. Several years ago at a high school math league event in St. John's, a question concerning the number of lattice points within a defined circle was posed. A number of teams quickly got the correct result of 61. Others struggled and typically did not correctly enumerate the amount. Why the discrepancy? Graphing calculators were being used by some teams but not by others. Of course, we had not previously encountered such a problem. Subsequent counting problems of this type have had to be accompanied by instructions to not allow calculator usage on a particular problem, or have been presented in a format where the number of lattice points may be provided and it is the curve that needs to be defined. The experience, however, was informative. The presence of high school teachers and university faculty at the math league games provided rich opportunities for conversation about transitional issues, curriculum, and math education, in general. The unexpected difficulties that may have arisen with a seemingly straightforward question or the brilliant solutions offered by students while presenting answers to particular questions provided valuable insight into the mathematical situations at hand. Surprises are healthy!

Teachers commonly seek out good problems for use in the teaching of mathematics. The plethora of contests combines with the care taken in preparing such problems to provide many excellent resources. The experience of preparing problems and bouncing ideas off one another can not help but make one a better problem poser while enhancing one's attention to the subtleties of language and context in such problem statements. Mathematics within and beyond the boundaries of curriculum are developed through experiences with mathematical contests. Many contests, including most math leagues, are not individually oriented in nature. Exchanges of ideas, conjectures, and seemingly risky responses are permitted in such team-based contexts. In essence, the experiences bear closer resemblance to what it means to do mathematics.

Models

Various models are used in mathematics contests. A few familiar models are highlighted here in that they provide some sense of the breadth. The selected examples reflect my own experience.

The Newfoundland and Labrador High School Math League has enjoyed a rich history of over fifteen years. The league is conducted in various sites throughout the province, usually over four Saturday mornings, culminating with a provincial gathering in the spring. The striking features include: the healthy gender balance; the presence and support of teachers; the cooperative format in which teams of four from a school work together; and, its recent adaptation to a web format for isolated schools. The model for the league has been successfully adapted and implemented by Richard Hoshino and Sarah McCurdy in Nova Scotia.

The Canadian Mathematics Competition has offered a wide range of contests under various names such as Fermat, Gauss, Euclid, and others. The national scope of the contests and the range from Grades 5 and 6 (in development) through secondary school are unique to the Canadian landscape. The University of Waterloo has been the host site for this venture. The voluntary spirit exemplified by the markers, the problem writers, and regional support personnel are likely the most striking feature of the unwritten story. For example, about 70 people gather for two weekends annually to produce, critique, revise, and ultimately, prepare the contests at the various levels. In addition, the publication of problem collections and the offering of seminars have served a range of interested people from classroom teachers through to International Mathematical Olympiad participants.

The British Columbia Colleges High School Math Contest has grown over thirty years from a local initiative in Kamloops to a provincial effort. The contests bring together students from various intermediate and secondary schools to regional institutions. The campuses serve as local sites for the contest, in addition to hosting mathematical events geared to teachers and students throughout the course of the day. The contest has become a valuable bridge between the various colleges/universities and the local math teaching community. The same contest is written at multiple sites at a common time. The prizes are awarded on a local basis. A similar model for junior high students is in place through postsecondary institutions in New Brunswick.

Seminars or camps are offered in various venues as a means of bringing together cohorts of interested math students. The students are usually invited based upon performance in a contest. For example, the Blundon Seminar hosted by Memorial University of Newfoundland (MUN) has been bringing together students, mainly from Grade 11 and 12, for over twenty years. The seminar features three days of problem solving sessions, lectures, and activities hosted by the math department at MUN. Such models have been emulated in the form of math camps that have been held in most provinces over the past few years.

In summary, a range of initiatives and models are currently in place throughout Canada and other nations. An inquiry or two to a local teacher association, university math department, or an individual interested in contests will likely result in learning more about the options available at an appropriate age or regional level. Websites for information on some contests are listed here. Those marked with an * offer contests in both French and English.

Newfoundland and Labrador High School Math League
http://www.math.mun.ca/~mleague

Canadian Mathematics Competition (*)
http://www.cemc.uwaterloo.ca

British Columbia Colleges High School Mathematics Contest
http://www.ouc.bc.ca/instructors/clee/bcchsmc/BCCHSMC.htm

Canadian Mathematics Society (*)
http://www.cms.math.ca/Competitions

New Brunswick Mathematics Competition (*)
http://erdos.math.unb.ca/mathcomp

Pythagoras, Fibonacci, and Byron-Germain (*)
http://www.mathematica.ca

Alberta High School Mathematics Competition
http://www.math.ualberta.ca/~ahsmc

Conclusion

While mathematics contests are commonplace in many schools, the topic is not one that has been discussed much at CMESG meetings. The invitation from Ralph Mason and David Reid (on behalf of the CMESG executive) to offer a topic group on this subject was appreciated. The intent of the topic group discussion in Wolfville and this subsequent written presentation has been twofold: to broaden the lens through which mathematical contests may be viewed and to offer ideas from which further discussion may ensue.

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Flanders Mathematical Olympiad (Vlaamse Wiskunde Olympiade): Report 1995–1996 & 1996–1997. *Mathematics Competitions*, 11(1), 53–94. (Reprinted with permission of Vlaamske Wiskunde Olympiade.)

New PhD Reports

Présentations de thèses de doctorat

Effects of Secondary Education on Mathematical Competencies: An Exploratory Study

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The rate at which we proceed with reforms in the high school mathematics curriculum suggests a need for long-term effects assessment of programs.

In particular, given a widespread perception across mathematics-intensive university sectors that the mathematical preparedness of the students entering their programs is lacking in many respects, it seems reasonable to want to assess the nature and magnitude of the problem and to look for possible relationships with the mathematics education received. Such work appears particularly pertinent when one considers both the variety of teaching approaches within a given mathematics curriculum and the increasing number of fields that make use of advanced mathematics and technology.

Examining the teaching of mathematics with the applications in mind often gets reduced to a utilitarian approach, opposed to the development of reasoning that should be the goal of mathematics education. But it was our hypothesis that, given the complexity of today's applications, there is at least as much reasoning involved in applying math in professional fields as there is in constructing abstract mathematical knowledge, and that theory and practice can only be seen as complementary.

My thesis therefore was developed around the following questions: How can we describe the high school mathematics education received by today's university students? What long-term effects of that education can be observed with students who make an extensive use of mathematics in their university programs? What kind of mathematics education seems to better prepare students for fields where mathematics is applied? What place should be given to technology and applications in the teaching and learning of mathematics?

This questioning led me to an exploratory study with first-year students in engineering, management, and computer science (Caron, 2001; 2003). In this study, the notion of competence, which I adopted very cautiously, became instrumental in allowing me to identify key elements of answers to these very open questions without having to submit the participants to artificial tasks.

Background

The capacity to bridge theory and practice in the teaching and learning of mathematics and science has been a growing topic of interest in the 1990s. This phenomenon has led to the popularization of the notion of *competence*, which originally comes from the industry where it has been used to capture the relationships between the work performed and the knowledge held by the individual (De Terssac, 1996). In his work in the field of science education, Orange (1997) presents *competence* as a more modest concept that allows translating into the classroom the stakes, from both societal and individual perspectives, of a *scientific culture*. And this idea conveys not only the sharing of technical skills, but also the development of mental structures, intellectual tools, and thinking abilities that open new possibilities of understanding. As many others (Vergnaud, 1981; Gascon-Perez, 1995), Orange considers that *competence* can be both developed and assessed through problem solving; in particular, the level of competence should be determined from the classes of problems being mastered by the individual. Using *competence* as an organizing element of a curriculum thus leads to the objective of progressively expanding the field of problems that can be solved by the students. Orange warns not to confuse this approach with a pragmatic orientation to teaching; rather, it should be seen as a reflection of the intent to open new intellectual possibilities for the students.

In the field of sociology of work, *competencies* which serve to distinguish between individuals in a competitive environment have been classified into three categories (De Terssac, 1996):

· *communication skills ("compétences d'explicitation")*: to translate, represent, interpret what the context is, what is to be done, and what has been done;
· *intervention skills ("compétences d'intervention")*: to act upon a situation by using available knowledge and by transforming encountered situations into reusable knowledge;
· *evaluation skills ("compétences d'évaluation")*: to identify, choose, and justify whatever is being engaged into action.

One of the original features of the thesis has been to combine this three-tier classification to the description of the different stages involved in mathematical problem solving in order to define a framework for assessing mathematical competencies.

Polya (1945) has described problem solving as a four-step process: *analyzing the problem, defining the plan, executing the plan,* and *looking back on the solution.* Schoenfeld (1985) has refined this description by identifying the existence of—and necessity for—a *control process* to manage the transition from one phase to another, in moving forward or going back.

These different phases and processes can be described in terms of competencies. The analysis stage relies heavily upon *communication skills* (to understand the problem and translate into a mathematical model) and *evaluation skills* (to identify the underlying mathematical concepts). The *evaluation skills* also play a key role in the planning stage, in the control process, and in looking back on the solution—that is, in every step where decisions have to be made. *Communication skills* also provide the necessary support for executing the plan and discussing the solution. *Intervention skills* are called upon mainly in the execution phase, but they also contribute to defining the plan, as one needs to know what can be done before being able to select an appropriate approach.

The mapping of competencies with the different stages of problem solving is summarized in Figure 1.

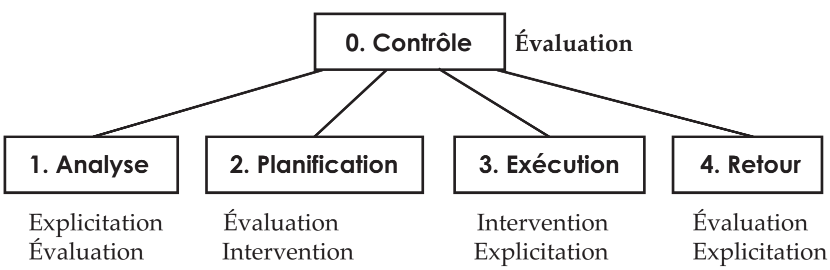

FIGURE 1. Competencies in problem solving

This model was used as reference to build a grid for classifying errors and assessing competencies of students in solving problems of applied mathematics. In order to be applicable across various fields of applications, competencies were defined in a generic fashion, without being linked to specific mathematical content (e.g., *Identify the objective*, *Identify an underlying mathematical concept*, *Write the (set of) equation(s)*, *Reason from a graph*, *Break the problem into smaller problems*, *Use a software function*, *Validate with particular cases*). For instance, failure to *identify an underlying mathematical concept* (e.g., derivative) would be associated with the evaluation skills in the analysis phase, whereas an error in *writing a system of equations* to model a situation would be associated with the communication skills in the analysis phase.

The Study

The objectives of the study were to clarify the relationships between mathematics education characteristics and mathematical competencies. In particular, we wanted to assess the place and role of technology and applications in math courses and their effects in preparing students to tackle applied math problems. The study was conducted with forty students in a Montreal university in their first year in engineering, business studies, or computer science.

In each of these sectors, a mandatory course where mathematics is applied was selected: *Mechanics* in engineering, *Discrete structures* in computer science, and two short 1-credit courses in business studies (*Financial mathematics* and *Modeling and Optimization*). Authentic problems proposed within these courses, which had to be solved in assignments or exams, were used to assess the mathematical competencies of the students. In adopting this approach, we wanted to maximize the "ecological validity" of the study by making use of real assessment material and involving the people responsible for these courses. In a first semester, with the help of the professors and teaching assistants of the courses, we validated and improved the grid through its application on sample student copies of exams or assignments. In the second semester, the grid was applied to what each of the forty participants produced when solving the selected problems; again, we were fortunate to have the collaboration of teaching assistants and professors as their experience could help locate and even explain the sources for some of the mistakes.

The educational history of each of the participants was gathered from a questionnaire that aimed, among other things, at quantifying the breadth and depth of mathematical knowledge, the exposure to proof, the level of integration of technology, and the presence of applications in the mathematics courses. Also, and this eventually proved to be a key element in the analysis, we wanted to describe their individual disposition towards mathematics. Some questions collected information with which to characterize students with respect to their way of relating to mathematics: procedural, theoretical, applied, reasoning, technological. Other questions helped provide the background of the participants: work experience, interest in reading, preferred games, and so on. The idea here was to capture the respective contributions of theory and practice in developing mathematical competencies.

Of the forty participants, twelve were selected for their contrasting profiles, in both their individual educational history and disposition towards mathematics, to allow for a finer qualitative analysis on a reduced sample of the students. To give an idea of the variety and richness of that sample, it included a philosophy major who was progressively moving toward a computer science diploma with a very theoretical disposition toward mathematics as well as a graduate from a technical 3-year course in accounting who was entering business undergraduate studies with a strongly procedural approach to mathematics. These twelve students accepted to give us a two-hour interview where we had them talk about their career choice, their perception of mathematics, their approach to learning mathematics, the education they had received from elementary school to university, especially in mathematics but also in all disciplines that could have influenced their educational path. In conjunction with the more detailed portraits that these interviews allowed, we went into a finer analysis of the solutions provided by these twelve students for the problems that we had selected.

The Results

Across all forty participants, who defined what we roughly called the "quantitative level" of the study, the math education received could be characterized globally as procedural: teaching of formulae, application in a series of exercises, and regulation through frequent assessment. The use of technology was limited to the scientific calculator in high school and roughly 15% of the math courses in CEGEP were making use of some other form of technology. The trajectory of these students actually preceded by a year or two the progressive implementation of the current math curriculum in Quebec where content has been enriched, regular use of technology has been advocated, and more emphasis has been put on problem solving. Though this situation meant that our results would become rapidly obsolete, the study still offered the opportunity to assess the long term effects of a teaching approach in its "purest" form, one that developed through many years of practice and that is unlikely to disappear instantaneously.

Our data indicate that 70% of the errors committed by the students in solving applied math problems (as encountered in their first year courses) were found in the stages of analysis (37%) or planning (33%). Using a "dual" perspective, we can also state that the errors found were linked first to evaluation (54%), second to communication (30%) and third to intervention (16%). The two most frequently observed errors had to do with difficulties in identifying the underlying mathematical concept(s) or an applicable solving method. These two categories alone accounted for 20% of the errors. Algebraic or calculus manipulations, often at the center of procedurally oriented teaching of mathematics, were responsible for only 2% of the errors.

We looked for relationships between the various indicators we had designed from the questionnaire and the difficulties observed in evaluation and communication. Though we could not find from our indicators a clear dependency on any of the factual components of the math education received (i.e., breadth or depth of math knowledge, level of exposure to proof, computer skills, etc.), the different dispositions towards mathematics, which may have been influenced by the teaching received, appeared more clearly as possible factors.

Intérêts en mathématiques	Erreurs commises dans les différents secteurs								
	Gestion			Informatique			Génie		
	EV	EX	IN	EV	EX	IN	EV	EX	IN
Théorie	-0,06	-0,28	-0,31	**-0,49**	-0,11	0,10	-0,29	-0,08	-0,15
Raisonnement	-0,04	-0,31	-0,16	**-0,54**	-0,31	0,17	**-0,62**	-0,12	0,31
Procédural	0,09	-0,05	0,14	0,35	-0,38	-0,19	**0,83**	0,14	**-0,67**
Application	0,08	-0,01	0,08	-0,31	-0,05	-0,26	0,12	0,29	-0,16

TABLE 1. Correlation between math disposition and type of errors made

As shown in Table 1, the students who approached mathematics with a favorable disposition towards reasoning tended to make fewer errors associated with *evaluation* and/or *communication* skills. Conversely, those who had a very procedural approach to learning and applying mathematics, though they generally displayed more than adequate *intervention* skills, showed a greater risk of making *evaluation* mistakes. This was particularly strong in the engineering sector (« génie ») where the same strong correlation factor (0.83) also characterized the relationship between procedural disposition and the grade obtained at the final exam; this led us to believe that our "procedural disposition indicator" could have served as early diagnosis tool. As both application and technology were almost non-existent in most of the mathematics education received, we observed no sign of technological disposition towards mathematics and no impact of application-oriented disposition.

At the qualitative level of the study, we looked for data to explain further or circumscribe some of these trends.

The interviews with students who showed a strongly procedural disposition to math made it very clear that it had been possible for them to obtain very good grades in a procedurally oriented curriculum without having to understand what they were doing or even distinguish between what they did or did not understand. They mainly relied on practice and memorization to associate questions and procedures and on external means (formative exams, pre-given answers) to validate what they were doing. Those who were still trying to apply that model of learning in their new field of application did not have the reflex of making use of the context to validate their approach or answers, and they also often found themselves lost when confronted with a problem for which a solving procedure was not readily available.

For one of the engineering students, solving a complex problem meant finding the right order for applying the different formulas he had learned. Modeling and reasoning were not part of the skills he had developed in his learning of mathematics, and this came out very clearly in the way he solved the problems encountered in his course of mechanics. He would use all possible formulas, irrespective of their applicability to the context of the problem, and would carefully arrange them so that only one unknown would appear at a time, allowing for its evaluation before the use of the next formula. Others who showed a procedural disposition and who had gone through a period without mathematics found it very hard to recover and use mathematical knowledge they had once acquired, as it never was structured properly.

The students who showed superior *evaluation* skills often took the initiative of spending less time in the repetition of exercises and more time in the *communication* of mathematical ideas: reading, questioning, debating, making use of definitions, rewriting or restructuring of a lesson, summarizing, and so on. This phenomenon was particularly striking with the strongest student of our sample from the computer science department. She confessed that she had always had memory problems and that practicing with exercises was of little help to her: She had to reduce what she had to learn by reorganizing drastically the content taught through the identification of the single key ideas which she could use to recover the other ones. That approach eventually led her to self-teaching of deductive reasoning, through the use of definitions and properties. As a consequence, she clearly distinguished herself in all of the problems where proof was required. Other reasons brought up by students who distanced themselves from the procedural influences of the math education received included individual quest for autonomy, rejection of the arbitrary, realization of the limits of learning by association, interest in applications, interest in abstract ideas.

The qualitative level of the analysis also allowed us to broadly qualify the impact of technology on both the required and developed skills. The impact of a given mathematical software tool on the mathematical skills appeared clearly dependent upon the amount of time for which the tool had been part of the mathematical activity of the student, the possibility of using that tool in regular assessment tasks and the level of analysis required by the proposed tasks. For most of the students, use of technology in math was very limited and when it was present, the learning activities almost always had an extra-curricular status. As a consequence, apart from some courses in applied statistics, very little was gained from these isolated attempts of integrating technology in mathematics.

Moreover, the study put in evidence the gaps that may exist between a traditional high school curriculum and a professional university program that relies on technology; these gaps may end up never being addressed. For instance, the numerical methods that are used to solve polynomial equations (and that form the core of many financial functions) remain completely unknown to the students: they do not learn about them prior to entering university and once they start their financial math course, they end up learning just when and how to call the appropriate preprogrammed functions. This does not help students develop critical judgment over the solution produced.

Finally, contrary to the generally accepted idea that learning programming can help develop logic and rigor which could be transferred to the practice of mathematics, the study revealed that the passage from the logic of programming to the logic of proof can be very difficult for many students. In particular, habits developed with conditional statements appear to affect initial understanding of the logical implication. Conversely, the students who had first mastered the logic of deductive reasoning showed superior programming skills: though they often had less programming experience, they were able to distinguish and treat adequately the different cases, they engaged into mental simulations to validate their pen-and-paper design of algorithms, and some of them even incorporated efficiency considerations in their design.

Conclusions

Through the operationalization of the notion of *competence*, we have been able to infer some of the long-term effects of a math curriculum on the mathematical preparedness of university students. But may be more than the results, it is the process by which our methodology was defined and applied that turned out to be of particular interest. This collaboration between different university sectors provided opportunities for mutually enriching discussion on curriculum, problem solving, and the role of errors. The development of a single tool for assessing *competencies* and its validation on authentic problems was perceived as a useful exercise by several of our collaborators: not only did it encourage them to look for the reasons behind the errors, it also helped them identify the recurring mistakes which they could try to address in their teaching. Despite these benefits, it has also become clear through the development of the project that assessment of competence is a time-consuming process that inevitably requires interpretation. The loss of absolute objectiveness could well be the price to pay for meaningful assessment. And, possibly, for better math education.

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Documenting Reform Instruction in the Classroom

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Introduction

There is an extensive body of research documenting the efficacy of reform instruction as a means of improving students' understanding of mathematics (Battista, 1999). There are also detailed but varied accounts of what reform instruction should look like in the classroom (e.g., National Council of Teachers of Mathematics, 2000; Schifter & Fosnot, 1993). However, with a few notable exceptions (e.g., Stigler, Gonzales, Kawanaka, Knoll, & Serrano, 1999) these accounts lack an overarching model giving a clear picture of what 'reform' or 'constructivist-oriented' teaching looks like. As Simon (1997) makes the case, although many teachers talk about 'constructivist teaching',

> this vocabulary has contributed to confusion about constructivism, implying that constructivism prescribes a particular model of teaching, thus obscuring the need to generate new models. (p. 60)

Because of the wide variation of description of reform instruction some researchers have begun to argue for the development of new theoretical models of instruction to bridge the gap between theory and practical classroom realities (Bauersfeld, 1995). A model is necessary to clearly delineate practice for teachers, and equally, for the purposes of research. What follows is the development of one model from theory to practical delineation of practice for the purpose of research.

The Integrated Mathematics Teaching Cycle Model

Simon (1995a, 1995b, 1997) takes up the challenge of bridging the gap between theory and practice by developing the *Integrated Mathematics Teaching Cycle* (Figure 1; Simon, 1997, p. 79). He makes an important distinction that the question is not, *Is this constructivist teaching?* but instead, *Is this a model of effective teaching and what does constructivist theory have to offer the model?* Simon draws on the theoretical and empirical work of Cobb and his colleagues (Cobb, 1989; Cobb, et al., 1991; Wood & Sellers, 1996) to create a concrete model of what effective 'constructivist' practice looks like in the classroom. He also incorporates views from the *interactionist* social perspective of Bauersfeld (1995), who sees mathematical learning as a process of enculturation into community in which knowledge is *taken-as-shared*.

Simon's model encapsulates and delineates his interpretation of the cycle of a reform-based or constructivist-oriented mathematical lesson. He begins with the factors that support and affect both the development and implementation of instruction, that is, the different types of knowledge a teacher must possess to teach this type of lesson. He cites five factors of teacher knowledge (see Figure 1) that affect their ability to teach within the reform-based teaching cycle. These are first, the *teacher's knowledge of mathematics*, which clearly affects his or her ability to understand the material and make connections amongst the concepts, and second, the *teacher's knowledge of sound and key mathematical activities and representations* basic to this instruction. Shulman (1987) defines this as pedagogical content knowledge, which includes "the most useful forms of representations of those ideas, the most powerful analogies, illustrations, examples, explanations and demonstrations …" (p. 8).

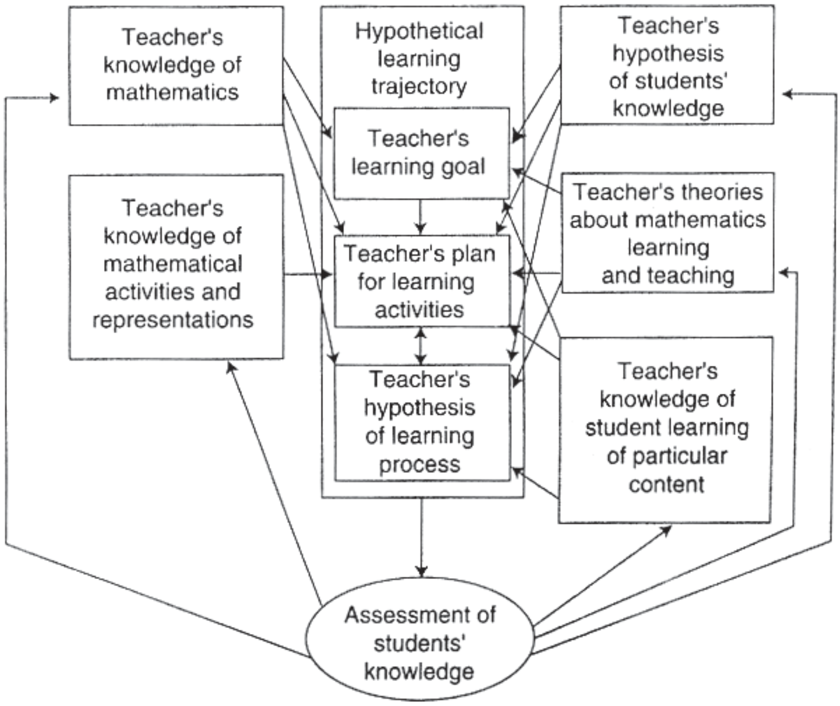

FIGURE 1. Simon's Model of the Integrated Mathematics Teaching Cycle (IMTC)[1]

This also includes the teacher's ability to use manipulatives appropriately. The third factor (which would also appear to be a type of pedagogical knowledge) is *knowledge of student learning,* that is, the areas students have difficulties with and the misconceptions they often develop when learning a specific concept. The fourth factor is the teacher's *model of students' knowledge in a given area*. The fifth factor in the model is *teacher's conceptions of mathematics and learning* which will affect the design and implementation of their lesson. Thus these five factors lay the groundwork for, and influence of instruction throughout the lesson.

Working with the Model in a Study of Reform Instruction

How useful is this coded model for delineating and analyzing reform instruction in the classroom? I used the instrument to document reform instruction in 18 classes of teachers participating in a professional development project called IMPACT MATH. The teachers were implementing both the new Ontario reform curriculum (MET, 1997) and lessons from the curricular replacement units supplied in the IMPACT MATH project. The data was drawn from classroom observation and teacher interviews.

I found three general patterns of teaching practice (see Table 2). The 18 teachers could be loosely grouped into patterns of mathematics teaching practice: *surface reform practice, mixed reform practice*, and *substantive reform practice*. These findings are similar, but not identical, to those of Spillane and Zeuli (1999) in the case of large-scale mathematical instructional reform in Michigan. Spillane and Zeuli describe the teaching patterns they found as follows. Teachers in Pattern 1 incorporated some reform instruction such as using manipulatives or used problems that were compatible with reform tenets. In Pattern 2, the teachers used tasks that were authentic or reform-based in design to help students explore 'principled knowledge' (as opposed to the development of rote or procedural knowledge). However, the emphasis was still on obtaining the right answer with limited discussion about thinking and reasoning. Finally, in Pattern 3 the teachers strongly reflected the main tenets

IMTC Model (See Figure 1)	Explanation	Expected Teacher and Student Practices (codes in bold)
Hypothetical Learning Trajectory	The goal for the student is an amalgam of curriculum set outside the classroom and knowledge of the student's present understanding.	The **teacher chooses a lesson that begins with the student's present knowledge** including what they already know and what areas are still weak (Simon, 1995).
Teacher's goal for the student	The teacher must have knowledge of the student's present level in order to set problems—while challen--ging—are not out of reach.	
Teacher's plan for student activities	Teacher has knowledge of available problems, investigations and necessary materials and methods to teach a reform-based lesson.	The **teacher chooses or develops a lesson lesson that is compatible with the reform tenets**. Tasks are chosen that have the potential to promote "conjectures, framing and solving mathematical problems, and justifying [their] mathematical procedures and solutions" (Spillane & Zeuli, 1999, p. 2). The teacher chooses **activities that require student autonomy.**
Teacher's hypothesis of student learning process	Teacher has an understanding of the possible directions that students may take as they learn a new concept; the typical misconceptions or stages they may achieve. E.g., in the teaching of multiplication students who have been encouraged to develop their own algorithms pically move through a series of increasingly sophisticated and elegant solutions beginning with repeated addition through to an algorithm approximating the one typically employed in North America (Baek, 1998; Kamii & Dominick, 1998).	The **teacher demonstrates an awareness of the different typical stages of learning** in the topic at hand (Fennema et al., 1996).
Problem Posing	The teacher poses a problem for which there is no immediate solution.	**Students work on a problem for which there is no immediate solution** (Spillane, 1999c).
Facilitation of the discourse	The lesson is structured to facilitate discussion amongst the students including the formulation explanation and defense of their thinking in small groups and whole class discussion.	**Errors are not directly corrected** but reworked in discussion with other students (Kamii & Housman 2000). That is the teacher highlights the conflicts between students' alternative solutions or interpretations (Cobb et al., 1991). Teacher asks **for alternative solutions** even when the correct answer has already been proffered.
Inquiry into students' mathematics		**Teacher continually assesses students' understanding.**
Interactive constitution of classroom practice		Students **work together to solve problems** (Spillane 1999c). Students **explain their reasoning to each other.** The teacher **establishes "social norms** that enable children to engage productively in small group work without constant monitoring from the teacher" (Cobb et al., 1991, p.7).
Teachers' knowledge of mathematical activities and representations		The teacher works from **a range of reform based resources** (Darling-Hammond & McLaughlin, 1995).

TABLE 1. Template of Reform-Based Instruction (Note: A partial table)

of the reform in that they used probing discussion of student explanations and thinking to help develop a strong understanding of essential mathematics concepts. It must be noted that these patterns of practice are loosely constructed so as not to lead to "inflexible labeling" (Brown, 1998, p. 273) in which any teacher is seen as precisely described and categorized. Instead, the patterns are meant to give a general outline in order to organize and deepen our understanding of teachers' practical interpretation and implementation of reform generally and the IMPACT MATH material specifically.

Pattern 1: *Surface Reform Practice* (n = 7, two categories 1a & 1b)	**Pattern 2: Mixed Practice (n = 8)**	**Pattern 3: *Substantive Reform Practice* (n = 3)**
Students work cooperatively	Students explain their thinking aloud	Students defend their ideas with each other
Teacher uses authentic problems or lessons compatible with reform tenets.	Students use mathematical language	Teacher uses perturbing questions
Students write up their explanations	Students use different methods to solve a problem	Teacher does not correct errors directly but uses them to foster discussion
	Manipulatives are available	Students generate their own algorithms
	Teacher establishes social norms for cooperative work.	Teacher deviates from the lesson to pursue a student idea
	Teacher uses authentic or alternative assessment	Teacher gives students time to think during discussion
	Teacher promotes student autonomy	Teacher chooses a lesson that begins with the students' present knowledge
	Students explain their thinking to each other in groups.	Students give alternative solutions to problems
		Teacher describes student explanation in more sophisticated mathematical language
		Students reflect on the reasonableness of their conjectures

TABLE 2. Codes for Teaching and Learning Patterns.

Results of Coding with the Instrument

The coding of the data using the model revealed one area of weakness in the data gathering process. The codes developed from the model worked well during the observation process but less well in analyzing the interview data. This was a result of lack of coherence between the interview questions and the model, rather than an insufficiency of the model and codes. There were factors in the model for which there were no interview questions and, therefore, no data for analysis. Although it was possible to draw inferences about teachers' understanding and knowledge as it applied to their lesson design and implementation, I found that questions directly addressing this area were necessary to build a substantive argument. Therefore, I developed a second questionnaire to address these factors in the second round of interviews. Most of these questions were related to the different types of teacher knowledge that Simon (1995b, 1997), includes in his model as necessary for the successful design and implementation of a reform-based lesson. Table 3 is a partial table (for purposes of space) of new questions and accompanying codes necessary to flesh out the model.

Stage 2: Questions derived from the model for post-lesson interview for Stage 2.	IMTC	Codes used in the data-analysis of the interview transcripts for Stage 2.
1. Where does this lesson fit into your plans?	Hypothetical Learning Trajectory	
2. What do students need to know before you begin this lesson? 3. What are your goals for the lesson?	Teacher's goal for the student. Teacher's knowledge of student learning in the particular content area.	**Teacher chooses a lesson that begins with students' understanding.**
4. Do you have set norms for your cooperative activities? (Describe) 5. What are you trying to accomplish?	Teacher's plan for student activities	**Teacher establishes social norms for cooperative work.** **Teacher's goal for student learning.**
6. Where did you anticipate this lesson might go? 7. What did you think the students would do?	Teacher's hypothesis of student learning process	**Teacher's projection of lesson.** **Teachers' knowledge of student's mathematical learning.**

TABLE 3. Interview Questions Developed from the Theoretical Model (a partial table)

A second-stage case analysis of three out of the eight of the *Mixed Practice* teachers was conducted using the coded model (with some modifications) and the revamped questionnaires. The coded model worked well in the in-depth case studies. I was able to use the coded model to analyze the new data to clearly delineate the nature of each teacher's instruction in comparison to the reform model. Moreover I was able to draw clear links between the teachers' reported understanding of reform, their own classroom situation and the observed classroom practice.

Note

1. Figure 1 (**Simon's Model of the Integrated Mathematics Teaching Cycle) was published by** Simon (1997, p. 79). It is reprinted with the permission of the author.

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Mathematics Content-Pedagogy Knowledge: A Psychoanalytic and Enactivist Approach

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The Question of Knowledge in Teacher Education

There is a singular lack of attention to subject matter in the literature on student teachers or beginning teachers. As research in teacher education evidences, teacher educators tend to take teachers' knowledge of the subject for granted and to treat pedagogy in isolation from content (Shulman, 1987). This observation led Shulman to focus on "pedagogical content knowledge", a particular amalgam of content and pedagogy, as a neglected but important way of understanding the knowledge base of teaching.

Shulman's ideas have sharpened our focus on disagreement on what counts as knowledge in general, and what counts specifically as pedagogical content knowledge. Two orientations seem to compete. One following Shulman, calls for a prescribed knowledge base for teacher education, and another challenges this notion of a prescribed knowledge base arguing that this neglects the fundamentally subjective, contextually-based, experiential, and personal-ethical character of teaching practice. Examples of the second orientation include: constructivism (teachers construct their own knowledge of practice); narrative inquiry (teachers' knowledge is personal woven in the stories that teachers tell of their practice); critical theory (knowledge depends on the context of practice: society's political, social, and economic interests pervade teaching practice); situational cognition (knowledge depends on particular practices); and postmodernism and poststructuralism (knowledge as social standpoints and power relationships that produce certain kinds of subjects). However by limiting the debate to arguments for or against a prescribed knowledge base, both orientations neglect the very kernel of knowledge in teacher education. My study attempted to move beyond this debate by exploring the nature and growth of knowledge of mathematics student teachers as they undertook their teacher education programs.

Research Methodology

Two questions guided the research:

a) What is the nature of mathematics pedagogical content knowledge?
b) What are the possibilities or spaces for its growth?

The research study involved four sites. Each site provided an opportunity to explore particular aspects of the research questions:

1) My own autobiographical recollection of memories and experiences as a student, student teacher, teacher, teacher educator, and researcher.
2) Data drawn from observation of an 8-week teaching practicum (Faculty of Education, University of Dar es Salaam, Tanzania).
3) Observation drawn from a 13-week Advanced Professional Term (curricula and methods) course at the University of Alberta.
4) School math classes involved in various research projects.

Data in my study were narratives of student teachers' experiences, their journal records, and notes where applicable—as well as field notes from conversational interviews and observation.

Theoretical Framework

I used two theoretical frameworks: enactivsm and psychoanalysis. These frameworks offer an alternative understanding of human knowledge. Central to these frameworks is the relationship between knowledge and human subjectivity. The starting place of understanding human subjectivity from an enactivist perspective is to begin with life itself, the bio-logic (Varela, 1992). In their seminal work on biology and cognition, Maturana and Varela (1971) introduced the idea of autopoiesis to show the organization of a minimal living system. According to Maturana and Varela, autopoiesis (from Gree for "self-producing") captures the mechanism or process that generates the identity of a living system and thus distinguishes living things from non-living things. Living things are characterized by an organization in which their only product is themselves (self-producing), with no separation between producer and product. For Maturana and Varela, cognition is at the center of life itself and arises in the activity of a living thing as it identifies itself as a unity by distinguishing itself from an environment at the same time (paradoxically) remaining coupled to this environment. Cognition is the bringing forth of a world of significance (that is missing or not pre-existing) (Maturana & Varela, 1992).

As well Maturana (1988) introduces the domain of language as a condition for our being as human subjects. For Maturana, as human beings we operate as observers; that is, we make distinctions in language. As observers we find ourselves in the praxis of living (or the happening of living) in language, in an experience, which as such just happens to us out of nowhere. Any explanation or description of how the praxis of living in language comes to be is operationally secondary to the praxis of living in language.

For Lacan (1977), like for Maturana, there is no subject prior to language but the subject cannot be reduced to language. Language introduces a division in the subject because the subject of enunciation (speaking being) and subject of statement (discourse) are not the same. It is because of this division that for Lacan the subject is a "divided subject". Furthermore, the division created by language creates the dimension of "desire" as the subject's need to conflate the two levels: subject of enunciation and the subject of statement. Because these two levels can not be conflated, this "desire" cannot be satisfied and this causes anxiety to the human subject. If this were the only case, human beings would experience reality as inconsistent. But Lacan introduces another dimension "fantasy" whose function is to conceal the division and allow us to experience reality as coherent. It is from this background that I analysed and interpreted student narratives focusing on how student teachers negotiate their identities and how their "desire" to be mathematics teachers is crucial to achieving this end. In the following section I offer two examples of this analysis. Lastly, using student teachers' narratives as a pedagogical space, I discuss how we might understand pedagogical content knowledge.

Becoming a Mathematics Teacher

Symbolic and Imaginary Identification

It is in the head

> *I remember the first time I was going to teach. I woke up early in the morning to prepare myself. This was an exciting day for me and my family. I never really wanted to be a teacher. I wanted to be an engineer. But here I was going to be a teacher. I made sure I read and understood well what I was going to teach. I took a very small piece of a paper and made a few points to remind myself during the teaching. I was not going to take any book or notes. This way, students would know that I knew what I was teaching. All was in my head. (Student teacher narrative)*

Reading this student teacher's narrative one might dismiss this as a typical first day experience that student teachers eventually overcome with practice. Yet, a close reading of this narrative reveals something more than just a typical first day teaching experience. As well from a constructivist perspective, one might read this as revealing one student teacher's belief about what it means to be a teacher. The student teacher believes that the teacher has to be one who knows. Also, this student teacher believes that knowledge is held in the head. In order to demonstrate this to the students he does not carry books or notes for reference precisely because all knowledge is in his head. One could also interpret this student narrative from a poststructuralist perspective where the concern might be how the student teacher is positioned by different discourses. Here one might invoke the "the teacher as an expert" myth discussed by Britzman (1991) and read the student teacher narrative as an example of how students live this myth.

From a psychoanalytic reading we must approach the question of what it means to be a teacher as a question of symbolic identification: identification with the signifier teacher. This identification with the signifier 'teacher' can be thought of as a question mark that troubles the subject (student teacher) and defies his or her attempts to discern its (teacher) meaning. This is because, in itself, the signifier teacher does not mean anything. The meaning and or identity of teacher acquires consistency only as a result of its entering into a differential relation with other signifiers. That is to say, its identity resides outside itself. In other words, the signifier serves as an enigma that promises meaning. Further, it is precisely because the signifier simultaneously does not mean anything and promises a meaning that the subject (student teacher) is engaged in a search for identity and struggle for meaning. This student teacher narrative can be read as an interplay between his symbolic and imaginary identification. The student teacher acquires an imaginary identity (self-identity) by misperceiving the meaning of the teacher as one that knows, one who holds knowledge in his head.

The Role of Fantasy and Emotions

The Student Who Knows Very Well

> *In my Math 20 class, there is a group of three students who sit at the front of the room, they understand the concepts extremely well. During a lecture a week ago, one of these students was bored with the lesson because he felt that he could do the work without any difficulties and so made a comment why should I bother going over the examples step by step if they understood what was to be done. Well to rectify the situation I gave the homework assignment to everyone and said that anyone who felt that they could do it, feel free to and those that still needed help could follow along the with the examples. I was continuing to do on the board. Half the class did start their homework however the other half asked me to do about four more examples. While I was walking around to see if anyone was having problems this student asked me why I felt it was necessary to follow all the steps. I found out he understood when I said others need those steps to understand how to do the homework, but he thought it was a waste of his time. Ever since that class he has sat at the front of the room and has this look on his face like he is better because he does not need extra help. It is very frustrating. How would anyone deal with students who sit directly in the front and stare out into space not because they don't understand but because they understand too well, and don't see the point? (Student teacher narrative)*

The subject symbolic identification with Other involves anxiety. Anxiety arises from the subject trying to discern the desire of the Other. That is to say the subject does not know what the Other wants from her, what object (*objet petit a*) she is for the Other. In the same way we might read the student teacher's experience above as an identification with the symbolic Other here epitomized by the students in the class. Thus her question to the students might be like this: "What do you want from me? You are telling me that I am your teacher, what object am I for you?" "What is more in me that makes me worthy of your desire?" Yet the students as the Other do not have the answers because they are desiring too. This may arouse anxiety for the student teacher.

It is fantasy that allows the student teacher to deal with this anxiety by constructing a narrative that provides for consistency and coherency in her experience. That is to say, through fantasy the student teacher constructs 'a formula' for her relationship with students or pedagogical relation thereby comes to recognize (desire) only those students who fit her 'formula': those students who will follow through the examples and then do their homework.

The student teacher does not recognize the three students who do not need the examples. The last sentence in the narrative shows how her relationship with these other students becomes so disturbing.

Content-Pedagogy Knowledge

> *If you give me a quadratic equation, any form of a quadratic equation, I know how to solve it. Yet when I think of teaching my students how to solve a quadratic equation, I don't know how I can do that. How can I stand in front of the class and teach quadratic equations? (Student teacher narrative)*

Reading the above student teacher narrative might evoke a number of reactions. A commonsensical reading might lead one to wonder: If this student teacher knows how to solve quadratic equations himself, how is it then that he finds it difficult to teach this to his students? Following this question a number of reasons might be given for this student teacher's experience, such as a lack of confidence on the part of the student teacher to stand in front of a class. In this way, one might suggest that what the student teacher needs is some 'how to' suggestions that might help him to gain confidence. Yet this reading misses what is at the heart of the student teacher's concern: How does one teach a particular subject like mathematics? Embedded in this question and the narrative itself is the question of the status of knowledge and its relation to the status of teaching and learning. In fact, these questions are already hinted in the ponderings above: If this student teacher knows how to solve quadratic equations and yet finds it difficult to know how to teach it, what is this knowledge of how to solve quadratic equations?

In the context of knowledge in teacher education these questions and student teachers' concerns raise the problem of the nature of pedagogical content knowledge. Rather than pedagogical content knowledge, I have chosen to use content-pedagogy knowledge. This is because I want to highlight and call our attention to what is at stake in the student teacher's concern: the space between content and pedagogy. This space indicated by hyphenation in content-pedagogy points to the dynamics of the teaching of a particular subject such as mathematics. From the enactivist and psychoanalytic perspective, I discuss how content-pedagogy knowledge might be understood, when it is a question of pedagogical relation, such as that between the teacher and students. I have developed three scenarios in which content-pedagogy knowledge might be understood in terms of knowledge and ignorance: when the teacher knows and the students are ignorant; when students are knowers; and finally when the teacher includes ignorance in her knowledge (pedagogical ignorance).

Content-Pedagogy Knowledge: Knowledge and Ignorance

During one of my conversations with the student teacher in the narrative at the beginning of this chapter, the student teacher explained to me how he came to figure out how to teach quadratic equations:

The teacher knows; the students are ignorant

> *At first I decided to observe Mr. Jabili, who has many years of teaching experience when he was teaching the same topic. I observed carefully the way he was explaining the steps of solving a quadratic equation. Then I went back and read the textbook again. It was then I realized that the book describes very well how to solve the quadratic equation. I noticed that there are several methods such as, using the general equation, sum-product or splitting the middle, completing the square etc. Lately I have figured out what I need to do in order to teach mathematics: I just ask myself, how do I express myself to someone who does not know what I am talking about. I think a good teacher is one who can express herself very well to her students, who can make them understand what they did not know. (Student teacher narrative)*

Reading this narrative we can imagine how this student teaches mathematics. Clearly it falls under what is known as the traditional way of teaching. It is a commonplace now in mathematics education to criticize this traditional model of teaching. Reform movements such as that envisioned in the NCTM Standards (NCTM, 1991) suggest ways of teaching different from this model. Yet, it is also common to find studies that report on how teachers continue to teach in the same way despite these reforms. How then are we to read this resistance of teachers to change?

The clue to this question lies in Maturana's understanding of the relationship between rationality and emotions: as human beings our decisions to act are based on our emotions and not rationality. From a psychoanalytic point of view the teacher-students relationship might be understood as the relationship between the subject (teacher) and Other (students). Also, that the teacher relates to the students by way of the object (*objet petit à*), the teacher thinks she is to the students. The student teacher in the narrative takes the Other, the students, as those who do not know: "I just ask myself, how do I express myself to someone who does not know what I am talking about." We might say that the student teacher takes the Other (students) as ignorant. By taking the students as ignorant, he desires to be 'knowledge' to the students who are ignorant. That is to say, knowledge functions as the object (*objet petit a)* that the teacher desires to be for the students. From psychoanalytic perspective this relationship is impossible. The student teacher cannot be 'knowledge' to the students. That is why he works hard, carefully, clearly, expresses himself well to be this object, 'knowledge'. It is important to note that this student teacher behavior is not just a simple problem that can be solved through experience. This behavior involves student teacher's emotions and feelings. The mathematics educator Mary Boole describes this pedagogical relation succinctly:

> The teacher … has desire to make those under [her] conform themselves to [her] ideals. Nations could not be built up, nor children preserved from ruin, if some such desire did not exist and exert itself in some degree. But it has its gamut of lusts, very similar to those run down by other faculties. First, the teacher wants to regulate the actions, conduct, and thought of other people in a way that does no obvious harm but is quite in excess both of normal rights and practical necessity. Next [she] wants to proselytise, convince, control, to arrest the spontaneous action of other minds, to an extent which ultimately defeats its own ends by making the pupils too feeble and automatic to carry on [her] teaching into the future with any vigour. Lastly, [she] acquires a sheer automatic lust for telling people to 'don't', for arresting spontaneous action in others in a way, which destroys their power even to learn at the time what [she] is trying to teach them. What is wanted is that we should … not go on fogging ourselves with any such foolish notion as that sex-passion is a lust of the flesh and teacher-lust a thing in itself pure and good, which may be legitimately indulged in to the uttermost.
>
> Few teachers now are so conceited as not to know that they have a great deal to learn, and that their methods need revising and improving, but the majority are seeking for improved methods of doing more of what they are already doing a great deal too much of. The improvement, which they most need is to … see their conduct, their aims, their whole attitude towards pupils … in the light reflected on them from those of the drunkard and the debauchee. (Boole, 1972, p. 11)

How then are we to seek this 'improvement' as Mary Boole suggests? An obvious possibility (logical or rational) is to reverse the notion of the students as ignorant and take them as knowers. What happens if we take student as knowers in a pedagogical relation?

Content-Pedagogy Knowledge: Students as Knowers

> *My grade 7 classes are exactly like Ann's Math 13 and 23 classes. They just have a stubborn attitude when it comes to being told to "sit still and work". I too have found that they, and perhaps especially them, simply can't do that. Perhaps what made the day a bit pleasanter compared to yesterday was that I got them even more involved.*
>
> *Yesterday I played "bio-jeopardy" with them in science (see the connection?), and they drove me wild with their wild excitement and exteme competitiveness. So today I tried to focus*

> *more on the classroom management aspect. But the same Grade 7 class gave me trouble as soon as they stepped into the classroom. I tried to "persuade" them to sit down, whatever that means. Not surprisingly that didn't work very well.*
>
> *So I seized the moment and mustered them out into the hallway. Since I was going to cover the topic on "ordering integers", I had them order themselves first from shortest to tallest. That took them about 5 minutes, to my amusement. Then I threatened to keep them in detention should they fail to beat that time. I jokingly asked them to line up according to I.Q. level, and that miraculously took only 3 minutes. Apparently there was a consensus among the students as to who was the smartest, although I did try a little philosophizing, arguing that they tuned me out. AGAIN. We did this drill a couple more times. A couple of them questioned the whole point of the "exercise", apparently because they have never done this kind of thing before.*
>
> *When I asked them what they thought the whole point of the exercise was, many of them gave me answers that were nice thoughts, such as "to show us how we're so uncooperative with me (the teacher) and with each other", which was absolutely true, of course. But they failed to see the math part until I gave them a hint. But in the end, I think I managed to show them that math doesn't have to be boring. I certainly didn't please all of them with what I did, but I'd like to think that I did please SOME of them at least. (Student teacher narrative)*

This student teacher narrative provides us with a pedagogical space to understand what is at stake when the Other (students) is taken as knower, the one who knows. In fact if we read it closely we see that the student teacher changed the pedagogical relation from that we have discussed above, that is taking the Other as ignorant to taking the Other as the knower. Earlier the student teacher had management problems with students. Then he decides to change the relation with the Other (see paragraph three). He takes the students out and has them do an exercise that might lead them to learn mathematics that is, ordering integers. Do we not see here the logic of taking the Other (students) as the knower? What is interesting in this narrative is that which is narrated by the student teacher in the last paragraph. To the student teacher's surprise, when he asks the students the point of the exercise, the students "fail to see" the math until he gives them a hint. Herein lies an impasse in this logic of taking the Other as the knower. An impasse manifested in the form of relation between knowledge and decision. Even though students are taken to be knowers the teacher is the one who decides this. We should not be tempted to take this experience as unique to the student teacher. Do we not see this logic in constructivist programs, where students are assumed to construct knowledge from their experiences, and yet it is the teacher who ultimately decides whether or which knowledge from students' experiences is mathematical.

How then are we to move out of the impasse? It is here that, I suggest another way of thinking about content-pedagogy knowledge. I suggest that we take ignorance to the subject (teacher) himself. I call this kind of ignorance "pedagogical ignorance" to highlight its context, the pedagogical relation.

Content-Pedagogy Knowledge: Pedagogical Ignorance

From psychoanalytic and enactivist perspectives, the dimension of ignorance is constitutive of our reality and our subjectivity as human beings in language. From an enactivist point of view, as human beings we find ourselves in the praxis of living or the experience in language, an experience which as such just happens to us out of nowhere. Reality, what we experience as objective, as a result of operating in language as human beings, includes illusions, fictions, and perceptions. We have no way of distinguishing what is illusion, fiction, or perception. In a Lacanian sense, reality is a result of fantasy. It is fantasy that provides a frame for us to experience our reality as coherent and consistent. But in order for this fantasy to be operative it has to be repressed. It follows that we cannot stand at a neutral place and distinguish illusion, fiction, or perception from our reality. Therein lies the paradox of knowledge and ignorance: What we know in reality is based on our ignorance (of repressed fantasy; not being able to distinguish illusion and perception). Rather than opposing knowledge and ignorance, I suggest we take the relationship between knowledge and ignorance as the relationship of the two sides of a mobius band. It is from this paradoxical relationship of knowledge and ignorance that pedagogical ignorance must be understood. Incidentally, this relationship of knowledge and ignorance is articulate in the notion of algebra as given by Mary Boole and articulated by Mason and Spence (1999) in the following:

> the origins of algebra lie in acknowledging ignorance of the answer to the problem. Knowing you do not know enables you to denote what you do not know by some symbol, and then to treat it as if it were known in order to write down expressions or properties, eventually arriving at knowledge of what previously you were ignorant. (p. 147)

What this means in a pedagogical relation is that as a teacher, my student(s) remain the Other, I cannot 'know' them. Therefore in my teaching, I must include my ignorance of my students and denote by some 'symbol' and then treat it as if it were known, in order to act. How then might we think of pedagogical ignorance in the teaching of a subject? Mary Boole (1972) gives us some direction:

> Mathematical certainty depends not on the subject matter of investigation but upon three conditions. The first is a constant recognition of the limits of our own knowledge and fact of our own ignorance. The second is reverence for the As-Yet-Unknown. The third is absolute fearlessness in meeting the *reductio ad absurdum*. In mathematics we are always delighted when we come to any such conclusions as 2 + 3 = 7. We feel that we have absolutely cleared out of the way one among the several possible hypotheses, and are ready to try another. (p. 44)

Even though Mary Boole speaks about mathematics, I suggest that the first and the second conditions apply to any subject. That is to say the function of pedagogical ignorance in the teaching of a subject might mean the following: the teacher recognizes the limits of her knowledge and her own ignorance (of fantasy); the teacher respects the 'as-yet-unknown'.

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Classroom Explorations of Mathematical Writing with Nine- and Ten-Year-Olds

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In my dissertation, written as a teacher-researcher, I presented my longitudinal explorations (1992–2002) with grade four pupils of the overlapping areas of mathematical and paramathematical writing (a term I discuss later). At different points during this period, all of these pupils were members of my classroom, a public elementary school in Vancouver, British Columbia.

There has been an extensive 'writing debate' in English concerning issues of teaching writing through 'creative process' or through explicitly teaching specific 'genre features' (see, for instance, Cope and Kalantzis, 1993). This profound and on-going site of tension has a particular connection with my work, although my study is not formulated precisely in these terms. Genre has emerged as a very influential notion in explorations of mathematical writing over the past decade. For example, Marks and Mousley (1990) strongly assert that the 'recount' narrative is over-used and that there is a need for greater use of expressive and factual genres. Solomon and O'Neill (1998) attempt to show how formal mathematical writing has specific features (including a primarily logical rather than temporal structuring) that are quite distinct from those of 'narrative'. In my study, I challenged myself to develop more contexts for writing mathematically and looked at different genres in terms of their syntactic and pragmatic features as suggested by these authors.

The work of Candia Morgan (1996, 1998) proved particularly significant to my study. She identified a need for more pupil experience with varied mathematical genres and made claims, supported by examples, about the lack of pupil knowledge of 'appropriate' written forms, specifically in relation to the examination-required report-writing of mathematical investigations. She also interviewed test-markers looking for qualities that are deemed valuable and necessary in mathematical writing. My work extends hers by looking at a variety of genres and ways in which pupils can gain experience with them. It differs from hers in that, particularly in relation to the age of pupils with whom I work, personal writing is actively encouraged as a way to connect meaning to mathematics.

Five main writing sites were used: mathematical journal writing, computer research journal writing, mathematical pen-pal letter writing (in conjunction with university pre-service students), different forms of in-class extended writing, including reports of mathematical investigations undertaken by the pupils, and (most significantly in terms of my dissertation) pupil textbook writing. The pupil writing from the last two sites came from one specific year, 1997–1998. With reference to the above-mentioned 'debate', during this intensive writing year, my pupils were certainly exposed to a variety of mathematical writing genres. This exposure contributed to their ability to produce the sophisticated textbook writing they did.

My analysis of their writing focused on aspects of five key and interrelated features of writing: audience, purpose, form (genre), content, and voice. Within these (increasingly overlapping and blurring) categories, I used tools of discourse analysis (in particular, attention to pronouns and general verb tense and mood) to identify and discuss specific features of their writing. In addition, I employed Eco's notion of *model reader* (for discussion of this notion in regard to mathematics texts, see Love and Pimm, 1996) and Bakhtin's (1952/1986) concept of *addressivity*, in order to examine larger-scale features of my pupils' writing. These ideas connected both to conventional textbook forms and a range of work reported in the research and professional literature, not least under the heading 'writing to learn mathematics' (e.g., Countryman, 1992).

I made use of the term *paramathematical* writing in order to discuss writing that supports mathematics even though it is not directly mathematical in itself. In the course of my dissertation, I identified two distinct forms of paramathematical writing: explicit personal text alongside more overtly mathematical writing; and certain syntactic choices (allied to the notion of 'voice') when writing text with the explicit intent of helping another pupil learn some mathematics. Finally, at a meta-level, throughout my dissertation, genuineness, caring and trust were themes that arose in the pupil writing and interleaved themselves through my discussion.

Opening context and research questions

The starting points for my interest in writing in mathematics were as follows:

- my belief that writing is a useful tool for communication and assessment;
- excitement about the potential for writing in mathematics;
- concern over the validity of offering either myself or the pupil as the primary or even sole audience;
- a curiosity about the connection between writing and learning in mathematics.

I took as my starting context the current proselytized use of writing in mathematics, including increasing numbers of text-book prompts and some high-stakes assessment items. I found little difficulty in problematising text-book instances like the following (from a Canadian grade 4 textbook):

- What did you find interesting about finding squares and parallelograms? [in reaction to a quilt picture]
- Design your own flag. Use fractions to describe its coloured sections. Tell what is important to you about your flag. [following an exercise on fractions and flags]
- Do you enjoy making and recording patterns? Explain. [following the use of T-table columns to show patterns]

I also included discussion of my own initial and tentative experiences with using journal writing in my own classroom.

Below are the three research questions I explored in my dissertation, although the third one only emerged relatively late on.

- What constitutes a sufficient understanding of the issues and practices surrounding writing in my mathematics classroom, so that I (as the class teacher) feel confident and informed about choosing, developing, analysing and criticising tasks and situations that I offer to my pupils?
- What are some effects of offering grade four pupils more explicit instruction and practice across a variety of written genres in the context of mathematical writing? In particular, how do the range and extent, as well as certain linguistic aspects of the form and voice, of their responses interact with the situated features of content, plausible purpose and audience?
- What can grade four pupils' paramathematical writing reveal that is not available in their straightforward mathematical writing?

Exploring five sites and five themes

Following is a tabular summary of the empirical work I carried out, classified along one side by the context of the writing task and along the other by the five organising themes I used to make sense of my pupils' writing.

	Audience	Content	Form	Purpose	Voice
Conventional Journals					
Computer Research Journals					
Mathematics Pen-Pal Letters					
Writing Year Investigations					
Textbook Writing					

TABLE 1. Tabular Summary of the Empirical Work

The possibility of pupil computer research journals arose from my classroom being used for a number of years as a research site by the E-GEMS group (headed by Prof. Maria Klawe) from UBC for the development of a sophisticated piece of Mathematics and Language Arts software called *Phoenix Quest* (see, for instance, Klawe et al., 2002). The pen-pal work arose in conjunction with Sandra Crespo (see Crespo, 1998; Phillips & Crespo, 1996). We paired pupils from my class with students from her elementary mathematics pre-service class at UBC. The work comprised a series of structured letter exchanges before the pen-pal pairs finally met one another.

In both of these instances, there was a genuine audience for the writing outside of the classroom, and in both cases my pupils were framed in the context as relative experts. In terms of the software developmental testing, they were expert on game play which the university team was studying; in terms of the pen-pals, they were expert in being grade fours studying mathematics, about which the undergraduates were studying.

I made explicit to my grade-four pupils my interest in and attention to the five themes. I explained the notions to them as follows. *Audience* is an awareness of who you are writing to or for; *purpose* tells you why you are doing this writing; *form* is the particular style or genre that your writing is to take; *content* is what you have to say or show; *voice* is both how you write the things you want to say and also how you place yourself in and in relation to the writing, as well as the ways you choose to invite the reader in.

Part of my work involved exploring aspects of audience and how it inter-related with the writing produced. I illustrate this here with regard to computer research journals. Despite the audience and purposes for writing being both more genuine and clearer than with conventional journals, I found with computer research journals that the mathematics was often circumvented and sub-ordinated to the game itself in the writing of my pupils. They (perfectly reasonably) assumed that the (insider) adult audience knew the subject better than they did.

I claim that the notion of insider audience and its connection to the perceived level of explicitness required in the writing accounts for the lack of specific mathematics in these computer research journal entries. When the mathematics was taken-as-shared between writer and reader (as it was in this case), it proved difficult to provide a purpose that the pupils would accept as legitimate to get the mathematics out of the game and into their written journals.

Even when explicitly teaching a concept resulted in mathematical knowledge that improved the game play, the pupils did not often report on the mathematics. This was because they presumed (usually rightly) the teacher and any other members of the research team who were present for the teaching already knew what mathematics the game/puzzle entailed and how to solve the specific problems.

In general, my work with this genre variant consequently extended the field of journal writing by means of careful attention to audience and, in particular, by distinguishing between an audience as being internal or external to the writing context (as well as being presumed to be at least as knowledgeable as the writer). When the audience is assumed already to know the topic and the solutions to problems, the pupil writers did not offer their written views about the mathematical nature of the work, though they were willing to discuss this orally. They were more inclined to write about game strategies or to ask questions about how to do a tricky bit of mathematics.

Finally, with regard to paramathematical elements of journal writing, I noticed use of personal elements being included alongside general mathematical claims in these computer journals. As well, pupils employed words and exclamatory symbols (punctuation) as expressions of excitement. In some of their writing, particularly their computer research journals, the writers used audience insider knowledge and emotives to express the pleasure/challenge of the problems they were working on. Combining these two areas made powerful paramathematical links to content that cannot be expressed in the detached or neutral technical words alone.

Next, I give a brief indication of the sort of outcomes from two other areas: mathematical pen-pal letters and textbook writing. With regard to the pen-pal letters, I presented them as a distinct genre, one that gave rise to the combining of mathematical and personal events. The pen-pal task generated very rich data. The quantity of the writing was far more extensive than that produced by either type of journal writing and much of the writing was explicitly mathematical in nature. The letters the pupils wrote were quite unlike any mathematical writing I had seen in a classroom setting before. Consequently, these texts were far more rewarding to examine and my attention was particularly drawn to features of voice and ways in which mutuality was generated and maintained by the writers.

With regard to the textbooks, innovation was there, but so well disguised by a standard content/standard format look to the page that it proved invisible to me at first glance. In contrast to the work of Shield and Galbraith (1998), who felt that pupil writing (even when writing letters) closely mimicked textbook writing— and considering Morgan's (1998) observation that "since the text book is the dominant model of mathematical writing available to school students, it is of interest to consider the extent to which students adopted text book language in their own writing" (p. 19)—my pupils' writing often did not fit this characterisation of textbook language, although its outward appearance mirrored a traditional textbook format. The grade-four pupils in my class, despite so relatively little textbook experience, demonstrated an understanding of how a chapter is written and how a textbook is put together better than many older students (and I have seen many examples of this) who are asked to analyse a text in class. In short, form and content features were similar to traditional textbooks, though voice aspects differed.

The bulk of my work centred on noticing the effects of a caring voice that invited the reader into the work, examining how these effects were created and citing examples that showed attempts to create a model reader. The linguistic tools I used to help me see these mechanisms in play included identifying how my pupils created authority (e.g., by using imperatives and unvoiced assertions, generalisations, negative assertions and counter-examples, and naming procedures). It also involved offering various features of voice that had been employed by my pupils (e.g., the particular use of pronouns; modal verbs, conditionals and hedges; directives - the use of 'have to', 'put', 'try' and 'now'; emotives; questioning strategies).

In addition to examining the nature and effect of these particular means present in their texts, I discussed some paramathematical features of the textbook writing apparent in preparing hints and in the pupils' presentation of textbook covers. I give two general instances of resources drawn on in relation to imputed purpose here:

- means for creating a trusted authority included the following: imperatives and assertions; generalizing, negative assertions and negative examples; headings and labels; procedures (numbers. steps, temporal words such as 'then' or 'after'); explanations; naming (e.g., author's first and last names).
- means for inviting the reader into the work and creating a model reader included: use of pronouns; modal verbs, such as 'should', 'might', 'could' or must'; hints; emotives; organisational strategies; visual techniques.

At the beginning of this report, I stated three questions that guided me through my thesis. Rather than try to summarise my work on all three of these questions, I choose to end this report with some discussion of paramathematical writing.

Throughout my dissertation work, I grew increasingly aware of a type of writing that sat alongside and supported mathematical writing. It was epitomised in structural elements of creating a text. Besides being a structural element, the paramathematical may be seen as a genre element too. In my thesis, I demonstrated that the paramathematical writing of my pupils was a caring way to immerse their readers in the mathematics that was being presented. They used the softening aspects of voice and the guiding elements of form to engage their readers in mathematics.

Paramathematical elements proved difficult to discuss on their own precisely because they related so closely to other features, especially to voice. Paramathematical writing often also entwined with mathematical writing to demonstrate caring for both the learner and the subject. Sometimes, rather than entwining, there was an overlapping. Paramathematical elements could additionally be seen in the tools of addressivity—a welcoming pronoun and an inviting phrase (e.g., *You are going to learn how to multiply—all you do is ...*) or a tone of reassurance (e.g., *Don't worry ... you'll get this ... There, that was easy*).

More generally, paramathematical writing showed me the levels of caring and commitment to learning that my pupils had. This was often blended with the concepts that they were teaching (explaining), thus there was a merging of form, content and purpose. Sometimes it ran alongside the mathematics, as in pen-pal writing, when the paramathematical elements helped to keep the non-mathematical world of the author in view, creating a wholeness that the audience could appreciate and perhaps identify with. Sometimes it was structural and ran inside the text's syntactic features. This was evident and discussed most thoroughly in voice-related topics. Although there are sentences that I feel I can separate and identify specifically as mathematical—for instance, ones that use:

- the imperatives 'solve' or 'match';
- the procedural 'Step 1, Step 2, ...';

and some that I can identify as personal:

- Did you get the right answer?
- That darn 4 again!

—it is often in the intersection and overlap of these two categories that the paramathematical exists.

Paramathematical writing is embedded in the full context of the writing—requiring purpose, audience, content, form and voice all working together. Somehow, it is in the author intentionality that the paramathematical resides: subtly and below the surface I found caring, trust and welcoming aspects of voice contributing to the atmosphere of learning that was being created in the pupil-authored textbooks and that was inseparable from the words being used in its creation.

In conclusion, using the concept of 'paramathematical writing' has allowed me to frame my work inside the ecological and ethical world-views of researchers such as Noddings (1984) and Jardine (1998). The conjoining of an ecological and ethical perspective within mathematical writing opened the door for me to more humanistic and personal writing of and about mathematics, while presenting a challenge to other mathematics educators to broaden their definition of what writing in mathematics *could* look like.

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Ad Hoc Sessions

Séances ad hoc

Empowering Math-Anxious, Upper Elementary Teachers in Overcoming Their Anxiety and Improving Their Teaching: An Exploratory Study

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Introduction

Elementary school teachers often exhibit math dislike and/or anxiety (e.g., Bush, 1989). It has been claimed that these teachers inadvertently pass on their negative attitudes toward math to their students (Karp, 1991) although these claims have not been supported by all studies (Bush, 1991). Recent math reforms and province-wide testing have added new dimensions in teachers' math-related fears. This presentation reported some findings from a study of 12 math-anxious teachers who participated in a series of *Math Empowerment Workshops*. The purpose of the study was twofold: (a) to gain a better understanding of these teachers' math-related anxieties, attitudes, and beliefs, and how they affected their math teaching practice; (b) through conducting the workshop series for these teachers, to explore a holistic approach to empowering these teachers in dealing with their anxieties and improving their math teaching.

Methodology

This was a qualitative case study of a group of twelve math anxious, Grades 4–8 teachers, six of whom taught at the intermediate level. The teachers participated in a series of eight 3-hour math empowerment workshops in early 2002. Data included pre- and post-study interviews, as well as data collected during the workshops: teachers' math work, journal entries, three questionnaires, and researchers' field notes. The workshops utilized a holistic approach, consisting of small group math explorations, problems solving, games, short teaching sessions, and discussions, along with emotional support activities such as group reflection, journal writing, and guided visualizations.

Findings and Discussion

Some aspects of the findings are summarized below, organized by themes.

Participants' Difficulties During Their Own Schooling: All participants reported serious problems with math during their schooling. Four of them developed math anxiety since elementary school. Often their math difficulties were reported in relation to specific teachers who were not particularly helpful for them. Their teachers taught math in a traditional, procedural and textbook-based way. The atmosphere in the class was often competitive which contributed to their anxiety. None of them dared asking questions in class and some felt so intimidated by particular teachers that they also did not dare asking them for help after class. Even those who did ask for teacher's help were often made to regret having done so.

Participants' Teaching Styles, Classroom Challenges and Coping Strategies: The teachers described how their limited math knowledge impaired their teaching and expressed concern about not giving their students the right skills they would need later. Their worst fear in teaching was being asked a difficult question in class that they would be unable to answer. They relied heavily on textbooks and spent a lot of time over-preparing for each lesson to make sure they knew how to solve each of the problems assigned. They often asked for help from colleagues or family members. A few courageous teachers have asked for help from their bright students during class. None of them allowed for alternative solutions to problems.

Teachers' Initial Math Knowledge: The teachers level of math knowledge was below our expectations. Many lacked basic concepts and skills such as multiplication, long division, fraction basics, operations on decimals, percentages and integers. All of them displayed *Algebra Phobia*, including the six intermediate teachers who were expected to teach algebra that year.

Workshop Experiences: The workshops provided a safe and supportive environment to encourage risk taking. The teachers were eager to learn and felt safe in sharing their anxieties and lack of competence. The workshops covered some of their knowledge gaps and focused mainly on rational numbers and problem solving. The intermediate teachers were also coached in developing algebra competence. While being highly anxious about problem solving and the math content, small group work helped alleviate some of the teachers' anxiety. When solving problems on their own, however, the teachers often got 'stuck' and were unable to proceed. To help alleviate their anxiety and overcome their mental blocks, use was made of the *divided page activity* (Tobias, 1976), where the left column on the page is used to record feelings, thoughts, and inner talk during problem solving and the right hand side column is used to record math work. This activity turned out to be highly successful in helping teachers get over their anxiety during problem solving and persevere with the problem in spite of difficulties. The presentation included a number of examples of how teachers made use of the 'divided page' in venting their frustrations and self doubts when feeling 'stuck' during problem solving, which often allowed them to get over their fears and mental blocks and continue with their problem solving efforts.

The teachers made every effort to put into use what they had learned in the workshops in their own classrooms, and then shared their experiences with the group in the following workshop. They were beaming with pride when reporting their successes, and unsuccessful attempts were discussed and analyzed with the whole group. Over time there was noticeable improvement in participants' knowledge and level of confidence. All but one of the teachers improved considerably in their basic math and problem solving skills and felt better about their math teaching. The six intermediate teachers reported having finally mastered the use of algebra in problem solving so they no longer dreaded teaching it. In general, the eight Math Empowerment workshops got these teachers started on their new path toward conquering math anxiety.

Note

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MOM, Sense Making and Rich Learning Tasks

Gary Flewelling

MOM is a tool that helps me to think about the major elements in the teaching/learning process. The following is a brief outline of MOM.

Police use MOM to screen for crime suspects. MOM assures them that a person will not/cannot commit a crime unless they have a **m**otive for committing the crime, have an **o**pportunity to commit the crime and possess the **m**eans to carry out the crime.

The general principle of MOM asserts that a creature will not/cannot commit a certain act unless it is motivated to do so, unless it has the opportunity to do so, and unless it has the means to do so. I find it very helpful to apply this principle to things pedagogical.

Applied to sense-making students, MOM asserts that students will not/cannot make sense unless they are motivated to do so, unless they have the opportunity to do so, and unless they have the means to do so. Applied to teachers, MOM asserts that teachers will not/cannot animate or facilitate student sense making unless they are motivated to do so, have the opportunity to do so, and possess the means to do so. (The learning environment within which students and teachers work can have both a positive and negative influence on the development of MOM for sense making.)

Motivation, opportunity, and means are connected dynamically. Each can feed on/shape/influence/generate one another. A major task of the teacher is to ensure that their students have the MOM to make sense (of/with their mathematics). Because the three dimensions of MOM are a necessary but not sufficient set of conditions for student sense making, another task of the teacher is to try to ensure that students, with the MOM to make sense, actually make sense.

MOM identifies suspects not culprits. Police need evidence that a suspect actually committed the crime. In the same way, teachers need evidence that the MOM for sense making is present within their classrooms and that their animating/facilitating actions have been effective. Teachers and students also need evidence that allows them to judge the quality of the sense making that is demonstrated and the quality of the sense that is made.

Most learning tasks generate lots of activity. Unfortunately the activity is typically not of the sense making kind. Tasks in texts and on tests typically fail to motivate sense making, fail to give students the opportunity to sense make and fail to encourage the development and use of sense making tools and processes. Rich learning tasks,[1] on the other hand, successfully designed/adapted/facilitated by the teacher and successfully engaged in by teachers and students provide students with the MOM to make sense, the MOM to learn with understanding, the MOM to use their knowledge authentically, the MOM to problem solve, the MOM to behave like a practitioner in the discipline, and the MOM to be true to their nature as sense makers.

Note

1. For a detailed treatment of rich learning tasks see G. Flewelling & Wm. Higginson, *Teaching With Rich Learning Tasks: A Handbook* (The Australian Association of Mathematics Teachers Inc., 2003).

A “Proof” that all Triangles are Equilateral: Drawing out the Mathematics from a Rich Investigative Problem

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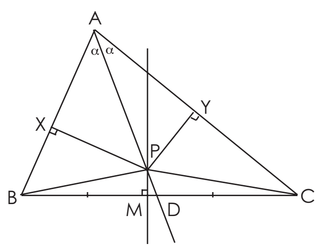
A pedagogically rich problem involves many components. First, the problem must be interesting and appealing. It must be accessible, yet not trivial. It must highlight and introduce powerful mathematical ideas, while revealing the beauty and intricacy of mathematics. Here is an example of such a problem. I have used this activity for many different audiences, from gifted Grade 9 students to a third-year undergraduate course in mathematical problem-solving.

To start, we draw any non-isosceles triangle ABC. Construct the perpendicular bisector of side BC, and the internal angle bisector of angle A. Let these two bisectors intersect at point P. Make sure that the bisectors are drawn so that the intersection point P lies *inside* the triangle. Now locate point X on AB and point Y on AC so that PX is perpendicular to AB, and PY is perpendicular to AC.



Then we analyze the diagram and come up with the following geometrical facts, all of which are easy to prove using basic congruency theorems:

1) PX = PY
2) AX = AY
3) PB = PC
4) XB = YC

We know that AX = AY and XB = YC. So adding these two equations, we find that AX+XB = AY+YC, or AB = AC. This proves that triangle ABC is isosceles!

Now we rotate the figure and apply the same argument on sides AC and BC, to prove that AC = BC. So now we have shown that all three sides have equal length. Therefore, we have a proof that all triangles are equilateral!

What went wrong? Starting with our diagram, we just used basic congruency theorems to prove that all triangles are equilateral. Of course, this proof is incorrect, but the critical question is, where does the proof break down? It turns out that the error occurs in the initial diagram: P is actually located outside the triangle. If we draw the diagram more carefully, we will see that P is always outside the triangle.

Why must P be outside the triangle? That opens up a new series of questions. We can prove this formally. In the proof of this result, we end up discovering and proving the Internal Angle Bisector Theorem, and learning powerful problem-solving techniques such as generality and symmetry. (For further analysis, see http://www.mscs.dal.ca/~hoshino/talks/cmesg03.pdf)

Multiple Meanings: Why Should We Encourage Their Use?

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Teachers resource books encourage use of multiple representations. Radford (2002) asks: What are the relationships between the representations and the concepts associated with them? This question is rarely addressed. In some elementary teachers' textbooks part-whole, quotient, and ratio approaches to fractions, for example, are outlined as multiple meanings that are needed for learning the fraction concept. Many teachers, however, appear to view models as material bases for abstract ideas. To some teachers the varied meanings are attempts to reach out to the tactile, visual, and other types of learners. The paper is an exploration of the challenges evoked by coupling an objectivist view of concepts with an interactive view of learning. I explore stances that talk about the ontology of mathematical concepts in ways that offer a strong case for the use of multi-embodiments. It is a theoretical paper; however, it is informed by teaching.

In a discussion on transformational geometry, Grade 7 students used the following action verbs: cut, fold, mirror, reflect, or draw. The imaginations evoked by each of these verbs were distinct; yet, there appeared to be invariants that afforded the students a meaningful discussion. Students appeared to have had multiple experiences with the concept of symmetry. But for a teacher the challenge is: Are the analogical experiences varied illustrations or just literal metaphors? Abstract ideas could vitally be layered by immediate actions and interactions. Drawing from Gadamer, Wittgenstein, Pierce, and other philosophers, researchers are gradually embracing the fluidity of concepts. Spence-Brown (1979) asserted, "If a different surface [clay, paper, or dynamical environment] is used, what is written on it, although identical in marking, may not be identical in meaning" (p. 86). Spence-Brown's statement that tools and mediums influence our conceptions may sound extreme. But could it not be the case that the sense made of concepts is dynamic and ever shifting? Gadamer (1992) asserts that concepts are constantly in the process of being formed, that the thing-in-itself is nothing but the continuity with which the various perceptual perspectives shade into one another. "Every 'shading' of the object of perception is exclusively distinct from every other, and each helps co-constitute the thing in itself as the continuum of these nuances" (p. 447). Gadamer's view is not far from the semiotics idea that concepts are the invariant of representations. From the enactivist perspective that frames this writing, concepts are nothing in themselves but tokens or eigen values for adequate mathematical actions that coordinate further mathematical behavior (Maturana, 2000 & von Foerster, 2003). Mathematical ideas arise with (and are meaningful in) mathematical actions such as counting, shaping, proving, and so on. Davis & Simmt (2002) illustrate that diversity and redundancy in models is essential for insights to develop. Concepts are emergent properties that arise from the interaction of lots of parts. There are varied stances that support the use of multiple meanings. But how do we package these details in ways that are pragmatic? It appears teachers would benefit from views about concepts that fit better than multiple intelligences with the use of multiple models.

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Panel Discussion

Table ronde

New Curriculum in Mathematics in Quebec: Continuities, Changes and Challenges

Nouveau curriculum d'études en mathématiques au Québec: Continuités et changements, questions qui se posent

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I will focus in this report on the intended curriculum and will highlight the new orientations and the important changes presented in this curriculum in terms of finalities, approaches, and school organization.

An important question for me, when we speak of a new curriculum, if we think about the teachers and the way they can give meaning to it, are the possibilities of taking account of their previous knowledge in relation to the past curriculum and to build from this experience. It seems to me that in education we too often have the tendency to consider that we must always change, without taking into account previous experience, without analyzing it. From that perspective, it seems to me important to return to the previous curriculum, and to see how the new ones can be articulated on the basis of past experience. After doing that, questions about some challenges and limits that we can now anticipate will be presented.

1. New orientations, important changes

The new curriculum in mathematics was implemented in elementary schools in 2000 and is now being elaborated for the secondary school. It will be implemented in this case in 2005.

This curriculum is part of a more global and systematic reform that involves all the curricula, not only mathematics. In all the fields (French, mathematics, sciences, human sciences, etc.), programs are changed simultaneously, and articulated on what we call "le programme des programmes" defined in terms of transversal competences.

1.1. Finalities of this new curriculum in mathematics

The new curriculum in mathematics is part of this reform and represented an important shift from a curriculum expressed in terms of content-to-be-covered to a curriculum expressed in terms of competences and types of educational experiences that should be considered for all students.

The choice of the concept of competence is a deliberate attempt to distinguish the orientation of the curriculum from content-to-be-covered, or basic skills, or minimal objectives approaches. The concept of competence has to be distinguished from skills or performance. Following Perrenoud (1997), it refers more to the process of activating resources (knowledge, skills, strategies, etc.) in a variety of contexts, namely "problematic situations."

Starting from the previous clarification of the concept of competence, a certain number of pedagogical and didactical issues can already be anticipated:

- The notion of "problematic situation" appears central. A competence is developed in situations.
- The competence is developed, built in context. A competence is contextualized.
- It is related to a reflexive use of knowledge.

- It intends to emphasize the integration of knowledge, skills and attitudes (an integration of different resources is a key idea).

What are more precisely the competences chosen by this new curriculum in mathematics, which are the same for the beginning of elementary schools to the end of secondary schools? The choices are different from those taken by other countries[1] where curriculum is also defined in terms of competences (see, e.g., Abrantes, 2002).

Three different competences (MEQ, 2000, 2003) are explained, with their components:

- Solve a problem situation
  - Decode elements that lend themselves to a mathematical treatment
  - Represent a problem situation with a mathematical model
  - Develop a mathematical solution
  - Validate a solution
  - Share solution information
- Reason mathematically
  - Establish conjectures
  - Produce demonstrations and proofs
  - Formulate and apply networks of mathematical concepts and processes
- Communicate using mathematical language
  - Choose an appropriate language for communication
  - Receive or send mathematical messages
  - Analyze mathematical communication system

1.2. Changements dans les approches que de telles orientations supposent

- *Le choix de situations-problèmes: un défi particulièrement important.*

  Le choix de situations-problèmes dans ce nouveau programme apparaît, on le voit dans ce qui suit, central.

  > " La résolution de situations-problèmes est au cœur des activités mathématiques comme des activités de la vie quotidienne. En tant que processus, elle constitue un objet d'apprentissage en soi, en tant que modalité pédagogique, elle soutient la plupart des démarches d'apprentissage du domaine. Elle revêt une importance toute particulière du fait que l'activité cognitive privilégiée dans le traitement des concepts mathématiques nécessite un raisonnement logique appliqué à des situations-problèmes » (MEQ, 2003, p. 5)

  Mais sur quelle base va se faire ce choix de situations problèmes par les enseignants? Qu'entend-on par situation problème? Ces questions centrales pour les enseignants, qui auront à mettre en œuvre ce nouveau programme, trouvent peu de réponses dans le curriculum tel que formulé.

- *Un accent mis sur le raisonnement et son développement chez les élèves.*

  Un accent particulier est également mis sur le développement du raisonnement, il s'agit là d'ailleurs d'une des compétences du programme.

  > « L'élève qui déploie un raisonnement en mathématiques structure sa pensée en intégrant un ensemble de savoirs et leurs interrelations » (MEQ, 2003, p. 6)

  Mais qu'a de spécifique le raisonnement en mathématiques ? Quels raisonnements cherche-t-on à développer? Le programme du secondaire est à cet effet plus explicite, on parle par exemple de raisonnement inductif, déductif, de la formulation de conjectures, du recours à l'analogie...

- *Le rôle joué par le langage et la communication.*

  Cette competence de communication apparaît intimement liée aux deux autres.

> « Le développement des compétences de résolution de problèmes et déployer un raisonnement nécessite le recours à la compétence communiquer à l'aide du langage mathématique » (MEQ, 2003, p. 6)

Le développement de cette compétence suppose de la part de l'enseignant qu'il mette en place de véritables situations de communication dans lesquelles le langage, le symbolisme ont un rôle à jouer en lien avec l'activité mathématique.

D'autres aspects présents dans le programme, et ne référant pas directement aux compétences, représentent également un défi pour les enseignants.

- *Le recours à la technologie.*

  La technologie (calculatrice à affichage graphique, logiciels de géométrie dynamique, internet,….) apparaît comme un moyen important pour supporter la résolution de problèmes, et la comprehension de concepts en mathématiques.

  > « La technologie s'avère précieuse pour soutenir la démarche de résolution de situations-problèmes. Elle favorise la compréhension de concepts et de processus en permettant l'exploration, la simulation et la représentation d'une plus grande quantité et diversité de cas. Elle augmente l'efficacité des élèves dans les tâches qui leur sont proposées et facilite la communication » (MEQ, 2003, p. 6)

  Cette intégration de la technologie dans les classes pose un certain nombre de défis aux enseignants.

- *Une dimension culturelle.*

  L'étude de l'évolution des mathématiques, y compris des mathématiques contemporaines, devrait former une partie importante de son enseignement.

  > « Le développement de la mathématique ayant été étroitement lié à l'évolution de l'humanité, la dimension historique de cette discipline devrait faire partie de son enseignement… » (MEQ, 2003, p. 6)

  À quoi réfère cette utilisation, si on veut qu'elle dépasse la simple anecdote à laquelle elle est souvent associée. Qu'exige t-elle de la part des enseignants?

- *Liens entre les mathématiques et les autres éléments du curriculum.*

  On retrouve ici deux idées clés : l'exploitation de situations provenant des domaines généraux de formation, tels la santé, l'environnement, les médias, la citoyenneté, le monde des affaires, et la participation de chaque programme de formation au développement d'un ensemble de compétences transversales.

  > « Grâce à une diversité de situations d'apprentissage, l'élève aura la possibilité d'établir des liens entre d'une part les compétences et les savoirs mathématiques et d'autre part certaines grandes questions issues des domaines généraux de formation….(et plus loin) L'élève qui exerce ses compétences en mathématiques développe l'ensemble des compétences transversales.. » (MEQ, 2003, p. 7)

  Comment cette mise en liens peut-elle se faire? Que suppose la réalisation de projets interdisciplinaires qui constituent une des formes possibles de cette mise en liens?

D'autres changements touchant à la culture de l'organisation scolaire, particulièrement importants pour le secondaire, sont également au cœur de cette réforme. Nous les reprendrons brièvement ci-dessous.

1.3. New school organization that such orientations suppose, challenging the teachers and the school

The development of competences and educational experiences should be considered by cycle (and not for each grade level). In the case of elementary schools, and more particularly of secondary schools, it is an important change. Learning is now considered as a process that continues on for two years, and that requires an articulation between teams of teachers. Other ways of thinking about teaching and assessment had then to be considered.

Teams of teachers for each cycle, in mathematics, or from different disciplines, have to work together to implement this reform.

Assessment has to be considered differently by cycle and in new ways. The assessment of competences supposes the elaboration of complex situations where the competences could be actualized.

2. Situation du nouveau programme en mathématiques par rapport aux programmes précédents

Dans l'évolution récente de l'éducation au Québec, le concept de compétence et le processus de développement curriculaire qui touche tous les programmes, de la maternelle à l'université, sont devenus les volets d'un mouvement important.

Dans le cas des mathématiques, cette évolution est, nous le verrons par la suite, en continuité avec les précédents programmes et le mouvement amorcé en 1980.

Nous reviendrons sur certains de ces aspects, en vue de montrer que l'évolution curriculaire actuelle prend son ancrage dans les précédents programmes. Ainsi l'inclusion de dimensions comme la résolution de problèmes, le développement du raisonnement, l'utilisation des nouvelles technologies, comme nous le verrons ci-dessous, n'est pas complètement nouvelle.

2.1. Le rôle de la résolution de problèmes

Ce mouvement à l'égard du rôle de la résolution de problèmes en mathématiques a été amorcé en 1980 avec le programme du primaire en mathématiques, dans lequel l'on retrouve une intention d'approche par résolution de problèmes.

> « On cherche à utiliser les notions mathématiques déjà acquises dans la résolution de problèmes issus de situations réelles mais non mathématiques…cela entraîne par exemple un choix approprié de situations d'apprentissage extraites de la vie réelle plutôt que la présentation directe des concepts ou habiletés visés, cela implique également une certaine insistance sur le développement d'une habileté générale à résoudre des problèmes » (MEQ, 1980, p 7)

Cette intention n'est toutefois pas à cette époque opérationnalisée, elle demeure au niveau des souhaits. Le contenu de ce programme n'est en effet nullement en accord avec cette intention. Celui-ci demeure caractérisé par un certain contenu mathématique à couvrir, prenant la forme d'un découpage en objectifs et sous-objectifs.

> « De façon particulière, dans l'implantation du programme, l'approche par résolution de problèmes a été escamotée, voire presque ignorée, laissant place à de multiples interprétations..Or plusieurs enseignants n'auraient vu dans cette option que la nécessité de présentation de problèmes raisonnés ou d'y consacrer plus de temps. Cette confusion est inquiétante pour la réalisation des changements fondamentaux auxquels ce type de démarche est si intimement lié » (MEQ, 1988, p 7)

Il fallait donc aller plus loin et clarifier ce que l'on entend par résolution de problèmes. C'est ainsi qu'est né en 1988 un guide pédagogique, portant sur la résolution de problèmes.

Des indices importants quant à un changement de paradigme apparaissent dans ce fascicule

- à travers la nature des problèmes considérés (problèmes réalistes, fantaisistes, proprement mathématiques, problèmes ouverts, problèmes à données incomplètes, superflues, contradictoires…)

- et à travers le rôle de la résolution de problèmes qui apparaît à la fois comme une habileté de base à développer et un moyen pour explorer, construire de nouvelles connaissances, en favoriser une meilleure compréhension. La résolution de problèmes n'apparaît plus ici comme une application de concepts préalablement étudiés, elle peut apparaître en amont de cet apprentissage ou au cours de celui-ci.

Le programme de mathématiques de 1994 au secondaire actualise de manière definitive cette orientation.

2.2. A problem-solving approach and the development of reasoning

The problem solving approach, as we have seen previously, had been precised in the "fascicule K" (1988), where we found indicators of changes

- On the nature of problems (different types of problems in relation to context, real problems, mathematical problems, fantasies, open problems, problems with contradictory data, with missing data…)
- On the role played by problems:
  - Explore, construct, broaden, increase, apply and integrate mathematical knowledge (concepts, properties, techniques, procedures, algorithms, etc.);
  - Develop intellectual skills (organizing, structuring, abstracting, estimating, generalizing, deducing, justifying, etc.);
  - Develop positive attitudes (becoming aware of one's potential, respecting the opinions of others, being imaginative and creative, rigorous and precise, etc.);
  - Apply problem-solving strategies (looking for patterns, representing a problem by a figure or a graph, using a formula, formulating an equation, etc.)

Problem solving in the 1994's curriculum for high school appears as one of the two major educational principles intended to guide teachers in their work with students: "Using problem solving at all stages of learning". As an instructional approach, problem solving may play a significant role in various stages of the learning of mathematics. It is not only seen as a way to review learning of some concepts or to apply them, but it is also a way to explore, construct new knowledge, to restructure it, and so on.

> Problem solving is both a basic skill that students should develop and an effective teaching approach that promotes the development of mathematical knowledge, thinking skills, socio-affective attitudes and problem solving strategies. (MEQ, 1994)

We found there, in this program, global objectives that summarize the role that mathematics plays, and a certain view of mathematics that we can put in relation easily with the new curriculum (MEQ, 2003).

- Problem solving appears central.
  The curriculum aims to increase the students' ability to analyze the data associated with a problem and to use appropriate strategies to arrive at a solution that they will be able to verify, interpret and generalize.
- Reasoning is also at the heart of learning of mathematics.
  The curriculum aims to increase the students' ability to formulate hypotheses and verify them using an inductive or a deductive method.
- Communicating supports reasoning and problem solving
  The curriculum aims to increase the students' ability to grasp and transmit information and to express their thoughts clearly, using mathematical language.
- Establishing links is also pointed out.
  The curriculum aims to increase the students' ability to establish links between knowledge they are acquiring and the knowledge they already have in mathematics, and with other disciplines, encouraging them to view their knowledge as a tool that can be useful to them in everyday life.

However, an important difference between this program (MEQ, 1994) and the new ones (MEQ, 2003) is the articulation between these global objectives of mathematics education and mathematical knowledge, clearly put in light.

For example, in relation with problem solving, we can find

· To solve problems involving several operations on natural numbers
· To solve problems involving rational numbers
· To solve problems involving straight lines or angles, triangles, quadrilaterals, the perimeter or area of certain polygons, etc.

In relation with reasoning, we can find

· To help students develop number and operation sense
· To develop proportional reasoning
· To justify an assertion used in solving a problem involving numbers, angles, etc.

The process of communicating will take the form of

· To present information about a situation by means of a table or a graph
· To express the relationships between quantities
· To express in their own words the rule relating two quantities, or to express in symbolic language this rule.

3. Défis et limites

On peut apercevoir à partir de ce qui précède les acquis antérieurs sur lesquels peut tabler l'implantation de ce nouveau curriculum québécois au primaire et au secondaire. On peut également en apercevoir les faiblesses et les limites.

Je reprendrai quelques unes des questions importante soulevées par ce curriculum et ses orientations, en lien avec son implantation.

· *Une de celles-ci, fondamentale, est liée à l'articulation entre compétences et savoirs mathématiques.*

La formulation du curriculum de mathématiques en termes de compétences et d'expériences signifiantes n'est pas une garantie automatique de succès. Le concept de compétence n'est en effet pas simple, et des incompréhensions diverses de celui-ci sont visibles dès maintenant. On l'associe par exemple à la performance de l'élève (être compétent dans la tâche) ou aux anciens objectifs et sous-objectifs, en cherchant à découper telle compétence en objectifs de comportement mesurables…Il apparaît très difficile pour les enseignants de donner sens, d'actualiser ce concept de compétence dans la pratique, de le rendre « opérationnel ». La même difficulté que celle que l'on avait observé en 1980 à propos de la résolution de problèmes ne risque t-elle donc pas de se produire?

La présentation de ce programme ne met pas non plus très clairement en évidence le lien avec les contenus, les compétences étant décrites avec toutes leurs composantes et les contenus d'autre part. Ne considère t-on pas ainsi dans une telle présentation que l'articulation entre les deux doit se faire comme par magie. Il y a là il me semble une question centrale à considérer. Les savoirs mathématiques, de l'ordre de ressources mobilisées en contexte, ont un rôle central à jouer dans le développement de compétences. Cette articulation doit être davantage développée.

· *Un deuxième défi important est relié aux dérives possibles de l'intégration des matières*

Dans l'idée de mise en liens entre compétences en mathématiques et autres éléments du curriculum (domaines généraux de formation, autres disciplines), prenant la forme entre autres de projets interdisciplinaires, quelque chose de fondamental m'apparaît important à interroger.

Ces projets ont en effet donné lieu par le passé et donnent encore lieu souvent à des pseudo intégrations dans lesquelles le rôle des mathématiques et les apprentissages qui s'y réalisent sont fortement questionnables. Dans ces projets, les mathématiques sont en effet souvent prises comme discipline de service (cela se réduit par exemple à une utilisation de mesures, de calculs,….) au détriment d'un véritable travail sur les concepts.

La question à se poser n'est pas ici de savoir quelles mathématiques les autres disciplines ont besoin et pour quel usage, mais de cerner le travail sur les concepts mathématiques qui peut être fait à l'intérieur de projets interdisciplinaires. Cela suppose un travail d'analyse préalable fondamental pour pouvoir élaborer de telles situations.

· *Un troisième aspect a trait aux tensions entre les orientations du curriculum et les pratiques évaluatives*

L'évaluation de compétences chez les élèves requière l'observation fine dans différentes situations plus ou moins complexes, la mise au point de nouveaux outils, la confiance dans le jugement professionnel de l'enseignant. Elle va à l'encontre des pratiques évaluatives qui se sont installées de manière forte dans le milieu scolaire depuis plusieurs années.

· *Enfin un dernier aspect à considérer, et non le moindre, concerne la manière même de penser le développement et l'implantation de ce curriculum.*

Un défi important est relié à la manière dont a été pensé le développement et l'implantation de ce curriculum. Quel rôle les écoles et les enseignants sont-ils appelés à jouer?

Ce processus continue à être envisagé de manière « top down », il laisse peu de marge de manœuvre dans les décisions qui concernent l'enseignement et l'apprentissage aux enseignants. Le projet n'est pas vu comme étant conçu, porté et développé par l'école.

Note

1. The curriculum in Portugal is, for example, defined in terms of: the disposition to think mathematically—that is to explore problematic situations, search for patterns, formulate and test conjectures, make generalizations, think logically; pleasure—involving mathematical reasoning and the conception that the validity of a statement is related to the consistence of the logical argumentation rather than to some external authority; the capacity to discuss with others and communicate mathematical thoughts through the use of both written and oral language adequate to the situation; the disposition to try to understand the structure of a problem and the capacity to develop problem solving processes, analyze errors and try alternative strategies (Abrantes, p. 52).

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Excerpts

The State of Mathematics Education in Ontario— Where we came from and where we are in March, 2003

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The teaching and learning of mathematics in Ontario has been in a constant state of flux over the last 50 years—the span of my learning and teaching experience. In the 1960s the "new" math emerged in which students encountered set theory and properties of number systems. For those students who operated well in an abstract world, this approach was quite fine. All other students would have to be satisfied with learning the "facts" and a few algorithms regardless of their use in any "real world" applications. With the passage of time, problem solving became the "buzz" word describing what ought to be the cornerstone of effective teaching in mathematics. In 1985, new guidelines were introduced. The front and back matter in these documents described approaches to teaching mathematics through inquiry, and problem solving while at the same time embracing the use technology. These statements mirror many of the tenets of the newest documents for curriculum renewal in Ontario.

Curriculum Reformation in Ontario—1997 to the Present

Elementary

It would be easy to concentrate on the changes in curriculum in the secondary years, but it was in 1997 that the renewal began with the implementation of *Mathematics, The Ontario Curriculum, Grades 1–8*. This policy document replaced *The Common Curriculum: Policies and Outcomes, Grades 1–9, 1995*. Teachers and parents wanted more clarity about the required learning outcomes for each and every grade. The province wanted a curriculum that was truly standard across the province. In other words, the mathematics delivered in a Grade 2 classroom in Thunder Bay would be the same as the mathematics delivered in Grade 2 classrooms in Sarnia or Ottawa. Moreover, the mantra of the provincial government was clearly stated in the fourth sentence of the document, "The mathematics curriculum set out in this document is significantly more *rigorous* (my italics) and demanding than previous curricula."

The main body of the curriculum is organized by strands—Number Sense and Numeration, Geometry and Spatial Sense, Measurement, Patterning and Algebra, and Data Management and Probability. Each strand includes general and specific expectations (a total for all strands of approximately 80–100 specific expectations per grade). The "front" matter addresses such issues as the role of parents, teachers, and students, the importance of Mathematics, the importance of problem solving, and pencil-and-paper skills, and the use of calculators and computers.

Leaders representing subject associations, OAME (Ontario Association for Mathematics Education) and OMCA (Ontario Mathematics Coordinator's Association), embraced the guiding principles in this document. They were buoyed by statements, such as, "Students engage in problem solving in all strands …"; "In analyzing problems and presenting solutions, students are expected to use technology effectively"; and "They should also use problem-solving methods extensively as a means of developing the full range of mathematical skills and knowledge in all strands." References to the importance of communication in learning mathematics and the importance of garnering positive attitudes towards students were well received. On the other hand, there was a certain degree of skepticism about the reference to the statement, "This provision of detail will eliminate the need for school boards to write their own expectations, will ensure consistency in curriculum across the province, and *will facilitate province-wide testing"* (my italics).

If there was any one aspect of this document that was truly at variance with previous practice, it was the change in the assessment and evaluation paradigm. Not only reporting where initially teachers were required to report on all five strands in each of the three terms (this was set to be phased in over a couple of years) but also, assessing and grading based on "Levels of Performance" across four "Categories". (Active lobbying by the OMCA and OAME led to a shift in Ministry policy regarding reporting. Thereafter teachers in elementary grades were required to report on each strand a minimum of twice per school year in conjunction with the requirement to report on at least 2 strands in each term.) The four Categories are: (1) Problem solving, (2) Understanding of concepts, (3) Application of mathematical procedures, and (4) Communication of required knowledge related to concepts, procedures, and problem solving. This shift could be characterized as the move from a "norm-referenced" to a "criterion-referenced" approach to assessment and evaluation.

By 1998, province-wide Grade 3 census testing of mathematics had begun. The EQAO (The Education Quality and Accountability Office), an arms lengthy agency of the provincial government, developed, field-tested, evaluated, and reported the results of the first Grade 3 assessment (initially a test that was administered over two weeks occupying a portion of each school day). In ensuing years, EQAO introduced both the Grade 6 assessments and the Grade 9 Applied and Academics Mathematics assessments.

Implementation of the Elementary Curriculum

On paper, the curriculum appeared to be a positive step forward. Nevertheless, new curriculum implementation is not an event, it is a process. Everyone from Ministry officials, to mathematics consultants and coordinators working centrally in boards, and teachers agreed that full implementation would take at least five years.

If there was a master plan for implementation, it was not readily evident. Nevertheless, some aspects of implementation did occur in a timely manner. Textbooks were provincially approved and funds were provided to schools for their purchase (two books for Grades 1–6 courses and four books for Grades 7 and 8). There has continued to be significant controversy over the effectiveness and usability of this new generation of texts. Although some teachers embraced these new resources, many teachers have relegated them to the shelves and resorted to previous books. For the new books to be used properly, teachers required high quality training that generally was not readily available. Inservice for teachers was most certainly uneven across the province. Those school boards that had centrally assigned curriculum specialist in mathematics and budgets for both after school and during school workshops were able to start to sow the seeds of change among their teachers. On the other hand, a significant number of school boards (especially rural and northern boards) in Ontario had no central mathematics subject specialists and / or had few resources to put to inservice. It is also critical to note that the implementation of this curriculum paralleled the implementation of all other subjects' curricula and was at a time when all boards were having to cut budgets (especially central budgets) to meet the provincial funding formula.

Is there evidence in 2003 that there have been shifts in what and how teachers deliver curriculum? The answer is, "Yes!" How can this be so given the spotty approach to teacher training? Organizations, such as, the OAME and OMCA worked very hard to support the implementation through their publications, conferences, and workshops. Perhaps the greatest influence, at least initially, was the role of province-wide EQAO testing of mathematics in Grades 3 and 6. EQAO ensured that all testing instruments were developed by teachers, field tested by teachers, and marked by teachers. Eighty percent of the tests were comprised of full response tasks that up until last year were marked by levels of performance holistically. The remaining 20% was multiple choice. The impact on teaching can be attributed to two influences. Firstly, all Grade 3 and 6 teachers administering the test would become familiar with the style and substance of full-response tasks that addressed all strands and categories. Secondly, great value for the implementation of curriculum derived from the massive training of developers and markers. Over the course of two weeks each summer, nearly 2000 teachers were engaged in the language of categories, strands, and levels of performance. Further, they discussed the qualities of hundreds of student responses to arrive at consensus for grading. Teachers more often than not returned to their classrooms in the fall better prepared to deliver the intended curriculum. Obviously, the results of EQAO tests were also badly used by the press and misinterpreted by the public especially where schools in various districts were ranked or real estate values were affected.

Recently, there have been two major developments in the implementation of the elementary mathematics curriculum. First, at least four publishers are actively preparing new textbooks to meet the needs of the new curriculum and respond to the concerns about the books that were originally released in the late nineties. Second, the Ministry of Education has entered the fray by initiating the *Early Math Strategy.* This well-funded program provides for in depth training for a lead primary mathematics teacher in every elementary school in Ontario. An average of $2000 will be provided to every elementary school in Ontario to purchase manipulatives, calculators, and other teacher resources. Teachers and administrators will also receive training to assist them in analyzing their school and EQAO data in order to set targets (improvement in EQAO scores).

Secondary

Secondary curriculum renewal in Ontario began in earnest in early 1997, Geoffrey Roulet from Queen's University wrote a background research paper entitled *Mathematics Curriculum for Ontario Secondary Schools: Issues, Choices and Options.* He identified three visions for teaching and learning mathematics—instrumentalism, formalism, and problem solving or social-constructivism. Roulet states that the curriculum writers must be clear about which philosophy will underpin their writing and be consistent and true to it.

Curriculum renewal in Ontario has been evolving since the publication of the NCTM *Curriculum and Evaluation Standards for School Mathematics* in 1989. OAME, OMCA, and more recently the Field's Institute Mathematics Education Forum have actively engaged in the conversation about secondary school mathematics curriculum. Building consensus among mathematics teachers, secondary mathematics heads, board consultants and coordinators, university and college professors, faculty of education professors, parents, Ministry officials, and other interested parties has been at the very least, difficult. Nevertheless, an expert panel was struck to construct a framework for a new curriculum even though it would not be necessarily binding to the writers. The panel was comprised of representatives from secondary schools, board coordinators and consultants, college and university professors, and workers from business and the professions. Their report highlighted the importance inquiry-based learning, investigations, mathematical communication, mathematical modeling, appropriate use of technology, and the importance of proper implementation. These tenets were and are very much in line with the NCTM *Standards.*

The Field's Institute successfully applied to the Ministry of Education to manage the secondary mathematics writing team. Two documents were produced:

Mathematics: The Ontario Curriculum, Grades 9 and 10 (1999)
Mathematics: The Ontario Curriculum, Grades 11 and 12 (2000)

Implementation of these policy statements would be staged—Grade 9 in 1999, Grade 10 in 2000, Grade 11 in 2001, and Grade 12 in 2002—unlike the elementary mathematics curriculum where all 8 grades were introduced simultaneously.

In replacing the 1985 documents, *Mathematics Part One (Basic), Mathematics Part Two (General),* and *Mathematics Part Three (Advanced),* and the 1995 *The Common Curriculum: Policies and Outcomes, Grades 1–9,* the mathematics program was changed from five years to four years. Many students under the old curriculum, in practice, took most of their Ontario Academic Credits in their fifth year (Grade 13). Very few topics in the five-year program were eliminated (the most notable omissions were trigonometric differentiation and integration from the new Calculus, Advanced Functions, and Introductory Calculus) so that one could characterize the new curriculum as a compressed version of the former program. The second noteworthy change was the shift to "destination-based" exit courses. That is, courses would be designated as "University", "University / College", "College", or "Workplace".

In general, each course was structured in a similar manner to the elementary curriculum. Each course is comprised of three or four strands. For example, Grade 10 Foundations of Mathematics is organized by the strands "Proportional Reasoning", "Linear Functions", and "Quadratic Functions". Each strand includes both overall and specific expectations.

As with the elementary curriculum, the most significant shift was in the area of assessment and evaluation. Student work must be judged using the criteria and levels of performance as outlined in the mathematics Achievement Chart. Each of four categories, Knowledge and Understanding; Thinking, Inquiry, and Problem Solving; Communication; and Application must be taken into account to determine the grade. This move from "norm-referencing" to "criterion-referencing" has been controversial in the mathematics community.

Implementation of the Secondary Curriculum

The greatest divergence between the implementation of the elementary and secondary curricula was in the area of Ministry support. Where there was little support for elementary mathematics teachers until more recently, secondary mathematics teachers were provided with a slew of resources and opportunities for inservice. There were, however, significant issues that went unaddressed.

A number of textbook publishers were quick off the mark to provide resource packages (i.e., textbooks, test banks, blackline masters, teacher resource, online support, etc.) for "main stream" mathematics courses (e.g., Grades 9 and 10 Principles of Mathematics (academic) and Advanced Functions and Introductory Calculus). The Ministry of Education provided funds to all high schools to purchase both textbooks and in 1999, graphing calculators, motion sensors, and other technology peripherals. The Ministry also commissioned the development of "course profiles". The contents of these resource documents included one approach to a scope and sequence by clustering expectations and providing a lesson plan with a student activity. Other useful resources were referenced as well. Full profiles (both public and Catholic school) were written for the Grade 9 Applied and Academic Mathematics, for example, but partial profiles (some, not all, of the units fully fleshed out) were written for all other courses (with the exception of the "locally developed Grade 10 Essentials Mathematics). "Grade 9 Exemplars" were also produced to inform assessment and evaluation. OAME and OMCA also published resources, such as, the CARE package that was created to assist teachers. Most recently, the OAME is producing their "vision" of mathematics that will include a statement of principles, a video to show the continuum of learning in the Mathematics classroom from Kindergarten to Grade 12, and teacher support materials. A number of school boards were also active in creating materials for teachers to use.

This partial list of workshops, inservices, and conferences is indicative of the effort to help secondary teachers implement the new curriculum:

- The Ministry of Education has provided funds to inservice teachers (primarily funds for release time) annually (1999—Grade 9 curriculum, 2000—Grade 10 curriculum, …);
- The OAME and OMCA developed and delivered Ministry-funded summer institutes in 13 locations across Ontario to support the use of graphing calculators in mathematics classrooms;
- Summer institutes to help mathematics teachers to integrate both dynamic geometry software (GSP—Geometer's Sketchpad) and dynamic statistics software (Fathom) into their programs were supported; and
- The OAME and OMCA annually sponsor mathematics conferences.

EQAO testing for Grade 9 students was introduced three years ago. The vision was to provide an instrument that included multiple choice, short response, and long response items. Students would also be required to carry out an "investigation" over 2 or 3 periods that would involve the use of technology (e.g., dynamic geometry software or graphing calculators). EQAO expressed concern that a technology-based investigation would not be fair because the implementation and availability of technology across Ontario was uneven. As a result the technology-based investigation was never implemented as a scored element of the test.

Is there evidence in 2003 that there have been shifts in what and how secondary teachers deliver curriculum? At very best, implementation has been spotty. Are teachers addressing the expectations in the new curriculum? In general, they are but as with the introduction of any new course the timing and the organization would require fine-tuning over time. Have the pedagogical and assessment tenets of the new curriculum been widely adopted? In some departments and by some individuals the answer is "yes". There is, however, substantial anecdotal evidence that teachers are facing challenges in adopting the use of technology, building investigations and inquiries into the their programs, employing an active learning approach, and implementing a criterion-based approach to assessment and evaluation.

One key reason for this sluggish implementation is the very low morale among most secondary school teachers. From their perspective, they see larger classes, inadequate resources, materials that do not meet the needs of all students (e.g., ESL students and "low-functioning" students), fewer department heads, fewer math "specialists" in their classrooms, tons of Ministry policies being "dumped" on them (Secondary School Reform policies that cut across all subject areas), and a massive change in the approach to assessment and evaluation. Many will say that pace and depth of change was unreasonable and unworkable. Lastly, teacher attitudes, for the most part, have been tainted by the political backdrop that has existed in Ontario for the last eight years. Most feel that they have been demeaned or, at the very least, unappreciated. Many teachers feel that they have been working in an environment of mutual mistrust.

Summary of Key Issues in Mathematics Education in Ontario

Implementation

Neither the elementary curriculum (as introduced in 1997) nor the secondary curriculum (as introduced in 1999) is anywhere near full implementation. Even though teachers have generally constructed their programs around the expectations listed in the policy documents, many teachers are struggling with the philosophical underpinnings and the practical implementation of the best current thought about learning and teaching mathematics. Learning through inquiry, using manipulatives and technology, and employing teacher-student or student-student discourse in large or small groups are all examples of techniques where practice has been shown to be effective in contrast to the traditional "chalk and talk" style that was so prevalent not so very long ago. The required change from norm-referenced to criterion-referenced assessment and evaluation has been the source of greatest angst and resistance by teachers across Ontario. Much of teacher reluctance to adopt new methods is either due to their lack of confidence in their own mathematical abilities (especially in the elementary panel) and/or their lack of current knowledge about how children learn mathematics.

Only recently the Ministry of Education has committed to significant resources to provide inservice for elementary school teachers (The Early Math Strategy). Support for secondary school mathematics has been ongoing in a rather piecemeal fashion.

Publishers and organizations such as the OMCA and OAME have attempted to fill the gaps by developing a wide variety of resources and providing workshops.

Technology

It is important to take a special look at the use of technology in the Ontario mathematics classroom. At all levels of instruction, calculators and other rich technologies can be used effectively to improve instruction. Students from the very youngest to the those about to graduate high school are universally "tuned" into the use of technology in their every day lives. Students are motivated to learn when they are invited into a technological world. They can "see" mathematics when using graphing calculators and "kinesthetically feel" mathematics when using motion detectors. They can sense the limitless possibilities when dynamically exploring geometry or statistics in a software environment.

One of the most positive developments in Ontario mathematics has been the province-wide licensing of dynamic geometry and statistics software packages (Geometer's Sketchpad and Fathom) by the Ministry of Education. Even though this is the case, there is evidence to suggest that the adoption rate of these and other powerful technologies has been relatively low. Not nearly enough teachers have received the in depth training required to not only be comfortable with the technology (i.e., pushing the buttons), but also how to effectively create an environment of learning mathematics using technology. Many teachers will also point to access problem—not enough dedicated computer time for mathematics instruction.

Teacher Shortage

The Ontario College of Teachers has publicly stated that there is a looming shortage of qualified secondary school mathematics teachers. Graduates from university mathematics and associated programs are opting for more lucrative opportunities in business. Even if remuneration is not the driving force behind their decision, in many cases the constant assault on teachers by governments and various factions in society certainly is. Teacher morale couldn't be worse and this has led many excellent teachers to retire early or seek alternative careers.

Although the shortage of teachers at the elementary level may not be so acute, the proportion of those who enjoy mathematics, understand the content, and know how children learn is disturbingly small. This is particularly problematic in middle schools in Ontario.

On-line Learning

Many boards and the Ministry of Education have been quick to jump on the bandwagon that students should have opportunities to learn mathematics on-line. The Field's Institute Mathematics Education Forum has sponsored two symposia to discuss issues that need to be addressed by any provider that wishes to offer on-line mathematics courses. The general consensus is that on-line learning environments may be beneficial in small doses (modules to enhance the mathematics classroom) but for the vast majority of students full courses would not be conducive to effective learning.

The Public

The public's view of the subject of mathematics and mathematics education is coloured by a wide array of influences. First and foremost, many perceive that the way they were taught mathematics is the way children should be taught today—not only in terms of the methodology (primarily a rote approach to learning), but content as well. Many of these people also profess to have had a great distaste for mathematics or are even "phobic" about the mathematics that exists in their day-to-day lives. Another significant influence is the media. Editorials, feature stories, and interviews quite often are biased toward the failure of mathematics education (based on ostensibly poor EQAO results or Ontario's ranking in international testing). Frequently, writers and broadcasters get on to the "back to the basics" bandwagon. Even when evidence suggests that their conclusions may be questionable, public policy is often influenced by the opinions they express. Although, those involved in mathematics education have been successful in convincing the public that mathematics is a highly important component of a child's education, they have done a poorer job at helping the public understand what it is to be a "numerate" citizen and how children best learn mathematics.

Conclusions

In Ontario there is a new curriculum. There is a strong belief that more students will become better problem solvers even though some of their arithmetic and algebraic skills may be somewhat weaker. Nevertheless it is believed that the keys to improvement in student learning include more comprehensive and systematic teacher training; improvements in and availability of text, manipulative, supplementary resources such as print copies of rich tasks and activities, calculators, and computer software; attracting more qualified teachers into the system; and changing the public's perception of the nature of modern mathematics, how it is practiced, and how it is best learned.

Atlantic Canada Mathematics Curriculum and Nova Scotia Implementation

Donna Karsten
Nova Scotia Department of Education

Curriculum Framework

History

1989	NCTM Standards
1993	Maritime Provinces Education Foundation (MPEF)
1994	Framework for high school documents started
1995	Atlantic Provinces Education Foundation (APEF)
	New GCO/ SCO Structure elementary guides started
1996	Foundation for the Atlantic Canada Mathematics Curriculum
1997	Implementation began for Grades Primary–6
1998	Grades 7–8 Guides and Implementation
1999	Grades 9–10 Guides and Implementation
2000	Mathematics 11 Guides and Implementation
2001	Mathematics 12 Guides and Implementation
2002	Pre-Calculus 12 Guides Implementation

Vision

The Atlantic Canada mathematics curriculum is shaped by a vision that fosters the development of mathematically literate students who can extend and apply their learning and who are effective participants in an increasingly technological society.

Outcomes Framework

- Essential Graduation Learnings (EGLs)
- General Curriculum Outcomes (GCOs) – broad mathematical expectations
- Key-stage Curriculum Outcomes (KSCOs) – at the end of Grades 3, 6, 9, and 12
- Specific Curriculum Outcomes (SCOs) – for each grade level and course

Outcomes Frameworks are based on the four strands advocated by the National Council of Teachers of Mathematics (NCTM):

- *Number Concept/Number and Relationship Operation* (GCO A: Students will demonstrate number sense and apply number theory concepts; GCO B: Students will demonstrate operation sense and apply operation principles and procedures in both numerical and algebraic situations.)
- *Patterns and Relations* (GCO C: Students will explore, recognize, represent, and apply patterns and relationships, both informally and formally.)
- *Shape and Space* (GCO D: Students will demonstrate an understanding of and apply concepts and skills associates with measurement; GCO E: Students will demonstrate spatial sense and apply geometric concepts, properties, and relationships.)

· *Data Management and Probability* (GCO F: Students will solve problems involving the collection, display, and analysis of data; GCO G: Students will represent and solve problems involving uncertainty.)

The Big Five

· Communication Standard
· Problem Solving Standard
· Connections Standard
· Reasoning and Proof Standard
· Representation Standard

Levels of Questions

· Level 1 – Knowledge and Procedure
· Level 2 – Comprehension of Concepts and Procedures
· Level 3 – Application and Problem Solving

Implementation

Model

The "train the trainer" model was use. The trainers received 5 days of in-service, and that worked well for them. They were then asked to support implementation within their boards.

Assessment

A Provincial Elementary Mathematics Program Assessment was administered in 2001 in Grade 5. Results were disappointing, about 42% average. This assessment, which addressed Grades 3, 4, and 5 outcomes, was administered again in May 2003.

A Provincial Junior High Mathematics Program Assessment was administered in Grade 8 in May 2002. Results were very disappointing, with an average of 32%. This assessment, which addressed Grades 6, 7, and 8 outcomes, will be administered again in May 2004.

Mathematics Strategy

In response to the Assessments results, in 2002–2003 the Nova Scotia Department of Education developed and began the implementation of the Mathematics Strategy. This strategy has several components:

· *Regional Leaders* – A strategy was put in place to provide target funding in school boards' profile sheet for a mathematics implementation support person. By August 2003, all seven boards will have at least one person in place whose only responsibilities centre on mathematics.

· *School-Based Leaders* – The "train the trainer" model was now brought into question. What were its shortcomings? The answer was quite apparent: The trainers only received two days to work with teachers in their district. Trainers were not given any release time to work with teachers at the classroom level. A new model was needed. A decision to identify and support leaders at the school-site was made and the process was started. To establish these school-based leaders, 880 teachers were given two days of PD.

· *"Mathematics: A Teaching Resource"* – Ten new support documents, *Mathematics: A Teaching Resource*, one for each grade Primary–9, were developed. They each contain four sections:

  1. Planning (year, unit, and lesson)
  2. Mathematical language
  3. Assessment
  4. Technology

  The documents are currently available in a draft version and are in the process of being finalized.

· *Time to Learn Strategy* – The Time to Learn Strategy was introduced to provide a minimum amount of mathematics instruction from Primary–9. A minimum of 45 minutes per day for Grades Primary–2. A minimum of 1 hour per day for Grades 3–9. A minimum of 5 minutes per day for mental math and estimation, Grades 1–9.

· *Resources* – A commitment has been made to providing instructional resources to support mathematics teaching and learning. Purchases in the past few years have included:

Core Resources, Grades 10–12
Technology: TI-83, Computers (Geometer's Sketchpad, Secondary Toolkit plus more)
Manipulatives: Fraction Blocks and Binder
Mental math: *Mental Math in the Primary Grades*, *Mental Math in the Middle Grades*, *Mental Math in Junior High*, *Daily Tune-Ups Grades K–6*.
Literacy Connections: *Side by Side* Library, *PM Maths* Series, *I Get It!*

Some purchases planned for next few years:

New Core Resources, Grades Primary–9

· *Outcomes Frameworks to Inform Professional Development – An Outcomes Framework to Inform Professional Development for Mathematics Teachers* and *An Outcomes Framework to Inform Professional Development for Administrators* are being developed.

Summer Institutes

Fourteen two-day summer institutes will be offered during the week of August 18–22, 2003.

Mathematics Strategy: Ongoing Initiatives

· Home/School Communication
· Homework Projects
· Geometer's Sketchpad Use
· Web-based question bank for secondary teachers
· Mathematics Strategy PD Grants

Professional Development Grant

These grants are intended to supplement existing funding provided by boards for professional development in mathematics. The purposes of these grants are to:

1. Firmly establish mathematics leaders at each school, Grades Primary–9.
2. Support school-based mathematics leaders by addressing their own professional growth needs.
3. Expand professional development opportunities for all teachers of mathematics, Grades Primary–9.
4. Provide opportunities for some school-based mathematics leaders to mentor one or more of their colleagues.

Mathematics Strategy: Up-coming Initiatives

· Professional development for administrators
· Increase co-operation among stakeholders in Mathematics education
· Math Links document
· Algebra task force
· Professional growth through use of Annenburg Project
· Design of a Pathways document

Mathematics Curriculum in Canada

Tom O'Shea
Simon Fraser University

Education in each province is controlled to a greater or lesser extent by a ministry or department of education of the provincial government. Each ministry has its own mandate to establish curricula and to evaluate student performance. In most provinces, the ministry provides detailed curriculum guides for each subject area; about half the provinces require students to write centrally set and marked examinations as part of the final evaluation for high school graduation. There are similarities in the ways the provinces develop and examine curricula, and some measure of common purpose can be ascertained by examining curriculum guides, research reports, provincial mathematics teacher journals, and, more recently, international studies of mathematics curriculum and achievement.

The curriculum model which served as an organizing framework for this presentation was adopted from the Second International Mathematics Study. In this model, curriculum can be thought of as intended (by curriculum developers), implemented (as realized in the classroom), and attained (as demonstrated by student achievement and attitudes). The focus of the presentation was on how the intended curriculum has developed over time, as determined through an examination of official documents and journal articles. Some attention was also given to the implemented curriculum, as described in research reports and assessment documents.

The Mathematics Curriculum

The NCTM *Standards* clearly have been a major influence in curriculum design. They formed the kernel about which curriculum reform coalesced. Their influence is evident in the provincial curriculum documents and within the frameworks set out by two consortia (The Western and Northern Canadian Protocol for Collaboration in Basic Education, and the Atlantic Provinces Education Foundation). The result is a de facto "Canadian" curriculum, at least for the grades prior to the senior level. The eventual construction of a national curriculum seems to be a natural extension of the moves to regionalize mathematics curricula. The forerunner of this possibility in mathematics may be seen in the "pan-Canadian" science curriculum developed under the auspices of the CMEC's 1995 Pan-Canadian Protocol for Collaboration in School Curriculum.

While a national curriculum would benefit students moving across jurisdictions and would assure post-secondary institutions and employers that students had been exposed to a common set of learning expectations, there has been little examination of the negative ramifications of such a move. One would expect, at least, some discussion of the potential elimination of much of the involvement and experimentation of teachers at the local and provincial level.

Concomitant with the move toward a standardized curriculum is the expectation that more students must take more mathematics during their school years. This increasing expectation of mathematical competence exacerbates the problem of designing suitable mathematical experiences for students with little interest in, or aptitude for, the subject.

The requirement for more mandatory mathematics education for more students also resulted in the extension of the common mathematics curriculum within provinces to a higher grade than formerly. Previously, most provinces allowed students to opt into different mathematics tracks starting in Grade 9. At present , however, differentiation of curriculum usually begins in Grade 10.

The Mathematics Curriculum Guide

The curriculum guide itself was transformed over the years. At mid-century, guides typically took the form of a syllabus that simply listed the mathematical topics that were to be discussed at or across grade levels. Now, however, curriculum guides can more generally be seen as teaching resources. The curriculum is usually defined by a number of increasingly specific learning expectations, each accompanied by a mathematical problem to illustrate what might be expected of students working at that grade level. Guides also contain suggested instructional and assessment strategies, glossaries of terms, and lists of recommended teacher resource materials. In general, these documents provide useful addenda to a curriculum increasingly dominated by atomistic behavioral objectives.

The Curriculum Development Process

What was once a secretive process, in which committee members were not allowed to discuss their deliberations prior to official approval, changed to one marked by openness and extensive consultation. At mid-century, when curriculum guides were centered on a single or a few authorized textbooks, advance notice of curriculum change conferred advantage to publishers seeking to have their material selected as the single authorized textbook. By the 1990s, the openness by which curriculum was negotiated was beneficial to two parties: publishers knew that they were able to produce materials congruent with the curriculum, and education authorities were confident that materials would be available for teachers when the changes were put into place.

The Match Between the Implemented and Intended Curriculum

The implemented curriculum can only be determined through classroom observation and student and teacher self-report. Although few studies in Canada have been undertaken in this regard, they suggest that teachers may have been more influenced by the textbooks they used than by the official curriculum guide. Program assessment reports in British Columbia in the 1980s consistently pointed out the discrepancy between teachers' self-reports of what they taught and what was suggested in official documents, and expressed concern about the lack of time teachers devoted to geometry, probability, and statistics, all topics new to the curriculum. On the other hand, the results of the 1990 British Columbia Mathematics Assessment indicated that Grades 4, 7, and 10 teachers covered or intended to cover at least 85 percent of the prescribed. This may have reflected the growing match between the content of authorized textbooks and the ministry's curriculum.

Summary

The changes in the mathematics curriculum in Canada over the second half of the twentieth century paralleled curriculum movements worldwide. Canadian provinces experienced the euphoria of the new Math movement, and the subsequent sobering reaction from those proposing to return to a stronger emphasis on the basics. They also responded to the calls for change contained in the NCTM's *Agenda for Action* and made problem solving central to the mathematics curriculum of the 1980s. The influence of the NCTM Standards in the curriculum of the 1990s was very strong.

The nature of mathematics curriculum making in Canada changed dramatically over the second half of the twentieth century. The changes took a number of different forms: from individual provincial curricula to a nascent national curriculum; from an almost secretive bureaucratic process to an open process in collaboration with teachers; and from a view of curriculum as a syllabus or list of content topics to the view of a curriculum as a guide to content, teaching, and evaluation. The content of the intended curriculum also changed, and those changes were consistent with changes taking place in other countries, but attenuated. New topics were added to the curriculum, most notably statistics and probability, and curriculum developers wrestled with the problem of technology and its role in teaching mathematics.

Curriculum developers struggled with the issues of what mathematics was suitable for which students. In general, more mathematics became required of all students. Furthermore, within each province the curriculum common to all students shifted from the end of elementary school to a later stage in school, typically to the end of Grade 9. The mathematical curriculum for students not intending to pursue post-secondary education continued to be a problem, and appeared to remain unresolved.

In spite of mandated and suggested changes in the intended curriculum there were fewer changes in the implemented curriculum. There is doubt, for example, as to the extent to which teachers embraced problem solving or the use of technology.

Appendices

Appendices

APPENDIX A

Working Groups at Each Annual Meeting

1977 *Queen's University, Kingston, Ontario*

- Teacher education programmes
- Undergraduate mathematics programmes and prospective teachers
- Research and mathematics education
- Learning and teaching mathematics

1978 *Queen's University, Kingston, Ontario*

- Mathematics courses for prospective elementary teachers
- Mathematization
- Research in mathematics education

1979 *Queen's University, Kingston, Ontario*

- Ratio and proportion: a study of a mathematical concept
- Minicalculators in the mathematics classroom
- Is there a mathematical method?
- Topics suitable for mathematics courses for elementary teachers

1980 *Université Laval, Québec, Québec*

- The teaching of calculus and analysis
- Applications of mathematics for high school students
- Geometry in the elementary and junior high school curriculum
- The diagnosis and remediation of common mathematical errors

1981 *University of Alberta, Edmonton, Alberta*

- Research and the classroom
- Computer education for teachers
- Issues in the teaching of calculus
- Revitalising mathematics in teacher education courses

1982 *Queen's University, Kingston, Ontario*

- The influence of computer science on undergraduate mathematics education
- Applications of research in mathematics education to teacher training programmes
- Problem solving in the curriculum

1983 *University of British Columbia, Vancouver, British Columbia*

- Developing statistical thinking
- Training in diagnosis and remediation of teachers
- Mathematics and language
- The influence of computer science on the mathematics curriculum

1984 *University of Waterloo, Waterloo, Ontario*

- Logo and the mathematics curriculum
- The impact of research and technology on school algebra
- Epistemology and mathematics
- Visual thinking in mathematics

1985 *Université Laval, Québec, Québec*

- Lessons from research about students' errors
- Logo activities for the high school
- Impact of symbolic manipulation software on the teaching of calculus

1986 *Memorial University of Newfoundland, St. John's, Newfoundland*

- The role of feelings in mathematics
- The problem of rigour in mathematics teaching
- Microcomputers in teacher education
- The role of microcomputers in developing statistical thinking

1987 *Queen's University, Kingston, Ontario*

- Methods courses for secondary teacher education
- The problem of formal reasoning in undergraduate programmes
- Small group work in the mathematics classroom

1988 *University of Manitoba, Winnipeg, Manitoba*

- Teacher education: what could it be?
- Natural learning and mathematics
- Using software for geometrical investigations
- A study of the remedial teaching of mathematics

1989 *Brock University, St. Catharines, Ontario*

- Using computers to investigate work with teachers
- Computers in the undergraduate mathematics curriculum
- Natural language and mathematical language
- Research strategies for pupils' conceptions in mathematics

1990 *Simon Fraser University, Vancouver, British Columbia*

- Reading and writing in the mathematics classroom
- The NCTM "Standards" and Canadian reality
- Explanatory models of children's mathematics
- Chaos and fractal geometry for high school students

1991 *University of New Brunswick, Fredericton, New Brunswick*

- Fractal geometry in the curriculum
- Socio-cultural aspects of mathematics
- Technology and understanding mathematics
- Constructivism: implications for teacher education in mathematics

1992 *ICME–7, Université Laval, Québec, Québec*

1993 *York University, Toronto, Ontario*

- Research in undergraduate teaching and learning of mathematics
- New ideas in assessment
- Computers in the classroom: mathematical and social implications

· Gender and mathematics
· Training pre-service teachers for creating mathematical communities in the classroom

1994 *University of Regina, Regina, Saskatchewan*

· Theories of mathematics education
· Pre-service mathematics teachers as purposeful learners: issues of enculturation
· Popularizing mathematics

1995 *University of Western Ontario, London, Ontario*

· Autonomy and authority in the design and conduct of learning activity
· Expanding the conversation: trying to talk about what our theories don't talk about
· Factors affecting the transition from high school to university mathematics
· Geometric proofs and knowledge without axioms

1996 *Mount Saint Vincent University, Halifax, Nova Scotia*

· Teacher education: challenges, opportunities and innovations
· Formation à l'enseignement des mathématiques au secondaire: nouvelles perspectives et défis
· What is dynamic algebra?
· The role of proof in post-secondary education

1997 *Lakehead University, Thunder Bay, Ontario*

· Awareness and expression of generality in teaching mathematics
· Communicating mathematics
· The crisis in school mathematics content

1998 *University of British Columbia, Vancouver, British Columbia*

· Assessing mathematical thinking
· From theory to observational data (and back again)
· Bringing Ethnomathematics into the classroom in a meaningful way
· Mathematical software for the undergraduate curriculum

1999 *Brock University, St. Catharines, Ontario*

· Information technology and mathematics education: What's out there and how can we use it?
· Applied mathematics in the secondary school curriculum
· Elementary mathematics
· Teaching practices and teacher education

2000 *Université du Québec à Montréal, Montréal, Québec*

· Des cours de mathématiques pour les futurs enseignants et enseignantes du primaire/Mathematics courses for prospective elementary teachers
· Crafting an algebraic mind: Intersections from history and the contemporary mathematics classroom
· Mathematics education et didactique des mathématiques : y a-t-il une raison pour vivre des vies séparées?/Mathematics education et didactique des mathématiques: Is there a reason for living separate lives?
· Teachers, technologies, and productive pedagogy

2001 *University of Alberta, Edmonton, Alberta*

· Considering how linear algebra is taught and learned
· Children's proving

· Inservice mathematics teacher education
· Where is the mathematics?

2002 *Queen's University, Kingston, Ontario*

· Mathematics and the arts
· Philosophy for children on mathematics
· The arithmetic/algebra interface: Implications for primary and secondary mathematics / Articulation arithmétique/algèbre : Implications pour l'enseignement des mathématiques au primaire et au secondaire
· Mathematics, the written and the drawn
· Des cours de mathémathiques pour les futurs (et actuels) maîtres au secondaire / Types and characteristics desired of courses in mathematics programs for future (and in-service) teachers

APPENDIX B

Plenary Lectures at Each Annual Meeting

1977	A.J. COLEMAN	The objectives of mathematics education
	C. GAULIN	Innovations in teacher education programmes
	T.E. KIEREN	The state of research in mathematics education
1978	G.R. RISING	The mathematician's contribution to curriculum development
	A.I. WEINZWEIG	The mathematician's contribution to pedagogy
1979	J. AGASSI	The Lakatosian revolution
	J.A. EASLEY	Formal and informal research methods and the cultural status of school mathematics
1980	C. GATTEGNO	Reflections on forty years of thinking about the teaching of mathematics
	D. HAWKINS	Understanding understanding mathematics
1981	K. IVERSON	Mathematics and computers
	J. KILPATRICK	The reasonable effectiveness of research in mathematics education
1982	P.J. DAVIS	Towards a philosophy of computation
	G. VERGNAUD	Cognitive and developmental psychology and research in mathematics education
1983	S.I. BROWN	The nature of problem generation and the mathematics curriculum
	P.J. HILTON	The nature of mathematics today and implications for mathematics teaching
1984	A.J. BISHOP	The social construction of meaning: A significant development for mathematics education?
	L. HENKIN	Linguistic aspects of mathematics and mathematics instruction
1985	H. BAUERSFELD	Contributions to a fundamental theory of mathematics learning and teaching
	H.O. POLLAK	On the relation between the applications of mathematics and the teaching of mathematics

1986	R. FINNEY	Professional applications of undergraduate mathematics
	A.H. SCHOENFELD	Confessions of an accidental theorist
1987	P. NESHER	Formulating instructional theory: the role of students' misconceptions
	H.S. WILF	The calculator with a college education
1988	C. KEITEL	Mathematics education and technology
	L.A. STEEN	All one system
1989	N. BALACHEFF	Teaching mathematical proof: The relevance and complexity of a social approach
	D. SCHATTSNEIDER	Geometry is alive and well
1990	U. D'AMBROSIO	Values in mathematics education
	A. SIERPINSKA	On understanding mathematics
1991	J .J. KAPUT	Mathematics and technology: Multiple visions of multiple futures
	C. LABORDE	Approches théoriques et méthodologiques des recherches françaises en didactique des mathématiques
1992	ICME-7	
1993	G.G. JOSEPH	What is a square root? A study of geometrical representation in different mathematical traditions
	J CONFREY	Forging a revised theory of intellectual development: Piaget, Vygotsky and beyond
1994	A. SFARD	Understanding = Doing + Seeing ?
	K. DEVLIN	Mathematics for the twenty-first century
1995	M. ARTIGUE	The role of epistemological analysis in a didactic approach to the phenomenon of mathematics learning and teaching
	K. MILLETT	Teaching and making certain it counts
1996	C. HOYLES	Beyond the classroom: The curriculum as a key factor in students' approaches to proof
	D. HENDERSON	Alive mathematical reasoning
1997	R. BORASSI	What does it really mean to teach mathematics through inquiry?
	P. TAYLOR	The high school math curriculum
	T. KIEREN	Triple embodiment: Studies of mathematical understanding-in-interaction in my work and in the work of CMESG/GCEDM
1998	J. MASON	Structure of attention in teaching mathematics
	K. HEINRICH	Communicating mathematics or mathematics storytelling

1999	J. BORWEIN	The impact of technology on the doing of mathematics
	W. WHITELEY	The decline and rise of geometry in 20^{th} century North America
	W. LANGFORD	Industrial mathematics for the 21^{st} century
	J. ADLER	Learning to understand mathematics teacher development and change: Researching resource availability and use in the context of formalised INSET in South Africa
	B. BARTON	An archaeology of mathematical concepts: Sifting languages for mathematical meanings
2000	G. LABELLE	Manipulating combinatorial structures
	M. BARTOLINI BUSSI	The theoretical dimension of mathematics: A challenge for didacticians
2001	O. SKOVSMOSE	Mathematics in action: A challenge for social theorising
	C. ROUSSEAU	Mathematics, a living discipline within science and technology
2002	D. BALL & H. BASS	Toward a practice-based theory of mathematical knowledge for teaching
	J. BORWEIN	The experimental mathematician: The pleasure of discovery and the role of proof

APPENDIX C

Proceedings of Annual Meetings

Past proceedings of CMESG/GCEDM annual meetings have been deposited in the ERIC documentation system with call numbers as follows:

Proceedings of the 1980 Annual Meeting	ED 204120
Proceedings of the 1981 Annual Meeting	ED 234988
Proceedings of the 1982 Annual Meeting	ED 234989
Proceedings of the 1983 Annual Meeting	ED 243653
Proceedings of the 1984 Annual Meeting	ED 257640
Proceedings of the 1985 Annual Meeting	ED 277573
Proceedings of the 1986 Annual Meeting	ED 297966
Proceedings of the 1987 Annual Meeting	ED 295842
Proceedings of the 1988 Annual Meeting	ED 306259
Proceedings of the 1989 Annual Meeting	ED 319606
Proceedings of the 1990 Annual Meeting	ED 344746
Proceedings of the 1991 Annual Meeting	ED 350161
Proceedings of the 1993 Annual Meeting	ED 407243
Proceedings of the 1994 Annual Meeting	ED 407242
Proceedings of the 1995 Annual Meeting	ED 407241
Proceedings of the 1996 Annual Meeting	ED 425054
Proceedings of the 1997 Annual Meeting	ED 423116
Proceedings of the 1998 Annual Meeting	ED 431624
Proceedings of the 1999 Annual Meeting	ED 445894
Proceedings of the 2000 Annual Meeting	ED 472094
Proceedings of the 2001 Annual Meeting	ED 472091
Proceedings of the 2002 Annual Meeting	*submitted*
Proceedings of the 2003 Annual Meeting	*submitted*

Note

There was no Annual Meeting in 1992 because Canada hosted the Seventh International Conference on Mathematical Education that year.