CANADIAN MATHEMATICS EDUCATION
STUDY GROUP

GROUPE CANADIEN D'ÉTUDE EN DIDACTIQUE
DES MATHÉMATIQUES

PROCEEDINGS / ACTES
2005 ANNUAL MEETING

University of Ottawa
May 27 – May 31, 2005

EDITED BY:
Peter Liljedahl, *Simon Fraser University*

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Proceedings of the 2005 Annual Meeting of the Canadian Mathematics Education Study Group / Groupe Canadien d'Étude en Didactique des Mathématiques

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Introduction

Frédéric Gourdeau – Président, CMESG/GCEDM
Université Laval

Débutons… par le commencement. La planification de la rencontre de 2005, tenue pour la première fois à Ottawa, avait débuté un an auparavant, à Québec. Nous étions en 2004, année d'ICME, et j'avais le plaisir de travailler avec Olive Chapman, Florence Glanfield, Joel Hillel, Ralph Mason et Luis Radford au sein de l'exécutif. Nous avions fait notre travail rondement et les semaines qui suivaient allaient nous permettre de concrétiser le programme tout en le modifiant un peu, comme il se doit.

C'est pendant cette période que Gila Hanna offrit un témoignage qui allait appuyer nos démarches. Lors d'une séance plénière à ICME-10, brillamment animée par Michèle Artigue, Gila Hanna affirma avec conviction que le groupe était *la meilleure organisation en enseignement des mathématiques au monde.* J'étais à ce moment assis à côté de Sandy Dawson, que nous avions invité à animer un groupe de travail. Malheureusement, Sandy avait prévu ne pas assister à notre rencontre de 2005… Profitant de la perche tendue par Gila, je me suis alors tourné vers Sandy en lui disant : *I would be honored if such an organization was asking me to lead a working group.* Sandy a bien ri et, un peu plus tard, lui et Arthur Powell acceptait notre invitation à animer un groupe portant sur le thème Mathématiques, Société et Paix.

And so it was for Sandy and Arthur, and so it was for many others. It is truly amazing to witness the level of commitment that many have for our group and for the betterment of mathematics education. Extremely busy people who agree to work on a topic that they might not have chosen, with people they often hardly know. And the magic operates, every year.

The 2005 meeting was a success. Our plenary speakers, Stephen Lerman and Jean Taylor, gave wonderful talks and took a very active part in our work, joining in as we always invite our plenary speakers to. The discussion period with Jean Taylor was particularly moving as she discussed issues she had to face as a woman in mathematics.

The working groups were diverse and led by people from all over the country, experienced members and newer ones. Each working group had a good number of participants and the coffee break were a time to hear a little of what was happening in the other groups. Many of us would have liked to be in more than one group, and this certainly was the case for me. Choosing between "Mathematics, Society and Peace", "Learning Mathematics in the Early Years", "Discrete Mathematics" and "Socio-Cultural Dimensions of Mathematics Learning" was not easy. And beyond all this, there is so much more: new PhD presentations, topic sessions and ad hoc sessions, many ad hoc sessions, and conversations! How wonderful.

En terminant, je tiens à dire un mot sur la dernière séance de notre programme académique, qui ne peut malheureusement pas être reproduite dans les Actes. Celle-ci consistait en un débat sur la proposition : que nulle mathématiques ne soit obligatoire. Le débat oratoire opposait Rina Zazkis et Bill Higginson à Carolyn Kieran et Ralph Mason, et l'auditoire tout entier s'est joint au débat que j'ai eu l'immense plaisir d'animer. Un délice !

I encourage you to read these proceedings, for there is a wealth of ideas and so much to reflect on. To all who took part, accepted difficult mandates, shared their ideas, and generally were part of it, thank you so much. To Barbra Graves and Chris Suurtam, who hosted us, thank you: we'd like to go back… And to Peter Liljedahl, the new editor of our proceedings, thanks for agreeing to take on this role and congratulations on a job well done.

Plenary Lectures

Conférences plénières

Learning Mathematics as Developing Identity in the Classroom

Stephen Lerman
Centre for Mathematics Education
London South Bank University

> *It's not a matter of understanding mathematics, it's a matter of getting used to it'*
> *(Von Neumann)*

For my talk I take inspiration from Von Neumann, whom I take to be suggesting that 'understanding' is not a useful description of the process of becoming mathematical, and that the idea of enculturation, getting used to mathematics, is more appropriate (I recognise that his words can also be taken otherwise). I also take inspiration from a small study carried out by a colleague of mine, Peter Winbourne, in which he examined the nature of secondary school students' experiences of learning across the range of curriculum areas by following a few students throughout their day into different classes. In each class the goal of the teacher could be said to be about enabling the students to develop their 'consciousness', as a speaker/reader/writer of English, as a school scientist, or indeed as a school mathematician. In each class the lives of the students, their views, their experiences and opinions formed a key element of their learning, except in the mathematics classroom. It was as if they had to leave their identity at the door in order to develop a mathematical consciousness.

My third source of inspiration is a sentence from Bernstein:

> If the culture of the teacher is to become part of the consciousness of the child, then the culture of the child must first be in the consciousness of the teacher (Bernstein 1990)

In this talk I will take further some of the ideas I discussed in Lerman (2000a) concerning what I proposed was a social turn in research in mathematics education since the mid to late 1980s. As teachers of mathematics we could describe our goal, as I mentioned above, as enabling students to acquire or align with the (school) mathematical consciousness of the teacher, in which case, as Bernstein suggests, teachers need to become aware of the culture of the child. I have been writing 'school mathematics' rather than 'mathematics' in recognition that what forms the school mathematics curriculum is a recontextualisation, a shift from one context to another, of 'academic mathematics', a process that has undergone changes over the years of course, but one that always involves a selection. That selection is governed by ideology, whether we are talking about the New Maths, reform mathematics or so-called traditional mathematics.

'Studying the culture of the child' and the process whereby the 'culture of the teacher' becomes 'part of the consciousness of the child' is a task that can perhaps be best carried out by researchers or teachers as researchers. I am interested in this area and intend to carry out a study in classrooms. This talk is intended as a mapping out of the theoretical field for such a study.

In my (Lerman, 2000a) paper I developed a unit of analysis, in an attempt to draw together affect, cognition and the regulating effects of social practices. I called that unit 'person-in-practice-in-person' and elsewhere (Lerman, 2000b) I exemplified how one might use that unit. Here, I want to work with the notion of identity.

My talk will consist of the following parts: an introduction, which I have already begun and will have just a few more remarks to make, followed by a review of some of the research in our field that has addressed notions of identity. I will then examine some of the ideas of Bernstein and work in the literature of late modernity. One cannot ignore, I believe, the effects of government policies on teachers and on students and this will be the next section. Studying identity calls for listening to students but there can be problems with voice studies, and I will discuss this briefly before moving to the final section, that of drawing together the elements of a 'toolkit' (Bartolini Bussi, 1991) for researching developing identity in the classroom.

INTRODUCTION

In a recent research project ("The production and use of theories of teaching and learning mathematics", see http://myweb.lsbu.ac.uk/~lermans/ ESRCProjectHOMEPAGE.html, and also Lerman, Tsatsaroni & Xu, 2003) we studied the identities of mathematics educators as interpreted from a study of a representative sample of research texts in journals and conference proceedings. Following a systematic study of those texts we developed a language of description to say something about the state of our community and the effects of a range of regulating agencies on identities. What we were interested in was to understand openings of the sub-field of mathematics education research to influences coming from the wider intellectual field, and by tracing any changes in the pattern of influences to analyse their consequences for how knowledge is defined; the latter taken to constitute the basis for identity formation. I see my present talk and intended research as an extension of that study and field of interest to another domain, that of students' and teachers' mathematical identities in the classroom.

One might ask why use the notion of identity? First, it has become a common focus of attention in the social sciences in general. In 1996 Stuart Hall said, "There has been a veritable explosion in recent years around the concept of 'identity'", to which Zygmunt Bauman (2001) added, "The explosion has triggered an avalanche."

The anthropological perspective of Jean Lave has become a powerful influence on research in our community and she shifted the language of learning from cognition to that of identity.

> We have argued that, from the perspective we have developed here, learning and a sense of identity are inseparable: They are the same phenomenon. (Lave & Wenger, 1991, p. 115)

Therefore I have taken learning as developing identity as the focus of my talk and its title, although it will be most important to contrast mathematical *identity* with mathematical *subjectivity* (e.g. Dowling, 2001). Subjectivity focuses on how individuals are both the *subject* in the sense of the actor in a discourse but are also *subjected to* the possibilities and limitations, the affordances and constraints, of that discourse. Identity is therefore produced in discourses and the notion of subjectivity captures that regulation. Post-structuralist theories have proved powerful tools for researchers in mathematics education (e.g. Walkerdine 1988; 1998; Evans 2000; Walshaw, 2004) for examining subjectivity.

THE FOCUS ON IDENTITY IN EDUCATION AND EDUCATIONAL RESEARCH

One cannot but be aware of the manifestations of identity in students' lives, whether it be in the clothing they wear, to conform, to identify with a sports team or sports star, or with a media star, or in the music they listen to, through which, in their choices, they express conformity to one group or another, or resistance to conformity. There are racial and cultural styles of dress, speech and gestures which students may adopt, sometimes independent of whether they 'belong' to that social group or not. 'Belonging' is perhaps best judged by the person, not by an observer. In the outward expression of religion, through dress, we are again strongly aware of identity and identification.

Research studies of gender, ethnicity, social class etc. demonstrate the struggle for identity, acceptance and, sometimes, just a peaceful path through childhood and adolescence and through schooling in particular. I will refer to some of these studies later.

Within our own field one can see certain trends in research that are, I argue, influencing us to widen our sense of the learning process. I have already mentioned the work of Jean Lave, but I would also mention the ways in which we have been talking about mathematical thinking and mathematical competence for many years now, these ideas indicating more than knowledge acquisition, more indeed of a state of being in relation to mathematical activity. So too the sub-fields of ethnomathematics, cultural psychology and socio-cultural theories are about mathematics as culture. It must be said, though, that we do not yet have a substantial body of research that builds on the literature of identity in social science and in education more generally. Perhaps Jo Boaler has done the most extensive work in a series of studies and publications and I will turn to an examination of her work first in looking at what has been done in our own field.

In her 1997 book (revised as Boaler 2002a) Boaler demonstrates very powerfully how different forms of pedagogy have the effect of producing different mathematical identities in the two schools in her study. In one the students are taught through the use of textbooks, practice of skills, and past examination papers. These students, to summarize a rich description rather crudely, see being mathematical as having successfully memorized a range of skills so that they can recognize, in an examination situation, which skill to apply. This was far from easy, as questions set often look quite different to those practiced in class. Indeed they were often unable to recognize what type of skill a question called for or failed to recall the necessary skill. Certainly their notion of mathematics was that it is something you do in school mathematics lessons and it bore no relation to the rest of their lives. When asked to solve realistic problems they had no way of thinking for themselves or adapting what they had learnt. The students in the other school learnt through problem solving. They could relate what they did in school to everyday problem solving. They saw doing mathematics as working with others, deciding how to tackle a new problem and in finding how to acquire the skills they needed to solve that problem. In terms of confidence, self-concept and enjoyment, and in equity terms too, the latter students' identities were quite different and certainly more effective in terms of mathematical success, than the students in the text-book based school.

In a later study Boaler & Greeno (2000), drawing on Holland *et al* (1998), interviewed students in advanced placement calculus classes. They characterised the schools as discussion-based and didactic. In the latter the "students presented their worlds as structured, individualized and ritualized, the other group as relational, communicative and connected" (p. 178). Their study showed how, in more traditional classrooms the image of a mathematical identity was not one that female students in particular would choose, it differed so greatly from how they described their own identities. "They talked not about their inability to do the mathematics, but about the kinds of person they wanted to be – creative, verbal, and humane." They saw the identity of a mathematician, in their perception, as one with which they did not

identify, indeed from which they felt alienated. They saw themselves as imaginative and creative, in stark contrast to how they perceived a successful mathematician.

In Boaler (2002b) she proposed a triangle of knowledge, identity and practice to represent the learning process, and she argued that some learners advance through the 'dance of agency' (Pickering, 1995), the interchange between human and disciplinary agency, others through a more passive relationship.

Boaler's current study of a school using Elizabeth Cohen's 'Complex Instruction' (Cohen, 1994) once again demonstrates the fruitfulness of the notion of identity to describe the effects of a particular form of pedagogy, in this case carefully researched and designed collaborative work on mathematical problems, with the teacher playing a fairly strong role in guiding and monitoring the learning of students.

There is a growing number of studies drawing on aspects of identity, including: Mellony Graven's (2004) study of in-service teacher education; Heather Mendick's (2003) study of gendered relationships to learning mathematics in advanced studies classrooms, arguing that "What [students] enjoy when doing mathematics is the identity work they do through it." and Hannah Bartholomew's (2005) study in which she says "The identity work in which students are engaging, and the associated emotional factors, are implicated at all levels, not as a background which may facilitate or hinder mathematical achievement, but as an inevitable part of what it means to do mathematics and regard oneself as mathematical." My brief review here is not intended to be exhaustive but indicative.

IDENTITY/SUBJECTIVITY

Thus far I have talked in terms of identity as a way of capturing a fuller sense of the process of development in mathematics classrooms and elsewhere. There is also a substantial literature on subjectivity seen, for example in our own field, as produced in the framing of pedagogic codes (Dowling, 2001) and in the production of regimes of truth (Walkerdine, 1998). In some senses identity and subjectivity are complementary, the one a focus on agency, carrying with it the dangers of fixed notions of identity, the other on structure, carrying with it the dangers of losing sight of the potential of the individual for choosing the discourse from which to speak out. 'Identity' seems to carry with it a sense of choice or decision: "I am who I choose to be." I will discuss here at some length research on identity as positioning in the work of Bernstein, and I will then briefly refer to how theories of late modernity engage with issues of identity.

Bernstein

Basil Bernstein's (2000) book, entitled *Pedagogy, Symbolic Control and Identity*, sets up a framework for discussing identity in terms of how those identities are produced in pedagogic discourse and, indeed, how different social groups are positioned differently by a pedagogic discourse. In particular, of great current significance, is his thesis that the shift from a traditional performance mode of pedagogy to a liberal-progressive, or as we might say a 'reform' mode of pedagogy, was also a shift from a visible pedagogy to an invisible one. A visible pedagogy is one where the rules of recognition and realization are explicit, an invisible one being where the rules are implicit. Bernstein's thesis, well supported by research, indicates that pupils from middle class backgrounds have acquired those rules in their home life whereas pupils from disadvantaged backgrounds have not. Two examples to illustrate Bernstein's thesis, the first from Cooper & Dunne (2000):

> A drink and a box of popcorn together cost 90p. 2 drinks and a box of popcorn together cost £1.45. What does a box of popcorn cost?

This was a question set in national tests. One child from a working class background answered:

> "*I said to myself that in a sweetshop a can of coke is normally 40p so I thought of a number and the number was 50p so I add 40p and 50p and it equalled 90p.*

This response was typical of children from similar backgrounds and typical of questions set in everyday contexts. The issue is that the everyday-ness misleads pupils into focusing on the everyday and not on the required mathematical meaning. In subsequent interviews, when the researchers explained what the question was asking, those students could provide the correct answer. Hence the everyday-ness is resulting in those students not being able to represent their mathematical knowledge.

Holland's (1981) research used the example of classifying foods. She gave young children pictures of food and asked them to classify the pictures in whichever way they wished. The working class children were more likely to offer local classification systems, such as what would be offered as Sunday lunch. The middle-class students were more likely to classify them according to food groups - a school-based classification system. She then asked them to put the pictures together and re-classify them. She noted that middle-class students were able to switch between codes, offering different everyday classifications, whereas this was not the case with the working-class students who tended to rely on local pedagogic codes. Two things are to be noted here. First, the middle-class children had a range of strategies for classification including the abstract 'scientific', school-based one, whereas the working-class children possessed only the local. Secondly, the middle class-students knew which was privileged in schools and knew to present the school-based classification first. This is why, Bernstein pointed out, schools reproduce the access to symbolic control in society at large. Pupils' mathematical identities are produced in the classroom with different effects on different social groups (see also Delpit, 1988).

In Morgan, Tsatsaroni & Lerman (2002) we revisited research carried out by Morgan (1998) in which she examined teachers' assessment practices in the context of written investigation tasks. Morgan used critical discourse analysis focusing on linguistic features of teachers' interviews to identify their *positioning* in discourses of assessment. Teachers are given official criteria buy the examination board and draw on other informal discourses too in making and in justifying their judgements. Morgan found 8 positions:

- examiner, using externally determined criteria
- examiner, setting and using her own criteria
- teacher/advocate, looking for opportunities to give credit to students
- teacher/adviser, suggesting ways of meeting the criteria
- teacher/pedagogue, suggesting ways students might improve their perceived levels of mathematical competence
- imaginary naïve reader
- interested mathematician
- interviewee

In our work re-examining the findings we first developed a model based on two dimensions of *voice* and *forms of practice*, elaborated by two other dimensions, *specialised/localised* and *focus on absence/presence*, drawing on Bernstein's later work in his (2000). We then re-examined the interviews and found that four positions emerged (in place of 8):

- Examiner: using externally determined criteria
- Examiner: setting his/her own (professional) criteria
- Teacher-adviser
- Teacher-advocate

Teachers' identities in processes of assessment are thus formed by drawing on different discourses in ways that they determine appropriate. Our re-examination, by also showing the origins of the discourses on which teachers are drawing enables one to see how changes may be effected, which is much more of a problem when the analysis is a grounded one.

As a final example to illustrate the task of pupils in recognising what is expected in terms of reading the task and in being able to produce the appropriate text, consider what should the student do when faced with the following question, again based on a question in a UK national test.

> *This is the sign in a lift at an office block:*

> This lift can carry up to14 people.

> *In the morning rush, 269 people want to go up in this lift.*

> *How many times must it go up?*

In test conditions the student is supposed to ignore the everyday setting, divide 269 by 14 and then realise that there can't be part of a journey, rounding up to 20 times. In classroom situations, though, the teacher might expect a range of problem-solving responses such as:

- If you are at the back of the queue will you wait for 19 journeys?
- No-one counts 14 people into the lift, so there might be 10, there might be 15 or 16.
- If someone has a child in a stroller, or is in a wheelchair, fewer people will be able to enter.

There are many others of course. These responses are appropriate and indeed required in a non-test problem-solving context, the key being that the pupils need to know what is expected of them.

Late modernity

The literature of sociologists Giddens, Beck, Lash, Bauman and others examine the notion of the process of doing the work of identity as one of self-determination in this time of late modernity. They describe three periods, from traditional society to first or early modernity to the present late modernity and they raise the question of whether structure has fallen away as a determinant of life choices. In traditional society identity was formed by the place one lived, one's family life and one's status in that society. In the period of first modernity identity was determined by social class and occupation. There was mobility in that period, leading to the move away from traditional society. In late modernity it is suggested that we can engage in the project of the self, writing our own identities. Young people can imagine who they will be and what they will do, irrespective of gender, social class, location, parental occupation or whatever. As one might imagine, there is as much disagreement as agreement, some theorists arguing, for example, that rather than seeing a de-traditionalisation of roles we are in fact seeing a re-traditionalisation. A student of mine, for example, in studying the course selections and subsequent career choices of mature women has found that for many women their choices are being managed within and in addition to more traditional gendered roles of child-care and household responsibilities.

Bernstein (2000) also responded to the literature on late modernity and presented an account that drew on a distinction between 'pedagogic identities' and 'local identities', the latter incorporating issues such as the marketing of education and the power of different narratives of identity such as religion and a 'better' society.

THE REGULATION OF EDUCATION

Government policies for education inevitably have effects on identities, first for teachers and administrators and consequently for pupils. Ball (2001) draws on the notion of performativity to describe its effects in the very highly regulated educational system in the UK. Elements of that regulation include:

- National tests for children at ages 7, 11, 14 and 16 have had the effect that schools teach to the test, distorting the curriculum and the pedagogy.
- League tables: school results on those tests are published school by school. The intention is to empower parents in their choices of schools for their children with the expectation that it will force schools to improve, based on what happens to supermarkets that under-perform.
- The government appointed Office for Standards in Education (OFSTED) inspects all schools and teacher training institutions. Poor inspections can result in schools being put into 'special measures' and potentially falling roles and even closure. Similar but more immediate effects can result from poor inspections in Universities' teacher training departments since student numbers are allocated on the basis of inspection performance year by year.
- Mathematics teaching in primary schools and now what is called Key Stage 3 (11 to 14 year olds) includes not just content but also teaching method. Given the inspections, teachers tend to follow the prescribed methods irrespective of their own tendencies and experience.
- Schools continually receive packages of materials from the Department for Education; teachers find it very difficult to keep up with the constant changes.
- The language of education changes, with effects that are more than changes in utterances. For example, teacher education became teacher *training*, which conveys a very different sense of the nature of teaching. The National Curriculum is *delivered* by teachers, learning goals are called targets, and every aspect of teaching is dominated by the notion of *quality* is spite of its meaning being highly disputed and endlessly open to interpretation.
- In universities heads of departments have to meet performance indicators and their pay is determined accordingly.
- All University Departments are subject to periodic peer evaluation of research output through the Research Assessment Exercise (RAE). Some resulting effects include: setting the length of time that PhD students must complete, irrespective of the nature of the research; pressures on researchers to publish their research often before time; Universities 'poaching' successful academics to boost their own level, etc.

This fairly lengthy (somewhat self-indulgent!) whinge is to indicate just how tightly regulated education is in the UK. Its result has profound consequences for identity. Ball presents some quotations from Jeffrey and Woods (1998) of teachers' feelings:

> I don't have the job satisfaction now I had once working with young kids because I feel every time I do something intuitive I just feel guilty about it. 'Is this right; am I doing it the right way; does this cover what I am supposed to be covering.

> My first reaction was 'I'm not going to play the game', but I am and they know I am. I don't respect myself for it; my own self respect goes down. Why aren't I making a stand?

> I've never compromised before and I feel ashamed. It's like licking their boots.

> She was the only year 6 teacher at Trafflon and after criticism of their SATs results she resolved to go down the path of 'improvement of results'. She changed her curriculum, and achieved her aim of getting the second best results the following year in her LEA. She justified this by saying that she was 'now just doing a job'; and

> had withdrawn her total involvement to preserve her 'sanity'. 'The results were better because I acted like a function machine'.

Ball talks of self-regulation, but not of the poststructuralists' panopticon, the internalisation of the all-seeing regulating eye of discursive practices.

> Instead it is the uncertainty and instability of being judged in different ways, by different means, through different agents; the 'bringing-off' of performances – the flow of changing demands, expectations and indicators that make us continually accountable and constantly recorded.

RESEARCHING VOICE

Pupils' own social lives are dominant for them, the social mores of their interactions have priority. As I wrote elsewhere (2000a),

> More important to students than learning what the teacher has to offer are aspects of their peer interactions such as gender roles, ethnic stereotypes, body shape and size, abilities valued by peers, relationship to school life, and others (McLaughlin, 1994). The ways in which individuals want to see themselves developing, perhaps as the classroom fool, perhaps as attractive to someone else in the classroom, perhaps as gaining praise and attention from the teacher or indirectly from their parents, leads to particular goals in the classroom and therefore particular ways of behaving and to different things being learned, certainly different from what the teacher may wish for the learners. (p. 31)

In contrast, and often in conflict, in the (mathematics) classroom we, as teachers, are concerned with imposing/encouraging a mathematical identity onto their already dominant (fragmentation) localization of identity.

Reay (2002) tells the story of Shaun who struggles to find a balance between succeeding academically in a challenging school whilst presenting himself as one of the tough young males. Things become more difficult towards the end of Shaun's first year at secondary school:

> *It's getting harder because like some boys, yeah, like a couple of my friends, yeah, they go 'Oh, you are the teacher's pet' and all that. Right? What? Am I a teacher's pet because I do my work and tell you lot to shut up when you are talking and miss is trying to talk? And they go, 'yeah so you're still a teacher's pet'. (p. 228)*

In another ethnographic study Kehily quotes Mike:

> *It's a sort of a stigma ain't it? A quiet person in the class would be called 'gay' or summat. I was for a time 'cos I was fairly quiet in the classroom and for a while everyone was calling me gay. (p. 120/121)*

There are dangers in researching 'voice', as Arnot & Reay (2004) discuss, drawing on Bernstein's work.

- There is a potentially unstoppable spiral of ever more fragmented voices. What we require as researchers is to be able to talk about how these voices are produced, if we are also to be able to see how things can change.
- One cannot ignore the relationship between the researcher and the person being interviewed, that is, the problem of the pedagogic relationship producing/regulating 'voice'.
- Seen as produced in pedagogic relations, voice is the power to constrain whereas message has the potential to transform.
- However it is not so easy to separate, since voice is realized in message.

SUMMARY

As I mentioned at the start of this talk, my interest is in the formation of (school) mathematical identities, how some students become enculturated into becoming school mathematicians whilst others do not. In the literature of our field we have identified a range of aspects of becoming mathematical such as disciplinary agency (Boaler, 2002b); discursive approaches in teaching and learning (Kieran, Forman & Sfard, 2003); and the aesthetics of mathematics, such as beauty, simplicity (Davis & Hersh, 1983). In this talk I have reviewed other bodies of literature from which have come notions, in addition to the mathematical identity, of: pedagogic identity; performative identity; social/localised identity in late modernity; and identity expressed as voice/message. I have also discussed subjectivity. I believe that what is needed to research identity is an ethnography, but one informed by the perspectives presented here.

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Formative Influences

Jean E. Taylor
Courant Institute, New York University

To see a public lecture similar to the plenary address on Soap Bubbles and Crystals which I gave to the CMESG in May 2005, go to http://www.psych.utoronto.ca/~rci/ (the website of the Royal Canadian Institute). You will find there, near the end of the fall 2005 talks, my PowerPoint presentation together with a streaming video. Since anyone interested in hearing the talk again, or in going through the slides, can visit that website, I will address here issues of my background and why I chose a career as a mathematician.

There were no mathematicians or scientists in my family, and I never met any mathematicians or scientists before I went to college; I grew up in rather ordinary suburban Sacramento. Yet logical thinking was part of my upbringing, as my father was a lawyer. I was always interested in the logic puzzles which used to be printed in my local newspaper – the kind where they would say: There are three shelves in the cupboard, with a total of 25 bottles; there are twice as many bottles on the second shelf as on the first, and one more bottle on the third shelf than the first shelf; how many bottles are on each shelf? I was delighted when I first learned algebra, and found out that by the simple expedient of using an x, such problems could be solved easily.

That first algebra class was notable for another reason. I was always asking the teacher if you couldn't solve a problem by another method I'd figured out. Finally, he got exasperated and told me that he was giving me a choice. Either I took the test on algebra he'd taken in the Navy, in which case I'd get an A if I passed and an F if I failed, or I had to shut up. I shut up.

I believe it was the summer after the ninth grade when my family went to a family church camp at Zephyr Point on Lake Tahoe which featured lectures from a Harvard professor. In one, he said that there had to be the possibility that something was true in order to talk about it. In discussing this later with my father, we came up with a counterexample: the statement "the moon is made of green cheese." So I talked to the professor about this, and then other issues. By the end of the conference, he told me I should think seriously about applying to Radcliffe College. That kind of direct suggestion was highly unusual for me and made a big impact.

When I was applying for college, "diversity" meant getting applicants from all over the United States. In particular, the "Seven Sisters" (including Radcliffe) had a Seven College Scholarship awarded to one applicant from each of three or four regions. In applying for this scholarship, you could apply to three of the seven for the price of one, which appealed to my Scottish background. So I read catalogues from all seven, and decided I liked best Mount Holyoke, Radcliffe, and Bryn Mawr. I think I rated Mount Holyoke first on the application, more or less on the basis that its catalogue said it had two lakes and a ski hill. I was all set to accept admission to Stanford University when I got a telegram (a telegram!) from Mount

Holyoke congratulating me on being awarded that Scholarship, and asking me to wire back collect (collect!) and stating that it was confidential (confidential!) for the time being. I wasn't even given the choice of the other two colleges; I guess they assumed my first choice of several months ago must still be my first choice. I was so impressed by the telegram that I decided to go to Mount Holyoke, sight unseen. Also, I'd never been east of the Rocky Mountains and this looked like a good way to see a bit more of the world.

At Mount Holyoke, I majored in chemistry. In high school, my chemistry teacher had made a point of telling me at the end of the year that I should consider going into chemistry – this in spite of the fact that I'd dropped and thereby destroyed the results of the main experiment of the year. He said I had a real talent for the subject, and that I'd get more used to lab work later. I actually don't think I did any better in chemistry than any other subject (except French – I always struggled with French). Still, I wrote mathematics and chemistry as possible majors on my application to MHC. During the summer before I enrolled, the math department at MHC told me I'd have to retake the first semester of calculus, because my high school course couldn't possibly have been as rigorous as their course. The chair of the chemistry department, on the other hand, contacted me and said I should skip the initial chemistry courses altogether and go straight into Qualitative Analysis, since my mathematics was so good. Go figure. In any case, chemistry seemed much more dynamic, and so I chose chemistry. My clumsiness in the lab continued, as due to too-vigorous stirring I broke the test tube containing the results of a semester's worth of effort in Qualitative Analysis (thereby becoming reduced to trying to guess the remaining unknowns by the colors produced by the solution as it dribbled over the lab bench), and then I knocked over several hundred dollars worth of glassware in organic chemistry lab. My professors changed from telling me that I'd get used to lab work to telling me about the possibilities of theoretical chemistry.

I applied to grad school in physical chemistry and in biophysics, eventually choosing physical chemistry at the University of California at Berkeley. I had taken all required course work, passed my Ph.D. qualifying exams, and started work on my thesis when I was seduced by mathematics. Many of my Hiking Club friends were math graduate students, and at their suggestion I audited a course in Differential Geometry taught by S.S. Chern. It was wonderful. Imagine, a mathematical language for talking about surfaces! About that time my chemistry thesis advisor told me that "a chemist who doesn't do experiments is like a man who deliberately cuts off one of his hands." So instead I cut off chemistry, and switched to mathematics.

I never worked harder in my life than that first semester, when with minimal background I plunged into graduate level mathematics courses. I learned for the first time what it meant to REALLY understand something (norms, in particular). I studied with friends (Wendy Teller and Dan Asimov – yes, related to THAT Teller and THAT Asimov). But I also became more involved in anti-war activities, and felt a major conflict between participating in the immediacy of anti-war protests in Sproul Plaza versus my long-term plan to study mathematics. A way out came in the form of a marriage proposal by letter from my long-term former boyfriend, who had definitively broken up with me just three months earlier when he got his Ph.D. and left for several months in Brazil. He suggested that we meet at the ski resort Val d'Isere in France and then I go with him to the University of Warwick, where he was about to start a post-doc, and marry him there. I jumped at the offer, managing within about a week to wrap up my course work, dispose of my extensive collection of topo maps of California mountains and most of my other belongings, make flight reservations, etc. etc.

I got my National Science Foundation graduate fellowship transferred to the University of Warwick (after having had it changed from chemistry to mathematics back at Berkeley), and I settled into married life and more mathematics graduate study, passing my Ph.D. qualifying exams at the end of the academic year. But my husband wanted to go back to the U.S., being concerned about keeping tabs on his draft status. He was going to the Institute for Advanced

Study, so I applied, way past the deadline, to transfer to Princeton University. After a few hitches, I was indeed accepted at Princeton. So in September 1970 I enrolled in my fourth graduate program, and even got my NSF graduate fellowship transferred once again. (I adore NSF.)

Just before leaving Europe, I went with my husband to the International Congress of Mathematicians in Nice, France, where at his suggestion I went to hear Fred Almgren give a talk on Geometric Measure Theory. I was delighted by it, and talked to Almgren afterwards. He was rather surprised to hear I was coming to Princeton, since when he'd left in February for a semester in Russia, I'd not even applied. But he believed me, and suggested a thesis problem which appealed to me very much.

And this takes me to the start of my lecture: in it, I describe the problem (that of proving that the triple junctions in soap films – or more particularly, in "two-dimensional flat chains modulo 3 in three space" – are smooth). And I tell how it led to my finishing my thesis (except for rewriting for style) by May1972, my discovery that the results could be extended to all soap films and soap bubble clusters during the summer of 1972, my Instructorship at MIT in 1972-73, where I found out that soap froths are a model for the internal grain structure in metals, and thus the rest of my career.

If there is any message in all of this to teachers, it must be that the encouragement you give to students individually can have a disproportionate effect. Even some very good students need to be told they are very good. Who knows, someone you encourage to study mathematics may even one day become a renowned chemist.

Working Groups

Groupes de travail

Mathematics Education, Society, and Peace

Arthur Powell, *Rutgers University*
A. J. (Sandy) Dawson, *University of Hawaii & PREL*

Opening Circle—Introduction of the Talking Stick

- Canadian Aboriginal Peoples—Eagle feather
- Micronesian societies—tribal governance
- New talking stick—insert inscription on the box
- Covenants of circle operation
  - What is said in circle belongs in the circle.
  - The circle is a practice in discernment.
  - Each person takes responsibility for asking the circle for the support s/he wants and needs.
  - Each person takes responsibility for agreeing or not agreeing to participate in specific requests.
  - Anyone in the circle may call for silence, time-out, to re-establish focus, to re-center, or to remember the need for…guidance.
  - Agreements are adaptable. If something is not working, revise the agreements and maintain the process.
  - Practice listening without interrupting.
- <u>At a minimum</u>, the stick will go round the circle twice—in fact it went around three times.

Mathematics and Peace: the Wisdom Circle—the first occasion

- How are you connected to the topic of the WG: mathematics and peace?
- How are mathematics and peace connected for you?
- <u>At conclusion of the final round</u>: later today or carrying over to tomorrow, we invite you to share a short vignette of when in your work life or in your personal life an occurrence that illustrates/shows/demonstrates a relationship between mathematics, peace and society

Notes from the co-leaders after the first circle.

The circle seemed to intensely engage participants, and what was anticipated to take at most one hour, consumed most of the first morning. Some participants attested to the fact that the use of circle was the most challenging and stimulating activity ever

engaged in at a professional conference. Later, other participants argued that the circle approach was to confining and time consuming.

In preparing for the meeting of the group, the co-leaders proposed that the group focus initially on two broad areas:

Mathematics education, philosophy and ethics
- One of the questions addressed in this area was whether and why mathematics and mathematics education can be or should be subjected to ethical considerations.
- An additional question was how awareness of the plurality of truth, as one—not yet well-understood—feature of modern mathematics, can provide a basis for the discovery of values (among them values which are preconditions for peace) and the shaping of interpretative frames.

The second area was the use of mathematics education to contribute to peace
- For example, the leaders thought the working group might address the question of whether or not the use mathematical modeling or analyses of peace and conflict situations can:
  - lead to a better understanding of situations and their dynamics - and thus to insight regarding the probability to avoid escalation
  - demonstrate that, in some cases, a mathematical model can lead to wrong or morally unacceptable conclusions
  - assist students in becoming aware that there are situations where decisions must not be based on mathematical considerations alone
  - illustrate that mathematics can be used to expose dangerous trends thus leading to the insight that counter-measures are badly needed
  - illustrate that mathematics itself can be part of the basis on which to build a "better" world.

In light of the issues introduced during the first circle as well as the focus areas and questions generated in the pre-planning done by the co-leaders, the themes given below were suggested for discussion. Based on these themes various subgroups were formed based on which of the themes the participants of the subgroup wished to address.

- What are the relationships between inner—within the self—and outer—within society—peace? Does one depend on the existence of the other? Are they interrelated, mutually constitutive of each other, or co-emergent? What are possible roles that mathematics education can, does, and should play?
- What are the characteristics of mathematics teaching practices that advance social justice (values education)? How does an awareness of the plurality of perspectives, as one—not yet well-understood—feature of modern mathematics, provide a basis for the discovery of values?
- In what ways does using mathematics to examine issues of social inequities and unjust distribution of social goods and services contribute to inner and outer peace?
- What are the characteristics of curriculum and other materials that contribute to social justice? How does an awareness of the plurality of perspectives, as one—not yet well-understood—feature of modern mathematics, provide a basis for the generation of curriculum and other materials?

The various subgroups generated issues, concerns, questions, and viewpoints, a sampling of which are provided below:

- Discussion in all subgroups was wide ranging, divergent on some issues, convergent on others
  - When focusing on teacher practices it seemed clear that a plurality of approaches were successfully used
  - There seemed to be consensus that both math and value systems are human constructs, and that mathematics teaching should open opportunities for a variety of perspectives
  - It was pointed out that there was a difference between the sharing of and the imposition of ideas.
- When discussing what issues could be addressed in a mathematics classroom, various subgroups made these points:
  - idea of balance in terms of social issues
    - life enjoyment and aesthetics
    - all ideas are not equal
  - emphasizing a plurality of algorithms
    - invented algorithms parallel invented framing of social issues
    - verifying algorithms parallel verifying framing of social issues
  - there is an issue of whether or not teachers should bring their own values into the classroom
    - kids and social issues: it is important for children to have 'choice' in the social issues they address
  - there is a continuum of kids/teachers view of math as being right answers to being shades of grey parallel again view of way social issues are viewed.
  - that teachers must consider the age of learners in terms of what can be addressed and what cannot
  - one subgroup talked about classroom abuses:
    - the imposition of the view that children are inadequate,
    - the rightness/wrongness of answers, but that
    - math had an authority structure that was independent of the teacher(s)
  - one subgroup talked about choice, contending that
    - teachers had as much choice as possible right down to selection of exercises
    - choice applies to teachers as well in terms of curriculum selection and methods
    - there are norms and social freedoms—inside of constraints there is freedom
    - exercise of choice can lead to empowerment or disempowerment of the students and the teachers
  - Another subgroup focused primarily on the teaching of mathematics and noted that:
    - the way and how of teaching mathematics is central, but questioned if the 'way' particular to math or is it general to all education
    - they value different algorithms because this can lead to a valuing of alternate world views
    - there were questions about how explicit should the teacher be when introducing values or frameworks to students?
    - There was a question as to how do to evaluate so that you don't subvert all that you have done during the course
  - Another subgroup addressed concerns about the misuse of and dis-ease caused by the teaching of mathematics; for example the
    - removal of the obstacle of the expectation of failure (e.g., Rutgers)

- removal of math as a classification system
- prevalence of fear and frustration for many
- implicit maintenance of status quo by not challenging it.
- oppressive tyranny of detail
- use of math as a tool
- math is imperialist (e.g., mathematization of biology)
- impression of the discipline as being inhumane
- math has been internationalized without respect for local context
- math as a sorting process: schools our children deserve

From these subgroup discussion, during one of the final sessions when meeting in circle, the group summarized many of the points noted above in the following fashion:

1. Classroom tension between
   imposition of teacher's perspective versus the 'offering' of alternate viewpoints
   the 'how' of engaging students in alternate perspectives
2. Challenges to including social justice in the teaching of mathematics
   tensions among the political nature of teaching, individual learning, and culture
   inculcation of society values versus challenging societal inequalities, changing the direction or path of the status quo
   tension among learners who have different objectives in their study of mathematics
   tension when the 'safety' of living in mathematics allows one to escape the political issue
   bringing issues into the classroom for consideration by learners when we as teachers have not dealt with the issues ourselves.
   if we have not reached inner peace ourselves, what are we doing when we bring social issues to learners?
   implicit that teaching social justice is GOOD. Does this necessarily lead to inner peace?
   Important to learn mathematics in order to learn to think properly. Doing mathematics allows a study within an atmosphere that is not socially loaded, is not as emotionally loaded.
   if we wish students to be socially active in mathematics then we must also be independently active socially in mathematics.
   very difficult to lead people to challenge their own 'privilege' yet is may be just such people who have an enormous moral responsibility to enact that challenge.
   lack of sufficiently detailed data to inform a discussion of social issues

The three days closed with Circle. The co-leaders reminded participants that we began with these questions on Saturday

- How are you connected to the topic of the WG: mathematics and peace?
- How are mathematics and peace connected for you?

They invited participants to close the group's circle by addressing these questions?

- What impact has the WG discussions had on you?
- Have new issues come up for you?
- Or have you seen old issues in new ways?

A week or two after the meeting, the co-leaders circulated a participant email list along with an invitation for participants to reflect back on the work of the group and offer whatever comments they wished. These were received from Walter Wheatley (email dated June 9, 2005):

Sandy

I experienced an interesting (and disturbing) clash of cultures during an address at the Canadian Mathematics Society meeting last weekend (Editor: the weekend after the GCEDM/CMESG meetings).

The talk in question was a teaching award winner from U.B.C., who was an unabashed Platonist in his approach to mathematics and mathematics research. However, having taken the position that mathematical objects exist and their study was of great importance, he proceeded to trash the position of critics who worry about the connections between the study of 'very real objects' of importance and the exercise of power in our society. (I did not record the sequence of adjectives, but he was poking fun at terms like "neo-colonialist, deconstructionist, patriarchal, ... ") In short, he turned from the claimed reality of mathematics to a position that there is no real responsibility for the use of math, by whom, for what purposed, with what impact on people's lives. The image that came to mind was of 'hand washing' - there is no responsibility attached to working with methods of such power, or with accepting direction for work from organizations of great power. This is a position I still associate with the philosophy attributed (perhaps incorrectly) to Bertrand Russell: "a Mathematician is a person with his (sic) feet planted firmly in mid-air". What we do is real for us, and its connections to other areas and to society pass without reflection and without moral responsibility.

I think that the current separation of 'pure math' from other areas of science and social science is an artifact of the 20th century and not something that is cognitively or socially sustainable. Many of the reflections on mathematics as a refugee from confusion and pressures of daily life (inner peace) direct our attention to 'pure mathematics' - not to its ability to 'make sense' of physical or social contexts. The act of making sense (or realizing that what is happening does not make sense) tends to be disturbing rather than leading to calm detachment. There is, of course, an inner peace which comes from this struggle to make sense of things, and that is the inner peace I work towards.

The examples cited by Geoff and by David seemed to be about the connections of mathematics and the world, gauged to engender questioning, reflection, and action (or at least struggle). I find this fits well with John Mason's comment that mathematics should be challenging and require some disequilibrium. Doing mathematics requires engaged emotions. ('That fits' is an emotional comment, and plays an essential role in problem solving - as described in the book Descartes' Error.)

Since the CMS talk was in a context without debate, with many colleagues around who shared his detachment, I found I could only stare out the window and wait for my own emotional and political response to pass. I find that mathematics educators do have a different culture, better informed by watching the impact of learning (or failing to learn) mathematics on children and on societies. This may be another case of the ongoing divide captured by the chapter of Anna Sfard: "Are mathematicians and mathematics educators talking about the same subject?" There is increasing evidence the answer is no - both as a description of the concepts and the cognition and human acts of doing mathematics, and as a description of the social context and impact.

Thanks for a leading us in a great session - a source of continuing reflection and perhaps of continuing support for these reflections.

Learning Mathematics in the Early Years (pre-K-3)

Ann Anderson, *University of British Columbia*
Laurent Theis, *Université de Sherbrooke*
Ruth Dawson, *Elementary Teachers' Federation of Ontario*

Participants

Ann Anderson (UBC)	Dave Hewitt (Bristol)
Analia Bergé (ConcordiaU)	Geri Lorway (Alberta)
Caroline Bardini (France)	Lily Moshé (YorkU)
Marianna Bogomolny (SFU)	Joan Moss (OISE)
Eli Brettler (YorkU)	Gladys Sterenberg (UAlberta)
Michelle Cordy (UWO)	Laurent Theis (USherbrooke)
George Gadanidis (UWO)	Kinley Wangdi

Introduction

There is electricity in the air as participants attending CMESG gather in the lecture theatre at Fauteux Hall. Old friends greet each other up close and from a distance; newcomers begin meeting others seated around them. Following announcements and introductions, leaders of the working groups are asked to augment the descriptions already available on the website and in conference handouts. We choose to capture the essence of our goals with a brief bilingual description. Basically, we indicate that Working Group B will deal with three interrelated issues or "big ideas": nature of early math experiences, complexity of early math concepts, and support/resources for "teachers". We point out that we plan to draw on our empirical research to offer examples captured in classroom and home settings as striking instances to serve as catalysts for our discussions. We reiterate that Working Group B is structured for and around participants' contributions. We also alert our CMESG colleagues that Ruth Dawson, our third team member is not able to join us.

From here, using a diary format, we want to share with readers our sense of the discussions and events that evolved over the next three days when 14 "strangers" quickly coalesced around a common interest and curiosity about early years mathematics education. We will document the essence of the conversations, the debates and the conclusions that evolved and whenever possible will use and acknowledge the participants' words as they contributed to the collective voice that we portray here.

Day 1, May 28, 2005 [9 –10:15; 10:45-12 pm]

After we welcome participants to the first session of Working Group B, we focus our attention on the proposed format for the coming three days. Basically, we intend to begin with

introductions, to use small groups for interactive discussions and to have participants record their discussions' highlights for us. We plan to dedicate Day 1 to Preschoolers' early mathematical engagement; Day 2 to Algebraic reasoning in the early years, and Day 3 to Professional development for early years teachers. We intend to open Days 2 and 3 with a short connections/reflections session and of course, whole group closure activity was planned for the end of each day's session.

In as timely a fashion as possible we move to introductions to discover who has joined us and what has brought them to this working group, The diversity both in terms of locations from which they come, the background experience they bring and the desires they have is enriching. Some bring elementary experiences while others are more familiar with secondary and post-secondary mathematics education, and others are mathematicians. Several are parents who see this session connecting with their need/ desire to understand their children's mathematics learning. Some are motivated by their current or future roles as pre-service mathematics educators for elementary and primary teachers. For some their interest in algebra has led them here; for others it is their interest in spatial/geometric understanding and its perceived understated role in classrooms. Another speaks of children's meta-cognition and others point to teachers' knowledge. Thus, by simply greeting our participants and coming to know them, the complexity of learning mathematics in the early years becomes apparent.

Our attention turns next to beginning the discussion of early mathematics experiences. To stimulate discussion and to contextualize our dialogue somewhat, we share two video clips of preschool children and their mothers "doing" mathematics at home. Space does not permit detailed verbatim transcripts, so we choose to describe the video excerpts for the reader in order that the references made to them within the discussions can be shared meaningfully.

Video clip #1: Wooden sticks, parking spaces and spatial sense

A mother and son sit on the living room floor, and the mother is pouring out materials for making a town (some houses, trees, "sidewalks", and so on). A large collection of toy cars that they had been playing with previously lie to one side. The son takes the wooden flat sticks, "sidewalks", and begins to place them in a formation that resembles an inverted U and two more "sticks" extending the sides. He exclaims "look what I made" and after his Mom inquires, he reveals he is making "parking spaces". He continues at her encouragement to make some others, and creates several different configurations, one that looks like a Z, T and inverted U shape touching one another. [We interrupt the excerpt here. We indicate that as the session continues the child moves to parking cars, and shows an understanding of parking lots summarized by his mother's comment "not blocking them-they'll all be able to get out". His play continues with his mother associating it with their experiences with parking at a local public market and shopping area.]

Video clip #2: Playdoh, pizza and fractions

A mother and daughter sit at a child-sized table to "play" with playdoh. The child wants to make pizza and begins to flatten the ball of playdoh; the mother assists with pressing the dough into a round thick pizza dough. They are chatting, mother asks things like what shape is a pizza and the child responds triangle. As time passes, the mother asks "what about if we had two friends join us, how many pieces of pizza would we need?" This quickly turns to how to cut the "pizza", "… Cut in half and cut in half again". The mother gestures a horizontal cutting motion towards the middle of the dough and then a vertical motion perpendicular to her first gesture. The child cuts close to the centre of the dough along the vertical. She then motions as if to cut vertically again to the right of the first cut. The mother reminds her to cut across the first line. When asked how many, the child then counts the pieces and the mother encourages her to check for size equivalence. More people are added to the visit, and 6 pieces are needed (child says 8, maybe including her mom and herself in the total). She proceeds to

cut two vertical cuts near the center of the pizza, and a sliver is created. Mother uses it to check for size equivalence again, commenting that one will be hungry, and that she needs to do it so pieces are the same. [We end the video clip about here.]

To focus the small group discussions around preschoolers' early mathematical engagement, we provided the following guiding questions with which participants may engage one another in response to the videos.

- What types of experiences (in the home) prior to school support mathematical learning?
- What do we know about the nature of mathematics learning prior to school?
- What role(s) does significant others play in early mathematical engagement?
- What might we learn from children's first "teachers" of mathematics?
- In what ways should research of out-of-school experiences inform our practices in mathematics classrooms during the early years of schooling?
- What impact do/can/should (our knowledge of) children's mathematical experiences prior to school have on their school experiences?

What follows is a synopsis of that discussion which attempts to give voice to our participants' impressions and perspectives. For the substance of the small group discussions we relied on their written records. In essence, the comments focused on the specifics of the video taped activities which in turn spurred discussion of broader issues. Interestingly, in small group discussions the focus shifted accordingly for different videos. For video #1 participants spent considerable time discussing the boy's actions and the strengths he demonstrated. There was mention of his high visualization skills, his abstract thinking and his knowledge of parking cars, whereas it was simply noted that the "girl was correct in counting, but not splitting equally". In contrast the focus on video #2 was drawn to the mother and not so much the daughter. Discussion here focused on the parent-directed nature of the activity and the sense of "pressure on the mother, the daughter giving wrong answers". In fact, some participants spoke of the boy's behaviour as natural and yet spoke of the mother-daughter activity as "not natural –in front of camera- behaviour". Some participants contrasted the two activities as "constructivist approach versus rule following". It was noted that the parking spaces activity was "boy initiated action" whereas in playdoh pizzas, "the girl followed the mother's teaching"; indeed some felt there was too much intervention in Video #2. Interestingly in response to Video #2, participants wondered if fractions have meaning at the girl's age and yet no such question arose about the age appropriateness of the spatial activity of the boy. Both videos gave rise to some speculative questions, such as "How would the boy actually park the cars? If no cars to try?; "Why should portions be equal?"; In the number patterns 2, 4, why 6? Why 8?". In addition, participants paid attention to the gendering of the activity or context (i.e. the girl was in a kitchen with pizza, the boy was in a living room with cars and blocks). It was also noted that the son was positioned as active, the daughter less so. Diagrams recorded by one small group showed rough sketches of 4 parking lot designs, and a rough sketch of a circle with three intersecting lines not quite meeting at the same point dividing the circle into 6 unequal portions. Such records suggest that participants may have reconstructed the children's mathematical experiences to orient their discussion. A rough square was also sketched in one group's notes, perhaps they may have wondered if alternative shapes would have led to different responses for the daughter.

As our first session came to an end, we co-generated a concept map of key points around our first guiding question regarding the nature of mathematics (preK-3) (Figure 1).
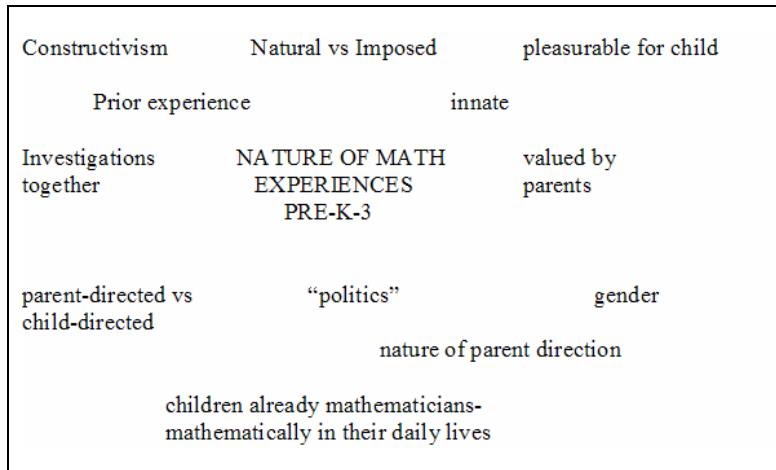
Figure 1: Concept map generated by participants, Day 1

In addition to the concept map, we kept notes on the whole group discussion that was generated. When we refocused as a whole group on the nature of mathematical experiences (PreK-3) and reflected on how such experiences should be, responses connected to the videos were juxtaposed alongside more general considerations. Participants felt that such mathematical experiences are valued by the parents as shown by the close connection between the parent and child and the pleasure for the child. As documented in the concept map, the amount of constructivism present in each video clip and the role of the parent in this process was expressed by some as a "parent directed versus child directed" dichotomy. The gendered nature of the activity in the videos was duly noted here as it had been in the small groups, but so too were speculative questions raised, such as "Would the pizza mom act differently with a boy? What would the dad do with the parking spaces?" Indeed, the nature of the pizza activity drew considerable commentary, including a strong statement that "Pizza activity gave every wrong message of what mathematics is all about. Mother modeled too much, procedures were pushed on the kid, not appropriate for a child of this age". In contrast to such specific critiques were statements like "children are already mathematicians, and they continuously operate mathematically during their whole life. No need for special math activities." Similarly, it was pointed out that for First Nation families (traditional family), many things are taken care of without recognition and thus the idea of being together and mathematics is occurring or not occurring seemed to resonate. On the other hand, it was felt by some that there are "some things children need to be told about". That there need to be "situations where names and conventions get said." Yet, the "importance of language and that mathematical objects are named by the right names" was followed by comments to the effect that "working mathematically is not only in the name". In addition, the political aspect was raised whereby policy makers seem to think differently about early mathematics in that they seem to focus more in terms of what mathematics children have to do when entering elementary school. Participants also wondered, "Are parents anxious themselves about what kind of mathematics they're supposed to know, or is the motivation more about the fun that is possible to have with these activities? Do parents have a better understanding of getting literacy started than to get mathematics learning started?" In contrast, some participants mused as to whether these early mathematics experiences might be more about "what can the parent learn from their kids rather than what can I teach them?" At this juncture permit us to digress briefly and add a reflection sparked by this last comment. Interestingly, since the I was not readily identified, it opens up multiple interpretations suggesting that "it might be more about what parents can learn from their children than what 'experts" can teach the parents"; or it could be that early

mathematical engagement may be more about "what children can 'teach" their parents rather than what parents can teach their children". Of course, "parents" are a proxy really for any significant others in children's lives.

Day 2, May 29, 2005 [9 –10:30; 11-12 pm]

We welcome our participants back and begin with a short whole group discussion focused on connections/reflections on our previous day's deliberations. As we generate a second concept map regarding how research on mathematics in the home might inform classroom practice in the early years (Figure 2), participants reiterate the value in thinking broadly about mathematical experiences for young children and reaffirm young children's strengths as mathematics learners.
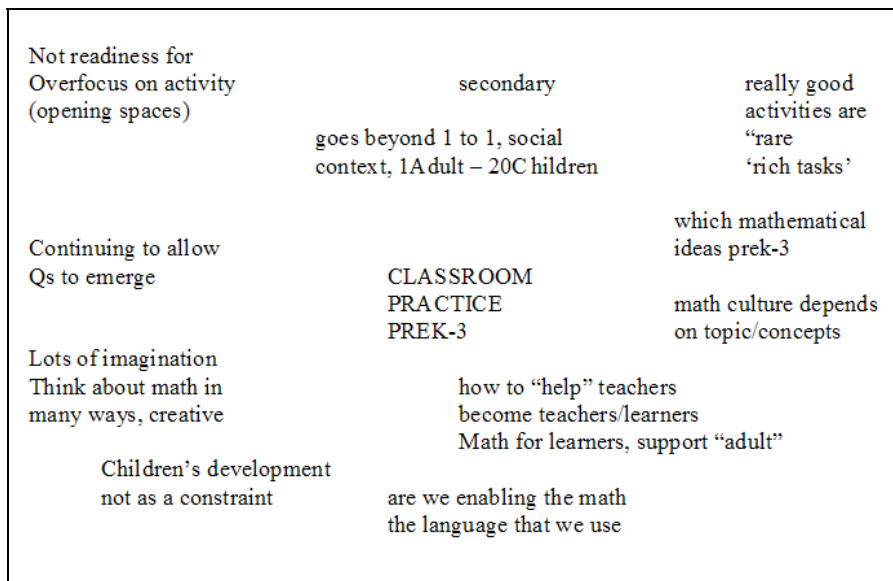
Figure 2: Concept Map generated by participants, Day 2

Although many of these key points may prove self-explanatory, we will elaborate on a few using the notes gathered during the whole group discussion. For instance, it was pointed out that we need to focus more on opening spaces for dialogue and listening and have confidence that the easiest things can turn out to be the richest. Thus, participants in Working Group B felt we should attend to "how to make mathematics creative and imaginative for children", so that there is room for play with mathematical concepts. It was also believed that whereas the discussion is often about what children cannot do, we can and should play very early on with big mathematical ideas. In a similar vein, although children's development in the early years needs to be considered, it was recognized that development might be cultural and should not be used to constrain possibilities for children. In addition, we began to question the support the interacting adult may need and the identities teachers may form.

We then (and now) turn our attention to algebraic reasoning in the early years, which we use to explore the complexity of early mathematics concepts. To begin on the subject of early algebra, we presented the following two situations, whose aim was to develop algebraic thinking in first graders.

The first situation was intended to show what kind of tasks can contribute to change first graders' view of the equals sign as a "do something" symbol. This conception is widely spread among elementary school pupils and prevents them from considering any equality that is different from an "a + b = c" structure as being wrong. We presented a task, represented in figure 3, that we used in our doctoral research, in which we tried to teach the equals sign as a relational symbol to a class of first graders. In these tasks, the children were asked to find out how many tokens are in the box if there is the same amount of token on the right side than on the left side. To solve this situation, the children necessarily had to consider the equals sign as a relational symbol.
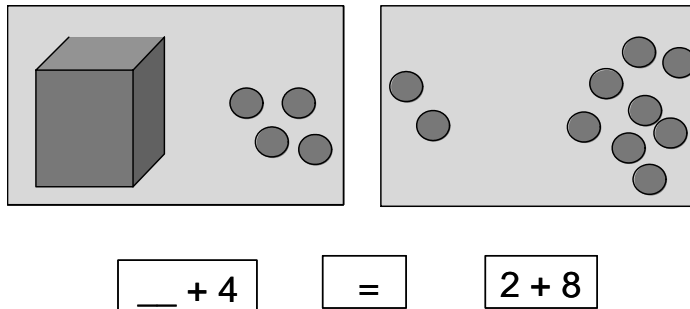
Figure 3: Representation of "__ + 4 = 2 + 8". (Theis, 2005)

We also briefly presented some results of our doctoral research, which showed that while first graders are able to make significant progress towards understanding the equals sign as a relational symbol, this shift is a significant cognitive obstacle for them.

The second task we presented was drawn from a classroom experience with first and second graders. In this task, the children were asked to find out the amount in each envelope, after being told that there is the same amount on both sides. Figure 4 shows two of the presented examples.
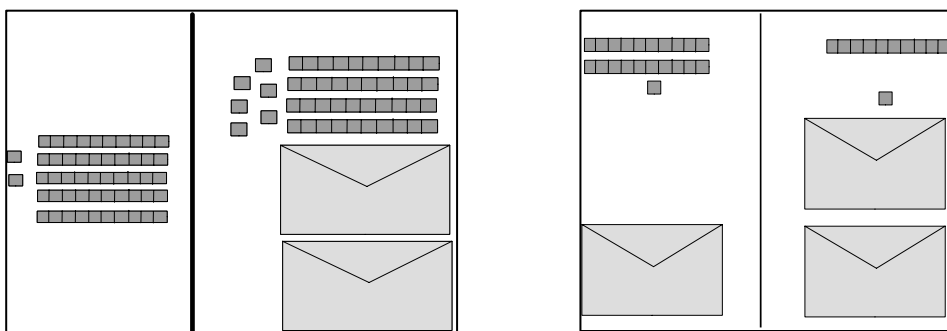
Figure 4: Two examples of tasks that contain more than one unknown.

This task differs from the one from our doctoral research, because it needs no work on the symbolic representation of numbers and because each example contains more than one unknown.

After our presentation, the following questions were submitted to the discussion group:

- How can we define "algebraic reasoning" in the early years of elementary school?
- What are your experiences with early algebra?
- Can algebraic reasoning be developed from the beginning of elementary school?
- Why develop algebraic reasoning from the beginning of elementary school?
- What are the benefits of developing algebraic reasoning from the beginning of elementary school?

The participants' discussion turned quickly to what early algebra (and even algebra) is. For some, algebra is necessarily about change, and for them, the tasks we presented were not of an algebraic nature, because they admit only one solution. The notion of "early algebra" was also questioned. If there is early algebra, can there be "just-in-time algebra" or late algebra? We then agreed to focus our discussion on the development of algebraic thinking (the term "algebraic reasoning" was also rejected, because it seems to imply some kind of justification that was not present in the presented tasks). For other participants however, algebra is much more about generalization and finding rules. For them, doing "early algebra" with first graders amounts to pushing generalizations further than what would normally be done. The discovery of patterns is also an important aspect of algebra and algebraic thinking in the early grades. For these participants, algebra does not necessarily imply algebraic notation either. As soon as children go further than just doing operations and think about operations, algebraic thinking is implied. This approach is similar to the one developed by Carpenter, Franke & Levi (2003), which has been mentioned by some participants. Others again described the early algebra more poetically as the equivalent of reading Shakespeare in the second grade in a language class. Just as for algebra, the curriculum does not promote reading Shakespeare in the early years. Early algebra is, then, about talking in the early years about a subject that is traditionally taught much later in the curriculum.

Algebraic thinking is also needed to understand the structure of a language or arithmetic. Dave Hewitt suggested that young children have to discover patterns and to generalize in order to understand and learn their first language. Therefore, some very common errors from young children (for instance, "he goed" in English), are errors of over-generalization. Generalization is also needed to understand our base ten number system. To be able to name the numbers orally, kids have to make certain generalizations, which are already a kind of algebraic thinking. This allowed some participants to state that, at this level, there was a genuine overlap between arithmetic and algebra.

The implications of the development of algebraic thinking in the early years on teaching and teacher training was also discussed in our working group. It seemed particularly important that teachers are able to recognize those situations in which they can push further a student's reasoning. The richness of an activity then seemed to depend more on the ability of the teacher to make the most of an activity than on the quality of the textbook that is used. An activity is rarely algebraic in itself. It is the teacher's ability to recognize opportunities to make pupils think algebraically that creates a rich activity.

Day 3, May 30, 2005 [9-10:30; 11-12pm]

We have reached Day 3 and our final Working Group session together. The discussion of early algebra has taken on a life of its own and continues to be animated as we meet in small groups at first to reflect on our Day 2 theme "Algebraic reasoning in the early years". Many of

the ideas discussed on Day 2 resurfaced and were interspersed with additional points that seemingly arose during today's reflections. Of particular

note was a comment about "considering the types of problems mathematicians work with" like "which numbers are more prime 8 or 6" say rather than "which numbers are prime" as opportunities for children to explore a middle ground. In addition, the brief description of a grade 4 activity where children explore "how to curve a line" was intriguing. The recurring theme of "algebra as capturing change" was at times connected with growing patterns, with shadows, with functions, and once again with "making generalizations beyond what children normally do". History of algebra, philosophy and epistemology were all touched on. A point regarding the very language we use gave us pause. It was noted that the label "early algebra" does us a disservice as it has potential to further segment our already very segmented mathematics curriculum. That said, there seem to be a collective agreement for algebra in the early years and that we as mathematics educators need to be advocates for it.

We next turned our attention to the focus for Day 3, the professional development for early years teachers and posed the following questions.

- How might we as educators support teachers' understanding of mathematics learning in the early years?
- What parameters would we establish as a community for pre-service and in-service education that supports such understanding?
- What strategies do we need to inform and influence policy writers and curriculum developers to support effective teaching and learning in the early years?
- What experiences, to align policy and curriculum at both preschool and primary (K-3) level, can we draw on to establish sound mathematics instruction?

Although it is difficult to distil from the notes when small group discussion around algebra moved to discussion around supporting teachers to teach algebra to a more general discussion of supporting teachers in the early years, key points that were captured in the notes are summarized here. Participants felt that we should "Let pre-service teachers play same games (do the same things) as we want them to play (do) with children". Many felt that teachers need more mathematical experiences but most qualified their statements in one way or another, with comments that ranged from "Get them to do mathematics that is worth doing", to "Allow them to experience mathematics differently- experience open-ended strategies" to "Emotions, promotions, frustrations – in communities – that deepen personal experience with mathematics". There was also a sense that we need to build teachers' knowledge of how mathematics is constructed by children and engage teachers in "more reflection – more on child, more on mathematics" such that the "Purpose isn't solely to do mathematics but evokes a discussion about learning/teaching mathematics". Some participants drew parallels with literacy, and raised questions of "What's literature of math – what should I (teacher) bring in" and spoke of "whole language versus phonics" and the "classics", challenging mathematics educators to reflect on what might be equivalent concepts/contexts for us. Participants also spoke of the need for the teacher to be a learner, to have experiences such as building a notation system, working with bases other than base ten, and so on; basically re-visiting those mathematical ideas that may be taken-for-granted as adults. Interestingly, drawing directly from the Working Group experience, it was felt that teachers need to grapple with questions (complex and simple) in the same dialogical ways we had over this time together. And we were reminded that teachers are immersed in the language of outcomes and the use of textbooks, and we would serve them well to assist them in coming to understand these phenomena in a way more compatible to the goals of which we speak.

Participants also voiced the need to create pedagogical models and the sense that there is no need to try to change everything at the same time. One pedagogical model included experiences where the teachers start to solve a problem themselves, then do this problem with

children, receiving and giving feedback. Teachers who thought they came up with all possible solutions learn the diversity problem solving supports. Another pedagogical model involves pre-service teachers hosting Math fairs for/in schools. Student teachers watch and interact with a large number of students engaging in their activity in small groups. These children are from different ages, grades and diverse backgrounds. Overall there was agreement that time is important, that change in practice does not occur quickly and requires sustained support. Finally, at our request, individuals specified their highlights of the conversations and we draw on those as we did on the final morning of the conference (See Figure 5) to close the deliberations of Working Group B.

Mathematical Engagement	• **Children are already mathematicians.** • **We need access to their mathematical thinking.** • **We value children's means of thinking and communicating.** • **We need to prompt children to name the "mathematical."**
Algebra	• **What is algebra? – No consensus** • **Algebra – algebraic reasoning – algebraic thinking** • **Role of numbers in algebra, role of structure debated** • **We sought the mathematical equivalent of Romeo & Juliet**
Professional development	• **Teachers need to "live" math, be math learners.** • **What math was qualified, and seemed to be related to what we want done with children.** • **Need to give teachers feedback, mentoring.**

Figure 5: "Final" Oral Report: Working Group B

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Discrete Mathematics

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Participants

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Introduction

In its edition of May 21, 2005, just prior to the CMESG annual meeting, the Globe and Mail ran a story under the heading "The puzzle London is mad about: You won't be able to resist." The article went on to describe Sudoku, a variant of the Latin square invented by Euler in the 18th Century, that has become very popular first in Japan and now also in England, where several newspapers run a regular Sudoku puzzle for their readers.

One of the virtues of Sudoku, according to the article in the Globe, is that it involves no mathematics. A Sudoku puzzle consists of a nine-by-nine square that is further subdivided into three-by-three squares. Some of the squares contain digits. The goal is to enter digits in the remaining squares in such a way that each row, each column, and each distinguished three-by-three square contains all nine digits between 1 and 9.

Of course, the puzzle does involve mathematics. However, the kind of mathematics needed is so accessible that it is not thought of as mathematics by the public: logic and patience are enough to equip the puzzler. The kind of mathematics represented by Sudoku is attractive precisely because it is engaging and challenging and yet accessible to all. Sudoku remains interesting even to participants at a CMESG conference because the surface question leads to other, deeper, mathematical issues of existence, uniqueness and enumeration. Sudoku is an instance of discrete mathematics, the topic of this working group. The questions raised by Sudoku kept us engaged over many lunches and dinners. They typify beautifully the characteristics that make discrete mathematics an appealing component of the discipline.

Discrete mathematics has been around for a long time. As far back as Fibonacci (1170-1250), whose sequence was introduced to solve an enumeration problem for counting rabbits descending from a single pair, proofs included methods that would today be described as belonging to discrete mathematics. However, as a separately designated area in mathematics it acquired its current label only when the rise of computers in the mid twentieth century suggested a greater significance for the processes and objects of discrete mathematics than had been assigned them in the past.

In mathematics research discrete mathematics has since become a separate area, with its own practitioners, courses, texts, and journals. The first research journal specifically set aside for the area ("Discrete Mathematics", published by Elsevier) began publication in 1971. Texts intended for courses at the upper undergraduate level began to appear around that time as well (Bondy and Murty 1976, Liu 1968, Roberts, 1976, Stanat and McAllister 1977, and Tucker 1980), along with suggestions that the growing importance of the subject indicated a need for its inclusion in undergraduate programs for mathematics majors (see Ralston 1981, Tucker 1981).

It comes as no surprise that discrete mathematics has also been part of the school curriculum long before it went by that name. At least as far back as the 1950's, many high school curricula included units on combinations and permutations in the senior year, though at the time these units were always embedded in courses whose titles (especially Geometry and Algebra) reflected other mathematical goals considered more significant.

The growth of the discrete mathematics school curriculum is well documented by Dossey (Dossey, 1991). In response to the emergence of discrete mathematics as a separate subject in university curricula, the Conference Board of the Mathematical Sciences in its report *The Mathematical Sciences Curriculum K-12: What Is Still Fundamental and What Is Not* (1983) as well as the NCTM in *Computing and Mathematics: The impact on Secondary School Curricula* (Fey 1984) both called for an increase in the discrete mathematics content in the school mathematics curriculum. In NCTM's *Curriculum and Evaluation Standards for School Mathematics* (1989) discrete mathematics was finally included as one of the standards for secondary level mathematics education. There we read the following summary on page 176:

In grades 9-12, the mathematics curriculum should include topics from discrete mathematics so that all students can

- represent problem situations using discrete structures such as finite graphs, matrices, sequences, and recurrence relations;
- represent and analyze finite graphs using matrices;
- develop and analyze algorithms;
- solve enumeration and finite probability problems;

and so that, in addition, college-intended students can

- represent and solve problems using linear programming and difference equations;

- investigate problem situations that arise in connection with computer validation and the application of algorithms.

The discussion of discrete mathematics in the NCTM emphasizes especially its significance for computer science, and for the world of information processing.

This identification in the 1980's of discrete mathematics as a curriculum area of growing importance, together with a sense that the subject had not entered the curriculum to the extent that might have been expected, given these early initiatives, led to the creation of this working group. "Whatever happened to discrete mathematics?" was the title suggested to us initially.

Our Activities

While all of the participants in our working group were somewhat familiar with discrete mathematics, and several had studied it quite extensively, a number of us were hard put to say precisely how it is distinguished from the other more traditional areas of mathematics. We decided, therefore, to focus the discussions around three questions:

- What do we mean by discrete mathematics?
- To what extent do we see it in the school curricula?
- Should there be a stronger emphasis on it, and why or why not?

For the first question it was clear from the outset that we were looking not for a precise definition of discrete mathematics, but rather a description of the main characteristics of the area, sufficient to address the other two questions. We began with three prototypical problems: The Towers of Hanoi; determining the lengths of the pickets in a fence whose profile is parabolic; and determining a schedule for preparation for a school play.

The Towers of Hanoi problem is canonical in discrete mathematics and likely so familiar that it may need little description. However, in the interest of completeness, the problem asks us to move a pile of disks from one of three locations to another subject to two rules. First, only one disk can move at each step. Secondly, a larger disk may not be placed on top of a smaller disk. Also, a goal is to use the least possible number of moves. Participants were given wooden models with which to engage in the problem (see picture).
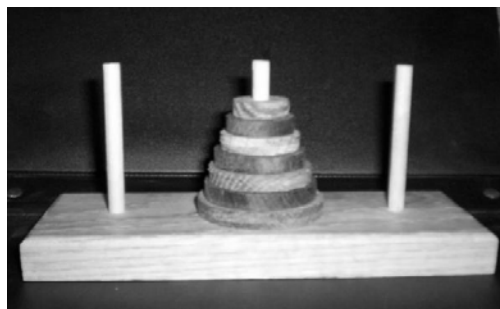
The picket fence problem involved the design of a picket fence (see picture). The tops of the pickets are to form a parabola such that the shortest picket, in the middle, is 90 cm and the longest pickets (at the two ends) 120 cm. If there are 21 pickets over a span of about 2.4 meters, and if the pickets are 7 cm wide with a 5 cm space between them, how long should we cut each of the pickets?

For the third problem the group was asked to plan the production of a musical play. The problem listed 13 tasks, some of which could not be started before others were completed. We were asked to determine how long it would take to do the planning, and the extent to which that time could be shortened if extra help were provided:

> Your class plans to produce a musical play. You and your friend Emilie are asked to do the organization. You have some experience directing and Emilie has a lot of experience with music. The play involves 5 actors and a chorus of 10 choristers. You sat down together, and made a list of all the tasks, together with the time required for each task (see table).

Task	Time Required	Person Responsible	Prior Task
A: advertising for actors, choristers	2 weeks		
B: auditions for actors	1 week	you	
C: auditions for choristers	2 weeks	Emilie	
D: rehearse with the actors	2 weeks	you	
E: rehearse with the choristers	3 weeks	Emilie	
F: joint rehearsals	1 week	both	
G: dress rehearsal	1 week	both	
H: set design	1 week		
I: construction of the set	2 weeks		
J: ordering tickets	2 weeks		
K: reserving the hall	4 weeks		
L: advertising the performance	3 weeks		
M: making costumes	4 weeks		

> Some of the entries in the third column are left blank, because you have not yet decided who is going to manage those jobs, and after some thinking you decided to add a fourth column because you realize that some of the jobs cannot be started until some others are completed. Here are some questions:

What tasks must be done before which (i.e. complete the last column)? Neither Emilie nor you have time to work on sets or costumes while you are rehearsing. Assuming you can find people to organize the design and construction of the sets, and the making of the costumes, how much time is needed before opening night?

You are having some trouble finding both a manager for set design and a manager for costumes. However, you have someone who can do both tasks. Will that delay the opening?

As we worked on the three problems, first by ourselves, and then in groups, we asked ourselves the following questions:

- How do you solve it?
- What tools are used/learned in doing this problem? Are these important tools?
- Do I think or feel differently doing this problem from when I do a problem from algebra, geometry or analysis?
- What age level is it good for?
- Is the problem engaging?
- Is the problem rich?
- Is it discrete mathematics?
- Does it fit into the curriculum, and where?

We noticed that the Towers of Hanoi represents a number of problems at varying levels of difficulty. You can simply try to move the disks according to the rules until a solution is obtained. You can try to minimize the number of moves needed to find the solution. You can try to find a system for solving the problem. You can try to code the solutions. The website http://www.cut-the-knot.org/Curriculum/Combinatorics/TowerOfHanoi.shtml is one of many that allow the user to interactively engage with the puzzle.

We noted that the solution need not only be understood recursively but can also be described iteratively. A discussion of the recursive solution is linked to the above website. For the iterative solution, David Poole described for us how it helps to arrange the three pegs of the game in a triangle. The puzzle is then solved by following the instructions

- on odd-numbered moves, move the smallest disk clockwise;
- on even-numbered moves, make the single other move that is possible.

The coding of the solutions can be done in several ways. One very visual way to code the problem is obtained by assigning a ternary integer to each position of the disks as follows: The first digit (from the left) is assigned to the largest disk, the next digit is assigned to the second-largest disk and so on. The value of the digit is then chosen to be 0, 1, or 2 depending on the peg on which the disk represented by that digit is located. Thus, for the three-disk game, the number 000 has all three disks on the first peg, while the number 122 has the large disk on the middle peg and the other two on the third. We can then associate the possible moves by means of a triangular graph. For the two-disk case the graph is as follows:
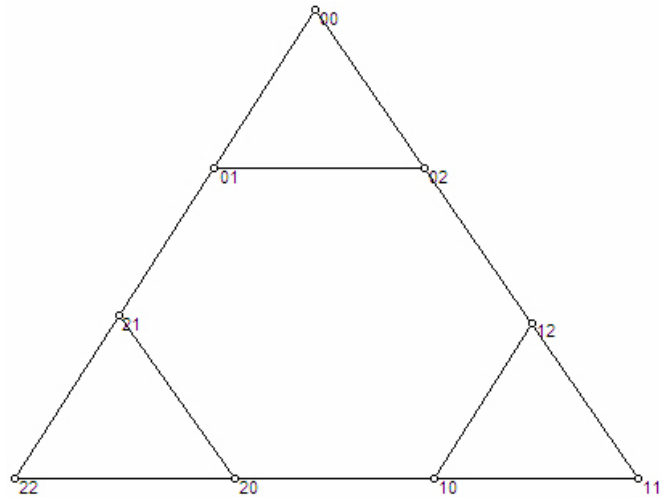
The allowable moves are indicated on the diagram by connecting line segments. The goal is to go from 00 to 22 in the smallest possible number of moves. Both the optimal solution and the number of its moves are immediately evident from the diagram. The recursive nature of the problem is revealed in the way the diagram for an n-disk puzzle is embedded in the diagram for the (n+1)-disk puzzle. Here we show these for n equal to 3:
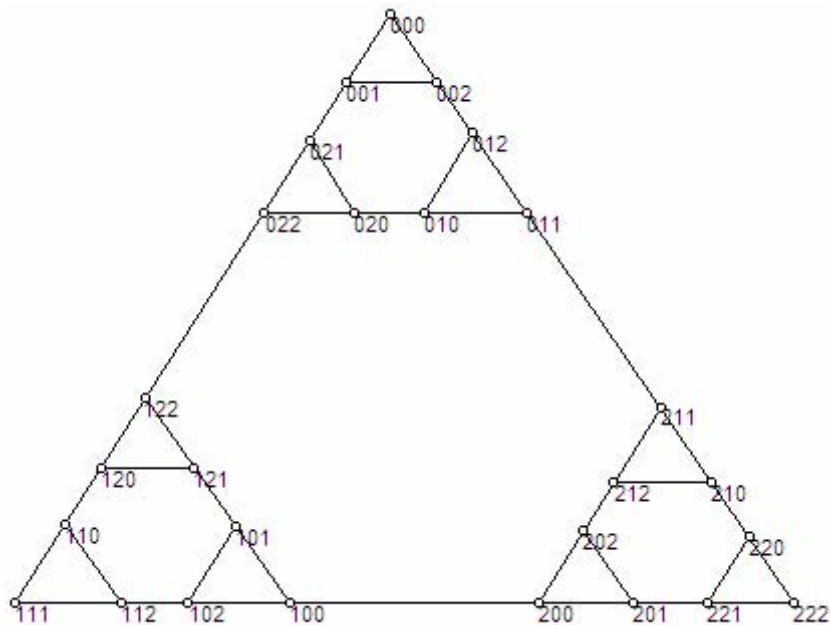
The coding indicated in this figure is related to Gray codes used in coding theory. Notice that the numbers at adjacent vertices differ only in one digit. This is the key idea for Gray codes. The possibility of creating errors in electronic coding is reduced when information is changed only one bit at a time.

As posed, the fence problem only requires that we consider the lengths of 10 pickets. A discrete mathematics perspective on this problem might use the fact that for a parabola second differences are constant, say k, and first differences are linear. As such, if the length of the ith picket is $90 + y_i$, then we have $30 = y_{10} = y_9 + 10k = y_8 + 9k + 10k = \ldots = k + 2k + 3k + \ldots + 10k = (k*10*11)/2$. This allows us to determine k and then readily compute the lengths.

Most of the group found it difficult not to consider the fence problem from a continuous perspective. We have been trained to think of parabolas in terms of algebra and analytic geometry. For most, the temptation was to write down the equation of a general parabola, and plug in the given information to evaluate the parameters. Bringing recursive ideas such as finites differences to such a problem was not part of our mind sets. In a presentation on using spreadsheets, France Caron illustrated programming the spreadsheet to determine the picket lengths with an arbitrary constant second difference. She then displayed how we could iteratively adjust the constant second difference until we had lengths that fit the endpoint criteria ($y_{10} = 30$).

Our discussion noted that much of Calculus can be done using finite differences, but not all were persuaded of the value of doing that. Indeed, there was considerable difference of opinion about the utility of a discrete perspective for understanding the Fundamental Theorem of Calculus.

The scheduling problem gave rise to two different technical representations, an Interval Graph and a Flow Graph, pictured below.
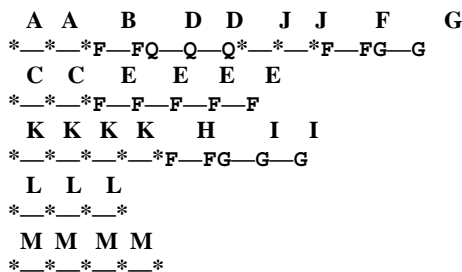
INTERVAL GRAPH   **FLOW GRAPH**
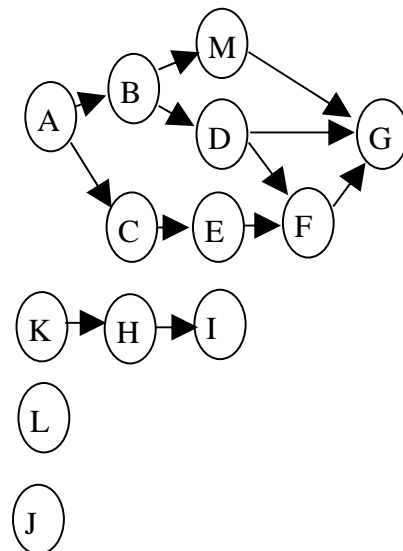
There was a distinct difference in how we felt about trying to solve this problem compared to how we might feel about solving an algebra or calculus problem. The scheduling problem has a very rich context, it is accessible without extensive training and it seems doable from the outset – one can offer a feasible solution and one doesn't feel compelled to offer an optimal solution.

The spreadsheet is the tool par excellence for demonstrating the power of discrete calculations, and this was wonderfully illustrated with a short presentation by France Caron. As mentioned above, she illustrated an iterative solution to the fence problem. Similarly, she offered an iterative model for fitting a sine curve to data. She also used a spreadsheet with great effect to demonstrate the discrete mathematics hidden behind the graphs displayed on computers and graphing calculators. In high school, students play with tables of function values, but when they ask a graphing calculator to produce a graph for the same function, it comes out looking continuous. Students can easily miss the fact that these pictures are generated by the same kinds of tables of function values. By instructing the spreadsheet software to plot the values in a table and discussing the discrepancies between the outcome and the expected picture, France shows students the relationship between step-size and apparent continuity. We were also given some examples of the ways spreadsheets can be used to demonstrate the relationship between finite differences and growth rates, and population dynamics for two populations.

David Reid was one of several people in the group who had brought Sudoku puzzles along to the discussion. He presented them to the whole group, inviting us to ask questions about the number of possible puzzles, and about the relationship between the number of solutions and the number of digits that are set at the start. In particular, he challenged us to ask some of those questions about a four-by-four analogue of the standard nine-by-nine puzzle. In the following figure there are three four-by-four Sudoku grids. We were challenged to determine the number of ways the first grid could be filled with digits between 1 and 4, respecting the rules enunciated earlier. This occupied us during breaks. Eventually, most of us reached agreement that this total is 288.

The second grid is a puzzle, found by David Reid, for which there is more than one solution even though nearly all the entries are given. The third, also provided by David, is a puzzle whose solution is unique even though only four entries are revealed at the start. These suggest extremes to the question of what initial conditions are required to generate a puzzle with a unique solution.

Our Deliberations and Conclusions

As we continued looking at these and other problems we progressed to a working understanding of what we mean by discrete mathematics. We put this in terms of the "big ideas in discrete mathematics" which we summed up as follows:

- Recursion (including spread sheet arithmetic)
- Iteration
- Pattern recognition
- Mathematical induction
- Algorithm
- Enumeration (organization of a set)

- Case analysis
- Combinatorial reasoning in proofs
- Structured reasoning as in computer programming
- Modeling and visualization using graphs and other combinatorial objects.

Of course, already a number of these concepts appear in school curricula (as written and as delivered). It is difficult to get away from discrete mathematics. In our discussion we tried to determine whether there are good reasons to increase the profile of discrete mathematics in schools. To provide focus for this part of the discussion we distributed and read a paper by Anthony Ralston "The Zero-Based Curriculum" (Ralston, 1994), in which he invites his readers to consider what they would put into a school curriculum if it were possible to create one from scratch without regard for conventions; or, as he puts it, "not constrained by the shackles of reality". He points out that "[in] zero-based budgeting all items in the budget must be justified *ab initio* and not because there was a similar item in last year's budget". As such, this perspective for curriculum development asks us to focus on why curriculum items should be included, and this became a focus for our discussion.

Often, as indicated in the short history outlined above, the growing significance of discrete mathematics is associated with the increasingly important role of computers in modern life. Ralston considers "the rapidly growing importance of calculators and computers is one of the major motivations for the zero-based approach to curriculum" (Ralston, 1994). The NCTM documents seem to emphasize the importance of discrete mathematics for applications and algorithmic methods of solving problems.

It is true that the presence of computers has not only created many new fields of application for discrete mathematics, but it has also opened new ways of doing mathematics which make use of discrete mathematics. For one thing, the possibility of efficiently implementing algorithms has radically expanded the classes of problems that can now be explored or solved, and therefore has provided a new status to recursion, iteration, and other aspects of discrete mathematics. This, among other things, offers valid and workable alternatives to explicit forms or analytical solutions (which are not always possible). In most textbooks on early algebra, there seems to be the implicit assumption that the explicit form is the only way to generalize. Without denying the fundamental value of enabling students to see patterns in such a way, we should also recognize that expressing $S(n)$ with a recurrence relation is yet another legitimate way of generalizing. Such relations often emerge naturally when modelling the governing principles of a situation as the discrete equivalent of a differential equation. They also allow for the generation of the sequence, and this can be accomplished very efficiently with the use of technology. As such the legitimacy of a recursive perspective should be made explicit to teachers.

However, citing the presence of computers for favouring discrete mathematics over other branches of the discipline would be short-sighted for a couple of reasons. In the first place, in many computer applications, even when the actual techniques employed are discrete, the intuition that guides much of the thinking is continuous. For example, in computer graphics, as well as in engineering and architecture, the ability to visualize three-dimensionally continues to be at least as important as a good understanding of the discrete mathematics underlying the computing processes that allow the detailed calculations and modelling of problems in these areas. In the same spirit that gave rise to the idea of multiple representations (graphical, symbolic, numerical), whenever possible, the combination of the different perspectives (three dimensional geometry vs. transformation matrices, continuous vs. discrete ideas , explicit vs. recursive solutions, differential vs. difference equations) could help provide a deeper understanding of the underlying mathematics. In the second place, an overly applied or technical approach to discrete mathematics could cause us to miss its inherent beauty, and its unique ability to generate problems that are intriguing, accessible to even very young

audiences, and challenging at a variety of levels--as were those we sampled in our working group.

That said, we also acknowledged the utility of discrete mathematics in settings other than those related to computers. We heard from our B.C. participants about the recent plebiscite on the issue of alternative voting schemes. We heard from Dave Lidstone, that along with his picket fence, many other projects around his house had used discrete mathematics in their construction. Some of us felt that ideas and practices of mathematical proof usually introduced in analysis courses are more readily understood by learners when presented in a discrete context. The outcome of our discussions could be summarized as follows:

Discrete mathematics:

- is accessible for learners;
- is an excellent source of problems that are engaging and that allow for multiple solutions;
- requires little technical language and therefore readily promotes communication skills;
- has the potential for drawing learners into other areas of mathematics;
- is particularly useful in the practice of responsible citizenship (e.g. understanding voting schemes);
- helps make our environment more pleasing (e.g. designing picket fences);
- helps promote the popular image of mathematics as part of human culture (e.g. playing Sudoku);
- is essential to the study of computer science.

Toward the end of our three days we tried to get a sense of the extent to which discrete mathematics is part of the curriculum already. The consensus was that while there is discrete mathematics at various levels, there should be more, it should be used better, and the connections between the discrete mathematics at different levels should be made more evident.

There is some discrete mathematics as early as kindergarten and primary school. For example the use of visual patterns constructed using manipulatives is an early instance of discrete mathematics. The Topic Group presentation by Joan Moss at this conference was an excellent example of early practice of discrete mathematics. The arrangement and re-arrangement of coloured blocks to produce patterned sequences are an important prequel to the study of counting procedures associated with probability in the middle grades and to the study of combinations and permutations in the senior year. This observation also highlights the importance of making these vertical connections transparent to the teachers. Otherwise these units run the risk of being just so many interesting but arbitrary activities that can be replaced when another more appealing activity comes along.

Not all jurisdictions pursue discrete mathematics topics to the same extent. In Nova Scotia there is some probability and combinatorics in middle school, and some discussion of finite differences in the context of experimental situations, as well as some discussion of matrices without operations in the senior year.

The Quebec curriculum is undergoing revision, which has been completed at the elementary level, and is about to begin at the secondary level. In this reformed curriculum number patterns are used to introduce algebra concepts. Probability, which used to be restricted almost entirely to grade 8 is now also part of the elementary curriculum, as a result of the reforms. An introduction to graph theory and its applications is available in the last year of high school, but functions there as a terminal mathematics course for students not taking the calculus sequence. The new curriculum is not expected to change this very much.

In Ontario, middle school discusses probability and counting, the grade 11 curriculum mandates a discussion of algebraic and geometric sequences, and the grade 12 curriculum has a separate course on Geometry and Discrete Mathematics. This course includes discussions of various kinds of proof, including proof by induction, as well as counting techniques involving combinations, permutations and the binomial theorem. This course is currently required as one of two senior year mathematics courses for entry into science and engineering programmes. However, many students find the course difficult, and some universities are starting to drop this course from its requirements

This last development indicates a trend that concerns us. Too often the topics chosen, especially for the high school curriculum, are chosen to facilitate early study of calculus. In the process of this headlong rush to get to a course that has a kind of mythical status but is not really that important or suitable for all students, a lot of really beautiful mathematics is missed.

Mathematics curricula in the four western provinces and the three territories follow a common framework under the Western and Northern Canada Protocol. The framework for mathematics curricula in grades ten through twelve includes discrete mathematics as part of the Patterns and Relations Strand, and the Statistics and Probability Strand. Specifically, general outcomes for these strands include "analyze number patterns", "apply principles of mathematical reasoning", "analyze recursive patterns" and "solve problems based on the counting of sets, using techniques such as the fundamental counting principle, and permutations and combinations" (see pages 25 and 35 of The Common Curriculum Framework for Grades 10-12 Mathematics at http://www.wncp.ca/). Of course, details of how this framework is followed vary across these jurisdictions.

In British Columbia, students follow a common curriculum in K through 9 and then choose one of three "pathways" in grades 10 through 12. These are pathways are labelled Essentials of Mathematics, Applications of Mathematics and Principles of Mathematics. Secondary school graduation requires one these courses at the grade 11 level. Entrance to post secondary institutions usually requires Applications of Math 11, but post-secondary mathematics courses ask for Principles of Math 12 as a prerequisite course. This course includes significant study of patterns in numerical sequences, permutations and combinations, and discrete probability. Indeed, questions addressing these topics typically constitute over 25% of the provincial examination for the course.

In summary, discrete mathematics is present in the curricula to some extent. However, it is not well-connected across the duration of the curriculum, it receives less attention than arithmetic and algebra, and it gets pushed aside in our eagerness to teach students calculus.

There are numerous resources for further reading, both on the significance of discrete mathematics and on ideas for introducing the subject into the classroom. Two that we recommend especially are Kenney and Hirsch (1991) and Rosenstein, Franzblau and Roberts (1997).

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Socio-Cultural Dimensions of Mathematics Learning

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> *… new knowledge, then, is constituted and arises in the social interaction of members of a social group (culture) whose accomplishments reproduce as well as transmute the culture (e.g., of the mathematical community, of teacher and students of a class, etc.).*
>
> - Bauersfeld

The goals of this working group were to explore socio-cultural dimensions of learning mathematics and their implications for teaching. Throughout the three days, participants were provided with opportunities to engage in explorations of socio-cultural theories and practices through small and large group discussions. The contexts for the discussions included the sharing of mathematical artefacts, readings of texts by socio-cultural researchers, analysis of a classroom video scenario, and pondering one's personal understandings of socio-cultural dimensions of mathematics education.

Day 1

To begin the working group, participants were immersed in an activity that emphasized the forming and communicating of relationships. Each participant was given a mathematical artefact to interpret through her/his own eyes and then to seek out others in the room with whom they felt 'connected'. The mathematical artefacts consisted of cards containing various mathematical relationships, diagrams, formulas, ideas, problems, etc. taken from junior high school textbooks. Participants were asked to become familiar with what was on their card and

then to look for four or five others in the room who had artefacts that related. No details of the types of relationships were provided as the activity was intended to have participants explore their own lenses for how they made sense of the artefacts. Once participants formed their groups, they were instructed that they would then be asked to communicate to the larger group the nature of the relationships that brought them together. In addition to placing the participants in the role of anthropologist, this activity was designed as a means for participants to introduce themselves to one another and to begin to create a community of learners within the working group.

As the participants walked around the room, sharing aspects of their artefacts, we could hear people saying such things as 'geometrical', 'linear relationship', 'a picture with four lines', 'developing ideas', 'I'm a loner', etc. After about 10-15 minutes, there were five groups formed of five to seven people in each group. Each group was then provided with a piece of chart paper to record an explanation of their artefacts' relationship/connection to each other. They were asked to respond to: What is it about the artefacts, and your interpretations of them, that brought you together? After approximately 15 minutes of rich discussions, the groups post their paper on the front board of the room. As part of the sharing, people first introduced themselves and their artefacts, and then explained their poster response.

It was interesting to note that, while the groups were simply asked to post their paper in 'a public space', once the first group posted their paper on the front board, each group followed by doing the same...
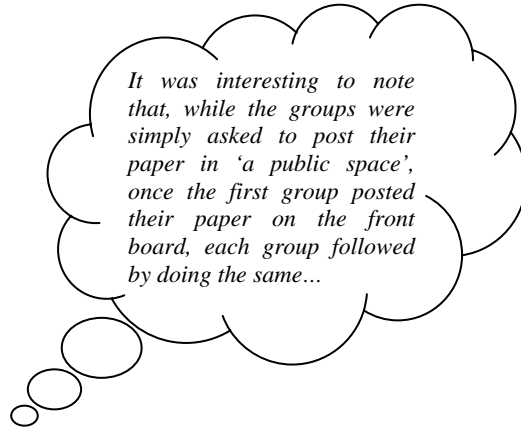
Group A used words/phrases such as content, context, empirical reasoning, data collection, small probability experiments, social, economic and cultural aspects (interpretation differences), approach to math (games vs. real life), teaching math in context, etc. in explaining their relationships. They commented, "we first looked at the mathematical content as a teacher would… and then we noticed so many socio-cultural aspects… we were starting to see these artefacts with the eyes of the children."
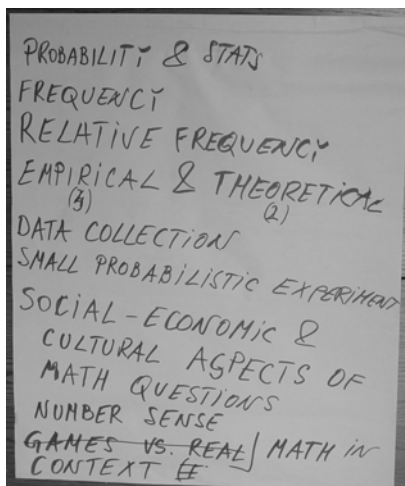
Group A
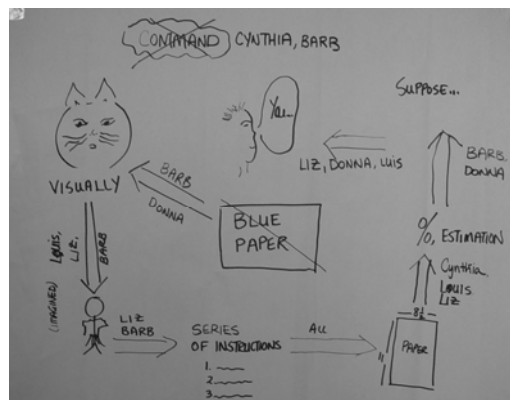
Group B

Group B began by saying, "there is no end to the discussion on what could connect us in this group." One participant stated: "I thought not having a picture might be a limitation… do I have something else in common?" Members of this group noted that there were many things not visible in the beginning that became more visible as they talked about different possible links. It is also interesting to note that one member commented that the group formation was partly based on which people she thought she would like to work with, and then she made would 'make it work' in terms of describing the artefacts' relationship to each other.
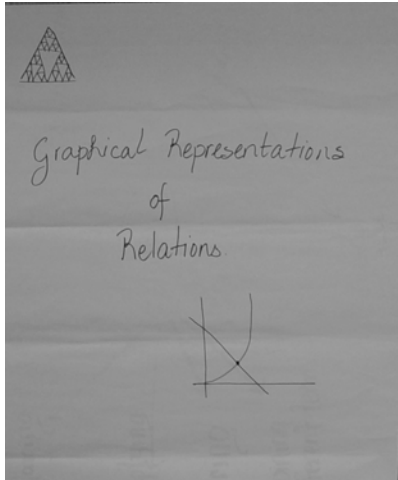
Group C
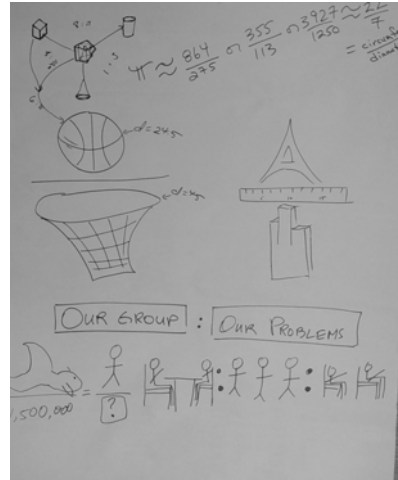
Group D

The language of Group C's explanation included such things as 'graphical', 'representations of relations', 'points on a Cartesian plane', 'getting to the core of our artefacts', while their poster had only the words 'graphical representation' on it. They commented: "if you look at our poster, it is minimal and bare… socio-culturally speaking, we tried to zero in on this core, or essence… maybe we chose graphical representation because we came to this group with such an interest, thinking about a common theme". This group also made comments about the importance of belonging and feeling comfortable first and then trying to find one thing in common between their artefacts.

Group D commented, "we identified ourselves in a rudimentary fashion by the type of mathematics problem we had." The group began to wonder out loud about how they could have formed a different group. Like a few of the other groups, this group seemed to find one another first based on comfort with, and interest in, each other, and then they focused on finding a connection. "It is interesting to see who you end up with and who you don't… we need a means to communicate but we approach [the activity] like students, without structure." They noted also how the large piece of paper was key in how it formed a "shared space" which became a gathering of ideas.

The poster created by Group E consisted of a concept web of ideas and connections. They explained how two small groups were brought together out of a need to create a larger group, to belong, to share, and to reconcile differences; for them, this meant that finding the mathematics connections was an afterthought. Even though they arrived at the concept of multiplication as the first commonality, they commented that "the more we looked, the more connections we saw", such as place value, algorithm, etc. It was interesting to note how the socio-cultural dimensions of interacting with, and listening to, other groups came into play when they commented that they began to notice more in their own group while listening to other groups explain their relationships.
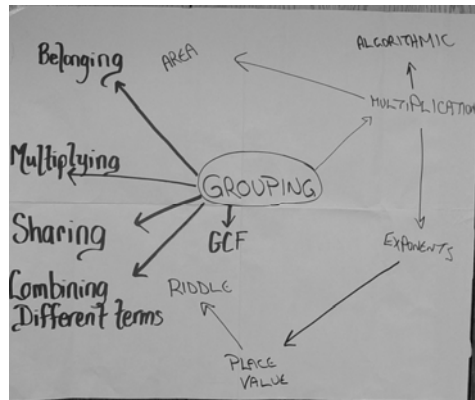
Group E
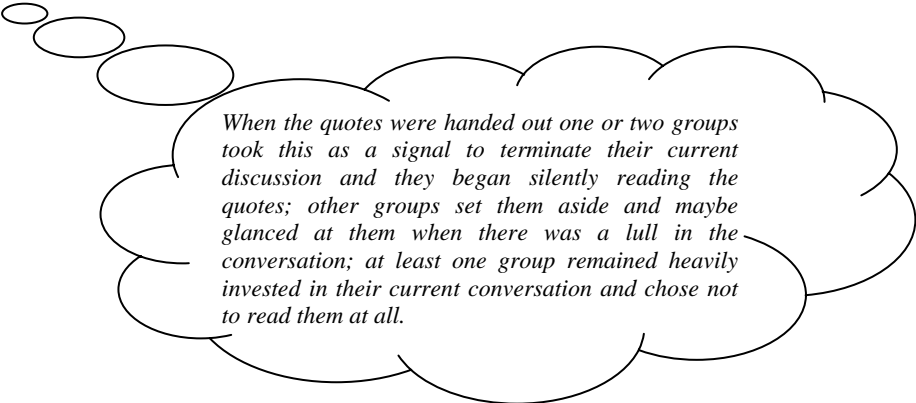
This activity was concluded by sharing with the group that the "artefacts" were randomly selected from the textbooks and that there was some concern in the planning stages whether the artefact cards would lead to any possible groupings at all. However, from the rich discussions taking place it seems that the concern was unnecessary. We wondered whether it might be that mathematics is a highly linked network hence relationships among many elements are inevitable.

At this point, new groups were formed and they were asked to engage in another level of analysis by observing the first activity and the interactions that took place from a different perspective. They were encouraged to ask the questions: who and what are we? What do the experiences of the first activity this morning say about us, about our enculturation, about the codes we use? Similarly, participants were asked to think about the students that come into their classrooms with their own codes and patterns of behaviour. How do we make visible those invisible structures that limit/exclude certain codes while reinforcing others?
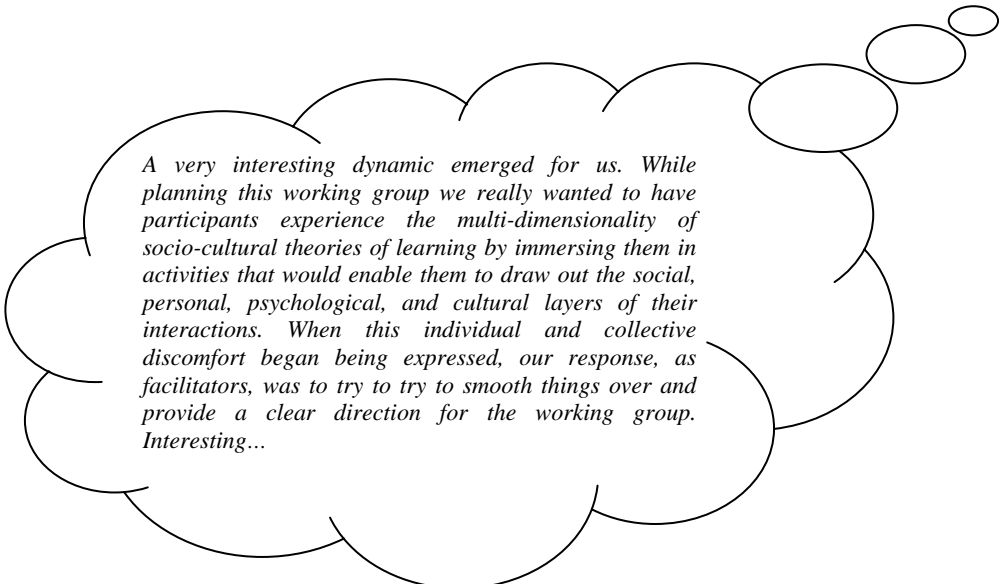
Some group members commented that such a level of analysis is less comfortable now. They felt they were being asked to speak/share from an individual perspective while trying to represent a collective. The expressed awkwardness led to comments and questions from the group members about where this working group was headed. There was some discomfort being expressed at not being clear about what, precisely, the theories of socio-cultural theories are.

When the groups entered into discussions, they generally began by sharing something about their math artefacts and their previous groups' discussion but then conversations moved in other directions, particularly into the language of socio-cultural theories and classroom experiences. Some groups were so wrapped up in their discussions that the closing activity of the morning was met with mixed enthusiasm.

In the closing activity, a sheet of paper containing one quote was presented to each of the six group members. These quotes were drawn from the writings of Ernest, Lerman, Cobb, Radford, Zevenbergen, Restivo, and Bauersfeld (see Appendix A). The task for each group member was described in quite general terms: read the quote and highlight words and/or terms that speak to you of socio-cultural aspects of learning. Since each group member received a different quote, they were asked to share their quote and briefly discuss what they highlighted and why. The working group's first day drew to a close with the suggestion that we take up the ideas and connections from the quote discussions first thing tomorrow.

When the quotes were handed out one or two groups took this as a signal to terminate their current discussion and they began silently reading the quotes; other groups set them aside and maybe glanced at them when there was a lull in the conversation; at least one group remained heavily invested in their current conversation and chose not to read them at all.

A very interesting dynamic emerged for us. While planning this working group we really wanted to have participants experience the multi-dimensionality of socio-cultural theories of learning by immersing them in activities that would enable them to draw out the social, personal, psychological, and cultural layers of their interactions. When this individual and collective discomfort began being expressed, our response, as facilitators, was to try to try to smooth things over and provide a clear direction for the working group. Interesting...

Day 2

We began Day 2 with a discussion on issues emerging from the quotes handed out yesterday. To focus the large group discussion, we asked that participants first spend approximately three minutes on an individual response to the question: What about your quote confused, excited, fascinated or surprised you? Once the participants began to share their thoughts on the quotes and their response to the above question, we noted some key ideas emerging:

The nature of social activity

- If we consider social activity in its broadest sense, then even reading a book must be social activity. Is everything that humans do considered social? If so, then even a cognitivist perspective is social.

- Socio-cultural is a dynamic process. Humans are born into a world. Nothing develops if we don't have interactions with other humans.
- Conversation, including how we read the body and gestures, is the essential form,

The role of language

- I want to accept a fully linguistic, social knowledge and that interactions with artefacts are always mediated by culture. Naming is culture. However, some of knowing is pre-linguistic, bodily feelings, for example. What if we notice things? What is the directness?
- Language is not the only means of mediation. A child's action may seem pre-linguistic and direct but is it mediated in/with language.
- Maybe there are things out there for which the only way we can talk about them is with language. By the time we realize it, it is too late. We only have language. We happen in language. That is our living as human beings. There is no way other in language.
- If a person had an experience and it was not spoken of it, then would it have occurred?
- Is contemplation cognitive?
- Would this conversation be the same if we just had e-mail? Perhaps there is something beyond the language, so we have to meet in person. Why can't you learn mathematics from texts? You need the teacher to do some things.
- There is no such thing as pure language; there is the body, chemistry. To learn from a book is very difficult.
- One element of what is going on here relates to when an individual has experiences and makes conjectures about it. Where does motivation, interest come from? We are endowed with possibilities.
- Interpretations of the baby's gestures are overlaid with the meaning. What something means must be mediated by culture. Mediation is at this stage. What they mean and their significance is culturally mediated.
- Many things happen unconsciously. How do children learn language? We can't pin down origins. Consciousness helps us learn about them.

On Theories

- In the quote by Cobb, he spoke about conflict in theory. The primacy of theories is questionable; primacy might be problematic. Also, is conflict in opposition to understanding?
- Cobb says the resolution to the debate is that the two perspectives [cognitive and social] are quite complimentary; to be cognitive is to be social.
- In the quote by Ernest, he speaks of the nature of knowledge and the nature of learning. It is important to understand both sides. What are the implications of these dichotomies? This acknowledgement has significant ethical implications (to which one participant says that the history of mathematics is one of these implications).
- Aren't we up to disrupting these binaries?
- The social bias of pedagogical practice has implications. Linguistic habits of students position them for success or failure. Students carry social baggage.
-
- On Mathematics
- Mathematics itself is a human story.
- Is there a real world out there and we are trying to construct it?

- There are many mathematical cultures. When I reflect on what I do, I behave in many ways. The child too behaves in many ways. We, as part of a culture, know how to behave within various domains.
- Mathematics is an interesting case in relation to the absolute. There is the sense that once you construct the integers, then there isn't a largest prime. Until it is framed by humans, it isn't out there. It appears there is an inevitability within mathematics.
- The creation of mathematics is a creative act.
- We don't construct mathematical meaning, we make sense of mathematics.
- Here's a thought experiment. If we could communicate with someone from another planet, I could understand their mathematics. (A response to this was: but your view of mathematics allowed you to recognize mathematics.)
- A socio-culturalist view cannot include an absolute view of knowledge. We construct mathematical systems in the same way we construct legal systems. We encounter objects that are already here and the problem is to make meaning. Vygotsky was clear that it is impossible to transmit knowledge. The importance of meaning… the importance of a concept is the making sense of it.
- We all have our own perspectives. Are we willing to look at other theories as a way to gain deeper insight?
- Here's another thought experiment: assume you are a socio-culturalist and you want to teach fractions. What would you do? How would you design it? How would you go about it?

At this point, it seemed apparent that the participants were heading toward a practical discussion of what it means to teach and learn from socio-culturalist perspectives. The quotes helped participants to understand a few key ideas of researchers and scholars who might label themselves as socio-cultural theorists, but now participants asked: What does one *see* in a mathematics classroom when it is viewed through socio-cultural lenses? At this point a research video of a mathematics lesson in a grade seven class was shown. In this episode the students put up posters with representations of a solution to a problem they had been working on and the teacher led the class in an activity of taking note of those solutions. The problem was simply stated.

Imagine a train made of toothpicks in the following shape. How many toothpicks for the next train? How many toothpicks for a train with 100 cars (squares)? Write an expression for the number of toothpicks in a train of any size.
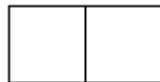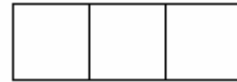
1 car	2 cars	3 cars
4 toothpicks	**7 toothpicks**	**10 toothpicks**

After the video, the participants in the working group were asked to comment on what they noticed. The working group participant comments will be expressed here as being related to three main themes: language, community of learners, and classroom culture.

Language

The participants commented that there seemed to be two different kinds of mathematics language: that of the teacher and that of the students. The teacher's language was fast-paced

and excited, while the student language was slow, sometimes unformulated, sometimes thoughtful and methodical. Part of the teacher language included using "I need you to…" when providing direction for the students. This was interpreted as creating a strong didactic contract, and contributed to perceived issues of power in the classroom. For example, the participants noticed that incorrect responses could have been followed up on to understand their meaning, thus valuing other sources of knowledge besides the teacher.

Other comments focused on how the teacher used language both as function and form. At one point in the video, the teacher asked if any of the students have a 'good thing to share' and one wonders if the students would know what this is. Within language there are clearly shared meanings but also many assumptions within the teacher's use of language.

In addition to the spoken language of the teacher, one of the participants noticed that the use of gestures in re-voicing student ideas was a very important part of the meaning making in the classroom. For example, there were gestures involved as part of articulating a distinction between 'ends' (of the train) and '*n*s' (a letter denoting a variable).

Community of Learners

One participant commented on the role of the teacher in attempting to create a community of learners. It was noticed that there was a clear move to create a community of learners but, due to the amount of teacher talk and direction that maybe some of this effort was lost. Was there a mistrust to let the students come forward and explain their own work?

Shaping and synthesizing these notes makes us wonder if we are capturing the socio-cultural flavour by settling on 'three themes'… who charged us with this task?

Several of the participants noted that they found themselves noticing whether the children were paying attention or not, and how this added to or detracted from participation in the community of learners. Students were multi-tasking— passing a note, reading, responding, etc. There was a tremendous amount of activity going on, with what seemed like minds in many places, yet at the same time apparently accomplishing what the teacher wanted them to. Participants wondered if the teacher was more tolerant of these side conversations because of her understanding of cognition from a socio-cultural perspective. Although there was some criticism of the teacher's tolerance of the diversity of those interactions, it was also noted that in a traditional classroom it is easy to mistake compliance for engagement.
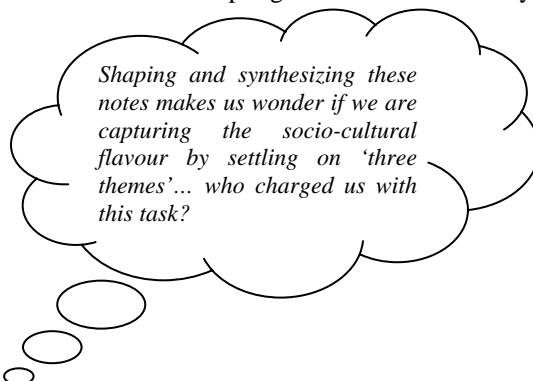
One suggestion was that one way to think about the conversation in the classroom is for the teacher to ensure that the topic of the classroom conversation is mathematics and if students want a conversation with her then it will be about mathematics. Her guiding question became based in asking, what does mathematics compel me to do in this classroom?

Classroom Culture

As a researcher, one can pay attention to different things, including the mathematics itself and how students were making sense of the mathematics. One of the participants commented that the teacher seemed to make an attempt to shift what the students did to explain *what mathematics is* (e.g., what is algebra and why we use it) to an understanding of the culture of mathematics. An example of the recognition of a micro-culture within the classroom was the

working group's discussion of the meaning of 'square trains'. The meaning was obvious within the classroom between the teacher and her students but not to use watching the events of the lesson.

A more subtle aspect of culture noted in the video related to cultural norms of who speaks up and when they speak up in the mathematics classroom. There are the 'whisperers' (Houssart, 2001) who carry on side conversations in a lesson. To what extent is their knowledge acknowledged? What is considered privileged knowledge? Where is the power if the teacher is at the front of the room? Even if the 'whispered' ideas are not validated by the teacher, they are still said and acknowledged by their peers. In other words, socio-cultural theories provide us with an opportunity to speak about what is whispered and often hidden from the teacher in the classroom/curriculum.

One participant noted that the teacher was very present, mediating as a way to make visible something that may not be visible to the students (i.e., the mathematical objects). From a constructivist perspective, this might be seen as interfering but, from a socio-cultural perspective, it seemed that the teacher was trying to refrain from addressing just one idea but making knowledge apparent for all by offering the students a way of reflecting on the (almost invisible) mathematical objects. It became apparent that some working group participants felt there was not enough focus on the linguistic aspects of mathematics, so that the language of math could be uncovered, whereas others wondered whether a focus on linguistics came at the expense of teaching and learning math. We wondered how can we marry the two better/more effectively?

Discussing the role of the teacher lead to a rather vibrant discussion on the didactic contract. One participant expressed a belief that students make a strong didactic contract in that they want to hear from the teacher, not the other students. To this participant, the concept of leadership from the front was a very serious issue; the multi-tasking that was visible in the classroom seemed to point toward a need for more discipline and focus.

At one point during this large group discussion of the video, a few participants expressed some frustration with trying to understand what **is/is not** socio-cultural. "Which things of these (on the video) are socio-cultural and which are not? "I am struggling with this… does *everything* belong to the set of socio-cultural?" This question then formed the basis for an exploration, in small groups, of two key questions:

- What are the questions we might ask from socio-cultural perspectives?
- What are the objects of analysis of socio-cultural research?

Trying to think of the 'good' questions that will create some sort of socio-cultural 'aha' moment just isn't working for us! ☺
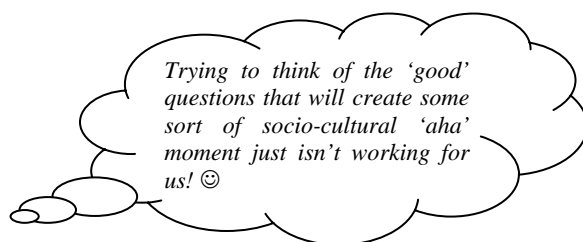
One object of analysis from a socio-cultural perspective could be identity— in the video we see one student resist (or at least show discomfort) at the idea that his work will become the focus for the class. What does this say about his identity within the mathematics class? How does the math teacher encourage (discourage) students to bring their identities into the classroom?

Asking the question, "what are the objects of study in socio-cultural research?" seems to bother a few of the participants. One commented that she/he was getting the sense that some questions are more legitimate/valid socio-cultural questions. Another participant attempted to address this concern by describing an interpretation whereby one is still viewed as asking the same questions about learning, teaching, the learner, the teacher, etc. but that the questions

change meaning when they are framed in this new socio-cultural paradigm. In this new paradigm, issues of semiotic mediation, the relationship between cognition and the body, and other issues not previously addressed become a significant part of reshaping the old questions.

With reference to an 'old' question, one participant asked a question from a didactic perspective: how do the students (in the video) know if their answers are correct? What kind of feedback is there for the students in this regard? This participant queried the group as to whether or not this question was also coming form a socio-cultural perspective. When the group responded that it was indeed an important socio-cultural question, the participant felt that there was more guesswork than validation of the mathematics happening in the video, and there was even a question as to whether any real learning could have occurred during that class if there were actually no wrong answers. This lead to much group discussion on what kind of validation is important and what dimensions of student understanding are being studied. The session concluded with some thoughts on standard symbolism in mathematics, translations from text to embodiment, and the recognition that, from socio-cultural perspectives, looking at the individual is not enough.

Day 3

The discussion on the third day picked up more-or-less where we left off from the previous day, with some very engaging questions emerging from reflections on the video and discussion:

- Where was the potential for mathematics in the lesson?
- Shouldn't the teacher wrap up the lesson by telling students what they learned?
- Does such a 'wrap-up' of a lesson impose on students what we think they learned?
- Why not have students move on to create new problems?
- How much chaos can we tolerate? Is it true that carefully thought out learning environments can be spaces where a lot of chaos is tolerated?
- Does the activity itself have occasions for the student to generalize or know he/she is right or wrong? Are there any occasions in the socio-cultural classroom room for that to happen?
- Does this occasion present itself more through the conversation than through the task itself?
- Do we provide sufficient activities to know what is coming next?
- Is there something to be gained if we are looking at the different curriculum we have in the classroom— the learning curriculum of the pupils, the teaching curriculum of the teacher? What are the students working at? What is the teacher intending?
- Do we create a tension when we have a goal (i.e., get them to a certain place) while valuing student meanings at the same time?
- What do we know/think about teacher image—teacher as facilitator, teacher as expert, teacher as learner? Do we value the moments of learning as a teacher, where the teacher does not already know?

Our conversation took at side journey as we became interested in teaching early algebra and algebraic syntax. The student work that framed this conversation included the following artefacts:
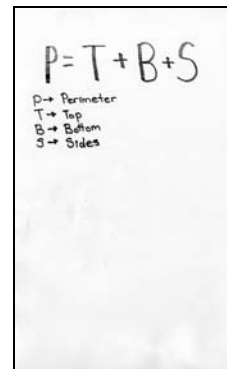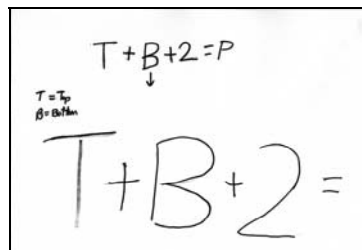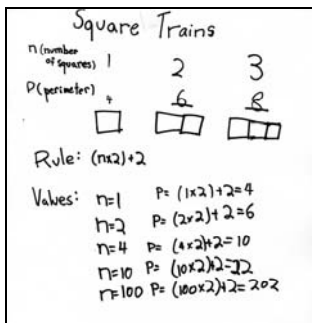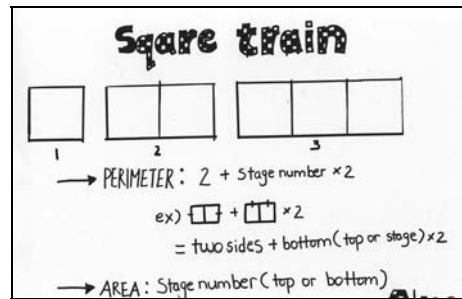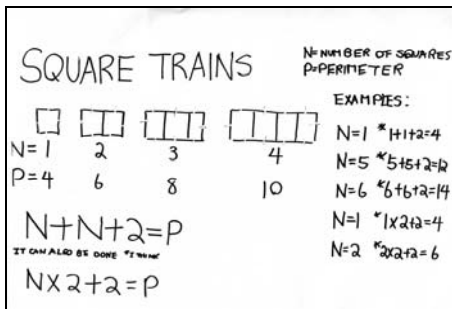
Trying to understand algebraic syntax of others provides the impetus for negotiation in the classroom. The mathematical content is already there. We are trying to make the students aware of the mathematical knowledge. There is content in the culture and it must be achieved. Here we are trying to help students make sense of the mathematical notation.

Notice how the children are using 't'. It is different from 'n' representing the number of squares. How is it that there is a favoured form? Is one form better than the other? If the child is able to use 't' as arbitrary then it may be a generalization. Is it possible to theorize if one is more abstract than the other?

We are investigating problems like this in our research. We have some mathematical texts where kids use other things to express generalization, e.g. natural language. The mathematical text is unfolding in time and space. It is made up of words and gestures. Even if it is written they are doing algebra. The 'written' plays a crucial role in expression generalization. Can those texts that express generality be more or less abstract? As long as the student remains thinking about the top and the bottom is there generality?

Other working group members expressed their interest in the student use of variables and/or generality (figure 1).

Necessary and arbitrary: the notion of arbitrary and necessary is helpful. Generalizing has nothing to do with what you write. It has nothing to do with the symbols. It helps you express but it is not the generalizing.

What was socio-cultural in what we were doing prior to what we are saying? We are situated within but we weren't talking about it. We were coming from it not talking about it.

We come into the world by how we interact. What is socio-cultural? The discussion of what the symbols mean etc.— that discussion is the application of socio-cultural theory to

education. So, what does it mean? It means that to study the discursive practices, to explain what is going on in this room, we do an analysis. We are the living theory.
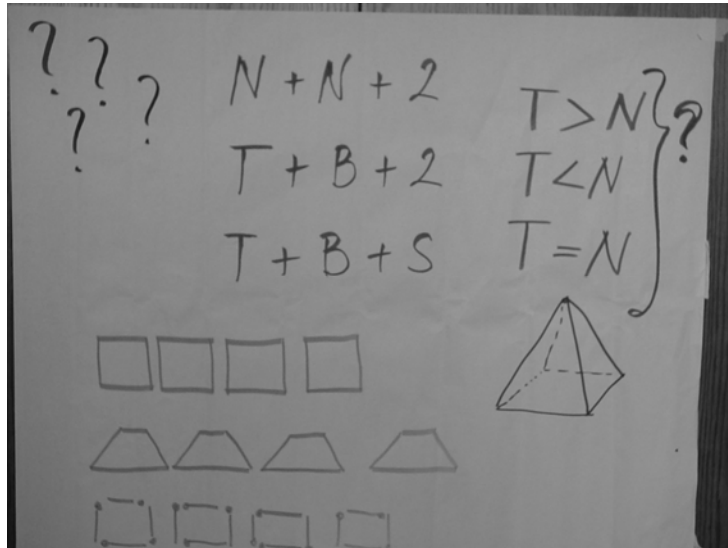
Figure 1**: produced by working group members**

This much dialogue over the variables. Have students thought about what they did? Did they engage in a discursive practice? What ought to happen in the classroom? There are norms. The explicit notion of what we are trying to work towards as a group/community.

It is normal that we don't have a single goal. It is part of our community that we develop our goal. It is valid to come with different goals. This is one of the most important aspects of socio-cultural study and theory.

At this point, the working group facilitators posed the following two questions for small group discussion:

- Did you come here *with* a socio-cultural lens or did you come here *looking* for one?
- How do you know (i.e., how do you know if you came *with* or if you found one?)

*While it appears that these two questions might have emerged for us on the spot— a teachable moment so to speak—nothing could be further from the truth. For three or more hours on the afternoon of day 2 we sat on a hallway bench, pondering **what to do**!*

The groups were asked to discuss these questions for 30 minutes, returning with reports from members who came with and members who came looking, and how they know.

Group 1:

I was looking and did not know that I already have it. I know because with the definitions of social constructivism I then connected what I am doing when I am doing research. This helps me to gather some other aspects for exploring teaching and learning and to put it together with

social constructivism. We extended the awareness and we want to ask: How can we help pre-service teachers be aware of the socio-cultural in our classrooms?

I came with a very strong lens (like gender) and I came to reflect on the socio-cultural difference. The discussions bring me back to my initial learning in "From Curriculum to Cognition." I made progress with this very practical thing and with looking at mathematical culture.

Group 2:
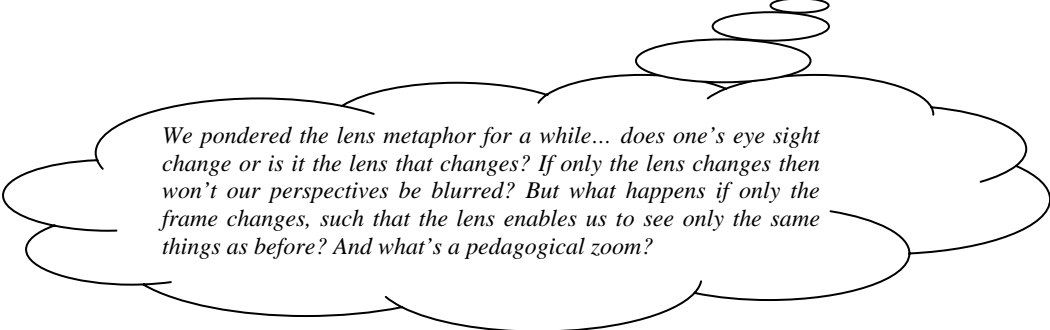
We had difficulty with the metaphor. It was like there was a lens out there. This is what comes from within it. I do things with my students and I know intuitively that it is good but not supported because there weren't these theories then. Socio-cultural theories offer justification and a language for knowing and extending what we are doing. Theory is energizing. You adopt the theories that resonate with your experience. To have explanatory principles about why something might be working. These socio-cultural discourses speak to how we are. It is not a lens but an implant.

Group 3:

We told each other about the problems that we experience in our practice and that they could be studied with socio-cultural theory. We came with theories/tools for sensitizing concepts. I came wanting to improve what I do, and I primarily teach. I have trouble working on my students' perceptions of math; they indicate 'math is not for me' (for different reasons) and I try to understand these reasons. And I thought this working group might be an avenue to work on that. I came with a lens but I didn't know it. I was searching for a language. I have some research questions that I might be able to answer using a socio-cultural theory. It wasn't the intentional but the incidental that allowed me to make sense of this.

Group 4:

When I shared with my group, I felt smart. I thought I came looking and then I found it within. I thought I had a unique discovery. But now I see others found it too. Perhaps the socio-cultural has been unpacked a lot more here. If we describe these as socio-cultural lenses, like glasses, then everything you are looking at is viewed through them. If there is a non-socio-cultural lens and you see it then you could give explanations for all the things they are looking at. Having this lens suggests that socio-cultural theories are fixed. The problem is that they are constantly changing. I have been trying to integrate socio-cultural theories and analyses into my research. But I always enjoy finding new things. Socio-cultural is not a fixed thing. It is constantly emerging.

We pondered the lens metaphor for a while… does one's eye sight change or is it the lens that changes? If only the lens changes then won't our perspectives be blurred? But what happens if only the frame changes, such that the lens enables us to see only the same things as before? And what's a pedagogical zoom?

At this point, a few participants reflected on what it means to be social and a social being, wondering if socio-cultural theories are actually focused on the individual's interactions. A response to this query directed attention to the idea that, while everything is social, socio-cultural theories try to answer the question: To what extent is the socio-cultural dimension constitutive of the self, of knowledge? This question segued into a request from the facilitators for participants to spend a few moments writing a reflection on their experience of the working group. The following collage attempts to illustrate a few of these voices and reflections…

Socio-cultural lens is a somewhat elusive concept. It varies among individuals, among groups and among theories. Moreover, each individual's socio-cultural lens is dynamic, constantly shifting and emerging. Same for group who adhere to specific socio-cultural theories or perspectives. While various socio-cultural theories have emerged in the literature also these theories are continuing to evolve.

Nearly all (or all) members of our group came into this group with some "slice" (at least) of a socio-cultural lens already in place. Some of our lenses were quite blurred at first and got somewhat more clarified during our group discussions and activities.

I now see a socio-cultural lens during analyzing qualitative data as a more holistic perspective on all aspects explicit and implicit in the data. This includes a focus on cognitive, affective, social, cultural and even spiritual aspects.

I think I know that as "Human Beings" (teachers, educators, students), we retroactively re-write our experience thru our interaction/conversation with others: "I realize thru our conversation I have been doing things that might fall under socio-cultural perspective."

$$\{\overline{SCT}\} = ?$$

It is important to note that the cognitive is not absent in socio-cultural perspectives. However, it is more than a zoom-in of the lens bringing the cognitive into focus. When the researcher zooms in on the cognitive, the cognitive is differently described by the socio-culturalist than by the Piagetian.

Cognition has been redefined by the socio-cultural theories. It goes "beyond the skin" according to Radford. The mind is distributed in the tools, the interaction, the gestures, the language used and so on. Cognition is viewed as being more social than in Piagetian perspectives. According to Piaget, cognition is individual. The social is just a backdrop for individual cognitive reorganization. But in social-cultural perspectives, the social plays a very different role with respect to an "individual's cognition.

Pre-soc cult
Classroom = maths + noise (gender, social, xxx class norm, etc)

Soc-cult – finding meaning + significance for learning of maths in the "noise"

O: What else is in the noise?

About the distinction between social constructivism and socio-cultural approaches: The difference between the social and the cultural is that the former deals with the interactive dimension of the individuals, patterns of action and doing, in short what frames my interaction with others. The latter deals with the beliefs and meaning that encompass my form of thinking and doing. It includes e.g. the historically constituted, aesthetic, conceptual knowledge that was already there before my appearing in the world. Socio-constructivism attends to the social but ignore the cultural dimension of being and knowing. This point makes both approaches different.

I come away from this working group delighted that I decided to stay in it rather than exploit the dispensation given to plenary speakers to visit every group. My delight is because I will take away some fresh insight and challenging ideas.
These include:
i) Anna's point about pushing knowledge forward, the responsibility of the teacher
ii) Luis's point that a socio-cultural view both calls for redefinitions and poses new questions
iii) A need to clarify "social" "cultural" and "socio-cultural" Internalisation is a creative process because it always leads to development.
iv) Finally, I note we have scarcely used the terms "community of practice" and "situated cognition" making me aware of the work we need to do here too. Thank you!!

The important thing for me is that we all came here with the idea of building something together. Of making sense of socio-cultural approaches and develop tools, gain practice, etc. So actually believing that it's relevant to work together on socio-cultural issues is an important point. And to me this is what we made. We talk about identity and well came here with identities and we all leave this working group with identities in evolution., questioning ourselves. The "problem" of focusing arise when looking at students work, but it also arise looking at our own goal in this group.

Je suis toujours sceptique au sujet de cette idée que tout est socio-culturel. J'ai discuté de cette idée avec plusieurs personnes du groupe au moment des pauses, mais je pense que ce point de vue ne permet pas de prendre en compte certaines expériences d'enfants et d'adultes qui me semblent très personnelles.

Certains ont décrit ces expériences à l'aide de termes comme « l'extase mathématique » ou la « comtemplation ». Même si ces expériences peuvent résulter d'apprentissages faits dans un contexte socio-culturel, il n'en demeure pas moins que leur caractère très personnel fait qu'elles sont souvent ignorées par l'enseignant et après coup par l'apprenant qui en vient à les considérer non pertinentes. L'expérience des enfants ne peut pas être mis uniquement sur le compte de l'expérience socio-culturel.

Comme plusieurs personnes du groupe, je suis arrivée ici en pensant que je n'avais pas de « socio-cultural lens », mais j'ai découvert que j'en avais et que je ne peux pas faire autrement que de me définit comme enseignante à partir de ces lunettes. J'ai beaucoup appris en écoutant les réactions des autres vis-à-vis la situation d'enseignement présentée. J'ai beaucoup appris de ces différents points de vue.

J'ai aussi pris conscience que lorsque je pense à ces lunettes socio-culturelles, je pense davantage aux enfants. Toutefois, je crois que c'est autant avec les futurs enseignants. Ils arrivent à la formation des enseignants avec un bagage dont on doit tenir compte dans notre enseignement.

Je vous remercie pour cet aterlier et je regrette de ne pas avoir pu y contribuer davantage (à cause du langage!).

I thought I may have acquired some understanding, I also have more questions than I had 3 days ago. "The socio-cultural" is usually (often_ followed by a noun in plural—theories or lenses. So, what are the theories? What are the different theories and how many of them are there? (3? 17?) Also, what would "cultural" approach (without "cultural") look like? Are they always together?
And—so what? questions still there. How does my awareness of socio-cultural aspects changes me as a researcher or as a teacher, or does it?

Appendix A - Quotes for Discussion (Day 1)

Just as the historical construction of mathematics and mathematical knowledge is central to social constructivism, so too is the social aspect of knowledge. Knowledge production is based on the deliberate choices and endeavors of mathematicians, elaborated through extensive processes of reasoning. Since both contingencies and choices are at work in the creation of mathematics, it cannot be claimed that the overall development is either necessary or arbitrary. Following Bloor (1984), Harding (1986), and others, mathematical knowledge is understood as social, cultural, and public, and not as external, absolute or otherwise extra-human. Mathematics is viewed as basically linguistic, textual, and semiotic, but embedded in the social world of human interaction. The form in which this is embodied in practice is in conversation, understood in the extended sense of Rorty (1979), Harré (1983), and many others who take conversation as a basic epistemological form. (Ernest, 2004, p. 25)

Taking conversation as an epistemological starting point has the effect of re-grounding mathematical knowledge in physically-embodied, socially-situated acts of human knowing and communication. It rejects the Cartesian dualism of mind versus body, and knowledge versus the world. It acknowledges that there are multiple valid voices and perspectives on knowledge. And, as Habermas (1981) notes, this acknowledgement also has significant ethical implications. (Ernest, 2004, p. 26)

Culture, language and meaning precede us. We are born into a world already formed discursively. The reality or otherwise of the world or the certainty of our knowledge of it are not the issue: the issue is that we receive all knowledge of the world through language and other forms of communication. What things signify is learned by us as we grow into our cultures, the plurality arising from the multiple situations that constitute us: gender; class; ethnicity; colour; religion, and so on. Although we experience physical interactions in addition to social ones and we learn to use artefacts, what the physical objects and the nature of those interactions mean and what are the purposes and functions (history) of the artefacts for the individual is always mediated by culture. Physical interactions and artefacts, therefore, are also inherently social. Knowledge contents are culture specific, and consequently so too are world-views. (Lerman, 2001, p. 91)

These cognitive and sociocultural perspectives at times appear to be in direct conflict, with adherents to each claiming hegemony for their view of what it means to know and learn (Steffe, 1995; Voigt, 1992). Thus, there is currently a dispute over both whether the mind is located in the head or in the individual-in-social-action, and whether learning is primarily a process of active cognitive reorganization or a process of enculturation into a community of practice (Minick, 1989). Similarly, the issue of whether social and cultural processes have primacy over individual processes, or vice versa, is the subject of intense debate (Fosnot, 1993; O'Loughlin, 1992; van Oers, 1990). Further, adherents to the two positions differ on the role that signs and symbols play in psychological development. Cognitive theorists tend to characterize them as a means by which students express and communicate their thinking, whereas sociocultural theorists typically treat them as carriers of either established meanings or of a practice's intellectual heritage. In general, the attempts of the two groups of theorists to understand the other's position are confounded by their differing usage of a variety of terms including *activity*, *setting*, *context*, *task*, *problem*, *goal*, *negotiation*, and *meaning*. (Cobb, 1996, p. 35)

I think that if, as the socio-cultural perspective suggests, knowledge is a process whose product is obtained through negotiation of meaning which results from the social activity of individuals and is encompassed by the cultural framework in which the individuals are embedded, the history of mathematics has a lot to offer to the epistemology of mathematics. Indeed, historico-epistemological analyses may provide us with interesting information about the development of mathematical knowledge within a culture and across different cultures and provide us with information about the way in which the meanings arose and changed; we need to understand the negotiations and the cultural conceptions that underlie these meanings.

A cultural historico-epistemological investigation may inform us about the way in which competitive research programs confronted each other at a certain moment in the development of mathematics and to better understand the issues of such confrontations, seeing the confrontations not only through the cognitive lenses of the victorious programs but also within the context of the sociocultural values and commitments at stake in such confrontations [see Glas, 1993]. (Radford, 1997, p. 32)

The role of classroom interactions in the construction of mathematics meaning has been well documented, particularly by those working in the area of constructivism. This body of literature has been powerful in illuminating the role and importance of interaction in the negotiation and development of mathematical meaning. What is less researched is the political dimension of such interactions whereby the competencies needed to participate effectively, as determined by the hegemonic culture embedded with such interactional practices, are closely aligned to the social background of the students. This chapter seeks to explore one aspect of interactional patterns in mathematics classrooms in terms of the social milieu within which such interactions occur and the subsequent potential for students to participate effectively within such contexts. In doing so, my purpose is to raise awareness of how some pedagogical practices can be socially biased in order that they may be identified as contributing to the successful (or failed) participation in classroom dialogue. As a consequence of this analysis, some of the apolitical assumptions that have been built into the constructivist writings may also be challenged… It is argued that students enter the school context with a linguistic habitus that predisposes students to interact and talk in ways that will be recognized or marginalized in and through the pedagogic practices of the classroom. Where students enter the classroom with a linguistic habitus congruous with the legitimate linguistic practices of the classroom, such habitus becomes a form of capital that can be exchanged for academic success. (Zevenbergen, 2001, pp. 201-202)

Take seriously the fact that you and your students are a collectivism, and that your communication is based on collective representations. Whether you stand at the front of the room facing your students, or sit among them, you are not dealing with a set of individuals but a collectivity... From a sociological perspective, there are no individuals. But this does not mean that there are no persons. And if agency, free will and responsibility are eliminated by a radically sociologized view of person, they come back to like in a political framework. Persons are real, and they are not simple cogs in the collective machine; but they are through and through social. This fundamental fact must my kept in mind whenever teachers and students interact on-on-one, face-to-face.... Given up the idea that the basic relationship in the classroom is between textbook and learner, or teacher and learner, or textbook/teacher and learning. Instead take seriously the epistemological potentials extant in the collectivity-- between and among students (including the teacher). And learn to seek the genesis of learning and knowledge in interpersonal relationships (Skovsmose, 1993). In adopting this approach, consider the dialectics of people and technologies (Katheryn Crawford, Chapter 6), but

without letting it obscure the fundamental emotional coupling that ties people together in social networks.... (Restivo, 1999, pp. 131-132)

… new knowledge, then, is constituted and arises in the social interaction of members of a social group (culture) whose accomplishments reproduce as well as transmute the culture (e.g., of the mathematical community, of teacher and students of a class, etc.). The notion of "negotiation is the art of constructing new meaning" (Bruner, 1986, p.149) goes beyond the limited psychological focus on the dichotomous relations between students and subject matter, or between teacher and students, and opens the field for sociological, interactionist perspectives respectively (cf. Mead, 1934; Mehan, 1975, 1979; Miller, 1986). (Bauersfeld, 1988, p.39)

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Topic Sessions

Sessions thématiques

Partition of Integers and Their Reciprocals as Hidden K-12 Mathematics Curriculum

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Introduction

Soon after the 1989 publication of the National Council of Teachers of Mathematics *Curriculum and Evaluation Standards for School Mathematics*, the importance of revealing conceptual linkages among topics and ideas within the traditional curriculum through the use of technology became evident (Kaput & Thompson, 1994). It was further argued (Balacheff & Kaput, 1996) that technology is having an ever-deepening impact on the curriculum dimension of mathematics education. The research cited above highlights the importance of inquiry into the interplay between technology and the mathematics curriculum, which, in particular, becomes an essential consideration in designing teacher education programs.

The intent of this paper is to introduce to the CMESG/GCEDM community the notion of hidden mathematics curriculum in teacher education proposed recently by the authors (Abramovich & Brouwer, 2003a, 2003b, 2004). The general notion of hidden curriculum can be traced back more than three decades (Jackson, 1968) and has received much attention in foundational educational research (Ginsburg & Clift, 1990). While a hidden curriculum framework in a traditional sense explores tacit features that structure life in schools, hidden mathematics curriculum includes tacit concepts and structures that underlie a variety of school mathematical activities. The notion of hidden mathematics curriculum attempts to bridge practice and research as it is based on the observation that many mathematical activities across the K-12 curriculum, seemingly disconnected from a naïve perspective, are, in fact, permeated by a common mathematical concept or structure, traditionally hidden from learners because of its complexity. Such complexity may be either procedural or conceptual in nature. It should be noted that while the traditional conception of hidden curriculum has a negative connotation for learning, the notion of hidden mathematics curriculum is proposed as a positive learning framework. The authors' approach to investigating the idea of hidden mathematics curriculum in teacher education is to find and work with a series of topics found across the curriculum that from a deeper perspective may be described by a common mathematical concept. Thus the hidden curriculum approach connects research and practice within mathematics teacher education. Technology has great potential to enhance this approach through appropriate pedagogical mediation.

It is often observed that teachers of mathematics do not have sufficient mathematical background to see the general concepts behind particular phenomena. This lack of understanding contributes to the communicating of mathematics to learners in a disconnected fashion, so that elementary students believe that the problem solving work that they are doing

is limited to their grade level. By the same token, secondary students cannot see the connection between problem solving they are currently engaged in and their earlier mathematical experience. Thus, teachers can use the knowledge of hidden concepts and structures in the mathematics curriculum to extend the curriculum in both directions. This suggests the importance of exploring a hidden curriculum framework across all levels of mathematics teacher education.

The notion of hidden mathematics curriculum may have a profound impact on mathematics teacher education provided that prospective K-12 teachers (hereafter referred to as pre-teachers) are given the opportunity to learn advanced ideas in a social context of competent guidance enhanced by appropriate technology tools. A pedagogic mediation of technology-integrated hidden mathematics curriculum framework supports the advancement of Freudenthal's (1983) theoretical construct of the didactical phenomenology of mathematics as "a way to show the teacher the places where the learner might step into the learning process of mankind" (p. ix). In other words, technology-enabled learning in a social milieu of expert-novice relationships opens windows into the hidden meanings of, otherwise perceived as elementary, mathematical concepts.

Such a focus on expertly assisted learning of mathematics by pre-teachers brings to mind one of the basic tenets of the Vygotskian theory of education which considers social interaction as the primary educative mechanism and conceptualizes learning as a transactional process of developing informed entries into a culture with the support of more capable members, or agents of this culture (Vygotsky, 1986). As far as a mathematical culture is concerned, the above notion of hidden mathematics curriculum may serve as a powerful intellectual link between the two concepts of Freudenthal's didactical phenomenology of mathematics and Vygotsky's zone of proximal development (ZPD) that learning by transaction creates. The latter concept was based on the assumption that human learning is essentially a social process in the sense that what one can do with the assistance of a more knowledgeable other fully characterizes one's cognitive development.

This combination of pedagogical and psychological theories provides theoretical underpinning for the didactical framework of hidden mathematics curriculum. More specifically, this paper argues that the pedagogy of revealing hidden curriculum messages to pre-teachers in a social milieu of computer-enabled learning creates the ZPD that, in turn, provides the basis for one's profound understanding of elementary mathematics. Motivated by work done with pre-teachers in various mathematics education courses, this paper shows how technology tools, such as spreadsheets, computer algebra systems (CAS), and dynamic geometry software, enable an informal journey into hidden aspects of the formal content of the school mathematics curriculum.

Partition of Integers as Hidden Mathematics Curriculum

One profound concept that unites many of the topics found across the school mathematics curriculum is the partition of integers. It may be due to the complexity of mathematics behind this concept that it has not been explicitly highlighted in the curriculum as such. Yet, partitioning problems hidden in mundane arithmetic are present in the curriculum from early grades and continue throughout the school curriculum. Consider the following problem: *It takes 37 cents in postage to mail a letter. A post office has stamps of denomination 1 cent, 7 cents, and 29 cents. In how many ways can one make this postage out of these three types of stamps?* Practicing addition in a situated learning environment is an early grade appropriate partitioning task. It appears that young children, and quite possibly their teachers, would not likely be able to find all eight solutions that the three stamp denominations provide. However, whereas it might not be important for the children to solve the problem completely, it would

be a reasonable expectation for the teachers to do so because some children may want to know how close their efforts are to the complete solution.

A way around the apparent complexity of the postage and like problems is to provide teachers with tools that allow them to model the problems electronically. It is through such modeling that teachers are able to develop correct expectations for the problem solution and create similar problems by changing appropriate parameters involved. Thus an informal investigation of the partitioning problems of the above type can be conducted through numerical approaches enhanced by technology. Being embedded in technology, advanced mathematics can be discussed in an applied context. In such a way, the use of technology allows for implicit knowledge about partitions to be extracted from the hidden mathematics curriculum and thus to become more explicit. Such a pedagogical alternative enables one to see how to explore partitions at various levels of the school curriculum, internalize ideas, concepts, and problem solving strategies used at different grade levels, view them from a common perspective, and appreciate mathematics in K-12 curriculum as an integrated whole. In addition, tools developed in the context of elementary curriculum can be appropriated for developing curriculum materials at the secondary level.

Technology and Partition of Integers

One technology tool that can be used effectively in exploring the above problem with stamps is a spreadsheet. Its computational and operational capability makes it possible to represent numerically a three-dimensional mathematical model given by a linear Diophantine equation in three variables. For the postage problem such an equation has the form $x+7y+29z=37$. Pedagogically speaking, the use of a spreadsheet enables a teacher in preparing for the lesson on problem solving with multiple solutions not only to find all those solutions, but better still, to explore many interesting questions arising from this and like situations on a level not accessible otherwise. For example, one may wonder if there is a combination of three stamps that uniquely makes 37 cents postage. Details on programming such a spreadsheet can be found elsewhere (Abramovich & Brouwer, 2003a).

Another form of technology that can be used effectively as a modeling tool in the context of problems of the postage type is a CAS such as Maple. Theoretically, the software can be used to solve partitioning problems of any arbitrary dimension. That allows one to extend the postage problem to more than three stamp denominations, for example, to find the ways of making 37 cents postage out of 1 cent, 7 cents, 10 cents, 19 cents, and 29 cents stamp denominations. A Maple solution based on a program described elsewhere (Abramovich & Brouwer, 2003a) gives an answer 22.

As an alternative to using Maple (or a spreadsheet), pre-teachers can be introduced to the Graphing Calculator 3.2 [*GC*] produced by Pacific Tech that, in addition to being a relation grapher, has symbolic manipulation capabilities, including the expansion of products of polynomials. The need for such an expansion in the context of partitioning problems arises if one considers the problems from a broader perspective. With this in mind note that, in general, the partition of a positive integer is its representation as a sum of counting numbers without regard to order. A formal approach to counting the number of such partitions deals with the method of generating functions that differentiates whether these numbers can enter a partition more than once or at most once. In the first case, this method enables one to find the number of partitions of 37 into the summands 1, 7, and 29 as the coefficient of r^{37} in the expansion of $(1+r+r^2+\ldots)(1+r^7+r^{14}+\ldots)(1+r^{29}+r^{58}+\ldots)$. Theoretically, by expanding this product one can find that eight terms have an exponent of 37. Each such term brings about a partition of 37 cents postage into a combination of stamps of 1 cent, 7 cents, and 29 cents. The situation is different for the second case where numbers can enter a partition at most once. For

example, for the above five stamp denominations, the generating function has the form of the product $(1+r)$ $(1+r^7)$ $(1+r^{10})$ $(1+r^{19})$ $(1+r^{29})$. Expanding this product of five binomials results in two terms of r^{37} thus confirming conclusion that can be obtained through simple arithmetic.

Finally, a combination of software can be put to work in the context of developing research-like experience in mathematics among secondary pre-teachers through technology-enabled problem posing. To this end, the teachers can be asked (by using a spreadsheet) to modify the problem of making postage out of three stamp denominations to fit the high school curriculum; that is, to pose a problem that leads to a system of three simultaneous equations in three variables. While the spreadsheet can be useful in posing a problem that leads to such a system, the *GC* can be used as a tool for solving the problem graphically by taking one variable as a slider-controlled parameter and manipulating the slider until the convergence of the graphs of the three equations to a single point on the plane is realized. More details about this approach can be found elsewhere (Abramovich & Brouwer, 2003b).

Partition of Unit Fractions as Hidden Mathematics Curriculum

A related but distinct concept that belongs to a hidden domain of K-12 mathematics curriculum is the partition of unit fractions into sums of like fractions. There are problems both within and outside mathematics in which the importance of such representations arises. It should be noted that there has been a recent increase of interest in problems involving unit fractions; however, the primary motivation to incorporate these ideas into mathematics teacher education was prompted by a hands-on activity of covering one-half of a circle with other fraction circles observed in an elementary classroom in rural upstate New York. A hidden mathematical depth of this activity resides in the fact that various seemingly unrelated problems found in K-12 curriculum are, in fact, equivalent to partitions of unit fractions (including those smaller than one-half) into the sums of like fractions.

Following is a set of geometric explorations that have been linked to the partition of unit fractions into the sum of like fractions. (1) *How many rectangles with integer sides whose areas are numerically equal to their perimeters are there?* (2) *What is the total number of rectangles with integer sides whose area, numerically, is n times as much as its semi-perimeter?* (3) *How many ternary tessellations (in which three polygons share the same point as vertex) are there?* (4) *How many right rectangular prisms are there with different integer sides and volume numerically equal to surface area?* (5) *What is the total number of right rectangular prisms with integer sides whose volume, numerically, is three (four, five, etc.) times as much as the half of its surface area?* In addition, one may explore: (6) *How many pairs (triples) of workers can be hired to complete a job in n days?* Note that none of these explorations requires partitions with more than three fractions, although the last one can be easily extended to more.

Technology, Unit Fractions, and Formal Mathematics

The activity of covering fraction circles representing one-half as well as those representing smaller fractions can be enhanced by the use of custom tools created within The Geometer's Sketchpad (*GSP*). For example, using these tools elementary pre-teachers found two ways that one-half (as well as one-third) is a sum of two unit fractions (Abramovich & Brouwer, 2004). At this point formal mathematics was introduced when they were asked to use mathematical reasoning to explain to the class (and in written form afterwards) the correctness of their hands-on finding in terms of fraction circles.

The sense that mathematics proof is concerned with the "public acceptability of the knowledge being discovered" (Bell, 1979, p. 368) is consistent with the Vygotskian notion of

learning as a social activity. Asking pre-teachers to communicate their proof schemata through written speech (as part of portfolio assessment) is a form of creating ZPD. Writing proofs may be seen as an elevation to higher ground what Vygotsky called "second-order symbolism, which involves the creation of written signs for the spoken symbols of words" (Vygotsky, 1978, p. 115). Vygotsky argued that such symbolism can be developed through a meaningful play which in the case of pre-teachers involved the use of multicoloured electronic fraction circles, a kind of toys for adult learners.

While *GSP* can be used to explore experimentally question (1) listed in the previous section, its equivalence to the partition of one-half can be established algebraically. Next, a spreadsheet environment can be introduced to enable pre-teachers' use of computing in finding answers to questions (2) and (6) for different values of *n*. Regarding question (3), *GSP* can be used to help pre-teachers to achieve conscious control over the conceptual system of rotation in the context of geometry of regular polygons. Finally, questions (4) and (5) about solids can be explored with another spreadsheet environment in which one can put to work its three-dimensional computational capacity. More details about technological mediation of hidden curriculum framework in the context of partition of fractions can be found elsewhere (Abramovich & Brouwer, 2004).

Conclusions

The notion of hidden mathematics curriculum advances several agendas within the realm of technology-enhanced mathematics teacher education. First, this notion enables one to introduce technology into mathematics teacher education programs in multiple ways. One option is to introduce various tools of technology into a program to support mathematical investigation and connection building without teaching special courses on technology. Another option is to develop a follow-up course on technology grounded in its educational application in which the pre-teachers concurrently learn technology and mathematics in the context of creating computational environments already familiar to them.

Second, the notion of hidden mathematics curriculum enables pre-teachers' learning of mathematical concepts (traditionally considered advanced) within the context of a mathematics education course. Pre-teachers develop an understanding of how these deeper concepts provide common structure for the explicit curriculum and connect its different ideas and representations. In addition, in a technology-mediated intellectual milieu, achieving control over a concept occurs in a socially created ZPD where intuitive understanding of the concept meets the logic and formalism needed for its representation through a computational medium. The product of this human-computer interaction aided by competent tutelage is a solution, which, once internalized, becomes a part of one's consciousness.

Finally, a hidden mathematics curriculum approach that identifies deep concepts and structures of mathematics makes it possible to elevate pre-teachers' learning of mathematics to higher ground, so that "the new higher concepts in turn transform the meaning of the lower" (Vygotsky, 1986, p. 202). Climbing to this new height creates in pre-teachers greater self-confidence in their abilities to teach mathematics. At the elementary level, traditionally poorly understood topics, like formal arithmetical operations with fractions, when highlighted from a different, sometimes advanced, perspective in which the pre-teachers experience success, leads to a greater understanding of and confidence in those topics. Indeed, classroom observation of pre-teachers working through a series of carefully graduated combination of hands-on and technology-enhanced investigations indicated that their confidence steadily grew as the activities went along. Using technology, the pre-teachers were able to make significant progress in connecting their informal explorations with formal symbolic mathematics. To put it differently, experiencing success at a higher conceptual level expanded their ZPDs, thus empowering them to tackle procedural details with confidence. It is through

the expansion of this zone that the notion of a hidden mathematics curriculum has the potential to significantly broaden pre-teachers' content knowledge at all levels, bring positive change in various teaching-related psychological phenomena, and eventually affect the way that mathematics is taught in the schools.

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Mathématiques et musique

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Résumé

L'exposition, ce sont des liens entre les mathématiques et la musique: Pythagore et Euler et la musique; le théoricien de la musique Forte et la *set theory* (musicale); le compositeur Xenakis et le groupe de rotation du cube sur lui-même. *Le développement*, c'est la *théorie mathématique de la musique* devenue discipline de recherche, exemplifiée par la présentation d'un modèle topologique de l'analyse mélodique. *La réexposition*, c'est tout simplement une citation de Leibniz sur la nature de la musique.

Abstract

The exposition: it is some links between mathematics and music including Pythagoras and Euler and music, the music theorist Forte and the (music) set theory, the composer Xenakis and the group of cube rotations on itself. *The development*: it's the *Mathematical Music Theory* as a research discipline and exemplified by the presentation of a topological model of melodic analysis. *The recapitulation*, it is simply a quote of Leibniz on the nature of music. See the website *Maths & Music* (ref.4).

Introduction

"*Mathématiques et musique*": ça peut créer un effet de surprise comme provoquer un commentaire tel "oui bien sûr, Pythagore!". Eh oui, il y a eu Pythagore et ses études de proportions d'intervalles musicaux. C'est certainement l'exemple le plus classique, même si antique (!), de liens entre la musique et les mathématiques. Mais y a-t-il plus? Y a-t-il eu d'autres mathématiciens qui sont venus mettre leurs *oreilles* dans la musique? Y a-t-il eu des musiciens qui ont inséré des équations dans leurs oeuvres? Quelles mathématiques *sont jouées* dans la musique?

Le but de cette communication est de présenter un sujet mathématique encore peu connu et d'en souligner les activités de recherche encore moins connues.

Nous présentons quelques exemples diversifiés de liens entre les mathématiques et la musique: l'étude de la musique par deux mathématiciens célèbres, l'introduction des mathématiques par un théoricien de la musique pour formaliser sa théorie analytique et l'utilisation des mathématiques par un compositeur du XXème siècle et un du XVIIème siècle. Ce qui nous mène ensuite à l'introduction de la *théorie mathématique de la musique* devenue

une discipline de recherche. C'est une théorie mathématique, donc à la fois cohérente et rigoureuse, pour l'analyse d'objets musicaux et leurs relations. Et c'est dans ce contexte que nous exposerons brièvement un modèle topologique de l'analyse mélodique de la musique, sujet principal de mes travaux de recherche.

Nous ne présentons pas ici de réflexions sur l'enseignement des mathématiques par l'utilisation de la musique. C'est un sujet qui, selon moi, est certainement riche en possibilités, faisant inévitablement partie de mes recherches envisagées, et qui a été, jusqu'à maintenant, très peu développé. Mentionnons seulement que quelques activités *mathé-musiques* aux niveaux élémentaires (réf.1), secondaires (réf.2) et universitaires (réf.3) se font de plus en plus présentes, au Canada et partout dans le monde. Citons par exemple l'activité "*Guitare, rapports et proportions*" de France Caron, didacticienne à l'Université de Montréal, qu'elle utilise principalement avec ses étudiants au programme de formation à l'enseignement des mathématiques au secondaire. Nous invitons le lecteur à visiter notre site internet *Maths & Music* (réf.4) présentant entre autres des listes de références et de cours en mathématiques et musique.

Quelques liens entre mathématiques et musique

Des mathématiciens et la musique

La légende veut qu'un jour Pythagore, se promenant devant la boutique d'un forgeron, observa que des sons consonants étaient produits lorsque le forgeron frappait l'enclume avec des marteaux dont les masses étaient dans des rapports simples. Ce qui l'emmena à créer un monocorde pour étudier systématiquement les rapports de fréquences des sons. Il observa que deux cordes pincées (de même densité et sous même tension) donnent un son plaisant si leurs longueurs sont dans un rapport de deux petits entiers. Par exemple, si les cordes sont dans un rapport de 2:1, elles produisent la même note mais à l'octave (la corde la plus courte produisant le son plus aigu). Les cordes dont les longueurs sont dans un rapport de 3:2 produisent la quinte juste, par exemple do – sol, qui est très agréable à l'écoute. Si à partir de ce rapport, on reproduit une quinte, le rapport suivant en relation avec la note originale sera $3^2 : 2^2$ que l'on ramène dans la même octave précédente soit à 9:8. Et ainsi de suite, si on reproduit successivement des quintes justes, et on obtient la gamme de Pythagore[1]:

Note	do	ré	mi	fa	sol	la	si	do
Ratio	1:1	9:8	81:64	4:3	3:2	27:16	243:128	2:1

Cette gamme de do ne s'entend pas comme celle jouée sur un piano, car sur ce dernier, les rapports de fréquences ont été réajustés pour faciliter les changements de clés. Plus

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[1] Il faut noter que, dans cette gamme, le rapport de fréquences 4:3 de la note fa se trouve en produisant une quinte inférieure au do, ce qui n'est pas exactement le même rapport qu'on obtiendrait en produisant une quinte supérieure au si*b* (le cycle complet des quintes étant do-sol-ré-la-mi-si-fa#-do#-sol#-mib-sib-fa-do) qui donnerait alors le rapport $3^{11} : 2^{17}$. L'impossibilité de refermer une gamme par progression de quintes justes vient du fait qu'on ne puisse trouver deux entiers *l* et *m* tels que $3^l = 2^m$.

précisément, on observe que les rapports entre les *tons* do – ré, ré – mi, fa – sol, sol – la, la – si sont tous de 9:8 et les rapports entre les *demi-tons* mi – fa et si – do sont tous deux de 256:243. Or $(256:243)^2 < 9:8$, et donc le demi-ton pythagoricien n'est pas exactement un *demi-ton* à proprement dit, alors que le demi-ton sur le piano a été ajusté pour être exactement la moitié d'un ton.

En se basant sur les calculs de Pythagore, Euler s'est intéressé à l'ordre de consonance d'un ensemble d'intervalles. Il a abordé le problème en utilisant le théorème fondamental de l'arithmétique, soit la factorisation unique, à ordre près, de tout entier positif en nombres premiers, i.e., pour tout $n \in N : n = p_1^{e_1} p_2^{e_2} ... p_k^{e_k}$ où les p_i sont des premiers et les e_i des entiers positifs. Euler a employé cette décomposition dans la définition de la fonction de *Gradus*

$$\Gamma(n) := 1 + \sum_i e_i(p_i - 1)$$

Puis, il a étendu sa fonction aux rationnels a:b, que l'on peut supposer réduit:

$$\Gamma(a:b) := \Gamma(ab)$$

Sa conjecture fut que plus la valeur du gradus du rapport d'entiers d'un intervalle est petite, plus l'intervalle est consonnant. Par exemple, en comparant les intervalles de la gamme de do majeure[2], on obtient l'ordre suivant de consonance:

Intervalles	Rapport d'entiers
Octave	2:1
Quinte	3:2
Quarte	4:3
Sixte majeure	5:3
Tierce majeure	5:4
Seconde majeure	9:8
Septième majeure	15:8

Un théoricien de la musique et la théorie des ensembles

Allen Forte, théoricien contemporain de la musique, a intégré dans les années soixante des concepts de la théorie des ensembles dans son approche d'analyse de musique atonale (réf.5). Des regroupements de notes, tels des accords, sont représentés par des ensembles d'entiers 0, 1, ..., 11 correspondant aux hauteurs des notes: 0 – do, 1 – do#, 2 – ré, ..., 11 – si. La musique atonale[3] est une musique contemporaine dans laquelle les règles de tonalité ne tiennent plus.

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[2] Dans cet exemple, nous considérons la gamme en intonation juste, c'est-à-dire dans laquelle les rapports de fréquences des accords majeurs, soient do-mi-sol, fa-la-do et sol-si-ré, sont tous 4 :5 :6.

[3] Un exemple de musique atonale est la musique sérielle du début du XXème siècle, i.e. de Schoenberg ou Webern. C'est une musique très différente de la musique classique. Les règles de tonalité ne tiennent plus et ce que nous concevons normalement d'"harmonieux" n'est plus recherché dans cette musique.

Par conséquent, les notes telles ré*b* et do# sont toutes deux identifiées par le même entier 1. Forte établit des relations entre les regroupements de notes en utilisant par exemple les concepts d'intersection, complément et sous-ensemble. La figure 1 présente un exemple d'identification de regroupements de notes "*à la Allen Forte*". Il faut noter que Forte n'était pas le premier à utiliser la représentation ensembliste dans son approche. Des théoriciens–compositeurs tels Milton Babbitt (né en 1916) (réf.6) et Anatol Vieru (1926 – 1998) (réf.7) l'ont précédé. Par contre, on peut affirmer qu'à partir de cette période *Forte*, l'approche mathématique dans l'analyse de la musique a commencé à s'établir parmi les théoriciens de la musique Nord-Américains. Et c'est dans cette période que la représentation du système chromatique dans le cercle, tel que représenté dans la figure 1, est véritablement devenue usuelle.
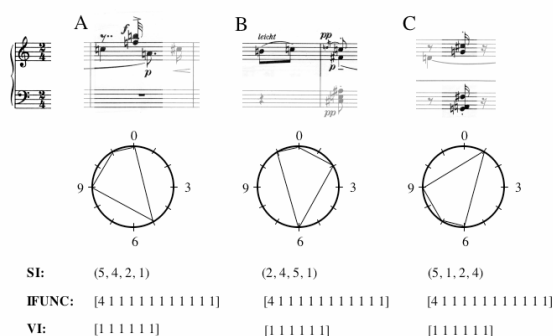
Figure 1: Exemple (moderne) d'identifications de regroupements de notes *À la Forte*. On y remarque que le système chromatique est représenté par $\mathbb{Z}/12\mathbb{Z}$.

Il est important de noter que le développement de cette approche ensembliste, souvent appelée *Set Theory* ou *American Set Theory*, dépasse aujourd'hui largement l'application à la musique atonale. Signalons que le théoricien David Lewin y a considérablement contribué (réf.8) par son approche catégorielle.

Des compositeurs et les mathématiques

Il n'y a pas que le mathématicien qui *joue de ses violons* dans la musique ou le théoricien de la musique qui *intègre* les mathématiques dans son analyse, il y a aussi le compositeur qui a son mot à dire, ou plutôt à *faire entendre*. Par exemple, Iannis Xenakis (1922-2001) a composé son oeuvre *Nomos Alpha* en suivant un algorithme très précis. Il a utilisé le groupe de rotations du cube sur lui-même, d'ordre 24, et à chacun des sommets du cube, il associa ce qu'il appela une cellule musicale. Par exemple, une répétition de notes ou un nuage de notes. Or, les rotations se laissent représenter par des permutations des 8 sommets. Puis, il considéra un procédé de Fibonnacci pour créer son œuvre: il engendra une suite de rotations en choisissant d'abord deux rotations g_1, g_2, puis en déterminant les suivantes par

$$g_{k+2} := g_{k+1} \circ g_k.$$

Étant donné que le groupe est fini, la suite est périodique. Il est à remarquer que Xenakis a en fait utilisé une suite aux propriétés maximales. En effet, Moreno Andreatta a démontré dans sa thèse de doctorat (réf.9), en 2003, que la suite possède le plus grand nombre d'éléments différents et qu'elle est de période maximale.

Xénakis a délibérément utilisé un groupe (mathématique) dans la composition de son oeuvre. Je me dois de faire remarquer que ce procédé était déjà présent dans les compositeurs baroques, mais qu'il était caché sous les règles de composition, plus précisément de fugue. L'idée de la fugue est de présenter une mélodie, puis de la répéter le plus souvent possible dans d'autres voix et décalées dans le temps: comme un canon. Ces répétitions, appelées imitations, sont strictes, transposées (e.g. une quinte plus haute), inversées (réflexion par rapport à l'axe horizontal) ou rétrogradées (réflexion par rapport à l'axe vertical). Et on y reconnaît bien la structure de groupe. La figure 2 présente les premières mesures de l'*Inventio I* de Johann Sebastian Bach (1685-1750) dans laquelle quelques symétries y sont soulignées.
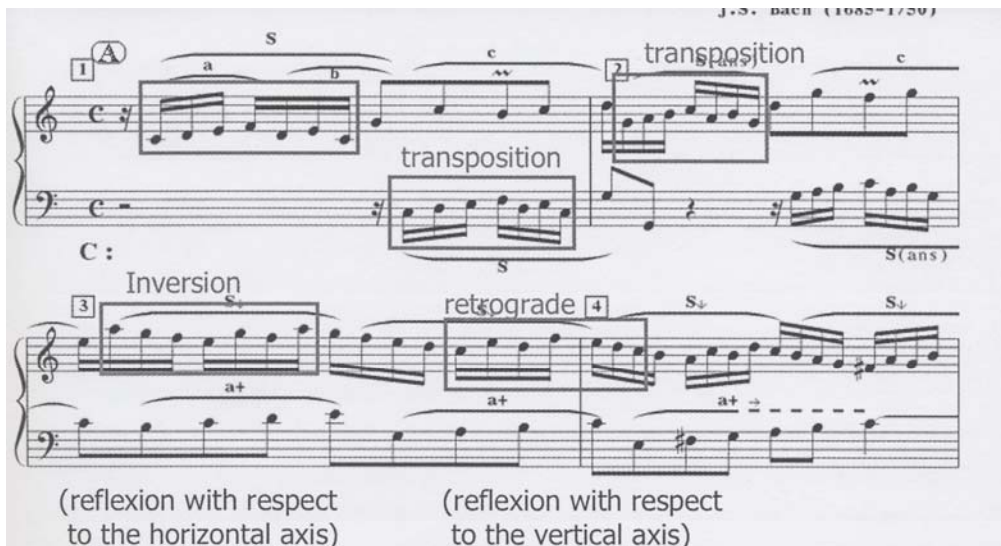
Figure 2: Quelques symétries de l'*Inventio I* de Johann Sebastian Bach.

Ces liens sélectionnés entre musique et mathématiques sillonnent différents aspects de la musique, plus précisément les aspects de composition et d'analyse et les aspects physique et mental. La musique est complexe et pour une étude systématique, il est important de bien identifier toutes ses dimensions et d'en mesurer alors les limites de l'approche. Ce qui nous mène à la théorie mathématique de la musique.

Théorie mathématique de la musique

Préludons avec une citation de Pierre Boulez (né en 1925), chef d'orchestre et compositeur Français: "Music cannot be degenerated or reduced to a section of Mathematics: Music is fundamentally rooted within physical, psychological and semiotic realities. But we need more sophisticated methods besides statistical and empirical data in order to formally describe musical instances". C'est dans ce contexte que la théorie mathématique de la musique,

abbrégée MaMuTh[4], se veut développer un cadre de travail scientifique pour la musicologie tout en respectant les limites imposées par la nature même de la musique.

Or, dans les années quatre-vingt, le mathématicien et musicien professionnel Guerino Mazzola a présenté (réf.10,11) le début du développement d'un cadre mathématique abstrait de la MaMuTh. Mazzola propose des structures abstraites dans lesquelles plusieurs approches se redéfinissent, se généralisent et, parfois, résolvent certaines implications incohérentes non-désirées. Avec des concepts de géométrie algébrique, théorie des modules et catégories, topologies algébriques et combinatoires, théorie des représentations et, enfin, théorie des topoï, et avec l'apport de diverses disciplines telles les sciences informatiques, la sémantique, la physique, les sciences cognitives, et les mathématiques, on étudie les objets musicaux et leurs relations dans les contextes de composition, analyse et interprétation.

Pour illustrer le type de recherche dans le domaine de la MaMuTh, je présente brièvement les espaces motiviques, modèle topologique de l'analyse et structure mélodique de la musique, qui constituent le sujet principal de mes travaux de recherche.

Espaces motiviques

Décrivons d'abord en quelques lignes ce qu'est l'analyse mélodique ou, plus précisément, l'analyse motivique. Cette dernière décrit la structure d'une composition musicale par le biais de son organisation hiérarchique de ses motifs (courtes mélodies d'environ deux à dix notes). Cette analyse se résume souvent à déterminer le motif générateur, appelé motif germinal, de la composition. Ce motif a la fonction particulière d'unifier toute la composition. Plus précisément, selon le théoricien Rudolph Réti (réf.12), c'est le motif dont le contour est répété tout au long de la composition, soit comme imitation (stricte répétition), soit comme variation ou comme transformation. L'exemple classique d'un motif germinal est le fameux motif formé des quatre notes sol – sol – sol – mi*b* (voir figure 3) de la *Cinquième Symphonie* de Beethoven que l'on entend, merveilleusement, du début à la fin de la symphonie.

Figure 3: Motif germinal à l'ouverture de la *Cinquième Symphonie* de Beethoven.

Nous utilisons l'approche de Réti pour la formalisation des concepts musicaux reliés à la structure motivique. Nous en présentons les idées dans cette communication et référons le lecteur à (réf.13,14) pour une description détaillée.

Les notes sont paramétrisées par leur temps d'attaque (*O*), hauteur (*P*), durée, articulation, crescendo et glissando. L'**espace de notes** est un espace réel $R^{\{O,P,...\}} \equiv R^n$ de dimension finie contenant au moins les paramètres *O* et *P* et possiblement quelques autres de la liste. Un **motif** est un ensemble fini non-vide de notes ayant toutes des temps d'attaque distincts. En d'autres mots, cette propriété exclut les accords. Le motif d'ouverture de la *Cinquième Symphonie* est un bon exemple de motif. Le **contour** des motifs est formalisé par une certaine application ensembliste sur l'ensemble de tous (oui, tous !) les motifs de la composition. Par

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[4] Originaire du nom allemand de la théorie, soit **Ma**thematische **Mu**sik**th**eorie, et qui concorde également avec *Mathematical Music Theory*.

exemple, le contour mélodique d'un motif M se représente par la matrice (δ_{ij}) où δ_{ij}=1 si la hauteur de la note j est plus élevée que la note i, δ_{ij}=0 si les notes ont la même hauteur et sinon, δ_{ij}=-1. Les imitations, telles la transposition (par exemple do-ré-mi devenant sol-la-si joué une à quinte supérieure) ou l'inversion (par exemple do-do#-ré devenant do-si-si*b*) sont formalisées par l'action d'un groupe sur l'ensemble des contours. Les classes de motifs résultant de l'action de groupe sont appelées **gestalt**.

L'introduction de métriques sur les contours de même cardinalité qui, sous certaines conditions, s'étendent sur les gestalt de même cardinalité, permet de formaliser la similarité mélodique *de même cardinalité*. Nous en sommes alors à l'étape cruciale de la construction du modèle: les variations et transformations de motifs impliquant la similarité de motifs de cardinalités différentes. Nous définissons le **voisinage** $V_\varepsilon(M)$ **de rayon** ε **du motif** M comme étant l'ensemble de tous les motifs contenant un sous-motif de même cardinalité que M et dont le gestalt est au plus à ε-distant du gestalt de *M*. Nous disons alors que le motif *N* est une variation de *M* si $N \in V_\varepsilon(M)$ ou $M \in V_\varepsilon(N)$. Et la transformation de motifs implique tout simplement des rayons plus larges.

Sous certaines conditions[5], la collection de tous les voisinages de motifs forme une base pour une topologie sur l'ensemble de tous les motifs de la composition. Cette structure topologique, appelée **topologie motivique**, correspond à la structure hiérarchique de motifs de la composition. Dans ces espaces de motifs, malheureusement que de type T_0 et donc pas Hausdorff comme le plan Euclidien, le motif le plus `dense'[6], c'est-à-dire ayant le plus de motifs voisins, représente le motif germinal.

Ce modèle a été appliqué à *l'Art de la Fugue* de Johann Sebastian Bach dans le but d'étudier le problème de la longueur de son thème principal: est-il formé des huit premières notes dans l'ouverture ou plutôt de ses douze premières notes? Or, du point de vue compositionel, il est bien accepté que le thème doit contenir toutes les douze notes. En utilisant notre modèle, nous avons conclu que le thème, du point de vue de sa structure motivique, n'est composé que de ses huit premières notes. Plus précisément, la structure motivique du thème est déjà présentée dans ses huit premières notes et l'ajout des quatre dernières notes appuie la structure motivique mais ne l'enrichit pas.

Quelques difficultés en théorie mathématique de la musique

Nous terminons la présentation de la MaMuTh par quelques questions qui se posent naturellement en raison de son caractère interdisciplinaire. Nous nous contentons de les énumérer en espérant qu'elles provoqueront quelques réflexions au lecteur.

- La terminologie de termes musicaux n'est pas universelle (même mot pour plusieurs concepts ou plusieurs mots pour un même concept). Comment travaille le mathématicien dans sa formalisation d'objets musicaux et de leurs relations?

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[5] C'est-à-dire dans tous les cas classiques à une exception près: la construction avec le contour mélodique de notes consécutives seulement est un cas pathologique. Il ne mène pas à des voisinages de motifs stables par rapport à leurs structures de sous-motifs.

[6] Le terme *dense* n'est pas tout à fait au sens topologique. Il est utilisé pour d´écrire l'idée de densité mais dans un espace de type T_0. Dans un espace motivique de propriété asymétrique originant des voisinages de la base, on doit plutôt considérer les deux relations "*être inclu dans*" et "*inclure dans*" son voisinage permettant de bien formaliser la fonction germinale d'un motif.

Comment est reçue cette terminologie par les musiciens et théoriciens de la musique?

- Comment valide-t-on un modèle mathématique en musique? Qu'est-ce que la *vérité* en musique?

- Est-ce que Beethoven a intentionnellement incorporé son motif "sol-sol-sol-mi*b*" dans sa *Cinquième Symphonie* chaque fois que *nous* l'entendons? En fait, avons-nous le droit d'examiner à la loupe, mathématique computationnelle du XXIème siècle, son oeuvre d'art?

Conclusion

"*La musique est une mathématique de l'âme qui compte sans savoir qu'elle compte*", a dit un jour le célèbre mathématicien Gottfried W. Leibniz (1646 – 1716).

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New PhD Reports

Présentations de théses de doctorat

Students' Understanding of the Completeness Property of the Set of Real Numbers

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This thesis was performed at University of Buenos Aires, from May 2000 to June 2004, under the supervision of Professor Michèle Artigue.

Introduction

The set of real numbers (ℝ) is the numerical domain in which calculus and mathematical analysis are done, beginning at the pre-university level. However, we can hypothesize that students who enter university do not have a clear idea of what this domain represents, nor of its properties.

In most universities, approaching the set of real numbers is officially done on a progressive basis along several courses. Indeed, students first contact real numbers early in their schooling path. They come across real numbers in the solutions to geometrical problems, as square roots or as polynomial roots, by studying decimal expansions, etc. Further on – when studies in mathematical analysis start, and the set of real numbers turns into the natural domain of functions – other properties become relevant.

How is this process reflected in the internal construction that students carry out? What are the ideas students have concerning ℝ and how do they evolve? What difficulties does this evolution pose, how are they taken into account from the teaching perspective? In this thesis I have studied upon these questions, focusing on a property that is essential for the work in mathematical analysis: the completeness property of ℝ. I will use the word *completeness* to refer to the property of ℝ that can be stated as *every non-empty and upper-bounded set of real numbers has a least upper bound that belongs to ℝ* – among other equivalent characterizations. I will use the word *continuity* for referring to the analogous property stated for the line.

This research was carried out within the undergraduate programs in Pure and Applied Mathematics at the University of Buenos Aires, Argentina. There, the set of real numbers is studied in the first years along four courses: Pre-university Analysis, Analysis I, Complements of Analysis II and Advanced Calculus. I have assumed that the problems students solve in Pre-university Analysis do not require them to consider completeness explicitly. Further on, in Analysis I, to what extent do the exercises students solve demand that they make explicit which numerical set they are using and what their properties are? Is it possible to do this whole course, having the correspondence between points of the line and numbers as a support? Is it possible for a student to consider that the numbers are the rational,

their square roots, and some irrational numbers (like π or *e*) without contradiction? As well, can students consider that the numbers are just the algebraic numbers without inconsistency? Is the statement *a real bounded-above non-decreasing sequence has a limit* taken by the students as a manifestation of completeness or as evidence that constitutes a powerful tool to show convergence? When students learn that a continuous function in a bounded closed interval reaches the maximum, or when they learn the Intermediate Value Theorem, do they relate these results to completeness or do they use them as evident theorems assuming completeness as something pre-existent?

The least upper bound axiom is usually taught in the courses Complements of Analysis II and Advanced Calculus, followed by the instruction of other equivalent expressions of completeness. Each expression opens different meanings of completeness. I was interested in analyzing how the students' ideas are modified starting by one expression of completeness or another.

The initial objectives of this research were:

- To study the relationship between the notion of the *set of real numbers* and the first years in undergraduate studies in Mathematics.
- To know how students elaborate the notion of completeness and to find under what conditions their understanding evolves.

Both objectives can be more precisely described using the notion of *rapport au savoir* (Chevallard, 1992): I was interested in characterizing the institutional and personal rapports towards completeness, and their evolution. A cross-analysis that relates institutional and personal rapports constitutes, for me, a central issue. Indeed, a study of the students' conceptions that is not linked with the students' learning, even if it gives a panorama of what they know, does not provide us with elements for changing something regarding to improving their performance.

The thesis is organized in introduction, six chapters and conclusions. The chapters are:

I. History and Epistemology
II. Synthesis of Related Didactic Research
III. Mathematical and Cognitive Panoramas
IV. Institutional Analysis
V. Personal Rapport I
VI. Personal Rapport II

Chapter I: History and Epistemology

The first chapter consists of a historic-epistemological study about the emergence of the set of real numbers. The analysis performed allows us to delineate a reconstruction of the historical genesis about the notion of the *set of real numbers.* The work was committed to linking problems and questions of specific historical periods with the state of knowledge and available tools at those times, and with the different generated conceptualizations. It allowed us also to understand the origin of the current formulations. The following questions were considered:

- How was the correspondence between numbers and points of a line performed in different historical periods?
- How was the work of mathematicians in analysis, before completeness was stated?
- Which conditions made necessary the formalization of this notion?
- Which were the different answers given to this problem?
- How have the present formulations been reached?

Different moments in the history of mathematics were examined; such as the Euclidean, the intermediate Arab and the medieval European periods, the development of calculus, the arithmetization of analysis, the constructions of a numerical system and finally the axiomatization of the set of real numbers.

The main ideas obtained from this chapter are:

- The same mathematical statement changes its statute related to different projects. In this particular topic, completeness, the rigor and the precision in the definitions are in the heart of the changes. Completeness, which was considered an implicit tool at determinate moment, was provided of a certain status taking into account the work of Cauchy and Bolzano. Cantor and Dedekind built a set such that this property could be proved, Hilbert considered this property as an axiom.

- The relation model-modeled situation can represent a support as well as an obstacle. In the early developments of calculus, working with problems of reality did not make completeness explicit. The existence of the solution in the model is what makes necessary the completeness of the numerical set, but that was not the problem being studied at that time. Analogously, for the learning of $\mathbb{R}$ it is necessary to present the students problems that are not in context. The notion of real number is not caught by problems that involve reading measures with instruments. For these situations, rational numbers are sufficient.

- The arguments that are considered sufficient to validate the mathematical work are modified through the history: only under a certain *level of validation* it is possible to see the existence of some numbers (like maximums or limits) as a problem, and consequently, to deal with completeness.

This chapter provides us with a frame for conceiving the existence of different types of cognitive balances and conceptualization levels regarding real numbers and completeness.

Chapter II: Synthesis of related didactic research

We may distinguish two levels linked to understanding $\mathbb{R}$. One concerns the existence of irrational numbers: their decimal, unlimited and non-periodical expansion, their representation one-by-one on the line, and their approximation in the solutions of calculations. This level is linked to the fact that rational numbers are not sufficient to answer school problems, and the existence of other numbers is therefore required. The other level concerns the properties of $\mathbb{R}$ when this set is seen as the natural domain of mathematical analysis. These properties arise with the study of functions and sequences and when the aims are to justify the statements and to characterize the properties $\mathbb{R}$ has as a set. We may say that a sign of studying mathematical analysis is the introduction of a *local point of view*. A local treatment is at the heart of analysis: what matters is what occurs in a neighbourhood of a point, without much concern for what occurs in a global scale or in the point as isolated. Completeness is not a property of one number but of a set of numbers; neither it is a completely global property. My interests concern the second mentioned level.

The reviewed literature contemplates the first level more and not the second; it provided me with a panorama of research mostly at the secondary level of schooling. That constituted for me a map of the didactical knowledge about this matter previous to this study – a valuable departure point. Some points of consensus among them can be stressed: the existence of the student's contradictory ideas regarding density, decimal expansions and non-decimality on one hand; the representation of numbers on the line, that not necessarily helps to overcome these difficulties, on the other hand.

Chapter III: Mathematical and Cognitive Panorama

The third chapter consists of the analysis of the mathematical and cognitive characteristics linked to the completeness of ℝ. I called "Mathematical Panorama" to an unfolding of the mathematical elements linked to completeness: continuity of the line, the real line, different characterizations of completeness (by means of least upper bound, of nested closed intervals, of cuts, of fundamental sequences, of connectenes, etc.) the Archimedean property, the completeness of more general spaces. My idea was to obtain a structure that served me to identify or link sectors of this panorama with different possible conceptualizations.

I elaborated also a cognitive model of completeness, taking as a reference the Mathematical Panorama, the epistemological analysis, my own experience as an instructor and the related research analyzed in the former chapter. I procured separate –necessarily in a fictitious way- the cognitive aspects involved, structuring them in six variable axes. My objective was to model different states of conceptualization of this notion. These axes, together with their initial state, were named *Cognitive Panorama*. These six axes have a common origin, as an initial state or starting point where completeness is seen as evident, with a strong support in the graphic or mental representation. At this level, completeness is not identified as a mathematical object. The initial state would admit, itself, different sublevels, but for this work it was considered as only one. The six axes are: Technical Availability, Tool/Object, Necessity, Validation, Flexibility and Position on the constructions of ℝ.

These axes are related not only to the technical availability of a "theoretical learner" towards these notions, but also to the degree of reflection of a learner: about the tool and object aspects of completeness, about the necessity of including completeness as a condition in the definition of ℝ, about the kind of validation and the level of conscience of a learner regarding validation in mathematics, about the flexible use of different aspects of completeness and about the role of constructions.

Reformulation of the questions

The completion of this first part of the thesis (epistemological analysis, literature review and the mathematical and cognitive panoramas) contributed to reformulate the original objectives and to turn them more precise:

- Related to the institutional analysis, I was interested in identifying what has been called by Chevallard *rapport institutionnel* (Chevallard, 1992). The following questions guided the work:
- To what extent completeness appears as a necessary condition or as a tool in the students' work?
- To what extent does the institution offer tasks that favor an evolution in the axis Tool/Object?
- Which axes of the Cognitive Panorama are stressed by the institution; which ones remain weaker? Which consequences do these options have in the conceptualization of the students?

Related to the cognitive/individual analysis, I was interested in identifying what has been called by Chevallard the *rapport personnel* of students, in this case also in terms of the Cognitive Panorama. The leading questions were:

- To what extent students think that it is necessary to prove a theorem that "can be seen"? (in the research I utilized Bolzano's theorem).
- To what extent do they consider that completeness is a necessary tool to prove it?
- What do students think about what completeness is, how do they express it, which problems, theorems, exercises do they relate it with?
- How does this view evolve as they progress through the different courses?

- How do students interpret the fact that there are different ways of introducing completeness? Are these ways available for them?
- How do students interpret the constructions by Dedekind cuts or by Cauchy sequences; how do they link the constructions with their old knowledge about real numbers?

Chapter IV: Institutional Analysis

Having as a reference frame the above questions and the notion of *praxeology* (Chevallard, 1998), I made an analysis of the exercises and problems that the students have to solve along the four courses we mentioned before.

Taking the introduced axes as a reference, I found that the tasks regarding completeness in Pre-university Analysis are generally on a technical level. Completeness occurs mostly as an implicit tool: its explicit explanation is neither required nor used in problems and exercises. Students at this level can operate as if the properties held naturally and its foundation were something external to them. Making completeness explicit seems to be necessary only for instructors, not for students. The majority of the exercises can be answered through seeing representations – that is why we may say that validation at this point is likely empirical.

There is a breach in the *didactic contract* that happens in the transition from Pre-university Analysis to Analysis I. In this course, representations are no longer allowed as a basis for argumentations. It seems that students demonstrate more due to their teachers' demands than for their interest from a mathematical viewpoint. They have not only to learn to make proofs, but also to accept proving as a genuine task. This is not usually made explicit. Completeness appears as an explicit instrument in the demonstrations that instructors make and students need to know for succeeding in the final exam. Completeness is still, somehow, encapsulated in the theorems. Regarding the validation axis there is a jump, provoked more for an external motivation than for an internal one.

In the course Complements of Analysis II there are exercises where the supremum and the infimum appear as objects of study and an instrumental role is considered too, in defining distances. This favours an evolution in the axis Tool/Object.

In Advanced Calculus, regarding completeness, the students have to prove the equivalence of different ways of defining this notion. This exercise potentially improves the performance of the technical skills and puts a student in optimal conditions for perceiving flexibility among the different expressions of completeness. Students have as well to solve some classical problems of complete metric spaces. Completeness of general metric spaces – and not only of the set of real numbers – is in the outline. The limitations of a non complete space and the potentialities of a complete space can be understood and acquire more sense in a general space, as they are studied in this course, and this labels an evolution in the axis Tool/Object.

From the analysis carried out, we affirm that three of the six axes of the Cognitive Panorama (those that are more related to a "meta" level: Necessity, Flexibility and Position on the constructions of ℝ) are left vacant from the offer of the institution. They are set aside, if so, for the students' private work.

Chapters V and VI: Personal *Rapport* I and II

Chapters fifth and sixth involve what we called the experimental part of this thesis. The fifth chapter consists of the *a priori* and *a posteriori* analyses of several clinical interviews carried out with students from Analysis I, Complements to Analysis II and Advanced Calculus, with the purpose of identifying their personal rapport regarding completeness in terms of the

Cognitive Panorama. We elaborated guides of interviews for each course with some of the questions in common, as we wished to study the evolution among the different courses. In the interviews I answered about Bolzano's Theorem, about completeness and different ways of defining it and about the constructions of ℝ.

The sixth chapter surveys the answers to a written questionnaire given voluntarily by the majority of the students from Analysis I (124 of 192 students), Complements of Analysis II (11 of 24) and Advanced Calculus (10 of 16) to the following two questions:

- If you wish to explain to a younger student that a real non-decreasing bounded above sequence has a limit, how would you do it?
- What does "ℝ is a complete set" mean for you?

For this question we decided to analyze only the answers given by students of Complements and Advanced Calculus, as we thought the youngest students would take the word *complete* in its daily meaning. An apriori and an aposteriori analyses of these questions constitute the core of this chapter.

Some of the main conclusions of these chapters, given in terms of the Cognitive Panorama, follow. Only the more significant axes are included in this report.

- Regarding the axis Validation, in the interviews I asked the students if they considered necessary to demonstrate Bolzano's Theorem and why. Students from the three courses expressed that at some point of their studies they considered this theorem obvious. This question allows us to know to what extent and how were they coming out of this conviction. The diversity of answers becomes difficult to structure this axis in levels by courses. Essentially we have found three kinds of answers: 1) demonstrate "because this is "what should be done" (by privileging the normative characteristic), 2) demonstrate to convince others or convince oneself by detaching from the geometrical representation, 3) to prove to understand the mathematics in play. We can assume that the order in this kind of answers shows an evolution in rationality, and we should expect that this tendency takes place passing through the courses. That was not the result we obtained. The majority of the answers from the three courses is distributed in the two first types (to prove because this is what should be done and to prove to convince). Only one answer of each course is placed in "proving for better understanding". There are several students, still the advanced ones, that do not know why it is necessary to demonstrate Bolzano's theorem.

- Regarding the Tool/Object axe, particularly speaking the *tool* aspect, I claim that the statement *"a real non-decreasing bounded above sequence has a limit"* is taken by the majority of students as something evident. A *proof* of that it is that almost all the answers consisted in explaining in detail the words involved (sequence, non decreasing sequence, bounded above sequence) and they include in some part of the answer "it must have" a limit, "it is clear" that it has a limit. A lot of students support their answer in a drawing. It is interesting to analyze which is the role of the drawing in the answer. To look whether it is the whole answer, if it appears as an explanation of a text, or if there is a text explaining the drawing. When the frame of reference of the situation is changed to a drawing, implicitly arise the identification of points and numbers. A problem of numbers is translated into a geometric problem. But the problem is no longer a problem if the frame is geometric. In a drawing, the existence of the limit is obvious, it is there: it can be drawn. Drawing the problem favours the perception that the sequence is a Cauchy sequence, as its terms press together. When students translate the question to the design and they think that obviously it has a limit, they use implicitly that such a sequence *must* have a limit (it is basically the idea of completeness of Cantor). I think that an answer that is mostly given by a

design would mean a tendency to validate empirically. Several students based their answers in non-mathematical terms or pictures: (a barrier, a man who walks up to a wall, a flask that is filled in with droplets of water, etc.). I would say that they take contexts where the existence of the limit cannot be doubted, sometimes in the chosen context the situations turns discrete. The answers to the questionnaire and the interviews of the three courses show that only a few students of Analysis see completeness as a tool to define some new elements. This can be related to the fact that students can use strong theorems, so that they do not face to completeness. Most of the students do not know which problems completeness solve.

- Regarding the object aspect, I think that this aspect is a bit too weak for students in Complements of Analysis II and Advanced Calculus. Most of the students of Complements (10 of 11) express completeness in a non-operational way: by means of the daily use of the word *complete* or by means of images. I link this weakness with the way in which students express themselves: the expression "complete means that it has no gaps" in referring to completeness and the expression "getting closer" are weak and non operational images of mathematical definitions that do not favor the mathematical work, even if in part this kind of expressions could be introduced by the non formal style of the interview. Indeed "complete means that it has no gaps" can be thought as a degenerate version of the statement "every cut of the set has an unique element of separation", as well "getting closer" is a deformation of "approximation" that is, posing the distance between two objects less than some positive number.

- Regarding the axis Position on the constructions of $\mathbb{R}$, for the students the constructions have different roles, as well as for mathematicians: for some of them, constructions seem to be out of what it is necessary to know, for some others the constructions are models for the axiomatic system that defines $\mathbb{R}$, while others understand why constructing a complete set makes sense. This latest case favours not to fall in a preconstructed view of $\mathbb{R}$ and to understand the role of the completeness axiom, which essentially allows defining numbers, no matter which version of it is used.

Conclusions

Questionnaires and interviews show cognitive aspects of students that are not accessible to instructors when they teach. It is interesting to observe in many cases the distance between the tasks they did to pass the exams and some conceptions that resist, expressed in *natural* answers which in a first moment beat the *rational* answers. The technical availabilities or the technical skills do not trigger automatically a reflection, an attentive consideration of the mathematical object in play. Researchers in Math Education have referred to this issue distinguishing *action* from *reflection on the action*. In the thesis it is shown that solving the exercises and passing the exams is not enough for making students see completeness as a necessary condition to develop mathematical analysis. To understand completeness as a property or an axiom that answers to a veritable problem demands to situate in a certain perspective which is not the most natural. For most students, doing the typical exercises of supremum does not have as a consequence to understand that $\mathbb{R}$ is the set that contains all the supremum of its bounded above subsets. Few students can perceive that Cauchy sequences come from the necessity of characterizing the kind of sequences that must converge to develop analysis, and that completeness enunciates that the limit belongs to the set. Very few find sense in studying the constructions by Dedekind's cuts or Cauchy sequences, and when they do it is to consider the role of model of the axiomatic system that define $\mathbb{R}$. Constructions, then, remain a bit empty of sense. All these aspects, that in general remain in

the sphere of private work of the student, are elements that I think are important to take in account when it is necessary to prepare an outline, a guide of exercises or a course.

The type of analysis and study carried out in this thesis in not a statistical one, it is a qualitative study that shows the subtleties of teaching and learning. The interviewed students are few in number but in their answers it is clear that they doubt when asked why to demonstrate, they mix in a first moment the notions of density and completeness, they hold that algebraic numbers suffice to develop analysis, they use expressions that do not help to operate, they do not recognize the equivalence between two statements after proving it… even if they are students that have succeeded in difficult exams, with good grades. This should not be understood as a criticism to the institution or the educational system, rather as a study that shows the *complexity of this system*, highlighting the existence of several stages in the acquisition of notions, and several types of cognitive balances coexisting in the same subject while learning takes place.

The thesis and all the bibliographic references is available at:

http://etd.bl.fcen.uba.ar/tede2/tde_busca/tde.php?id=7&id2=14&id

Study of Two Teaching Approaches Focusing on Spatial Sense with Three Different Profiles of High School Students

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Context

This thesis was supervised by Louise Poirier and was triggered by three factors. Firstly, the interaction between two different teaching practices : mathematics teaching and figure skating teaching. These two teaching approaches seem, *a priori*, far apart from one another, but they involve common spatial knowledge : in both, students need to learn how to oriented themselves in space and do three dimensional transformations (translation and rotation). But, learning wise, results of students vary depending on their discipline. In classrooms, we observed that students had difficulties learning geometrics and spatial knowledge, as other researches have shown (Bessot, 1983; Izard, 1990; Parzysz, 1991). For their part, skaters had more facility with spatial knowledge in general, even athletes who were having difficulties in other field of mathematics (Marchand, 2004). Secondly, the lack of continuity and specificity in the curriculum for spatial knowledge from one level to another (Marchand, 2004) : in Quebec, spatial knowledge in three dimensional space is treated throughout the elementary school classes and then students have to wait two years before reworking in three dimensional space (even if the new curriculum mention three dimensional objects in the first two years, it treats it in two dimensional space by focusing on their developments and representations). In the mathematic curriculum of the elementary school and of the third year of the secondary school, spatial knowledge represents one of the major objectives, but when we look at the specific goals, spatial knowledge is absent (all goals are related to geometric knowledge). And, the third factor is the similarity of the task proposed by handbooks to students from one grade to another : four out of ten objectives from the end of elementary school are similar to the secondary level objectives. For example, we asked to describe a solid in terms of number of faces, vertexes and edges to eleven years old students and again, three years later, with fourteen years old students (Marchand, 2004).

With this interaction between sport and school, the action of pupils plays an important role in our project but action is not something new in teaching methods (Andrew, 1996; Bednarz & Garnier, 1991; Bishop, 1980; Clements & Battista, 1992; Ducret, 1984; Larochelle & Bednarz, 1994; 1996; Musick, 1978; Pallascio, 1995; Pêcheux, 1990; Piaget, 1948; 1970; 1972; Rigal, 1996; Shaw, 1990; Stanic & Owens, 1990). These considerations led to our research questions:

- How can we characterize a teaching approach on spatial knowledge aimed on concrete and abstract action of pupils compared to a more conventional approach?
- How do different profiles of students react to these approaches?

The findings for the first question are the focus of this article. But, before answering this question, we expose a summary of our theoretical framework and method.

Theoretical framework

Our interdisciplinary framework takes its roots in didactic, psychology and sports' concepts. For each discipline, we extract these key elements:

- The researches in didactic allowed us to specify the definition of spatial and geometric knowledge. These are distinct but linked, and give us guidelines for teaching spatial knowledge. By geometric knowledge, we mean the process that interiorize geometric (instituted) aspects of an object and formalises them to create an ideal mathematical object. By spatial knowledge we mean the process that interiorize physical aspects of an object and allows us to create and manipulate mental images. These definitions are our own but were inspired by several others (Berthelot & Salin, 1992; Chevallard & Jullien, 1990; Clements & Battista, 1992; Laborde, 1988 and Piaget & Inhelder, 1948). Mental images are not often explored in classrooms, but Slee (1987), Hutton and Lescohier (1983) confirm the utility of this tool not only in several disciplines but also in mathematics learning. Denis (1989) showed that visualization (the activation and manipulating of mental images) is an asset to students in mathematics problems resolving. Finally, specifically for spatial knowledge, Yackel and Whealtey (1990) recommended to develop the visualization with students by presenting Tangram activities and questioning students explicitly on their mental images. One of their activities is to present an image constructed of different geometric forms for three seconds and ask students to reconstruct the image shown with Tangram pieces.

- The researches in psychology gave us the basis for the development of spatial and geometric knowledge. From the analysis of four development model (Hoffer, 1977 *in* Del Grande 1990; Dion, Pallascio & Papillon, 1985; van Hiele, 1959 *in* Lunkenbein, 1982 and Piaget & Inhelder, 1948) we chose one that expresses particularly the development of spatial knowledge for elementary and secondary students in terms of actions. Here is the model of Piaget and Inhelder (1948) on the development of spatial knowledge :
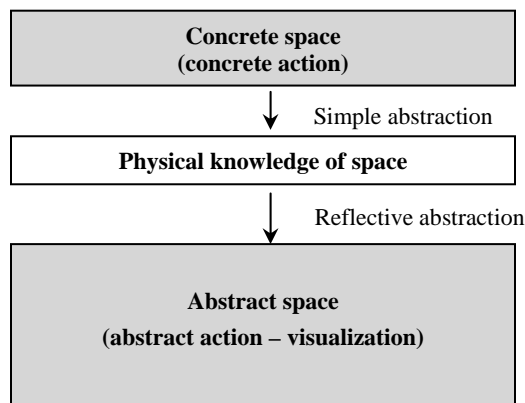
- Finally, the researches in sports showed us some concrete ways to apply visualization. For the last twenty years, the sports domain developed several mental programs that increase athletes' level of the performance (Duda, 1995; Pelissier & Billouin, 1989; Orlick, 1990; Porter & Foster, 1990 and Zhang, Ma, Orlick & Zitzelsberger, 1992). They found that the most efficient element was visualization, which means asking athletes to visualize their technical abilities and questioning these mental images. These results are consistent with the didactic's ones. In addition, the sports researches showed that to develop spatial knowledge, the athlete had not only to go from concrete to abstract space but also from abstract to concrete space and do several passages from one to the other. To complete the Piaget and Inhelder model, we add this sport finding that the development is circular and goes also from abstract space to concrete space.

Method

All aspects of our theoretical framework were put together to create a learning approach based on actions of pupils and also to develop grids that were used to analyze our research results.

To answer our first research question, we created two didactic interventions, one based on our framework and another based on the conventional approach elaborated by an external teacher. The comparison between the two will allow us, *a posteriori*, to characterize each one. To construct those interventions we fixed curtains variables:

- The grade of students : secondary 3 (14 and 15 years old) which is the first year where the program deals with spatial knowledge in three-dimensional space since the elementary school.
- The length of the intervention : one hour lesson.
- The mathematical objective inspired from the curriculum :
  - Goal 1 : Construct, starting from a written description, a house composed of a rectangular prism and a rectangular pyramid. (with straws and pipes cleaner)
  - Goal 2 : Transform, by a 360˚ rotation, a polygon to a cone or a cylinder.

Each intervention, planned on paper, was experimented with four small groups of students (between 2 and 8 students), sixty students in total. After each lesson, we went back and interviewed each group on their reasoning and difficulties by watching a videotape of the lesson.

All of the lessons and interviews were transcribed and analyzed. Two grids were created : one for the point of view of the teacher and the other one for the student. For this paper, we focus our attention on the teacher. The grid used to analyze the planning and the teaching approach is based on our framework, inspired by anterior grids (Gauthier, 1997; Garner & Cass, 1965; Perkins, 1965; Piaget, 1973; Taba, 1965 and Waimon, 1968) and composed of five elements : objectives and intentions of the teacher, type of intervention (explaining, giving instructions, reformulating a student answer, validating a answer,…), demanded tasks (observe, describe, construct, represent, search or justify), questioning (close or open, on a fact or a reasoning, on geometry or spatial knowledge, on mental images, on a result…) and the action reference (concrete, abstract or anterior).

Results

The first step to characterize our teaching approach centered on the pupils' actions (concrete and abstract) is to compare it to the conventional approach. Here, we present the differences between the two for each objective :

Goal 1

Firstly, the instructions given to construct the houses were very different. In the case of the conventional approach, the student, individually, had to built a house with this description written on the board : "construct a house composed of a rectangular prism and a rectangular pyramid." For the approach centered on pupil's actions, the construction necessitated teamwork and the written instruction was: "construct a house, where the main body is a rectangular prism in which the side rectangles are bigger than the base. The roof is a rectangular pyramid with its vertex on the extension of one of the prism's edge.

In addition of this description, students had to anticipate the amount and the length (small, medium and long) of all the straws they needed before starting the construction. Clearly, this activity was simple for fourteen and fifteen years old students in the conventional approach; in fact, all the students succeeded in building the house. However, in our approach aimed on their actions, only one student (out of 17 students in total) correctly build the house but he did not adequately anticipate the amount of straws; all the others were not able to anticipate or correctly build the house.

In these activities, we observe the divergence in the use of the material : in the conventional approach, the student always had access to the material and could cut straws as he pleased; in the action approach, the student did not have access to the material at first and could not cut the straws as three lengths had already been established.

The intention of both approaches for this activity was not the same : in the conventional approach, the teacher wants to observe students in action to know if they use geometric knowledge to build their house or if they act randomly. In the action approach, the teacher wants students to develop and make explicit their mental images, which is very different.

Finally, both approaches do not aim their questioning on the same type of action and knowledge : the conventional approach aims on the concrete action (what is different from one house to another? Why is our house leaning? Is the base of the house a rectangle?) and on geometric knowledge (is a square a rectangle? Does your house respect the description?) the action approach centers on the abstract action and spatial knowledge (from the reading of the description, what did you see in your head? Did you see the same house that you built? What did you see that made you choose eight big straws? Can you make the house turn in your head?).

Goal 2

The instructions were similar for the two approaches; students had to transform, by a 360° rotation, a polygon to a cone or a cylinder.

The differences were in the teacher's intentions, in the use of the material and in the questioning. The teacher for the conventional approach wanted to observe if students were able to identify rotations in their everyday objects while the other teacher's intentions were to practise student's kinetic visualization and to increase awareness of their mental images.

Secondly, the teacher of the conventional approach used straws and concrete figures to realize the different rotations. In that case, students were ask to take the correct figure (ex. : a triangle) and place a straw on the figure to represent the axis of rotation and make the figure

turn. In the case of the action approach, students did not have any material, they had to look at an image of the figure and of the axis on a cardboard. The action approach also presented more complex cases of rotation by involving oblique axis and by putting the axis inside the figures. In doing so, as for the first activity, the conventional task and questioning aimed on concrete action and geometric knowledge by asking students to build and draw the situation while the action task and questioning centered on abstract action by asking pupils to anticipate the result and make explicit their mental images.

Summary

These compared results gave us tools to answer our first research question on the teaching of spatial knowledge : How to characterize a teaching approach on spatial knowledge aimed on concrete and abstract action of the pupils compared to a more conventional approach?

- The tasks asked of students have to be more difficult and have to include tasks like description, observation, anticipation (research) and construction.
- The questioning of the teacher plays an important role and needs to be focused on spatial knowledge, not geometric knowledge, and on the mental process and not primarily on the construction result.
- Consequently, the lessons have to be aimed on abstract actions and not concrete action, especially for fourteen and fifteen years old students.

To create a more favourable and potential activity to develop spatial knowledge, the teacher has to consider these three aspects as a whole; their combination will permit the evolution of spatial knowledge in term of mental images.

Conclusion

If we go back to our *a priori* analysis and combine it with our findings, we can extract several recommendations for future studies. Going back to our curriculum and books analysis, we can accurately say that the lack of continuity and specificity creates difficulties and differences in the interpretation of the curriculum resulting in a lack of tools for teachers and a shift towards geometric knowledge. To counter these gaps, we recommend the following:

- Raise the awareness of teachers concerning the possible shift from spatial knowledge to geometric knowledge and vice versa and develop a more suited teaching approach for these know ledges.
- Look more intently at the teaching and learning of spatial knowledge from elementary school through high school and eliminate the actual lack of continuity in the curriculum.
- Create tools for teachers like technology support and a series of activities focusing on spatial knowledge.
- Vary the tasks and not limit our interventions to the contents of schoolbooks centered primarily on observation.
- Find more relevant activities for high school students by going further into the analysis of the activities presented in the actual schoolbooks.
- Make the students manipulate and construct, even in high school, to help create mental images by aiming the teaching and questioning on abstract actions resulting from concrete actions.

This study gives us an overview of the situation and a guideline for one lesson; it is to be considered a starting block for the construction of a sequence of activities favourable to the development of spatial knowledge.

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Communicating Mathematics Online: The Case of Online Help

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Introduction

This thesis investigated a particular type of online tutoring environment in the domain of mathematics. Its focus was on mathematics help sites where professional or peer tutors answer questions online. Communication there is asynchronous, text-based, and service is provided without charge. Questions and answers are public and therefore available for the review of all visitors to the site. Such service provides a rare opportunity for students to anonymously satisfy their genuine knowledge needs. As Silver noted in the "1990 Yearbook of the National Council of Teachers of Mathematics":

> Much can be learned when students are invited to make their thinking and reasoning public. Moreover, it is also clear that student verbalization can not only help teachers to gain insight into the knowledge and thinking of their students but also furnish a powerful way for students to learn from each other. (1990, p. 7)

This thesis described and analyzed the communication on mathematics help Web sites in order to increase our understanding of the characteristics, benefits and drawbacks of tutoring mathematics online. I essentially argued that the main value of online help in mathematics is the provision of educational environments where students are taking an active role in their learning and setting up an agenda in the tutorial discourse. Therefore it gives educators the opportunity to learn the kind of questions students ask in an environment that provides for the anonymity of communication. Since, on such Web sites, help is offered by peers or by expert tutors, I looked into their tutoring habits and for evidence that these two categories of tutors teach differently in an online environment.

Definitions

Student in this study is any person who poses one or more questions on mathematics help Web site.

Peer tutoring is tutoring by volunteer learners on the Web site. According to Goodlad and Hirst (1989, p. 13), "peer tutoring is the system of instruction in which learners help each other and learn by teaching." Here it is essential that the peer is someone who has the same status as other visitors to the site.

Expert tutor is an experienced, qualified person who is recognized by the administration of the Web site as such.

Theoretical Framework

Based on the constructivist approach (LeJeune & Richardson, 1998; Jonassen, 1995) mathematics online help sites are the environments where students may be taking active role in their learning. Before they pose the questions they may go through the process of self-diagnostics, and after they receive the answer, they may go through the process of self-explanation, which are both important strategies for learning (Chi, 1998). However, based only on questions, while missing all the visual and many other clues about the student, it is very difficult for the tutors to come up with the proper model of a student.

Taking socio-cultural perspective, mathematics online help sites provide support for a student in knowledge building within a community of learners. Tutor is there to help students by using scaffolding techniques and, as a more knowledgeable peer, to support transformation of potential into actual in the student's zone of proximal development (Vygotsky, 1978). Online communication on public help sites also provides visitors with variety of topics presented in argumentative form and in semiformal mathematics language. Therefore, they agree with Resnick's (1988) recommendation that mathematics should be taught as if it were an ill-structured discipline: a domain in which multiple interpretations, arguments, and debate are called for and natural. Furthermore, online help sites have important role as meeting places for people who want to participate in mathematics discourse.

Research Problem and Research Questions

This study addressed the following problem: *What are the characteristics of asynchronous online mathematics help environments and do they provide conditions for learning?*

Based on this problem, this descriptive and analytic study was designed to answer the following questions that are subsumed under three themes: a) Characteristics of online mathematics help interaction and comparison to the interaction in face-to-face tutoring (based on results of other research); b) Finding out to what extent and how do tutors in online help model students with whom they interact with and comparison between expert and peer tutors' answers; and, c) Finding if there is a noticeable difference between questions posted on different sites, even in the same discipline and to what factors can these differences be attributed?

Method

The mixture of quantitative and qualitative research methods were used here. The discourse on the mathematics help Web sites was analyzed according to the Verbal Data Analysis (Chi, 1997) as a method of quantifying qualitative data. Each question and answer in the sample was categorized according to its communicative goal, content and form. Cognitive level together with the level of specification and attitude for both questions and answers were also determined. Taxonomies of tutorial discourse developed by researchers in intelligent tutoring systems (Graesser, Person, & Huber, 1992; Shah, Evens, Michael, & Rovick, 2002) were adapted and extended to fit the data thus resulting in developing Taxonomy of Online Tutoring (Martinovic, 2004, p. 63-66). Each student utterance was classified along the following dimensions: Question Form (Short-answer and Long-answer question with subcategories), Communicative Goal (Request for Confirmation, Challenge, Conversational Repair, Establishing Common Ground, Request for Information, Acknowledgement with subcategories, Social Coordination, Conversation Control), Degree of Certainty (Does the student hedge or not?), and the Degree of Specification (High, Medium, Low).

Categorization of tutor answers was done according to what they teach, how they teach, do they hedge, and how helpful are their answers. Each tutor's utterance was classified along the following dimensions: Communicative Goal (Providing Information, Causal Explanation, Acknowledgement, Conversational Repair, Instructions in the Rules of the Game, Teaching Problem Solving Algorithm, Probing the Inference, Brushing Off, Teaching the Language of Mathematics, Establishing Common Ground, Social Coordination, and Conversation Control), Delivery Mode (Hint, Elaboration, Rephrasing, Analogy, Request/Directive, and Multiple Answers), Degree of Certainty (Does the tutor hedge or not?), and the Degree of Specification (High, Medium, Low).

After the data were first categorized, the methods of descriptive statistics and statistical tests for categorical data provided for the numerical characteristics and comparisons between different classes. More qualitative approach was applied on the transcribed interviews with online tutors and their tutoring logs. Field notes were then summarized on a grid with categories. Examples of particular questions and answers from the Web sites were compared to general observations.

Data, in the form of 200 threads of communication, were collected from each of the three purposely selected Web sites (A-C) that differed in the type of tutoring offered. This was expected to provide the greatest variability between the discourses on these sites. None of the sites was related to some course. The first site (A), used in the initial phase of the study, offered both expert and peer tutoring and was used to develop and test instruments. The other two sites, one with peer tutoring (B) and the other with expert tutoring (C), were used for data collection. During the scope of two months or five answered questions, five volunteer expert tutors kept logs of their online communication with students, where they recorded their thoughts about the tutoring process. These tutors were also interviewed, thus providing relevant information necessary for obtaining a fuller understanding of mathematics online help.

These different instruments for gathering the data and combined methods of inquiry insured the best blend of structured as well as unstructured techniques for the investigation of such a complex topic.

General Findings

Peer tutors generally needed less time than expert tutors to respond to a question and to complete communication with a student. The peer tutoring site had a smaller transitional distance between interlocutors, since the frequency of messages in average thread was higher than on the expert tutoring site. A sense of community was evident on the peer tutoring site from the form of the requests and directives in the student discourse and the frequency of first-person pronouns.

All three analyzed sites were of the exploratory nature, thus letting students find their way around, which not all of them did. Also, students' poor writing skills emerged as an important deficiency in the text-based communication, which was obvious from the number of spelling and grammatical errors; as well as the number of incomplete or unclear questions.

Tutoring online proved to be very different from tutoring face-to-face, since otherwise experienced tutors and educators felt frustrated and overwhelmed when they first started their online service. The following factors emerged from the interviews and discourse data: (a) It is difficult to communicate mathematics in the text-based medium; (b) A simple two-step dialogue process (ask a question, answer the question) becomes a multi-step process in a written form; (c) The visual and symbolic mathematics forms, if used, lose their original purpose (to support, clarify) since the diagrams and mathematics symbols are not clear, uniform, or efficient in online environment; and, (d) Since the whole exchange of messages

can take too long to finish, students may lose interest or hope and turn to other topics or resources.

Tutors' answers have certain credibility in public communication, so errors in tutors' answers that do not get repaired can cause students to lose trust in this service. Luckily, errors on the expert tutoring site were rare and the majority of errors on the peer tutoring site did get fixed. Another problem, especially on the expert tutoring site, present the questions that do not get answered, simply as a consequence of the tutors not being able to catch up with their constantly growing numbers (in 2004, site C reported receiving over 300 questions/day).

Comparison Between Peer and Expert Tutoring

Expert tutors reported that they usually select questions they feel most competent about and among such they take the most recently posted (such "warm" questions most likely have students eager to receive an answer). In their tutoring, experts followed some general rules like the policies of the Web site or some rules that emerged from their previous experiences in teaching or tutoring. They also followed some specific rules that they shaped according to each student (i.e., based on the sophistication of the language, infer the age of a student). Expert tutors put a lot of emphasis on teaching the language of mathematics and when unable to model a student they often used multiple answers. In mathematics online help, providing alternative answers in one message makes the message longer and for the inexperienced or unmotivated student more difficult to comprehend. On the other hand, it makes communication more efficient, since the tutor does not have to wait (often in vain) for the student to refine the question, but tries to anticipate possible problems in advance and therefore provides Multiple Answers. This shows to students a variety of optional paths that they can take in solving some problem and presents mathematics as a discursive subject.

Experts preferred providing hints rather than complete solutions and motivating students to show their work. They tried to help without revealing too much, especially when they felt that a question is part of a homework assignment. In their tutoring, they used scaffolding techniques of rephrasing, analogies, and summarizing, as well as elaboration as external self-explanation.

Peer tutors focused on mathematics problems in questions and, if they found a problem incomplete to the point that they cannot repair it, they brushed off the student. They advised others how to communicate online and what the elements of fair play are on the Web site. Their instructions did not go beyond the scope of the question, thus predominantly using concise explanations. Also, peer tutors were inclined more than experts to provide straight answers to students. It was obvious that the peer tutors' answers were more to the point than experts', and conversely, expert tutors tried not to answer directly for various reasons: (a) Ethical—the Web site may have a "not doing your homework" policy; (b) Pedagogical—they believe that by doing so they would not be really helpful to students; and (c) Path of Least Resistance—they spend less time answering, to the point that they do not need to do the mathematics problem.

Although the differences in peer tutoring and expert tutoring scores on sites B and C may be partially the consequence of their different cultures and practices, I was careful to make plausible generalizations in my conclusions—supported by the results from the site with both types of tutoring, and evidence obtained from the expert tutors' logs and interviews. Such a broad approach resulted in the conclusion that peer tutoring also differs from expert tutoring in the following aspects (see Table 1).

Table 1: Summary of Differences Between Peer and Expert Tutors

Peer Tutors:	Expert Tutors:
Write less.	Write more.
Focus on mathematical problem.	Look at the message holistically.
Provide detailed account of steps. Use more directives.	Answer in general terms. Reinforce the concepts. Use more hints.
Establish close relation with a student.	Express personal involvement with the subject. Speak with the authority of the mathematical community.
Answer as if they are doing the exercises from the textbook.	Try to model a student.
Pass over syntax errors.	Are more helpful. Are more direct in their feedback.

Overall, peer tutoring sites may be used differently than expert tutoring sites. If one needs a quick answer to a specific, typical question, peer tutoring sites are certainly more beneficial. If one needs advice, reinforcement of ideas, or an answer to some esoteric question, expert tutoring sites are more useful.

Students in Mathematics Online Help

Although the online medium provides for anonymity of communication, students still do not use this service to their full benefit, since they post many messages without almost any context. This affects tutoring online by making modelling an online student even more difficult.

Students are not very helpful to tutors, they rarely: (a) explain what their problem is; (b) provide background information; and (c) provide their work. On all three sites the students mostly asked knowledge-deficit questions, but also asked a fair amount of deep-reasoning questions (between levels 2 and 6 on the Bloom's scale). They also asked more long-answer than short-answer questions.

Obviously, students, in general, tend to leave to tutors to figure out what they wanted to ask. Even if the mathematical problem is stated clearly, it may still be missing the background information about the student and what is that the student wants to know. Although researchers believe that students lack questioning skills, in online cases there are three other likely causes for posting telegraphic (low specification) messages: (a) the nature of the medium is such that it takes much time and effort to type a mathematics question; (b) students may bring bad practices from their schooling, such as hiding their ignorance, and (c) students may believe that since there is only one way to answer mathematics questions, all other background information that they may provide to online tutors is redundant.

The feedback that students gave to tutors was mostly general (conveyed to the tutor that the student read the answer). All other types of acknowledgements were rare, especially those that show if a student went through the process of self-explaining. These cases were named Comprehended and Fully Comprehended. Such acknowledgements express satisfaction with an answer and explain how such answer affected the student. Chi (1998) cautioned that poor learners think they understand most of the time when in fact they do not, implying that they do not detect any conflicts between their understanding and what the text says. This means that even in the category Comprehended the cases may be that the student thought the conflict was resolved, while in fact it was not. Except for the acknowledgements, the only other way for an observer to detect the cases of evident resolution of problems that students had, would

be to follow the communication in threads and from the content of students' messages, determine if there is evidence that the students did or did not comprehend, which is exactly how tutors judge the success of their help.

There were detected cases of abuse of online help, but they were relatively rare and taken care of by other participants in the discourse: in less drastic cases the student was taught a lesson on "fair play," while in more drastic cases such questions were simply ignored by the peers.

Contributions of the Research

One contribution of the present thesis is that it provides a broad overview of the relatively new phenomenon of public asynchronous online help in mathematics. By investigating this rather uncharted territory this thesis set up the groundwork for additional research in this field.

Another contribution is towards the development of taxonomy applicable to the online exchange of questions and answers between many students and many tutors. As noted earlier, existing taxonomies were combined using categories applicable to this type of communication and this thesis' research problem, and further extended by new categories or refinements.

Benefits and Drawbacks of Mathematics Online Help

While it is necessary for instructional methodologies to investigate the use of modern technologies in mathematics education, providing powerful tools does not result in educational benefits per se. As Hanna wrote:

> While technology has the potential to be a vehicle through which students can see, negotiate, evaluate and debate mathematics, they are remarkably entrenched in their beliefs and practices; thus students do not include experimentation and inquiry as routine experiences in the teaching and learning of mathematics. These instructional approaches will need to be introduced, nurtured and refined if students are to explore data and processes fundamental to a contemporary curriculum. (Hanna, 1990, p. 27)

Online help in mathematics should not be solely looked at as a service to students when their teacher is not around. It is also an excellent model of how students can efficiently and successfully communicate in mathematics. If properly used, it can help students to reflect, self-explain, and build their confidence. It can give them the opportunity, once when they feel ready, to also provide assistance to others. Online help has great value as a learner support feature**,** and as such, should warrant more recognition from institutions at all levels of schooling. Its impact extends beyond the direct benefit to its participants. I see in it the realization of Clements (1997, p. 750) proclamation that "all people, regardless of age, gender, race, or creed, should be free to participate fully in formal and informal mathematics education programs," but also the indirect benefit for teachers as Cohen explained:

> Learner support features, it is suggested, actually empower teachers, who are released from the mundane technical support and tutoring tasks, and allowed to contribute to their more specialized, multi-faceted role as students' guide, Socratic questioner, role model, collaborator, and motivator. (Cohen, 1990, p. 334)

For educators, online help sites provide a view into students' uncertainties and misconceptions. The peer tutoring sites are to some extent similar to other online models for collaborative inquiry investigated by Scardamalia, Bereiter, and Burtis (1994) who concluded that:

> Getting students to explain, justify and argue for their conceptions is useful because it helps students to elaborate and engage in deeper processing. On the other hand, some difficulties may exist with ideational confrontation because students often do not know enough to debate alternative explanations. (Appendix B, p. 18)

As such, online help sites fall somewhere in between communication-oriented and information-oriented computer supported collaborative learning tools, thus supporting both dialogue and monologue forms of the discourse, which are important to nurture in learners as noted by Hoadley and Enyedy (1999).

Implications for Theory

This study took a very broad approach to investigating characteristics of asynchronous online mathematics help and their relation to learning mathematics. Such an approach was necessary to the understanding of this relatively new phenomenon. In preparation for this research, several disciplines were studied and their results incorporated and built upon in this thesis. The areas of distance learning, specifically computer-mediated communication; mathematics learning; and natural and computer tutoring were the closest parent disciplines for the present study.

This study illustrated the development of a mixed (quantitative and qualitative) method for the research, and how the research methods from one discipline (Intelligent Tutoring Systems) can be modified and applied in another field (Computer Mediated Communication).

The approach taken in this thesis can be used as a model in further research in parent disciplines while its findings can be used in any research related to communicating mathematical ideas.

Implications for Further Research

This study provided detailed analysis and conclusions regarding communicating mathematics online in the case of online help. However, there are other areas that remain to be explored.

- In this study students' questioning and peer tutoring were analyzed through the transcripts of their communication. Future researchers can fine-tune the findings by providing a controlled environment where the effects of online help on students and peer tutors can be further investigated.
- One can investigate the communication of other visual mathematics disciplines, like geometry, in a text-based online medium.
- This study addressed the frustration that new expert tutors feel when they start tutoring online. This topic certainly deserves more attention in future research.
- In order to preserve the role of an observer and the privacy of participants, the researcher in this study did not approach peer tutors. Some future study can explore how peer tutors in online help answer the questions and how they understand their role.
- This study provided some answers and asked some new questions. Here, one of the findings was that students in asynchronous mathematics online help have different questioning habits as compared to results of studies done in other learning environments. One plausible explanation for that was that the online medium which frees students from didactical contract, presents determinant factor, for the type of questions students ask. This hypothesis should be further investigated in a controlled environment.

Limitations of the Study

The limitation of the study is in its inability to reliably access and analyze background information (non measurable factors) about visitors to the Web sites. However, I believe that these factors did not affect the findings, since all precautions were taken not to over generalize the results. The study focused deliberately on public mathematics Web sites that provide asynchronous help by human tutors who do the tutoring on a voluntary basis.

Therefore, the results of this research cover the scope of similar services on the Internet. Random selection of threads from the two sites (B and C) used in the main study and big sample size provides for the generalizability of the results obtained through discourse analysis. Two other details may also limit the generalizability of the results from site C, their archives consisted of the threads which were recommended by the tutors and further edited by the site staff. Furthermore, the small sample of five online expert tutors from two sites and the fact that they volunteered to participate in the study may also limit the generalizability of the findings.

Conclusions

This study attempted to provide an insight into mathematics online help sites that provide asynchronous, public, and free-of-charge tutoring. It gave the description, analysis and comparison of these phenomena to other educational environments. It also established distinctions between peer and expert tutoring and provided evidence that online help supports and results in learning. It also cautioned about some deficiencies of this service.

The value of both peer and expert tutoring sites is evident, but our learners need to be instructed as to how to ask questions. Even in an environment where they are almost or completely anonymous, they still ask a lot of questions that are shallow, not clear, or not informative enough for tutors. Other researchers already pointed to the importance of discourse in mathematics. This study shows that providing a discursive environment has to go together with efforts to educate students on how to use this medium to its full potential and to the students' full benefit. On all levels of schooling mathematics educators should help students to develop their discourse skills.

One factor that affects how students use online help sites is the novelty of such a service. With the spread and further development of online communication technologies, we can expect more students and tutors to become used to their features. It is my hope that this study will promote mathematics online help, contribute to the design of such Web sites, and positively affect the tutoring process on the Web sites.

Acknowledgements

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Attending in Mathematics: A Dynamic View About Students' Thinking

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My area of research interest is students' *mathematical thinking*. In part, I can trace this interest to my own experience, first as a student and later as a teacher of school mathematics, especially in one-on-one experiences tutoring students. I began to develop a hunch that whereas explaining content to students helped, it was not all that was vital for students to make sense of mathematics. In my dissertation I explore what it means for students to bring forth their own mathematical worlds in which the mathematics makes deep sense. I study what students attend to and how they attend.

My interest in matters of attention was evoked by the work of John Mason and a few others who suggest that mathematical thinking is synonymous with *mathematical ways of perceiving*. During activity, different students may attend differently. To investigate mathematical attentiveness, I focus on the variations in students' engagement. In some frameworks variations might be interpreted as erroneous, immature or partial ways of thinking. I interpret variations as indications of the dynamic nature of the perceived and the perceiver. I use theoretical frameworks that problematize the assumption of pre-existing objects of attention. Ecological and complexity—ecological complexity—frameworks emphasize ways of *acting* and *being* in bringing forth perceptible worlds.

To guide the study I initially asked:

> What do students attend to in mathematical tasks?
> When do shifts in attention to that which is mathematically relevant occur?

The question evolved as I sought to make my observations coherent. I began to ask a layered question:

> In what ways do students, not only as individual beings with mental and psychological structures, but also as social and organic systems embedded in social collectives and enabled by cultures, extended by language, artefacts and technological systems, and embodied with a neuro-motor system and body, attend in mathematical tasks?

This shift was prompted by issues such as collective and distributed sense-making. Rather than focusing solely on particular mathematical and cognitive structures, I began to attend to students' actions and interactions—their written work and utterances, the materials they worked with, and the collectives that sprang up during their interaction. Also, this shift involved a drift toward considering attention as a participatory act. In attending, it appears, the attendee also enacts what is attended to. Mathematical objects, tasks and environments, being more dynamic in nature, participate in *inclining* students to attend mathematically.

Theoretical Explorations

In the mathematics education community, there is an ongoing conversation that aims at triggering new ways of thinking about, talking about and acting on students' learning. I adopt ecological complexity approaches to contribute to this conversation. I draw from and extend the work of complexity researchers Kieren, Pirie and Gordon Calvert (1999), and Davis and Simmt (2003). Drawing from the work of *social biologists* such as Maturana (1988) and complexity theorists such as Johnson (2001), I elaborate on the *embodied*, *embedded* and *extended* nature of students' thinking (this is referred to as an enactive perspective). I specifically adopt complexity approaches, such as observing layers of interaction and using organic, geographic and evolutionary dynamics to investigate learning.

Complexity and enactivist research considers learning to involve many interacting systems. These may include the learning agents—the individual and groups of students—and learning environments. Learning is part of a broader phenomenon, cognition. Cognition, action and perception involve the bringing forth of a world that is significant and that co-evolves with the learner. Defined broadly, thinking includes not only the mental but also the bodily, the social, the formal and the symbolic-technological domain. Thinking mathematically may then be understood as expanding mathematical possibilities in these domains (Kieren, 1999). To Kieren, thinking arises in action as the learner co-adapts to an ever-adapting world.

In the study I rephrase thinking in terms of observation and attentiveness. I work with the notion that observation is a fundamental human operation. I draw from the work of complexity researchers Luhmann (2002), Maturana (1988) and von Foerster (2003), and of mathematician Spencer-Brown (1979), who explore how humans enact objects. I attempt to relate this work to the work of William James and to hermeneutic-phenomenology views about perception. By studying students' mathematical attentiveness, I investigate the ways in which mathematical objects, concepts and tools arise, and the distinctions which lead to the enactment of mathematical concepts. Theories of observation motivate me to attend to students as observers whose worlds, being and states are precisely the distinctions they make and the operations they perform (Maturana, 1988). Mathematical observers interact in a specific domain—the mathematical world. They attend to the mathematical objects that they enact. Objects such as numbers, lines and functions are brought forth when mathematical observers—mathematicians, mathematics users, teachers and learners—act and interact in ways that mark mathematical distinctions. In this context, to teach mathematics involves occasioning students to become mathematical observers, making it more probable that they will attend mathematically.

Research Data For The Study

Data was gathered through interaction with and observation of secondary school students, both in extra-curricular research and in a classroom project. In a year-long classroom project (in 2001/2002) the main researcher taught a grade 7 class of 27 students. I participated as a research observer. In the extra-curricular sessions, 28 paired students from different schools participated in solving problems (summer 2001, 2002, 2003). An example of an extra-curricular task is the consecutive terms property (CTP), for example $1 + 2 = 3$, $2 + 3 + 4 = 9$ and $5 + 6 = 11$ (from Mason, Burton & Stacey, 1986). It follows that the numbers 3, 9 and 11 have the CTP. Numbers such as 4 do not have the property. Given a few examples, students were asked to describe the numbers that had the property.

During the empirical investigations I closely observed students' mathematical activities. Of their engagement, I asked: What do students attend to and how do they attend? When do shifts in their attention happen? How can teachers occasion them to attend in mathematical ways? The data I gathered included video clips, written work, transcripts and artefacts of sessions.

Moments when student made varied sense, had their understanding shift, or worked in divergent, discontinuous and novel ways were of particular interest to me. For example, students' written solutions for the CTP task varied. Some students wrote extensively, others did not. Some of those who wrote used the equals sign whereas those who used a number line did not. Some listed and crossed out numbers that did not have the property whereas others did not write these numbers at all. Some students' written work evidently drifted through these variations as they *wrote to solve* the problem. The salience of the shifts and differences, for me, is centered on the following questions: (a) What is the role of activity and artefacts in orienting students' attention? (b) When students act and interact differently, what do they see and or not see? (c) What distinctions does a particular form of interaction enact? and (d) How do these activities and interactions expand the space of the possible? In my dissertation, I explore data gathered by observing an activity using fractions (Fraction Kit Activity). Before exploring this data, I will briefly outline the research methodology.

Research Methodology

My methodology is interpretive. I read hermeneutics research alongside complexity theories of observation. A comparison can be made between the recursion in the hermeneutic circle and orders of observation. Cyberneticians claim that there are (at least) three orders of observation that create layered observations: zero-order, first-order and second-order.

At the level of the *zero-order observing*, observers act, activity becomes structured and purposeful behaviour emerges. The actors do not ask about the *why* and *how* of their behaviour. When learners or researchers do mathematics but do not seek to comprehend the nature of their activity, they are engaging in zero-order observation. *First-order observing* involves observing the what, how or why of behaviours. For instance, the research community studies the characteristics of learning. Students are encouraged to reflect on their actions. It is in first-order observing that one reflects on how students think. This may lead to the development of notions like thinking *stages* or *strategies*. In first-order observation mathematical thinking is observed as a phenomenon with fixed properties. Every observation at the first-order level is not aware of itself and the distinctions it makes. It is not even aware that it is making distinctions. Thus Luhmann (2002) and von Foerster (2003) argue that there is a need for observation to ascend above itself and attempt to interrogate its conditions or, at least, to reflect on the consequences of its observations. Observations are operations that shape the world. Thus people who build theories about learning ought to let the underlying ways of talking about aspects such as learning styles and thinking strategies become explicit. *Second-order cybernetic* observing deals with observing *observing* (von Foerster, 2003). It illuminates the conditions, properties and blind spots of prior first-order observations. For instance, one asks: When we enquire about what students attend to and how they attend, what do we focus on and why? This observing illuminates observing systems and seeks useful ones. In these *theories of observation* and *acts of distinction* the enquirer is no longer an independent observer who watches the world go by, but a participant in the circularity of human conditions (Namukasa, 2005).

The data analysis involved observing how students made distinctions, how they marked mathematical states and how they enacted mathematical worlds. In the analysis and writing, I specifically worked with moments that happened to be salient for me. The moments at times arose as questions, hunches or moments of surprise. For example, as I participated in a lesson on fractions I noticed that different students were approaching the same exercise differently. In that moment I made a zero-order observation. I dubbed such moments *interpretive moments*. To listen for, to *work around* and *to remark* on interpretive moments, most of which are salient only for a short time, is an act of first-order observation. No single interpretation is static, done once and for all and in a way that rules out other interpretations. Second-order

observation is helpful in hermeneutically *evaluating* interpretations and in seeking alternative interpretations.

Some Research Results: The Fraction Kit Activity

In a seventh grade lesson the teacher introduced a new manipulative for thinking about fractions. She gave each student a Fraction Kit (Kieren, Pirie & Gordon, 1999)—an envelope containing rectangular pieces of different colours. After explaining that the kits had been assembled by cutting sheets of paper into specific sizes (as shown in Figure 1), the teacher took a white piece and two red pieces out of her kit. Holding the two red pieces against the white piece she asserted, "If this white piece is a whole, then each of these two reds will be?" "A half," students responded. The teacher then affirmed, "If this *white* piece is a *whole* and it takes *two reds* to cover it, then *a red* is a *half*." Next she asked the students to find the sizes of the rest of the pieces in the kit. All students appeared to be engaged in the task. Some of them talked to a nearby student about their findings, but most students worked independently.
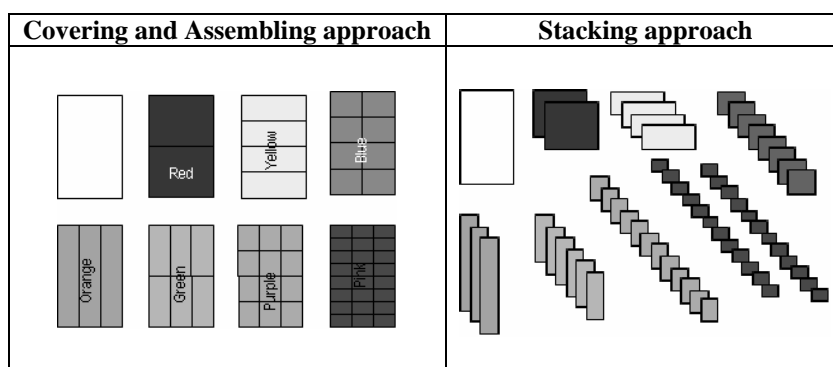
Figure 1: Fraction Kit: Covering and Stacking

The teacher reminded the students to record their results. Most of their records were of this form: 1 white—a whole (1), 1 red—a half (½), 1 orange—a third (⅓), and so on. A brief glance around the room indicated that the students were working on the task in somewhat different ways. In addition to the approach directly prompted by the teacher—that of placing the smaller pieces against the whole—another approach was evident. Some members of the class were neatly arranging pieces of a given colour in a stack without covering the whole (Column 2, Figure 1). A few students covered smaller pieces; they had quarters covering halves, sixths covering thirds, and so on. There might have been other ways, such as assembling wholes of different colours but I did not notice the students using them.

The teacher-researcher and I began to wonder about the ways in which students were able to figure out the sizes of the pieces without covering a whole. The relevance of this vignette for me is centred on the following questions: What did the students see, or not see when the teacher held two red pieces against a white piece? What distinctions did the students need to make in order to perceive, for example, each of the reds as a half? In the paragraphs that follow I illustrate how research questions posed about such an activity could be layered. In Figure 2, I illustrate five layers of analysis. To show that questioning remains unfinished; I include ellipses for further layers of questioning beneath and beyond.

What Observing Systems Are At Work?
Inter-System and -Object Analysis
Appreciate *Distinctions Made*
Seek *Conditions of Possibility*
Ecological Complexity Metaphors

What Worlds Of Significance?
Multi-Domain Analysis
Understand Significant *Actions & Interactions*
Encourage *Collectively Enacted Worlds*
Hermeneutic & Enactive Discourses

How Do Humans Selectively Attend?
Inter-Subjective Analysis
Generate *Pools of Meaning*
Appreciate *Difference & Essences*
Structural & Philosophical Phenomenology

How Do Children Attend?
Cognitive Processes Analysis
Understand Sources of Children's *Errors*
Seek Ways of *Guiding Children*
Child & Developmental Psychology

What Content Is There?
Task and Subject Matter Analysis
Appreciate *Mathematical Structures*
Seek Ways of *Explaining Thoroughly*
Subject Matter Structuralism

Figure 2: A Layering of Questions and Units of Analysis

The results of the study included a drift. During my teaching I examined the structural aspects of the mathematical task (what is there to be attended to?). My focus drifted at the beginning of the research to the psychological aspects of children (what mental mechanism do students attend with?) and my personal experiences (what do I attend to and how?). As the research progressed I began looking at two ontological and observational questions: How do students enact worlds? In what ways do observing bodies and systems make distinctions?

A Layering of Research Attention

What is mathematically present to be attended to? Task Analysis

In looking at the Fraction Kit we might ask, what is mathematically present to be attended to? This question calls for analysing the task for subject matter. We might conclude that the Fraction Kit activity involves part-whole and part-part relationships as well as equivalent fractions. The question of what mathematical concepts a particular activity illuminates is insufficient in analysing students' activity. As a structural question it may not seek to study

the nature of novel and divergent understandings, such as stacking, iterative covering and assembling. Historically, there has been a shift from solely analysing tasks for structure and from considering students' unexpected methods as errors, toward looking more closely at children's divergent interpretations (Confrey, 1994). This involves a shift from what is there to be attended to, towards focusing on *what children attend to and how they attend.*

What processes underlie children's thinking? Cognitive Analysis

Many researchers identify the origin of common errors so as to eliminate their sources or to circumnavigate them (Sierpinska 1990; Zazkis & Liljedahl, 2004). Cognitive analysis raises questions about structures—psychological and structural—which evoke in children non-standard perspectives on mathematical objects. It might be noted that the students who stacked the pieces faced a cognitive obstacle. They worked with the fraction pieces as part of a discrete rather than of a continuous, area space. These students might see each pink piece as $\frac{1}{23}$ instead of $\frac{1}{24}$ if they happen to see only 23 instead of all 24 pink pieces. Some students in this grade 7 class had difficulty with the task that followed when the teacher asked them to explore ways of covering a half piece. "What do you mean by covering? Should we use same colour pieces?" a few students asked anxiously as they shifted in their chairs. The teacher explained individually to those who asked while the rest of the class proceeded to generate combinations for a half. In a conversation after the lesson the teacher commented, "I noticed that the students who used the stacking approach rather than the intended—covering—approach on the first task were the ones who found difficulty with the second task." Indeed, a student acting in the stacking world might later think that there are eleven-twelfths $\left(\frac{11}{12}\right)$ in two wholes and three quarters $\left(2\frac{3}{4}\right)$. Arlene did so in a lesson that followed (see Figure 3). Beyond seeing divergent methods as obstacles to be avoided, we may look at them as various mathematical worlds that students are disposed to bring forth.
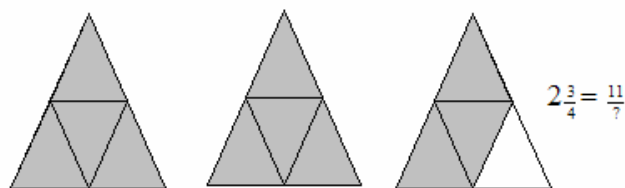
Figure 3: Diagram used by Arlene to write $2\frac{3}{4}$ as a fraction

What do I attend to? Inter-subjective Analysis

According to Mason (1994), in order to answer the question, "What do students attend to in a mathematical task?" one ought to ask, "What do we, as researchers, philosophers or mathematicians, attend to as we think mathematically?" This question adds a phenomenological flavour to both subject matter and cognitive analysis. In interpretive/phenomenological inquiry, children's mathematical conceptions are not viewed as inferior to adults' conceptions. What both adults and children already know is viewed as pre-understanding, a condition of possibility and grounds for further attending. In the frame of phenomenography (Marton, 1997) in particular, varied actions with the Fraction Kit—covering, stacking or assembling—would be the multiple categories or interpretations of the concept of fractions. Whereas phenomenographers identify sets of invariants as the essences

of what is attended to, the enactivist does not want to fix the object of attention. It is not always assumed that a mathematical task is independent of the mathematician, the teacher or the student who acts on it.

A focus on enacted worlds that are perceiver-dependent involves *multi-world analysis,* sometimes called *inter-subjective analysis*. Students who stacked appeared to attend to numerical aspects of the kit. They asked, "A single pink piece is one of how many total pink pieces in the kit?" They did not ask, "What portion of the whole does a single pink piece cover or assemble?" "One of how many" is more likely to evoke a chain of thoughts from numerical aspects, to multiplicative ratios, and finally to multiplicative relationships. It is least likely to evoke thoughts from size aspects, to portions of a whole, to combining and comparing of fractions. Each approach involves a different fractional world.

How Do Students Enact Worlds? Multi-domain Analysis

At this layer of enactivist questioning, a teacher or researcher tries to make sense of what students may be possibly attending to, the mathematical worlds brought forth. Some researchers, in addition to analysing tasks and speculating about how to overcome sources of errors, seek to determine the domain of validity of a perceived error (Balacheff, 1990). For the students who stacked I speculated about the ratio worlds they might have enacted. In a world brought forth by stacking fraction pieces, from the perspective of a learner, it might make perfect sense to conclude that $2\frac{3}{4} = \frac{11}{12}$. To Arlene it was 11 small (but whole) shaded triangles out of a collection of 12. It is plausible that Arlene attended to the collection, the numerosity of the smaller triangles. She stacked the small triangles, rather than covered the whole big triangle. The challenge becomes "In what ways can we invite students to bring forth a space in which the conventional and general is highlighted?" This analysis has implications for helping students make sense of mathematics. Orienting mathematical attentiveness involves paying attention to conditions of learning at many integrated levels: the structural, the psychological, the experiential, the collective, the institutional and the ecological. Therefore it appears useful that we study learners as systems that attend— observing systems at layered levels, individual, formal and so on. Drawing from theories of observing and acts of distinction, humans may be studied as observing systems who, in the operations of sensing, perceiving and observing, enact worlds and mark possible states.

How Do People Make Distinctions? Analysing Observing Systems

Maturana (1988) explains that as cognitive beings continually interact they enact objects, entities or relations that make up their worlds. Errors point to the existence of a different observing system, a co-constituted ontology from which the observer may view the actions of another observer as mistaken. In this case, conventional mathematical distinctions could also be construed as mistakes when observed, transiently or otherwise, from another observing system. *Inter-objective* analysts ponder ways in which students' mathematical concepts make perfect sense. Moreover, one need not only seek to understand the idiosyncratic mathematical worlds that students bring forth, but also to investigate the conventional mathematical domain as one of many possible worlds. What are the conditions of possibility for enacting a covering, part-whole (rational number as compared to ratio) fraction world? Subject matter, cognitive, phenomenological and enactivist analyses must be part of a broadened appreciation of embodied, extended and embedded dynamical contexts.

The inter-objective stance considers humans to be observing systems nested within larger observing systems and interacting with other observing systems. By considering students, teachers, mathematicians, collectives of students and the culture of mathematics, each and all as observing systems, we may see the distinctions those systems make, the objects they enact and the states they mark as they attend. This is a radical interpretation of multiple *points of*

view. When a student views fraction kit pieces as a covering to be measured this is a point of view—a distinction by an observing system with specific conditions of possibilities.

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Silence and Voice in the Mathematics Classroom

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Because language is the primary medium through which mathematical understandings are shared, the form of the discourse in mathematics classrooms is significant. For fluid communication, it is necessary that teachers and students use language as though it accurately represents their mathematical ideas. However, when we direct our attention specifically toward language, new possibilities can be opened up for seeing more clearly the nature of the classroom discourse and the nature of mathematics itself. To investigate such possibilities, I engaged a class of 15- and 16-year old mathematics students in an extended conversation about their mathematics communication.

Language Counts in Mathematics: Scholarly Context

Since Pimm (1987) introduced discourse analysis to mathematics education scholarship, there has been growing interest in the nature and form of mathematics classroom discourse. In particular, this interest has been focussed on lexical and grammatical features (e.g. Rowland, 2000; Morgan, 1998; Weingrad, 1998; L. Bills, 2000; Herbel-Eisenmann, 2000; Phillips, 2002; Gerofsky, 2004). Discourse can also be approached in other ways. Mathematics educators have considered mathematics learning discourses with an interest in semiotics (e.g. Duval, 1999; Radford, 2002), genre (e.g. Gerofsky, 2004), post-structuralist hermeneutics (e.g. Brown, 2001), conversation analysis (e.g. Barwell, 2003) and socio-cultural milieux (e.g. Cobb et al, 1992). Mathematical signs are unique because mathematical objects themselves are inaccessible and because language plays a central role in both the development of these objects and in the mediation of shared understandings of them.

In my research, I initiated conversation with a focus on lexico-grammatical features of the classroom language practice because of my sense that recent scholarly analysis of such features could help with the decisions mathematics educators and learners face. While I appreciate the value of this new stream of research within mathematics education, my investigation differs from most of the other studies of classroom discourse because it is oriented more toward critical exploration than to description.

Seeing What is Not There: Method

In this investigation of possibilities, I used Norman Fairclough's *critical discourse analysis* (CDA) to uncover alternative possibilities for participation in mathematics classroom discourse (see Fairclough, 1995). When CDA is introduced in an educational setting, Fairclough (1992) calls it *critical language awareness* (CLA). Attempts to include CLA in school curricula typically involve language arts and language-acquisition classes. My research

investigates the introduction of CLA in a mathematics classroom. While co-teaching pure mathematics to a class of grade 11 pure mathematics students (a class of students aiming for university matriculation), I had a nineteen-week conversation with the students and their regular teacher. As I directed these participants' attention to features of their language practice in the classroom, I drew out a student perspective on mathematics learning.

The CDA language-analysis frame I brought to the classroom was complemented by Skovsmose and Borba's (2000; 2004) new framework for critical mathematics educational research. Following their guidelines, I directed my attention to aspects of the discourse that were *not there* as much as to aspects that were present. In this case, this meant attending to things that were *not said* as well as to what *was* spoken. Indeed, investigation of possibilities within a discourse must imagine alternatives to present language practice. I planned to discover a range of possibilities available to mathematics learners and teachers for participation within their classroom discourses.

Hearing Silence: Student Response to Language Awareness

Though I was interested in developing with the participant students a sense of the range of alternative language forms that could fit in the classroom discourse, our critical awareness also exposed silences – silences that seem to be endemic to mathematics education discourse. Most obvious among these is the tendency to remain silent about language, to ignore the significance of the role language plays in mathematics and in mathematics learning.

As an instance of this kind of silence, the participants in this research found various ways to resist my attempts to raise their awareness of language. I characterize their general lack of cooperation as *passive resistance* because their response to my interventions could not be described as outright rebellion. Rather, the participants typically responded to my prompts with either literal silence or with shallow, disinterested but compliant answers. Furthermore, most of the students evaded interviews even though they expressed a willingness to spend their free time in interviews. In classroom discussions, although I varied the form and context of my prompts throughout the term of our engagement, I did not find a strategy that I would consider generally successful for raising language awareness.

Valero and Vithal (1998) illustrate the importance of disruption in research settings and argue against typical research methodologies that assume and promote stability. It was important for me to attend to the participant students' apparently disruptive resistance to my research agenda. While the participant students' resistance was significant in itself, it became especially important as a context in which to view the exceptions to their general passive resistance. The exceptions, in which students became engaged and persistent, reflect some of these students' real concerns about this discourse that had been a part of their daily lives for 11 years now – their mathematics classroom discourse. Three streams of the larger conversation exemplify some possibilities associated with attention to language.

Agency in Mathematics: The Silent Human Voice

In one of the three exemplar streams of conversation about language, Joey, a 17-year-old boy, resisted the idea of personal agency. In an interview that followed a number of conversations about voice he articulated a viewpoint that he and his classmates expressed repeatedly: "You shouldn't use any voice, you should use the general voice. I've termed it the general voice because I'm cool and I can make my own terms." (For elaboration on and further interpretation regarding this stream of conversation, see Wagner, 2004a.)

This part of the conversation began when I asked students to identify who had agency in various texts, including a newspaper article and a mathematics textbook page. In the conversations about agency that followed, we focused our attention on the subjects in our sentences. For example, if the subject of a sentence is *I*, then the speaker is likely to be taking initiative in some way. In these initial conversations, I thought it significant that the students could not find evidence of human agency in their mathematics textbook language, but I did not raise the issue with them because I wanted to listen to what *they* noticed and found significant.

They seemed to think that the absence of human agency in the language practice suggested that such agency is insignificant in mathematics. I would argue that this absence is *not* insignificant. Absences are significant. This particular absence is an essential part of mathematics discourse and practice, an absence that has been noted by researchers who have approached mathematics learning in diverse ways (e.g. Balacheff, 1988). Awareness of this aspect of the discourse can help students find their place within the discourse, though it may appear from the language practice that there is no place for human agency at all.

As the conversation progressed, students resisted my suggestion that a general *you* voice could be appropriate in mathematics – as in, "If you add a constant, the curve translates vertically." Rowland (2000) and Pimm (1987) both comment on this form of the *you* voice in mathematics learning environments. It was not the sense of generalization the students resisted: it was the inclusion of personal pronouns. The students seemed to share the opinion that mathematics has no place for human initiative, though I tried with provocation and examples to direct them to recognize human agency in mathematics.

Joey's proclamation about what he deemed the "general voice" (language that masks human agency) is significant because with these words he recognized his capacity for invention in language practice. At the same time, he argued against the linguistic recognition of invention and initiative in mathematics. When he moved from envisaging particular perspectives in mathematics to envisaging a general, conventional perspective, he exemplified a tension that is at the heart of mathematics (c.f. Boaler, 2003; Pickering, 1995). Every student of mathematics must come to terms with this tension, either explicitly or implicitly.

Facing the Mathematics: The Turned-Away Human Face

In another stream of my larger conversation with the students in this classroom, our attention was drawn to paralinguistic communication practices. Two students in the class, Arwa and Tharshini, became fascinated with the way they directed their gaze in mathematics class. They drew others, including myself, into their fascination. Arwa and Tharshini noticed that they and their classmates did not look at each other when communicating mathematics. These two girls, who were close friends, became interested in the direction of students' gazes when they noted similarities between Tharshini's family dog and the dog in a cartoon I had shown to the class. In the cartoon, a man scolds his dog while pointing at a mess on the floor (Larson, 1983/1989, p. 230). The dog looks at the man's finger, not at the mess. As they chatted about the cartoon (perhaps while doing mathematics homework together), Arwa and Tharshini realized that when they communicated mathematics they were like the dog, looking at the symbols of mathematics, not at the actual objects of mathematics, and not at each other.

The cartoon resembles stories of the Buddha scolding his disciples for looking at his finger instead of at the moon (e.g. Nhât Hanh, 2002). Like the Buddha, I showed the participant students the cartoon to illustrate my sense of the oddness of our critical attention to language. We were looking at language, which, like a finger, is a medium for pointing at ideas and objects. In our discussions about language, instead of paying attention to mathematics itself, I

was asking them to pay attention to the language with which we talked about, symbolized or otherwise pointed at the mathematics.

As Arwa and Tharshini paid attention to their language practice and, in this case, to paralinguistic aspects of communication, they noticed that mathematics is seen through symbols and that no two people can see any symbol in the same way. With these observations, they developed a sense of the inaccessibility of mathematical objects, a characteristic of the discourse noted by numerous scholars interested in semiotics (e.g. Sfard, 2000; Duval, 1999).

There is an irony in this stream of conversation. A few times, these two good friends reported to me their fascination with the transcripts of our discussions, because these transcripts seemed to support their belief that each thought the same thoughts as the other. Their observations about language use and pointing reveal their awareness of the problematic nature of this belief. Ironically, they spoke as though they knew what the other person was thinking even as they articulated their awareness that each could not possibly know what the other person was thinking. This sense of shared meaning suggests an intimacy associated with effective communication (c.f. Gordon Calvert, 2001).

Despite this intimacy, the students in this class typically did not look at each other when communicating mathematics. Rather, they shared their gaze at the same mathematical objects, looking through the symbols they were creating on paper. Ironically, the intimacy engendered by shared mathematics typically does not seem to be represented in the form of the communication – we avoid each other's gazes and we mask human agency by avoiding personal pronouns. This relates to the tension Adler (2001) calls the dilemma of transparency, which is faced by mathematics educators: Should we draw students' attention to the mathematics or to the language we use to talk about the mathematics. The tendency to look at the symbols rather than at the person is another aspect of the linguistic manifestation of the abstract, general nature of mathematics.

These two girls turned their faces to look *at* some paralinguistic features of their communication – the nature of their attention and their eye movement – and became aware of the way they looked *through* their mathematical symbols to see the mathematics itself. With critical attention to language, they were enabled to "see" and discuss things they were not literally seeing in their mathematics, namely their mathematical objects and each other's faces. Seeing the invisible is like hearing silence.

Articulating Insignificance: De-Emphasis and Mathematics

In a third stream of my larger conversation with the students in this classroom, we considered the effects of a word often used in mathematics classrooms, the word *just*. Our interest in the word began when I showed the class a transcript from a previous day when Jessye had said, "And you just change it to two square root five, right?" I circled the word *just* in the transcript and asked the class, "What does that mean when she says *just*?" (For elaboration on and further interpretation regarding this stream of conversation, see Wagner, 2004b.)

Gary paraphrased his interpretation of Jessye's meaning: "You simply change it." He also suggested that when teachers use the word *just* in a mathematical explanation, it is insulting because the teacher is suggesting that something is simple, when for students the process might be difficult.

Many of Gary's classmates agreed with him. None of the students in this class said that they themselves felt insulted. Rather, they seemed to be worried that others would feel insulted. Their concern was pedagogical. The passion they displayed when talking about this word pointed to the importance of the word in this mathematics class and their previous

experiences. These students were indeed demonstrating an awareness of language practice. However, critical awareness demands an exploration of a *range* of possibilities. In the early stages of this stream of conversation, most of them seemed to be fixated on *one* account of the effects of teachers using *just*.

Though I felt partially responsible for the students' vein of worry about mathematics teachers suggesting simplicity by using the word *just*, I resisted their complaint. The participant teacher and I continued to use the word *just* regularly when we taught. In order to convince the students of another more positive perspective on the use of the word, I wrote a single-page essay for them, referring to the adverbs *just* and *simply* as "diminutives" (although they are not actually diminutives) because they suggest that the actions they describe are unimportant or trivial. This essay marked the beginning of my public disagreement with the students and the emergence of their clear voice.

The de-emphasis that is made possible with words like *just* is necessary for emphasis. Gattegno (1984) asserts that every circumstance of life involves stressing and ignoring, which relates to relative emphasis. He adds that the process of stressing and ignoring is especially important in mathematics education because it is the process of abstraction.

Diminutives like *just* can be used for pointing, I said in the essay for students. The de-emphasis of one procedure can emphasize another procedure or another aspect of the reasoning. With such emphasis and de-emphasis, we point attention to the important ideas we are talking about. Presenting this reasoning in the essay, I thought I had made a clear point about a positive effect of a teacher or student using de-emphasis. However, the students were not convinced.

In response to my essay, the students continued to express their concern that words like *just* can be insulting, that these adverbs suggest a procedure is obvious when it may not be so obvious to students. Though I considered their interest in this pedagogical issue a significant revelation, I felt frustrated that these students seemed uninterested in my suggestion that teachers and students use de-emphasis to point in mathematics communication. While their resistance to the alternative possibilities exposed a deficit in their *critical* language awareness, the resistance clarified that the concern they were expressing was important from their perspective. The students' resistance verified the role critical language awareness can play in drawing out the authentic voice of students, the articulation of their unique perspective on mathematics classroom discourse.

Late in the term, I resurrected the stream of conversation. This time, Jocelyn expressed another concern: she resented it when her teachers glossed over any aspect of their mathematics in an explanation. Her concern pointed at another aspect of the language practice in question. When the words *just* or *simply* are used to indicate simplicity, they actually replace a more careful explanation of the procedure indicated by the verb. For example, when a teacher says, "and we just solve that," the adverb *just* suggests that the solving is straightforward, unremarkable. The teacher has the option to describe the solving procedure in great detail, but chooses not to do this.

When Jocelyn expressed her contempt for teachers who are vague, Tharshini argued against her concern by noting the time constraints teachers face. Jocelyn's concern relates to a typical conversation norm often noted by linguists – that people ought to and try to speak with sufficient but not excessive detail. This norm was first recognized by philosopher Paul Grice (see Levinson, 1983). Jocelyn and Tharshini were arguing about the amount of detail teachers ought to provide, an issue teachers confront every day. Different students want and need different degrees of explanation and vagueness. Even teachers who think they explain their mathematical examples fully cannot possibly do so. Rowland's (2000) extensive study of vagueness in mathematics discourse and Channell's (1994) study of vagueness in more general situations both overlook the role adverbs like *just* play in facilitating vagueness,

In this stream of conversation, we were talking about glossed-over mathematics, but the principle can be extended. In any discourse, it is natural to *just* fit in, to follow the language and behaviour patterns of the people around us. In mathematics class, it is understandable that students would think, "this is *just* how it is done." Alternative mathematical possibilities can become accessible to students when they realize that certain language patterns can actually mask these alternatives. And with critical attention to language, which implies consideration of a range of possibilities, we open for ourselves the possibility of seeing what others do not see, of hearing what is not normally said. We afford ourselves the possibility of listening to the silences.

Responding to Silence

Silence is a factor in each of these three streams of conversation about, respectively, silencing a person, avoiding a person's face, and skirting explanation. An awareness of the tendency to silence the human agent in mathematics classroom discourse can help teachers attune themselves to this and other silences, and can provide researchers with new insight into the nature of the discipline and students' relationship with the discipline.

With my exploration of language, I sought to look at something that actually exists – secondary school mathematics classroom discourse. In the exploration, I developed an interest in things that are not there (or not recognized): the silences endemic in the discursive practice of the classroom. I have been led to wonder about other possible ways of researching silence and voids in the mathematics classroom. In this study, my awareness of silences *emerged* from the experience of attending critically to speech. How might we enter an experience with the *intention* of listening to silence?

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Ad Hoc Sessions

Séances ad hoc

What Can We Learn From Learner-Generated Examples: A Case of Linear Algebra?

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Linear algebra is one of the most applicable subjects in the field of mathematics and the sciences. It has become a required course for many disciplines. Students' understanding of linear algebra and the effects of different teaching methods on students' understanding have been investigated from a variety of perspectives. However, these questions have not been examined through the lens of student-generated examples. Since, in order to achieve understanding, students have to be engaged in a mathematical task, the present study investigates whether and in what way the example-generation tasks influence students' understanding of linear algebra. In particular, the study examines students' ability to construct examples for mathematical statements and objects in the undergraduate linear algebra course. It aims to analyze and describe what difficulties students encounter when constructing examples, and how example-generation tasks can inform researchers about students' understanding of linear algebra.

Research has shown that linear algebra is one of the postsecondary mathematics courses that students are having difficulty with (Dorier, 2000; Carlson et al, 1997). Part of the difficulty is due to the abstract nature of the subject. Dubinsky (1997) points out that there is a lack of pedagogical strategies that give students a chance to construct their own ideas about concepts in the subject. As research shows (Hazzan & Zazkis, 1999; Watson & Mason, 2004), the construction of examples by students contributes to the development of understanding of the mathematical concepts. Simultaneously, learner-generated examples may highlight difficulties that students experience.

The study discussed students' difficulties with constructing examples, and also suggested possible correlations of students' understanding with the generated examples. Furthermore, it showed that the example-generation tasks reveal students' (mis)understanding of the mathematical concepts. In particular, generating examples for the mathematical statements require more than just procedural understanding of the topic. This research provides a better understanding of the role of example-generation tasks in students' understanding of linear algebra. It analyzes students' difficulties involved in generating examples and how students' examples correlate with their understanding.

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Reconstructing Foundational Mathematical Knowledge: Experiences of Math-Anxious Elementary Teachers in a Graduate Course

Rina Cohen
OISE/UT

Introduction

Math anxiety is prevalent in our society in general and particularly among pre-service and in-service elementary teachers (Hembree, 1990; Ashcraft, 2002). Teachers with higher levels of math anxiety often lack mastery of fundamental math concepts, which has a detrimental effect on their teaching (Cohen & Green, 2002). It has been claimed that highly anxious teachers may unintentionally pass on their negative feelings and attitudes to their students (Bulmahn & Young, 1982; Jackson & Leffingwell, 1999; Karp, 1991), but this claim has been debated by other researchers (e.g. Bush, 1989).

Based on a pilot study with a group of twelve math-anxious elementary teachers who participated in a series of eight Math Empowerment Workshops (Cohen & Green, 2002), a new Graduate course was developed by the author at OISE/UT, titled: *Gaining Confidence In Mathematics: A Holistic Approach to Overcoming Mathematics Anxiety.* This one-term course has been regularly taught since the summer of 2003 and is also open to pre-service teachers in one of the initial teacher education programs at OISE/UT. Usually 30%-50% of the course population consists of pre-service teachers. The course provides a reform-based learning environment in which hands-on math explorations, mental math with invented strategies and creative problem solving by individuals and groups are combined with journal writing, reflection, relaxation and guided visualization activities.

Initial Study

The initial study took place when the course was taught for the first time in summer 2003 to a class of 13 in-service and 5 pre-service teachers. Data consisted of participants' journal entries, math work and final reflection papers, three questionnaires, and researchers' field notes. In-depth, qualitative data analysis focusing on the experiences of nine of the most anxious class members was conducted. Findings indicated that the teachers went through significant positive changes in their math related affect (McLeod; 1992) during the course. By reflecting on their usually negative, early math experiences, and on assigned readings on math anxiety such as Ashcraft (2002), teachers gained a better understanding of how their anxiety interfered with their mental functioning during problem solving activities. Such reflection, along with various relaxation, sensitising and guided visualization activities, helped them learn how to recognize when they were experiencing anxiety and mental blocks while solving problems. The group mental math and problem solving sessions helped unleash the teachers' mathematical creativity as they finally overcame their blocks and started inventing their own strategies. As the course progressed, teachers' sense of self efficacy and confidence grew significantly. For a more detailed discussion of the findings see Cohen & Leung (2004).

Reconnecting Fractured Knowledge Schemas

The current paper focused on course participants' construction of math knowledge during the course. Teachers' knowledge building occurred through the various math activities and particularly through inventing their own strategies during mental math and problem solving. As one of the teachers expressed in her journal: "students will learn more meaningfully if allowed to use creative thinking and processes".

Teachers' understanding of the developmental nature of their own math knowledge building processes was illuminated by reading Skemp's historical article *Schematic Learning* (1972) (based on Piaget's theory.) As one of the teachers who wrote in her journal reflection: "there are topics which we cannot learn effectively unless we know something else already… if students have gaps in their knowledge or are unable to connect their schemas, then learning difficulties arise". Early in the course she recognized her own gaps in math knowledge and worked hard during the course to fill those gaps. At the end of the course she wrote: " I am pleased that the holistic techniques used in this course helped me unlearn concepts that I found confusing, and *reconnected my fractured schemas*. I was supported by working in groups, and because I had manipulatives which helped me see the relationships between concepts". The concept of reconnecting "fractured knowledge schemas", coined by this teacher, has since been one of the central themes in the course.

The author has been in touch with some of the previously math-anxious teachers who have taken this course and was pleased learn that most of them have developed a certain level of confidence and pride in their math teaching. Nearly all of them continue their professional development in math and several have gone on to become Math Lead teachers in their schools.

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Roadkill, Skeletons, and Other Mathematical Metaphors

Julie Long, *University of Alberta*
Gladys Sterenberg, *University of Alberta[1]*

Background experiences as learners of mathematics and beliefs about the nature of mathematics are related to pedagogy. Understanding new views of mathematics is very challenging and can contribute to teacher resistance. As teacher educators, we have become aware of the fruitfulness of considering metaphors of mathematics with our students. From our initial teaching experiences, we wondered about the generative promise of using metaphors. To what extent can experiences and images of mathematics be better understood, and perhaps altered, through metaphoric analysis?

Dans cette étude, nous avons interviewé huit personnes aux sujets de la pédagogie et de l'apprentissage des mathématiques. Julie a interviewé quatre futures enseignantes qui venaient de completer un cours de quatre mois en didactique des mathématiques. Gladys a interviewé quatre enseignantes du niveau élementaire. Ces dernières entrevues ont eu lieu au début et à la fin d'une étude de quatre mois à propos de la métaphore *les mathématiques sont une histoire*.

Pre-service teachers suggested that mathematics is roadkill, a skeleton in the closet, calypso, a mosquito, marching ants, a journey, a spider's web, classical music, and hip-hop/rap music. In-service teachers offered metaphors of mathematics is a puzzle, a story, a mountain, a pattern, a battle, a language, and a construction site.

The metaphors presented in this study illustrate how pre-service and in-service teachers experience mathematics. What we find interesting is the tendency by the participants to migrate to uni-dimensional interpretations of metaphors; teachers offered one dimension of the metaphor, either a positive or a negative dimension, suggesting a dichotomy. Perhaps mathematics is not dichotomous; perhaps it includes aspects that are simultaneously connected and sticky (web), adventurous and arduous (journey), as well as changing and stable (language). We believe that a comprehensive study of the particular metaphors may offer a fruitful way of talking about mathematics with teachers, especially in this era of reform where the sole focus seems to be on the more positive aspects of mathematics.

Les résultats de cette étude nous rappellent la gravité de certaines questions en éducation et en mathématiques. Nous pensons surtout aux perceptions des mathématiques, à l'importance de l'appui dans l'apprentissage des mathématiques, aux conséquences de nos expériences précédantes, et à l'influence de nos anciens enseignants et anciennes enseignantes sur nos idées à propos de l'enseignement.

Mathematics in Waldorf Education

John Grant McLoughlin
University of New Brunswick

The limbs, the heart, and the head - that is the order through which mathematics is introduced in Waldorf education. The respective representations of willing, feeling, and thinking are encountered in a manner that brings forth the teaching and learning of mathematics as one part of a general education. While it is unreasonable to examine the tenets of Waldorf education in a short period, the ad hoc offers an avenue for opening discussion of how mathematics is approached in the Waldorf model. My personal connection with the topic weaves into various facets of my life including professional development, home schooling, and ongoing discussions, readings, and collaborations with others actively interested in Waldorf education.

The Association of Waldorf Schools of North America (AWSNA) has a website that contains many links to other aspects of Waldorf education. Established Waldorf schools or initiatives can be found in many places worldwide, including various locations in Canada, particularly in BC, Ontario, and Quebec. Further discussion on this subject is welcomed. Feel free to contact John Grant McLoughlin. A selection of relevant readings or resources, in mathematics, appears below.

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Using Mathematics as a Source When Creating Metaphors or Images for Teaching and Learning

Joyce Mgombelo, *Brock University*
Dave Hewitt, *University of Birmingham, UK*

It is customary, for mathematics education researchers and practitioners to use borrowed metaphors from other perspectives such as psychology, socio-cultural studies, cognitive science, philosophy and sociology when thinking about teaching and learning. However, despite the fact that mathematics education researchers and practitioners very often use mathematical metaphors and images for teaching and learning the role of mathematics as a source of metaphors and images for teaching and learning is underrepresented. This ad-hoc session was prompted by the presenters' shared interest in the use of mathematics as a source of metaphors for teaching and learning and the need to bring to the awareness of mathematics educators this important role of mathematics. Participants were invited to share mathematical metaphors which come to their mind. Dave Hewitt offered two mathematical metaphors as a starter:

The perimeter of a fractal

Seeing something complex and zooming in to find that new complexities appear where he previously thought the perimeter might become smooth. For Dave this symbolised the challenge of teaching – a beginning teacher can feel that there are so many things to think about with the new challenge of teaching but that each of these might become easier and perhaps even straight forward once certain awareness and skills are developed. However, the more someone learns about an aspect of teaching, rather than it becoming simpler, new issues and complexities arise of which the beginning teacher was not originally aware. So developed awareness of an issue brings new sub-issues which were previously unknown and the level of complexity seems to be maintained.

The solution of a quadratic equation

Seeing the two roots as symbolizing making a choice between one action and another by a teacher, both of which could be seen as 'solutions' to a classroom situation, but where the particular context might mean one action is preferred over the other.

The following are some of the metaphors that were offered by participants:

Rubic cube

Can I get to where I want to go? The journey may not be straight forward even if possible. If I swap two coloured squares then maybe I cannot get to where I want to go. It is not always possible (David Pimm).

A cusp catastrophe

If I do something small then sometimes something dramatic can occur. Looking the other way round, for something dramatic to occur I might only need to do something small to assist this (David Pimm).

Escher's stairs

Sometimes you feel like you are going up and then find that you are back at the beginning (especially with PhD studies!) (Shabnam Kavousian).

Networks

Students expect there to be a 'best' way. Sometimes there are not clear answers but maybe you are still not immobilised, you can still work on the problem (Marty Hoffman).

There can be disconnected networks which later change into one connected one (Elaine Simmt).

Tension between a curriculum document which offers a linear model of learning versus learning as a complexly connected network (David Reid).

Catastrophe theory

Sensitivity to initial conditions – I can say something to different people but it doesn't have the same effect (Elaine Simmt).

Relationships

These can become the objects of our attention when really they are not there. An issue for a teacher is to know what to put together so that another person might see a pattern (which is not there as an object itself) (Immaculate Namukasa-Kizito).

In working together we can have a community which works towards something but which is not about 'getting there'. 'Truth' is not something I buy, but within a group we don't have to say the same to know what each other means (Laurinda Brown).

$\sqrt{2}$ – it knows all its decimal places but won't tell us, and never will! Once we are in the middle of a problem and we are stuck, it can help to get out of this 'stuck' state by assuming the problem was solved (John Mason).

Expressing a generality for the number of matchsticks in an n by n square, such as this 3 by 3 example:
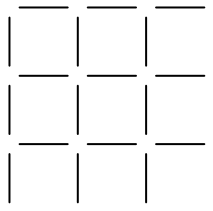
People can come up with quite different algebraic expressions – everybody can seem as if they are saying different things but really they are saying the same (John Mason).

Refining the Canadian Survey Questions for the *Census at School* Survey to Provide Richer Mathematical Learning

Joel Yan, *University Liaison Program, Statistics Canada*
Mary Townsend, *Education Outreach Coordinator, Statistics Canada*
Florence Glanfield, *University of Saskatchewan*

What is the Census at School project?

Students aged 8 to 18 from across Canada are getting involved in the international Census at School project. They respond in class to an online survey, covering topics such as their height, pets or favorite school subject. Then they 'play detective' with the anonymous results, discovering interesting patterns and comparisons that bring their lessons to life. Students have fun experiencing this survey about themselves, while gaining important skills in using information technology to understand their world and make informed decisions. They become aware of the important role of the national census in collecting information to help us understand our country and its people.

The Census at School international project began in the United Kingdom in 2000. It now contains database of results from Australia, New Zealand, South Africa and, as of summer 2004, Canada! Statistics Canada is responsible for the Canadian component of the project. To develop the survey questions we started with the international questionnaire and then consulted with a pan-Canadian teacher advisory committee to adapt this questionnaire for classrooms across Canada. The resulting Canadian survey includes some questions that are common to all participating countries and others, on topics such as bullying and allergies, which were developed by an advisory board of mathematics teachers from across Canada. These teachers have also created over 20 online learning activities for grades 4 to 12. Members of the Census at School teacher advisory panel are:

- Anna Spanik (Halifax Regional School Board)
- France Caron (Université du Québec à Montréal)
- Tom Steinke(Ottawa-Carleton Catholic DSB)
- Florence Glanfield (University of Saskatchewan)
- Bradd Hart (McMaster University)

How teachers are using the Census at School project in their schools

Depending on their grade level, students can use these results to:

- create different types of graphs to answer questions such as "Do boys and girls eat different breakfast foods?"
- explore relationships between variables: "Does foot size increase with height?"
- analyze a phenomenon like bullying and determine which age group is most at risk: "Bullying - studying to curb it".
- compare their class with typical students their age in Canada, or other countries around the world using the activity "How weird is our class?"

Students can compare their class results to provincial, national and international data available on the Canadian website www.censusatschoool.ca under Results and Data. Based on positive teacher feedback on the project, the Ontario Ministry of Education has listed Census at School as a recommended example of a primary data source for teaching data management in the new mathematics curriculum for grades 6, 7 and 8 (Ontario, 2005).

What teachers are saying about *Census at School*

The project is a success as demonstrated by comments from participating teachers listed on the website (http://www19.statcan.ca/05/05_000_e.htm#02) and in the paper presented at the Canadian Math Education Forum (Statistics Canada, May 2005).

> "My students got more out of this project than any text book or teacher could communicate." - Larry Scanlon, primary-intermediate special education teacher, Waterloo, Ontario

> "Kids connect best with data they can see themselves in. Census at School makes it painless to collect data in electronic format, so that students can spend more time analyzing the data. Census at School data has a nice balance of numeric and categorical variables, which allow for a rich array of representations and analyses." - Tom Steinke, Educational Consultant, Ottawa-Carleton Catholic School Board

> "Through this project, students see statistics (and math in general) as a set of conceptual tools that help them better understand the complexity of the world in which they live in." -France Caron, Education professor, Université du Québec à Montréal, Québec.

Over 8,000 Canadian students completed the survey in the 2003-2004 school year. The project was even more popular in the 2004-2005 school year with over 20,000 students completing the survey. A series of summary results tables at the Canada and provincial are posted on the www.censusatschool.ca site under "Data and results". The survey found that math was the second favourite subject of high school students after physical education, while elementary school students preferred physical education followed by art. In high school, 24% of girls and 22% of boys said they don't eat breakfast.

Students and teachers find working with data about their class and their peers an interesting way to learn data analysis skills. Join in at http://www.censusatschool.ca.

Summary of Discussion at the CMESG Workshop

At this ad-hoc workshop at the CMESG we were seeking input and proposed new questions from mathematics educators for our 2005-06 *Census at School* questionnaire. As a result of this session, detailed input given in a post-CMESG meeting at Statistics Canada on May 31st,

and earlier feedback from mathematics teachers and the advisory panel, the following major changes were incorporated in the questionnaire that is being used this school year:

- Two new questions were added on student attitudes towards social issues, thanks to Ralph Mason and others. (See text box below)
- Several new questions were added on physical measurement of students, so that there are now 6 measurement questions.
- Several questions were reworded so as to provide more extensive continuous numeric data (e.g. the time use and smoking questions).

There are now 30 questions for secondary students and 28 for primary students. As of December 2005,over 7,000 students in Canada have already entered their data using this new questionnaire into the Census at school database. The questionnaire benefited from the input received from CMESG members.

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Ontario Ministry of Education, *The Ontario Curriculum, Grades 1-8 Mathematics*, Revised, 2005. http://www.edu.gov.on.ca

Canadian Census at School website – http://www.censusatschool.ca

International Census at School website – http://www.censusatschool.org/

Undergraduate Students' Errors That May Be Related to Confusing a Set With its Elements[2]

Kalifa Traoré, *Université de Ouagadougou*
Caroline Lajoie, *Université du Québec à Montréal*
Roberta Mura, *Université Laval*

In a recent study of students' difficulties with the ideas of normal subgroup and quotient group (Lajoie and Mura, 2004), it was observed that several students misunderstood the very nature of the elements of a quotient group: they seemed to think that the elements of a quotient group *G/N* were elements of *G* instead of subsets of *G*. Believing that a weakness in set theoretical prerequisites might be a factor contributing to this misunderstanding, Lajoie and Mura have started a new research project aimed at examining students' difficulties with the first concepts of set theory and their use in elementary group theory. In the ad hoc group we discussed a few types of errors observed during a pilot study for this project and argued that all the types of errors presented could be related to confusing a set with its elements. All the data were collected from the work of students in one course on logics and set theory. The students were mathematics or computer science majors and had already passed a first course in abstract algebra. Some of the errors observed may seem rather surprising among such a population.

In the ad hoc group, excerpts from students' work were presented that illustrate the following types of errors.

> T1: when there are two pairs of nested braces, one of the pairs can be deleted.
> Example: $\cup (\{\phi, \{\{x_1, x_2\}\}, \{\{y_1, y_2, y_3\}\}, \{\{x_1, x_2, y_1, y_2, y_3\}\}\}) = \{x_1, x_2, y_1, y_2, y_3\}$.

> T2: confusion between belonging and inclusion.

> T2.1: if all the elements of *A* belong to *B*, then *A* too belongs to *B*.
> Example: since each of ϕ, $\{x\}$ and $\{y\}$ belong to $\{\phi, \{x\}, \{y\}, \{x, y\}\}$, then $\{\phi, \{x\}, \{y\}\}$ also belongs to $\{\phi, \{x\}, \{y\}, \{x, y\}\}$.

> T2.2: $A \in B \Rightarrow A \subset B$.
> Example: since $F \not\subset G$, thus *F* is certainly not an element of *G*.

> T2.3: if all the elements of *A* are subsets of *B*, then *A* too is a subset of *B*.

Example: we see immediately that $\{\{x\}, \{y\}, \{x, y\}, \phi\} \subset \{x, y\}$.

T3: confusion between the union of two sets and the set consisting of those two sets $(A \cup B = \{A, B\})$.
Example: $a \cup c \cup x \cup a = \{a, c, x\}$.

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Appendices

Appendices

APPENDIX A

Working Groups at Each Annual Meeting

1977   *Queen's University, Kingston, Ontario*

· Teacher education programmes
· Undergraduate mathematics programmes and prospective teachers
· Research and mathematics education
· Learning and teaching mathematics

1978   *Queen's University, Kingston, Ontario*

· Mathematics courses for prospective elementary teachers
· Mathematization
· Research in mathematics education

1979   *Queen's University, Kingston, Ontario*

· Ratio and proportion: a study of a mathematical concept
· Minicalculators in the mathematics classroom
· Is there a mathematical method?
· Topics suitable for mathematics courses for elementary teachers

1980   *Université Laval, Québec, Québec*

· The teaching of calculus and analysis
· Applications of mathematics for high school students
· Geometry in the elementary and junior high school curriculum
· The diagnosis and remediation of common mathematical errors

1981   *University of Alberta, Edmonton, Alberta*

· Research and the classroom
· Computer education for teachers
· Issues in the teaching of calculus
· Revitalising mathematics in teacher education courses

1982   *Queen's University, Kingston, Ontario*

· The influence of computer science on undergraduate mathematics education
· Applications of research in mathematics education to teacher training programmes
· Problem solving in the curriculum

1983   *University of British Columbia, Vancouver, British Columbia*

· Developing statistical thinking
· Training in diagnosis and remediation of teachers
· Mathematics and language
· The influence of computer science on the mathematics curriculum

1984   *University of Waterloo, Waterloo, Ontario*

· Logo and the mathematics curriculum
· The impact of research and technology on school algebra
· Epistemology and mathematics
· Visual thinking in mathematics

1985   *Université Laval, Québec, Québec*

· Lessons from research about students' errors
· Logo activities for the high school
· Impact of symbolic manipulation software on the teaching of calculus

1986   *Memorial University of Newfoundland, St. John's, Newfoundland*

· The role of feelings in mathematics
· The problem of rigour in mathematics teaching
· Microcomputers in teacher education
· The role of microcomputers in developing statistical thinking

1987   *Queen's University, Kingston, Ontario*

· Methods courses for secondary teacher education
· The problem of formal reasoning in undergraduate programmes
· Small group work in the mathematics classroom

1988   *University of Manitoba, Winnipeg, Manitoba*

· Teacher education: what could it be?
· Natural learning and mathematics
· Using software for geometrical investigations
· A study of the remedial teaching of mathematics

1989   *Brock University, St. Catharines, Ontario*

· Using computers to investigate work with teachers
· Computers in the undergraduate mathematics curriculum
· Natural language and mathematical language
· Research strategies for pupils' conceptions in mathematics

1990   *Simon Fraser University, Vancouver, British Columbia*

· Reading and writing in the mathematics classroom
· The NCTM "Standards" and Canadian reality
· Explanatory models of children's mathematics
· Chaos and fractal geometry for high school students

1991   *University of New Brunswick, Fredericton, New Brunswick*

· Fractal geometry in the curriculum
· Socio-cultural aspects of mathematics
· Technology and understanding mathematics
· Constructivism: implications for teacher education in mathematics

1992   *ICME–7, Université Laval, Québec, Québec*

1993   *York University, Toronto, Ontario*

· Research in undergraduate teaching and learning of mathematics
· New ideas in assessment
· Computers in the classroom: mathematical and social implications
· Gender and mathematics
· Training pre-service teachers for creating mathematical communities in the classroom

1994   *University of Regina, Regina, Saskatchewan*

· Theories of mathematics education
· Pre-service mathematics teachers as purposeful learners: issues of enculturation
· Popularizing mathematics

1995   *University of Western Ontario, London, Ontario*

· Autonomy and authority in the design and conduct of learning activity
· Expanding the conversation: trying to talk about what our theories don't talk about
· Factors affecting the transition from high school to university mathematics
· Geometric proofs and knowledge without axioms

1996   *Mount Saint Vincent University, Halifax, Nova Scotia*

· Teacher education: challenges, opportunities and innovations
· Formation à l'enseignement des mathématiques au secondaire: nouvelles perspectives et défis
· What is dynamic algebra?
· The role of proof in post-secondary education

1997   *Lakehead University, Thunder Bay, Ontario*

· Awareness and expression of generality in teaching mathematics
· Communicating mathematics
· The crisis in school mathematics content

1998   *University of British Columbia, Vancouver, British Columbia*

- · Assessing mathematical thinking
- · From theory to observational data (and back again)
- · Bringing Ethnomathematics into the classroom in a meaningful way
- · Mathematical software for the undergraduate curriculum

1999   *Brock University, St. Catharines, Ontario*

- · Information technology and mathematics education: What's out there and how can we use it?
- · Applied mathematics in the secondary school curriculum
- · Elementary mathematics
- · Teaching practices and teacher education

2000   *Université du Québec à Montréal, Montréal, Québec*

- · Des cours de mathématiques pour les futurs enseignants et enseignantes du primaire/Mathematics courses for prospective elementary teachers
- · Crafting an algebraic mind: Intersections from history and the contemporary mathematics classroom
- · Mathematics education et didactique des mathématiques : y a-t-il une raison pour vivre des vies séparées?/Mathematics education et didactique des mathématiques: Is there a reason for living separate lives?
- · Teachers, technologies, and productive pedagogy

2001   *University of Alberta, Edmonton, Alberta*

- · Considering how linear algebra is taught and learned
- · Children's proving
- · Inservice mathematics teacher education
- · Where is the mathematics?

2002   *Queen's University, Kingston, Ontario*

- · Mathematics and the arts
- · Philosophy for children on mathematics
- · The arithmetic/algebra interface: Implications for primary and secondary mathematics / Articulation arithmétique/algèbre: Implications pour l'enseignement des mathématiques au primaire et au secondaire
- · Mathematics, the written and the drawn
- · Des cours de mathémathiques pour les futurs (et actuels) maîtres au secondaire / Types and characteristics desired of courses in mathematics programs for future (and in-service) teachers

2003   *Acadia University, Wolfville, Nova Scotia*

- · L'histoire des mathématiques en tant que levier pédagogique au primaire et au secondaire / The history of mathematics as a pedagogic tool in Grades K–12
- · Teacher research: An empowering practice?
- · Images of undergraduate mathematics
- · A mathematics curriculum manifesto

2004   *Univerité Laval, Québec City, Québec*

· Learner generated examples as space for mathematical learning
· Transition to university mathematics
· Integrating applications and modeling in secondary and post secondary mathematics
· Elementary teacher education - Defining the crucial experiences
· A critical look at the language and practice of mathematics education technology

2005   *University of Ottawa, Ottawa, Ontario*

· Mathematics, Education, Society, and Peace
· Learning Mathematics in the Early Years (pre-K – 3)
· Dicrete Mathematics in Secondary School Curriculum
· Socio-Cultural Dimensions of Mathematics Learning

APPENDIX B

Plenary Lectures at Each Annual Meeting

1977	A.J. COLEMAN	The objectives of mathematics education
	C. GAULIN	Innovations in teacher education programmes
	T.E. KIEREN	The state of research in mathematics education

1978   G.R. RISING   The mathematician's contribution to curriculum development

A.I. WEINZWEIG   The mathematician's contribution to pedagogy

1979   J. AGASSI   The Lakatosian revolution

J.A. EASLEY   Formal and informal research methods and the cultural status of school mathematics

1980   C. GATTEGNO   Reflections on forty years of thinking about the teaching of mathematics

D. HAWKINS   Understanding understanding mathematics

1981   K. IVERSON   Mathematics and computers

J. KILPATRICK   The reasonable effectiveness of research in mathematics education

1982   P.J. DAVIS   Towards a philosophy of computation

G. VERGNAUD   Cognitive and developmental psychology and research in mathematics education

1983   S.I. BROWN   The nature of problem generation and the mathematics curriculum

P.J. HILTON   The nature of mathematics today and implications for mathematics teaching

1984	A.J. BISHOP	The social construction of meaning: A significant development for mathematics education?
	L. HENKIN	Linguistic aspects of mathematics and mathematics instruction
1985	H. BAUERSFELD	Contributions to a fundamental theory of mathematics learning and teaching
	H.O. POLLAK	On the relation between the applications of mathematics and the teaching of mathematics
1986	R. FINNEY	Professional applications of undergraduate mathematics
	A.H. SCHOENFELD	Confessions of an accidental theorist
1987	P. NESHER	Formulating instructional theory: the role of students' misconceptions
	H.S. WILF	The calculator with a college education
1988	C. KEITEL	Mathematics education and technology
	L.A. STEEN	All one system
1989	N. BALACHEFF	Teaching mathematical proof: The relevance and complexity of a social approach
	D. SCHATTSNEIDER	Geometry is alive and well
1990	U. D'AMBROSIO	Values in mathematics education
	A. SIERPINSKA	On understanding mathematics
1991	J .J. KAPUT	Mathematics and technology: Multiple visions of multiple futures
	C. LABORDE	Approches théoriques et méthodologiques des recherches françaises en didactique des mathématiques
1992	ICME-7	
1993	G.G. JOSEPH	What is a square root? A study of geometrical representation in different mathematical traditions
	J CONFREY	Forging a revised theory of intellectual development: Piaget, Vygotsky and beyond
1994	A. SFARD	Understanding = Doing + Seeing ?
	K. DEVLIN	Mathematics for the twenty-first century
1995	M. ARTIGUE	The role of epistemological analysis in a didactic approach to the phenomenon of mathematics learning and teaching
	K. MILLETT	Teaching and making certain it counts
1996	C. HOYLES	Beyond the classroom: The curriculum as a key factor in students' approaches to proof
	D. HENDERSON	Alive mathematical reasoning

1997	R. BORASSI	What does it really mean to teach mathematics through inquiry?
	P. TAYLOR	The high school math curriculum
	T. KIEREN	Triple embodiment: Studies of mathematical understanding-in-interaction in my work and in the work of CMESG/GCEDM
1998	J. MASON	Structure of attention in teaching mathematics
	K. HEINRICH	Communicating mathematics or mathematics storytelling
1999	J. BORWEIN	The impact of technology on the doing of mathematics
	W. WHITELEY	The decline and rise of geometry in 20th century North America
	W. LANGFORD	Industrial mathematics for the 21st century
	J. ADLER	Learning to understand mathematics teacher development and change: Researching resource availability and use in the context of formalised INSET in South Africa
	B. BARTON	An archaeology of mathematical concepts: Sifting languages for mathematical meanings
2000	G. LABELLE	Manipulating combinatorial structures
	M. B. BUSSI	The theoretical dimension of mathematics: A challenge for didacticians
2001	O. SKOVSMOSE	Mathematics in action: A challenge for social theorising
	C. ROUSSEAU	Mathematics, a living discipline within science and technology
2002	D. BALL & H. BASS	Toward a practice-based theory of mathematical knowledge for teaching
	J. BORWEIN	The experimental mathematician: The pleasure of discovery and the role of proof
2003	T. ARCHIBALD	Using history of mathematics in the classroom: Prospects and problems
	A. SIERPINSKA	Research in mathematics education through a keyhole
2004	C. MARGOLINAS	The teacher's situation and knowledge as enacted in mathematics classroom activity
	N. BOULEAU	Evariste Galois's personality: the psychological context of an unusual fondness for abstract mathematics
2005	S. LERMAN	Learning as developing identity in the mathematics classroom
	J. TAYLOR	Soap bubbles and crystals

APPENDIX C

Proceedings of Annual Meetings

Past proceedings of CMESG/GCEDM annual meetings have been deposited in the ERIC documentation system with call numbers as follows:

Proceedings of the 1980 Annual Meeting . . . . ED 204120

Proceedings of the 1981 Annual Meeting . . . . ED 234988

Proceedings of the 1982 Annual Meeting . . . . ED 234989

Proceedings of the 1983 Annual Meeting . . . . ED 243653

Proceedings of the 1984 Annual Meeting . . . . ED 257640

Proceedings of the 1985 Annual Meeting . . . . ED 277573

Proceedings of the 1986 Annual Meeting . . . . ED 297966

Proceedings of the 1987 Annual Meeting . . . . ED 295842

Proceedings of the 1988 Annual Meeting . . . . ED 306259

Proceedings of the 1989 Annual Meeting . . . . ED 319606

Proceedings of the 1990 Annual Meeting . . . . ED 344746

Proceedings of the 1991 Annual Meeting . . . . ED 350161

Proceedings of the 1993 Annual Meeting . . . . ED 407243

Proceedings of the 1994 Annual Meeting . . . . ED 407242

Proceedings of the 1995 Annual Meeting . . . .   ED 407241

Proceedings of the 1996 Annual Meeting . . . .   ED 425054

Proceedings of the 1997 Annual Meeting . . . .   ED 423116

Proceedings of the 1998 Annual Meeting . . . .   ED 431624

Proceedings of the 1999 Annual Meeting . . . .   ED 445894

Proceedings of the 2000 Annual Meeting . . . .   ED 472094

Proceedings of the 2001 Annual Meeting . . . .   ED 472091

Proceedings of the 2002 Annual Meeting . . . .   *submitted*

Proceedings of the 2003 Annual Meeting . . . .   *submitted*

Proceedings of the 2004 Annual Meeting . . . .   *submitted*

Proceedings of the 2005 Annual Meeting . . . .   *submitted*

Note

There was no Annual Meeting in 1992 because Canada hosted the Seventh International Conference on Mathematical Education that year.