CANADIAN MATHEMATICS EDUCATION STUDY GROUP

GROUPE CANADIEN D'ÉTUDE EN DIDACTIQUE DES MATHÉMATIQUES

PROCEEDINGS / ACTES
2006 ANNUAL MEETING / RENCONTRE ANNUELLE 2006

University of Calgary
June 3 – June 7, 2006

EDITED BY:
Peter Liljedahl, *Simon Fraser University*

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30th Annual Meeting
University of Calgary
June 3 – June 7, 2006

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On behalf of the members, the CMESG/GCEDM Executive would like to take this opportunity to thank our local hosts for a wonderful 2006 Annual Meeting and Conference. Specifically, we thank Olive Chapman, Sandy Orsten, Valeen Chow, Carolyn Blum, and Jo Towers for their hard work in making the 2006 meeting a memorable and enjoyable experience. We would also like to thank the guest speakers, working group leaders, topic group and ad hoc presenters, and all the participants for making the 2006 meeting intellectually stimulating and worthwhile.

L'organisation d'une rencontre annuelle du Groupe canadien d'étude en didactique des mathématiques demande beaucoup de travail. Au nom de ses membres, l'exécutif du GCEDM/CMESG tient à remercier les hôtes de la rencontre de 2006. Grâce à leur travail et à leur dévouement, Olive Chapman, Sandy Orsten, Valeen Chow, Carolyn Blum et Jo Towers ont contribué à faire de le rencontre un grand succès. Nous tenons aussi à remercier les conférenciers, présentateurs, animateurs, ainsi que tous les participants qui ont contribué à faire de cette rencontre un lieu d'échanges riches et stimulants.

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Introduction

Frédéric Gourdeau – Président, CMESG/GCEDM
Université Laval

Calgary marked the 30th anniversary of CMESG, counting ICME in 1992 as part of our history. It was our first visit to Calgary and our hosts made sure it would be a success. A pre-conference workshop for teachers on the Saturday started things for many, and some activities were coordinated with the Canadian Mathematical Society. Through this and the vitality of mathematics education graduate programs in Calgary as well as Edmonton and Vancouver, attendance numbers soared above expectations, including a very large number of participants who were at their first CMESG meeting. Our local hosts did wonders. Olive Chapman, who liaised with the executive as numbers were rising, did an amazing job, and thanks are due to her and her team.

And the spirit of CMESG strived once more. Our two plenary speakers, Barbara Jaworski and Ed Doolittle, joined in and became part of the group, providing us with food for the intellect and the spiritual, in their talks and in the conversation that ensued. Chris Breen, invited to co-lead a working group entitled *Developing Trust and Respect When Working with Teachers of Mathematics* with Julie Long and Cynthia Nichol, was also there with us, adding to our collective reflection. I joined the working group on *Secondary Mathematics Teacher Development* facilitated by Joyce Mgombelo, Morris Orzech, David Poole and Sophie René de Cotret, listening as intently as I could, learning and reflecting with the other participants.

Pendant ce temps, Susan Gerofsky et Patrica Marchand exploraient *Le corps, le sens et l'apprentissage des maths* dans leur groupe de travail, alors que Stewart Craven, Linda Gattuso et Cynthia Nicolson unissaient leurs forces pour travailler sur les liens à développer entre la pensée probabiliste et la pensée statistique dans l'enseignement des mathématiques. Je ne pouvais y être, ne pouvant être partout à la fois, mais je peux lire et les Actes de la rencontre me permettent donc de revivre une partie de ce qui s'est passé.

Il y a aussi eu plus d'une douzaine de séances ad hoc, impliquant directement plus d'une vingtaine de présentateurs, en plus des cinq présentations de thèses de doctorat récentes, des deux séances thématiques, et du panel organisé conjointement avec la Société mathématique du Canada à l'initiative de Peter Taylor.

Our celebration of the 30th anniversary will remain a high point for me and many. There was true joy in our celebrations, a friendship so invigorating in our scholarly endeavours. As you can see in these proceedings, the pursuit of our collective quest for the betterment of mathematical education goes hand in hand with the quality and sincerity of our conversations, both in our official sessions and in our less official ones, be they *Ad hocs* or bus rides, dinner conversations or pizza and beer outings – these last few do not figure in the proceedings, but they are part of our deliberations!

Travailler à la planification de cette rencontre en compagnie des membres de l'exécutif a été un plaisir : je remercie sincèrement France Caron, Sandy Dawson, Doug Franks, Florence Glanfield et Leo Jonker. Je remercie aussi Peter Liljedahl, éditeur des Actes, pour son magnifique travail. The best way to thank him, however, is to conclude here and encourage you to read on.

Plenary Lectures

Conférences plénières

Developmental Research in Mathematics Teaching and Learning: *Developing Learning Communities Based on Inquiry and Design*

Barbara Jaworski
Agder University College, Norway

This paper focuses on a developmental research project entitled *Learning Communities in Mathematics*[1] which involves a collaboration between teachers and didacticians[2] to create *communities of inquiry* to explore development of mathematics teaching and learning. A developmental aim is that students' learning of mathematics will improve as teachers and didacticians come to know more about learning processes and the tasks and tools that promote learning.

The paper is in three parts: an introductory section whose purpose is to set the scene and create some images on which to base the theory that follows; the central section which is a theoretical account; a final section in which particular details of the project are given, together with questions and issues emerging from it.

Setting the Scene: Mathematics in Learning and Teaching

In this section I give two examples from the project to set the scene for the sections which follow. The first example is from video data recorded in a first year classroom with pupils of age six years. The second is from input at a workshop from an upper secondary teacher and the mathematical thinking that emerged.

In Egil's classroom

Egil is a teacher in Grade 1 at one of the primary schools in the project. He invited one of the didacticians, Roy, to his classroom to film some activity on number. Together these two discussed ideas for classroom tasks and Roy filmed one lesson. Subsequently, Egil, Roy and Eli, another didactician, watched the video together and discussed aspects and issues related to the pupils and the teaching. During this discussion, further ideas for tasks were offered; Egil designed further activity for the classroom, based on this discussion and again Roy filmed the lesson. The following account is taken from episodes in this lesson.

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[1] The LCM Project is supported within the KUL Programme (Kunskap, Utdanning og Laering – Knowledge, Education and Learning) of the Norwegian Research Council (Norges Forskningsraad, NFR). Project number 157949/S20.

[2] The term *didacticians* means those professionals with responsibility for theorising teaching. We avoid the term 'educator' since it is ambiguous – teachers are also educators – although didacticians may also be *teacher-educator*s; i.e., professionals with responsibility for teacher education.

In the classroom, children sit on low benches, forming three sides of a square with Egil sitting beside a flipchart at the fourth side. He writes on the flipchart as follows:

$$\boxed{} + \boxed{} = \boxed{} + \boxed{} = \boxed{8}$$

Figure 1

He holds up the pen and a boy comes forward, writing 2 and 6 in the first two boxes.

$$\boxed{2} + \boxed{6} = \boxed{} + \boxed{} = \boxed{8}$$

Figure 2

He holds up the pen again and a girl comes forward, writing 4 and 4 in the next boxes.

$$\boxed{2} + \boxed{6} = \boxed{4} + \boxed{4} = \boxed{8}$$

Figure 3

Then Egil draws another set of boxes, as in Figure 1. This time however, he fills them in himself – not with figurate numbers as in Figures 2 and 3 above, but with 'dice numbers' – familiar patterns of spots to show 6+2=4+4=8

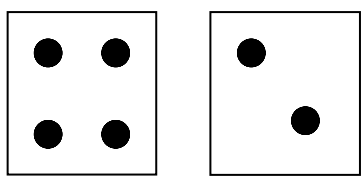

Figure 4

In the activity that follows, the camera shows several pupils working with sheets of paper in which they fill in boxes either with figurate numbers or with dice numbers so that the sums equal the given total. One girl is making 8 by inserting dice numbers. She starts confidently entering five in the first box and then pauses, counting on her fingers before entering three correctly in the second box. She repeated the process entering dice numbers four and four in the third and fourth boxes. Viewing the video tape later, Roy and Eli were fascinated by the finger counting and wondered about the girl's thinking as she so carefully counted before entering the number.

Errors in Algebra

In one of our workshops, focusing on algebra, Stefan, a teacher from an upper secondary school offered input to a plenary session as feedback from a small group activity session. He indicated that one of the problems he and his colleagues face at upper secondary level is that students make very basic algebraic mistakes which hinders their work on more advanced topics. He gave several examples of which the following is just one:

$$\frac{x+4}{x} = ?$$

Figure 5

$$\frac{x+4}{x} = 4$$

Figure 6

$$\frac{\cancel{x}+4}{\cancel{x}} = 4$$

Figure 7

Given the statement in Figure 5, students typically offered the solution given in Figure 6 which is incorrect because they cancelled the x top and bottom as shown in Figure 7. According to Stefan, it was important that students should know that cancelling in this case is inappropriate.

When didacticians discussed this later, it was recognised that students possibly confuse (x+4) with x.4, or 4x, in which case it would be acceptable to cancel the fours. How can students come to know that when the sign is multiplication it is acceptable to cancel, but when the sign is addition, it is not acceptable? Perhaps just stating the rules and asking students to remember the two cases is not a teaching strategy that works.

As a result of such thinking, didacticians devised a task, as in Figure 8, which was included in a subsequent workshop focusing on algebra. The task was designed to encourage participants in the workshop to try out values of x to see if it is possible to find a solution by trial and error, ultimately to solve an equation $x+4=4x$, getting a solution of $x=4/3$. The task was designed to promote discussion around the algebraic equation and its solution to open up the nature of the algebraic object and its various meanings.

$$\frac{x+4}{x} = 4$$

What does this mean? Is it true?
For what values of x?

Figure 8

In the two examples above, we see first a teacher designing classroom activity in response to input from didacticians, and, second, didacticians designing workshop activity in response to input from teachers. In both cases, *design* is for *learning*: in the first case learning of number relationships for Year 1 pupils; in the second, learning about didactical treatment of algebraic relationships for teachers at a variety of levels of experience. In both cases, teachers and didacticians engaged fundamentally in inquiry as they explored the nature of activity that could contribute to the desired learning. In designing activity for learning for others, they learned themselves. The teacher reflected on outcomes in pupils' activity and thinking, together with didacticians, and became more knowledgeable about the possibilities afforded by the tasks offered and pupils' responses to them. Didacticians, in the first case, reflected on the teacher's activity and gained insights into their collaborative development of awareness about the developmental process in classroom design. In the second case, didacticians learned themselves from a design process in which there task was to design activity for focusing on didactical thinking in algebraic teaching.

Questions for the Project

A developmental aim in our work is that students' learning of mathematics will improve as teachers and didacticians come to know more about learning processes and the tasks and tools that promote learning. We are interested in developing the teaching and learning of mathematics so that pupils have better opportunities to learn mathematics with understanding and fluency. We see this involving three layers of attention:

- How can teachers *and pupils* create a mathematical environment in their classrooms with suitable opportunity for pupils to learn mathematics with understanding and fluency?
- How can didacticians *and teachers* create a didactical environment in their interactive space (in schools and college) with suitable opportunity for teachers to develop mathematics teaching with understanding and fluency?
- How can xxxxxxx *and didacticians* create a (supra-didactical?) environment … with suitable opportunity for didacticians to learn (didacting?) with understanding and fluency

The pattern in these layers is probably clear – we do not know who the xxxxxxx are who work with didacticians, but perhaps we might consider it to be the Mathematics Education Research Community.

The questions above talk about "creating" an environment. In order to do this, we all have to learn about what is involved and how it can work out in practice. So we might recast these questions in terms of the *learning* of all involved

- How can didacticians learn (about) effective ways of working with teachers to enable teachers to conceptualise approaches to teaching that will result in principled learning of mathematics for their pupils?
- How can teachers learn (about) effective ways of working with pupils to enable pupils' principled learning of mathematics
- How can pupils learn mathematics?

Hence the project is called *Learning Communities in Mathematics*, LCM. We seek to create communities between teachers and didacticians that allow us to address questions about improving pupils' opportunity to learn mathematics. In doing so, we aim to learn ourselves according to the questions above. We see ourselves working as co-learning partners as pointed out by Wagner (1997) who writes

> In a co-learning agreement, *researchers and practitioners* are both participants in processes of education and systems of schooling. Both are engaged in action and reflection. By working together, each might learn something about the world of the other. Of equal importance, however, each may learn something more about his or her own world and its connections to institutions and schooling. (Wagner, 1997, p 16 – my emphasis)

We would change the wording slightly to read

> In a co-learning agreement, *teachers and didacticians* are both participants in processes of education and systems of schooling. Both are engaged in action and reflection. *Both can be engaged in research.* By working together, each might learn something about the world of the other. Of equal importance, however, each may learn something more about his or her own world and its connections to institutions and schooling.

An important developmental aim for the project is that mathematics teachers and didacticians work together, with joint responsibility and complementary knowledge, towards promoting growth of understanding of mathematics learning and teaching in classrooms. We can *all* engage in research as I indicate below.

Theoretical Formulation

Theoretical aims of LCM

Two key theoretical ideas in our work are those of *developmental research* and *inquiry community*. We seek to link developmental research and inquiry communities, based on

- a considerable literature on inquiry in mathematics learning and teaching,
- an extension of communities of practice to communities of inquiry,
- an emergent paradigm known as design research, and
- an elaboration of links between design, inquiry, reflection and research.

Community

The term ‘community’ designates a group of people identifiable by who they are in terms of how they relate to each other, their common activities and ways of thinking, beliefs and values. Activities are likely to be explicit, whereas ways of thinking, beliefs and values are more implicit. Wenger describes community as “a way of talking about the social configurations in which our enterprises are defined as worth pursuing and our participation is recognisable as competence” (1998, p. 5). Any community has common purposes and activities and established norms of activity. According to Rogoff and colleagues, in a learning community, “learning involves transformation of participation in collaborative endeavour” (1996, p. 388).

We draw particularly on the work of Lave and Wenger (1991) who speak of situated learning within *communities of practice* in which participants engage in well defined practice and knowledge is *in* the practice. Wenger speaks of “modes of belonging” to a community of practice including engagement, imagination and alignment (1998, 174, f.f.). We engage with ideas through participation in communicative practice, develop those ideas through exercising imagination and align ourselves “with respect to a broad and rich picture of the world” (p.218). *Alignment* within a community of practice involves participants aligning themselves with conditions or characteristics of the practice through their *engagement*. Through the exercise of *imagination* during *engagement,* participants develop personal trajectories and hence an identity related to their practice. However, alignment can result in perpetuation of modes of practice that can be seen as unhelpful, unsuccessful or ineffective (Jaworski, 2006).

Alternatively, alignment can be a critical process in which the individual questions the purposes and implications of aligning with norms of practice – *critical alignment* in which it is possible for participants to align with aspects of practice while critically questioning roles and purposes as a part of their participation for ongoing regeneration of the practice. Wenger presents learning as “a process of becoming”. “It is in that formation of identity that learning can become a source of meaningfulness and of personal and social energy” (1998, p. 215). Concepts of critical alignment and a process of becoming are related to our central theoretical idea of inquiry in practice.

Inquiry

To inquire means to ask a question, to make an investigation, to acquire information, or to search for knowledge (Chambers’ dictionary). Wells (1999) describes *dialogic inquiry* as “a willingness to wonder, to ask questions, and to seek to understand by collaborating with others in the attempt to make answers to them” (p. 122).

We have a long history of inquiry in mathematics learning and teaching, reflected in the literature. For example, the problem solving movement, perhaps deriving from the work of Polya (e.g., 1945) saw problem solving processes and heuristics as centrally important to mathematics learning and teaching and the development of understandings in mathematic (e.g., Mason, Burton and Stacy 1981; Schoenfeld, 1985). Problem solving, investigation and inquiry in classroom mathematics were promoted in the United States through the NCTM standards (NCTM, 1989) and the empirical work of many researchers (e.g., Confrey, 1991; Lampert, 2001). In the UK, *investigations* offered a mode of inquiry in classroom mathematics (e.g., Jaworski, 1994; Ruthven, 2001). In the Netherlands, The Freudental Institute introduced the notion of *Realistic Mathematics Education* incorporating a strong element of investigation and inquiry to make mathematics *real* for pupils (Gravemeijer, 1994). Developing from the earlier work in problem solving, Mason (2001) proposed a *Discipline of Noticing* in which practitioners or researchers explored their own practice through an overtly reflective and questioning approach to learning and teaching which can lead to a deeper awareness of decisions in practice.

Community of Inquiry

Notions of *community of practice* and *inquiry* together lead to the formulation of a concept of *community of inquiry*. A community of inquiry refers to a community of practice in which inquiry is central to activity. So, for example, if the practice is mathematics teaching, a community of inquiry in mathematics teaching develops the teaching of mathematics in inquiry ways. In LCM, we think of inquiry communities as being more than communities of practice. In a community of inquiry, inquiry is not the practice of a community of practice: rather, we see inquiry both as a tool for developing practice, and as *a way of being in* practice (Jaworski, 2004a).

According to Wells (1999), inquiry communities are a special form of communities of practice. They are distinguished by forms of 'metaknowing' that develop from inquiry in reflective and reflexive processes. I suggest that formation of a community of inquiry requires "critical alignment" within a community of practice (Jaworski, 2006).

Design Research

In LCM we use inquiry overtly to *design* activity at a number of levels: for example mathematical activity for pupils in classrooms, or activity in workshops where teachers and didacticians can explore together aspects of mathematics and its learning and teaching. The snapshots provided in the opening section above offered examples of such design processes. A mode of research called *design research* has been described as

> An emerging research dialect … contrasting with dialects of confirmation or description … attempts to support arguments constructed around the results of active innovation and intervention in classrooms. The operative grammar, which draws upon models from design and engineering, is generative and transformative. It is directed primarily at understanding learning and teaching processes when the researcher is active as an educator. (Kelly, 2003. My emphasis)

Kelly's brief description of design research fits very well the research we engage with in LCM. Here we seek less to design a product than a process: our design is of ways of approaching teaching in various modes. Wood and Berry (2003) speak of a design-based research paradigm in teacher education which involves, or emphasizes the following points:

- The development of a physical or theoretical artifact. For the researcher/teacher educator the product being developed and tested is the professional development model itself.
- Cycles of iterations

- The design and revision of products are rooted in multiple models and theories
- Contextual setting of the research and the need for shareable and generalizeable results
- The role of the teacher educator/researcher -- acting both as a researcher and as a didactician (adapted from Wood & Berry, 2003).

In our research, we include also teachers as potential researchers and partners in a design/inquiry-based approach to teaching (Jaworski, 2004b).

In accordance with this characterization of design research, we link design, inquiry and research to conceptualise four kinds of theory-based activity:

- Creating Partnerships:
    - Didacticians and teachers work together for mutual benefit and support – both should be involved in design and implementation at conceptual levels for the success of innovation. This is our community of inquiry.
- Designing Materials and Approaches:
    - Design of tasks for workshops and classrooms; design of approaches to learning and teaching; design of research/inquiry to learn about developmental processes and learning outcomes. This activity is fundamentally inquiry-based.
- Reflective Action
    - in the use of designed materials and approaches and (critically) reflective questioning of outcomes. Our reflection is action oriented as in Dewey (1933), Schön (1987), Mason (2001). See also Jaworski (1994 chapter 11).
- Research
    - into all of the above in relation to research questions about the realization of inquiry communities and their contribution to improved learning.

We interpret the above though an overt design cycle consisting of a cycle of stages:

Plan →Act→Observe→Reflect→Feedback→Plan … .

So, for example, didacticians plan activity for workshops, organize and participate in the workshops and bring back data (audio or video). Through study of the data and reflection on the events, further workshops are designed. Despite its seemingly linear nature, the elements of the design cycle are inter-linked with reflection being important at all stages and observation being central to the research nature of the whole activity. We see such a cycle being similar to the kind of cycle that is central to *action research* (e.g., McNiff, 1988; Elliott, 1991) and a basis for programme of *lesson study* (e.g. Stigler and Hiebert, 1999) and *learning study* (Marton and Tsui, 2003).

However, we see both design and inquiry as tools and styles of activity, ways of being in activity that have the purpose of promoting development in mathematics learning and teaching. Thus, in LCM, we talk rather of a *developmental paradigm* in which the operative grammar is “generative and transformative”, and ‘design’ is one factor in development. Thus, we see “arguments constructed around the results of active innovation and intervention in classrooms” (Kelly, 2003) as being fundamentally developmental since they are rooted in emergent thinking in inquiry communities, and themselves promote development of thinking and practice (Cestari, Daland, Eriksen & Jaworski, 2005). Such a process involves deep reflexivity between research and development and it is the basis of the LCM project.

The LCM Project

As mentioned earlier, a developmental aim of the project is that students' learning of mathematics will improve as teachers and didacticians come to know more about learning processes and the tasks and tools that promote learning. In this section, I provide short details of the project to indicate how the theoretical principles discussed above related to project activity, and the issues that arose from relating theory and practice in the project.

The Project team

Within the LCM project, we are a team of 14 didacticians (including 5 doctoral students) working with 8 schools (including primary, lower and upper secondary) with a minimum of three teachers from each school. Schools volunteered to be part of the project as a result of an invitation from the college. One condition of joining was that the school principal should be committed to the aims and activity of the project; a second was that at least three teachers from the school would participate in the project, forming a school team who would attend project workshops and work together in school to design activity for the classroom. The project budget included modest funding to cover schools' expenses in participation.

Workshops

Central to activity in the project have been our *workshops* at the college. Workshops have had an important community-building purpose bringing teachers and didacticians together with opportunity to work on mathematics and develop joint thinking on how to create opportunity for pupils to engage with mathematics fruitfully. We had six workshops per year during the first two years, and have planned for four in our current year. This has involved three *phases* of activity: In Year 1 we had a phase largely of community building; in Year 2, Phase 2, we focused on planning for the classroom, bringing inquiry into planning and into classroom teaching; in Year 3, Phase 3, where we are now, our focus is on schools setting their own goals for activity and undertaking activity through an inquiry/design cycle.

Workshops have each been three and a half hours in length and have included both plenary and small group activity. Plenary input has come from both didacticians and teachers in a number of forms including introduction to mathematical topics or tasks (chiefly by didacticians), reporting from classroom activity (mainly by teachers) and reporting from small group activity (by all). Small group activity has included working on a mathematical task, usually followed by didactical and pedagogic discussion (didacticians and teachers together). Small groups have been formed sometimes with teachers across the educational levels (early primary to upper secondary) and sometimes with those teaching at the same level of school.

Designing activity for the classroom

Design of the project suggested that teachers in a school team would work together in school to design activity for the classroom. Teachers themselves would decide on the focus and nature of activity, although didacticians would be available to discuss ideas and to observe classroom activity. From the didactician team, three didacticians were associated with each school to liaise with activity, provide support and collect data. Four of the doctoral students in the project provided the first line of contact with schools and became an important part of a school team. Typically, school activity built on workshop activity with workshop tasks being redesigned for the classroom, or classroom activity being designed specially according to teachers' or school goals in liaison with didacticians. Pupils' achievement was surveyed periodically in a specially designed longitudinal study based on national and international tests.

Three layers of inquiry

In the project we engage in inquiry in three layers or levels:

- Inquiry in mathematics:
  - Teachers and didacticians exploring mathematics together in tasks and problems in workshops;
  - Pupils in schools learning mathematics through exploration in tasks and problems in classrooms.
- Inquiry in teaching mathematics:
  - Teachers using inquiry in the design and implementation of tasks, problems and mathematical activity in classrooms in association with didacticians.
- Inquiry in developing the teaching of mathematics:
  - Teachers and didacticians researching the processes of using inquiry in mathematics and in the teaching and learning of mathematics.

Classroom innovation

An important aspect of inquiry in the project was an exploration of classroom innovation relating to activity in workshops and of school teams in schools. The nature of innovation depended on the level of school and the particular focuses and interests of the teacher team. For example, one upper secondary school team designed a set of tasks relating to the learning of linear functions, and trialled them in three classrooms; in one of the primary schools, individual teachers modified workshop tasks directly for their own classroom and designed further tasks as a result of dialogue with didacticians. In the first case, the teacher team worked closely together and with didacticians; in the second, it was individual teachers who planned and carried out classroom innovation with didacticians support. In Phases 1 and 2, didacticians visited schools regularly, talked with teachers about classroom activity, joined school team meetings and recorded classroom activity on video.

Research questions and associated data

Video recordings of classrooms were just one source of data in the project. Didacticians recorded all meetings at the college, usually in audio format, with workshops recorded largely in video format. In schools, data was collected largely in the form of classroom video recordings, with audio recordings of meetings attended by didacticians. There were very few recordings, in any format, of school meetings at which didacticians were not present. Data and its analysis was largely owned by didacticians, with video data also providing a source for teachers to review classroom activity and reflect on teaching. Didacticians obtained permission for data collection from the national data protection agency and schools assisted in collecting permissions from parents or students as appropriate.

The project was guided by a main set of research questions stated in the original project proposal and modified and amplified during the life of the project. Doctoral students had their own research questions related to the particular focus of their research. The data collected was as comprehensive as possible to allow research questions to be addressed, although rarely was data directly related to particular research questions. The project produced a large quantity of data, and a data bank was carefully organized for security and access by all didacticians. All data was available to all didacticians and didacticians linked to a school provided access to data for teachers if this was desired. Analysis of data took place at the university college and involved didacticians singly or in groups, sometimes involving teachers, and related to specific research questions.

Reflection in college, schools and in workshops

An important aspect of didacticians' roles within the project was to provide opportunity for reflection on developing thinking about classroom activity. This occurred both in school and in workshop settings in different forms and in special meetings between didacticians and teachers at the university college. Video data was an important resource in stimulating reflection and providing a basis for discussion about a range of issues relating to planning for the classroom and to classroom activity with pupils.

In workshops, both plenary and small group activity encouraged reflection around what was being done and learned. Such reflection was an important part of community building, providing opportunity for common ways of thinking to emerge and for developmental concepts to grow within the community. The nature and focus of such reflective activity was an important source of learning within the project, for both didacticians and teachers. The issues it raised form important contributions to knowledge about possibilities and constraints of such activity for learning and teaching development.

In summary

The project has brought together teachers and didacticians to create one or several communities of inquiry to allow inquiry into processes of learning and teaching that can promote pupils' learning of mathematics. Inquiry has been a fundamental process in our work together on mathematics and design of mathematical tasks for workshops and classrooms. Teachers have worked with didacticians in workshops in the university college and didacticians have worked with teachers in schools. An important aim of our work together has been to generate inquiry *as a way of being*, such that inquiry comes to underpin our activity and thinking at all levels. This allows development of "critical alignment" such that while we work within our own community, system and culture we can do so with a critical stance through which we ask questions about what we do and seek ways of understanding this better and perceiving viable alternatives. Over time, we expect changed ways of thinking to lead to changed modes of practice, but recognize the many issues that challenge this expectation.

Areas of Issues

Theory within the LCM project has both guided activity and developed with activity. Notions of inquiry and community led the project with references particularly to the work of Wagner, Wenger and Wells. Research questions have particularly addressed the nature of community and of inquiry within the project from their practical manifestations. What has it meant for us to develop community? What community or communities do we see in the project? What issues are recognizable in such communities? How has inquiry emerged in the project? What do different groups see inquiry to mean for them?

Here I mention some of the issues that have emerged from the project, relating them to the theoretical perspectives I have discussed.

Community and Partnership

The LCM project was initiated and designed by didacticians. We sought collaboration with teachers and have succeeded in building a project community in which we all work together with respect and trust. Our data has many examples to support this claim. As well as a project community, we have separate communities of teachers in a school, didacticians in the college with expectations from our own community, system and culture, ways of thinking, roles and goals. We have tried to characterize our community of didacticians, recognizing many of the issues we have faced in the developing project (Cestari, et al, 2006). Within the teachers' community, each school has its own identity with specific features related to the level of the school and to particular focus and emphasis in each school. For example, the upper secondary schools are distinguished by a close emphasis on the higher secondary mathematics curriculum and topics within a certain text book. One of the primary schools has its own "Phase Model", in which teachers at all levels within the school operate according to certain defined phases of activity. Our data from schools is not rich enough to enable a deep characterization of each school community as a whole. However, we gain special insights into the activity of project teachers that provides glimpses of the special nature of particular schools.

Teachers and didacticians bring different knowledge and expertise to the project. The project draws fundamentally on this joint knowledge. Activity takes place both in college and in schools. We all meet together in workshops. Teachers then work in their own schools and didacticians visit schools. Development is supposed to take place in schools. There are clear and unavoidable asymmetries in these roles and relationships. So an important question for our project is what exactly this means for partnership.

The project has been largely controlled by didacticians, who set the pace of events, although responding as much as possible to teachers' input. We have talked about co-learning in partnerships between teachers and didacticians, and we can provide evidence to show that mutual understandings have grown through the time of the project. However, we are only just beginning to be at a stage where teachers take leadership within the project (Jaworski, 2005). Exploration of these issues takes us into a wide area of research into partnerships in educational settings – something we can consider for the future.

Systems and culture

We all act and think within our own systems and culture as set out by Lave and Wenger, 1991 and Wenger, 1998. When we come together, we learn to see things from other perspectives to some extent. Didacticians, guided by theory and the literature on pedagogic practice in mathematics teaching and learning, have idealized notions of possibilities for promoting pupils' learning in classrooms. Teachers have established ways of being and doing within their own schools that are resistant to change. These both constrain what is possible and give structure to it. Didacticians, from a college perspective, seeing the school system from the outside have to understand the power of the system. For example, many schools have a horizontal structure in which teachers work together in year or grade teams. Each team may have only one teacher who has expertise in or responsibility for mathematics. The project team of teachers in mathematics is often a vertical group crossing the year teams. It is thus extremely difficult for the school project team to find time to meet in the ordinary school day. Also, planning for classroom activity across the year teams poses both curricular and organizational problems. Many of the project teachers, with ideas for tasks or activity of value for their classroom have designed activity for their own pupils. Often, such activity has been related just to their one class, rather than design attempting to cross classroom and year boundaries as didacticians have wished. Nevertheless, there are examples of the teacher team in a school really working together, and with didacticians to organise and effect joint activity (Fuglestad, Goodchild & Jaworski, in press; Hundeland, Erfjord, Grevholm & Breiteig, 2005).

We have worked hard in our joint activity to develop respect and trust between teachers and didacticians and we can trace development through activity and contributions in workshops. We work towards mutual understandings of the cultures that influence, constrain and make possible the educational process in schools. As didacticians, we see our own growth of knowledge and understanding where practices and possibilities for schools are concerned. However, it is more difficult to gain access to such growth of knowledge for teachers.

Ways of thinking about teaching and learning in mathematics

Working with schools across 13 years of schooling presents its own challenges, not least in terms of the mathematical expectations of teachers at different levels. Although didacticians have tried to foster cross-level discussions that enable understandings teachers, perhaps unsurprisingly, prefer to work with their peers at the same level. One reason for this, for which we have evidence, is to do with mathematical confidence and expectation. Some teachers at higher secondary level want to be able to work with mathematics at their level and are reluctant to spend time focusing on lower level mathematics, even though this focus might provide insights to their students' mathematical experience when they arrive at higher secondary school. Some teachers at primary school feel unconfident with mathematics, and threatened by the mathematical knowledge of their colleagues at higher levels. These two positions have to be acknowledged, and worked with, if cross-level didactical/pedagogic understandings are to develop. Although this is not a surprise for didacticians, we can see more clearly how this divide manifests itself, and presents real challenges for community development.

The didacticians' roles have appeared different with respect to the two groups. With higher secondary teachers an important role has been to suggest ways of opening up the traditional mathematics as presented in their text book, to allow student more active participation with mathematics. With primary teachers, who have often been the most innovative in the project in introducing inquiry activity in the classroom, it is a case of finding respectful ways of offering opportunity to develop mathematical knowledge, for example in the very challenging domain of algebra. Didacticians, in their own inquiry community, are learning at deep levels how to recognize issues for teachers and the serious challenges that such recognition brings with it. We persevere, in our design of workshops, with generating practice that brings the whole community together while respecting the differing needs of teachers at different levels. This is an important area of inquiry for us.

Concluding Words

In this paper I have tried to present a view of didactic/pedagogic practice that spans school-college communities and tries to build fruitfully on the diverse knowledge and experience that teachers and didacticians bring to the developmental process. The snapshots at the beginning were offered to set the scene for the discussion of theory and practice that followed. In many ways typical of our activity as a whole, they took us into the classroom and collaborative design of classroom tasks; they took us into the workshop where teachers at one level offered important elements from their experience which led to further design of workshops tasks by didacticians. The design/inquiry process has been central to our joint activity, manifested at various levels. It has been an important aim that teachers should be centrally involved in inquiry and design. So far this has been more implicit than explicit, but activity in Phase 3 is making the design process for teachers more explicit with an overt focus on school goals and the activity related to realising these goals. Other papers, some of them referred to here, take us into different aspects of the project in more detail, articulating issues in greater depth than has been possible here. Those interested might access our further work through the website (http://fag.hia.no/lcm/)

I end with a quotation from one teacher, Agnes, translated from the Norwegian by one of my colleagues (Daland, in press) which typifies an optimistic note for the project:

> *Agnes:* *…in the beginning I struggled, had a bit of a problem with this because then I thought very much about <u>you should come and tell us how we should run the mathematics teaching</u>. That was how I thought, you are the great teachers … but now I see that <u>my view has gradually changed because I see that you are participants in this as much as we are</u> even*

though it is you that organize. Nevertheless I experience that you are participating and are just as interested as we are to solve the tasks on our level and find possibilities, find tasks that may be appropriate for the pupils, and that I think is very nice. So I have changed my view during this time. (FG_060313 my underlining) (Adapted from Daland, in press)

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Mathematics as Medicine

Edward Doolittle
University of Regina

I remember well my first visit to the Navajo reservation in Arizona. I was traveling with a dear friend who had been a few times before. Driving through the desert in near total darkness, I spotted some strange coloured lights flashing on the horizon near the place where the town in the reservation should be, pulses which would stop and then start again in an irregular pattern. "What's that?" I asked. "I don't know," said my friend, clearly disturbed. "I've never seen that before." Thoughts of UFO abduction began to form in my mind. "You're pulling my leg," I said. "No really," she said, "I have no idea what those coloured lights are!" We continued to drive through the darkness, perplexed and staring in wonder at the coloured lights, for what seemed like ages. We were closer, but I still couldn't make sense of the experience. The lights disappeared behind a hill. As we climbed the hill I held my breath, knowing that the truth was on the other side. We reached the top and it finally unfolded clearly before me. The circus had come to the Navajo reservation.

I am an Indian. I am a mathematician. Those two aspects of my identity seem to be in constant opposition, yet I cannot let go of either.

My father, the late Edward Lorne Doolittle, was a Mohawk Indian from the Six Nations reservation in southern Ontario. My father's mother, Belda Brant, attended residential school where she lost her language and learned how to clean hotel rooms. My father's father, Clifford Doolittle, was killed in a railroad construction accident when my father was five years old. The settlement offered by the railroad company was $35 a month. My grandfather's spirit came to my grandmother to tell her that she should take the family off the reserve to find work, which she did. Although that helped to keep the family fed, it had the effect of further distancing them from their culture.

By the time I was a teenager, my father and mother, Eleanor Naylor, third generation English in Canada, had managed to pull together comfortable middle class existence. I grew up in the suburbs of Hamilton, Ontario, knowing almost nothing of my Indigenous culture. Aside from occasional weekend visits with my aunts and uncles, I had no idea what it meant to be Indian. I learned Latin instead of Mohawk, the Bible instead of *Kayanerekowa* and *Karihwiyo*, fairy tales instead of Coyote tales. I also enjoyed solving puzzles.

Despite all the advantages and privileges and resources available to me, I still felt there was something missing in my life. I tried to find what was missing in religion, but I stopped attending Baptist church at the age of 14. My Sunday school teacher had used some twine to tie my wrist to that of the other student in my age group, a sweet young woman whose name I can't recall, in order to illustrate some point about sin. I thought the problem might have been more with the teacher than with me or the girl. Instead I read about Zen Buddhism and searched for enlightenment. I studied religion instead of biology in high school, disappointing those who wanted to be able to call me Dr. Doolittle. My religion teacher invited a friend of his who we only knew as Krishna to come and talk about Hinduism. “Ask him whatever you want,” said the teacher. So I asked Krishna, “Is it possible to convert to Hinduism?”

“Yes,” said Krishna reluctantly, “But I discourage it.”

What?

“If you can’t find what you’re looking for in your own tradition, you won’t be able to find it in ours.”

Enlightenment. I don’t even know what my tradition is, I wanted to say. How can I find anything within my own tradition if I don’t even have a tradition? That was my problem, though, not Krishna’s. He could not tell me where to find the something that was missing.

I was accepted to university, and my general intention was to study Artificial Intelligence at the University of Toronto. My parents had not planned for my university education. Instead they relied on the fact that status Indians received education funding from the federal government. While there was no denying that I was a status Indian—I had the card to prove it—I felt that the benefit was for real Indians, not privileged, pale, suburban half-Indians. At first I balked at the suggestion that I should accept Indian Affairs funding for my education, but that upset my parents greatly. I reconsidered, and (in a deal only with myself) accepted on the condition that I would do my best to deserve the benefit offered to me. I resolved to become an Indian.

At the University of Toronto I connected with the Indian Health Careers Program, a program designed to help to increase the representation of Aboriginal people in medicine and other health-related careers. Dianne Longboat, the director of the program, hosted gatherings with traditional teachers and elders, and invited me to attend. The experience of hearing elders speak turned me inside-out. For the first time I directly experienced a powerful tradition of thought and experience which stood completely outside of the Western tradition in which I had been educated. The power and wisdom of the words of the elders were like a streak of lightning shooting through my brain.

It felt like my whole life had been a preparation for those moments, when I understood that there really are different ways of thinking and being. Ways which were not only different, but truly powerful; a tradition that stood on its own, entirely independent from European thought, and had great gifts to offer. Even better, all this could somehow be mine. The search for that which was missing had ended. That is when I really started to become an Indigenous person. The work of bringing it in, filling myself with it had begun, continues to this day, and will not end until my life is over.

I am unbelievably fortunate. Something I could do to satisfy my obligations became something I could do for my own benefit. And not just a way to self-discovery and to fill a hole, but a way to power and strength, a way to change the world. The opportunity to become an Indigenous person is one of the greatest gifts I have ever been granted.

I am not wise, nor deeply knowledgeable about my culture, nor gifted in oratory like the elders. How can I, with my lack of gifts of expression, convince you that our way is a powerful way? I often think about three simple words spoken by Chief John Snow: “We have survived.” Our ways must be powerful if they have helped people survive through one of the greatest holocausts in human history: one hundred million dead of disease, starvation, and warfare; loss of land, wealth, culture, and knowledge; injustice and wanton destruction all around. Through it all, we (the survivors) have survived.

Back in the regular world, I abandoned the study of Artificial Intelligence, which seemed to be reaching a dead end at the time, possibly because its mechanistic approach was just too simpleminded to approach something as complex as the human mind, and took up mathematics, which is what I seemed to do best and which always seemed fun and natural to me. I studied mathematics at the University of Toronto for twelve years, ultimately earning a PhD in pure mathematics under Peter Greiner; my dissertation was on the topic of hypoelliptic partial differential operators. Peter is like a father to me in a certain sense. He is also a great mathematician, and wise in his own way and in his own tradition. He is my mathematical father, connecting me to another strong lineage which includes Solomon Lefschetz and Carl Neumann. My two fathers have never met.

One of my major life goals is to resolve the apparent incompatibility between the two aspects of my identity, being a mathematician and being an Indigenous person.

To that end, I would like to explore various interfaces between mathematics and Indigenous thought. At this point I am more interested in searching for possibilities than organizing my thoughts in any particular way. I have tried to identify the main sources for my thinking, but I have neglected making exact references to the literature. I hope that you will forgive my poor scholarship, but the need for references is reduced because of the availability of such information in this modern age. In any case, I don't always remember things the way they were said or written, but I remember the impression they made on me.

Perhaps the most common, most straightforward, and simplest interface between mathematics and Indigenous people is the proposal that mathematics is a requirement for Indigenous people to succeed in the job market. The problem is often stated in terms of the desperate state of education of Aboriginal people in terms of math and science. Many researchers have attempted to quantify or otherwise justify that assessment and then conclude that we must find ways to improve outcomes and achievement indicators for the benefit of the students.

I am skeptical of that approach. For one thing, we have heard such talk before, in connection with residential schooling for example. I don't doubt the sincerity and desire to do good of those who take that point of view, but the concern that I have, partly from history, partly from personal experience, is that as something is gained, something might be lost too. We have some idea of the benefit, but do we know anything at all about the cost?

The complexity of the situation seems to expand endlessly the more it is examined. It is tempting to search for simple solutions to complex problems and to offer simple responses to complex situations; that is what Western thought (mathematics included) teaches us to do. However, such responses have not been adequate as we can see from the continuing nature of the problem (whatever that problem really is).

As examples of the surprising and complex nature of Indigenous mathematics education, I would like to offer some impressions taken from a paper by William Leap on the mathematics education of the Ute Indians.

> Q: If he gets four dollars a day, how many is he going to have in two days?
> A: Six.
>
> Q: Let's imagine you have 72 pennies right here in a pile, and there's one boy sitting here, one boy there, one boy there, and one boy there. What would you do to make certain everybody got the same number of pennies?
> A: Pass them out until they are all gone.
>
> Q: If your brother took his truck to Salt Lake City, how much would he have to spend on gas?
> A: My brother doesn't have a truck.

Another approach to the apparent incompatibility between Indigenous thought and mathematics is ethnomathematics. Roughly speaking, ethnomathematics expands the meaning of "mathematics" to include very general notions of counting, measuring, locating, designing, playing, and explaining. From the perspective of mathematics education, the task is to identify examples of such activities within a culture and use those examples to teach mathematics. Many different examples of Native American ethnomathematics have been discussed by authors such as Marcia Ascher, Michael Closs, and many others. For example, the peach pit bowl game of my people is discussed in Ascher's book *Ethnomathematics*.

Some of the most interesting examples of ethnomathematics in North America, in my opinion, involve the idea of mapping in an extended sense. The feeling I get from Native American maps and diagrams is that they are not static maps of locations and spatial relationships, but maps of processes, like how to get from one place to another, how to make a caribou dinner from scratch, how to give thanks and show respect to everything that's good, or how to mourn.

Ethnomathematics is far more reflective and respectful to Indigenous traditions of thought than the simpler reflex to help Indians succeed at improving their outcomes on standardized tests. However, the danger of oversimplification remains, perhaps more insidious because the motives are put forward as purer. An example of such oversimplification which I have encountered repeatedly in discussions with well-meaning people I call Cone on the Range. "The tipi is a cone," I have heard countless times. But that is surely wrong; the tipi is not a cone. Just look at a tipi with open eyes. It bulges here, sinks in there, has holes for people and smoke and bugs to pass, a floor made of dirt and grass, various smells and sounds and textures. There is a body of tradition and ceremony attached to the tipi which is completely different from and rivals that of the cone. Similarly, there is a ceremonial and spiritual tradition connected with the peach pit bowl game that is completely lost in Ascher's treatment.

Aside from being wrong, oversimplifications such as calling a tipi a cone or analyzing the peach pit bowl game only in terms of probabilities and odds may have other serious implications in an educational context. My feeling is that Indigenous students who are presented with such oversimplifications feel that their culture has been appropriated by a powerful force for the purpose of leading them away from the culture. The starting point (tipi, game) may be reasonable but the direction is away from the culture and toward some strange and uncomfortable place. Students may, implicitly or explicitly, come to question the motives of teachers who lead them away from the true complexities of their cultures.

There is a more pervasive and insidious example which I call Squaring the Circle. Of course, Squaring the Circle is one of the unsolvable mathematical problems of antiquity, but the term is also used by blues musicians such as Sterling "Satan" Magee for the process of reasoning too much about something that one should be feeling; I believe the term "square" is meant in a pejorative sense in that context.

In modern Indigenous thought, a tool called the medicine wheel is often used to divide complex situations into four simpler categories. Many Indigenous people will staunchly defend the process of dividing wholes into four aspects, such as the person into the physical, emotional, spiritual, and mental. However, I feel, based on personal experience, that such analyses square the circle; they are pale oversimplifications of complex and powerful traditions which have gone underground. One revelatory experience for me took place at a meeting with teachers, an elder, and a number of well-meaning researchers at the University of Saskatchewan. After the presentation of a rather complicated example of the use of the medicine wheel in the theory of science education, Elder Betty McKenna of Moose Jaw was asked what she thought about it. Betty responded: "I have worked on a real medicine wheel."

The implication, of course, is that a geometrical, abstract medicine wheel is not real. But what then, is a real medicine wheel? It is an approximately circular arrangement of stones on the ground, often with spokes radiating from a centre, sometimes with loops of stones occurring at irregular intervals around the perimeter. There are many pictures available on the Web of real medicine wheels such as the Bighorn Medicine Wheel. Note that they blend with the landscape as it rises and falls; they are not regular. The stones used to mark them are of different sizes and shapes and colours; the number of spokes is not necessarily a multiple of four and not clearly meaningful in any way at all. The purpose and meaning of such wheels is to some extent lost, or more likely has gone underground. My belief is that they were used not to divide and analyze, but as "maps" of processes of ceremony, thanksgiving, timekeeping, and communication. Or maybe not.

Notwithstanding the concerns I have about ethnomathematics in math education, I feel that ethnomathematics is a worthwhile pursuit. I would like to propose another example for the body knowledge of the ethnomathematics of Native North America. However, before I do so, I can't resist telling a joke which I first heard from Eber Hampton at a barbecue sponsored by Luther College on the occasion of the opening of First Nations University.

> When the astronauts first landed on the moon, they saw a strange sight: a teepee sitting right there on the lunar surface some distance from the landing craft. The astronauts bounced over in their spacesuits to marvel at the sight. Finally, one of them got the nerve to knock on the hide covering the entrance. An old man parted the doorway and looked out, just as surprised to see the astronauts as they were to see him. They stared at each other for a few moments, and then the old man noticed the American flag planted some distance behind the astronauts. Seeing the flag, the old man exclaimed, "Oh no! Not you guys again!"

The capture of Detroit is one of the highlights of Canadian military history. Near the beginning of the War of 1812, the government of Canada and its wartime leader, Isaac Brock, were concerned about its ability to fight a war on three fronts: the Detroit river, the Niagara river, and the St. Lawrence River. Brock decided to try to neutralize the threat in Detroit quickly by launching an immediate, overwhelming attack on the American forces stationed in Fort Lernoult, Detroit. Short of manpower, he gathered as many militia as possible and dressed them in red jackets to make them look like regulars, and recruited as many Indians as he could to the cause. Key to those recruitment efforts was the great chief Tecumseh, who was impressed with Brock and willing to support Brock's fight against the Americans.

In the decisive tactic of the attack on Fort Lernoult, Tecumseh had his Indians march past a point which the Americans could see, change their clothing somewhat, sneak back around to their starting point, and march again and again through the Americans' field of vision. "One little, two little, three little Indians … ." Several thousand non-existent Indians later the Americans thought they were severely outnumbered and surrendered without firing a shot.

That, I would say, is a fine example of the Native American use of mathematics. It is something which we own, something of which we can be proud. That is what is missing, from most of the examples of ethnomathematics used in education. In ethnomathematics, there is usually a sense that there is something larger behind the scene, let us call it "real mathematics", which is not ours. That perceived lack has the effect of making us feel ashamed rather than proud.

Passion was a major key to Tecumseh's success in the opinion of his biographer John Sugden. In the Indigenous world view, perhaps feelings like passion and pride are more valuable than the knowledge of facts, ideas, rules, regulations, and methods. We need to follow Tecumseh's example and instill a sense of pride and passion in our students, not shame and apathy.

Apropos are historian William Wood's words on the impact of the death of Brock at Niagara-on-the-Lake shortly after the capture of Detroit: "Genius is a thing apart from mere addition and subtraction." Brock was just one man, but his life and death changed the course of history. Arithmetic is not always the best tool to use.

One good example notwithstanding, we are still left with the question of what we can do to resolve the apparent incompatibilities between Indigenous thought and mathematics. I would like to make two suggestions about how we might be able to proceed from here.

First, I would like to consider the question of how we might be able to pull mathematics into Indigenous culture rather than how mathematics might be pushed onto Indigenous people or how Indigenous culture might be pulled onto mathematics. What might be the difference between thought which is authentic to the culture rather than a simulacrum of an idea from elsewhere?

Let us consider how foreign words and concepts are introduced into the Mohawk. Some words are simply borrowed, in a process familiar to English speakers, from a European language as in *Kabatsya* = garbage, *Ti* = tea, and *Takós* = cat (probably from Dutch *de poes*, i.e., the puss). There are obvious signs that those words are not originally Mohawk words: the presence of strange sounds (the b sound in garbage), single syllable words, or words with stress on the wrong syllable. Some borrowed words have the overall style of Mohawk (e.g., begin with "*ra-*") but lack the internal structure of Mohawk words, as in *Rasanya* = lasagna which, if it were really a Mohawk word, would mean something like "he sanyaed", whatever sanyaing would be. A similar example is *Rasohs* = sauce, apparently from the French *la sauce*. All of those examples lack the nuance, complexity, and internal structure that Mohawk words typically have. If there is any connotation, it is ridiculous, as in "he sanyaed".

On the other hand, there are new Mohawk words to describe new concepts, words which developed within the Mohawk tradition. For example, we have *kaya'tarha* = television, literally "it has bodies on its surface"; *teyothyatatken* = banana, literally "the fruit that has bent itself"; *kawennokwas* = radio, literally "it throws out songs"; and *kawennarha*, literally "it has words on its surface", a word proposed, but not (yet?) generally accepted, for describing a computer. Those words really mean something and are not just dry tokens the way English nouns are. They are better because they ours, but it is not simply a matter of pride. Since they are ours, they are consistent and coherent with the rest of the language; they strengthen the language just as the language strengthens them; and they can be modified and built upon to add further complexity and sophistication to the language.

New words are coined constantly within the Mohawk tradition. The spirit of the language is inventive and playful, not acquisitive like the spirit of English. I myself have coined a few new words, for example *kahnekahontsi* = cola drink, literally "black water" or "black drink", and *Kwiskwis nikawahràsas* = bacon bits, literally "little pig meats". The latter made Kahnekotsyentha kenha laugh and is now regularly used by a small group in Six Nations. Some day it may come into general use.

Second, I would say we need to recognize that mathematics is an essentially simple (not complex, although often complicated) way of thinking. Mathematics is all about simplifying, clarifying, analyzing, and breaking down. On the other hand, Indigenous thought is all about developing and building up sophisticated, complex responses to complex phenomena such as the weather, animal migratory patterns, healing, and human behaviour. A colleague at First Nations House at the University of Toronto told me about one occasion on which her grandmother held a baby. "There's something wrong with this baby," said the grandmother. It turned out that the child had a serious illness, but the child's parents and doctor had all missed the problem until the grandmother felt that something was wrong. We can weigh and measure and test, but true complexity cannot be handled by simple means.

Time for another joke. This one I heard at the Sakewewak Storytelling Festival in Regina several years ago. I'm afraid I can't remember the name of the storyteller; if anyone out there knows, please tell me so I can credit him properly in the future.

> In a town in a certain reserve in Saskatchewan, some young boys were breaking into houses. The RCMP investigated. They came into town and asked the first person they see, an old man sitting in front of his house, whether he knew anything about the break-ins. "Yup," said the old man. "Do you know who's been doing it?" asked the police. "Yup," said the old man, "those four boys." "Would you be prepared to testify in court?" asked the police. "Yup," said the old man. So the RCMP arrested the boys and charged them with break and enter.
>
> Court day arrived, and the old man took the stand. The prosecutor asked him, "Do you know who's being doing those break-ins?" "Yes," said the old man. The prosecutor asked, "Can you point to the individuals in question?" "Yes," said the old man, "it's those four boys sitting over there." "Thank you," said the prosecutor, "those are all the questions I have."
>
> Then the defence lawyer began his cross-examination. "Have you actually seen those boys breaking in to a house?" "No," said the old man, "I haven't actually seen it myself." "Then how do you know it's them?" asked the defence lawyer. "I have my ways of knowing," said the old man. "I'm sorry, your evidence is hearsay. We can't accept it," the defence lawyer said. The judge agreed, and dismissed the witness.
>
> Well, the old man was not too happy about being dismissed like that, so as he walked past the judge on the way back to his seat, he let out a fart. A long, loud one. A big one. The judge banged on his gavel and said, "I could have you charged with contempt of court for that!"
>
> The old man turned to face the judge and asked, "Did you see anything?"

Given the apparent incompatibilities between Indigenous thought and mathematics, I suggest that instead of asking "What is Indigenous mathematics," it may be helpful to start with the following question instead: "What are the Indigenous analogues to mathematics?"

For example, we might ask what the role of mathematics is in non-Indigenous culture. I believe that one function mathematics plays is as a source of power, which is one reason people are so concerned about learning it or seeing that it is taught to their children. Power is also an important concept in my culture. In fact, the core message of the *Kayanerekowa*, the Great Good Way, is *Skennen, Kahsha'sten'tshera, Ka'nikonhriyo* = Peace, Power, and Good Mind. (The word "righteousness" is often seen in place of "good mind", but the latter is a better translation.) Power is central to our understanding of following a good way.

> *Seeing me in my patched-up, faded shirt, my down-at-heels cowboy boots, the hearing aid whistling in my ear, looking at the flimsy shack with its bad-smelling outhouse—it all doesn't add up to a white man's idea of a holy man. You've seen me drunk and broke. You've heard me curse and tell a sexy joke. You know I'm no better or wiser than other men. But I've been up on the hilltop, got my vision and my power, the rest is just trimmings. That vision never leaves me.* —Lame Deer

All this talk about power tends to make some people nervous. However, *kahsha'sten'tshera* in this context is not power in isolation, rather power within a strong ethical tradition, if "ethical" is the right word. Another aspect of the tradition in which power sits is humility. As Black Elk said,

> *I cured with the power that came through me. Of course, it was not I who cured, it was the power from the Outer World, the visions and the ceremonies had only made me like a hole through which the power could come to the two leggeds. If I thought that I was doing it myself, the hole would close up and no power could come through. Then everything I could do would be foolish.*

Black Elk's reference to power coming through him reminds me of Ramanujan, a great inspiration to me, one of the finest mathematical minds of the 20th century. Ramanujan could not describe the source of his mathematical insight, but believed it did not come from him personally; instead it came through him in dreams from his family goddess, Namakkal. Ramanujan had a morning ritual of writing down the thoughts that came to him in dreams shortly after awakening.

Indigenous spiritual traditions and mathematics are perhaps not really so far apart after all. Perhaps. Perhaps we can think of mathematics as a kind of medicine, a healing power. But can it make our lives better as a people, or are its benefits restricted to just a few fortunate individuals?

I would like to finish with the Blackfoot horse creation story. This version of the story is taken from Ted Chamberlin's most recent book, *Horse*.

> A long time ago there was a poor boy who tried to obtain secret power so that he might be able to get some of the things he wanted but did not have. He went out from his camp and slept alone on the mountains, near great rocks, beside rivers. He wandered until he came to a large lake northeast of the Sweetgrass Hills. By the side of that lake he broke down and cried. The powerful water spirit—an old man—who lived in that lake heard him and told his son to go to the boy and find out why he was crying. The son went to the sorrowing boy and told him that his father wished to see him. 'But how can I go to him?' the lad asked. 'Hold onto my shoulders and close your eyes,' the son replied. 'Don't look until I tell you to do so.' They started into the water. As they moved along the son told the boy, 'My father will offer you your choice of animals in this lake. Be sure to choose the old mallard and its little ones.'
>
> When they reached his father's lodge on the bottom of the lake, the son told the boy to open his eyes. They entered the lodge, and the old man said, 'Come sit over here.' Then he asked, 'My boy, what did you come for?' The boy explained, 'I have been a very poor boy. I left my camp to look for secret power so that I may be able to start out for myself.' The old man then said, 'Now, son, you are going to become the leader of your tribe. You will have plenty of everything. Do you see all the animals in this lake? They are all mine.' The boy, remembering the son's advice, said, 'I should thank you for giving me as many of them as you can.' Then the old man offered him his choice. The boy asked for the mallard and its young. The old man replied, 'Don't take that one. It is old and of no value.' But the boy insisted. Four times he asked for the mallard. Then the old man said, 'You are a wise boy. When you leave my lodge my son will take you to the edge of the lake, and there in the darkness, he will catch the mallard for you. When you leave the lake don't look back.'
>
> The boy did as he was told. At the edge of the lake the water spirit's son collected some marsh grass and braided it into a rope. With the rope he caught the old mallard and led it ashore. He placed the rope in the boy's hand and told him to walk on, but not to look back until daybreak. As the boy walked along he heard the duck's feathers flapping on the ground. Later he could no longer hear that sound. As he proceeded he heard the sound of heavy feet behind him, and a strange noise, the cry of an animal. The braided marsh grass turned into a rawhide rope in his hand. but he did not look back until dawn.
>
> At daybreak he turned and saw a strange animal at the end of the line—a horse. He mounted it and, using the rawhide rope as a bridle, rode back to camp. Then he found that many horses had followed him.
>
> The people of the camp were afraid of the strange animals. But the boy signed to them not to fear. He dismounted and tied a knot in the tail of his horse. Then he gave everybody horses; there were plenty for everyone and he had quite a herd left

> over for himself. Five of the older men in camp gave their daughters to him in return for the horses. They gave him a fine lodge also.
>
> Until that time the people had had only dogs. But the boy told them how to handle the strange horses. He showed them how to use them for packing, how to break them for riding and for the travois, and he gave the horse its name, elk dog. One day the men asked him, 'These elk dogs, would they be of any use in hunting buffalo?' 'They are fine for that,' the boy replied. 'Let me show you.' Whereupon he taught his people how to chase the buffalo on horseback. He also showed them how to make whips and other gear for their horses. Once when they came to a river the boy's friends asked him, 'These elk dogs, are they of any use to us in the water?' He replied, 'That is where they are best. I got them from the water.' So they learned how to use horses in crossing streams.
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> The boy grew older and became a great chief, the leader of his people. Since that time every chief has owned a lot of horses.

Given the frustrations and difficulties of the task facing us, it is reasonable to ask, "Do we really need this stuff anyway?" As a response I offer the completion of the earlier quotation by Chief John Snow: "We have survived, but survival by itself is not enough. A people must also grow and flourish."

Working Groups

Groupes de travail

Secondary Mathematics Teacher Development
La formation des enseignants de mathématiques du secondaire

Joyce Mgombelo, *Brock University*
Morris Orzech, *Queen's University*
David Poole, *Trent University*
Sophie René de Cotret, *Université de Montréal*

Participants

Mary Beisiegel
Tanya Berezovski
Bob Berglind
Marianna Bogomolny
Peter Brouwer
France Caron
Rosalind Carson
Natasha Davidson
Brent Davis
Malgorzata Dubiel
Gary Flewelling
Holly Fraser
Frédéric Gourdeau
Gordon Hamilton
Tom Kieren
Dasha Kinelovsky
Ralph Mason
Irene Meglis
Mark Mercer
Krista Poscente
Jérôme Proulx
Tina Rapke
Geoff Roulet
Bouchra Sabbagh
Chris Surtamm

Description of the working group theme

This is what the working group leaders wrote in advance of the meeting as a description of the working group theme:

The aim of this group is to generate a healthy and productive dialogue between mathematicians and mathematics educators about secondary mathematics teacher development. We hope that working group participants will contribute and take away intellectual or physical resources of practical use in their work on the professional development of pre-service and in-service secondary mathematics teachers. The topics below focus on revealing or generating such practical resources.

- "Best-practice" examples of (pre-service and in-service) courses and programs for secondary mathematics teachers; what they provide that less satisfactory models do not.
- Some characteristics of potential and actual collaborations between mathematicians and mathematics educators that enhance the effectiveness of their individual efforts towards the professional development of secondary mathematics teachers.
- The extent to which high school curricula, and their formulation in terms of topics, goals, or proficiencies, should affect the university experience of secondary mathematics teachers.

- Exploration of the idea that mathematics for secondary teaching is akin to an area of applied mathematics[3] and examples of how this idea can influence course design (activities, tasks, curricular choices, etc.).

We expect that curricular topics by themselves will not be the focus of the working group. However, it seems inevitable that topics and courses will arise in illustrating how to bridge between identifying desirable outcomes (be they teacher characteristics and teacher views of mathematics, or interactions between teachers, mathematicians and mathematics educators) and implementing tasks and activities that foster them.

Some concerns that may not affect classroom practice in the short run can be fundamental in shaping our understanding of what we do. These include questions about societal goals implicit or explicit in why we teach mathematics in high school, and how these goals affect the high school curriculum. If participants wish the working group will engage with such questions.

La formation des enseignants de mathématiques du secondaire

Le but de ce groupe de travail est de générer un dialogue riche et fécond entre mathématiciens et didacticiens des mathématiques à propos de la formation des enseignants de mathématiques du secondaire. Nous espérons que les participants au groupe de travail pourront échanger diverses ressources (matérielles ou intellectuelles) qu'ils utilisent dans le cadre de la formation initiale ou continue des enseignants de mathématiques du secondaire. Les thèmes de travail suivants visent à mettre en évidence ou à générer de telles ressources.

- Examen de cours et programmes jugés « exemplaires » pour la formation initiale et continue des enseignants de mathématiques du secondaire; qu'apportent-ils que des d'autres moins satisfaisants n'apportent pas?
- Caractéristiques de collaborations, potentielles ou effectives, entre mathématiciens et didacticiens des mathématiques qui accroissent l'efficacité de leurs efforts individuels pour la formation des enseignants de mathématiques du secondaire.
- Dans quelle mesure les curriculums du secondaire et leur formulation en termes de contenus, objectifs ou compétences, devraient-ils avoir une incidence sur l'expérience universitaire des enseignants de mathématiques du secondaire.
- Exploration de l'idée selon laquelle les mathématiques pour l'enseignement secondaire seraient apparentées à un domaine des mathématiques appliquées[1] et présentation d'exemples de la façon dont cette idée peut influencer la conception de cours (activités, tâches, choix curriculaires, etc.)

Nous ne souhaitions pas centrer nos discussions sur la définition de contenus du curriculum. Toutefois, ces derniers entreront nécessairement en jeu lors du travail sur les thèmes précédents, notamment pour la description de visées de formation (caractéristiques des enseignants, interactions enseignants-mathématiciens-didacticiens) et de tâches et activités qui pourraient les alimenter.

Enfin, nous sommes conscients que certains facteurs qui ne semblent pas avoir une influence directe sur les pratiques de classe jouent tout de même un rôle fondamental dans notre façon de les concevoir. On peut penser par exemple aux finalités sociales, implicites ou explicites,

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[3]See in particular pp.13-15 of Usiskin, Z., 2001, Teachers' Mathematics: A Collection of Content Deserving To Be A Field, in National Summit on the Mathematical Education of Teachers, Nov. 2-3, 2001.
(Voir aussi http://www.cbmsweb.org/NationalSummit/WG_Speakers/usiskin.pdf)

qui justifient l'enseignement des mathématiques au secondaire, et à leurs incidences sur les curriculums du secondaire. Si les participants le souhaitent, le groupe de travail pourrait aussi explorer ces questions.

Background/overview of participants' Interests/ questions/ views

What drives the sense (implicit in the organization of this working group) that reform is needed in how pre-service secondary mathematics teachers are prepared? Is it just part of a "buzz" about reform of mathematics teaching, spurred, e.g., by widely publicized projects in calculus reform? The evidence for that is far from convincing. It is true that there is considerable interest in alternative models for teaching undergraduate mathematics, but there is no sense of urgency or pronounced dissatisfaction driving change in curricula and teaching methods for most math majors (including those *not* intending graduate work in mathematics) – except for pre-service mathematics teachers. If this perception of more widespread and more intense dissatisfaction in how we prepare secondary mathematics teachers is correct, it is interesting to ask what drives it. Is it a feeling that in our math major programs the special needs of teachers (say as compared to the special needs of students wanting to become actuaries) are an afterthought? Has this feeling been in part generated by an emerging awareness or belief that "mathematics for teaching" is an identifiable entity that can be given substance through the mathematics curriculum? Some of the interest in reforming teacher preparation likely comes from the U.S.-based sense of crisis about the mathematics and science achievement of students. Both curriculum and teaching have become targets for reform. Does the Canadian situation justify the same drive towards reform? Some insight into these questions will be implicit in our description of the discussion in our Working Group.

Our opening roundtable, at which Working Group participants introduced themselves to the group, revealed a wide range of practice and experience related to secondary mathematics teaching. This included past experience as high school teachers; current school board work as a resource person for teachers; university education of pre-service teachers in mathematics or education departments; and graduate study and research related to the secondary mathematics classroom. The motivations of participants for joining our group were correspondingly diverse.

Some participants expressed interested in presenting mathematics to students in better ways. What might better mean? It included characteristics such as being *more engaging*, *more meaningful*, and *less abstract*. For some participants the expression of these desiderata was not focused on courses for pre-service teachers. For others it was, including implicit or explicit questions such as whether tying the mathematics to the students' future work might help generate engagement, and whether our goals and approaches in preparing teachers can be separated from broad issues of how mathematics *is* taught and how *it should be* taught.

A number of participants had special responsibilities for teacher preparation, involving design and instruction of courses in mathematics and/or mathematical methods and/or didactics of mathematics. The frameworks in which this work is carried out vary considerably between educational systems. Some models (typified by Ontario and British Columbia universities) involve virtually no consultation, and often little contact, between instructors of mathematics courses and instructors of methods courses for secondary mathematics teachers (to the point where someone in one group may be unsure what the other is doing or how it fits in to the overall teacher preparation). In Québec the mathematical formation and the math education formation are usually done in their respective departments – except for UQAM where both are done into a math department which includes didacticians – and each set of teachers seems to have an understanding of the overall teacher preparation package. According to the participants from Québec universities there is consultation between the mathematicians and didacticians, and a few mathematical courses are even specially developed for pre-service teachers. As in UQAM, pre-service teachers at State University of New York colleges may take their mathematics and mathematical methods courses in the same department, and from some of the same teachers.

Of the three models mentioned above, that of Québec seemed the one in which people are most comfortable with their role in the preparation of secondary mathematics teachers. Although the SUNY approach might seem to offer maximum flexibility in designing and delivering a program for teachers, the ferment mentioned in our opening paragraph creates uncertainty as to what the design principles should be, and to what extent there should be migration of items traditionally treated in "methods" courses into "curricular" courses, and vice-versa. In both of these models prospective teachers share some of their mathematics courses with the other math majors, but also take courses dedicated to their intended profession. The Ontario examples we were able to sample shared the Québec and SUNY characteristic that mathematics programs for prospective secondary teacher include courses designed with an eye to professional goals. But the existence of these tailored mathematics courses is tenuous, and the design principles behind the "tailoring," as well as the connections with mathematical methods courses, are often ad-hoc.

A common feature of our participants' various contexts for mathematics teacher formation is that mathematicians involved in the process do have an interest in the question of what they can or should do to prepare teachers. In past decades the typical mathematician's answer to such questions would have centred around exposing pre-service teachers to more mathematics, which would usually have meant more advanced and abstract mathematics. This viewpoint is not prevalent in today's discussions of the same questions. Most people interested in the issue are aware that there has been no success in documenting that more advanced mathematics courses contribute to the formation of better secondary teachers. With this as background it is reasonable to question whether there is in fact much that mathematics departments can contribute to what secondary mathematics teachers need to know. We heard the view that what mathematics teachers need to know is "embedded in the moment of teaching," and is hence is something that cannot be taught in the university classroom. This notion surfaced in a different form, through comments suggesting the efficacy of in-service development that included mathematical as well as pedagogical components.

Given the structure of current programs, it is not obvious how instructors of pre-service teachers might take into account that what teachers need to know comes through teaching. It seems that mathematics departments have limited scope for systematic interaction with in-service mathematics teachers. Pre-service teachers, except those in Québec or those in concurrent education programs, do not bring classroom teaching experience into their upper-division mathematics courses. Moreover, the interest of participants in "mathematics for teaching" included questions that would not disappear even if one accepts that much of what goes into shaping a teacher occurs in the practice of teaching. Examples of these questions include: how to set a good foundation for ongoing professional development; how to lead pre-service teachers to see themselves and their future students as having the potential to be creative problem solvers and formulators of mathematics; how to convey interest in what makes it possible for people to create, shape, and communicate mathematics, and what allows mathematics to be doable by humans.

The questions raised in the last paragraph do not connect in an obvious way with the traditional curriculum in either mathematics or mathematics education courses. To what extent should fretting about that curriculum be part of our effort to "do the best we can" for pre-service teachers? The Working Group co-leaders, respecting guidance from the meeting organizers, had avoided plans for turning the discussion towards curricular specifics. However, other participants were not hesitant to bring up curriculum-related issues. Three views about desirable course content for prospective teachers were voiced in varying ways by several participants (and were likely shared by quite a few others). One was the value of relating mathematics to historical and cultural perspectives. Another was the importance of bringing students to see mathematics as having internal coherence, both within a subject and between levels. A third was the benefit of having pre-service teachers see elementary mathematics from a more advanced perspective.

Although clearly related to curriculum decisions, these suggestions were not always framed that way. Some emerged as questions about what a curriculum would look like that blended inquiry with traditional instruction. Some were presented as ideas for changing *what* we teach so as to introduce a variety of instructional techniques in place of over-reliance on a lecture-based approach. The desire we heard from participants for a more diversified font of classroom instruction techniques had two aspects: personal professional enrichment, and disciplinary improvement.

Curricular suggestions are likely to be perceived as directed to the mathematics rather than the education component of the pre-service teacher's program. However, in most cases this intention would have had to be inferred from what was said, and drawing the inference that everything is just fine in the Teacher's College curriculum might lead us to overlook opportunities for improving the overall preparation of pre-service teachers. For example, we mentioned earlier the SUNY attempt to introduce more mathematical components into methods courses. The Québec teacher formation program, which incorporates considerations of mathematical coherence into the design of the mathematics education curriculum, might be a source for philosophically agreeable and practically implementable ideas for those who teach methods courses in other jurisdictions.

We should not leave the impression that mathematics departments did not come in for explicit criticism. In the opening roundtable we heard of frustration, experienced by pre-service teachers and their education program instructors, at how a mathematics degree had simply not prepared its recipients for what they would need as a teacher. The background for this criticism was the Ontario model where intending teachers obtain their B.Ed. degree following a B.A. or B.Sc.

Participants raised other questions that bear on the interaction of mathematics instruction and mathematics education instruction in the formation of secondary mathematics teachers. Can research in mathematics education beneficially influence the mathematics classroom experience of pre-service teachers? How can the mathematics and mathematics education communities share insights each has into activities that engage students and help make sense of mathematics? Does the seemingly common vocabulary that these communities use to talk about mathematics teaching mean the same thing to each group? Are there significant differences in the way each group perceives the transition from mathematics as a collection of facts to something more, and do they have the same "something more" in view when they talk about secondary teachers and their students?

Summary of the Working Group's Quest for Insight

The group worked in an iterative way. Directly following our opening roundtable we identified four large issues that seemed to emerge from participant interests. In the second part of our first day we broke up into four groups, each assigned the task of identifying questions relevant to one issue. These questions were brought back to the whole group for clustering into themes to be analyzed in more depth. The initial four issues from which these themes were to be distilled were:

- Pedagogy and content as a shared responsibility
- Ways of learning math – what do we want to achieve?

- Interesting words: rigourous, meaningful, conceptual, deep, abstract, coherent, contextual.
- The "affect/attitudes" component of math education

The group distilled the following themes (each accompanied here by a synopsis of what emerged when we discussed it).

Theme 1. Identifying desirable mathematics experiences

Synopsis of the discussion

What characteristics and goals should we keep in mind in the mathematical training and education of prospective secondary mathematics teachers? We should try to foster a non-constrained view of mathematics through:

- breadth of subject matter extending beyond traditional material in calculus, linear algebra, and introductory statistics, to include experience in numerical methods, modeling, and discrete mathematics;
- experience in *doing* mathematics, extending beyond solving template problems, to include such things as formulating definitions and explanations, constructing and using mathematical models, etc.;
- connecting mathematics taught at university to that taught in schools; helping students to think about "where things come from, where they go;"
- exposure to topics that require non-routine perspectives, such as non-Euclidean geometry;
- linking mathematics to historical and cultural phenomena.

Theme 2. Questioning assumptions

Synopsis of the discussion

Some conventional perceptions that influence the preparation of mathematics teachers can restrict the educational vision of the pre-service teacher, and the experience of his or her students. Designing training regimens for teachers should involve critical review of what mathematics is, of the purpose of education, of how we conceptualize the learner, and of what teaching is about. Some suggestions for fresh points of view include:

- Mathematics is a way of thinking and learning rather than a set of rules.
- The learner and the subject should be seen as evolving and contextualized.
- Teaching and learning should include elements that perturb comfort, mastery, equilibrium.
- Learner autonomy and actualization should complement the layered view mathematics.

Theme 3. Rich vehicles for mathematical development – what and where?

Synopsis of the discussion

A common model of pre-service teacher development involves first studying in a mathematics department (often through courses common to all math majors), and then taking an education degree, which includes exposure to pedagogically-oriented "rich" ways to understand and teach mathematics. Do these components have to be separate? Could "rich experiences" of value to a prospective teacher be part of her mathematics program? Some tension is inevitable. Constraints of time and subject matter may not respect the interests of prospective teachers in math major courses. However, non-traditional classroom activities that are good for prospective teachers can also be good for prospective researchers. For example, mathematics history can be used to help develop appreciation and mastery of proof. So can activity such as serious writing (e.g., producing "parts of textbooks" from which other students could learn). The latter task, as well as others that involve reading mathematics, can provide valuable modeling for continuing professional development, be it as teacher or researcher. Technology can also be a vehicle for the rich mathematical activities we would like students to experience. Knowledge of resources that support learning "mathematics for teaching" would be a valuable addition to the professional toolkit of mathematicians educating pre-service teachers, and to their students.

Theme 4. Affect in the mathematical education of teachers

Synopsis of the discussion

We would like secondary mathematics teachers to have passion and curiosity for mathematics and its teaching: about the subject, about how people learn it, about pedagogical approaches to teaching it. We would like teachers to develop lifelong learning habits, and to think of themselves and their students as doers of math, rather than passive receivers trying to master template problems.

Balanced use of disruption and reinforcement should be used as a counter to an algorithms-and-templates view of mathematics that is too often a legacy of the overall mathematics education that pre-service teachers experience. The disruption in outlook must be accompanied by reinforcement of the pre-service teacher's confidence in what he/she *does* know, and in his/her command of materials and approaches supporting broadened conceptions of mathematics.

The Denouement

For our first two days we divided our time between working in breakout groups and as a whole group. The smaller groups were asked to consider specific questions, and their reports became the basis for plenary discussion within the time available. Our closing whole-group session was reserved for individual reflection on our Working Group experience. It was clear that on the "physical" side we fell well short of our goal of having participants "take away intellectual or physical resources of practical use in their work on the professional development of pre-service and in-service secondary mathematics teachers." On the other hand, the reflective and cordial ways in which participants revealed this were probably indicative that the Working Group had probably done better with respect to "intellectual" resources, and in its aim to "generate a healthy and productive dialogue between mathematicians and mathematics educators about secondary mathematics teacher development." In one form or another we heard that although what happened in the Working Group was not what the person had expected, or not directly germane to his or her professional situation, there was appreciation of the ideas and opinions encountered – of the opportunity they presented for a broadened perspective, and of the sense they engendered of kindred thoughts among people playing different roles towards the common mission of broadening the appreciation of what mathematics is, and of improving the effectiveness of our teaching work.

We will review some closing comments by participants not only to record some of the interesting observations made, but because they provide insight into questions raised in the previous section of this report. Although some of these remarks were voiced by only one or two people, it was our sense that the sentiment behind them had considerable support from others.

Some of the observations we heard seemed linked to the organizers' decision, for reasons mentioned earlier, to shy away from curricular specifics. People who had hoped to leave the Working Group with new tools and resources ready for classroom use did not fare well in having this expectation met. This was particularly true for those whose focus as teachers was on development of their ideas on how to teach math, rather than on "how to teach how to teach math." Although this "how to teach how to teach" motif was appropriate to the Working Group's theme, had there been less abstraction in its treatment there might have been more benefit for those participants who do not teach mathematics or mathematics education courses specifically intended for pre-service teachers. Another summary comment indicating a need for concrete materials was the observation that there was a need for textbooks and similar resources to support the rich experiences we want for our pre-service teachers, and ultimately for *their* students.

Even if concrete classroom materials did not surface through our discussions, some applicable *perspectives* did. One was that each of us can, as a teacher, have a role in defining what mathematics *is*, without distancing ourselves entirely from community standards and curriculum guidelines for the courses we teach. We heard various forms of an appreciation for the polite exchange between people with different assumptions and curricular missions. We were reminded of the need to think globally while acting locally. For some of us the discourse in our Working Group was seen as something one could refer to later – as a source of encouragement and as a model for civil disagreement – in dealing with tensions that can arise when we depart from traditional teaching. One participant saw in our discussion an illustration that in confronting assumed barriers the "monsters under the bed" might turn out not to be where or what we thought they would be. One person likened our Working Group discourse to *The Life of Pi*, reminding her of the novel's presentation of different stories accounting for the same reality.

The shared appreciation of our amicable and thought-provoking dialogue did not prevent critical views from surfacing during our closing roundtable. The aforementioned dearth of specific proposals made some of our discussion seem superficial to people already aware of the problems – but not of solutions. Some participants sensed a quest for a "holy grail" in the tone of our some discussion, or in the goals for it (such as the identification of "best practice" models). This was seen as potentially leading to frustration and defeatism. Some of the closing comments echoed opening remarks on a divide between the implicit perspectives of mathematics departments and mathematics education departments on what pre-service teachers (and by extension, on what mathematics students) need. In our closing session we heard the view that mathematicians had "catching up" to do in learning about, developing, and communicating a richer vision of mathematics teaching and learning.

Two perspectives that came up during our breakout and plenary sessions were referred to during our closing debriefing as new considerations that would be food for future thought. One was the view of the secondary teacher as a lifelong learner. The other was the possibility of the teacher leading students to assume some of the responsibility for what transpired in a classroom or learning community, hoping to make the student's experience less like "hopping on an elevator" (with no choice of destination and no opportunity to explore). The mathematics student (in high school and in our pre-service classes) should be seen as a potential artisan, not just a keeper of artifacts. The word "devolution" was proposed to capture the idea of transferring scope for self-direction to students. It is clear that for this concept to gain currency in our discourse we will need examples of what it can mean in practice. If nothing else, many (perhaps most) of us have too little insight into teacher thinking and development on which to build a principled approach to course and curriculum development that includes "devolution," or even more familiar ideas such as lifelong learning, or scaffolding of mathematical understanding, or the use of disequilibrium to generate engagement and to deepen understanding.

Our final reflection on our closing roundtable is about an issue core to the CMESG. Something the Working Group leaders had hoped for was a list of existing or potential ways to bridge the separation between mathematics department people and mathematics education people in the preparation of pre-service secondary teachers. No list emerged. Although during our meetings we did hear occasional references to attempts at inter-faculty cooperation or consultation or communication, most examples seemed too ad-hoc, or too new, or too locally oriented, to be transferable models. It might be that other settings, such as the series of Canada-wide forums initiated with the 1995 Québec City meeting, may be more appropriate vehicles for dealing with the structural issues that limit ongoing cooperation and consultation between mathematicians and mathematics educators in areas where they have shared responsibility. On the other hand, even people who had hoped for the emergence of explicit bridging models did not criticize that this did not happen. Rather, they felt that what *did* happen through our Working Group discourse provided compensatory benefit that we could each apply in our own sphere of work and influence.

Developing Links Between Statistical and Probabilistic Thinking in School Mathematics Education / *Le développement de liens entre la pensée statistique et la pensée probabiliste dans l'apprentissage des mathématiques à l'école*

Stewart Craven, *Toronto District School Board*
Linda Gattuso, *Université du Québec à Montréal*
Cynthia Nicolson, *University of British Columbia*

Introduction

The value and significance of the discussions initiated in our working group might best be illustrated by the following example of student learning.

The setting was a summer school classroom in which students in Grades 7 and 8 had been enrolled to improve their skills in mathematics. The instruction was innovative—at least in terms of summer school instruction. Students were engaged in an inquiry that required the use of statistical methods to predict the likely distance a paper airplane would fly, based on the results of tests of the prototype over a number of trials. One group of students presented its findings by using a graph and by providing accurate figures for the mean, median, and mode of the distances recorded for all of the trials. Although the students in this group found the mean and median to be very close to 9m, they concluded that the airplane would fly 15m the next time it was launched! Having completed their statistical analysis, the students' prediction was in no way influenced by the graphs and numbers they had generated. They had, instead, based their prediction on hope, recording and calculating the statistics only to satisfy the requirements of the teacher.

Comme on le voit dans cette anecdote, les connaissances acquises à l'école ne sont pas toujours utilisées. Ce phénomène est bien documenté dans la didactique des sciences en particulier (Joshua S., Dupin J.J., 1993)[4]

Session One

Our working group comprised individuals who brought with them a multitude of perspectives. Included in the group were statistics professors, mathematicians, faculty of education professors, graduate students, high school and elementary school educators, and a representative of Statistics Canada. Following introductions, we began the process of making sense of the key issues affecting the teaching and learning of statistics and probability in

[4] Introduction à la didactique des sciences et des mathématiques, Joshua S., Dupin J.J., PUF, 1993, Paris.

schools. We subdivided into small groups to consider the question, "What do we mean by *statistical thinking*?"

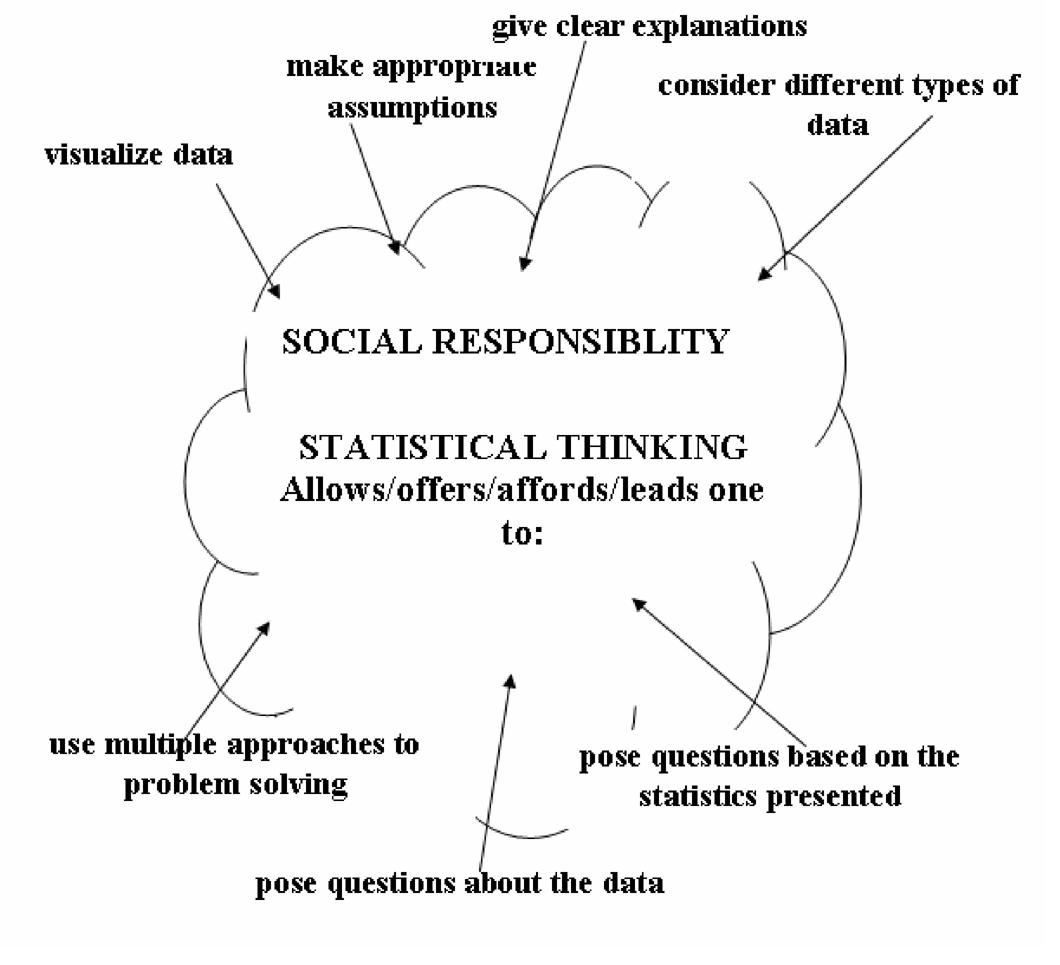
Le groupe était formé de statisticiens, de mathématiciens, de professeurs des sciences de l'éducation, de formateurs d'enseignants du primaire et du secondaire, ainsi que d'étudiants gradués. De prime abord, le groupe divisé en petites équipes s'est posé la question: "Qu'est-ce que la pensée statistique?" et voici les réponses obtenues dans chaque équipe.

The first group generated the following ideas:

Statistical thinking occurs when one can:

- understand poll results and margins of error;
- analyze distributions by considering centres and spreads;
- use statistical formulas appropriately; and
- make connections to applications in areas such as economics, the environment, sociology, the pure sciences, and political science.

The second group decided to use a schematic diagram to convey its ideas.



The remainder of the first session was devoted to identifying the "big ideas" in statistics and probability. These ideas are summarized in the following table:

"Big Ideas" in Statistics	**"Big Ideas" in Probability**
Use statistics to seek patterns and make predictions	Understand the applications of the additive and multiplicative principals when determining probabilities
Recognize the difference between qualitative and quantitative data	Understand the difference between dependent and independent events
Use and construct graphs appropriate to the type of data	Understand expected value
Recognize the difference between a sample and a population	Understand counting (combinations and permutations)
Understand that a carefully designed sample can be representative of a population and can be used to make predictions	In experimentation, understand the relative predictive quality of doing a few trials versus many trials
Understand notions of statistical significance and sampling variability	Understand the difference between theoretical and empirical probability
Use distributions to reveal patterns/trends/central tendencies and deviations from patterns/trends/central tendencies	Understand relative frequency
Understand that statistics can be used to deal with uncertainty and can answer relevant and important questions	Understand randomness
	Understand the uses of simulations
	Understand variation
	Understand sample space

The following table summarizes the processes of doing statistics or probability:

Think Critically	Interpret	Make Conclusions (Decisions)
Confront/Address	Compare	Communicate
Pose Hypotheses	Understand Fallacies	

Session Two

Lors de la deuxième session, ***Cynthia Nicolson*** *a proposé un jeu de dés appelé le « petit cochon ». Les participants après avoir joué à ce jeu, ont tenté de trouver une stratégie gagnante et de voir si cette activité pouvait mettre en oeuvre la pensée probabiliste.*

The second session on June 5 opened with a lively game of "PIG." The rules of the game require that participants make decisions based on strategies that may or may not be based on probabilistic or statistical thinking. At first, all participants are standing as the dice roller starts the game by rolling a pair of dice. Given that neither of the two dice roll a "one," all standing score the total on the dice. After each roll, the participants must decide whether to remain standing or sit down. If a participant sits down, his or her score is "locked in" for that round, but if a participant remains standing, three consequences may occur:

- a "one" is not rolled, and the participant can accumulate the additional points for the roll
- a "one" is rolled, and all points for that round are forfeited
- double "ones" are rolled and all points for <u>all</u> previous rounds are forfeited

A round comes to an end when all participants have decided to sit or when at least one "1" has been rolled. The following chart shows a possible scoring scenario:

		Participant #1		Participant #2		Participant #3	
Dice Rolls		**Round #1**	**Round #2**	**Round #1**	**Round #2**	**Round #1**	**Round #2**
(4,4)	(6,3)	Stand 8	Stand 9	Stand 8	~~Stand 9~~	~~Stand 8~~	Stand 9
(2,5)	(2,3)	Stand 7	Stand 5	Stand 7	~~Stand 5~~	~~Stand 7~~	Stand 5
(5,6)	(2,2)	Sit	Stand 4	Stand 11	~~Stand 4~~	~~Stand 11~~	Sit
(1,6)	(3,4)		Sit	Sit	~~Stand 7~~	Stand	
	(1,1)				~~Stand~~		
Totals		15	18	0 ~~26~~	0	0	14

We asked participants to think about their own thoughts and feelings while they took part in the activity, and we proposed the following three questions:

i. How does this activity promote probabilistic thinking?
ii. What possible student misconceptions might arise?
iii. What prior understandings are required?

Some paraphrased responses follow:

> *One participant said that he would sit down after a given number of trials because it was likely that a "one" was due to come up.*
>
> *Another suggested that he had played this game a lot with students and wondered whether this was truly a probability game.*
>
> *In reply, another asked, "Is it not still probabilistic thinking, even though the ideas may be misconceived?" In other words, students might be aware of the existence of underlying probability ideas, even though they misapply them.*
>
> *One participant mentioned the psychological aspect of the activity (e.g., everyone will approach the activity differently depending upon how "conservative" he or she might be). In a similar vein, another person added, "There was a point at which we had 34, and I could feel myself getting to 40, so I didn't want to sit down.*

Game Strategies

Participants discussed strategies and then played three or more rounds in pairs to test their strategies. Strategies included: a) stop after a certain number of rolls, b) stop after a certain number of points, or c) play according to one's intuition. One participant noticed that some students adopted a strategy based on their observation that winning scores were always around 75 in a five-round game. In each round, they would quit once they reached 15, or, if unsuccessful, compensate in the next round (e.g., quit when they reach 30). One individual stated that the optimal score for any round is 18.

Reference was made to the "gambler's fallacy" (i.e., a "one" has not been rolled for a while; therefore, it is due to come up). Understanding that dice don't have a memory is an example of valuable probabilistic thinking. Another said, "It depended on who was rolling the dice.

She was not a hot dice roller. She got ones repeatedly." One might ask if this feeling is sound probabilistic thinking or the idea that the dice themselves could be "lucky."

At this point, we readdressed the central question based on the activity, "PIG." Does this activity demand statistical or probabilistic thinking? Certainly, this game is a simulation where one accumulates data and uses this data to predict future results. Nevertheless, further questions can be asked: Are games such as PIG valuable in helping students learn about statistics? Will students who are engaged in such games realize that they can use statistical analysis to predict future outcomes? Does it matter which comes first —computing a theoretical result, or conducting experiments to collect data? If one were to draw a conclusion at this point, one can see that this type of activity addresses both statistical and probabilistic thinking and that statistical analysis appears to lead ideas in probability. Certainly, if approached carefully, this activity does provide an excellent leaning activity for students because it provides an environment where cognitive dissonance surely exists.

Après la pause, ***Stewart Craven*** *a proposé aux équipes de préparer une activité et pour les inspirer, il a distribué à chaque équipe un sac de croustilles.*

After the break, we asked the members of the working group to rearrange themselves to consider a different activity. Each small group was given a package of potato chips and were asked to:

- use the bag of potato chips as an inspiration for a lesson,
- design such an activity for any grade level, and
- state clearly what the students would learn

This turned out to quite a challenging task as some groups became bogged down with possible mathematical connections. Eventually, all of the groups completed the task.

The first group reported the following steps:

- We dumped out the chips and sorted them into approximate sizes.
- We made a display (a concrete graph).
- We compared the heights of columns of chips.
- We asked questions, "What if you went to a standard unit of measure to try to measure chips?" "Where is probability in all of this?" " What makes a good chip?

In response to this group, it was suggested that it was not necessary to force probability into an activity where descriptive statistics might well be the focus of the learning.

Another group looked at the maximum diameter of the chips, using a grid. They asked, "What questions does this answer?" They added, "We ate our outliers" much to the whole group's amusement.

The third group reported, "When a potato is cut into chips, the bigger ones come from the centre…but the bigger ones are more likely to break. You can count how many are whole but you can't count the broken ones…if this is true, the distribution would drop off very quickly at one end." And they concluded, "We have a worthwhile question." Other people chimed in that this could lead to understanding of the concept "quality control" and that one might be able to ascertain a "potato signature curve."

At this point, a member of the group shared his experience with secondary classes that completed an activity based on the use of bags of potato chips.

- All students were provided with identical bags of potato chips.
- Students used a very sensitive scale to find the mass of the full and empty bags of potato chips.
- Students sketched the distribution of the masses of all the bags of chips.

- The students decided (in advance) on a tolerance level; (e.g., at least 80% of the bags must have a net weight equal to or greater than the net weight signified on the package).
- Where the results were below the tolerance level, the students wrote to the potato chip company to apprise it of their concerns.
- The students learned about distributions, variance, and probability.
- The students discovered how mathematics could support consumer activism.

The last group reported that they also started with the idea of consumer awareness. Students could:

- bring in bags of different brands
- examine each bag to see how many chips were whole, how many were bite sized, and how many were not eatable
- do an analysis of the different brands

Certain unresolved issues arose with respect to this kind of analysis.

Again, we see that the statistical thinking appears to precede the probabilistic thinking. Another significant observation from this activity was that teachers and professors found it difficult to create and structure authentic learning situations in which statistical and probabilistic thinking would be nurtured. It became apparent that a great deal of thought was required to connect appropriate learning to meaningful contexts.

La plupart des équipes en sont restées à des activités élémentaires. Une d'entre elles, par contre, est allée plus loin en proposant une activité initiant au contrôle de qualité.

Session Three

La troisième session a commencé par une présentation très intéressante du "Recensement à l'école" et du logiciel « Thinker Plot » par Joël Z. Yan de Statistique Canada.

Our third session started with a presentation about "Census at School." "Census at School" is an online environment in which students in Grades 4 through 12 from around the world engage in statistical inquiry. Students fill out anonymous surveys, download sample of data, ask research questions, analyze their data, and draw conclusions. Schools from across Canada complete the online surveys to build the database and then draw upon that database to answer important questions.

The ensuing discussion opened with a comment that the data was "spawning" the questions. This is a reversal of the common notion that researchers start with a question and then find appropriate data to answer that question. In other words, it is like saying, "Here's the data, what questions do you want to ask?" Other individuals in the group responded quite positively to this idea, "It makes math beautiful." Another participant surmised that the students should be generating the questions, but "most often, teachers generate the questions. It's a huge difference!"

In the example from the demonstration, the children were learning about the prevalence of bullying in their school. This led to the comment, " The children are learning so much more than mathematics—about bullying, differences between boys and girls. It's great!" Another participant observed, "I saw that the Grade 7 and 8 kids were interested in data about themselves."

What do children need to be able to engage in these kinds of statistical inquiries? Teachers need permission to teach in this manner, they need to know how to use and teach with technology (e.g., completing online surveys, downloading data, using spreadsheets, and using dynamic statistics software such as Tinkerplots and Fathom), and they must have appropriate access to technology. To these ends, teachers must have access to professional development that provides opportunities to learn in depth.

We then asked, “ What do you personally feel about teaching statistics? Is it done appropriately? Why or why not?”

This conversation started with the comment, “I personally believe that no child should leave our school system without statistical thinking experience. …data is present in everyday life.” If statistics is only for some students, then this builds the notion of the privileged or the not privileged.

What are the key issues? The responses are organized under four subheadings.

Par la suite, une discussion s’est engagée et des questions fort pertinentes ont été traitées. Il a été surtout question de l’enseignement des statistiques à l’école, pourquoi, quand et comment les enseigner et les problèmes auxquels on fait face relativement à cet enseignement. Entre autres, le peu de place accordée dans la plupart des programmes scolaires des provinces canadiennes sauf au Québec qui a inclus ces notions à partir du primaire.

Un problème qui a également été soulevé est le manque de préparation et de soutien des futurs enseignants. Ceci est dû à la fois au peu de temps accordé à leur formation et aussi au fait que les formateurs eux-mêmes se sentent démunis face à l’enseignement des probabilités et des statistiques.

Why Should Statistics Be Taught?

The general feeling of the group is summed up in this statement, “I agree that to be a numerate citizen you need to think statistically. Should we be doing it? Of course we should. Statistics certainly speaks to the utility side of math. I think everyone’s going to say, ‘of course we should do it.’” Another participant provided an example to support the importance of teaching children to think statistically. He described the problem where many citizens could not make sense of the proposed changes to the provincial electoral system. This can in part be attributed to our inability to engage and teach children how to think statistically. We cannot allow students to graduate from our schools without such skills if we expect them to be fully functioning citizens.

When Should Statistics Be Taught?

More than one participant stated that it was inappropriate to teach probability before Grade 5, and that developmentally, ideas such as confidence levels should not be addressed until the senior years of high school.

How Should Statistics Be Taught?

An example of a place where statistics is embedded in mathematics learning was drawn from the development of the Applied curriculum in Western Canada. The intention was that the data would be collected and an analysis would come out of the data. This shifted the focus of the traditional mathematics lesson away from “I teach you something, you practice, and then you apply it to something called problem solving.” Another participant stated that it is critical that data analysis be based on authentic and interesting contexts, such as global warming, obesity, polling results, or demographics. To quote yet another participant, “The real life aspect is really an important part. [The context] isn’t just a cloak—a faking of reality.”

It was also noted that many curricula use statistical ideas and procedures to support the teaching of other areas of mathematics, such as number sense and numeration.

The discussion moved to a very interesting notion. It started with statements like, “Students need to learn statistics conceptually, not formally.” This led to the idea that there is a developmental continuum for learning concepts in statistics and that perhaps there is a less formal, “pre-statistical thinking” phase for elementary school students. One member of the group wondered whether “pre-statistical thinking” is analogous to “pre-algebraic thinking.” What does less formal mean? Does it mean a more descriptive approach to statistics? Is it about encouraging data conversations among students with less concern about formal analysis?

What are the Obstacles Encountered in Teaching Statistics Well?

The first concern related to pedagogical approaches that often lead to conceptions that must be corrected at some point. Often, teachers’ background and/or knowledge in mathematics, and in particular, statistics, is very weak or even non-existent at a university level. This point was reiterated several times. Limited access to, and insufficient knowledge about the use of technology was a clear concern. Teachers need good professional development to know how to conduct a statistics lesson that takes advantage of the tremendous power found through technology. While this problem is not insurmountable, solving it does take a concerted effort.

To draw our working group session to a close, we decided to break into two groups. We proposed the following themes:

i. Statistical and probabilistic thinking in the elementary school curriculum
ii. Statistical and probabilistic thinking in the secondary school curriculum
iii. Using technology to teach statistics and probability
iv. Teacher education

One group chose to meld the first two topics together, and the other group chose teacher education for consideration. When we reviewed the notes from both groups, it was evident that the deliberations had taken similar paths. The following synopsis is an amalgamation of the ideas generated by the two groups.

What are the key elements of a rich program or curriculum in statistics and/or probability?

Any curriculum in mathematics must be written so that the learning trajectory is developmentally appropriate. The notion that writing a curriculum that begins with “pre-statistical” thinking in early grades and builds towards more formal “statistical thinking” by the end of secondary school was agreed upon.

Secondly, all concurred that appropriate/authentic contexts are essential when designing a rich statistics program. Students are far more likely to engage if they “own” the question. There was a suggestion that it might be possible to introduce a context in early elementary school that could be revisited across the grades with increasing degrees of sophistication.

Also, students should be able to avail themselves of the power of technology to gather data (e.g., Census at School), represent the data (e.g., spreadsheet programs), and analyze the data (e.g., dynamic statistical software such as Tinkerplots™ or Fathom™).

What training and background do teachers bring to the statistics/probability classroom?

Requirements to gain admission to faculties of education across Canada vary significantly. For example, a teacher wishing to teach elementary school does not need a single university course in mathematics to gain entrance to many Ontario faculties of education. In Saskatchewan, however, all teacher candidates must have a math content course at the university level. In the province of Quebec, teacher certification is done over four years and includes six half-year courses related to mathematics. Across Canada, most certified secondary teachers teach statistics without ever having taken a single course in statistics at university.

Generally, participants agreed that most teachers do not have sufficient background or experience in statistics or probability before they start their teaching careers.

What support do teachers receive/require after they have been hired by a school board?

Teachers need opportunities to deeply learn the content of statistics and probability. The group identified some examples of the kinds of topics where they found teachers struggling. Included in this list were concepts around the use of percentages; applications of representations such as bar graphs, histograms, stem and leaf plots, and box and whisker plots; uncertainty; and randomness.

Teachers need opportunities to learn about how children come to understand/learn statistics and probability, and they need to be introduced to current research in this area. This understanding must inform how teachers teach statistics and probability. Our group identified some aspects of good teaching:

- using inquiry/problem-solving approaches to teach statistics and probability,
- engaging students by using tasks based on authentic contexts that are interesting to the students,
- allowing students to pose their own questions,
- using technology and concrete materials appropriately, and
- using tasks that have multiple entry points and are open-middled (with multiple solution routes) and/or open-ended

When professional development opportunities are provided for teachers, it is critical that excellent methodologies be modelled; in other words, teachers should be taught as they are expected to teach.

Teachers need opportunities to learn how to design good lessons and long-term plans. Many in our group suggested that resources need to be provided to teachers—especially beginning teachers. These might include specific lesson plans, rich statistics/probability tasks, and exemplars of student performance for assessment and evaluation. There were concerns raised that it is difficult for a teacher to implement someone else’s lesson plan because the environment and audience for the developer may not align with the other teacher’s circumstances. This is where a small group of teachers working together locally (ideally, within a single school), possibly with the assistance of a “knowledgeable other,” could co-develop lessons, tasks, and long-term trajectories.

Who is responsible for supporting professional learning? Some suggestions included Board/District lead teachers, curriculum leaders, or consultants; outside experts from organizations such Statistics Canada; university/college professors from mathematics, statistics, or education departments; and provincial teacher associations or the NCTM (National Council of Teachers of Mathematics).

Conclusions

This working group recognized, overwhelmingly, that teachers are critical to successful student learning in statistics and probability. However, teachers must first be comfortable and confident with their own statistical and probabilistic thinking. They must also understand that they should teach statistics and probability through the use of authentic tasks based on issues of interest to their students. Teachers must feel that they have permission and support when undertaking inquiries of this nature (e.g., statistics involving social issues). To this end, provisions must be made by faculties of education across Canada to ensure that teacher candidates graduate with an appropriate background in statistical and probabilistic thinking, either through carefully defined entrance courses or through courses offered within the faculty. Boards and Districts of Education have a responsibility to provide professional learning at the system level as well as providing local, in-school opportunities. Teachers must have access to journals (e.g., NCTM, provincial associations, and research publications), print materials, websites (e.g., IASE (International Association for Statistical Education), The Math Forum, or MathCentral, which gives teachers the chance to have their questions answered by mathematicians), and other on-line resources. It was further proposed that there should be an ongoing group/committee, organized (perhaps under the auspices of the CMS or CMESG) to continue the investigation into the effective teaching of statistics and probability.

Suggested Readings That Informed Our Discussions

Cobb, P. (1999). Individual and collective mathematical learning: The case of statistical data analysis. *Mathematical Thinking and Learning*, *1*, 5–44.

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Developing Trust and Respect When Working with Teachers of Mathematics

Chris Breen, University of Cape Town South Africa
Julie Long, University of Alberta
Cynthia Nicol, University of British Columbia

Participants

Ruth Beatty	Ann Kajander	Joan Moss
Paul Betts	Donna Kotsopoulos	Craig Newell
Cathy Bruce	Cecilia Kutas	Sandy Orsten
Mary Cameron	Dave Lidstone	Jamie Pyper
Sandy Dawson	Lisa Lunney	Riaz Saloojee
Thomas Falkenberg	Janelle McFeetors	Elaine Simmt
Claude Gaulin	Anna McQuillan	Dalene Swanson
Beth Herbel-Eisenmann	Helena Miranda	Jennifer Thom
Kim Hunter	Ami Mamolo	Rina Zazkis
Barbara Jaworski	Marie-Pier Morin	

> In systems of trust, people are free to create the relationships they need. Trust enables the system to open. The system expands to include those it had excluded. More conversations–more diverse and diverging views–become important. People decide to work with those from whom they had been separate (Wheatley & Kellner-Rogers, 1996, p. 83).

Beginnings

In this working group we (the group facilitators and participants) explored the nature of trust and respect with the goal of continuing and strengthening our understanding of relationships with those we encounter in mathematics education. We began with the assumption that perhaps the nature and importance of respect and trust differ across the situations that bring us together in mathematics education: preservice teacher education, teacher professional development, academic research, and graduate studies. Our three working group sessions were organized around these different situations, and we asked ourselves: How do we, and might we, create relationships that are respectful and trusting and what possibilities arise when this happens?

The nature of respect and trust is not easily articulated. Nor is it easily explored. Respect could be an expression of value, appreciation, or high regard of one to another. Trust on the other hand could depend on the extent to which we experience respect. As a group we used our time together and our experiences as places to study, notice, and become more aware of the nature of respect and trust–and in the process, perhaps, would become more respectful and trustful ourselves.

There are multilayers of trust and respect. At the first night during CMESG working group introductions we signal the likelihood that in this working group we will have to move from our possible comfort zones and enact our own understandings of respect and trust when dealing with each other. In a similar way the three facilitators are going to have to face the same challenge in their interactions with each other.

Our three days together were organized around a series of activities designed to provide experiences related to trust and respect. We began as a group by sitting in a circle, introducing ourselves to one another, and sharing a word or phrase for trust and/or respect.

unconditional, opening space for possibility, safety to risk, challenge power structures, discipline competence, can't break faith, comfort, honouring dignity, exposure, earned, valuing positions, acknowledging other, valuing other, believe, ethic of care, openness to new ideas, dignity of self/other, pedagogical relationships, safety, space for other, being reliable/available, decentering self, learning to listen, confidence, relational, deep listening, risk taking

We have coupled the words "trust" and "respect" without wondering why they are together. Perhaps there is value in differentiating between the two words, perhaps even defining the two words as separate concepts for consideration, so that we can unpack why they are coupled together

The words are illusive and murky … but are dripping with context and meaning

Sharing Secrets

We moved into small groups of three or four to learn more about each other. Given three cards each, we were asked to select and secretly write something about ourselves on each card–something that other group members would not likely know. Keeping the messages hidden from others, the cards were collected and shuffled. Each group member selected a card and read it out loud to the group. The task was to decide, as a group, who had written the message.

I decided to write three secrets that reflected three levels of deepness, and to offer them in order of deepness. My first secret was easy to offer even to a complete stranger–I felt no risk in offering it. The second secret told more about myself, and certainly might influence how *others perceived me–I felt a small risk in offering it to people I had just met. The third secret was deep and personal; it was an admitting of weakness concerning my teaching practice–I felt great risk in sharing it. I was surprised when the game did not allow me to control the revealing of my secrets, but found comfort in the outward reactions of the other group members when the most deepest of my secrets was revealed.*

The activity was intended to be an introductory "get to know each other" kind of task. But it also served to reveal the sometimes hidden assumptions and biases we hold. Some group members mentioned being surprised and uncomfortable that they did not get to read their own writing to the group. Others reflected on stereotypes, such as matching 'I like fishing' with a male member of the group or 'I have a cantankerous old cat' to a female member. Many were curious about their group members, having never met them before. Some took the exercise as a game and tried to guess, through handwriting and process of elimination, which person belonged to each slip. As we came together in a large group again, we discussed how we had decided what to write and what the decision, as well as the process, had to do with respect and trust.

- Deciding what to write depended on who was in the group and the possible value judgments that might be made with what was shared. Offering a secret and publicly owning it involves trusting the group members to receive it with care.
- Deciding who wrote the secret message revealed assumptions and biases that might prevent us from really getting to know each other. *Are we aware of the way in which our own beliefs and life experiences influence our realities?*
- Deciding what to do with what was offered varied among groups. *Is it a chance to learn more about each other or a task we need to complete?*

Stories of Trust and Respect as Teacher Educators

We re-arranged the circle to chairs around small table groups, providing a space for us to write and share stories of trust and respect in our work from or with pre-service teachers. At the small table groups, we share stories and begin to wonder how to develop respect and trust with preservice teachers. And, what is it about teaching and learning mathematics that makes developing trust and respect challenging (or not)?

The whisperers

> My classroom is similar to this conference space. There are seven hexagonal tables. The pre-service teachers sit in groups for much of our 'Communication through mathematics education' class. And there is one group of students, one table, that is a bit noisy. They are the whisperers.
>
> During whole group discussions, they elbow one another and speak in low voices. I know that they are helping one another to understand and asking for clarification because I have eavesdropped and know the expressions on their faces, the bend of their heads. But they disturb other students who are trying to follow the larger conversation. Those other students are vocal: asking them to quiet down and to show professional courtesy. All eyes turn toward me and I see the hurt and flushed cheeks of the whisperers.

Circle talk–Wisdom circle

> I told the students (pre-service teachers) of how aboriginal peoples often used circle as a means of building trust and making decisions. I related how I had used Circle in my work with teachers across the Pacific. I laid out some ground rules for working in circle: (1) only the person holding the talking stick was allowed to speak, all others had to listen, (2) what was said in circle stayed in the circle, (3) when the stick was passed to you, the option was to speak or to pass it along with no negative judgments being made if one

chose not to speak, and (4) one was to speak from one's own center, not trying to challenge or correct or denigrate what had been said previously.

The Circle and the talking stick seemed to touch a chord in most class members. Very personal items were shared, and the group grew in their appreciation and compassion for each other. After the first session, I promised that we'd have circle at least twice more, once in the middle of the term, and during the final class, and we did just that. The Circle conveyed a sense of trust and respect for what students said by providing a non-judgmental situation where student's inner most thoughts and feelings could be revealed to others. This approach–sharing in circle, along with sharing of course evaluation, and sharing the students' school experience–had a very powerful impact on the students. And it turned my teaching around–it connected me to the students and them to me, and that was good.

Trusting selves

Through an interactive activity in a small group setting, students concluded that a square was also a rectangle. Despite having drawn the conclusions themselves, they were reluctant to trust their own findings and concluded that the activity itself, as constructed by instructor, may have drawn them to false conclusions. Students were asked to further research the persisting question, is a square also a triangle, on the internet at which time they concluded that indeed this was true. This story highlights the mistrust pre-service teachers have of their own content knowledge, and potentially the content knowledge of another. *How do pre-service teachers come to trust their own and another's content knowledge?*

Trusting others

The pre-service teachers in this story had a mindset that prevented them from contemplating the potential of new and alternative methods of teaching being explored within the pre-service methods course. The individual sharing this story felt that trust had to be gained from the pre-service teachers in order to enable them to venture into unfamiliar teaching and learning experiences. One factor that was important in gaining the trust was the instructor's authenticity as a classroom teacher, who *had* practiced alternative ways of teaching and could draw from personal experience and success. *How do pre-service teachers come to trust unfamiliar pedagogy?*

Intervening

I hover around the tables, listening to snippets of conversation. The discussions are lively and require little of me. It is in reflecting during this lull in the session that I realize that the story I shared and the story I am living are about respect and trust in general, but more particularly the two stories are about intervening.

Intervening has a number of definitions, but I think the meanings 'getting involved' and 'interfering' are the two that help me understand respect and trust. In the classroom as well as in the working group, there are situations that seem to demand intervention or my involvement. Action is expected of me (and I expect it of myself). Fulfilling expectations (of teaching, of leading a session, of participating in a working group) is one way to develop trust with another person, though I realize that living simply to fulfill another's desires would show a lack of respect for myself.

There are other times in my work with people that I feel a space open for intervening, but I am careful and cautious. I do not want to interfere in a way that would break trust or show disrespect. Instead, I choose to listen, but as I sit next to a table suddenly surrounded with laughter and smiling faces glance in my direction, I wonder if I can find respectful and trusting ways to interfere.

Angst

I think the most common theme of the stories we told was angst. This is the emotion I felt as people told their stories, even the ones with "happy endings." Why do we experience such uncertainty and angst? Do these stories suggest a lack of trust for the other? I really appreciated the comment that you need not earn my respect, I respect you (my interpretation–I respect you because you are human?) but you must earn my trust, I do not trust you. I need to think about this more.

Silence

After everyone in our group had shared their writing, there was a sustained silence. I imagine that the reason for remaining silent was different for each person, and yet there was group consensus to remain silent for a short time. What assumptions might I make about this silence, given that such a silence had not previously happened during the working group? Was this silence an expression of a respect that had developed, much like a minute every year at the 11th minute of the11th hour of the 11th day of the 11th month? Was this silence a respect for, trust in, or coercion by words spoken by another? Had all the people in the group who were usually quick to speak, like me, suddenly decided not to speak?

Questions

The stories shared within our group are all different. The first story illuminated trust issues around technology use in the classroom–a story that may well resonate with many instructors as institutions increasingly move towards encouraging the use of laptops in instructional settings. The second story brought attention to the issues of trust associated with content knowledge, while the third story highlighted issues related pedagogical knowledge. The final story brought voice to pre-service students. The stories, although different, are likely reminiscent of similar stories from others in Faculties of Education. In summary we conclude again with the three persistent questions arising from these stories: How do we come to trust the use of technology in pre-service teacher education? How do pre-service teachers come to trust unfamiliar pedagogy? How can pre-service teacher educators come to trust the lived experiences of pre-service teachers?

Zoom: Trust and Respect in Working With Practing Teachers

Our second day begins with a request from a group leader to slow things down and hold the space in such a way that we could each be aware of our assumptions through listening for what disturbed or surprised us (Wheatley, 2005), and to really respect the contributions of each person.

The next activity focused around excerpts from Istvan Banyai's (1995) book titled *Zoom*. This is a wordless picture book that tells a sequential story with pictures embedded within pictures. Sitting in a large whole group circle each person received one page from the book and was asked to study only that page and to keep their page hidden from the view of others. The task for the group was to sequence the pictures to tell a story without revealing their picture to others or looking at another's picture.

It doesn't take long for someone to make a suggestion as to how we might start and my colleagues rise from their seats. I say something about a cruise ship and very quickly find myself among a couple of people who also have a cruise ship featured in some way in their picture. We are trying to make sense of how what appears to be an advertisement for a cruise is related to other scenes that seem to be of a cruise ship.

People group themselves with others whose pictures seem to be connected to their own. But it is clear that some are not able to connect themselves and their pictures to any group. And some are not as certain what their pictures are pictures of in order to decide which group to join.

Someone in the larger group calls for our attention. We sit down and try to organize again. There is some sense that at least some of us are satisfied with the progress we are beginning to make but others express some frustration, or at least confusion. We once again look for connections but this time we are opening up to find more connections. I personally feel as though the task is going very well. There is a sense of something grander emerging among the participants. The "story" is taking shape.

Someone calls for us to take our seats to reflect the order we have come to. I sit with the few people who have pictures that fit with mine. I am very confident, dare I say certain?, that we have worked out at least some local coherence and I believe that if everyone else in the room has local coherence we will be very close to a resolution to the problem posed to us by the working group leader. Indeed, I am so certain that if people have local coherence we will have a solution that I say so out loud to the whole group. I think there are others in the group who understand my argument: we are locally coherent then by transitivity we will have global coherence. But there are others who do not seem to either understand or trust my reasoning. "Trust me" I think to myself. "Just do it. You'll see, it'll work."

The idea of zooming is offered. But there is disagreement whether the pictures are zooming in or zooming out and where the story might start. The more people get involved in the activity of trying to solve the problem correctly, the less they seemed to listen to each other.

Respecting task–Respecting relationship

> It was not until the reflections from the group emerged that I found some things that delighted me, bothered me, and interrupted what I take for granted. I was bothered by my abrupt response to another who asked a question for more information before I was able to take my turn in the story telling. I was not surprised that the story, more or less, worked out. And I was delighted that the local coherence strategy worked. (Especially because it is such a strong personal interpretation of how things generally work in the world.) I was taken by the comment about respecting the task and respecting the relationship (or person) and found myself bothered by my emerging awareness that I respect the task over the relationship. I have thought about this much more since the session. I know that I appreciate it and feel respected when students respond well to the tasks I offer them in my classes. I am deliberate about the tasks I offer and I need the students to take the tasks on wholeheartedly if I am to be a teacher to them (develop a relationship with them). I feel respected then when they respect my tasks. But I see a problem with my stance. It is illustrated by my abrupt response to the person who wanted more information while I revealed my place in the story. In respecting the task (insisting it not be interrupted even though another person clearly needed/desired more information) I did not respect the other. I did not make space for the other. This bothers me but I am appreciative of the gift of awareness.

Losing the path of respect

> I start thinking about the several moments where I had felt we had lost the path of respect and trust. And how many additional moments were there that I did not notice? How does one know that one is offending? It seems safe to assume that I am doing this all the time. It's presumably much easier to be aware of other people's lack of respect than it is of one's own. How do I keep the conversation open to maximise the possibility of my learning?

> It seems to me that maths teacher educators are faced with a challenging paradox. The traditional stress in doing mathematics has been on getting the solution as quickly and efficiently as possible. The traditional method of control in the classroom is to encourage competition as a means of keeping the class focus. In our teaching we encourage teachers to plan each lesson very carefully to make sure that topics are covered efficiently. However, a focus on the above seems to detract from the quality of listening and communication that are required to build respect and trust in the classroom.

How do we reconcile these conflicting demands? What do we need to stress and ignore form our traditional practices if we want to foreground respect and trust in our encounters with teachers of mathematics?

Nature Trust Walk: Working As Graduate Advisors

For this activity the senior researchers paired themselves with junior researchers. Once outside the junior researchers gathered to receive their instructions. The senior researchers waited with eyes closed. The junior researchers selected a senior researcher and without verbal communication lead the researcher following strict constraints such as: objects could be explored only with the left hand while walking started only with the right leg first. After a few minutes of guiding around stairs, trees, grass and pathways the junior researchers met again as a group leaving their partners to wait with eyes closed. Their instructions were now to guide, still nonverbally, in a more exploratory way with less restriction but increased awareness of the interests and moves of the senior researcher.

One of the important realizations this experience reinforced was that once I had tried something, I did it with more confidence. For instance, when I was taken down some steps I proceeded haltingly, resting both feet on each step. When we turned around to come back up the same steps, I walked with speed and confidence, having counted the number of steps on the way down.

On the second round I was encouraged to shake hands with a guided colleague, and took the liberty of touching her arm and necklace, exploring until I thought I knew whom I had met.

These affective experiences translate quit literally to the research experience, in terms of the relationship between graduate student and research director. For example, reliance on the guide is significant: he or she determines the path we take, just as the research director does. However, the charge is most satisfied if free exploration is allowed along the way. As the charge gains experience, familiarity with the process, and confidence in his/her ability to cope, he/she is emboldened, and becomes more effective at exploring. Use of new skills imparts confidence and the motivation to explore more possibilities. The increased motivation to try new things as one becomes more experienced and confident translates to the experience of the teacher who is attempting to implement new practices (e.g. reform methods in mathematics). It takes time and experience to gain competence and confidence, so the expectation that new teachers implement reform methods precipitately and immediately may not be realistic. They must be allowed the time and experience necessary to build their skills and confidence.

During the first walk, I did lots of thinking about trust and respect. The activity was obviously about trust, and I could easily develop parallels with advisor-student relationships. At times, I needed to work at trusting my leader. My leader commented after the first walk that I was easy to lead.

Perhaps these assumptions can be paralleled to the norms of practice for a relationship between advisor-as-expert and student-as-novice.

During the second walk, I continued to assume that I was being lead by my partner. I found ways to see my partner's actions as leading but now providing me with some freedom to explore. I was surprised to hear that I was supposed to lead.

At the start of the third session, we faced the challenge of noticing moments of lack of respect present in our own interactions during the Zoom activity. Then we faced a challenging image, a dream experienced by a group leader the previous night.

Dream

I am South African political leader in hiding at a friend's house. I am playing with some children who seem to know who I am. Suddenly there is a noise and there are a whole lot of police on motor-bikes outside and the house seems to be surrounded. We run. I leap over garden fences but then trip and am captured…

I come to in captivity were comrades are being beaten up by the police. I am very drowsy. There has been an unexpected catastrophe with many people having been killed. It turns out that the police had fired bullets containing a drugged serum that had previously been tested on animals. They had got the dose all wrong and the overdose had killed many.

I am at police briefing on the latest police action. The Brigadier looks at me and reminds me that I am not allowed to speak or make comments on the situation. I accepted this. The people leave and the two of us are left together. Something happens that I find humorous. I tell this to the Brigadier and he starts laughing too. I say to him "You see, even communists have a sense of humour" and we walk off together in the same direction.

I am completing this on the day after the former President of apartheid South Africa, P.W.Botha was buried. This is the same P.W.Botha, who ruled between 1978 and 1989, yet refused to testify to the TRC. Despite this, the TRC found PW Botha responsible for gross human rights violations, including all violence sanctioned by the State Security Council, or SSC, an executive organ of his apartheid regime. "By virtue of his position as head of state and chairperson of the State Security Council (SSC), Botha contributed to and facilitated a climate in which ... gross violations of human rights did occur, and as such is accountable for such violations," the report said. (BBC News Special Report 10/98).

The funeral of PW Botha, the last South African leader to staunchly defend the apartheid system, has taken place with President Thabo Mbeki in attendance…. At the service, Mr Mbeki and his wife, Zanele, sat alongside the last white president of South Africa, FW De Klerk who oversaw apartheid's dismantling. … President Mbeki said on Tuesday that a balanced appraisal was needed of Mr Botha's life "to promote national reconciliation"… Mr Mbeki's eldest son, Kwanda, is believed to have been killed by agents of the apartheid government under Mr Botha. (BBC News 8 November 2006).

> I am very aware that neither the dream nor the funereal reality will allow me to consider the terms of respect and trust to be interchangeable. I can allow respect to be present in both, but trust seems to be a very long distance away!

I found the Nature Walk experience profound and I must thank my guide—she was wonderful. I wrote the piece below when we went back to the room.

When we were asked to share what we wrote with each other in the small group, I found myself moved by thinking about how much my guides have taught me and couldn't speak. Another person read the piece for me.

I wish to be attentive to the ways I am in the world.

I am blind but not without light.

I have my guide–my mentors;
I look them in the eye and see possibilities, recognize implications.
They walk the halls with me;
their lessons echo in my minds ear when I read, when I listen, when I am still.

I extend my arms and stretch my fingers into the world.

I worry the world is void,
there is nothing,
not that there is danger but that there is nothing.
What could be more dangerous than there being nothing in the world for me.

Ah, it is not void at all; I feel the sun.

Is this what the pre-service teacher experiences? Is this what my students fear?
Are they reaching out grasping for something and panicking that they will leave with nothing?

What can I do to make a difference?

What do I touch when I extend my arm and reach out my fingers?

Summary

In this working group a series of activities over the three days gave us shared experiences to learn more about each other and the nature of trust and respect. From sharing secrets or mysteries about ourselves in small groups (revealing how our own beliefs and experiences frame what we share, how we listen, and our presence with one another), to writing and sharing stories of trust and mistrust (sharing our success and failures; thus being vulnerable in the presence of others), to collectively re-creating a story from a set of sequential pictures as a way of learning more about communication, perspective taking, collaborating, listening, and valuing others' contributions, to a nature trust walk as space for learning about ourselves and how we might lead or be lead. These activities were invitations to learn about our own actions, how they were respectful or trusting, disrespectful and mistrusting, and how they might be related to working in mathematics education.

We also attempted to explore actions around what we needed to do to develop trust and respect in working with mathematics teachers within the three contexts: work involving pre-service teachers, practicing teachers, and graduate students. For example, some individuals formed a group to discuss their future actions in working with practicing teachers or in-service teachers collectively decided to:

- Take up ethics more fully in our own work
- Watch assumptions when entering the space of working with others
- Subject ourselves to the same or similar processes as those we work with in research relationships
- Expose the researcher-self as learner
- Consider the value of multi-voices and multi-visions

As part of our purpose for being together we consciously agreed on listening better–to not just share experiences but to try to dwell with what was offered and be willing to be disturbed. We were not always very good at this and we needed to remind each other when we were disrespectful–and of instances of disrespect, as well as instances of respect.

There were moments of emotional intensity that provided opportunities to learn more about each other and ourselves. We noticed that over the three days a space was created where we could risk vulnerability, share personal insights, and reveal assumptions.

Listening requires taking on or seeing other perspectives. This however can be difficult to do. It is comfortable and easy to align ourselves with others who think like us–certainly more difficult, but more rewarding to listen to those whose perspective is different. It is also difficult if we are not aware that another frame or perspective is available. With attention on developing trust and respect possibilities may arise that can lead to new insights, awareness, and ways of being.

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The Body, the Senses and Mathematics Learning
Le corps, les sens et l'apprentissage des mathématiques

Susan Gerofsky, *University of British Columbia*
Patricia Marchand, *Université du Québec à Rimouski*

Participants

Aldona Businskas	Amy Hackenberg	Anna Dutfield
Annie Corriveau	Barbara Graves	Bijoy Hangmo Subba
Carlos Zerpa	Dave Hewitt	Dave Wagner
Dragana Martinovic	Edward Doolittle	Elena Halmaghi
Eli Brettler	George Gadanidis	Geri Lorway
Gladys Sterenberg	Grace Orzech	Irene Percival
Jo Towers	Kate Mackrell	Katie Northcott Susan
London McNab	Kerry Handscomb	Laurinda Brown
Lily Moshe	Margaret Sinclair	Nathalie Sinclair
Nicola Benton	Olive Chapman	Pamela Hagen
Pat Margerm	Petra Menz	Stephen Campbell
Steven Khan	Kathy Kubota-Zarivnij	

Introduction

In this working group we intended to go beyond manipulatives to consider the link between concrete and abstract action in terms of broader questions of embodiment and the effectiveness of sensory learning in mathematics. We considered justifications for embodied, concrete actions including:

- the idea of *multiple entry points* to mathematics learning through the senses;
- using multimodal, multisensory means to offer *a variety of mathematically equivalent representations* of algebraic relations;
- various embodiments of mathematical relationships as a *stage in concept development* (moving from concrete to abstract);
- embodied representations as a way to *materialize the mental construction* of a concept (moving from abstract to concrete);
- multisensory approaches as a way to serve students with *varied learning styles*;
- as an entry point to the *aesthetics of mathematics*; and
- as a way to access *different types of mental "visualization", manipulation, and memory* (of visual, auditory, tangible, kinesthetic and symbolic objects).

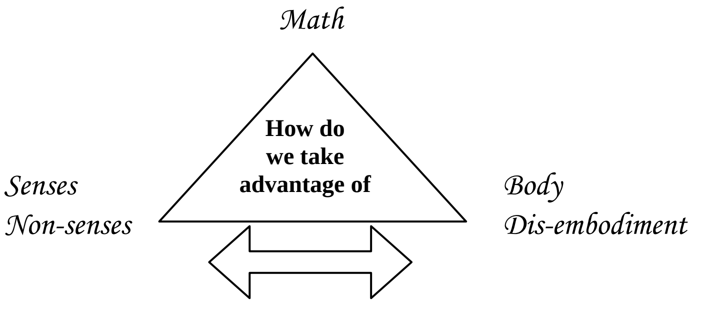
We worked to find a balance of shared experiential activities and discussion of theoretical issues grounded in those activities throughout the three days of our working group. This report will outline aspects of both the activities and our discussions.

Day 1: Expectations, visualization & kinesthetic linear graphs

On the first day we wanted to develop a degree of shared understanding of our working group's topic, in an area that is relatively new in the field of mathematics education, and to build relationships within our large group. To discuss the group members' expectations, we asked participants in small groups to produce a poster with their answers to two questions. Here are some answers from the group:

Question 1: *If you could choose one question for the working group to address, what would it be?*

- We may value diversity and learning styles, but do ideas of embodiment offer a common ground?
- How do I assess so that I can formalize? How do I increase awareness of connections between body and mathematics to make them accessible?
- How do visualization and physical activity form bases for mathematics learning? How can they be supported, so that students' abilities in visualization and physical activities are developed rather than suppressed in a variety of mathematical areas (and not just geometry)?
- What are the relationships between visualization and mathematics understandings?
- How can we bring the senses back into mathematics?



Question 2: *Do you see a link between the body, the senses and mathematics learning? If so, can you give a concrete example?*

- Counting numbers with our hands.
- Transformations: rotation, reflection and translation.
- Creating 3D models.
- Fractions: Slicing bars.
- The use of the body to create different shapes and solids.
- Balance and riding a bike.
- Solving equations.
- Right-left.

Emergent themes in our working group included a desire to work on the relationship between the senses and mathematical understanding through questions of educating awareness, development (rather than suppression) of bodily way of knowing in mathematics, and ways to access formalized mathematics and assessment in such a context.

Some initial ideas and questions paraphrased from our group discussion at this point included the following:

- Would differently-abled learners or those with very diverse bodily experiences develop or experience a different mathematics?
- There are different ways of visualizing related to conceptualization. For example, kids may not "see" or conceive of numbers on a number line, in which case teachers' gestures referring to a number line may not make sense.
- How to make gestures, embodied concepts accessible so we can work on them?
- How can we bring the senses back into mathematics? How to bring the abstract down to earth, into our lived experience? We should not only consider the visual, but multisensory experiences.
- Can mathematical sensory experiences involve hearing as well as (or instead of) vision?
- Music: is it linear?
- Small children work on balance, experiencing "up/ down" and repetition, and constantly question "why". Children are mathematical from earliest age.
- We experience velocity and acceleration as forces through our bodies.
- Our concept of geometric rotation comes from our experiences of body rotation.
- When kids "form a circle" as part of school activities, we can draw attention to experiencing the circular.
- We can draw kids' attention to seeing/ sensing math outside the classroom.
- Thinking about embodied experiences in ice skating: Skaters experience lines, circles, spirals. Taking a body slide on ice is an experience of the plane.
- Balancing on a moving bus offers an experience of a triangle.
- Small-scale physical movements offer potential mathematical experiences as well. For example, embroidery and stitching can offer experiences of linearity and circularity.
- We take issue with "abstract" as a category – the highest levels of abstract math use visualization tools like diagrams, arrows and other visuals.

On the first day we engaged in two activities designed to create some shared experiences of visualization, memory and manipulation of mental objects.

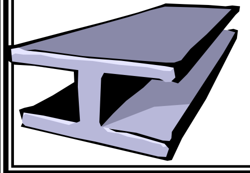
Visualization introduction

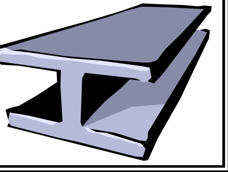
Before exploring activities involving mental visualization, the following statements were presented to the group:

⇒ Visualization is accessible to everybody.
⇒ Visualization is a very powerful mental tool.
⇒ Visualization can be learned and controlled.

Our first shared activity related to the second statement, on the power of visualization. Teams of three were formed; in each team, a static person had to hold their arm straight and the two others had to try and bend the arm of the static person at elbow level. After a first try, the static person had to read a textual description on a handout card, kept secret from the others in the group, before trying again. Once again, two members of the group had to try again to bend the arm of the static person and observe if there was a difference between the two attempts. The group reaction was immediate; in almost every case, people felt a significant difference, and noticed that the arm of the static person could not be bent, no matter how much force was applied the second time. Here is the description on the handout card read between the two trials:

Please, don't tell anyone about what is on this card!
Now, for the second time, you have to visualize that your arm is an **IRON BAR** and that it cannot be bent by anything. Concentrate and tell the others when you are ready to start. Think hard!!





A few participants had tried similar experiments before, but for most of the group, this was a new and revealing experience. This activity is a typical one used in sports visualization training to introduce young athletes to the power of their mental abilities in affecting their physical performance. The next step, for athletes, is to understand that they can learn to control this mental power. In our working group, we next tried a mathematical activity that relies on mental visualization abilities.

Tangram activity

The Tangram is a manipulative commonly used in mathematics learning, but Wheatley (1990) uses it in an uncommon way. He presents an image constructed from pieces of the puzzle for only three seconds, and then pupils have the task of reproducing the shape from their own Tangram pieces. Wheatley's research showed that small activities like this one can help pupils to develop a better spatial sense which can later be used in mathematical tasks. In our first day's session, the group experienced this new activity for the first time with three initial figures. After the activity we discussed the strategies people used to memorize the figures and their mental reference points.

Figure 1

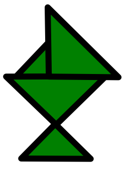


Figure 2

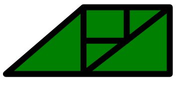
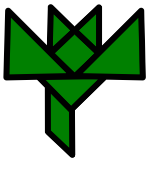


Figure 3



Most members of our group had an initial reaction similar to that of school pupils and preservice teachers that Patricia had worked with earlier: participants thought that they had the image firmly in mind while looking at it, but the mental image seemed to vanish when the three seconds of viewing ended, and it was very difficult to reproduce the shape with tangram pieces. This observation raised questions about the importance of the ways we use vision and our awareness of visual mental images. We noticed that demands on our modes of visualization may differ according to the task, and that it might be helpful to develop a repertoire of ways to visualize images.

In terms of strategies, some people in the group saw part of the image in their head, but not the global image; others saw the global image but not the details of the figure. These two strategies rely on different part of the brain according to neuroscience research (Houdé, 2004): the global vision is part of the function of the right hemisphere of the brain and the partial or local vision is a function of the left hemisphere. The group references were mostly to geometric figures in two dimensions (triangle, square, trapezium…) and a few had three dimensional references (boat, flower, bird …) depending on the figure presented. Research findings show that majority of elementary school pupils use three dimensional objects from their environment to build their mental images (Wheatley, 1990).

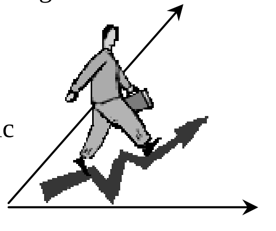
We revisited and further developed this visualization activity on Day 2 of the working group.

Here are some paraphrased excerpts from our group discussion following the visualization exercises*:*

- What do we mean by visualization? Is it like a TV screen in your head? Is it like a dream? What is imagining? Are images just visual, or physical? If I imagine it is cold, it gives me goose bumps. I feel the cold rather than seeing it.
- Images are not necessarily visual pictures.
- What about the colours of music?
- How do people who are visually handicapped imagine things? Are there tactile imaginings?
- Imagining can involve feeling in the emotional sense as well.
- Even in mathematical visualizations, emotional reactions come into play. Some students might not "like" a parallelogram because it seems skewed, so they might look at a parallelogram and describe a rectangle instead, ignoring the evidence of their senses.
- Some might not like to recognize a diamond-shape as a square because of its orientation on one point. A square in diamond position is less stable – not so satisfying.
- Are there better and worse images? Do we want to replace one image with another in class? Why or why not?
- Can we control our own imagery?
- Why work so hard to control things? What if the world doesn't cooperate? We have the opportunity to live in moment & be attentive. Many of us have seen new teachers imagining student reactions when planning lessons, and being very confused when students react differently.
- Perhaps it would be more productive imagining oneself to be a better teacher.
- Aboriginal cultural traditions use imagery, but see it coming from outside. The important thing is to be open to all possible images.

Making sense of linear graphs

We tried the following activity, designed to give a kinesthetic experience of linear relationships in two variables:



Two long intersecting perpendicular lines of masking tape were set up on the floor and labeled as the *x*- and *y*- axes. A chair with arms was set at the origin (the intersection of the two tape lines). Two volunteers came up and held onto the back or arm of the chair; one was facing in the positive direction along the *y*- axis, the other in the positive direction along the *x*-axis. The two volunteers were instructed to push equally hard, one in the horizontal direction, the other in the vertical direction on the count of three. Those observing were asked to notice the path the chair took. The activity was articulated as an embodied way of experiencing $y = x$, as the force in the y direction was equal to the force in the x direction. (The set-up could perhaps be read as the resolution of two equal orthogonal force vectors as a diagonal force vector.) Then we worked on "doing" $y=2x$ and $y = x + 2$ through similar chair-pushing activities.

Here are the questions that the group was asked to focus on during this activity:

1. Can linear functions be represented in a kinesthetic way? In a sensory way?

2. What new understandings could come from "reading the graph with your body"? Is anything lost in this process?
3. How could multisensory representations interact with other representations of linear functions? Is a deeper understanding achieved?

Some notes on this very controversial activity:

> ***y =x*** *was easy for the group, but some reported that they already know where it would go and that influenced their body movement. The case of* ***y = 2x*** *was a key moment here: the group proposed having two people push the chair in one direction and one in the other, and there was a lot of discussion for this example. Do we have to put two persons for the or for the y? What is the impact? If for each y, I have two x, does it mean that I have twice as many people pushing in the y direction (vertical), or that x (the horizontal) is twice as strong? Difficulties in communicating and explaining this simple linear relationship recalled some of the difficulties involved for student learners working with concepts and notations that are not transparent for beginners.*
>
> *For* ***y = x+2****, we talked about ways to express the linear relationship in everyday, non-technical language: «with an extra two at the end», «with two on top». Do we say (and think of) the extra two as coming in at the beginning or at the end? Is it added each time or only at the beginning when we are moving on the grid?*
>
> *There was also a distinction in the discussion between static embodiment (using coordinates of ordered pairs) and dynamic movement (physically tracing the graph. How can we connect the two? Is movement useful here? The group then discussed whether we should start by showing pupils coordinates or directions; the group seemed to lean towards teaching the relation between the two axes and do the coordinates afterwards. The issue of the parametric representation of time was also discussed, since both the x and y forces in the activity moved in relation to a constant time scale. A question was raised: Can a «kinetic» approach allow a better understanding of the concept of time involved in the creation of the graphic? Can this kinesthetic approach provide a useful representation of y = x, since time is always implicated as a parameter in the equation?*

Finally, some variations to the activity were proposed:

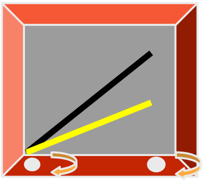
- creating a static grid of points with the whole group, with each person standing on the grid at a point that occurs on the table of values for the linear function.
- representing a particular point on the linear function using two people, one representing the x-value and one the y-value of the ordered pair. So, for example, if the desired linear function were $y = 3x$, if the "x-person" stood at 2 on the x-axis, the "y-person" would have to stand at 6 on the y-axis. Then the two people would walk towards one another along the grid lines on the floor (the lines of floor tiles), and meet at the point (2,6), which satisfies the equation.
- changing the placement of the axis to represent a vertical translation of the equation. So, for example, once the class had represented $y = x$ in one of the ways described, the x-axis could be shifted down two units to represent $y = x + 2$.

It was noted that static representations lost the sense of the x and y elements interacting dynamically over time, and gained an independence from notions of a constant time scale or a connection to physical forces and vectors.

Homework readings, Day One - Dickens (1996), Nuñez (2004), Schiralli & Sinclair (2003)

Day 2: Graphs and Gestures, Visualization in Sports & the Body

Linear functions with the Etch-a-sketch



Following up on our first day's work on kinesthetic linear functions, we engaged in an activity using Etch-a-sketch drawing toys to explore linear and other functions. The Etch-a-sketch cursor leaves a trace on the screen, and is operated by two knobs, a horizontal and a vertical control.

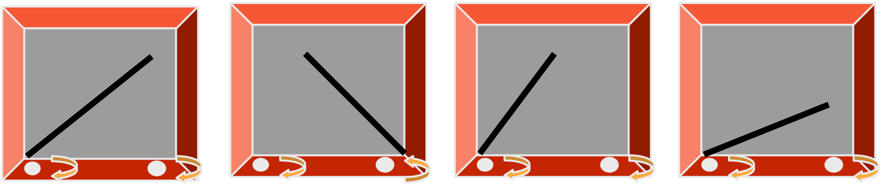
Working in small groups, participants first tried working with the following sets of instructions, which produced images of straight lines of varying slopes:

a) «Turn the knobs at the same rate, same direction»
b) «turn the knob at the same rate in opposite direction»
c) «turn y knob two turns for every one turn of the x knob»
c) «turn x knob two turns for every one turn of the y knob»

We discussed whether or not these kinesthetic/ visual representations could be said to correspond with:

a) $y = x$
b) $y = -x$
c) $y = 2x$
d) $x = 2y$

(a) (b) (c) (d)



After this brief introduction to the possibilities of the Etch-a-sketch as a manipulative for representing functions in two variables, participants were offered the chance to experiment with extending these ideas. Starting suggestions included:

- working in pairs, coordinating the actions of two people, where each is controlling one of the two knobs;
- contrasting the effect of working with hand-eye coordination with the effect of working "blind", covering up the screen and using only the kinesthetic mode;
- working on other non-linear functions and relations.
- some very interesting ideas emerged from these open-ended explorations; and in fact, we came to the conclusion that the Etch-a-sketch was an ideal manipulative for learning calculus! Entrepreneurially literate members of the group suggested that Susan contact the Ohio Arts Company, manufacturers of the Etch-a-sketch with ideas to market the toy to math educators (www.etch-sketch.com).

Notes from the discussion generated by this activity:

It was more difficult to draw shapes by coordinating the actions of two people but this "decoupling" of the left and right hand movements had the advantage of allowing people to focus on the relationship between the horizontal and vertical elements of a function in terms of both speed and direction.

Teams found it interesting to try and go back on their tracks (backwards), to retrace the function in reverse. A particular team decided to try and understand the functioning of the Etch-a-sketch by blacking it entirely and watching what would happen afterwards.

Some people tried to draw their function with their eyes closed and for a large majority, it was easier. Others felt the need to close their eyes very early in the task as it allowed them to internalize and formalize their action. There an interesting discussion that compared drawing on the Etch-a-sketch with driving a car along a winding road. If a person or pair watched the screen while drawing, they would treat the desired drawing like the winding road, and (unconsciously) steer the two knobs "by eye" to produce the desired product. On the other hand, if they worked blind (with eyes closed or the screen covered up), the drawers were able to focus on the tactile, kinesthetic sensations connected with turning the knobs, and much more interesting mathematical connections and "insights" resulted.

One of the interesting things that happened in this activity was the creation of other functions by the participants; this activity permitted a lot of creation by some teams. They felt like they were playing and could do what they wanted. It could open to more complicated mathematics (for example, sinusoidal functions were particularly interesting). One team wanted to draw a circle, but did not know how to turn the knobs, so they drew a diamond shape to observe the movement involved, then "bulged the sides out" to draw a circle. This activity allows better motor functions, «motricité», and representation of the function and we can also do the steps moving the knobs in a sequential way.

The group then treated questions like: Do we act before we think, or the opposite? Where is the embodiment? Where is the" magic"? (Mathematics is not the perfect representation of the reality, but an abstraction…)

Visualization practice

We continued by doing a visualization practice with the activity presented the first day with the Tangram and three new figures:

Figure 4

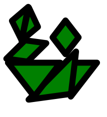


Figure 5

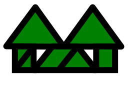


Figure 6

With these new figures, the group started to think about the factors that make a figure easier or harder then another one. So we discussed, what we call in French «les variables didactiques», and these are the factors that the group extracted:

- The numbers of pieces involved (figure 1 vs. figure 3).
- The orientation of the pieces (usual or not, similar to one another or not: figure 2 vs. figure 4).
- The symmetry of the figure (figure 4 vs. figure 5).
- The general form of the figure (knowledge of it and denseness or spacing out: figure 2 vs. 3).
- The presence of holes in the figure (this can change the perspective: figure 2 vs. figure 5).
- The positioning of a piece in relation to another one (vertex to vertex or not: figure 2 vs. figure 6).

We also discussed whether to retain the time limit of three seconds to look at the figure. We experimented with observing the figure without a given time limit before looking away and constructing it. Some (but not all) of the group found this process easier or less stressful. The time limit offered a specific constraint for this activity and changing it would change the goals of the activity. Teachers often use time restrictions for mental calculation activities We know that changing the time factor also changes the strategies used to solve the calculation.

A final variation proposed for this activity was to extend it to three-dimensional objects, by showing a three dimensional construction to pupils for three seconds and then putting a box over it. Pupils would have 3-D construction pieces to work with.

Presentation of a video on sports visualization and the possible transfer to mathematics classrooms

We watched and analyzed a video of figure skaters in practice sessions. The goal here was to present beginners and more advanced skaters and observe the tools developed to master their technical skills. In the beginning, skaters try to apply their technique in the learning of new jumps, but when they get to double jumps they cannot rely on their vision anymore. They have to develop other skills, learn to internalize their action by «walking» all of the position of the jumps on the ice, visualizing themselves doing the jump and correcting what they see in their head, reacting in action, and comparing what they see on a video of themselves and in their head. We had originally intended to look for parallels with the mathematics classroom, in linking concrete and abstract actions, but the ensuing discussion took some unexpected and interesting turns. Here are some elements of the discussion:

- Observing skaters who fell, got up and tried again, a question came up about the perception of error. In skating, the first thing that we learn is to fall and to get up again. The «falling» is seen as a useful concrete manifestation of what went wrong, and we observe, analyze and build on it to correct each element individually day by day. In mathematics classrooms, in contrast, errors are often treated as undesirable. How could we translate the idea of "building learning from our falls" in the mathematics classroom?
- Another discussion took place on the definition of mental visualization. Some participants saw mental visualization as something unique, ideal and rigid and others saw it as a process and a tool. From a sports point of view, «imagerie mentale» is the process by which skaters learn to create, manipulate and transform abstract actions, not only visual images but sounds, feelings, words, etc., to develop a better understanding of their body position and tend towards a successful technique (PNCE, 1991).

Further paraphrases from the discussion:

- *In math, can we slow everything down in a way that we can't do in skating.*
- *As teachers, we can play with time. For example, I might want to use chanting to speed up time. Sometimes I might want to restrict time.*
- *Do skaters think of visualization or just do the moves with their bodies? Is thinking with the body? With the brain?*
- *One thing about learning – you're taking in more and seeing things. In effect things are slowing down. It's useful to work at it on a perceptual level, to help people see more, which slows experienced time. I call it slow space, which I experience, for example, in tennis playing. I don't think math is fast. For the kids it's fast if the teacher is doing something complicated.*
- *Mihaly Csikszentmihalyi theorized the idea of "flow – time slowed down through intense engagement".*
- *It's about being present in the moment, moment by moment tracking.*
- *It's about creating image, even though the actual movement is fast. In one jump, the skater has to feel heavy, but we don't see that. With another, she imagined she had a pole in her grasp, and she could do the sequence.*
- *Is visualization necessary in skating?*
- *Not all skaters visualize.*

- *There's a difference between acting as a spectator watching someone jumping, and feeling or seeing yourself jumping.*
- *Do skaters practice on the ground without ice (to control speed)? In learning to play piano it sometimes helps to practice without a keyboard.*
- *Fear of falling impedes skating. Is this similar in math?*
- *Skaters first of all practice falling. They are not afraid of falling. Sometimes if they don't fall, I ask them to fall. Questions like "which side did you fall on?" give useful information.*
- *If skaters know all the "pieces", can they focus on all of them? Or is skating a complex of embodied movement?*
- *It seems there is some channeling of "magic" ability – by putting the fingers together the skater is able to do a jump.*
- *Some mathematicians don't visualize, or see an equation in their head.*
- *Some skaters don't need to visualize because they "have" the knowledge in their bodies.*
- *Dancers who "have it in their bodies" may still be using imagery but may not be bringing it to consciousness, until they "hit a wall" and can't do something. They may be using implicit imagery that they might not be aware of. A teacher might help them become more aware of this to help them bring it to consciousness and use it.*

Homework reading: Whiteley (2004).

Day 3: Movement and mathematics learning

On this last day, we focused primarily on group discussions rather than activities. We began the third day's session with a brief presentation on the last activity of the skating video to gather ideas that had scattered the day before, and offered an (optional) opportunity to participate in a demonstration of a new haptic computer interface for mathematics learning. The group then split into small groups and chose a subject of interest that they wanted to discuss further, with groups reporting back at the end of the session.

«Wrap up» on visualization and sports

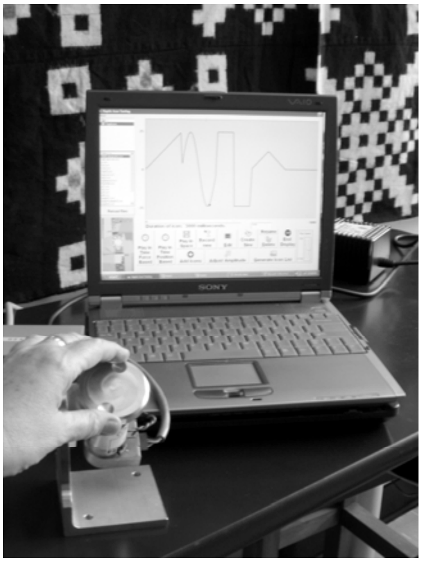
To summarize what we wanted to explore with the skating video, the context of this research was explained. In terms of spatial sense and visualization, Patricia had observed a marked contrast between teaching approaches in mathematics and skating. In mathematics education, curricula and textbooks lack the tools to teach visualization and spatial sense teaching. In sports, mental training has been a central focus of research for the last twenty years, and researchers have elaborated, tested and validated programs to develop mental skills including visualization, often involving isometric transformations (Marchand, 2006). In sports, it has been established that the most efficient link between concrete and abstract action was mental visualization (Orlick, 1990; Porter & Foster, 1990).

Here are some key elements from sports research, which could be used to think about in the learning of mathematics (from Marchand, 2006):

- Mental visualization can be learned, and images can be created and manipulated, so the teacher has the control of this learning.
- To develop mental visualization, concrete and abstract actions have to be in interaction, as seen in the skating video.
- It takes small activities to develop mental visualization (for example, the Tangram activities).

- In these activities, there has to be a key moment, where the student cannot rely on vision to solve the problem.
- Teachers have to make abstract actions and mental images explicit in their teaching by questioning the students about it, as a coach does in sport.

Demonstration available: haptic computer interface and functions



Susan made available a demonstration of “the Twiddler”, an inexpensive haptic interface that allows users to input a function or sequence of functions into an Excel spreadsheet, produce a graph, and then “feel their way” along the surface of the graph as if acting against gravity. Susan is working collaboratively with Karon MacLean and Mario Enriquez of Computer Science at the University of British Columbia to develop and test this and other haptic interfaces for use in school mathematics classrooms.

Susan also offered an adhoc session later in the CMESG meeting to introduce a topic related to our working group’s area of interest (but too long to include in the working group session). The adhoc session, on graphs and gestures, analyzed videotaped data of people describing the graphs of functions using only gestures and sounds. This pilot study found links between gestures involving the *x*- and *y*-axes and participants’ embodied sense of the vertical and horizontal; relationships between gestural speed and temporal interpretations of static graphs; and culturally-bound interpretations of left-right and up-down and imagined embodiments of the graphs.

Small groups framing and working on their areas of intense interest on body, senses and learning mathematics

To start the discussion, we listed several interests and questions that had come up during the first two days:

- Imagery versus visualization
- «Easy» vs. «hard» math
- Dynamic versus static representations
- Conscious versus unconscious knowing
- Bodily knowing
- Time
- Magic and the origins of imagery
- Is visualization necessary for mathematics?
- Is there a 1-to-1 matching between embodiment and formalization?

Each group chose a topic of intense interest as its focus. Three of these discussion groups sent us a written report of their discussions. Although these can only partially capture the richness of the contextualized discussion, and although other groups engaged in equally rich experiences, time constrains us to report on those discussion summaries which were submitted in writing.

Theme 1: Link between learning skating and learning mathematics

To explore the question broadly, part of the working group examined the situation linking these two contexts for learning by identifying seven aspects involved in each context:

Figure Skating learning	Mathematics learning
Special space: What is the effect of this contrast in the learning of spatial senses?	
➢ The rink represents a concrete and designated skating space where the roles of coaches and skaters are identified. ➢ Pupils learn skating in a «macro» space.	➢ Does such a space exist in our class for the learning of mathematics? ➢ They learn mathematics in a «micro» space (Berthelot & Salin, 1993-94).
Individualized learning: What would be the effect of a more individualized teaching approach in our classrooms? Without doing solely individual teaching, can we explore this avenue?	
➢ Here, we do not have any choice to start where the skater is and to build from there. The teaching is, consequently, more individualized. ➢ This gap might seem more significant, but in a framework of 2 or 3 years every skater is more or less at the same level, as in school.	➢ In math classrooms, whole group teaching implies a more equal progression (everyone is expected to arrive at more or less the same place at the same time). But is this realistic?
Age and skill range: What advantage can we get by mixing age and skill ranges in our classroom?	
➢ We have skaters of different levels and ages at on the ice at the same time and this gives them opportunities to see others performing the skills that the skater is trying to master and to develop mutual aid between skaters.	➢ We do not have a large range of age in the same classroom, but we often have diverse levels of expertise and we could benefit by creating more interactions between pupils.
Constant bodily performance: When do pupils perform mathematical activities in our classrooms?	
➢ Learners are constantly physically and mentally active in their learning process.	➢ Here, we are unsure about when and how «performance» actually take place? Can our pupils see and enjoy a mathematics performance? Can the joy of using mind, body and emotion be experienced in mathematics class, and if so, how?
Understanding: What tools the teachers uses to understand pupils reasoning?	
➢ The coach uses one tool that we don't often use in school: they frequently ask the skaters about their internal actions and emotions when trying and learning skills.	➢ In class, we concentrate on concrete actions like written solutions or explanations. But are these adequate to translate pupils' understandings and states of mind? Could we try to "get inside pupils' minds" to better understand their way of learning and to engage them in a reflection process?
Extra-curricular: How can mathematics become an extra-curricular activity?	
➢ Skating is an activity where pupils come to enjoy the sport and learn new tricks in a recreational setting for the most part. Parental involvement plays an important role.	➢ Can we create such an environment for mathematics? Can math club act in a similar way, where pupils could have a space to learn and enjoy mathematics in a more friendly and individual approach?
Learning to fail: Do we learn to fall down and get up in mathematics?	
➢ The first thing skaters learn is how to fall on the ice and how to get up from that situation. So, we provoke a fall and teach them the technique to get up. ➢ Error is viewed positively, giving information to the skaters on their abilities and used to construct and correct their skills.	➢ In the mathematics classroom, we lack similar experiences where the pupils would be in trouble and we teach them to overcome this obstacle. ➢ In mathematics, we do not often view errors in a positive way, as there seems to be less room for error.

The learning process in skating involves different phases: learning basic techniques (instruction, observation and repetition), trying new skills, coaches' interventions (including questions like: What do you feel? (body), and What do you see in your head? (mind)), analysis of the answers, adjustment, trying again, and so on, to repeat the cycle and gradually gain learning experience. Do we have a similar process in the construction of mathematics knowledge?

Theme 2: Visualization skills young children bring to primary school

The second group was interested in visualizing and image-making among elementary school students and ways in which to capitalize and build on what students bring to school (rather than allow it to become suppressed). The group raised following questions:

- What do children come to school with in terms of images and visualization? What research has been done in this area?
- What are differences and similarities between the ways that children and adults visualize and make images?
- What are the links between visualizing in the development of early literacy and early numeracy in young children?
- Some young children don't form images when they read (and therefore struggle to read). Are these children also perceptual counters (i.e., children who must have perceptual items in their visual field in order to count—they do not yet re-present perceptual items in visualized imagination so as to count hidden items; see Steffe, von Glasersfeld, Richards, and Cobb, 1983)?
- If we notice that 1st grade students like to combine shapes, and 3rd grade students like to decompose them, does that have any parallels in how young students construct number? What does that mean in terms of how they visualize and make images?
- Visual literacy…is there comparable work on visual mathematics?
- What helps children visualize? (Interactive applets, translation between sensory modalities…?)
- Which outside-of-school experiences (computer games, music, sports…) affect the ways that children visualize and make images?
- What is the connection between cultural/social experience and visualization?

The group suggested some activities that could be explored in elementary school mathematics classes to begin to address these questions:

- To help pupils enhance and develop their abilities to visualize and make images in mathematics activities, we could experiment with the tangram activity (Wheatley, 1990) or Clements' work (Clements & Battista, 1992).
- To enhance and develop students' visualization abilities, Anna Dutfield focuses on pupils orienting, constructing, and operating either on or with shapes and figures in a learning trajectory developed for elementary school mathematics teachers.
- To develop counting abilities and the construction of number we can refer to Steffe (1988) and Steffe, von Glasersfeld, Richard & Cobb (1983).
- To develop 3D visualization, we can talk about the example of a (3X3X3) cube where we mentally imagine it cut up (2+1 by 2+1 by 2+1) or painted. Can we develop a strategy for counting the number of cubes in a stack of cubes? How do we do it?
- To develop visualization, educators believed for a long time that we should start with points, lines and planes figures, but research (Andrews, 1996) proved that it would be better to start by imagining oneself in space. For example: Imagine you are wearing magical shoes that allow you to walk anywhere. You are inside a sphere, and there is a pole in the sphere. You climb up the pole and there is a hole at the top of the sphere, so you go up through that hole and look down through it—what do you see? Now you get off the sphere and there is a plane, like a floor, perched on top of the sphere for you to walk on. You walk for a while and then meet a wall. You are

> wearing magical shoes, so you can walk up the wall—imagine doing that, to where you meet another wall. What is the relationship of this new wall with the plane on top of the sphere?

This group added further references to explore their questions: Arcavi, 2003; Eisenberg & Dreyfus, 1991; Ginsburg, 1999; Greenes, Ginsgurg & Balfanz, 2004; Reynolds & Wheatley, 1992; Wheatley, 1998.

Theme 3: Embodiment, consciousness and intentionality.

This group's discussion centred on the relationship of embodied learning with conscious and unconscious knowing, and with intended and unintended effects. This group's discussion was speculative, raising issues and putting forth conjectures related to group members' experiences and readings. Here are some summary notes from the group:

External vs. internal impetus for learning: Eli wanted to explore the distinction between what is imposed from outside and what is internal. An example of something for which we have an internal impetus is the acquisition of language. We did not explore the acquisition of language, but as a group we explored the question of whether children learn to walk because they see other humans walking or because they have an inborn drive to learn it. Someone mentioned that babies find ways to move around even before they learn to walk and that the methods they develop may be quite individualized.

Moving beyond narrow interpretations of learning and knowing: Susan stopped by to listen to our discussion and mentioned an experience where she had a cut and after it healed, she found herself asking, "How did I know how to do that"? Grace recounted a related experience of having a serious case of flu and at some point having the thought "Oh, finally we are winning". Dave Hewitt raised the point that babies sleep a great deal in order to allow for serious work to happen. Sick people do this too. We also thought about the way that solutions to mathematical problems that we are working on often come to us after we have a good sleep. Susan noted that by focusing on intentionality and defining the boundaries of self in particular ways, we limit ourselves to a narrow band of possibility. Nathalie recounted a story of a child in nursery school who was holding a block and pointed it at a chair, using it as one would a TV remote. The teacher was saddened that children are so involved with watching TV, but Nathalie was fascinated by the child's invention.

Intentionality and education: Susan recalled McLuhan's comment that unintended effects are always greater than intended effects. Aldona recounted hearing a representative of the British Columbia government speaking to a group of educators and saying that "if it can't be measured it can't be improved".

Conscious vs. unconscious knowing: Eli raised the question of when consciousness begins. Nathalie mentioned a book called *The origin of consciousness in the breakdown of the bicameral mind* by Julian Jaynes. Katie thought about what it is like to play a memorized piece of music on the piano. Her experience was that interventions of consciousness seemed to correspond to interruptions in the flow and naturalness of the playing.

Developing an embodied mathematical automaticity through games, rhythm, movement: David raised the issue of automaticity in doing multiplication. What happens when one has to compute 7 times 8? In some cases, the answer is automatic. Sometimes we lose confidence in the automatic response and have to check the answer.

He talked about the difference between lesson's objectives and the tasks provided for learners. He mentioned a game called "Meet Lulu", in which Lulu and the player are represented as figures on a grid. When the player moves, Lulu moves as well, according to some rule. (For example, a one-unit move vertically by the player results in a two unit vertical move by Lulu, but in the opposite direction.) David had given his class the task of having the player and Lulu meet at a specific location in order to meet the lesson's objective of having students learn to use vector notation.

Nathalie mentioned the Frog game. She described playing this game with people replacing signs on a computer screen or marbles on a board. She said that at first the people had a lot of discussion about how to achieve the objectives of the game, but gradually, each player learned their part as thought it were a bit of choreography. You could see them swaying and learning the rhythm of play.

David mentioned a related game experience in which a number of players stood in a line, and only the leftmost player was allowed to sit or stand at will. The others were bound by the following rule: change state only when the player on your left is standing and all players to the left of that player are sitting. He said that after a while, he did not have to pay attention to what the others were doing. He eventually learned to simply count the moves and knew what to do. He compared this experience to the experience of playing a similar game using 1's to represent standing and 0's for sitting. In that case, one would have an overarching experience of the patterns as opposed to experiencing the game from the point of view of a single position.

David also remarked on the game of horse race with dice. He commented about students having the experience of feeling what it is like to be horse #7 versus that of being horse # 12.

We also talked about the availability of embodied experiences. Grace mentioned some students who had difficulty with a written exercise that involved different ways to tear four stamps from a five-by-five sheet. When the students later talked to her about this exercise, they suddenly recalled the game of Tetris and had no difficulty with the exercise.

Grace also described seeing a puzzle called the "binomial square" at a Montessori School. The puzzle consists of an (a X a) square, a (b X b) square and two (a X b) rectangles that the child can arrange as an (a + b) by (a + b) square. We wondered how teacher or learner might use this experience at a later time when algebra is being learned.

Conclusion

This working group offered a place to begin questioning the relationship between the senses and the learning of mathematical understandings. We assessed questions of awareness, experience and concept development by sharing experiences of concrete activities discussing links between these experiences and mathematics learning, particularly in terms of concept formation.

This new field of research within mathematics education is still in an explorative phase, but it is a promising new area. There is growing interest among mathematics educators in researching bodily, multisensory and multimodal entry points to mathematical learning and observing the effects of these modes of learning on learners' mathematics.

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Topic Sessions

Séances thématiques

Imagination and Digital Mathematical Performance

George Gadanidis
University of Western Ontario

Mathematical Imagination

I recently did a search in the Alberta mathematics curriculum documents for K-12 for the word ‘imagination’. I also did this search in Ontario mathematics curriculum documents for K-12. I could not find the word “imagination” in these documents.

The absence of the word ‘imagination’ in mathematics curriculum documents is troubling. Children have incredible imaginations, and it is their imaginations that lead them to explore, to question, to wonder, to look for interesting connections. Egan (1997b) suggests that there is a myth in education that teaching should start with what a child already knows and understands – rather than also with what a child can imagine.

Mathematical Performance

Approximately one year ago, Janette Hughes said to me: “I’m interviewing Penn Kemp. She’s a performance poet” (Hughes, 2006). Immediately I wondered: “What is a performance mathematician?” There are performances of poetry, of screenplays, of art. If we move outside of the domain of assessment, and use an artistic lens, what might it mean to ‘perform’ mathematics? What might it mean to perform mathematics in a classroom setting? In a teacher education setting? For the public?

By mathematical performance I refer to digital, artistic productions of mathematical discourse, which do two things. First, they draw our attention to complex mathematical ideas, offering us opportunities to flex our imaginations in fresh, experimental ways, and the potential of experiencing the pleasure of mathematical insight. In so doing, such performances also do a second thing: they disrupt and potentially reorganize our view of what mathematics is and what it means to do, learn and teach mathematics by challenging the following mathematical ‘truths’ typically assumed in school:

- mathematics is a cold science – rather than an aesthetic, human experience (Gadanidis & Hoogland, 2003);
- mathematics is about learning procedures for getting correct answers – rather than attending to and gaining insights about the complexity of mathematical ideas (Gadanidis, 2004);
- a good teacher makes learning easy – rather than creating situations where students have to think hard (Jonnassen, 2000); and

- teaching should start with what a child already knows and understands – rather than also with what a child can imagine (Egan, 1997b).

EXAMPLE #1: "IF PARALLEL LINES NEVER MEET"

The digital performances that I create are usually based on personal, critical experiences I've had with mathematics or mathematics education. For example, the digital performance of "If parallel lines never meet" (see Figure 1) was prompted by the story of one our sons (Timothy) when he was in grade 4 (Gadanidis, 2005a). Timothy was sitting at the kitchen table studying for a geometry test when we had the following conversation:

> *What's a parallelogram, Timothy?*
>
> *A quadrilateral with opposite sides equal and opposite sides parallel.*
>
> *What about opposite angles?*
>
> *They're equal too.*
>
> *Does a quadrilateral with opposite sides equal have to be a parallelogram?*

After some discussion and playing with straws, Timothy came to the following conclusions:

- A quadrilateral with opposite sides equal is a parallelogram.
- A quadrilateral with opposite sides parallel is a parallelogram.
- A quadrilateral with opposite angles equal is a parallelogram.

Although all of the above are properties of a parallelogram, not all of them are needed to define a parallelogram.

Then, Timothy said:

> *Daddy, is it OK if on the test I define a parallelogram as a quadrilateral with opposite sides equal and opposite sides parallel?*
>
> *Why Timothy?*
>
> *I don't think my teacher knows that it's redundant to say both.*

In response to Timothy's story, I wrote the song in Figure 1 about 'parallelness'.

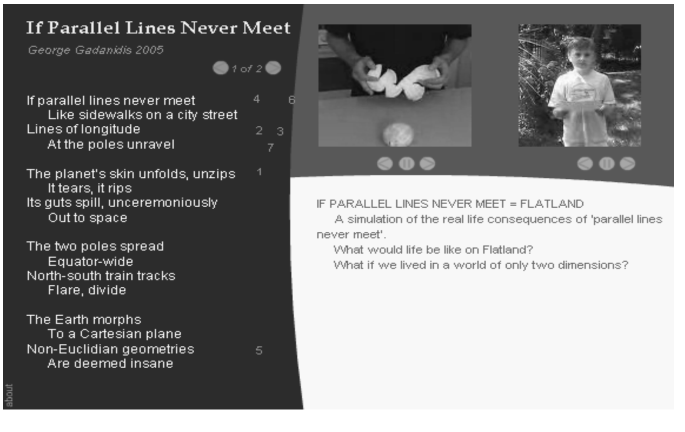

Figure 1. If parallel lines never meet

When I have asked teachers and students what they know about parallel lines, the typical response is that parallel lines are straight lines that run side-by-side and never meet. However, very few students (or teachers) understand that *parallelness* can be a much more complex concept. For example, for about two thousand years mathematicians tried to prove Euclid's parallel postulate as a theorem, but were unable to do so. The postulate turned out to be an assumption rather than a theorem, and different assumptions can lead to different geometries. For example, take a straight line and place a point beside it. On a Euclidean plane, there is one and only one line through this point that will not cross the first line (see Figure 2); however, on a spherical surface such a line is not possible; and, on a hyperbolic plane, there are an infinite number of lines possible.

From an extrinsic point of view, "there are no straight lines on a sphere"; but from an intrinsic point of view, from the view of a bug crawling on a sphere, a great circle is a straight line (Henderson, 2001; Henderson & Taimina, 2006, 64). If we look closely at to lines of longitude crossing the Equator, we notice that they both intersect at an angle of 90^{o}. Based on what students learn about lines and transversals in school, they should conclude that lines of longitude are parallel – yet they meet at the poles. If we tried to make lines of longitude behave like the parallel lines we studied in school then the planet's skin would tear, it would rip, and its guts, so to speak, would spill out to space. This idea is explored in the poem embedded in the online activity shown in Figure 1 (available at http://publish.edu.uwo.ca/george.gadanidis/parallel), which also includes a math music video and a number of video annotations that extend the problem. We also explored the sum of the angles in a 'triangle' drawn on a sphere, and discover that there is more than one possible answer. Complexities such as these make mathematics interesting and help engage students' imaginations. Exploring these complexities opens the door for experiencing a variety of meanings of straight and of parallel. For example, Henderson & Taimina (2006) talk about straight as meaning symmetric and straight as meaning the shortest path, and about extrinsic and intrinsic straightness. These different meanings or perspectives, which use "non-formal and experience and geometric imagery" (59), add layers of complexity that enrich the meaning of straightness. Taimina & Henderson (2006) suggest through such informal approaches "many levels of meaning in mathematics can be opened up in a way that most people can experience and find intellectually challenging and simulating" (59).

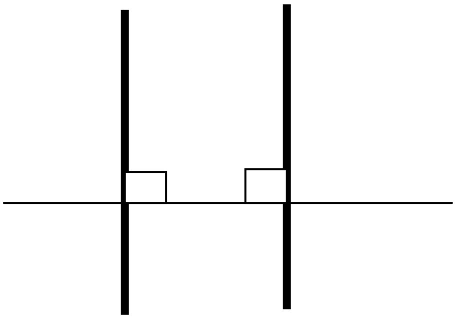

Figure 3. Parallel lines.

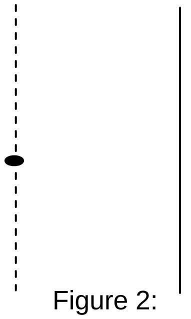

Figure 2: Parallel lines.

In the mathematics-for-teachers courses I teach, we explore parallel lines on a sphere. We initially explore lines of latitude and longitude. Are lines of latitude parallel? Are lines of longitude parallel? Which are more parallel, lines of longitude or lines of latitude? Then we investigate geodesics (the shortest paths between two points). For example, is the line of latitude shown in Figure 4 the shortest path between the two points shown? Using string to trace the lengths of various paths we draw on inflated balloons, we discover that lines of latitude are not geodesics, but great circles, like lines of longitude and the Equator, indeed are geodesics. One teacher commented,

Figure 4. Two points.

> *I feel like I was misled, misguided, told the half-truth about parallel lines. It is the first time that I have realised/felt that math isn't just black & white and can cause quite creative outcomes/discussions. Imagine if we are having these conversations about parallel lines, what a child could come up with (because they haven't seen the black & white picture yet).*

Another teacher reflected,

> *I felt lost at first as I struggled to remember math concepts from childhood and adolescence. I felt confused. What did a poem have to do with math? I was perplexed. Was there not only one answer to a mathematical question? I felt apprehensive. How would I discuss a mathematical concept that I did not fully understand? Then as I got into the swing of things, I felt more confident with my opinions, my answers and most importantly myself. I felt cheerful that I was experiencing math as a student and that I would hopefully be able to empathize with my future students. I felt happy that math instruction could be made to be engaging. Finally, I was giddy that I was thinking about math, actually thinking about math and not doing everything else to avoid it.*

Figures 5 and 6 show the explorations of parallel lines of two teachers participating in a fully online math course (where the discussion forum used had a tool that allowed users to draw in their postings). Figure 7 shows a poem authored by a group of middle school students after exploring the above parallel lines ideas.

Figure 5: A teacher's exploration of parallel lines

Figure 6: A teacher's exploration of curved lines as parallel lines

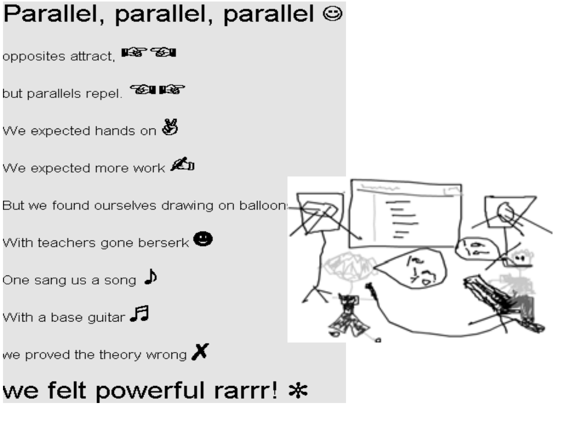

Figure 7: An illustrated poem authored by middle school students

MATHEMATICAL PERFORMANCE
~~THEATRE~~ ^ OF THE OPPRESSED

Pineau (2005, 15) suggests that "[t]he claim that teaching is a performance is at once self-evident and oxymoronic." However, as a theoretical claim, it is highly problematic. Pineau suggests that the typical interpretations of teaching-as-performance as (a) *teacher-as-actor* and (b) *teacher-as-artist* are weak, as the former reduces teaching to "teaching like an actor", and the latter equates it with "intuition, instinct, and innate creativity" (18-21). The alternative presented by Pineau raises issues of power and authority and sees performance as political struggle and resistance.

Performance as a form of political struggle and resistance has been the centrepiece of the work of Boal (1985), namely his book *Theatre of the Oppressed*. In one of Boal's Forum Theatre performances, a person in poverty shops for groceries and is confronted by the cashier as he does not have the money with which to pay for the food his family needs to survive. As the play unfolds, members of the audience (spect-actors) may at any time replace an actor and navigate the play in directions they deem appropriate. There are at least two important things *at play* in such a performance. First, the common script of "shop, pay, take home" is disrupted. A simple social convention is made complex and real – there *are* people in poverty that *cannot* pay for the food their family needs to survive. The digital performance shown in Figure 1 also follows a similar disruptive pattern, in a mathematical setting. The convention in a mathematics classroom is to assume that "parallel lines never meet". But very few students (or teachers) understand that this as an assumption. Despite mathematicians' attempts, there is no mathematical proof that parallel lines never meet. We may *assume* that parallel lines never meet (in the sense taught in school) and create a Euclidean geometry, a geometry of flat surfaces. We may just as easily assume that parallel lines *do* meet and generate other legitimate geometries, like that of the sphere. A second important thing at play in Boal's performances is the agency of the audience. A spect-actor has the same right as the actor to be a part of the play. How might such agency be produced in a digital mathematical performance? (Gadanidis, 2006)

POETRY

I chose to use poetry as the *centrepiece* of the digital mathematical performance because, as the poet Molly Peacock (1999) suggests, poetry is *screen-sized*. It is compact enough and cohesive enough to be held in one's mind as a whole. Poetry also makes use of image and metaphor, both of which help the reader sense deeper relationships to be explored. Also, seeing how Janette Hughes (2005) annotated the poems in her digital performances with videos, images, text, and interactive explorations of poetry, I felt a sense of aesthetic fit, and I felt compelled to explore a form of multimodal expression I found intriguing. The poetry coupled with the digital annotations offer multimodal performance and experience of mathematical ideas. Traditional school mathematics discourse has tended to prefer monomodality or bimodality (in cases where diagrams or graphs are employed). Kress and van Leeuwen (2001) suggest that in a digital environment "meaning is made in many different ways, always, in the many different modes and media which are co-present in a communicational ensemble" (111).

Example #2: "Out the Door"

In her book *The Cloister Walk*, the poet Kathleen Norris (1996) tells the story of her fourth-grade mathematics experience where her teacher laughed when she suggested that two plus two can't always be four. "But if two plus two was always four, then numbers were too literal, too boring, to be worth much attention. I wrote math off right then and there, and, of course, ended up with a classic case of math anxiety" (Norris 1996). Norris' story captured my mathematical imagination and I consequently wrote a song called "Out the door" (available at http://joyofx.com), where I explored (a) mathematical contexts where 2 + 2 may have other answers – for example, in base 3, 2 + 2 = 11, and when the sets {2} and {2} are joined (or added), the result is {2} – and (b) the pedagogical implications when children's mathematical imagination is (not) nurtured (Gadanidis, 2005b). This song (see Figure 8) is currently being developed as a digital performance similar to the one shown in Figure1.

I have used this song as a starting point for motivating teachers to reflect on mathematics teaching and learning, The song reminds me of the "bad teaching" that I and many other teachers I have known inflicted on students, by keeping interesting mathematics away from students because of the perceived need to move on with a lesson, or because of the lack of knowledge by the teacher, or because it's not in the curriculum, and so forth. I have wondered whether such songs are appropriate for students. Initially I hesitated. However, recently reflecting on the following story has helped me realize that if we don't talk with students about such issues while they are in school, then when will we?

> *About two years ago I wrote a short adventure novel (about math and artificial intelligence) which had a scene involving a pedophile. I remember chatting with a colleague who read the book and asking her whether she thought it was appropriate for our oldest son (then in grade 4) to read the book. She reminded me that it is in these school years that young children are often abused by adults, and if we don't tell them about pedophiles and other predators when they are most vulnerable then when will we?*

I have began to see this song (as well as the song called "In the closet" shown in Figure 9, also available at http://joyofx.com) as a song of the "mathematically oppressed". I wonder whether such songs might be used to empower students to become more like Boal's spect-actors and less like spectators in the math classroom and to motivate teachers to offer such opportunities for students.

Out the door

My teacher made me think
That math is such a bore
Said she was certain
Two twos are always 4

When I suggested
This cannot be
My teacher laughed
She *laughed* at me

It's not like her to say sorry
It's not like her to help me see
Show me what it's like to see
And I'll show you what my mind can really be

If two plus two
Is always four
Math lacks imagination
I'm out the door

But in base 3
I'm in math heaven
It turns out that
Two twos make 11

And when you add two sets
Sometimes the answer is not new
Did you know
{2} + {2} = {2}?

It's not like her to say sorry
It's not like her to help me see
Show me what it's like to see
And I'll show you what my mind can really be

What else, I wonder
Might the answer be
What other math
Could I see?

If the answer is four
And there's nothing else to mention
Then math's too literal
Not worthy of my attention

It's not like her to say sorry
It's not like her to help me see
Show me what it's like to see
And I'll show you what my mind can really be

Figure 8. "Out the door" lyrics.

Example #3: "In the Closet"

One day, our son Timothy came home from school quite upset - he was in grade 2 at the time. They were studying subtraction.

> *A student in Timothy's class had asked what the answer to "2 minus 6" would be.*
>
> *"You can't take a bigger number away from a smaller number," explained the teacher.*
>
> *"Isn't the answer negative?" asked Timothy.*
>
> *"You can't take a bigger number away from a smaller number," repeated the teacher.*
>
> *Timothy was upset. My wife Janette explained that "the teacher probably meant to say that he'd study negative numbers in another grade."*
>
> *"But why didn't she just say that the answer is negative 4?" asked Timothy.*

In response to Timothy's story, I wrote a song (Gadanidis, 2005c). called "In the closet" (see Figure 9).

In the Closet

I raised my hand
I wondered and I said
If we take six from two
What would be left?

My teacher he professed
That's interesting, but
It's not possible
To subtract like that

My Mommy suggested
As she wiped away my tears
He probably meant to say
You'd learn it in future years

I was not sure
Not absolutely positive
I dared, I asked
Sir, isn't the answer negative?

My teacher repeated
Interesting, but
It's just not possible
To subtract like that

But why didn't he say
I blurted as I sobbed
That the answer, the answer
Is negative four?

The answer is right there
In the thermometer
I may only be in grade two
But I can think too

My Mommy says math is a treat
That one day it'll make my mind
really kind of neat
Next time she says let's give math a look
I think I'll hide in my closet,
read a real good book

Figure 9: "In the Closet" Lyrics

Mathematical Imagination and Performance

Given the continuing educational/political trend of focusing on "standards, assessment, outcomes, and achievement", it is not surprising that imagination – children's and teachers' imaginations – gets overlooked (Greene, 1995, p.9). A focus on artistic performance is one way of disrupting and expanding what might be viewed as performance at the classroom level. Such a focus also brings attention to considerations of what makes for a good mathematical performance and what mathematics is worthy of performance. McKee (1997, p.5) suggests that good performances offer opportunities "to use our minds in fresh, experimental ways, to flex our emotions, to enjoy, to learn, to add depth to our days."`

Seeing mathematics as performance – using performance as a lens for looking at mathematics education – involves using metaphorical, imaginative thinking. Egan (1997a) suggests that young children are much more capable of such thinking than we typically assume and much more capable than adults. The ability to create and understand metaphor seems to decrease with age, which I believe makes it all the more important for us to try to see mathematics through imaginative lenses. Looking at mathematics through performance is, to use Maxine Greene's (1995, p.17) words, "a search for openings without which our lives narrow and our pathways become cul-de-sacs."

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Developing a Serious Enrichment Program

Keith F. Taylor
Department of Mathematics and Statistics
Dalhousie University

Environmental Scan

An informal survey that I have conducted over many years leads me to believe that approximately 10% of students from grade 5 onwards know the content before it is taught in school mathematics classes. I think that this is a terrible loss of opportunity for the individual students and for the country. Ten percent of students in grades 5 through 12 in Canada represent about 400,000 young people with incredible unrealized potential for learning additional mathematics. However, I recognize that it is not practical to address this issue through curriculum reform because there are fundamental obstacles.

Schools must teach all students and there are practical, as well as developmental, reasons why the amount of streaming of students according to their abilities is limited. Moreover, the amount of instructional time devoted to mathematics is limited, so core curriculum must be focused and of effective use for the bulk of the student body. As it turns out, curriculum is designed so that only the topics necessary for life skills, and to prepare prerequisites for calculus, linear algebra and statistics are delivered in a structured way. The result is that the regular curriculum can be compared to the grammar component of the language arts curriculum. Very little of the *poetry* of mathematics can be included.

There are wonderful people, many of them active in the Canadian Mathematics Education Study Group (CMESG), who work tirelessly to provide enrichment to under-challenged school students. My own meager efforts over 30 years have touched about 100 children. For many of them, it was a one-time contact. So I averaged about 3 enrichment engagements per year. Many others would have a higher average.

Also, many classroom teachers have built up a stockpile of challenging questions and activities for the mathematically capable student and do their best to set aside some time to encourage such students.

I would argue two points: Most of these enrichment activities focus on problem challenges with no development of coherent topic structure and at most 10,000 students per year are exposed to such activities, less than 1 in 50 of my top 10% target.

Look at the case of music education. The Royal Conservatory of Music has developed a coherent curriculum. They claim that there are more than 500,000 young people being served annually by their program in Canada. Imagine the impact if we had similar numbers expanding their knowledge base into the beautiful parts of mathematics that are kept secret from most of society.

Here is a thought provoking statistic. In the 2005 Canadian Mathematics Open competition, of the top **108** competitors, **84** had surnames that were of Asia origin. This does not represent the distribution in the general population. Either this means that encouragement and enrichment can have an enormous impact at the top end of the distribution or there is a natural target group for a serious enrichment program.

All of this suggests to me that there is an opportunity to introduce a new extra-curricular activity for young people – mathematics enrichment. This activity would simply be another choice available along side music lessons, sports or chess. If this activity became as popular as piano lessons, swimming lessons or gymnastics, there would be a profoundly positive impact on society.

Topics Under the Rubric of Mathematics

This following list is taken from the American Math Society Subject Classification scheme: **History of mathematics, logic, foundations, combinatorics, ordered structures, lattices, graph theory, number theory, polynomials**, rings, algebras, algebraic geometry, **matrix theory**, non-associative structures, category theory, homological algebra, K-theory, **group theory**, Lie groups, **real numbers, complex numbers**, measure theory, potential theory, analytic functions, special functions, differential equations, **difference equations, dynamical systems, ergodic theory, sequences, series, summability, approximations, Fourier analysis**, harmonic analysis, integral transformations, functional analysis, calculus of variations, **geometry, convex geometry, discrete geometry, differential geometry, topology, manifolds**, cell complexes, **probability**, stochastic processes, **numerical analysis, mechanics**, fluid mechanics, **optics**, electromagnetic theory, thermodynamics, **quantum theory, relativity**, astrophysics, geophysics, operations research, **game theory**.

The topics in boldface are ones where I can easily imagine the development of courses appropriate for bright children in the 10-17 age range; courses that are largely independent of the standard curriculum. I do not suggest that trying to cover all of these is reasonable. However, it is clear that a vast amount of intellectual material exists and creative people with an understanding of age-appropriate pedagogy could make aspects of this intellectual material available.

Concept of an Enrichment Program

I believe we should adapt the Royal Conservatory of Music model of program structure to develop enrichment materials, an examination system, and a set of credentials for mathematics. We could deliver content through web-based courses, supplemented where convenient with weekend workshops. We would not have the counterpart to the private music teacher. This should make a mathematics enrichment program considerably cheaper for the student than music lessons are.

Parents could enroll their children, children could ask to be enrolled, teachers could recommend to capable children that they enroll, adults could enroll, teachers could enroll, and senior citizens could enroll. Indeed, for those attracted to symmetry, patterns, logic and causality, a well-structured mathematics program that develops an understanding of many of the topics listed in section 2 could provide lifelong enjoyment and enrichment.

Initial Thoughts on Program Structure

There could be two initial streams, *number* and *shape*, that extend from beginner to advanced (roughly think ages 10 through 17). Additional streams would be made accessible to students who have reached an identified stage in an initial stream and/or a certain grade level in school. Care should be taken to not anticipate standard school curriculum.

To illustrate, I will discuss some possible topics in the number stream. The natural numbers are named as such for good reason. Bright young people are fascinated by prime numbers – the atoms of the integers – and become engrossed when introduced to some deeper properties. A sequence of courses that builds up to the study of encryption schemes would include modular (circular) arithmetic at length, large prime numbers, and the Euclidean algorithm. It would give a compelling introduction to the theory of secret codes. This area also provides a wonderful illustration of the power of *proof*. I have delivered courses on encryption to bright 13 year olds several times. They can understand the Euler-Fermat theorem that says the following: (Think of the integers involved as being huge, greater than 10^{200}.)

> Let n be a natural number. Let $\phi(n)$ denote the total number of integers from 1 to n that are relatively prime to n. If a is any integer that is relatively prime to n, then a raised to the power $\phi(n)$ is congruent to 1 modulo n.

This is the fundamental theorem on which the RSA encryption scheme is based and is widely used for security of internet-based financial transactions. It is easy to get the children developing their own unbreakable codes after they have this theorem. We only know this statement is true through deductive reasoning and proof. The children who come to understand this are empowered in a manner that cannot be overstated

The real line could be explored at length to build a profound understanding of the arithmetic rules mastered in earlier grades followed by an exploration of the decimal representation of numbers to the point where the *continuum* becomes internalized.

Beautiful items such as *Buddha's Pathway to Enlightenment* would build on binary and ternary representations. The complex numbers could lead to the power of the circle group (tying back to modular arithmetic) and the generation of fractals.

The stream on Shape could be described at length as well. Imagine one of the strands culminating in classifying the 17 wallpaper groups and the two dimensional surfaces, related topics that can be introduced through variations on the pacman video game.

Upon mastering the material to a certain level in a stream, the student would apply to be tested and receive a credential such as *Gauss Level 3, Canadian Mathematics Enrichment.* Such credentials could then become part of the resume when applying to university or for a job in the same way that having Grade 8 piano from the Royal Conservatory or a certain certificate in life saving from the Red Cross is listed by individuals.

Concluding Remarks

I am proposing an enrichment program in mathematics that will join activities such as music lessons or sports as an extracurricular option for bright children. The program should be structured to develop lasting understanding of those parts of mathematical knowledge that get little or no coverage in standard curricula. It also should be designed to develop the maturity of mathematical thinking that is possible for the top 10% of the population. It is this maturity of thinking that is the real goal. Graduates from this program need not go on to be mathematicians. Whatever they do in life from running a business to looking after the health of people or the environment, being skilled at recognizing patterns, inductive and deductive reasoning, and systematic analysis of complex situations will enrich their lives.

New PhD Reports

Présentations de thèses de doctorat

Transforming Images of Mathematics/Teaching: A Study of Pre-service Teachers/Teaching

Paul Betts
University of Winnipeg

Introduction

The purpose of this research was to explore (pre-service) teacher's images of learning, teaching and the nature of mathematics. This research was part of a larger reform movement in mathematics education (e.g., NCTM, 2000), oriented by socio-constructivist assumptions concerning learning and knowledge. Teaching a mathematics curriculum-content-based course seemed an ideal opportunity to expose pre-service teachers to alternative images of mathematics/teaching, with the goal of changing their views away from traditional conceptions.

My goal of changing pre-service teacher's images of mathematics/teaching was framed by complexity theory. In brief, complexity theory views knowing as entangled by and embedded in the biology, history, community and culture of a system. During the mathematics curriculum course, students were exposed to some uncertainties of mathematical knowledge, based implicitly on ideas from complexity (e.g., mathematics as embodied). Ideas from complexity theory are also evident in data analysis, where I developed a lens for interpreting the experiences of participants during the course.

The methodology I used was based on a coupling of narrative and reflective practice. Narrative (cf. Bruner, 1990) was seen as a metaphor for the processes by which a knower organizes experience. Reflective practice refers to the "research" of practitioners in action, which generates contextually bound truths. Schon's notion of reflective practice can be interpreted as teaching-is-an-experiment, which is a position supported by others (e.g., Wilson, 1995). Further, Wilson (1995) suggests that the imperatives of research (e.g., observation) improve teaching practice, and the imperatives of teaching (e.g., action that makes a difference) improve research.

During my doctoral dissertation, my research commitments changed. Before teaching the mathematics curriculum course that was the source of student participant data, I wanted to discover and articulate the knowledge of effective mathematics teachers. During the course, I was committed to exploring how pre-service teachers could embrace alternative and reform oriented images of mathematics/teaching. After the course, as a result of data analysis, I began to interpret the changing identity of pre-service teachers in relation to mathematics/teachers, based on my interpretations of their experiences. In the end, I realized a need to interpret my changing identity in relation to mathematics/teaching. In what follows, I endeavor to chronicle the salient features and events of my doctoral journey.

Before The Course

I wanted to know what (pre-service) teacher's needed to know about mathematics in order to effectively teach mathematics. There is a plethora of research suggesting that teachers need to know the subjects they teach deeply (e.g., Ma, 1999), but are inadequately trained in the subjects they teach (e.g., Darling-Hammond, 1996). To quote one education professor: "Science teacher's need to know that bears do not hibernate in winter." At the institution where this research was conducted, several content-of-the-curriculum courses were developed to ensure that pre-service elementary teachers understood the subjects they would teach. On the other hand, it could be argued that teachers should understand more than the curriculum content of the subjects they will teach, in order to be effective teachers.

The above analysis lead me to consider the subject knowledge of mathematics teachers that was anchored by pedagogical issues, labeled pedagogical content knowledge (PCK) (cf. Shulman, 1987). A brief survey of the literature concerning the PCK of mathematics teachers suggests that teachers with a more sophisticated PCK will be more effective teachers (e.g., Ball & Bass, 2002). But clarifying the nature of PCK is a problematic task. For example, PCK cannot be framed by an objective orientation since teaching acts are bound by value-laden assumptions concerning knowledge, learning and teaching (Ball & Wilson, 1996), and PCK may not adequately account for the dynamic, contextual and complex personal professional knowing of teachers (e.g., Clandinin & Connelly, 1995).

It became clear to me that I needed to unpack my notions of knowledge and knowing, which lead to an exploration of complexity theory. Knowing can be seen as emerging from the biology of a knower, and not merely as a one-to-one match between knower and the environment (Maturana & Varela, 1987). The biological roots of knowing and the reciprocal phenomenology of Merleau Ponty (1962) can be used to support the thesis that knower and known are inseparable (Varela, Thompson, & Rosch, 1991). I experience and organize reality as if it is an objective phenomenon. But, in organizing reality, I create reality. In other words, knower and known are co-interacting and inseparable: an "entre-deux" (Merleau-Ponty, 1962) between subjectivity and objectivity. In particular, Merleau-Ponty's "entre-deux," which Varela, Thompson and Rosch refer to as double embodiment, is the focus of attention in bringing together knower and known.

Further, a fundamental notion of complexity theory is the ecological co-dependence between/within a knowing system. Knowing is bounded by the biology of a system, and by a system's prior history of interactions with an environment, where environment includes entanglements with/in social communities and cultures. "Each layer or body can be simultaneously seen as a whole, a part of a whole, or as a complex compilation of smaller wholes" (Davis, Sumara, & Luce-Kapler, 2000, p. 73). Each system, or body, is part of larger systems and contains smaller systems, but these are living systems, so that the parts cannot be viewed as the pieces of a jigsaw puzzle. The whole emerges as something greater than the sum of its parts. In fact, systems should be viewed as blended, rather than as distinct parts blending.

Based on my readings concerning complexity, I rejected the notion of knowledge as a static body of propositions, facts and laws. I began to question my perceptions of the certainty of mathematics. Godel's uncertainty principle, ethnomathematics (e.g., Ascher, 1991), mathematics as socially constructed (e.g., Ernest, 1998), and mathematics as embodied and metaphorical (e.g., Lakoff & Nunez, 2000) became important ideas in my new perceptions of the nature of mathematics. I began to question the assumption that it is possible to clearly articulate exactly what knowledge and skills every teacher must know. And yet, I believed that mathematics teachers should embrace the alternative images of mathematics I was exploring.

During the Course

While planning and teaching the mathematics curriculum-content course that was the source of data for this inquiry, I sought to develop experiences for students that fostered an appreciation of mathematics as socially constructed. I focused on what it means to do mathematics, which I referred to as the nature of mathematics. I wanted to explore with students the following possibilities: alternative visions for the nature of mathematics; debate and critical thinking as part of the creation and learning of mathematics; and mathematics as a socially constructed body of knowledge. I also suggested to students several uncertainties concerning the nature, teaching and learning of mathematics. I hoped to challenge beliefs about mathematics rooted in the traditional school mathematics experiences common among pre-service teachers.

For example, one of the activities in the course involved folding and cutting paper, and was intended as a hands-on and enjoyable method to foster spatial sense. During this activity, one student found a way to cut a five-sided planar shape, which seemed mathematically impossible based on the given folding and cutting rules. The question became, did cutting the five-sided planar shape break the rules? I thought this question was an excellent opportunity to explore how mathematics was a socially contested body of knowledge involving uncertainties and interpretation, so I implemented an impromptu "court of law" debate. Some students prosecuted (i.e., provided arguments that the student had broken the rules), others defended, and others were the jury. I was the judge. After conducting a simulated court trial, the jury decided, almost unanimously, that the student had broken the rules. Despite the ruling of the jury, there was significant disagreement among students. I closed the class with a short lecture on how the activity illustrates what it means to do mathematics and some implied possibilities for learning and teaching mathematics. Student reactions to this activity served as a journal entry topic.

Course assignments, such as weekly interactive journals and problem solving events, were a main source of student participant data. For example, when responding to the interactive journals, I often challenged student beliefs concerning the nature of math and tried to detect their reactions to my challenges. For example, the following is a student response to one of my comments:

> *You said that arguing is an important part of math. I thought about that. I personally do not enjoy this but I see your point. Arguing is an important part of math. If no one ever argued, no one would bend the rules or try to disprove anything that was said. I wouldn't say that I don't like math. I guess there are just some parts that I just don't care for. I agree that debate and "why" are connected. I get the point. I know that we did not do very much asking why in school. We were told that something was right and we accepted it as the truth because it came from someone who we thought knew it all. It's funny that something I just realized is a huge part of math was not taught to us at a younger age.*

The above journal acknowledges the idea of "debate" but the student also seems to be frustrated. A majority of students were frustrated by my challenging journal responses, and with other aspects of the course. It was not until much later, when I rigorously examined my own role during the course, that I realized why frustration was a common experience among students enrolled in the course.

After the Course

During the process of analyzing student participant data, I began to notice tensions in the data for two of the participant cases. At this point, I also consciously re-oriented myself in terms of narratives rather than themes. Student quotes and my a-priori theoretical tools and personal understandings were the basis for the labels used to name narratives. For example, many students in my course wrote about math problems as always having a right answer, about mathematics as a certain body of knowledge, and about mathematics as black and white, which seems to correspond to an Absolutist epistemology of mathematics (cf. Ernest, 1998). Hence, I posited a math-as-black-and-white narrative to at least partially organize student experiences of mathematics/teaching. All narrative labels were developed based on interplay between student data and my theoretical commitments.

Lydia was one of my case studies. Lydia did well in school mathematics, and thought math was beautiful, logical and black-and-white. During the course, as part of her efforts to find answers to the question of the nature of mathematics, Lydia tried to notice other possibilities for the nature of mathematics. In reaction to the story of the development of non-Euclidean geometries, Lydia wrote, “All right, so if Euclidean geometry is wrong, then we have a problem. Do we not base our entire math on Euclidean geometry, or at least a great majority of it?” In this case, it seems that her efforts to find an answer are conflicting with her initial conceptions of mathematics. Lydia also viewed her role as student as one of finding and reproducing the concepts transmitted by an expert teacher. I interpreted Lydia’s experience as a conflicting interaction between math-as-black-and-white and student-as-reproducer narratives, because the course challenged black-and-white images of mathematics.

Similarly, in reaction to the paper cutting and folding activity, Lydia wrote:

> *Today’s class was a little frustrating for me. Trying to get the point across throughout the debate was frustrating. I found that is an aspect of the nature of math. I believe that the nature of math is all about emotions and feelings. It includes the appreciation, love and enjoyment of math. Along with the love of math comes the frustration that is also included…I truly feel that the nature of math has a great deal to do with the emotions that math brings. I also think that the nature of math is an area that is gray. Is there even a true meaning to the nature of math? I don’t think so?*

I had been discussing interpretation during class, and Lydia uses the word gray. Lydia loves doing math but must adapt to her frustration with the course. Lydia is looking for a right answer. In mathematics, these right answers are generated through logic. But she posits an emotional content to math. Again, I interpreted Lydia’s experience as another example of a conflicting interaction between narratives - in this case between math-as-black-and-white, positive-personal-mathematical-experiences, math-as-interpretation and student-as-reproducer.

In analyzing the cases of Lydia and one other student, I developed a lens, called “I|We Fractured,” for noticing and interpreting student participant’s changing images of mathematics/teaching. This lens was grounded in data analysis, but was also built from notions found within complexity theory and narrative-as-methodology. The lens emerged from initial attempts at data analysis, but then became the lens of data analysis. In what follows, I clarify the meaning of I|We Fractured, setting the stage for an application of this interpretive lens to a summary analysis of Lydia.

“I|We” is used to highlight the embodied and mutually emergent relation between an individual person and their social reality. The “I” of I|We refers to an individual’s experience. But “I” cannot be defined objectively as separate from social reality, and there are many different, competing and conflicting narratives available for interpreting an individual’s experience, so an individual’s identity is not interpreted as autonomous, rational and coherent. The “We” of I|We refers to the social reality of an individual, which is embodied and mutually embedded with/in a living net of individual, social community, and cultural systems. The boundary line (i.e., “|”) in I|We represents a deliberate attempt to notice and question any assumption concerning the distinction between I and We. I|We reflects the process of co-

construction of narratives for organizing meaning, and that identity is a complex relation between self and social reality.

The "Fractured" of I|We Fractured is intended to remind me that any interpretation of experience does not generate universal principles for understanding social phenomena. Rather, I wish to recognize that there are many different, competing and conflicting narratives available for co-constructing I|We. I|We is fractured, where each fragment represents one possible coherent use of narratives to organize social phenomena. The notion of individual as an autonomous, rational and coherent agent becomes problematic because narratives available for organizing a sense of identity are entangled, co-constructed and fractured. Narrating an individual identity as an autonomous, rational and coherent agent reflects the organization of experience with one possible fragment of available narratives.

The use of I|We Fractured involves a process of noticing and interpreting initial coherent narratives available for organizing a person's mathematics/teaching experiences, seeing new available narratives emerge and conflict with initial narratives, and seeing how these narrative interactions change as a person reconstructs a coherent collection of narratives for organizing her/his images of mathematics/teaching.

In the case of Lydia, she entered the course with positive and black-and-white images of mathematics. During the first half of the course, Lydia's frustration seemed to increase because I was not forthcoming with a clear and well-defined answer to the question of the nature of mathematics. As I made alternative images of mathematics available (e.g., interpretation), Lydia's frustration was seen as tension and conflict between several narratives used to organize her experiences. As the course proceeds, narratives become available and continue to interact. For example, a math-as-interpretation narrative seems to be available when solving one of the weekly problems:

> *I did the problem as I thought it should be done, but there was still a 'gray area' of the meaning of a couple of words. Instead of being satisfied with my product, I decided to explore the gray area of what was meant by that word…Maybe I was dwelling on the black and white, but I like to think that I was exploring my curiosity of what would happen if I tried it a different way.*

Lydia seemed to make a connection between problem solving and the role of interpretation within the nature of mathematics. I interpret this as an interaction between math-as-black-and-white and math-as-interpretation narratives, although now the interaction is complementary rather than conflicting.

At the end of the course, Lydia seemed to solidify a new possibility for mathematics/teaching. In the end-of-course portfolio assignment she wrote:

> *…it was my statement about taking the gray into consideration that really piqued something in me now after reading it again…I think that through all the other journals, I discovered things, but in this one, I discovered that it is really okay not to always get a right answer and that red check mark at the end of the solution to that tough question.*

Lydia was realizing the importance of process and interpretation, and in retrospect views this realization as a Eureka.

At the end of the course, Lydia seems to see process as her answer for the course. She wrote in the end-of-course portfolio assignment:

> *The part I found so frustrating was that after over fourteen years, you somewhat expected us to realize this in six weeks or so. That was the most frustrating part for me. But, at the same time, I began to realize it very quickly. It actually only took one comment from you in my journal [#3] to realize that math was not about product and more about process. You said something to encourage me, but at the*

> *same time it was like a boulder you dropped on my head and I realized how I have been seeing things. You said, "this" (meaning what is going on in my head) "is more important than this" (meaning answering the question that you asked for the journal response). When I read that, I began to realize that it is the process that is going on through my mind that is what really matters. I know the right answer matters in standards and exams, but if the process is completely wrong in my head, how am I supposed to get the right answer to begin with?*

Lydia is trying to incorporate her new ideas concerning math by using the idea of process. Process is attached to the gray of mathematics, and is an opportunity for learning as exploration. Product is attached to the black-and-white of mathematics, and is part of the measurable outcomes of math classrooms. I believe Lydia's images of mathematics can be re-constructed with a new collection of coherent and mutually supporting narratives, which include the initial narratives as well as math-as-interpretation and math-as-process. This re-construction is part of a process of re-establishing a sense of coherence, where narratives that once conflicted are now mutually supporting.

Nearing The End Of A Journey

After completing several case analyses of student participants, I made a deliberate attempt to uncover my own role as a teacher and researcher during my dissertation journey. I used I|We Fractured to make sense of my own changing images of mathematics/teaching. In the following, I will focus on articulating interactions between different, competing and contradictory narratives, in the context of my pedagogic decision making during the course. I will describe a tension between Reason and Facilitation narratives and a subsequent emergence of alternative narratives such as Teacher-as-expert.

During my doctoral studies, a reason narrative was a primal tool for justifying knowledge claims. Despite my increased insistence on the uncertainty of knowledge, reason was privileged. For example, I organized the notion of ways of knowing (e.g., logical and emotional delineation by Eisner) with a reason narrative, so that a rational way of knowing organized my understanding. When I began teaching the curriculum course that was the location of data collection for this inquiry, a reason narrative continued to at least partially organize pedagogical decisions. For example, during in-class activities, I encouraged debates. I wanted students to be critical of the nature of mathematics, and perhaps even notice that mathematical knowledge is socially constructed. Notions of reasoned and critical argumentation organized the notion of debate. It seems, based on student perceptions, that debate was not a nurturing discussion since a reason narrative organized it.

On the other hand, I also organized pedagogic decisions with a facilitation narrative. Rather than tell students what to think about the nature of mathematics, I sought to scaffold their thinking during in-class discussions and within the interactive journals. My facilitation narrative was based on two constructivist learning principles. First, I was attempting to build on the prior experiences of students, which reflected an acceptance of cognition based on adaptation, disequilibria and schema (cf. von Glasersfeld, 1995). Second, I was attempting to scaffold each student's zone of proximal development (cf. Vygotsky, 1978). For example, rather than tell the students that mathematics was socially constructed, I sought opportunities, such as the planar shapes debate, for students to experience mathematics as socially constructed. In other words, I oriented pedagogic decisions with a facilitation, rather than transmission narrative of teaching.

My reason and facilitation narratives competed. For example, my goal to facilitate a shift in student's perceptions of the nature of mathematics seemed to result in a propagation of the status-quo. I had hoped to promote a shift from worksheets/right answers to investigation/debate answers. But a conception of right answers still existed in my narratives for organizing teaching, namely an uncertainty of knowledge narrative, and hence the requirement that all knowledge claims be critiqued. Because my pedagogy was organized by uncertainty/inquiry and by reason/critique, an alternative normalization process emerged. Just like in a traditional mathematics course, "papers were collected and ticks and crosses distributed" (Klein, 2001, p. 268), except those students who insisted on mathematics as black and white received crosses and those students who entertained mathematics as interpretation received checkmarks. Students experiencing the course with frustration are suggestive of my contradiction between education as usual in terms of labeling students and a desire to get beyond education as usual in terms of the nature of math.

The fracturing of pedagogy by reason and facilitation narratives is a space for students who have been constituted as below normal to resist. For example, one student accused me of attacking her (ideas) but not providing an alternative. At one point, she remarked that she would teach in ways that worked for her (i.e., transmission pedagogy and absolutist mathematics epistemology). I chose to remain silent, even though I knew provincial mandates for differentiated instruction, predicated on the assumption of learner diversity, provided ample reason to disagree with her position, and place me in the position of having the correct answer according to provincial policy. A reason narrative organized my thoughts concerning this student, whereas a facilitation narrative organized my reaction. But she perceived that I did not nurture her ideas and beliefs. I did not notice the competing interaction between reason and facilitation narratives. Rather, I found alternative narratives, such as teacher-as-expert, to organize my experiences with this student.

My observations of student frustration during the course lead to a consideration of the notion of resistance. Based on theoretical constructs developed by Lather, I began to wonder if I, despite my intentions and efforts, presented the nature of mathematics' "emergent, multiply-sited, contradictory movements" (Lather, 1991, p.1) as a totalized and fixed monolithic. Even though I wanted my students to view mathematics as fallible and be critical of traditional views of mathematics, did I not create a binary space in which my view became the master voice? In other words, did my students experience the nature of mathematics in the same way as school mathematics – fixed, immutable, and inaccessible? It seems possible to present mathematics as not fixed, not immutable and not inaccessible, but still be fixed, immutable and inaccessible in presentation. And did the students in my class who noticed the contradiction "get smart" when they resisted?

Conclusions

Given the assumption that a knowing system is entangled with/in biology, history, social community and culture, the use of narrative is a tool for interpreting these entanglements. Within a knowing system and narrative framework, it seems possible to interpret a person's adaptation to frustration in terms of negotiating differing, competing and conflicting narratives in order to re-construct a sense of coherence. I|We Fractured represents the process and interpretive lens by which a narrator notices and interprets these narrative interactions. I have based this claim on the experiences of students and myself while teaching a mathematics curriculum course to pre-service elementary teachers.

It is reasonable to speculate whether an interpretation of pre-service teacher identity in relation to mathematics could be transferred to other contexts. I have used I|We Fractured to notice competing narratives within subsequent courses I have taught, and allowed this noticing to re-organize my teaching at least partially in terms of nurturing goals. Perhaps new understandings can be developed for the experiences of teachers across disciplines. Teacher educators could use this tool to interpret the experiences of their students. Further, since I was initially oriented by student's changing images, perhaps some insight can be developed concerning the complex question of teachers changing and changing teachers. Finally, perhaps the processes of self-study inherent in this inquiry would be beneficial to other teachers or educational researchers seeking to improve their own practice.

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Efficacy Shifts of Preservice Teachers Learning to Teach Mathematics

Cathy Bruce
Trent University

Abstract

In a qualitative study, the mathematics teaching efficacy of elementary teacher candidates was examined. Participants in the study were 10 preservice teachers enrolled in a mathematics methods course at a Canadian university. Data sources included open-ended surveys, observations, focus group and individual interviews, and math logs. Qualitative methods of Grounded Theory were used for data collection and analysis: an emergent zig-zag approach of data collection and analysis included open coding of transcripts and texts, active coding, theory notes and visual maps. The study found that preservice teachers followed a common learning trajectory but experienced that trajectory in different ways. Those candidates who's first classroom placements were positive, were able to maintain a consistently high level of efficacy. Those candidates who experienced challenging first placements, had varying lower levels of efficacy. This is important because once teacher efficacy is established, it is difficult to change and high efficacy is a predictor of both reform implementation and high student achievement.

Theoretical Framework

What is Teacher Efficacy?

Teacher efficacy is the teacher's self-assessment that he or she has the ability to impact student learning (Bandura, 1997). A teacher with high efficacy has the confidence that she or he can positively influence student achievement despite a range of challenging circumstances. A teacher with low efficacy believes that there are mitigating circumstances (such as low socio-economic status of the students or lack of resources) which are significantly more powerful influences over student learning and achievement than his or her teaching. Research in the area of teacher efficacy has produced a comprehensive body of literature over the past twenty-five years (Gibson & Dembo, 1984; Bandura, 1986, 1997; Ross, 1998; Tschannen-Moran, Woolfolk Hoy & Hoy, 1998, Tschannen-Moran, & Woolfolk Hoy, 2001, Goddard, Hoy & Woolfolk Hoy, 2004) indicating that teachers who believe they are effective are more likely to persist in meeting teaching goals when faced with obstacles and, more likely to experiment with effective but challenging instructional strategies such as student-directed methods (Riggs & Enochs, 1990) and performance assessments (Vitali, 1993). Teachers with high expectations about their ability to teach produce higher student achievement (Herman, Meece, & McCombs, 2000; Mascall, 2003; Moore & Esselman, 1994; Muijs & Reynolds, 2001; Ross, 1992; Ross & Bruce, in press a; Ross & Cousins, 1993; Watson, 1991), provided there is access to powerful innovations such as professional development and adequate resources. Teacher efficacy contributes to achievement because high efficacy teachers are innovative, use effective management strategies, support low ability students more fully, and build student confidence (evidence reviewed in Ross, 1998).

Bandura (1997) identified four sources of teacher efficacy information, the most powerful of which is mastery experience – having successful teaching experiences in the classroom. Self-efficacy generally rises with positive early experience, particularly following practice teaching opportunities for preservice teachers (Woolfolk Hoy, A., & Spero, R. B., 2005; Housego, 1990; Hoy & Woolfolk, 1990). Increasing confidence is the result of mastery experiences combined with classroom events that demonstrate the impact of the instructional strategies used. Efficacious beginning teachers are also more confident that they will remain in the teaching profession (Hall, Burley, Villeme, & Brockmeier, 1992). The three other sources of efficacy information identified by Bandura (1997) are: social and verbal persuasion (the influence of peers and others in conversation with the teacher); vicarious experience (opportunities to watch others of similar situation or skill level); and, physiological and emotional cues (feelings and reactions the teacher has during and after teaching experiences).

Challenges of Implementation of Math Reform

Mathematics is an illustrative area of research for understanding theories of teacher efficacy because there is a clear distinction between reform-based teaching and traditional methods, with evidence that reform practices produce greater student achievement effects than traditional teaching. With the implementation of reform-based mathematics teaching, problem solving and conceptual understanding improve (Boaler, 1998; Brenner, et al., 1997; Huntley, Rasmussen, Villarubi, Sangtong, & Fey, 2000; Schoen, Fey, Hirsch, & Coxford, 1999) without the loss of computational mastery (Hamilton et al., 2003; Mayer, 1999; Reys, Reys, & Koyama, 1996; Riordan & Noyce, 2001; Romberg, 1997; Villasenor & Kepner, 1993; Wood & Sellers, 1997). Yet the traditional model, with an emphasis on basic computational procedures, continues to be the common instructional approach. Students spend the majority of their time doing rote seatwork rather than constructing mathematical ideas. After a decade of mathematics reform in the 1960s, the Conference Board of the Mathematical Sciences (1975) found that "Teachers are essentially teaching the same way they were taught in school (p.77). And in the midst of current reforms, the average classroom shows little change." (Hiebert, p. 11, 1999) The same method of teaching persists even in the face of international pressure to change (Desimone, Smith, Baker & Ueno, 2005; D'Ambrosio, Boone, & Harkness, 2004; Ross, McDougall, Hogaboam-Gray, 2002).

The catalogue of barriers to reform implementation is lengthy. To begin, many preservice teachers are faced with the complexities of teaching mathematics in ways they have not experienced as students (Anderson & Piazza, 1996; Bruce 2006). Researchers have also found that teachers' beliefs about math teaching and math learning can be barriers to reform implementation (Barlow & Reddish, 2006; Brosnan et al., 1996; Rouseau, 2004). The depressed level of teacher comfort with disciplinary knowledge (Bibby, 2000), lack of sustained professional learning opportunities, and limited personal experiences with reform-based practice (Ross, 1999) all contribute to non-implementation. Innovative mathematics instruction requires high ratings of teacher self-efficacy combined with sound teacher knowledge about reform efforts and about reform-based teaching strategies (Borko & Putnam, 1995; Haney, Czerniak, & Lumpe, 1996; Enochs, Smith, & Huinker, 2000).

Qualitative research on teaching efficacy in mathematics is still at an early stage. Leading researchers in the field of teacher efficacy (such as Woolfolk Hoy & Spero, 2005) have called for further research to better understand how teacher efficacy is enhanced. New theories and models are needed to identify and examine the obstacles preservice teachers face, as well as the factors that enable efficacy enhancement in mathematics teaching as careers progress. These new models require constant comparison in the largest sense of the term: new models must be interwoven with existing models as informants and refiners to improve our understanding of how teachers increase their efficacy teaching mathematics.

Research Questions

The essential guiding research question for this study was: *What challenges do preservice teachers face in reform-based mathematics teaching, and what strategies enhance teacher efficacy?* Three key findings are reported in this paper: (i) teacher candidates experienced a similar trajectory as learning math teachers, but each candidate experienced the various stages in the trajectory differently; (ii) effective modeling of reform practices was crucial to candidate use of these methods on placement; and, (iii) the order and quality of the placements had a tremendous impact on efficacy outcomes for each teacher candidate.

Method

This study falls within the qualitative research tradition with a focus on one particular Bachelor of Education site and the experiences of a small group of participants. As an instructor, I was directly involved in the study. After significant ethical measures were implemented (such as undergoing 5 stages of ethical review and employing external assignment evaluators) I engaged in this 2 year research study with participants while teaching mathematics methods courses. Because of my immersion in the study site with participants, complexities and subtleties of the learning trajectories of teachers candidates were uncovered.

The elementary methods course

The elementary mathematics methods course was a mandatory 36 hour component of an intensive 10 month post graduate Bachelor of Education program in Canada. In the program, all elementary preservice teachers (approximately 200) were required to complete 10 half credits with approximately 23 hours of classes weekly. Teacher candidates completed 61 days of teaching placement in elementary grade (Kindergarten-Grade 8) classrooms in blocks of 3 weeks, 5 weeks and 6 weeks.

The mathematics methods course offered a reform-based perspective on mathematics pedagogy that focused on developing a mathematics learning community. For example, the use of manipulatives-based tasks in small groups each week supported knowledge construction and modeled how to engage groups of students in exploring concepts using hands-on materials. This approach was given explicit attention through 'stop-in-action' discussions of learning activities.

Participants

Beginning with 50 voluntary participants, a funnel strategy was applied to narrow the sample to 10 participants in a stratified random sample. Finally, a detailed analysis of 2 participants as extreme sampling case studies (Creswell, 2005) was conducted. Participants indicated a range of levels of confidence (from low to high confidence) and prior experiences (from negative to positive) in mathematics. Participants ranged in age from 23 to 52 years old. The study also employed an external moderator, who was an experienced teacher with no evaluative role, to moderate the focus group interviews. In this paper, the two extreme cases will be featured as bookends to detail experiences.

Sources of data

The study included two distinct phases: a pilot study and a full study. Data collection methods were: an open ended inventory (administered at the beginning of the program); interviews (both one-to-one and focus groups); written journal entries of participants (referred to as Math Logs); methods class observations (field notes of classroom activity and discussions); and, The Teachers' Sense of Efficacy Scale (short form: Tschannen-Moran & Woolfolk Hoy, 2001) administered on four occasions.

A series of data collection and analysis steps were taken using a grounded theory zig-zag method ensuring a representative selection of participants as well as opportunities to refine the quality of data collection throughout the study, increasing the depth of findings and saturation of themes.

Methodological triangulation was ensured through the multiple sources of data analysed for consistency and contradictory evidence from source to source. The external voluntary moderator provided independent open coding of data to enhance theory triangulation (Stake, 1995). In addition, member checks were conducted regularly with three participants of the study (1 from the pilot and 2 from the full study) to further ensure that interpretations were accurate.

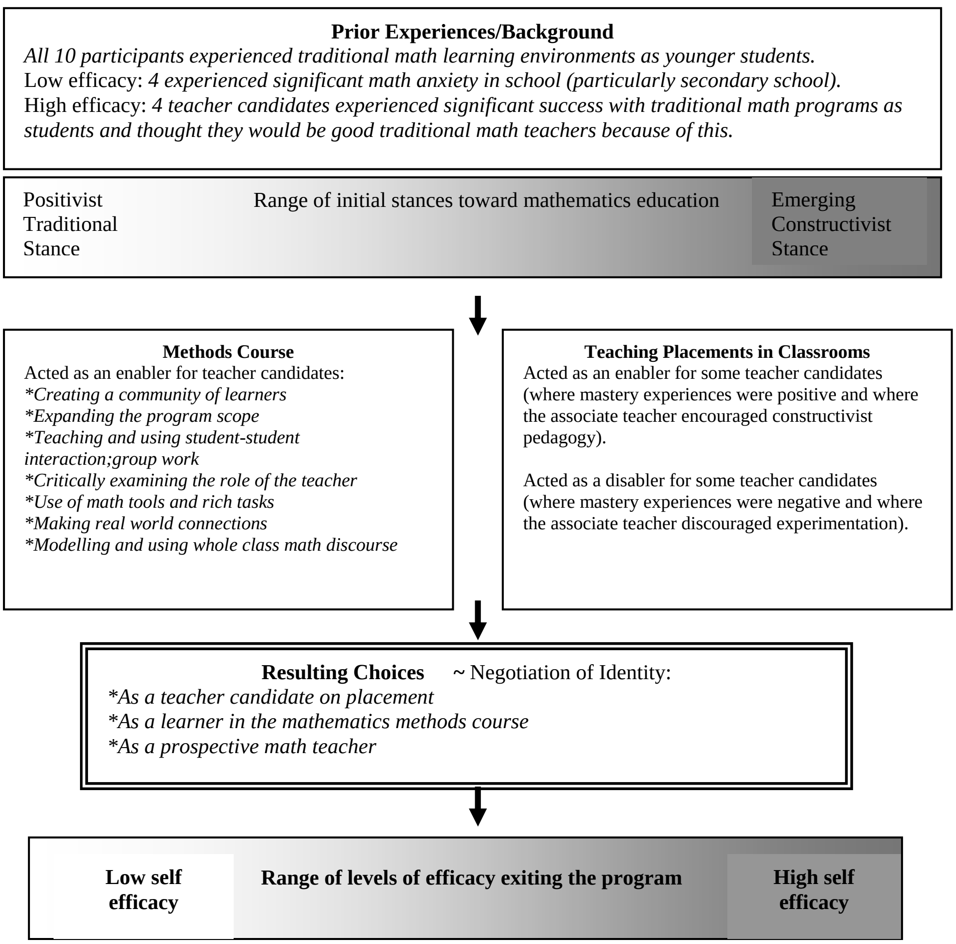

Figure 1. Learning trajectories of elementary mathematics preservice teachers

Findings

The data, by case and across the 10 cases, clearly describes a series of complex and critical stages in the program that all participants experienced. The analysis of findings resulted in a model for examining preservice teacher efficacy (see figure 1). The model in Figure 1 represents a cumulative understanding from all data sources. This model is temporal in nature because it reflects the findings of this particular study time frame with these specific participants. The greatest danger of including a summary model is that it oversimplifies the findings. The linear diagram gives the impression that the stages are perfectly sequential and do not overlap when in reality there was a flexing and moving back and forth through the stages.

In this paper, the two extreme case samples are described in detail as bookend examples of how participant efficacy changed, identifying the influences of change. The two extreme cases describe the trajectories of Maureen and Brian.

The Case of Maureen

Maureen's case represents an example of a most positive outcome.

Prior Experiences: Stage 1
Maureen experienced a traditional program throughout her elementary and secondary education. "Most of my experiences were drill and kill. Pretty dull and lifeless" (Math inventory, 2005). Later on, as an Educational Assistant, Maureen sometimes worked with individual students on mathematics tasks. She used base ten materials with students and "found it to be a lot of fun. Kids loved it" (Math Inventory, 2004). Maureen was developing an emerging constructivist stance from the onset of the program. Her conditions were ideal for entering a transition from traditional past experiences in mathematics as a student to reform-based pedagogy as a teacher.

Experiences with Coursework: Stage 2a
Overall, Maureen found the coursework to be an efficacy enhancer. She discussed several specific features of the coursework that supported her confidence building as a math teacher. These included engaging in group work, participating in a hands-on/ minds-on environment and building a math community. Maureen learned by "doing math". She was able to build her confidence to teach math through this process:

> *Maureen: Actually what taught me, what gave me confidence was actually doing the math, doing it in the class and working in the groups. I know I was on the other side of it. (She laughs)*
>
> *Cathy: What do you mean?*
>
> *Maureen: In terms of - I was the student learning. I was basically being the student. But I think being on that side of it gives you a lot of confidence about how it's going to be received because you are receiving it. (Ind. Interview, 1)*

Placements: Stage 2b
Maureen had the opportunity to work in two reform-based math programs while on placement. The two teachers were expert level associate teachers using reform-based pedagogy. The first associate teacher became a district level consultant shortly after Maureen's placement. The second was selected as a Ministry of Education trainer at her local district school board for a large-scale math initiative. Maureen had positive experiences in which she was both immersed in a reform-based program and encouraged to take risks. On her third placement, Maureen experienced a traditional math program. Although the teacher expected Maureen to follow her program, Maureen was confident enough to use some reform based practices in that placement:

I had the kids going off and doing problem solving on their own and the teacher was really struggling with it. And [the host teacher] wanted to answer the questions and I was walking around and the teacher said (whispers) “Shouldn’t we tell them?”…I said “Nooo, just let them try to figure it out”. The kids were getting a little frustrated and I kept supporting them and reinforcing and that was really interesting. That was a bit of a dance that I had to do. (Individual interview, 1)

Maureen was pushing against the existing norms of the math program, both with the students and the teacher. Indeed, Maureen was influencing the learning environment, the teacher, and the students to construct mathematical meaning. Because of her increased high level of efficacy, Maureen had the confidence to advocate for reform-based strategies in her final placement.

Identity Formation: Stage 3

Maureen’s identities as a general teacher and as a math teacher were intertwined. She was able to teach math on all placements, experimenting with reform-based methods and engaging deeply with coursework: “My identity has been changing so much this year, and I do feel I have grown and learned but my identity as a teacher is still tentatively emerging. I feel like I have been reinventing myself this year” (Math log, entry 16). This ever-changing, dynamic sense of self, although unsure at times, is essentially at the heart of what drives teachers toward on-going professional development. It is reasonable to predict that Maureen will continue to seek professional development opportunities in the future and that her identity formation as a math teacher will continue to grow and change.

Maureen’s Efficacy

On the twelve item, nine point Teacher’s Sense of Efficacy Scale where 1 is low and 9 is high, Maureen’s initial self ratings consisted of two 1’s, one 2, seven 3’s and two 4’s. Seven months later, her ratings changed to one 7.5, five 8’s and six 9’s. The self-assessment scores had increased steadily from the beginning of the program to the end.

The Case of Brian

The case of Brian represents an interesting juxtaposition to that of Maureen.

Prior Experiences: Stage 1

Overall, Brian was a successful math student throughout elementary school, high school and in university. “All of my careers have included math and I love to do it and pass it on to others. (Math log, entry 1). Brian also recognized issues related to math phobias in students: “I hope that new teaching methods and theories help to overcome the early phobias that are developed” (Math log, entry 2).

Experiences with Coursework: Stage 2a

When Brian entered the mathematics methods course, he was surprised and confused. His beliefs about mathematics, and about the teaching and learning of mathematics, were challenged.

What I began to realize is, is that much of my learning of math was nothing but rote and I really had no understanding of what I was doing. I just knew how to do it on an intuitive level. So you know suddenly being in this context where I’m trying to teach it and…you, not so much relearn math, but understand math. (Individual interview, 7)

Brian described his dissonance further: “So there were points I think probably in every class where I was sort of sitting there going: What did Cathy just say?” (Individual interview, 7). Wheately (2002) argues that “profound change requires substantial disequilibrium…through the discovery that ‘what I thought I knew isn’t enough to deal with this new situation’” (p. 9). But just how much disequilibria is manageable?

Brian found the modeling of constructivist pedagogy in the course and questioning techniques built his understanding of how to teach mathematics:

So what the math course did was it challenged the way I thought about how people learn, how you go about teaching: Start from the problem, start with something real, something meaningful. Turn a fan on and ask them what's happening? 'My hair's blowing!' 'Well why?' 'What is wind? Where does wind come from?' (Individual interview, 7)

Brian's self-efficacy scores moved from a mid-high range in September to the high range as the course progressed prior to the first placement. He reconciled his prior traditional experiences with current reform-based experiences increasing his confidence in his abilities to teach math. Brian was eager to try using reform-based pedagogy on placement.

<u>*Experiences on Placement: Stage 2b*</u>

Brian experienced a series of very traditional practicum placements and found them extremely challenging. In the first placement, Brian was actively discouraged from applying reform-based math strategies. "When I think of the first placement working with money with Grade five's, and everything was almost pure paper pencil with at least four kids who couldn't even handle paper pencil. It was just pure torture." (Focus group interview, 1) Brian received a negative evaluation from his associate teacher despite his attempts to follow the program as the teacher expected, which was very discouraging:

I am very confident in myself, and I went out [on placement] and I got a lousy evaluation and I went out [of the school] and balled my eyes out in the car. I went back into the school into the library, hit the computer, sent an email off to my advisor, and started off by saying "I'm good. I know I am, and I will succeed" (Focus group interview, 1).

On his second placement, Brian was again faced with a traditional math program where opportunities for him to take risks with reform-based practices were not fostered. In fact, Brian reached a low point both physically (becoming quite ill) and emotionally and left the placement with an incomplete standing. The physiological and emotional complexities overpowered all other sources of efficacy information for Brian at this time.

On Brian's third placement, the teacher was carefully selected, and agreed to support a more problem-based math program with Brian, but he still encountered a predominantly traditional environment where risks were not encouraged. Brian also believed that there were cumulative effects of the difficult series of placements.

Findings indicated that the order of the placements was crucial. In Maureen's case, the sequence worked in her favour to positively influence her overall efficacy as a math teacher. For Brian, the sequence worked against him to negatively influence his efficacy. As Brian entered his final (makeup) placement, he shifted his expectations of himself and of the placement.

My goal in the last placement is that I am going to do my best. I am going to recognize that mentally and physically I am still recovering. So I'm not trying to be the top level teacher here. I'm not trying to be the champion here. One day at a time, stick to the learning. (Individual interview, 7)

By lowering his standards and in having some positive mastery experiences on the final placement, Brian was able to achieve greater success. This contributed to an increase in hislevel of efficacy at the end of the program.

<u>*Identity Formation: Stage 3*</u>

Similar to Maureen, Brian explained how his identity as a math learner was deeply connected to his identity as a math teacher.

There was a lot of making us do math! I think the largest piece was that I don't learn any differently than those grade 1's do. If I'm really going to understand the concepts of teaching, at some point I am going to have to do it myself and understand what it is I am doing before I can actually start trying to explain that to someone else. (Individual interview, 7)

Brian's Efficacy

Brian saw the first placement as crucial to his wavering confidence. Brian's efficacy waved from low to high during the program. During his final placement, Brian had managed to teach mathematics with a focus on problem solving and as he gained some positive mastery experiences, his efficacy increased. It is difficult to predict stability of high efficacy ratings for Brian given the ongoing fluctuation throughout his preservice year.

Discussion and Implications

Each of the 10 participants experienced a series of relatively complex stages that were common. The framework of stages might be useful to other teacher induction programs when trying to understand the learning trajectories of preservice teachers.

Two participants experienced extreme trajectories, one being consistent and positive, the other being inconsistent and challenging. In the first case, the coursework supported Maureen's emerging social constructivist beliefs about teaching and learning and the first two placements were exceptionally positive mastery experiences. The third placement, although not as positive, was an opportunity for Maureen to advocate for the implementation of reform-based practices that the associate teacher had not used.

In the second case, the coursework was jarring and moved away from Brian's successful traditional prior experiences. Yet Brian was eager to try out new strategies. Unfortunately, most of his placements were unsupportive of the use of reform-based mathematics strategies. Brian's efficacy dropped to a very low level, along with his overall health and well-being. On the final placement, Brian lowered his expectations of himself, met with success, and was supported in implementing some elements of a reform-based program. This led to an increase in efficacy due to successful mastery experiences. The stability of this increase is questionable.

The remaining cases of the cross case analysis confirmed that the two extreme cases were, indeed, the extremes. All remaining cases also illustrated the three main findings of the study: first, teacher candidates experienced similar but varied trajectories as learning math teachers; second, effective modeling of reform practices was crucial to implementation; and third, the order and type of classroom placements had a dominant impact on efficacy outcomes for teacher candidates.

Three resulting recommendations follow: 1) As much as possible, candidates require high quality, supportive, reform-based classroom environments for their first placements. Unfortunately, this is not a likely scenario for all preservice teachers, given the depressed use of reform-based math implementation in North America (Bracey, 1989; Ross, 1999). This leads to a second recommendation: 2) Consecutive Bachelor of Education programs conduct focus group interviews, or focused conversations with all candidates immediately after their first placements. Subsequent placement decisions could then be made with the necessary information about which candidates are most in need of supportive math placements that encourage risk-taking and a range of teaching methods; and, 3) Preservice programs model reform practices with teacher candidates on an on-going basis so that aspiring teachers who have had limited prior experience with reform-based programs can experience the related teaching and learning strategies first-hand. In experiencing the strategies, preservice teachers have clearly indicated that they were more eager, confident, and capable of implementing these same strategies on placements.

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Composition of Functions and Inverse Function of a Function: Main Ideas, as Perceived by Teachers and Preservice Teachers

Calin Lucas
Simon Fraser University

This study investigates what teachers and preservice teachers consider being the main ideas behind two subtopics of the larger mathematical topic of function, namely the composition of functions and the inverse function of a function. The study also investigates what is teachers' and preservice teachers' content knowledge regarding the topics of composition of functions and the inverse function of a function and investigates the connections between the three components of content knowledge: subject matter knowledge, pedagogical content knowledge, and curricular knowledge. Furthermore, the study examines the influence of the teaching experience on content knowledge.

The results of the study suggest that the two groups of participants were not remarkably different in terms of their subject matter knowledge and their pedagogical content knowledge. Majority of participants presented a mainly procedural approach to both topics and disregarded some essential components, such as conditions for existence of inverse. However, the experienced teachers presented a higher competency in the area of curricular knowledge. With regards to the relation between the three components of content knowledge, subject matter knowledge and pedagogical knowledge appear to be interconnected, whereas curricular knowledge appeared to be independent of the previous two.

Introduction

Mathematically, the topic of composition of functions encompasses the following subtopics: the definition of the concept, the algorithm of finding the composite function, the non-commutative property, and the conditions that the domains and ranges of the functions have to meet in order for the composite function to exist. The inverse function of a function contains the following subtopics: the mathematical definition of the inverse function, the algorithm for calculating the inverse function, the relation between the graphs of the function and its inverse, and the concept of one-to-one function. All these subtopics can be defined as essential features of the two mathematical topics.

In addition to the components of each of the two topics, a display of adequate knowledge of the composition of functions and the inverse function of a function needs to encompass the connection between the two topics. The inverse function is defined only in relation with the operation of composing functions, similarly to the definition of the opposite number, in relation to the operation of addition of numbers, or the definition of reciprocal numbers, related to the multiplication of numbers.

This study focuses on what Shulman (1986) called the missing paradigm from the study of teaching, namely teachers' and preservice teachers' content knowledge regarding the two mathematical topics of composition of functions and the inverse function of a function.

The importance of the subject matter knowledge, in general, is the main focus of Shulman's (1986) study. Shulman argued for the importance of teachers' subject matter knowledge in the general process of teaching. Shulman divided content knowledge into three categories: subject matter knowledge (SMK), pedagogical content knowledge (PCK), and curricular knowledge (CK). In his description of the subject matter knowledge, the author emphasized that teachers need to know not just *it is so*, but also *why it is so*. In Shulman's view, pedagogical content knowledge does not refer to how a lesson is delivered (individual work, lecture, group work etc.). It refers to the most useful forms of representing ideas, the most powerful analogies, illustrations, examples, explanations, and demonstrations -- the ways of making the subject comprehensible to others. Implicit within the PCK is the knowledge of why a certain topic is difficult for learners. Further, Shulman defined three components of curricular knowledge: alternative curricular materials, lateral curricular connections and vertical curricular connections.

Teachers' content knowledge in general is also the focus of Ma's (1999) study. She introduced the notion of teaching with profound understanding of fundamental mathematics (PUFM) and she described what PUFM would encompass: an understanding of the terrain of fundamental mathematics that is deep, broad, and thorough: "It is the awareness of the conceptual structure and basic attitudes of mathematics inherent in elementary mathematics and the ability to provide a foundation for that conceptual structure and instil those basic attitudes in students" (Ma, 1999, p. 124).

The research done on teachers' content knowledge of the topic of functions is quite extensive. For example, Norman (1992) used different terms to describe teachers' knowledge related to the topic of functions: practical knowledge, pedagogical knowledge and content knowledge. These categories are very similar to the categories defined by Shulman and they contain the same notions as the attributes that defined Ma's PUFM.

Even (1993), tried to determine the subject matter knowledge, the pedagogical content knowledge and the connection between the two of them for a group of preservice mathematics teachers. Even found many of the participants in the study who lacked the necessary subject matter knowledge. This fact had influenced their pedagogical thinking.

Loyd and Wilson studied the understanding of the concept of function of a secondary mathematics teacher, and its impact on the teaching of functions. The researchers defined some categories to focus on when they studied the function concept in their subject's understanding: definition and image of the function concept; repertoire of functions in the high school curriculum; the importance and use of functions in varying contexts; and multiple representations and connections among them. These themes are part of the themes described by Ma and Shulman. The findings of the study suggested that "teachers' comprehensive and well-organized conceptions contribute to instruction characterized by emphases on conceptual connections, powerful representations and meaningful discussions" (Loyd and Wilson, 1998, p. 270).

Another relevant study is one conducted by Stein, Baxter and Leinhardt (1990). This study focuses on the subject matter knowledge of an elementary teacher, and the impact of this knowledge on class instruction.

A different kind of study that involved the scrutiny of the practice of an individual teacher is the study conducted by Haimes (1999). The researcher studied the teacher from a pedagogical point of view, and less from a subject matter knowledge point of view. Haimes' article illustrates a gap between the intended teaching approach and practice, and explains it by the fact that the teacher is unfamiliar with the "function" curriculum, in other terms, that the teacher is lacking Shulman's CK.

As mentioned above, this study focuses on the specific topics of composition of functions and the inverse function of a function. These topics are two subjects for which there are quite a limited number of references in the literature. One of the few research papers on students' understanding of composition of functions or the inverse function of a function is an article by Vidakovic (1996). She investigates how university students enrolled in a calculus class are able to work with the concept of inverse function, and how a computer environment might enhance students' ability to understand this concept.

Based on general theory, observations of students, and her own understanding of the inverse function of a function concept, the researcher derived a description of a construction processes for the developing schema of the inverse function of a function. The author calls the construction process for the developing schema "genetic decomposition". As a result of the study, Vidakovic designed an instructional treatment that might help the students "to go through the steps of reflective abstractions which appear in a genetic decomposition of the inverse function" (Vidakovic, 1992, p. 311).

Methods

Goals of the Study

This study has three goals, stated below:

1. What is teachers' and preservice teachers' content knowledge regarding the topics of composition of functions and the inverse function of a function?
2. How are the components of the content knowledge -- subject matter knowledge, pedagogical content knowledge and curricular knowledge -- interrelated?
3. How are the three components of the content knowledge influenced by the teaching experience?

Theoretical Framework

For the data analysis I used a framework derived from the theoretical framework proposed by Even (1990). The theoretical framework is suited to investigate the subject matter knowledge as well as the pedagogical subject knowledge of teachers. It contains the following seven aspects: essential features; different representations; alternative ways of approaching the concept; the strength of the concept; basic repertoire; knowledge and understanding of the concept; knowledge about mathematics.

The preliminary data analysis prompted me to modify the theoretical framework proposed by Even. The modified framework that I used for analyzing the field data contains the following criteria: essential features, the knowledge and understanding of the mathematical concept; different representations and alternative ways of approaching the topic; basic repertoire; knowledge of the mathematics curriculum; knowledge about mathematics.

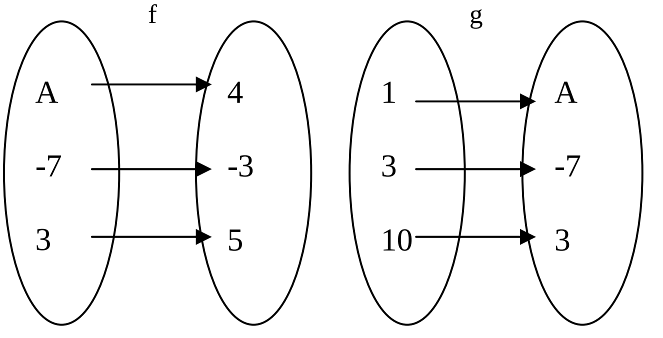
Participants

The participants in the study were two groups of students from Simon Fraser University (SFU). The first group was composed of 10 preservice teachers enrolled in Designs for Learning Secondary Mathematics course. The second group of participants consisted of practicing teachers with a minimum of 5 years experience in teaching mathematics.

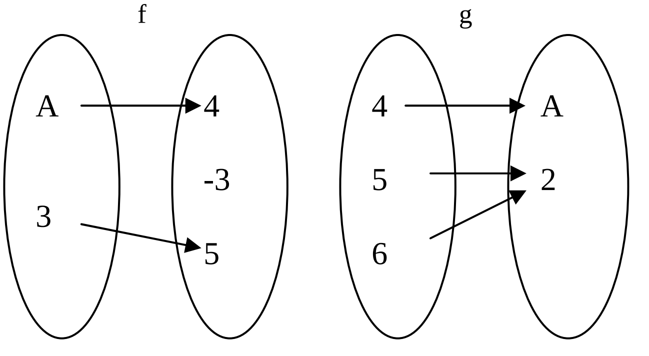
Data Collection

As data I collected a set of lesson plans addressing the teaching of the topics of composition of functions and the inverse function of a function, and I conducted a series of clinical, semi-structured interviews. During the interviews the participants were asked to clarify their arguments, using some specific tasks. Below I present these tasks:

A. Compose the following pairs of functions:
   1. $f(x) = \sqrt{4 - x^2}$ and $g(x) = \sqrt{x^2 - 9}$
   2.



B. Find the **inverse function** of the following functions
   1. $f(x) = 4, x \in \Re$ (follow the algebraic algorithm).
   2.



C. A student of yours calculates the inverse function of the function $f(x) = 3x - 4$ and the answer obtained is $f^{-1}(x) = -2x + 4$. The student checks his work, and when he "does" $f(x)$ with $g(x)$ he gets *x*. Are these two functions inverses for each other? Explain.

Figure 1: Problematic examples used in interviews

Results and Conclusion

Shulman (1986) defined three components of teachers' content knowledge: subject matter knowledge (SMK), pedagogical content knowledge (PCK) and curricular knowledge (CK). The framework designed and used for data analysis addresses the individual components of the content knowledge. The first and fifth criteria of the framework, essential features and understanding of mathematics and knowledge about mathematics, address the SMK. From the data collected it can be concluded that both groups, teachers and preservice teachers, displayed a relatively weak SMK. With regards to this component of the content knowledge, the two groups were different in some aspects and similar in others. The inservice teachers displayed a better grasp of the concept of function, but on the composition of functions they did not show a higher SMK than the preservice teachers. The SMK regarding the topic of inverse function seemed to be higher than the SMK related to the composition of function, for both groups. Among the members of the groups, the variations were fairly high. For example, some participants were not able to enunciate any definition for the inverse function, where others were able to give a correct definition for this concept.

The minor differences between the two groups of participants, regarding the SMK of the topics of composition of functions and the inverse function, suggests that in the case of this study teaching experience did not enhance the SMK of teachers.

The second component of the content knowledge, the PCK, is illustrated by the second and third criteria of the theoretical framework – different representations and alternative ways of approaching the topic and basic repertoire. PCK refers to the most appropriate forms of representing ideas, the most powerful analogies, illustrations, examples, explanations, and demonstrations, and it does not refer to how a lesson is delivered. Implicit within the PCK is the knowledge of why a certain topic is difficult for learners. My expectation was that the two components of the content knowledge, PCK and SMK, would be interrelated. In the case of the participants in this study, the above statement seems to be true. Both groups of participants showed a narrow PCK, reflected by their ways of approaching the topics of composition of functions and the inverse function, as well as by their ways of representing the functions during the process of teaching. The basic repertoire of the participants in the study was limited, but it reflected the SMK of the two mentioned topics. In the following paragraphs I elaborate on the link between the SMK and PCK in the case of the participants in this study.

The basic repertoire of the participants in the study reflected their SMK. For example, most of the participants were not aware of the role of domain and range for the composition of functions. As a direct consequence, their basic repertoire did not contain any problematic examples that would emphasize the importance of domains and ranges in composing functions. Continuing with the topic of composite functions, the participants did not provide specific examples for showing the exceptions from the non-commutative rule. Some of the members of the two groups were aware of the non-commutative property, but they did not push the issue to the end.

Regarding the inverse function, the basic repertoire of most of the teachers and preservice teachers contained for example, quadratic functions. The role of these examples is to show that not all the “elementary” functions have inverse functions. The basic repertoire of all the participants missed the example of constant functions, where the algebraic algorithm fails to provide an answer. This gap in the basic repertoire can be explained by the fact that the SMK of most of the participants did not contain a clear idea about the notion of one-to-one function.

With respect to the different representations and the ways of approaching the topic, the participants presented divided ideas on the order of teaching the two topics. Since there were teachers and preservice teachers who argued for composition first, and there were participants who argued for inverse function first, it can be said that experience does not have much to do with the ways of approaching the two topics. This argument is sustained by the reasons presented by the interviewees: the same argument, algebraic “easiness”, was presented as a decision factor in teaching composition first by one person, and then it was presented as a decisive factor for teaching inverse function first by a different person. There were a few participants with arguments rooted in their SMK or PCK. An example of this kind is an argument relying on the mathematical fact that there is no inverse without defining first the composition of functions.

The different representations for the functions exhibited by the members of both groups illustrated a limited PCK. For example, only one person used arrow diagrams in his explanations. In previous chapters, I mentioned the pedagogical advantages presented by the use of this representation. The most frequent representations presented in the study were the algebraic formulae and the Cartesian graphs, representations that are not necessarily the best representations to use when introducing essential features of composition of functions and the inverse function of a function. The almost exclusive use of these two representations denotes the limited capability of the participants in this study in using the most illustrative examples for making the mathematics comprehensible to students.

Contrary to expectations, this study shows that experience did not necessarily make a difference in the PCK displayed by the inservice and preservice teachers. The PCK seems to be influenced or related more tightly to the SMK than to the years of teaching experience that the individuals possessed.

The "knowledge of the curriculum" criterion of the framework addresses directly the CK, specifically, the vertical CK. The vertical CK deals with connections between different high school mathematics topics. The connections between high school mathematical topics are made possible by underlying mathematical concepts that are common through the curriculum. Related to this criterion, the two groups were different. The inservice teachers were able to make more connections between the topics of composition of functions and inverse function, and other topics of the curriculum. In the case of CK, this study shows that the teaching experience plays a significant role.

To summarize, the participants in my study displayed a relatively weak content knowledge of the topics of composition of functions and the inverse function of a function, illustrated by the weak SMK and PCK exhibited by both inservice and preservice teachers. The study shows that CMK and PCK are strongly related to each other, and, what is surprising, they are minimally influenced by experience. The CK is positively influenced by the teaching experience, and it seems to be relatively independent of SMK and PCK.

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Relations Among *I*, *Thou,* and *It* in an Elementary Mathematics Professional Development Setting

Gladys Sterenberg
University of Lethbridge

Mathematics is often viewed as a set of invariant prepositions, theorems, and axioms. However, from a historical perspective, these mathematical objects and results have evolved and are contingent on cultural developments within society. Some possibilities for learning mathematics within a historical context have been considered by researchers as they investigate the extent to which this can prompt teachers to consider the human agency in the discipline. Ernest (1991), Fauvel (2000), and Lakatos (1978) suggest that the study of the history of mathematics can provide a means for altering mathematical experiences.

The purpose of this research is to investigate experiences of four elementary teachers as they studied the history of mathematics, created a story based on this study, and implemented a series of lessons rooted in historical stories of mathematics. Three research questions frame my study: (1) How did engagement with the history of mathematics impact images of mathematics? (2) What images of mathematics were envisioned by the teachers in the study? (3) How did relationships among a teacher education, the teachers, and mathematics influence teacher learning about the nature of mathematics? Drawing on Martin Buber's notions of *I-Thou* and *I-It* relations and on David Hawkins' related triad of *I-Thou-It* relations, I primarily explore relations between the teachers and the mathematics being studied. In addition, I consider my relation to the mathematics and relations among the teachers and myself.

An *I*, a *Thou*, and an *It*

The philosophical ideas of Buber (1923/1970) on relations inform my thinking about interactions in professional development settings. He proposes that humans enter into relationships with their whole beings. In describing the nature of these relationships, he offers what he calls two 'primary words' or states: *I-Thou* and *I-It*.

According to Buber, the world of experience belongs to the primary word *I-It*. The primary word *I-Thou* determines the world of relation. Only *I-Thou* relations are enacted with our whole beings. For Buber, those who experience are not entering into relationships, since *It* does not participate. The world of *It* allows itself to be experienced but contributes nothing and nothing happens to it. In contrast, relations are actual: the relation acts on me as I act on it. Actual relations are dialogic, reciprocal, mutual meetings. Actual relations are genuine and authentic, developing through enactive encounters.

What is noticeably different from Buber's proposal is Hawkins' (1974) emphasis on relationships with the mathematics. He suggests that relations among the teacher, the student, and mathematics are dialogic and reciprocal; the interaction between these three "beings" is described in Buber's terms of *I-Thou* relations of mutual encounter. Hawkins explains these relationships in this way: "Without a Thou, there is no I evolving. Without an It there is no content for the context, no figure and no heat, but only an affair of mirrors confronting each other" (p. 52). Thus, Without an *It*, *I-Thou* is merely a reflection of relations. I suggest that the three dyads (*I-Thou*, *I-It*, *Thou-It*) presented by Hawkins can be used to describe relations within professional development settings, where *I* represents the teacher educator, *Thou* represents the teachers, and *It* represents mathematics. When relations among *I*, *Thou*, and *It* are encountered as the mutual relations Buber describes as *I-Thou*, professional learning becomes enactive.

Four elementary teachers within a rural school district were invited to participate in the study: Barb, Martha, Denise, and Christine. Each teacher was interviewed at the beginning and end of the project. During the first phase of the study, we identified the curricular area of number concepts as our focus. To begin our investigation, I offered selected readings on the history of numbers. After this study, each teacher wrote a short story about numbers and shared these with the group. We selected one of these because it seemed to resonate with each of us. We worked closely together as we planned student tasks and created lesson plans that incorporated the story.

The second phase of our study focused on the implementation of the story. Throughout this phase I provided release time for teachers so they could visit another member's classroom. While one person was teaching, the other made observations. In the third phase, we used a presentation software program to format the story and series of lessons to be included on a compact disk.

During the study, we met bi-weekly as a group eleven times. Bi-weekly teacher reflections were written by all participants and collected during this study. Collective and individual conversations were recorded and transcribed. Curriculum artefacts and observations of students were gathered and I wrote detailed field notes after each group meeting. My analysis examines the conversational discourses that seem to offer expanded relational possibilities and insights within our learning community.

Relations of Thou-It

To begin our investigation, I offered selected readings on the history of numbers from *Zero: Biography of a dangerous idea* (Seife, 2000) and we discussed our interpretation of this literature. Each of the teachers wrote a short story about the origin of numbers by placing characters into a setting and developing a story line. We chose to work with Denise's story because it offered many possibilities for investigating mathematics. She wrote:

> *Long, long ago, before there were numbers, things were quite different and much more difficult. One day a hard-working peasant named Matthew took potatoes to the market to trade for other services. He heard that Joseph, a merchant at the market, was very fair when it came to trading. So Matthew rushed with his potatoes to make a trade with Joseph. "I have SOME potatoes to trade Joseph. Can you trade my potatoes for SOME milk?" "Yes, Matthew, I can trade with you but, tell me, what does SOME mean?" Matthew thought. "I don't know what SOME is Joseph but this is what SOME looks like." He handed a dirty burlap bag to Joseph. Joseph peered into Matthew's deep, dark bag and snapped at him by saying, "I can't see SOME. Your bag is too deep and too dark." "The potatoes are all at the bottom." "You need to show me SOME." So Matthew turned the bag upside down and SOME potatoes dropped to the ground with a thud. "Here is SOME," Matthew said with*

> *pride. Joseph laughed and said, "I traded SOME potatoes with Alexi yesterday and his SOME was much more than yours. I cannot trade SOME milk with you today. Come back tomorrow with SOME MORE potatoes." Matthew understood what Joseph wanted so he went home and collected SOME MORE potatoes. When he returned to the market the following day, he gave his SOME MORE potatoes to Joseph and received the milk in return.*

The pedagogic intention of this particular story was to make mathematics more meaningful or relevant for students and to allow new concepts to be coupled with prior knowledge or experience. The connection to everyday experiences was significant to the teachers.

At the beginning of the project, I believed that through the process of creating and implementing a mathematical story, teachers might relate more positively to *It*. However, an initial preoccupation with the story dominated our conversations and the teachers appeared rooted firmly in a narrative mode of thinking. For me, this was problematic because I was interested in finding a bridge between story and mathematics. I believed that a consideration of the history of mathematics would encourage teachers to focus more on the mathematics. In other words, I wanted the teachers to think both narratively and mathematically about the story.

Bruner (1986) distinguishes between *narrative* and *paradigmatic* ways of knowing:

Narrative	Paradigmatic
Focuses on particular experiences	Emphasizes general causes
Establishes verisimilitude	Uses procedures to verify empirical truth
Seeks connections of lifelikeness	Driven by logically-generated hypothesis
Rooted in renascent (resonant) truth	Strives to identify universal truth

For Bruner, narrative thought is rooted in particular yet resonant experiences. By contrast, he believes that the paradigmatic mode seeks further and further abstraction from time and place. Mathematics is traditionally regarded as archetypal paradigmatic thought.

During the implementation stage, most of the teachers chose to offer story problems to their students. For example, in Denise's lesson, students were given a diagram of a field containing nine hills of potatoes, told that each hill yielded eight potatoes, and asked to calculate the number of potatoes that could be grown in the field. These choices exemplified a focus on the paradigmatic mode of knowing mathematics.

Although Denise mathematized the story by presenting her students with a paradigmatic task that involved solving a story problem, finding equivalencies, and constructing T-charts, she was particularly interested in how the students experienced the story. She decided to have them illustrate it. Her initial paradigmatic approach to the mathematics shifted to the narrative mode as she focused on how the students were imagining the narrative details of the story:

> *And I wanted to relate it back to [the context], because not much focus was done on Matthew and Mary's field. So I wanted to take the story to their field and to show their hills of potatoes and things like that. And we talked lots about how potatoes grow and things. And I think there's a lot of experience out of that. You could see on one of their pictures, they drew all these fields. Which was kind of neat because they brought that back themselves. And like I said, I chose to illustrate the story because I wanted to see what they would remember.*

The two tasks that Denise presented to her students, one of solving a story problem and the other of illustrating the story, demonstrate the two different modes of knowing mathematics. While many of the teachers included paradigmatic tasks of solving story problems, most became preoccupied with the narrative dimensions of the students' activity. Denise was surprised that "a lot of them went modern day even though we said long, long ago before there were numbers." Martha noticed that some of her students thought the story took place in pioneer times. Christine had also asked her students about how they were viewing the story and found it odd when one of them pictured the characters in white and brown. She suggested that this particular student could not form an image of the setting of the story.

The teachers' focus on the particularity of the images points to their preoccupation with the narrative dimension of the story. What happens when we become rooted in the narrative dimensions of mathematical stories? How are we then to turn to the paradigmatic mode of thinking? In what ways do these two modes of thought interact?

I had expected the teachers to encounter some tension between reality and fiction, between truth and imagination. However, this did not arise in our interactions. Throughout our conversations during the fifth, sixth, and seventh group meetings, the teachers had resisted my efforts to encourage them to depict the origins of numbers accurately in a historically correct setting. Until the eighth meeting, historical accuracy was not seriously considered. Even during the eighth meeting, it seemed that the teachers wanted to retain the folktale quality of their story but were challenged because their students were inventing a historically incorrect setting. The teachers continued to reject the categorization of the story as 'historical fiction'. Perhaps they feared that the significance of their story would be obscured if they focused on issues and questions of historical fact. In their story, mathematical "truths" involved philosophical ideas not factual data.

The teachers did not want the story set in the present time and chose a biblical era. Why was this seen as important? I was perplexed by the biblical quality of the story, particularly evident in the Denise's choice of character names. During the sixth group meeting, I drew her attention to her selection. Both Denise and Barb resisted renaming the characters, even when I offered historical evidence for an alternative place and time.

I began to investigate reasons for the biblical references. Bateson (1994) suggests that when a story resonates, there is a translation from simple narrative to parable. Traditionally, parables are related to a biblical context. I wondered if the qualities of parables were cued by the biblical tone of the story. During the seventh group meeting, I offered the possibility that their story might be a parable. This seemed to resonate with the teachers as we investigated the relationship between philosophy and parable. All of the teachers identified an important feature of their story as leaving the meaning open for discussion and interpretation. Martha suggested that in their story, "there's no lesson like a right or wrong lesson or a moral lesson, as we call it." Christine proposed that their story contained a concept, a kind of lesson about trade. Still unsure of the characteristics of parables, Martha asked, "Does a parable leave the reader asking questions?" The other teachers expressed agreement, whereupon Martha asked, "Is that what we're doing here? Wanting kids to ask and provoke the discussion about what is *some*? And what is *some more*?" This seemed to resonate with the teachers.

I decided to seek the teachers' reactions to Gerofsky's (1999) perspective on parables. She writes:

> *Unlike word problems in mathematics education, parables, koans and other teaching tales demand no solution, and often ask no direct question. They are not meant to be solved, and generally present an insoluble, paradoxical dilemma. Teaching with parables and other teaching tales traditionally involves a discussion of the contingencies of the story, of the problems of human life that relate to the story, of the sources of its paradox. Teaching tales are certainly not meant to be disposable exercises. They are made to be held onto, to irritate, to resonate. A good parable will inspire contemplation, and will be recalled in times of difficulty as a way of trying to make sense of a seeming impasse. (pp. 39-40, italics in original)*

When I presented this paragraph to the teachers, Denise immediately replied, "That's exactly it. This isn't meant to be solved, it's a question, a thought."

In what ways might parables be considered mathematical stories? In order to turn the conversation back to a paradigmatic mode of thinking, I asked the teachers to reflect on the *mathematical* value of the lesson that they implemented.

Initially, Denise identified student activities of constructing graphs and T-charts and performing multiplication operations and Barb extended the list: “There’s the number concept, there’s division, there’s measurement, mass, capacity. There is data management, making graphs. There’s problem solving, there’s patterning. There was tons [of mathematics].”

Christine mentioned that thinking about mathematics was evident and Denise commented that freedom to think was important. She highlighted the mathematical significance of argument and justification.

While the other teachers emphasized the value of mathematical processes, Martha felt that the story was powerful because it integrated language arts, mathematics, and health:

> *[The students and I] were getting into almost an ethical discussion and the health curriculum and how everything was so intertwined. I found it really hard when I was planning to pull out, because you had said make sure that it’s a math lesson, and I really struggled with that. Because it’s an LA lesson, it’s a math lesson, it’s a health lesson. It’s everything truly embedded together and to have to hang out one label on it, I struggled with [that].*

Martha’s struggle to separate the mathematical content from an ethical discussion was similar to the difficulty Barb experienced when trying to focus solely on one mathematical concept. On reflection, Barb recognized that this might not be problematic because “having this conversation with you, I’m thinking isn’t that great because that’s what math is all about. Math can be open-ended.” The focus on open-endedness and ethical pondering points towards to the parabolic dimensions of this story.

When Christine implemented the lesson, she admitted that she was not totally comfortable with the open-endedness of it. Yet she recognized the value of this experience for her students. She was willing to practice being comfortable with the open-endedness because she was doing it for a purpose that involved increasing student confidence in mathematics.

Parables are characterized by their concise narration, by their use of repetition to maintain the action, and by their lack of conclusion where certain issues are left unresolved. Many of this particular story’s features, as identified by the teachers, are similar to these characteristics of parables. Their story includes narrative illustrations of philosophical ideas of mathematics encountered in everyday life. It is a teaching tale and gets its truth, as well as its potential to educate, not from an exposition of facts but from the open-ended and metaphorical way it provokes interpretation and contemplation of the relationship between people and mathematics.

In this study, I offered an alternative to viewing mathematics as arithmetic procedures by engaging four elementary teachers in the study of the history of mathematics. For those teachers in my study who had previous negative experiences in mathematics, and for those who tended to focus their teaching on the direct instruction of algorithms, I prompted reflections on views of mathematics as a humanity. This perspective of mathematics was very challenging but story and history allowed teachers to engage in mathematics in a comfortable and familiar setting. I believe this lowered their resistance and contributed to more positive images of mathematics. While this encounter seemed to strengthen *Thou-It* relations, did this study shift *I-It* relations?

Relations Of I-It

As I began to better understand how the teachers were viewing the value of writing this type of story for a mathematics classroom, I reconsidered my view that teachers may be trying to escape mathematics through the narrative. My approach to writing historical stories was to begin with the paradigmatic and move towards the narrative way of knowing. The teachers were beginning from the narrative and were now with my pressure attempting to move toward the paradigmatic. This produced tensions because I tended to turn mathematics into a narrative rather than mathematize the story, the approach favoured by the teachers. Clearly, our approaches were not meeting halfway.

Thus, I reconsidered my view of how the two approaches need to meet. I intended to turn the mathematics into a story. Teachers were mathematizing the story through tasks; they were applying the mathematics. Perhaps these are not the only approaches. Pimm (1991, 1992, 1995) reminds me that the gap between mathematics and story can be bridged through language. In the teachers' story, language and mathematics are implicitly related. But it is possible not to see the mathematics in the language. Whereas I am seeing the mathematics in the everyday word of *some,* and the teachers are perhaps seeing the mathematics in this word, the students most likely cannot yet see these relations. Therefore, the story written by the teachers can promote discussion about mathematical discourse.

So how does the story created by the teachers align the narrative and paradigmatic worlds? Perhaps its appeal lies in the way it helps its reader and hearer engage in the story and wrestle with what the story means. This story involves significant philosophical problems arising throughout the history of mathematics. These are humanly contextualized issues: What are numbers? Why do we use numbers? How might they have been invented? What would a world be like without numbers? How do we engage in mathematics? What is mathematics? The story provokes thought about mathematics as a humanity and as a science. Pondering the philosophical aspects of the story and engaging in paradigmatic tasks of solving related story problems or creating number equivalencies can develop robust relations to mathematics.

My intention was to blend two dimensions of mathematics: logical deduction and analogical truth. When this story was used as a teaching tale, the teachers mathematized it in a variety of ways that emphasized the paradigmatic mode. By creating and implementing this story, the teachers seemed to engage in both dimensions of thinking.

Relations of I-Thou

The robust relations within the group enabled the teachers to support each other in their professional growth. This interaction continued outside of the group meetings and classroom visits as teachers became friends. Drawing on such conversations with the other teachers, Denise said:

> *I've just really enjoyed the mutual idea of all of us coming together for the same thing. Not really knowing, I guess, what I was getting into when we started and thinking just how […] we don't want to stop. This has been really good. More than met our expectations.*

An unwillingness to stop participating in the project seemed to indicate that the teachers were finding the experience worthwhile. They wanted to extend the study. Clearly, teacher learning occurred in this professional development setting because relations among *I*, *Thou*, and *It* were robust.

The presumed ease of co-construction as reported in the literature was problematic for me. Throughout my study, I grappled with my role as teacher educator within the group. I was aware of both my need to be seen as a teacher and the teachers' resistance to view me in this way. I was reluctant to demonstrate my mathematical expertise because I did not want to disrupt the developing relations of *I-Thou* or *Thou-It*. However, my choice to do this may have limited the depth of study. I could have, and perhaps should have, participated more actively as a voice of mathematics and its history.

It is clear to me that teacher educators with expertise in mathematics education have a unique role to play, especially when working in an elementary setting. A teacher educator's role could include offering an idea for the initial topic, demonstrating responsiveness, flexibility, and expertise in offering mathematical tasks for teacher learning, administering the study, and engaging teachers in conversations about theory of mathematics pedagogy as an outcome of their observations.

It is also apparent to me that academic researchers fulfill a necessary role in the group if the goal of the study group is to increase student learning and create teacher-centred theories. In the absence of experienced teacher leaders in elementary schools, administrators often get involved. This can be problematic if the agenda of the study becomes imposed as evident by the change in teachers' autonomy within professional learning communities that have become mandated by the government in this particular province. When participation becomes compulsory, relations of *I-Thou-It* can become strained. If researchers were to initiate the study groups, teachers' agendas would not necessarily be tied to district and school agendas. Strong *I-Thou-It* relations between the academy and the field could be fostered and theories of education could be robustly linked with classroom examples. Differentiated expertise within the group could be embraced.

Nested Relations of I-Thou-It

I am aware that fostering conversations about mathematics as a humanity can be done effectively in the context of the history of mathematics. However, it is the interplay of relations among *I*, *Thou*, and *It* that enhances professional learning. I claim that participation in this project fostered alternative images of mathematics as teachers grappled with philosophical tensions in mathematics. Engagement with the history of mathematics by itself was insufficient to move teachers toward greater involvement with the paradigmatic dimensions of mathematics. Nevertheless, teacher learning about mathematics occurred in this professional development setting because relations among the teacher educator, the teachers, and mathematics were robust.

Throughout this study I grappled with how mathematics could be viewed as a humanity. The creation of a parable fostered conversations about the philosophy of mathematics, an area of thought often omitted in most school experiences. Teachers' disembodied images of mathematics as a collection of disjointed facts were challenged by the integration of mathematics into a lifelike, narrative setting. However, seeking a middle ground between paradigmatic and narrative dimensions of mathematics proved difficult because of the teachers' tendency to attend to what they were comfortable with.

At the beginning of the study I was interested in how the teachers' experiences of mathematics, their views of mathematics, and their content knowledge of mathematics could be altered. While I believe the teachers encountered mathematics in a new way and I have evidence that their images of mathematics changed, I am challenged by their limited ease with the paradigmatic dimension of mathematics. Perhaps emphasizing mathematical stories constrained the possibilities for viewing mathematics as both narrative and paradigmatic. I have come to understand that my role might be to provoke some discomfort with the narrative aspects of mathematics within the dialogical community. I believe that my persistent presentation of relevant mathematical tasks to enhance teacher learning will not prompt relations of distance if I am encountering relations of *I-Thou*.

At the end of the study, I was able to envision myself as a researcher and teacher educator. My experience as a teacher provides me with expertise in teaching, but I also now realize that I have expertise in these other areas. I am confident I can now enter into meetings with my whole self and that my evolving self can find a place within a professional development community as a researcher and as a mathematics teacher of teachers.

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Voices in The Silence: Narratives of Disadvantage, Social Context and School Mathematics in Post-Apartheid South Africa.

Dalene M. Swanson
University of British Columbia

Introduction

Voices in the Silence is a critical exploration of the construction of disadvantage in school mathematics in social context. It provides a reflexive, narrative account of a pedagogic journey towards understanding the pedagogizing of difference in mathematics classrooms and its realizations as pedagogized disadvantage in and across diverse socio-political, economic, cultural, and pedagogic contexts.

The fieldwork mostly occurred within the Cape Province of South Africa, in schooling communities with socio-economic, cultural and historical differences. Research took the form of interviews, discussions, narrative-sharing, and participant observation, in a recent post-apartheid context.

In resistance to perpetuating hierarchized, linear or scientistic approaches to research within traditional social sciences and mathematics education, I embrace an arts-based methodology. Through narrative and poetic form, I engage with socio-political, cultural and pedagogic implications of the social construction of disadvantage in school mathematics discourse and practice. The dissertation, therefore, offers interdisciplinary approaches to critical concerns of inequity and access, calling on the emotive, spiritual, embodied, and personal domains of experience in problematizing the (re)production of disadvantage and certain socio-cultural practices that school mathematics supports.

The concept of *silence* is introduced to interrogate the interstitial/intertextu(r)al places of ‘lack’ and ‘deficit’, and competing ideological positions and discourses of power, which inform the pedagogic and lived realities of “disadvantage” in mathematics classrooms within different contexts. *Moments of articulation* within fieldwork define utterances and somatic performances embedded within narrative contexts and their attending discourses, and instigate investigation, deliberation and engagement in analyzing the multiple ways in which disadvantage takes root/route. These *signpost* where ‘voices in the silence’, in discourse, context, and the subjectivities they (re)produce, may be recognized, problematized and rearticulated through narrative.

This dissertation’s major contribution is to open up spaces for dialogue with(in) silence through a reflexive narratizing. Ultimately, *Voices in the Silence* is an invitation to a dialogical pedagogic journey that seeks to provide roots/routes of engagement with the ideals of social justice and an egalitarian society. It attempts to find narrative moments within the difficult terrain of research work and lived experience where constructed and pedagogized disadvantage can be re-imagined and transformed into transcendent pedagogies of empowerment and hope.

Given the narrative format of the work, theoretical perspectives and methodological approaches are purposefully integrated. It is helpful, therefore, to provide a research description of the 'Dissertation Structure', which follows.

Dissertation Structure

This dissertation is comprised of four sections, represented by the quarters of a circle. These sections represent four phases of a cyclical journey, metaphorically represented by the four phases, or quarters, of the moon. This is also in keeping with a more 'circular', or 'elliptical', narrative-based approach synonymous with some African indigenous epistemologies.

Each *section/phase* begins with a preface, walking the reader through that phase of the journey. This provides the reader with some background as to what to expect. It is metaphorically similar to explanatory travel notes in a photograph album or an entry in a journal of an expedition.

PHASE ONE: AN INTRODUCTION relates to the process of deciding on travel, where and how to travel, what the traveler might be looking for, the obstacles she might expect to encounter, and the way of seeing (or not seeing) that the traveler might bring to bear on the way in which she embarks on her travels. This section walks the reader through the introductory phase.

> **SILENCE** sets the tone for the dissertation by introducing the metaphor and theme of silence for debate. This debate congregates around *silence* as it invests in the social construction of disadvantage and the way it may be lived out in relation to school mathematics discourse in different contexts. It also offers an interpretation of silence as living and operating within the interstices and intertextuality of discourses, agents, and ideologies of power within the social domain, and how this (re)produces disjunctures, paradoxes and dilemmas within fieldwork, research writing and lived experience. While it interrogates the many slippery forms and interpretations of silence, it provides it with metaphorical significance through the theoretical feature of voice.

PHASE TWO: THEORETICAL DISCUSSION AND CRITIQUE represents organizing the trip; making travel arrangements; packing for travel; deciding what to take, what is needed, and checking one's itinerary. It walks the reader through some theoretical discussions and critique of narrative. It lays out a framework and reference points to enable the reading traveler to proceed. It also unfolds the map of the journey, as planned and experienced.

> **THE TELLING OF TALES** provides a background on narrative inquiry and presents a critique on the advantages and disadvantages of narrative within an arts-based framework. It also offers reasons for my embracing of narrative for this research project.
>
> **UNFOLDING THE MAP** tells how the pedagogic journey unfolds; what to expect for the rest of the dissertation; some details on data collection and the research journey; and a brief map of the journey.

PHASE THREE: JOURNEY ACROSS CONTEXTS represents the ever-emergent state of travel and the research journey itself. It walks the reader through the actual physical and pedagogic journey through the 'telling of tales'. The four narratives describe the narrative intricacies of the research journey, but follow a chronology of writing, rather than the physical route. In this way, the pedagogic journey is fore-grounded.

STATES OF NATURE is a reflexive account of a visit to a farm school in rural post-apartheid South Africa. It focuses on issues of normalization, localization, and proceduralism. It looks at the importance of context, prevailing ethos, and the political disjunctures between the local and global. Conservativism and white governmentality are problematized in how they create 'the normal.'

FISHES AND LOAVES addresses the philosophy of Africanisation, its incompatibility with the ideology of neo-liberalism, how Africanisation can become subsumed within neo-liberalism, and how this plays out in a mathematics classroom in a context of 'poverty'. Issues of neo-colonialism, and how these inform poverty education and disempowerment within a mathematics education context, are at the fore.

ROOTS/ROUTES explores concepts of rootedness and routedness. It addresses notions of performance and rhizomatic journeying as they inform research. It highlights dilemmas, disjunctures and paradoxes within mathematics education discourses and the mythologies produced, as informed by progressivism, neo-liberalism and globalization. Contradictions in local and global contexts are manifest in lived experiences as sites of struggle between competing ideologies. This narrative weaves a critical and reflexive account of research moments as lived experience.

CULTURAL BEADS AND MATHEMATICAL A.I.D.S. explores critical issues in mathematics education and highlights further contradictions and dilemmas within different research contexts. It addresses issues of universalism, pedagogic constructivism, and progressivism in mathematics education, and how these are recontextualized in local contexts which contribute to the construction of disadvantage. In particular, progressive education rhetoric of 'relevance' in mathematics education is interrogated in terms of its recontextualization across pedagogic contexts, and how it might facilitate pedagogic disempowerment rather than liberation.

PHASE FOUR: TOWARDS JOURNEY'S END; A RETURNING is an 'unpacking' phase. This is a time when photographs of memories are placed in an album, and reflective journal entries written. I walk the reader through the returning phase and through finding 'stopping places' to pause, reflect on proximities and distances to research relationships; to stand back; and to allow the voices of the journey to come together, to collide or coalesce in finding new meaning in the way they shape experience and create emergent identities. It is also a time to ponder and seek a new way ahead, perhaps embark on another post-travel journeying. This notion of *returning* is a double entendre in the dual senses of 'going back' and 'giving back'. There are three pieces. VOICES OF SILENCE is a poem describing the many voices of silence as they infuse themselves within research texts. VOICES explores disjunctions, paradoxes and ironies, bringing the voices from different research contexts into one coalescing text in examining how disadvantage is constructed and pedagogized within school mathematics. Lastly, VOICES IN THE SILENCE offers some closure and a re-opening.

Throughout the dissertation, **FOOTNOTES** and **ENDNOTES** appear. Footnotes (referenced *alphabetically*) refer to shorter commentaries, explanations, translations, or definitions. Endnotes (referenced *numerically*) refer to more in-depth theoretical discussions and critiques, or offer some alternative perspectives to parallel/ divergent/ convergent discussions or routes to the pedagogic journey. While the footnotes and endnotes provide context and theoretical grounding, the narratives can stand alone.

Significance of the Research

Voices in the Silence, through its critical sociological focus, narrative expression, theoretical complexity and interdisciplinarity, breaks new ground in the mathematics education field, and offers a significant contribution to qualitative inquiry in its theoretical-yet-grounded perspectives; non-traditional and aesthetic modes of delivery; multiple forms of engagement; integration of different methodological approaches, and commitment to egalitarianism and social justice.

The dissertation broadens the scope of interpretive possibilities to encompass interrogation of dominant discourses and universalizing ideologies within the social domain, which colonize meanings. Ideological positions such as globalization, neo-liberalism, neo-colonialism, and aspects of progressivism compete for hegemony within mathematics classroom contexts as sites of struggle for meaning, informing discursive positions of disadvantage, delimiting practice and disempowering students constructed in terms of social difference discourses such as ethnicity, gender, class, race, poverty, and ability, amongst other positions. The incommensurability of certain social domain discourses produce disjunctions, contradictions and dilemmas, experienced as a lived curriculum of *pedagogic disadvantage* in the lives of students and teachers within contexts of *constructed disadvantage*.

The research places a strong emphasis on local, situated and marginalized contexts, and examines the way in which hegemonic social domain discourses are reconfigured in pedagogic practice in these locations. It develops a critical position of education as situated within a complex of global politics, cultural knowledge, local values and identities.

While research took place in diverse South African schooling contexts during a period of unprecedented socio-political change, this helps to reveal certain oppressive pedagogies and practices that often are obscured by a veneer of overall greater socio-economic ‘wellbeing’ and ‘stability’ in other contexts. The implications for international contexts are made explicit, and have particular relevance for marginalized, multicultural, and aboriginal schooling contexts. The research contributes to policy initiatives that are directed at democratic education principles and social justice. Nevertheless, the dissertation emphasizes *resolution* over the traditional research objectives of proposing *solutions*.

Narrative and arts-informed approaches to mathematics education issues are rare, and the dissertation makes an important contribution in this respect and in its critical, socio-political focus, and moral and ethical commitments. It offers perspectives drawn from post-structuralism, post-modernism, and post-colonialism. It resists psychologistic interpretations of learning disadvantage that objectifies, constructs, pathologizes, and participates in deficit discourses. Through narrative, it deconstructs and reconstructs interpretations of pedagogic disadvantage and lived experience in ways that advocate for community empowerment, offer hope of renewal and transformative practice, and move us to consider our moral and ethical obligations to educational justice in a global context, while deeply engaging in reflexive and interdisciplinary research as living inquiry. Much of its contribution lies in the way it crosses difficult domains of knowledge and ways of knowing, and how it challenges certain accepted interpretations of mathematics education research, offering potential for future inquiry and practice.

Ad Hoc Sessions

Séances ad hoc

Students, Teachers, Logarithms and I

Tanya Berezovski
Simon Fraser University

The mathematical concept of a logarithm plays a crucial role in advanced mathematics courses, including calculus, differential equations, number theory, and complex analysis. However, there is no significant body of research in mathematics education addressing these concepts. The focus of my research is to examine pre-service and in-service teachers' understanding of logarithms and logarithmic functions and to explore how their understanding influences their choices of the approaches to teaching these concepts. Research attention to students' understanding of logarithms and logarithmic functions is slim (Berezovski, 2004; Berezovski & Zazkis, 2006; Kenney, 2005; Weber, 2002) and, to my best knowledge, no study to date has focused on the understanding of these concepts by teachers.

Research supplies consistent evidence that teachers' conceptions of mathematics strongly impact their instructional practice (Brophy, 1991; Fennema & Franke, 1992; Thompson, 1992). In addition, research findings (Pepin, 1999) provide evidence that teachers' instructional practices, especially in mathematics, reflect the teachers' conceptions of the subject matter.

To ground my assessment of teaching practices, I will look at the teachers' use of examples in the classroom and how it is related to their content knowledge of mathematics. Examples play an important role in mathematics education. They have been used in teaching throughout the ages. The teacher therefore has a role in providing a model of mathematical questioning and example creation, as well as organizing learning in such a way as to encourage it. Research showed (Rowland et al., 2003; Watson & Mason, 2002) that the teacher's choice of examples reflects his/her awareness of the nature of the concept and the dimensions of variations within that concept. Therefore, examples will reveal strengths and weaknesses in participants' *pedagogical* content knowledge (Shulman, 1987; Zazkis, 2006).

Clinical interviews are the chosen methodology for the examination of in-service and pre-service teachers' understanding of the concept of logarithms and logarithmic functions. In clinical interviews, I observe teachers' work on several standard tasks as well as on non-standard and challenging tasks. To accomplish this, I adopt the system of interpretive frameworks developed for investigating students' understanding of logarithms to probe teachers' understanding of the concept (Berezovski, 2004). This results in revealing of teachers' *mathematical* content knowledge of logarithms and logarithmic functions, and also suggests possible explanations of the sources of the misconceptions that they may have.

This research provides some recommendations for the curricular developers and instructors of the pre-service teacher education courses as well as to the teachers of the undergraduate and secondary mathematics. In developing implications for teaching practice, the study focuses on those which involve the initial introduction of the concept of logarithms. In the traditional curriculum the concept of logarithm is introduced and defined as an exponent. However, historically, logarithms were developed completely independently from exponents. Contemporary Mathematics Education research supports the notion of the introduction of mathematical concepts in the classroom through history lessons of their beginnings and development down through the centuries. This could be a beneficial teaching practice and a valuable enrichment activity (Davis, 2000). Furthermore, given the centrality of examples, this research identifies both helpful and unhelpful examples and tasks in teaching the topic.

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Guitars & Mathematics – Mathématiques et guitares

France Caron, *Université de Montréal*
Dave Lidstone, *Langara College*

Dans cette séance, nous avons exploré quelques-uns des liens qui unissent les mathématiques aux guitares. Après un bref survol des termes musicaux dont nous aurions besoin pour la séance, nous avons présenté une activité de modélisation de l'espacement entre les frettes d'une guitare (voir l'annexe A). Plusieurs modèles peuvent émerger dans une classe, surtout si l'on dispose d'outils de régression. On peut alors aborder la question de l'équivalence de certains de ces modèles (en faisant appel aux propriétés des logarithmes notamment) et faire ressortir la relation de récurrence qui permet d'expliquer le tempérament égal et la perte des harmonies pures. Nous avons enchaîné avec la présentation d'autres configurations pour la disposition des frettes, qui utilisent les rapports définis à partir d'autres tempéraments ; cela constitue un bel exercice de multiplication de fractions.

We looked at sums of sine curves as models of harmony, in particular for the octave, the fifth, and the third. Such considerations are aided using angle sum identities from trigonometry. This provides a wonderful theoretical explanation of the phenomenon of beats that we experience when two tones are slightly out of tune.

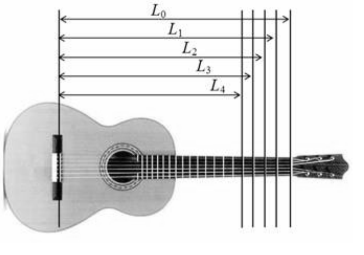
Lastly, we considered some symmetries of the fingerboard as modeled by an image of a quilt (see Appendix B). Horizontal translations along the fingerboard allow a player to transcribe keys by semi-tones (for example play a phrase in the scale of G and then translate it up two frets to play the same phrase in the key of A). Vertical translations across the fingerboard on the bottom four strings allow a player to transcribe keys by fourths (for example play a phrase in the scale of G and then translate it across one string to play the same phrase in the key of C). Finally, the diagonal translations evident in the quilt can be applied to managing the B string which is tuned a third above the G string rather than a fourth as with the rest of the strings.

Discussion from participants included the issue of presenting an appropriate level of instruction about musical issues to give students a context for the mathematics. In addition to giving up theoretical perfection when modelling from actual measurements, we also came across the "devil's interval" with the augmented fourth and its frequency ratio of $(2^{1/12})^6 = \sqrt{2}$

Annexe / Appendix A

n	Note	L_n (cm)
0	mi	65,5
1	fa	61,9
2	fa #	58,4
3	sol	55,1
4	sol #	52,1
5	la	49,2
6	la #	46,4
7	si	43,8
8	do	41,4
9	do #	39,1
10	ré	36,9
11	ré #	34,8
12	mi	32,8

Peut-on dégager de cette suite de mesures une régularité forte? Pouvez-vous l'expliquer ?



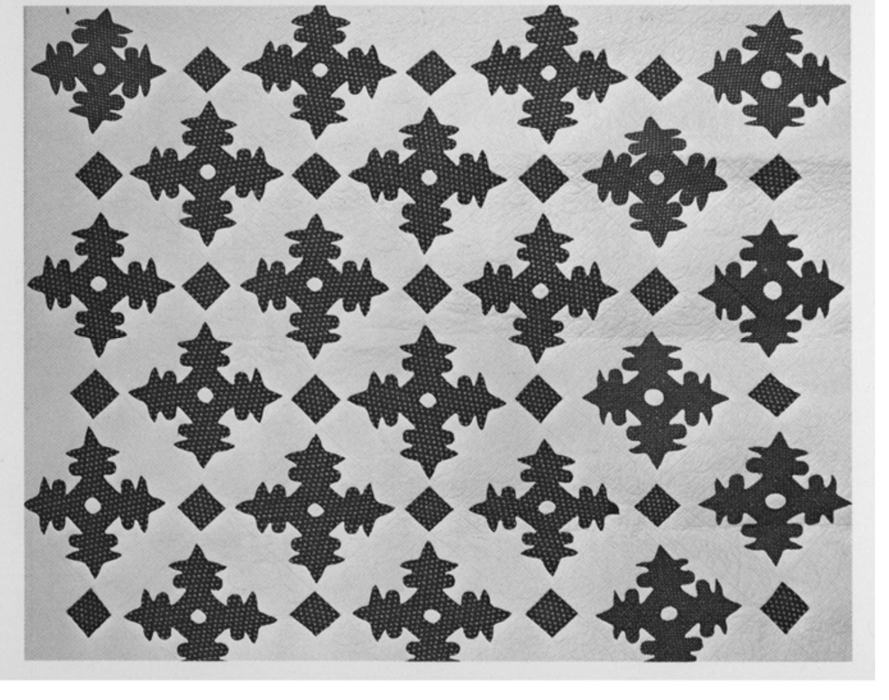
Comparez les longueurs associées aux deux notes de l'octave*. Qu'observez-vous ? Et pour la quinte* ? La tierce*? La quarte* ?

Qu'est-ce qui caractériserait ces accords qu'on trouve *harmonieux* à l'oreille ?

Cette analyse vous permet-elle de mieux caractériser la régularité observée?

* **Rappel :** on effectue une montée de 12 demi-tons pour l'octave (ex. do-do), de 7 pour la quinte (ex. do-sol), de 4 pour la tierce (ex. do-mi) et de 5 pour la quarte (ex. do-fa).

Annexe / Appendix B



Elluminating Over Great Distances: Words and Images Cross the Waters[5]

A. J. (Sandy) Dawson
University of Hawaii-Mãnoa
Pacific Resources for Education and Learning

This session tells the story of how ten graduate students separated by four time zones and millions of square miles of the Pacific Ocean met weekly to study geometry and measurement. The students were enrolled in a Masters of Education graduate program, and the mathematics course they were taking was one of five required by the program. Four of the ten students had previous experience with the *Elluminate* technology. The lead professor had no such experience, but he was being assisted by a colleague who had made limited use of *Elluminate* in a previous course.

Elluminate is a web-based instructional medium that provides a synchronous voice and visual instructional environment. Combined with the use of WebCT, the two media provide opportunities for students very remote from university/college centers to engage in stimulating and challenging academic work. The students reported on here would not be able to take graduate level work if not for these electronic media.

The presentation described the students and instructor settings. Issues around the use of the internet and computers on remote Pacific Islands were introduced, and discussion held as to how the issues were addressed. If the technology is available during the presentation, those in attendance will be able to become part of one of the course sessions, as all *Elluminate* class sessions are recorded and archived. In particular, an early February 2006 class will be 'played' for the participants. This was a class when the instructor was in Vancouver, four students were on Hawaii, five students were in American Samoa, and one student was on Pohnpei, one of the States of the Federation of Micronesia. The physical separation in this instance was seven time zones roughly a third of the way around the globe. The instructor, Samoan and Hawaiian students experienced the course on a Wednesday evening, while the student on Pohnpei experienced it on Thursday afternoon—yet it was all synchronous.

The presentation provided time for discussion of how the use media such as *Elluminate* and WebCT alter one's instruction: what are the strengths of the approach, what are the challenges, how does one have to change one's way of interaction with students—these are just some of the questions opened for examination.

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Mathematics Teaching and the Larger Purpose of Schooling: Developing ‘Moving Ideas’

Thomas Falkenber
University of Manitoba

K-12 teaching of mathematics should be embedded into a vision of a larger purpose of schooling. Any vision of a larger purpose of schooling makes assumptions about the human condition. Discussions about schooling, curriculum, (mathematics) teaching, etc. are framed by such assumptions. One way of conceptualizing (part of) the human condition is as follows (see Martin et al., 2003). Humans are born with the developmental potential for sophisticated memory (intelligence) and imagination. The emerging and developing capability to reflect upon the actual and possible (using memory and imagination) allows the developing child to engage in more and more sophisticated acts of selecting and choosing: *Human agency* emerges, which manifests itself in the practiced capability of making choices in self-determination and, thus, control instinct, impulse and even habituation to some degree. With human agency an existential concern for how to live one’s life emerges. Since this *existential concern* is so central to human life, general education should address this emerging existential concern. Then, the most general purpose of general education is to help students with their emerging existential concern of how to live their life. One way of framing the problem of addressing this general educational purpose is through Dewey’s notion of ‘moving ideas’: “The business of the educator – whether parent or teacher – is to see to it that the greatest possible number of ideas acquired by children and youth are acquired in such a vital way that they become *moving* ideas, motive-forces in the guidance of conduct. This demand and this opportunity make the moral purpose universal and dominant in all instruction – whatsoever the topic.” (1905/1975, p. 2) Ideas like ‘justice’, ‘sustainable living’, ‘democracy’, ‘citizenship’, ‘caring’, ‘friendship’, ‘education’, ‘beauty’, etc. could be considered worthwhile becoming moving ideas (motive-forces in the guidance of conduct) in general education and, thus, address students’ developing existential concern about how to live their life. For the teaching of mathematics – embedded within this larger purpose of schooling – two questions arise: (1) How can the teaching of mathematics contribute to the development of non-mathematics-specific moving ideas (see examples above), and (2) How can mathematical ideas become moving ideas for students? The current traditions and practices of teaching and learning school mathematics in North America do not have either question in focus.

Here are some approaches for addressing each question. To (1): (a) through a particular type of engagement with mathematics and mathematics learning in the classroom; e. g., politicizing of the mathematics classroom (Noddings, 1993), mathematics education as part of citizenship education (Simmt, 2001); (b) doing and studying mathematics to give purpose and meaning to one’s life; e.g., mathematics as a cultural product like literature is seen, doing mathematics as recreational activity. To (2): (a) the ‘mathematisation’ of one’s life-world (Wheeler, 1975); (b) mathematics as a tool for engagement with the world, e.g., the use and misuse of statistics.

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A Framework for Reflecting on Any 'Learning Game'

Gary Flewelling

The following truism (which I call the 'General Principle of MOM') is the basis for a framework that I use to help me reflect on both 'a learning game' and the players participating in that learning game. The General Principle of MOM : An entity will not (and cannot) act unless it is **m**-otivated to act, unless it has the **o**-pportunity to act, and unless it possesses the **m**-eans to act. (The entities I am interested in are learners and communities of learners. The act I am primarily interested in is the act of sense-making, including making sense of sense-making.) These three 'necessary conditions for sense making,' when fleshed out by significant factors associated with each condition, provide a generic framework for reflecting on any learning game.

Gary's Generic Framework for Reflecting on a Learning Game

1. Reflect on the Players' Desire / Willingness / Motivation to Play the Chosen Learning Game.
   a. previous learning experiences
   b. learning / classroom environment
   c. values
   d. dispositions / attitudes
   e. expectations; beliefs about / sense of self and other players
   f. beliefs about / sense of discipline
   g. motivation / incentives / rewards
2. Reflect on the Players' Opportunity to Play the Chosen Learning Game
   a. classroom / learning environment
   b. learning tasks
   c. interaction amongst players
3. Reflect on the Players' Ability / Wherewithal to Play the Chosen Learning Game
   a. prior knowledge
   b. previous learning experiences
   c. inclinations
   d. thinking / learning habits
   e. teaching practices / tools/strategies to facilitate learning
   f. interaction within the community of learners

I use the phrase 'play the chosen learning game' in the above framework. I do so convinced that there are really only two learning games to choose from, the 'knowledge game' (with its emphasis on acquisition of knowledge) and the 'sense-making game' (with its emphasis on action, using knowledge.) Though action and acquisition are a part of both games, they differ both qualitatively and quantitatively, making different demands on the players, fostering the development of different behaviours, skills, attitudes and beliefs. These differences become clearer with a careful application of the above framework.

Dialogical Relations in a Mathematics Classroom

Steven Khan
Queen's University

This case study investigated the polyphonic discourse in a beginning secondary school (Form 1) mathematics classroom in Trinidad. It relates how classroom and research interview 'talk' contributed to students', their teacher's and the researcher's developing conceptions of mathematics, themselves and each other. The study was approached from dialogical and socio-constructivist orientations.

Students and their teacher professed a diverse set of prior conceptions of mathematics. Several congruencies between their discourse models were identified. For example, students and their teacher essentially viewed mathematics as a rule based sequence of operations that are used to solve problems; both posited a simple economy in the learning of mathematics in which student attention and effort at home are rewarded with knowledge and understanding; both saw the teacher's responsibility as more than the transmission of information and perceived that the consequences of mathematics teaching include not only how one comes to view mathematics, but how one comes to view one's self in relation to mathematics and the degree to which one desires to engage with mathematics. Several cases described authoritative elements of the classroom discourse which included a well defined structure to lessons of ***Explanation, Example, Exercise and Evaluation***, mirroring the textbook, the use of 'cloze' type questions in IRE sequences, the ambiguous use of the collective pronoun 'we' and a reliance on rules and absent historical referents as justifications for mathematical activities and substitution for mathematical reasoning.

The result of immersion in this authoritative milieu was that both teacher and students seemed to be learning a discourse of *doing without understanding*.

Student and teacher questions and their desire to understand, however, served to interrupt the monological discourse and functioned as internally persuasive elements in the classroom. Questions such as those of *Marian, "Why is two to the zero one?", Katija: "You could have base one?"* and *Nadine: "Could you have a base number that has two digits in it?"* were all directed towards the teacher following her authoritative explanation of the topic. Their teacher, Saraswati, through her pedagogical acts also worked to create internally persuasive moments that served to counter some of the authoritative aspects of the discourse she represented, transmitted and worked within. In Saraswati's case these included a gentle reassuring touch to the forearm, an awareness of students' sensitivity to their peers' gaze, a movement and descent into that intimate space around a students' desk to answer individual questions, laughter and playfulness, an acknowledgement and addressing of students' questions and concerns that did not demean their contribution and a generosity in allowing students to share in the responsibilities for instructing each other. These actions influenced students' conception of Saraswati as a "nice", "good", "friendly" or caring teacher. Students' responses to pedagogy were internally persuasive for the teacher and precipitated ideological assessment. Both discourse types contributed to the formation of individual as well as social identities. Student and teacher utterances were internally persuasive for the researcher.

I recommend that research needs to attend more meaningfully to what is internally persuasive for students and teachers in mathematics teaching and learning. In addition, I theorize on the need for a dialogical relationship between dialogue and pedagogy that is attentive to the ambiguities in communication, which I have termed pedilogy, and which reveals our interdependencies, ‘intervulnerabilities’ and thus responsibilities to and for ourselves, our utterances, the discourses, the communities and the identities which we shape and which have shaped us.

Ethnomathematics and Audience

Lisa Lunney & David Wagner
University of New Brunswick

In this ad hoc discussion we described an idea for a contest designed to get students to explore mathematics in informal contexts. The contest will be part of a research project exploring connections between informal mathematical knowledge and formal classroom/academic mathematics.

In the first phase of this project we are working with elders from Mi'kmaq communities to collectively discover the informal mathematics of the community in both modern and historic times. We use Bishop's (1988) list of mathematical activities (counting, measuring, locating, designing, playing, and explaining) and find that the elders easily come up with examples of informal practices that could be seen as mathematical. The discussion has been rich, yet we are still unsure how to bring this into classrooms.

During an informal discussion of our own regarding an article relating to problem posing, we asked who is doing the mathematics in typical classrooms. This question led us to reflect on our ethnomathematical work with the elders. Who is doing the ethnomathematical work and for whom? Thus far, we and the elders in the group have been the ones doing the mathematics and we wondered how we could get children to investigate mathematical practices outside of the school context. Dave suggested that we need to find a way to invite the students to ask the questions of the elders. This had been his intention beginning this project. Lisa suggested a contest through the Atlantic Canada First Nation Help Desk. She knew that many schools participated in the contests and she knew that it would probably be received well by the schools.

We decided that we would provide the prizes, work with elders and teachers to establish criteria for categories, and provide professional development workshops for teachers when requested. We will be interested to note the values articulated in these discussions. We also determined that we would create a video of the instructions for the contest so that it too could be posted on the Help Desk website for students to view. The working title for this contest is "Show me your Math!"

We were particularly interested in Doolittle's CMESG plenary this year as he had brought attention to many of the questions we had been asking of ourselves. We raised some of these questions using the context of our ethnomathematical work to generate some discussion. Some of the questions we posed were:

> Should we be using the word 'ethnomathematics' at all?
> How do we do this work in a way that does not trivialize the Indigenous knowledge by simply applying Western mathematics to cultural artefacts?
> How can we ensure that there is 'pulling in' rather than 'pushing in'? (Doolittle, 2006)

The thoughtful participation of the assembled group has been helpful to us as we continue to struggle with these questions. Thank you to those who attended.

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Doolittle, E. (2006). Mathematics as medicine. In the *Proceedings of the Canadian Mathematics Education Study Group*.

Team-Teaching Mathematics: Reflections on the Collaboration between a Mathematician and a Mathematics Educator

Ami Mamolo, Malgorzata Dubiel, & Peter Liljedahl
Simon Fraser University

Pre-service elementary school teachers often experience powerful feelings of anxiety connected to the ability to do and communicate mathematics. An important part of educating in mathematics involves enriching students' experiences and knowledge while helping them overcome such deep-rooted anxieties. This past year, an opportunity to team-teach a mathematics course for pre-service elementary school teachers resulted in the unique collaboration between a mathematician, M. Dubiel, and a mathematics educator, P. Liljedahl.

The course ran for 13 weeks with two two-hour lectures a week. Instructors alternated teaching days for the most part, however a few classes were taught by both. A teaching assistant for the course, A. Mamolo, was present throughout the term to observe class dynamics as students adapted and responded to this singular environment. Both instructors maintained similar aspirations for the class and attitudes about teaching, which contributed to the overall coherence and fluidity of the course. The majority of students had no prior experience with team-teaching, however they quickly embraced the goals of the course and were generally enthusiastic about the different and complementary approaches of the team.

Prior to the start of the course, Liljedahl and Dubiel decided to take on separate roles in the classroom. While Liljedahl aimed to establish immediate, intuitive understanding of concepts, Dubiel worked towards promoting long-range, in-depth understanding. Part of my role as observer and assistant was to help link the two faces of the course by providing feedback of lectures and student responses. While on the whole the course was well received, an unfortunate drawback seemed to stem from the alternate teaching days. For some students this accentuated the dichotomy between instructors and seemed to hamper their likelihood of linking lecture material. A few students expressed frustration at the disconnect they felt existed between concepts discussed from one class to the next. However, when they were able to make connections between the two approaches, students boasted of having a more "complete" understanding, expressing confidence in their abilities to communicate both colloquially and mathematically the concepts at hand.

Many benefits of the distinct perspectives of a mathematician and mathematics educator came through during this course. The versatility and importance of mathematics, problem solving, and logical and critical thinking were emphasized by both instructors in different ways, and these carried well throughout. Some changes in this team-teaching model might help establish stronger connections between the two approaches, in particular, for bridging the gap between an intuitive and a formal understanding of mathematics.

What is Fundamental about the Fundamental Theorem of Calculus?

Bouchra Sabbagh & Olive Chapman
University of Calgary

The title of this ad hoc session was prompted by questions students ask in first-year university calculus tutorials currently experienced by the first author. We decided to focus on the question about the fundamental theorem of calculus [FTC] because our experiences and the research literature suggest that it is one of the most difficult topic for students to understand and apply and we are interested in considering ways of demystifying it for students. As a starting point, the first author conducted a review of the historical development of the FTC as part of her thesis. We considered this knowledge of history to be a potentially meaningful basis to help students to see that there is a humanistic component to the concept and that it embodies an underlying foundation of mathematics – to solve real-life problems. We present here a summary of this history that formed the basis of our discussion in this session.

The FTC states that if *f* is a function continuous on [a, b] and F is an anti derivative of *f*, then:

(i) $\int_a^b f(x)dx = F(b) - F(a)$; (ii) $\frac{d}{dx}\int_a^x f(t)dt = f(x)$.

The FTC, then, is the central concept of calculus because it connects the two main branches of calculus; integration and differentiation where integration originated from area and volume problems and differentiation from tangent problems and a speed of a moving point.

The notion of finding the area of a region bounded by curves is an old one. Archimedes (287-212 BC) successfully solved one case, but his method could not be applied generally. However, he is given credit for giving birth to the idea of integral calculus through the use of the method of exhaustion. The idea of differentiation also has its roots in the work of ancient Greeks, but it was not until 1629 when it began to take form through Fermat's use of Kepler's observation that the increment of a function becomes vanishingly small near its extreme values to solve maximum/minimum problems. He used the infinitesimal method to determine maximum/minimum problems, which is the equivalent to the modern definition of derivative $f'(x) = 0 = \lim_{h\to 0} (f(x + h) - f(x))/h = 0$.

Specific to the FTC, the basic relationship between the area under the curve y = *f*(*x*) and the antiderivative of *f* was noted first by Newton's teacher, Isaac Barrow (1630-1677). Barrow was a mathematician and a theologian. He held a mathematics chair at Cambridge and in 1669 he relinquished his post to the young Newton. Barrow in his "geometrical lectures" that dealt with motion concepts and curves, described his intuition to the FTC, but did not state it explicitly. It was not until the late 17th century that Newton made numerous and remarkable applications of his method of "fluxions" and formed his differential or "fluxional" calculus. Simultaneously, Leibniz was working on the same topics and, together, their works became associated with the creation of differential and integral calculus.

During the 19th century, the works of Cauchy and Riemann added to further development of integrals in terms of applications to continuous functions (for Cauchy) and to a limited amount of discontinuous functions (for Riemann). Riemann's doctoral dissertation: "Foundations of a general theory of functions of a complex variable" impressed even Gauss.

Our discussion during the ad hoc session raised some interesting questions. For example, could technology lead to a shift or further development to what is considered fundamental in calculus? How useful is the infinitesimal method to learning first year calculus?

"Why I'm Unhappy with Competition, Exams, Marks(!) and All That Stuff".

Peter Taylor
Queen's University

It was a lively discussion—also well attended which suggests that this is a significant issue. I did not take notes (caught up as I was in the heat of the discussion!) but I remember the main points addressed. I interpret them here through my own particular lens. There were two interwoven themes, around exams and testing, and cooperation or even better, *community*.

Exams. **Our current practice of setting time-limited high stakes exams at the end of every course causes considerable stress, sends the wrong message in terms of what's important, and wastes time and energy.**

Assessment serves two roles, feedback to the student and feedback to the community (certification) and in many cases these two functions interfere with one another. Though the students always need good, timely feedback, they do not need as much formal certification as we currently impose. What about deciding that for the first two years of study almost all our assessment should be solely for student feedback? This would simplify our lives (issues of "cheating" and collaboration would disappear) and it would make teaching simpler, easier to manage, more honest and faithful to the subject(!), freer and more enjoyable for both teacher and student. Even more importantly, it would give the student a significant message about the nature of learning and what it means to be a student. I believe that most students would respond to this *and would be changed.* A number of course would fail to respond and would run into serious trouble in their senior years. Some of these will be salvageable (in different ways) but in the meantime they will have learned an essential lesson about themselves.

Under this scheme, the other type of assessment, certification, would be mainly confined to the senior years, and should ideally take place between professor and student who know something about one another. Furthermore, it should be done under conditions that allow the students to do their best work, with enough time, proper space, and access to resources and to professors where appropriate.

Academic community. **Our students yearn to have a better or fuller sense of academic community, more opportunity (and encouragement) to interact with one another in a significant intellectual (and moral) forum.**

There are many ways we can try to facilitate this. We could give them some good investigations to work on in groups, or the challenge to fill in some of the spaces we leave open in our curriculum. I have had good success in senior courses with final exams to be done as group projects together with the requirement that an essay be written by each group describing in some detail how the group worked together, and where the strengths and weaknesses of the collaboration lay. Preparation for life?

E-STAT and Census at School as Resources for Inquiry-based Mathematics Learning Using Authentic Data

Joel Yan
Statistics Canada

> *"I have found E-STAT extremely useful in teaching statistics in the classroom. The statistical information that is available allows me to find material that my students can easily relate to. By connecting a classroom activity to a headline in the news media students can see the importance of understanding and interpreting statistics."*
>
> David MacFarlane, Dr. J.H. Gillis High School, Antigonish, Nova Scotia

To help bring math classes to life by using real Canadian and Nova Scotia data dealing with current issues that students care about, teachers are increasingly using E-STAT. E-STAT contains a wealth of local data source, with over 1,700 census characteristics available for each town and county across the country. As well as census data, E-STAT contains millions of tables on a range of subjects from over 250 ongoing surveys. And E-STAT lets teachers display this fascinating statistical information as scatter graphs, pie charts, bar charts, line graphs, histograms, and maps.

To help teachers start using E-STAT, the Nova Scotia Department of Education developed a unit for Grade 9 called "*Data Management Using E-STAT*". This 70-page unit is available at <http://lrt.ednet.ns.ca/PD/pdf/gr_9_statistics_canada.pdf>. Its five lessons were developed by the department's mathematics consultants, Sharon McCready, Donna Karsten and Nancy Chisholm, in collaboration with Statistics Canada. Lesson 5 summarizes the unit using a series of project activities where students make decisions, draw inferences and form conclusions. For each project, students examine a data set and analyse an accompanying statement.

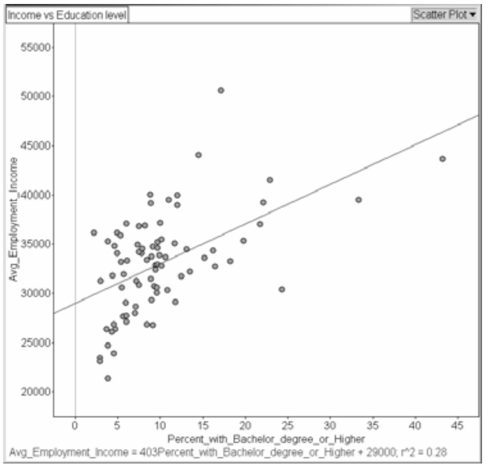
Project statements include:

Students can no longer afford to go to university. (see Example C)
Is Canada's population increasing linearly?
People in Nova Scotia consumed more potatoes in 1908 than they do now.
Halifax electricians need a raise
Canadians have become addicted to television

This unit has been extended to use New Brunswick data by Prof. Dave Wagner from University of New Brunswick. The unit will be loaded on the NB education portal <https://portal.nbed.nb.ca> and is being translated with support from the NB Ministry of Education. E-STAT is accessible on the <www.statcan.ca> site under Learning Resources (on the left side bar near the bottom).

In this ad-hoc workshop participants worked in a computer lab exploring the Statistics Canada website and E-STAT and Census at School in terms of the usefulness of these resources for the teaching and learning of mathematics.

Sample graph from the Data Management Unit to illustrate weak positive correlation



Question: For Nova Scotians is it worth staying in school and completing university?

Each point represents the data for one municipality in Nova Scotia.

Census at School: students learn data skills using data about themselves!

Quite a few education faculties across Canada have already introduced their mathematics teacher-candidates to the *Census at School* online survey. Completing the survey makes for an interesting and informative math education class for teacher candidates. *Census at School is* an exciting way for kids in grade 4 to 12 to learn data management skills while using data about themselves - their height, eating habits, use of time and much more! The Canadian survey was developed from the international survey based on input from expert mathematics educators across Canada, including several active CMESG members.

There are more than 20 exemplary learning activities offered on the Census at School site, as well as Canadian, provincial and international data based on the responses of students who completed the survey, including over 30,000 students across Canada last year.

Census at School probably works best for students in grade 6 to 9. They have fun analyzing and researching their own class data. Advanced students can also compare themselves to other cohorts in their age group.

Check out <www.censusatschool.ca> and the international site <www.censusatschool.org> for details.

A Teacher Education Success Story: A Collaborative Math and Health Assignment for Pre-service Teachers Using the E-STAT Database

Joel Yan, *Statistics Canada*
Doug Franks, Barb Olmsted, & Mike McCabe,
Nipissing University,

The Project

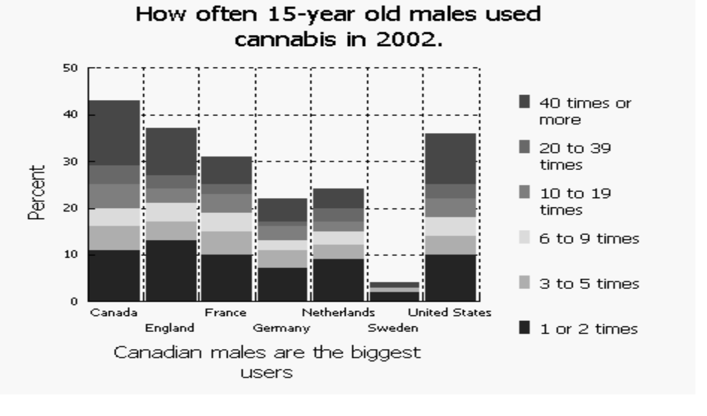
In the summer of 2005, a collaborative project was initiated between Nipissing Faculty of Education and Statistics Canada. The goal of the project was to develop a joint assignment for approximately 300 teacher-candidates taking health and math education courses based on current Canadian data in the E-STAT database from Statistics Canada and to research the effectiveness of the resource and assignment. Approximately 250 students completed a questionnaire on their experience with this assignment and the follow-up in school practicum experiences. In addition, students provided further in-depth feedback through focus group interviews.

In this workshop we reported on the status of the project, on the research results and future plans. The assignment generally worked well with teacher-candidates developing a presentation (including statistical graphs from E-STAT) on a selected topic for the health curriculum as listed at <http://www.nipissingu.ca/education/barbo/E-STAT.htm>. Pre-service teachers had to demonstrate technological competence and an understanding of health issues and mathematics data management concepts and skills in order to create meaningful classroom material for their students.

In this second year of the project we plan to continue our research using the new improved E-STAT and a revised version of the inter-disciplinary assignment.

Choosing E-STAT

E-STAT proved to be an excellent resource because of the richness of its data content, the relevance of its data to the health curriculum and its powerful graphing capabilities. Below is an example of the type of graphs students generated in their assignments. The graph below shows the relatively high use of marijuana by Canadian students.



Benefits of the Project

The following benefits have resulted from this project:

- Faculty had the opportunity to work collaboratively in designing a joint assignment and a shared research experience.
- Pre-service teacher candidates gained experience with inter-disciplinary teaching, in particular, experiencing the connection of mathematics, through technology, with other school subjects.
- Useful feedback was provided to Statistics Canada which has already resulted in improvements to E-STAT
- Teaching materials have been developed which will be shared with mathematics and health educators at other education faculties across Canada through a new website that will be housed at Nipissing University. <http://www.nipissingu.ca/iteachhealth/index.htm>

Interested persons are encouraged to contact any of the authors of this report.

Reference websites:

Barbara Olmsted, Nipissing University, Faculty of Education, joint health and math assignment: <http://www.nipissingu.ca/education/barbo/E-Stat.htm>

Statistics Canada. E-STAT database: <http://estat.statcan.ca>

Public Health Agency of Canada: information on the survey on the Health Behaviour of School-aged Children: <http://www.phac-aspc.gc.ca/dca-dea/7-18yrs-ans/index_e.html>

CMS/CMESG Joint Session

Activité conjointe SMC/GCEDM

Does a Math Education PhD Program Belong in a Math Dept?

Peter Taylor, *Queen's University*
Pamela Hagen, *University of British Columbia*
Peter Liljedahl, *Simon Fraser University*
Lily Moshe, *York University*

A Joint CMS/CMESG Education Session (to my knowledge the first such joint session) was held at the Westin Hotel on June 3, 2006. The timing was during the CMS meetings and just prior to the start of the CMESG meeting, and it was held at the CMS site rather than the CMESG site (The University of Calgary). Nevertheless there was excellent participation from members of both societies. The session was promoted with the following summary:

A lot of attention has been paid to the need for mathematicians to take education more seriously, even "professionally." Questions arise, such as what kind of research in education a mathematician might do, where this might be published etc. But here we are interested in a more specific question: "is there a place in a math dept for a PhD student who is working in math education?" I have a feeling that the answer is yes, as there are some important, intellectually challenging (and possibly even profound) problems around university level mathematics education. But there are some important questions that arise.

1. *What does such a student learn?* The first half of the answer is easy: lots of math, certainly enough to pass the PhD comprehensive exam or take the core courses. But what else?
2. *What are the problem areas that a student's research might focus on?* How do these differ from the comparable degree obtained in a Faculty of Education? Certainly there might be some overlap here, e.g. math anxiety, gender studies, the high school-university interface. But I feel that the difference would be the extent to which the student focused on the university (as opposed to the school) learning experience. There are other problems that I believe are intellectually important and possibly(!) unique to university, e.g. the relationship between teaching and research. [Actually even here there might be analogues at the high school level.]
3. *What is the future career path of the graduate?* Those who get their degree from a Faculty of Education often wind up teaching in a Faculty of Education. Those who get their degree from a Math Dept might wish to wind up teaching in a Math Dept. Will there be jobs for such graduates? Increasingly there is a consensus that Math Departments need to be hiring Math Education researchers. There's a whole climate of change around this issue, but there are questions too, for example concerning the priorities of the principal granting councils.
4. *Finally what do you say to this?* There is an interesting argument that such a program shouldn't exist, that a potential student would be better off doing a standard PhD in Math (or possibly something like History of Math) getting a productive mathematics research program going, getting a job in a good Math Dept in the normal way, and *then*, armed with mathematical experience and credibility(?), starting to work in Mathematics Education. Certainly a number of significant leaders in the field today have gone this route.

Facilitators:

Peter Taylor, Dept Mathematics and Statistics, Queen's University
Peter Liljedahl, Faculty of Education, Simon Fraser University
Lily Moshe, PhD student in Math Education. Dept Mathematics, York University
Pamela Hagen, PhD student in Math Education. Faculty of Education, UBC

Summary of the Session.

The facilitators made short presentations and a lively discussion followed. Probably the most significant point made was that a PhD program in mathematics education needs to be informed by both mathematics and mathematics education and it therefore needs the active counsel of both mathematicians and math educators. Given that there would be significant differences in content and culture between programs housed in the two separate departments (Faculties?) except possibly in those few places where mathematics and mathematics education are found in the same unit. A significant factor seemed to be where the student intended to work or teach in the future. Students who expected to spend their career in a Department of Mathematics might be well served by a program in such a department. A strong view put forward was that students who expected to spend their career in the school system needed the education that could only be provided in Faculty of Education, and this seemed to be generally accepted.

It was observed that at a number of Canadian Universities there are examples of PhD students in Mathematics Departments doing their thesis work in Mathematics Education (e.g. Concordia, York, but no official PhD programs in Mathematics Education seem to exist. [Though we may have missed something.] Lily Moshe gave a list of American Universities (from Walter Whiteley) which do seem to have such programs, though a study of the web sites reveals considerable variation in the nature of the programs.

Oklahoma State University <http://mathgrad.okstate.edu/pg5_PhD_mathedu.html>
Michigan State University <http://www.mth.msu.edu/Graduate/handbook_part3.html>
University of New Hampshire <http://www.gradschool.unh.edu/catalog/programs/math.html>
Oregon State University <http://www.math.oregonstate.edu/node/view/43>
University of Arizona <http://math.arizona.edu/gradprogram/handbook/phdrequirements.html>
Portland State University <http://www.mth.pdx.edu/programs/Mth_Ed_PHD_INFO.asp>
Northern Illinois University <http://www.math.niu.edu/programs/grad/docplans.html>

To the summary above, Peter, Pamela and Lily have added the following (edited) commentaries.

Peter Liljedahl:

"*Mathematics education is not mathematics*. It is a domain of professional work" (Bass, 2005). It has its own history, its own canon of knowledge, its own theories, traditions, and practices, and it has its own requirements for PhD students. The core of these requirements is preparation – preparation in research (Lester & Carpenter, 2001), mathematics (Dossey & Lappan, 2001), mathematics education (Presmeg & Wagner, 2001), teaching (Lambdin & Wilson, 2001), and policy (Long, 2001). This preparation is achieved through indoctrination into the communities of practice that comprises the field of mathematics education through course work, exposure to literature, apprenticeship (Schoenfeld, 1999), and most importantly, **collegial immersion into the mathematics education community**. If a *particular* mathematics department can provide the community within which such core preparation can be achieved then that *particular* mathematics department could, indeed, house a doctoral program in mathematics education.

Having stated this the next question is – are there any particular advantages to a doctoral program in mathematics education residing in a mathematics department? The answer to this question is also yes – and in this case there are good reasons why such a program would be of benefit to mathematics education community as a whole. First, regardless of the good work that has been done in mathematics education by mathematics educators, mathematicians still have a privileged place in discussion around curriculum, policy, and assessment – having mathematicians as spokespersons for mathematics education is unavoidable. As such, mathematicians steeped in the history, theories, traditions, and practices of mathematics education would make powerful and effectual spokespersons (c.f. Bass, 2005). Second, "there are some important, intellectually challenging (and possibly even profound) problems around university level mathematics" (from the above promotion) that need attention. The context of university mathematics is traditionally the domain of mathematicians. A mathematics educator, educated in a mathematics department, would have greater intellectual and cultural capital to do their work in such a context. These aforementioned advantages, however, do not trump the requirements laid out in the first paragraph – an inadequately prepared mathematics educator is not of benefit to the mathematics education community.

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Schoenfeld, A. (1999). The core, the canon, and the development of research skills: Issues in the preparation of education researchers. In E. Lagemann & L. Schulman (eds.), Issues in Education Research: Problems and Possibilities, pp. 166-202. New York, NY: Jossey-Bass.

Pam Hagen

Learning mathematics should not be an either/or enterprise but a collaborative undertaking between mathematics educators AND mathematicians. It must be said that such a collaboration would require (or lead to) changes in how we currently teach and what we teach on both sides.

Indeed there might be four necessary conditions for successful collaboration (Sultan & Artzt, 2005):

i. Motivation to collaborate. Acknowledgement of the strengths of each collaborator.
iii. Trust that the motives of each collaborator involve improving student learning.
iv. Helpfulness of both collaborators in reaching the mutual goals. Sultan, A & Artz, A (2005), Mathematicians are from Mars: Math Educators are From Venus: The Story of a Successful Collaboration, American Mathematical Society, January Notice, 52, 1, p. 48-53. Downloaded from <http://www.ams.org/notices/200501/comm-artzt.pdf>

Lily Moshe:

Regarding advantages of doing math education from a math department:

Having done comprehensive exams in math, someone with a math ed PhD would find it easier to teach a variety of undergraduate math courses, than someone who had graduated from an education department/faculty. Math majors who are planning on becoming teachers, or undergraduates coming from liberal arts programs may especially benefit from such instructors, particularly as their thesis might well be based on school- rather than university-level math. In addition someone with a Math Ed PhD from a mathematics department might have more influence in undergraduate program development and professor/TA teaching development.

Lastly 'math education' has a number in math reviews, so why shouldn't it be part of math? (This was one of the arguments that was raised at York in approving my own program.) This was a lively discussion at an important time of transition. I am convinced that over the next few years we will continue to see significant shifts in mathematics education at all levels.

Appendices

Annexes

APPENDIX A / ANNEXE A

Working Groups at Each Annual Meeting / Groupes de travail des rencontres annuelles

1977 *Queen's University, Kingston, Ontario*

· Teacher education programmes
· Undergraduate mathematics programmes and prospective teachers
· Research and mathematics education
· Learning and teaching mathematics

1978 *Queen's University, Kingston, Ontario*

· Mathematics courses for prospective elementary teachers
· Mathematization
· Research in mathematics education

1979 *Queen's University, Kingston, Ontario*

· Ratio and proportion: a study of a mathematical concept
· Minicalculators in the mathematics classroom
· Is there a mathematical method?
· Topics suitable for mathematics courses for elementary teachers

1980 *Université Laval, Québec, Québec*

· The teaching of calculus and analysis
· Applications of mathematics for high school students
· Geometry in the elementary and junior high school curriculum
· The diagnosis and remediation of common mathematical errors

1981 *University of Alberta, Edmonton, Alberta*

· Research and the classroom
· Computer education for teachers
· Issues in the teaching of calculus
· Revitalising mathematics in teacher education courses

1982 *Queen's University, Kingston, Ontario*

- The influence of computer science on undergraduate mathematics education
- Applications of research in mathematics education to teacher training programmes
- Problem solving in the curriculum

1983 *University of British Columbia, Vancouver, British Columbia*

- Developing statistical thinking
- Training in diagnosis and remediation of teachers
- Mathematics and language
- The influence of computer science on the mathematics curriculum

1984 *University of Waterloo, Waterloo, Ontario*

- Logo and the mathematics curriculum
- The impact of research and technology on school algebra
- Epistemology and mathematics
- Visual thinking in mathematics

1985 *Université Laval, Québec, Québec*

- Lessons from research about students' errors
- Logo activities for the high school
- Impact of symbolic manipulation software on the teaching of calculus

1986 *Memorial University of Newfoundland, St. John's, Newfoundland*

- The role of feelings in mathematics
- The problem of rigour in mathematics teaching
- Microcomputers in teacher education
- The role of microcomputers in developing statistical thinking

1987 *Queen's University, Kingston, Ontario*

- Methods courses for secondary teacher education
- The problem of formal reasoning in undergraduate programmes
- Small group work in the mathematics classroom

1988 *University of Manitoba, Winnipeg, Manitoba*

- Teacher education: what could it be?
- Natural learning and mathematics
- Using software for geometrical investigations
- A study of the remedial teaching of mathematics

1989 *Brock University, St. Catharines, Ontario*

- Using computers to investigate work with teachers
- Computers in the undergraduate mathematics curriculum
- Natural language and mathematical language
- Research strategies for pupils' conceptions in mathematics

1990 *Simon Fraser University, Vancouver, British Columbia*

- Reading and writing in the mathematics classroom
- The NCTM "Standards" and Canadian reality
- Explanatory models of children's mathematics
- Chaos and fractal geometry for high school students

1991 *University of New Brunswick, Fredericton, New Brunswick*

- Fractal geometry in the curriculum
- Socio-cultural aspects of mathematics
- Technology and understanding mathematics
- Constructivism: implications for teacher education in mathematics

1992 *ICME–7, Université Laval, Québec, Québec*

1993 *York University, Toronto, Ontario*

- Research in undergraduate teaching and learning of mathematics
- New ideas in assessment
- Computers in the classroom: mathematical and social implications
- Gender and mathematics
- Training pre-service teachers for creating mathematical communities in the classroom

1994 *University of Regina, Regina, Saskatchewan*

- Theories of mathematics education
- Pre-service mathematics teachers as purposeful learners: issues of enculturation
- Popularizing mathematics

1995 *University of Western Ontario, London, Ontario*

- Autonomy and authority in the design and conduct of learning activity
- Expanding the conversation: trying to talk about what our theories don't talk about
- Factors affecting the transition from high school to university mathematics
- Geometric proofs and knowledge without axioms

1996 *Mount Saint Vincent University, Halifax, Nova Scotia*

- Teacher education: challenges, opportunities and innovations
- Formation à l'enseignement des mathématiques au secondaire: nouvelles perspectives et défis
- What is dynamic algebra?
- The role of proof in post-secondary education

1997 *Lakehead University, Thunder Bay, Ontario*

- Awareness and expression of generality in teaching mathematics
- Communicating mathematics
- The crisis in school mathematics content

1998 *University of British Columbia, Vancouver, British Columbia*

- Assessing mathematical thinking
- From theory to observational data (and back again)
- Bringing Ethnomathematics into the classroom in a meaningful way
- Mathematical software for the undergraduate curriculum

1999 *Brock University, St. Catharines, Ontario*

- Information technology and mathematics education: What's out there and how can we use it?
- Applied mathematics in the secondary school curriculum
- Elementary mathematics
- Teaching practices and teacher education

2000 *Université du Québec à Montréal, Montréal, Québec*

- Des cours de mathématiques pour les futurs enseignants et enseignantes du primaire/Mathematics courses for prospective elementary teachers
- Crafting an algebraic mind: Intersections from history and the contemporary mathematics classroom
- Mathematics education et didactique des mathématiques : y a-t-il une raison pour vivre des vies séparées?/Mathematics education et didactique des mathématiques: Is there a reason for living separate lives?
- Teachers, technologies, and productive pedagogy

2001 *University of Alberta, Edmonton, Alberta*

- Considering how linear algebra is taught and learned
- Children's proving
- Inservice mathematics teacher education
- Where is the mathematics?

2002 *Queen's University, Kingston, Ontario*

- Mathematics and the arts
- Philosophy for children on mathematics
- The arithmetic/algebra interface: Implications for primary and secondary mathematics / Articulation arithmétique/algèbre: Implications pour l'enseignement des mathématiques au primaire et au secondaire
- Mathematics, the written and the drawn
- Des cours de mathémathiques pour les futurs (et actuels) maîtres au secondaire / Types and characteristics desired of courses in mathematics programs for future (and in-service) teachers

2003 *Acadia University, Wolfville, Nova Scotia*

- L'histoire des mathématiques en tant que levier pédagogique au primaire et au secondaire / The history of mathematics as a pedagogic tool in Grades K–12
- Teacher research: An empowering practice?
- Images of undergraduate mathematics
- A mathematics curriculum manifesto

2004 *Univerité Laval, Québec, Québec*

- Learner generated examples as space for mathematical learning
- Transition to university mathematics
- Integrating applications and modeling in secondary and post secondary mathematics
- Elementary teacher education - Defining the crucial experiences
- A critical look at the language and practice of mathematics education technology

2005 *University of Ottawa, Ottawa, Ontario*

- Mathematics, Education, Society, and Peace
- Learning Mathematics in the Early Years (pre-K – 3)
- Dicrete Mathematics in Secondary School Curriculum
- Socio-Cultural Dimensions of Mathematics Learning

2006 *University of Calgary, Alberta*

- Secondary Mathematics Teacher Development
- Developing Links Between Statistical and Probabilistic Thinking in School Mathematics Education
- Developing Trust and Respect When Working with Teachers of Mathematics
- The Body, the Sense, and Mathematics Learning

APPENDIX B / ANNEXE B

Plenary Lectures at Each Annual Meeting / Conférences plénières des rencontres annuelles

1977	A.J. COLEMAN	The objectives of mathematics education
	C. GAULIN	Innovations in teacher education programmes
	T.E. KIEREN	The state of research in mathematics education
1978	G.R. RISING	The mathematician's contribution to curriculum development
	A.I. WEINZWEIG	The mathematician's contribution to pedagogy
1979	J. AGASSI	The Lakatosian revolution
	J.A. EASLEY	Formal and informal research methods and the cultural status of school mathematics
1980	C. GATTEGNO	Reflections on forty years of thinking about the teaching of mathematics
	D. HAWKINS	Understanding understanding mathematics
1981	K. IVERSON	Mathematics and computers
	J. KILPATRICK	The reasonable effectiveness of research in mathematics education
1982	P.J. DAVIS	Towards a philosophy of computation
	G. VERGNAUD	Cognitive and developmental psychology and research in mathematics education
1983	S.I. BROWN	The nature of problem generation and the mathematics curriculum
	P.J. HILTON	The nature of mathematics today and implications for mathematics teaching

1984	A.J. BISHOP	The social construction of meaning: A significant development for mathematics education?
	L. HENKIN	Linguistic aspects of mathematics and mathematics instruction
1985	H. BAUERSFELD	Contributions to a fundamental theory of mathematics learning and teaching
	H.O. POLLAK	On the relation between the applications of mathematics and the teaching of mathematics
1986	R. FINNEY	Professional applications of undergraduate mathematics
	A.H. SCHOENFELD	Confessions of an accidental theorist
1987	P. NESHER	Formulating instructional theory: the role of students' misconceptions
	H.S. WILF	The calculator with a college education
1988	C. KEITEL	Mathematics education and technology
	L.A. STEEN	All one system
1989	N. BALACHEFF	Teaching mathematical proof: The relevance and complexity of a social approach
	D. SCHATTSNEIDER	Geometry is alive and well
1990	U. D'AMBROSIO	Values in mathematics education
	A. SIERPINSKA	On understanding mathematics
1991	J .J. KAPUT	Mathematics and technology: Multiple visions of multiple futures
	C. LABORDE	Approches théoriques et méthodologiques des recherches françaises en didactique des mathématiques
1992	ICME-7	
1993	G.G. JOSEPH	What is a square root? A study of geometrical representation in different mathematical traditions
	J CONFREY	Forging a revised theory of intellectual development: Piaget, Vygotsky and beyond
1994	A. SFARD	Understanding = Doing + Seeing ?
	K. DEVLIN	Mathematics for the twenty-first century
1995	M. ARTIGUE	The role of epistemological analysis in a didactic approach to the phenomenon of mathematics learning and teaching
	K. MILLETT	Teaching and making certain it counts
1996	C. HOYLES	Beyond the classroom: The curriculum as a key factor in students' approaches to proof
	D. HENDERSON	Alive mathematical reasoning

1997	R. BORASSI	What does it really mean to teach mathematics through inquiry?
	P. TAYLOR	The high school math curriculum
	T. KIEREN	Triple embodiment: Studies of mathematical understanding-in-interaction in my work and in the work of CMESG/GCEDM
1998	J. MASON	Structure of attention in teaching mathematics
	K. HEINRICH	Communicating mathematics or mathematics storytelling
1999	J. BORWEIN	The impact of technology on the doing of mathematics
	W. WHITELEY	The decline and rise of geometry in 20th century North America
	W. LANGFORD	Industrial mathematics for the 21st century
	J. ADLER	Learning to understand mathematics teacher development and change: Researching resource availability and use in the context of formalised INSET in South Africa
	B. BARTON	An archaeology of mathematical concepts: Sifting languages for mathematical meanings
2000	G. LABELLE	Manipulating combinatorial structures
	M. B. BUSSI	The theoretical dimension of mathematics: A challenge for didacticians
2001	O. SKOVSMOSE	Mathematics in action: A challenge for social theorising
	C. ROUSSEAU	Mathematics, a living discipline within science and technology
2002	D. BALL & H. BASS	Toward a practice-based theory of mathematical knowledge for teaching
	J. BORWEIN	The experimental mathematician: The pleasure of discovery and the role of proof
2003	T. ARCHIBALD	Using history of mathematics in the classroom: Prospects and problems
	A. SIERPINSKA	Research in mathematics education through a keyhole
2004	C. MARGOLINAS	La situation du professeur et les connaissances en jeu au cours de l'activité mathématique en classe
	N. BOULEAU	La personnalité d'Evariste Galois: le contexte psychologique d'un goût prononcé pour les mathématique abstraites
2005	S. LERMAN	Learning as developing identity in the mathematics classroom
	J. TAYLOR	Soap bubbles and crystals
2006	B. JAWORSKI	Developmental research in mathematics teaching and learning: Developing learning communities based on inquiry and design
	E. DOOLITTLE	Mathematics as medicine

APPENDIX C / ANNEXE C

Proceedings of Annual Meetings / Actes des rencontres annuelles

Past proceedings of CMESG/GCEDM annual meetings have been deposited in the ERIC documentation system with call numbers as follows:

Proceedings of the 1980 Annual Meeting	ED 204120
Proceedings of the 1981 Annual Meeting	ED 234988
Proceedings of the 1982 Annual Meeting	ED 234989
Proceedings of the 1983 Annual Meeting	ED 243653
Proceedings of the 1984 Annual Meeting	ED 257640
Proceedings of the 1985 Annual Meeting	ED 277573
Proceedings of the 1986 Annual Meeting	ED 297966
Proceedings of the 1987 Annual Meeting	ED 295842
Proceedings of the 1988 Annual Meeting	ED 306259
Proceedings of the 1989 Annual Meeting	ED 319606
Proceedings of the 1990 Annual Meeting	ED 344746
Proceedings of the 1991 Annual Meeting	ED 350161
Proceedings of the 1993 Annual Meeting	ED 407243
Proceedings of the 1994 Annual Meeting	ED 407242

Proceedings of the 1995 Annual Meeting	ED 407241
Proceedings of the 1996 Annual Meeting	ED 425054
Proceedings of the 1997 Annual Meeting	ED 423116
Proceedings of the 1998 Annual Meeting	ED 431624
Proceedings of the 1999 Annual Meeting	ED 445894
Proceedings of the 2000 Annual Meeting	ED 472094
Proceedings of the 2001 Annual Meeting	ED 472091
Proceedings of the 2002 Annual Meeting	*submitted*
Proceedings of the 2003 Annual Meeting	*submitted*
Proceedings of the 2004 Annual Meeting	*submitted*
Proceedings of the 2005 Annual Meeting	*submitted*
Proceedings of the 2006 Annual Meeting	*submitted*

Note

There was no Annual Meeting in 1992 because Canada hosted the Seventh International Conference on Mathematical Education that year.