CANADIAN MATHEMATICS EDUCATION
STUDY GROUP

GROUPE CANADIEN D'ÉTUDE EN DIDACTIQUE
DES MATHÉMATIQUES

PROCEEDINGS / ACTES
2008 ANNUAL MEETING /
RENCONTRE ANNUELLE 2008

CMESG
GCEDM

Université de Sherbrooke
May 23 – May 27, 2008

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32nd Annual Meeting
Université de Sherbrooke
May 23 – May 27, 2008

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The organisational work for our 2008 annual meeting at the Université de Sherbrooke enabled us to have another memorable meeting. Our local organiser, Laurent Theis, with the expert support of the CRYSTAL centre students (David Benoit, Guylaine Cotnoir, Annie Corriveau, Alexandre Ducharme Rivard, and Vincent Martin) as well as the CRYSTAL centre secretary, Catherine Kenny, managed to do everything with utmost efficiency and pleasantries, and we thank them. We would also like to thank the guest speakers, working group leaders, topic session and ad hoc presenters, and all the participants for making the 2008 meeting a stimulating and worthwhile experience.

L'organisation locale de notre rencontre annuelle de 2008 à l'Université de Sherbrooke nous a permis d'avoir une autre rencontre mémorable. Notre organisateur local, Laurent Theis, avec le support des étudiants du CREAS (David Benoit, Guylaine, Cotnoir, Annie Corriveau, Alexandre Ducharme Rivard et Vincent Martin) ainsi que de la secrétaire du CREAS, Catherine Kenny, ont réussi à organiser la rencontre avec une grande efficacité et nous les en remercions. Nous aimerions aussi remercier les conférenciers invités, les animateurs de groupes de travail, les présentateurs de séances thématiques et d'ateliers ad hoc, ainsi que tous les participants pour avoir fait de la rencontre 2008 une expérience stimulante et mémorable.

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Vendredi 23 mai	Samedi 24 mai	Dimanche 25 mai	Lundi 26 mai	Mardi 27 mai
	DEJEUNER	DEJEUNER	DEJEUNER	DEJEUNER
	9:00 – 11:00 Groupes de travail	9:00 – 11:00 Groupes de travail	9:00 – 11:00 Groupes de travail	9:00 – 10:00 Séance thématique
				10:00 – 11:00 Table ronde
	11:00-11:20 Pause café	11:00-11:20 Pause café	11:00-11:20 Pause café	11:00-11:30 Pause café
	11:20 – 12:30 Groupes de travail	11:20 – 12:30 Groupes de travail	11:20 – 12:30 Groupes de travail	11:30 – 12:30 Session de clôture
	12:30 – 1:30 Dîner	12:30 – 1:30 Dîner	12:30 – 1: 30 Dîner	
	1:30 – 2:00 Discussion de la plénière 1 en petit groupe	1:30 – 2:30 Plénière 2 Watson	1:30-2:00 Discussion de la plénière 2 en petit groupe	
	2:15 – 3:15 Discussion de la plénière 1	2:40-3:40 Séance thématique	2:15 – 3:15 Discussion de la plénière 2	
4:00 – 6:00 Inscription	3:15 – 3:45 Séances ad hoc (1)		3:15 – 3:45 Séances ad hoc (3)	
	3:45 – 4:15 Pause		3:45 – 4:15 Pause	
	4:15 – 4:45 Nouvelles thèses (1)	Excursion aux Mines de Capelton suivi du repas à North Hatley: départ à 3:50	4:15 – 4:45 Nouvelles thèses (2)	
5:30 – 6:45 BBQ	5:00 – 5:30 Séances ad hoc (2)		5:00 – 6:15 Assemblée générale annuelle	
6:45 – 7:45 Session d'ouverture	Souper libre réservation au Pub le Cartier à 7h00 pour les intéressé(e)s		7:00 Repas à l'université	
7:45 – 8:45 Plénière 1: Djebbar				
8:45 Réception				

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Working Groups</td><td>9:00 – 10:00
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<tr><td>10:00 – 11:00
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Coffee Break</td></tr>
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Watson</td><td>1:30-2:00
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<tr><td>2:15 – 3:15
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<tr><td rowspan="3">4:00 – 6:00
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Break</td><td>3:45 – 4:15
Break</td></tr>
<tr><td>4:15 – 4:45
New PhDs (1)</td><td rowspan="5">Excursion:
Capelton mines and supper at North Hatley
3:50 departure</td><td>4:15 – 4:45
New PhDs (2)</td></tr>
<tr><td>5:30 – 6:45
BBQ</td><td>5:00 – 5:30
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Opening session</td><td rowspan="3">Dinner On Your Own reservations at Pub le Cartier at 7:00 for those interested</td><td rowspan="3">7:00
Dinner
at the university</td></tr>
<tr><td>7:45 – 8:45
Plenary 1: Djebbar</td></tr>
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Reception</td></tr>
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Introduction

Florence Glanfield – President, CMESG/GCEDM
University of Alberta

It is with pleasure that I write the introduction to the 2008 proceedings of our 32nd Annual Meeting. It forces me to think back to a wonderful meeting, to the planning and dedication that went into its organization, and to the meeting itself, the conversations, the sharing which occurred.

Firstly, I remember the beauty of Sherbooke – in the heart of the Quebec's Eastern Townships – an area full of rich history – a history that we were able to embrace with the tour to the Capelton Mines – and dinner in North Hatley. We also enjoyed a fun evening of entertainment – it was so interesting to participate in an activity where EVERYONE was dancing within a short period of time. A special thanks to our local organizer, Laurent Theis and his colleagues at the Université de Sherbrooke for so warmly welcoming us! In addition to serving as the local organizer for our Annual Meeting, Laurent hosted the 2008 Annual Meeting of the Groupe des didacticiens des mathématiques du Québec (GDM), to which the CMESG/GCEDM community was invited to attend.

In addition to the social activities and socializing, the CMESG/GCEDM annual meetings are about the rich conversations that we have with colleagues – the program is designed for many opportunities to come to know and understand the multiple perspectives in mathematics teaching and learning.

I remember the initial planning in Fredericton in 2007. After two years of record-breaking attendance at our annual meetings – we decided to plan for 5 working groups. Attendance at the Sherbrooke meeting again surpassed our expectations and for the third year, we noticed that the number of participants at our annual meeting was closer to 100 than to 75 – definitely a need for 5 working groups – mathematical reasoning of young children; mathematics-in-and-for-teaching: the case of algebra; mathematics and human alienation; communication and mathematical technology use throughout the post-secondary curriculum; and cultures of generality and their associated pedagogies. A huge thank you to each of the working group leaders for their leadership – we are so very fortunate to have people in our community who say 'yes' when they are invited to facilitate our conversations!

We were also joined by our two plenaries – Ahmed Djebbar and Anne Watson. We started our 2008 Annual Meeting with Ahmed taking us on a journey to explore the early mathematical contributions of current day Islamic countries; two days later we were able to learn from Anne Watson's insightful lecture around adolescent learning and secondary mathematics. Our topic sessions – virtual problem solving opportunities to meet the needs of the net generation: knowledge building, knowledge sharing and being a part of the community; towards the 2009 Canadian Mathematics Education Forum; snowflakes serving mathematics; and dilemmas of equity and reform in mathematics education: rethinking equity in an increasingly diverse world; celebrating 5 new PhD's in mathematics education; and the panel discussion around the theme, "Rupture and coherence in advocacy in public policy" ensured that we left the meeting with new insights and new wonders about the way in which we engage in our research and teaching. Thanks to each of the presenters for your contribution to the 2008 meeting.

Finally, I wish to thank the 2007 – 2008 Executive, Frédéric Gourdeau, Brent Davis, Doug Franks, Dave Wagner, and Walter Whiteley who planned the program; Eva Knoll and Viktor Freiman who helped with the translation of the 2008 programme; and Peter Liljedahl, Christian Bernèche, and Susan Oesterle, editors of these proceedings, for their patience and dedication in the production process. Each year we are reminded of the generosity of individuals within the CMESG/GCEDM community as it is essential to the success of our annual meeting.

We will be reminded of each contribution and of our conversations as we read on...

Plenary Lectures

Conférences plénières

Art, culture et mathématiques en pays d'Islam (IX^{e}-XV^{e} s.)

Ahmed Djebbar
Université des Sciences et des Technologies de Lille

Lorsque, à partir de 632, les premiers cavaliers arabes se sont lancés à la conquête de nouveaux territoires, ils avaient une culture bien identifiée, essentiellement orale, constituée de nombreux poèmes, de chants, de récits de faits d'armes et de chroniques sur le passé de leurs tribus. Ils avaient aussi un art architectural dans les cités du sud et un art pictural qui variait en fonction des conditions de vie des populations de l'Arabie. Mais, à notre connaissance, ils n'avaient pas de science et donc pas de mathématiques écrites et consignées dans un corpus clairement identifié. Ce qui ne signifie pas qu'ils ne savaient pas calculer ou représenter des figures géométriques du moins pour les citadins qui avaient à réaliser des transactions commerciales ou à pratiquer un minimum d'activités artistiques.

Cette situation va progressivement changer, à partir du milieu du VIII^{e} siècle, au contact des populations qui vont être gouvernées par le pouvoir musulman et sous l'effet, à la fois, des conditions économiques et du nouveau mode de vie dans les villes conquises. L'élite des conquérants prendra connaissance alors des savoirs des civilisations antérieures et, au contact des anciennes élites des pays nouvellement soumis, elle s'initiera rapidement à leur raffinement dans l'habillement, la nourriture et l'art. Quant à la culture qui s'exprimait en langue arabe, et qui était, après l'Islam, le patrimoine le plus cher aux yeux des premiers conquérants, ce sont les nouveaux musulmans de l'empire qui, sans renier leurs propres cultures, vont contribuer à la valoriser, soit comme composante essentielle de la nouvelle religion à laquelle ils adhéraient désormais, soit par souci de promotion sociale.

Un art sans mathématique

C'est donc dans un contexte culturel dominé par la langue arabe et par les productions religieuses, littéraires et historiques, s'exprimant à travers elle, que vont naître et se développer de nouvelles pratiques mathématiques et que vont se constituer des passerelles entre, d'un côté, les activités scientifiques au sens large et, de l'autre, les multiples expressions culturelles et artistiques. Nous ne connaissons pas dans le détail tous les aspects de ces relations et leur évolution dans le temps. Mais les éléments que nous allons exposer constituent, à défaut d'une analyse complète encore prématurée, des exemples significatifs qui peuvent illustrer quelques facettes d'un phénomène original, à défaut de l'expliquer.

Aux premiers temps de la civilisation arabo-musulmane, les activités artistiques étaient dans le prolongement de ce qui se pratiquait dans chacune des régions conquises. Au niveau du continu, elles se limitaient aux techniques et aux thèmes pratiqués par les artistes des différentes régions soumises au nouveau pouvoir. Ces derniers faisaient donc, en fonction du lieu où ils exerçaient leur métier, de l'art persan en Asie Centrale, byzantin dans le Croissant fertile, roman dans la péninsule ibérique, berbère au Maghreb.

C'est ce que l'on peut constater encore lorsqu'on visite certaines mosquées de la période omeyyade (661-750), comme celle de Damas ou celle de Kairouan en Tunisie, ou certaines habitations de princes et de califes de la même période, comme le fameux *Qusayr 'Amra*, un petit palais, qui est aujourd'hui en Jordanie, et où les décorations sont empreintes d'un grand réalisme. L'analyse du style de ces peintures et des thèmes qu'elles représentent a révélé les traits essentiels et le savoir-faire des traditions artistiques gréco-romaines et persanes, avec des survivances des civilisations du Croissant fertile.

On y trouve, en particulier, des éléments de symétrie, à travers le dessin de certaines plantes et de certains animaux. Mais ce qui frappe le plus, ce n'est pas cette touche « mathématique » qui n'a rien d'original au vu des traditions antérieures. C'est surtout la représentation anthropomorphique qui est déclinée de différentes manières. Ainsi, malgré des versets du Coran et des paroles attribuées au Prophète (paroles qui ont été interprétées par un certain nombre de théologiens comme exprimant l'interdiction de toute représentation vivante et surtout humaine), les artistes des VIII^e^ et IX^e^ siècles n'ont pas modifié leur style. Ils ont continué à pratiquer un art figuratif reproduisant des scènes de la vie. C'est ainsi que dans le petit palais déjà évoqué, on peut admirer une scène de bain dans un hammam où les femmes, presque nues, sont représentées avec des maintiens et une gestuelle toute naturelle. On peut également admirer l'intérieur d'une coupole décorée comme une voûte céleste où ont été dessinés les différents personnages et animaux représentant les signes du zodiaque. Ce qui est une autre preuve du statut que possédait encore l'astrologie et ce malgré l'interdiction de sa pratique, plusieurs fois exprimée dans les textes de la nouvelle religion.

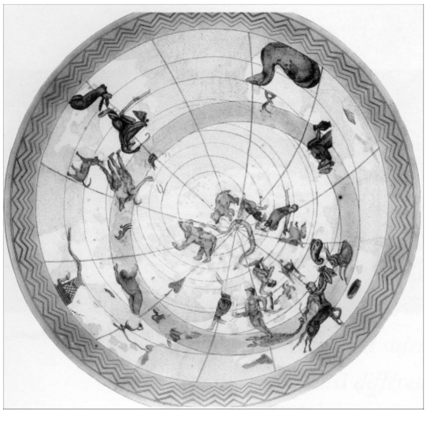


Figure 1: La voûte céleste de Qusayr 'Amra

Il semble que dans le domaine de l'art, la situation n'a pas beaucoup évolué au cours du premier siècle de la dynastie abbasside (751-850). Mais avec le développement des écoles théologiques et l'intensification des débats sur les problèmes de société, à la lumière des textes fondateurs de l'Islam, des attitudes nouvelles se font jour, en particulier à propos du contenu de l'art. Les causes de ce changement ne sont pas faciles à démêler et il n'est pas sûr qu'elles puissent être ramenées toutes à des considérations théologiques. Toujours est-il qu'à partir de la seconde moitié du IX^e^ siècle, une lecture plus dogmatique et plus restrictive de certains versets coraniques ou propos attribués au Prophètes commence à s'imposer. C'est le cas d'un verset qui réprouve les sculptures et les représentations humaines.

Art et mathématiques

A partir de ce moment-là, on observe une disparition progressive de l'art anthropomorphique dans les régions où il était pratiqué. Mais cela n'a pas concerné pas toutes les régions de l'empire musulman. Ce fut le cas pour une partie de la Perse où cet art a continué à se pratiquer même se cela n'a pas abouti, entre le IX^{e} et le XII^{e} siècle, à l'éclosion d'écoles artistiques. Parallèlement, on voit se développer des pratiques artistiques nouvelles faisant intervenir, en plus des motifs floraux dont la réalisation ne s'est jamais interrompue, des lignes, des figures et des solides géométriques. C'est ainsi que des transformations mathématiques simples, comme la symétrie axiale et centrale et les différents types de rotation, vont alimenter l'inspiration des artistes et des décorateurs.

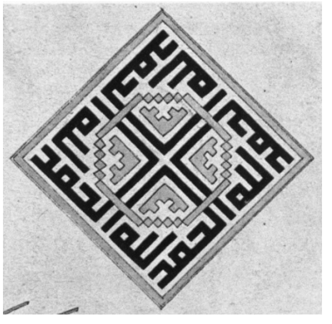


Figure 2: Symétrie et art

Jouant sur les possibilités offertes par les symétries et les rotations, certains artistes vont aller plus loin en développant tout un art du pavage. Comme ils avaient à décorer des surfaces relativement grandes, cela leur offrait la possibilité de répéter des motifs élémentaires qui n'ont pas toujours des propriétés de symétrie mais qui permettent d'en créer par le pavage de l'espace. Nous n'avons, à ce jour, aucune information sur les conditions dans lesquelles les artistes pionniers ont eu l'idée de ces décorations, comme nous ne savons pas qu'elle est la formation mathématique de ces artistes et dans quelle mesure, ils avaient pris conscience, progressivement, de la difficulté du problème et, par conséquent, du défi que représentaient la recherche et la découverte de nouveaux motifs permettant de paver l'espace. Quoi qu'il en soit, les réalisations qui nous sont parvenues, même si elles n'exigent pas de connaissances sophistiquée en géométrie révèlent tout au moins une culture solide dans ce domaine et, surtout, une imagination féconde et beaucoup d'ingéniosité.

Quant à la démarche qui aurait permis aux artistes des pays d'Islam de découvrir les 17 motifs pouvant recouvrir un plan, elle ne pouvait être le résultat de recherches mathématiques ayant eu lieu dans le cadre de la tradition scientifique arabe. Comme il est désormais bien connu, le théorème qui assure l'existence de ces 17 groupes de pavage n'a été établi qu'à la fin du XIX^{e} siècle. Et les outils mathématiques qui ont permis de le démontrer n'existaient pas à l'époque des artistes de l'*Alhambra* de Grenade (où l'on a répertorié le maximum de motifs élémentaires permettant de paver un plan). On est donc bien en présence d'une démarche d'artiste, faite d'intuition, de savoir-faire, d'expérimentation et d'élimination et non de l'application d'un résultat théorique établi par des mathématiciens chevronnés.

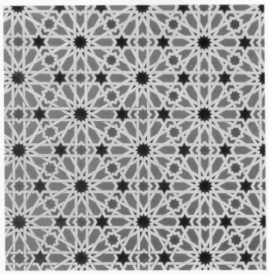


Figure 3: Pavage et arabesques

En architecture, des innovations sont apparues à différents niveaux : plan de masse, adjonction de volumes, introduction de nouvelles formes, etc. Dans ces différentes démarches, la symétrie est abondamment sollicitée pour accentuer l'harmonie. C'est ainsi que des architectes ont eu l'idée d'ajouter, dans leur conception des mosquées, certaines formes (coupoles, minarets) pour « équilibrer » l'ensemble. Il y eut ainsi l'apparition d'un second minaret et d'une coupole centrale puis on a multiplié les coupoles et on a varié leur forme à la fois pour des raisons pratiques et esthétiques. Avec les réalisations du grand architecte ottoman Sinân (m. 1588), on a aboutit à des édifices imposants et sophistiqués flanqués de quatre minarets et de nombreuse coupoles ou demi coupoles et de balcons.

Nous n'avons pas de témoignage sur la réalisation du fameux minaret de la ville de Samarra qui a la particularité d'être hélicoïdal et qui a donc nécessité un minimum de connaissance sur certaines courbes de l'espace et sur la forme de certains solides que l'on ne rencontre même pas dans les ouvrages classiques qui enseignaient à cette époque la géométrie.

Figure 4a. Minaret de Samarra

Pour les coupoles, Nous avons un témoignage précieux, d'un mathématicien cette fois, qui expose les procédés de réalisation des « *patrons* » qui servent à la réalisation de certains modèles. Il s'agit du grand savant persan al-Kâshî (m. 1429) qui a vécu a Samarkand et qui a travaillé comme principal astronome à l'observatoire de cette ville, fondé par le prince mongol Ulug Beg (1394-1449), petit fils du fameux Tamerlan (1336-1405). On trouve, dans son livre *Miftâh al-hisâb* [La clé du calcul], des constructions à la règle et au compas qui sont à la portée d'un artisan ayant un minimum de formation en géométrie euclidienne.

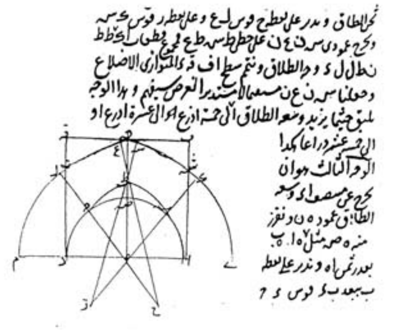
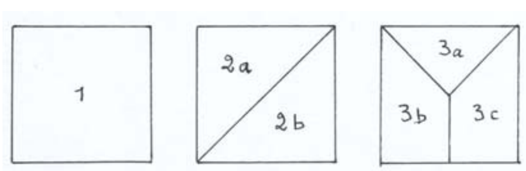


Figure 4b: Construction de coupoles par al-Kâshî

Au X^{e} siècle déjà, le mathématicien Abû l-Wafâ' (m. 997) évoquait cette catégorie de maîtres artisans qui avaient acquis cette formation. Il nous a même conservé, dans son livre intitulé *Kitâb fî mâ yahtâju ilayhi as-Sânic min a^{c}mâl al-handasa* [Livre sur ce qui est nécessaire à l'artisan en constructions géométriques], des procédés géométriques de découpage et de recomposition de figures que les décorateurs de son époque utilisaient, et il y a ajouté ses propres solutions jugées plus exactes. Voici un exemple de ces procédures : Il s'agit de découper trois carrés, de côté *c*, pour obtenir, après recomposition, un seul carré de côté *C*. Cela revient à construire, géométriquement un côté de longueur $\sqrt{3}\,c$. Une des solutions des artisans décorateurs est la suivante:

$\Rightarrow$



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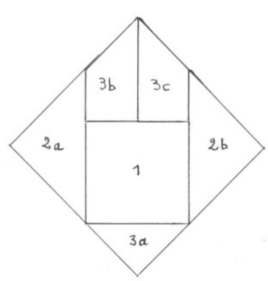
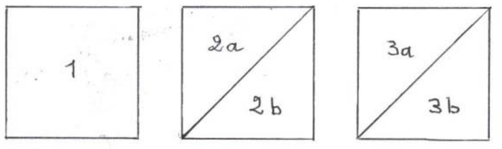


La solution donnée par Abû l-Wafâ est la suivante :



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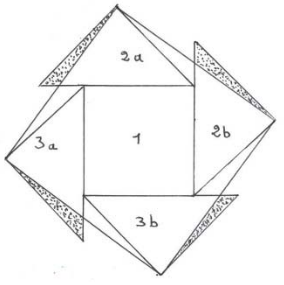

Puis, à partir du XI^{e} siècle le souci de décorer les intérieurs et peut-être aussi celui d'éliminer les aspérités des angles, ont amené les décorateurs des palais et des édifices religieux à concevoir des motifs à trois dimensions qui, en s'imbriquant rigoureusement, comblaient certains vides et ajoutait un élément décoratif original. Il s'agit d'une juxtaposition de petits solides en forme d'alvéoles qui portent le nom de *musardas*.

Figure 5a: Muqarnas

Nous ne savons rien encore sur l'origine de ces muqarnas, sur les premiers artistes qui les ont utilisés et sur la première phase de leur développement. Mais il semble bien que nous soyons là en présence d'une pratique artistique qui a été accompagnée, dès le début, de la maîtrise d'un savoir-faire géométrique plus élaboré que celui qui aurait servi à concevoir et à réaliser les arabesques. Nous avons en effet un témoignage précieux, d'un mathématicien cette fois, qui expose les procédés de réalisation des motifs élémentaires entrant dans la composition d'un muqarnas. Il s'agit encore une fois d'al-Kâshî.

Dans un des chapitres de *Miftâh al-hisâb* que nous avons déjà évoqué, il décrit des motifs de muqarnas qui, collés les uns aux autres, permettent de réaliser une sorte de pavage à trois dimensions recouvrant des coins de mur (intérieurs ou extérieurs), des plafond plats et même le creux des coupoles. Il fournit également les procédures géométriques pour réaliser ces motifs.

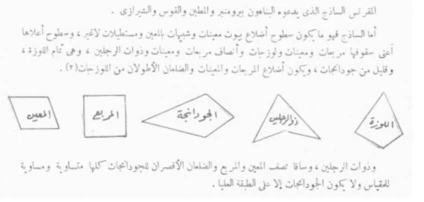

Figure 5b: Muqarnas

Il faut enfin évoquer un vaste domaine, spécifiquement arabe et musulman, où la démarche artistique est intervenue relativement tôt et où « l’esprit » géométrique a fini par avoir sa place et avoir une certaine influence dans certaines orientations artistiques de cette pratique. Il s’agit de la calligraphie, appréhendée à la fois comme un élément d’expression et comme un instrument de décoration aux multiples facettes. Sur le plan strictement décoratif, des artistes ont exploité toutes les particularités des lettres et de l’écriture arabes pour réaliser des compositions où l’esthétique de la calligraphie choisie est rehaussée par la symétrie ou la pseudo symétrie.

Pour faire double emploi, ces artistes introduisent parfois du sens dans leur composition en calligraphiant un ou deux vers d’un poème de circonstance. Mais souvent, ils se limitent à styliser le mot *Allah*, ou une formule à sa gloire, ou bien le nom du Prophète *Muhammad* ou enfin celui de son cousin et gendre *Ali*. Il y a même des tableaux à signification religieuse où la symétrie est au service d’une idée mystique, celle de la quête perpétuelle et son aboutissement par une sorte d’état de grâce où le fidèle se sent comme le reflet de Dieu.

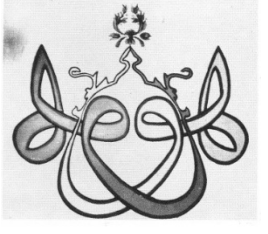

Figure 6a: Calligraphie mystique

Toutes les compositions que nous venons d’évoquer ont été, pendant longtemps le produit de la seule inspiration de l’artiste. Puis, à partir de l’époque du calife al-Muqtadir (908-932), une tendance à la « géométrisation » de la calligraphie s’est faite jour et s’est développée, avec des spécialistes éminents, comme Ibn Muqla (l’initiateur de ce nouveau courant) et son élève Ibn al-Bawwâb. Cette géométrisation a consisté à introduire dans l’enseignement de la calligraphie des proportions que l’artiste devait respecter et qui étaient exprimés à l’aide de cercles de losanges.

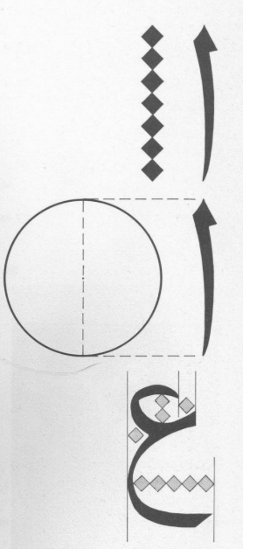

Figure 6b: calligraphie & proportion

Poésie et mathématique

Les relations entre les mathématiques et la poésie ont eu deux facettes distinctes et ont connu deux moments particuliers. Dans une première période, c'est la poésie, ou plutôt la métrique régissant l'art poétique, qui a eu besoin des mathématiques. Dans une seconde période, ce sont des mathématiciens qui ont eu recours à l'expression poétique pour présenter autrement un discours purement scientifique.

Pendant des siècles, et bien avant l'avènement de l'Islam, les poètes composaient leurs vers en choisissant, en fonction du thème traité, l'une des formes consacrées par la tradition. Ce sont les 15 mètres classiques de la poésie arabe. A la fin du VIII^{e} siècle, le premier linguiste et lexicographe, al-Khalîl Ibn Ahmad (m. vers 795) a entrepris d'analyser la structure interne de cette poésie essentiellement orale. Il a abouti à sa fameuse théorie des 5 cercles qui permettent d'engendrer, presque mécaniquement, chacun des 15 mètres. Pour cela, il est parti des deux éléments rythmiques qui sont à la base de la musique et de la poésie : le mouvement et l'absence de mouvement. Il a symbolisé le premier par 0 et le second par 1. Puis il a composé des rythmes élémentaires comme, par exemple, 01 (qui se prononce « pam »), 001 (= papam), 010 (= pampa), etc. Si on adopte les notations suivantes : S = 01, W = 001, M = 010 et T = 00, on obtient, par composition, SW, SSW, …, qui sont les rythmes fondamentaux. Puis, chacun d'eux fournit, par permutation circulaire, WS, SWS, WSS, ... A partir de là, al-Khalîl est arrivé à exprimer, au niveau rythmique, la structure interne de chacun des quinze mètres comme combinaisons de ces rythmes fondamentaux. A titre d'exemple, un des deux hémistiches du mètre dit « *long* » a la configuration combinatoire suivante : WS WSS WS WSS.

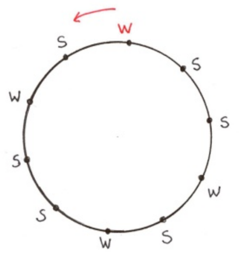

Figure 7a: Cercle métrique

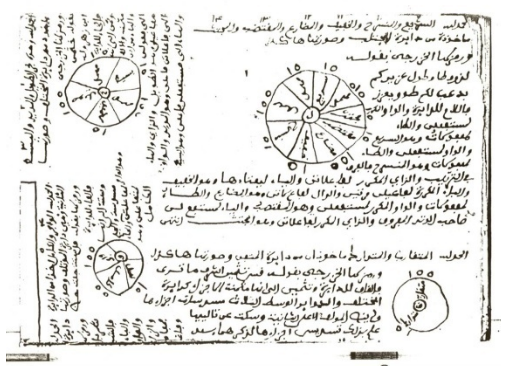

Figure 7b: Manuscrit contenant des cercles métriques

Nous ne savons pas encore si cette contribution d'al-Khalîl Ibn Ahmad a eu des prolongements mathématiques. En revanche, sur le plan de la pratique poétique, des métriciens du IX^{e} siècle ont « fabriqué » de nouveaux mètres en application de sa théorie. C'est ce qu'a fait al-Akhfash (m. 830), un des étudiants d'al-Khalîl. Mais, à notre connaissance, et en dehors d'un seul cas (qui deviendra le seizième mètre), les versificateurs ont boudé les nouveaux mètres proposés par les métriciens, probablement pour des raisons purement esthétiques.

En ce qui concerne le second aspect des relations entre mathématique et poésie, c'est-à-dire la versification de textes techniques, les manuscrits qui nous sont parvenus et qui ont été analysés fournissent des exemples variés du traitement de la matière de différentes disciplines. Il faut préciser que tous les poèmes mathématiques que nous avons pu analyser utilisent un seul et même mètre, le *rajaz,* que les versificateurs qualifie, ironiquement, de « *mètre pour les ânes* » parce que n'importe quelle personne, même la plus démunie en inspiration poétique, est sensée produire des vers avec ce mètre. De nombreux mathématiciens font bien sûr partie de cette catégorie. Mais le *rajaz* a un autre avantage : La longueur réduite de ses deux hémistiches et son rythme alerte en font un outil mnémotechnique tout à fait adapté au discours scientifique en général dans la mesure où l'étudiant peut facilement mémoriser des dizaines et parfois même des centaines de vers traitant d'un même sujet.

Parmi les textes mathématiques qui nous sont parvenus, il y a des exposés d'arithmétique, d'algèbre ou de géométrie, tous de niveaux élémentaires, même pour les étudiants de l'époque. Le plus célèbre des poèmes d'algèbre est la *Yasamîniyya* du nom de son auteur, Ibn al-Yâsamîn (m. 1204). En 57 vers, l'auteur expose les éléments de base de l'algèbre d'al-Khwârizmî (m. 850), avec les définitions des premiers objets de l'algèbre et de ses outils, suivies de la résolution des six équations canoniques. Au XIII^{e} siècle, Ibn Liyyûn, un polygraphe andalou, a publié un poème de plus de 200 vers sur la géométrie du mesurage. En arithmétique, il nous est parvenu aussi un certain nombre de poèmes exposant le système décimal positionnel et les algorithmes principaux (addition, soustraction, multiplication, division et extraction de racine).

La seconde catégorie de poèmes n'a pas été conçue par ses auteurs comme un ensemble d'outils pédagogiques mais plutôt comme une production culturelle et ludique à la fois. On y trouve, en particulier, des discours amoureux, présentés sous forme de problème ou d'énigmes arithmétiques. La plupart d'entre eux sont de petits sonnets dont l'écriture mathématique aboutit à une équation du premier degré. Ce sont donc autant d'exercices attrayants pour initier les adolescents aux techniques algébriques élémentaires. Voici un exemple représentatif de cette catégorie :

Je lui ai fait don d'un tiers de l'existence,
Et d'un quart et un sixième puis d'un huitième. Mais elle a refusé
Et elle a dit : « c'est peu ». J'ai dit : « j'ai un ajout »,
Et je lui ai donné deux tiers du septième de ce qui est déjà passé,
Et j'ai gardé, pour moi, vingt années pour vivre.
Or cela est si peu pour un jeune homme affaibli.

Si x est l'âge de la personne, on a :

$$(\frac{1}{3}+\frac{1}{4}+\frac{1}{6}+\frac{1}{8})x+\frac{2}{3}(\frac{1}{7})(\frac{1}{3}+\frac{1}{3}+\frac{1}{3}+\frac{1}{3})x+20=x$$. D'où : $x = 480$.

Mais on y trouve parfois des poèmes très osés, en particulier pour la société maghrébine de l'époque où le rite malékite avait favorisé un certain puritanisme. C'est le cas, par exemple, de cette déclaration d'amour arithmétiques mais non moins enflammée que l'on peut lire dans un ouvrage très sérieux rédigé par le mathématicien de Marrakech, Ibn al-Bannâ (m. 1321), qui était aussi savant en religion :

Les trois septièmes du cœur pour son regard
Un septième est offert pour le rose de <ses> deux joues
Un septième et la moitié d'un septième et le quart
Pour le refus d'un désir inassouvi
Un septième et un sixième d'un quart sont la part de seins bien arrondis
Qui se sont refusés au pêché de mon étreinte et qui m'ont repoussé
Le reste, qui est cinq parts, est pour des paroles d'elle
Qui étancheraient ma soif si elles étaient entendues

Si on note x le cœur tout entier, on a :

$$[\frac{3}{7}+\frac{1}{7}+\frac{1}{7}+\frac{1}{2}(\frac{1}{7})+\frac{1}{4}(\frac{1}{7})+\frac{1}{7}+\frac{1}{6}(\frac{1}{4}(\frac{1}{7}))]x+5=x$$. D'où : $x = 168$.

L'ouvrage en question est intitulé *Tanbîh al-albâb ^{c}alâ masâ'il al-hisâb* [Avertissement aux gens intelligents sur les problèmes du calcul]. La seconde partie de cette épître regroupe des petits poèmes de deux à six ou sept vers, qualifiés par l'auteur « *d'énigmes* » et dont les solutions sont soit des prénoms et des noms célèbres soit des nombres. Au niveau mathématique, chacune des énigmes s'exprime sous forme d'une équation du premier degré à coefficients fractionnaires dont la solution peut s'obtenir en utilisant soit la méthode de double fausse position, soit la méthode de l'inverse, soit le procédé algébrique. Mais, l'auteur n'expose aucun type de résolution, se contentant de donner la solution numérique ou le nom cherché. Voici un exemple dont la difficulté de résolution n'est pas de nature technique mais purement culturelle. Ce qui nous renvoie au contexte dans lequel ces problèmes ont été produits et diffusés :

Le nom de celui que Dieu m'a fait aimer,
C'est un prince qui le cherche, sache-le,
Son premier est un nombre égal à la valeur du troisième,
Son second est le cinquième de celui qui le suit,
Et le cinquième du premier inconnu ;
Son dernier est le dixième de celui qui le précède
Et une partie de sa somme, sache-le.

Il s'agit de trouver nom *Muhammad*, c'est-à-dire celui du Prophète. Il faut d'abord savoir que chaque phonème qui compose le nom a une valeur numérique, précisément celle qu'il possède dans la numération alphabétique en usage chez les astronomes de cette époque (qui, comme on le sait, n'est pas positionnelle puisqu'elle utilise 27 symboles pour désigner, respectivement, les 9 unités, les 9 dizaines et les 9 centaines). Ainsi, dans notre exemple, le nom cherché est constitué des quatre phonèmes M, H, M, D (puisque les voyelles *u* et *a* ne sont pas prises en compte). Or les valeurs numériques de ces lettres sont, dans l'ordre de leur position dans le prénom : 40, 8, 40, 4. Pour résoudre l'énigme, il suffit juste de se rappeler que l'arabe s'écrit de droite à gauche, M devenant la première lettre, H la seconde et D la dernière.

Ibn al-Bannâ ne donne aucune information sur l'origine de ces problèmes et sur leurs auteurs. Compte tenu de leur forme qui s'apparente plus à une énigme qu'à un véritable énoncé mathématique, leur composition est à la portée de n'importe quel versificateur connaissant la numération alphabétique. Il est donc vraisemblable que l'auteur n'a fait que les rassembler après les avoir entendus dans les milieux cultivés qu'il a fréquentés à Marrakech et à Fez. Il semble d'ailleurs que le répertoire de ces énigmes ait été plus large que ce contient son petit livre.

Il faut enfin évoquer une dernière catégorie de problèmes ludiques, dont on ne connaît pas l'origine exacte et qui a circulé à travers des épîtres puis des manuels. Ils sont tous en prose et ressemblent parfois à des énigmes. Une première famille consiste en des petites histoires qui contiennent l'évocation de nombres. Ce sont en fait des exercices de calcul habillés d'une histoire pour les rendre attrayants. La seconde consiste en des problèmes qui sont regroupés sous l'appellation « *problèmes de nombres cachés* ». Ils ont participé à l'animation de soirées culturelles avant d'être insérés dans des manuels de calcul, comme exercices ludiques. Il s'agit de choisir un interlocuteur qui sache calculer mentalement, à lui ordonner de faire une succession d'opérations arithmétiques et à l'interroger sur certains résultats partiels. A la fin du questionnaire, celui qui a posé les questions est capable de révéler le nombre pensé par la personne qui a été interrogée. Voici un exemple de ce type de jeux arithmétiques :

> *On demande à celui qui a pensé d'ajouter à ce qu'il pensé son égal puis il multiplie ce qui résulte de cela par deux puis il le double puis il retire de cela huit après huit; et chaque qu'il ôte de cela huit, il garde <un> jusqu'à épuisement de ce qu'il y a chez celui qui a pensé. Ce qui résulte comme nombre dans sa main sera ce qui a été pensé par celui qui a pensé.*

On trouve, dans les manuels qui nous sont parvenus, des variantes encore plus culturelles ou plus ludiques de ce jeu mathématique, comme la détermination deux ou trois nombres pensés ou d'un nom pensé, la recherche d'une ou de deux bagues cachées, ou du doigt qui porte la bague, ou d'une personne absente parmi trois, etc.

Il est intéressant de constater, à travers les documents qui nous sont parvenus, que ces jeux mathématiques ont traversé les frontières politiques et culturelles puisqu'on en trouve des spécimens en Europe dès le XII^e siècle, d'abord dans le *Liber Abaci* de Fibonacci (m. après 1240) puis dans divers manuels d'enseignement. C'est probablement le succès de ces problèmes récréatifs qui a amené Bachet de Méziriac (m. 1638) à leur consacrer tout un ouvrage intitulé « *Problèmes plaisants et délectables qui se font par les nombres*».

Linguistique et combinatoire

L'analyse des documents qui nous sont parvenus montrent que la pratique combinatoire a connu deux périodes distinctes : la première, antérieure au XIIe siècle, est celle où la combinatoire est cantonnée à des procédés d'énumération et de dénombrement, arithmétiques ou mécaniques, qui ne débouchent jamais, au vu des textes connus, sur une "modélisation" et donc sur des propositions générales ou sur des formules applicables à des problèmes de nature différente. La seconde période, qui commence peut-être dans la seconde moitié du XIIe siècle, est celle d'un retour de certaines préoccupations combinatoires accompagnant un renouveau des études linguistiques. On observe alors l'apparition de véritables propositions, énoncées et démontrées, un ensemble de procédés de calcul, soit à l'aide de tableaux soit exprimés par des formules arithmétiques et, enfin, un souci d'application des outils nouveaux pour résoudre des problèmes appartenant à différents domaines.

Mais, revenons un peu en arrière : Avec l'avènement du phénomène de traduction, le statut privilégié de l'arabe a favorisé le développement de plusieurs « *sciences de la langue* ». Dans ce cadre, les lexicographes ont énuméré, et parfois dénombré, des configurations de lettres de l'alphabet soumises à certaines contraintes dans le but de confectionner des lexiques. On sait, par exemple, qu'al-Khalîl Ibn Ahmad, que nous avons déjà évoqué pour sa contribution en métrique, avait donné le nombre de combinaisons p à p ($2 \leq p \leq 5$) des 28 lettres de l'alphabet arabe et, après lui, le grammairien Sîbawayh (m. 796) avait déterminé le nombre d'arrangements p à p ($2 \leq p \leq 5$) de ces mêmes lettres, mais en tenant compte des incompatibilités de prononciation de la langue arabe.

Cette tradition combinatoire s'est maintenue, avec des variantes et dans le cadre de préoccupations nouvelles, chez les linguistes des siècles suivants. On peut citer, en particulier, Hamza al-Isfahânî (m. 970) qui a repris, dans un de ses livres, les dénombrements effectués par al-Khalîl. Après lui, Ibn Jinnî (m. 1005) a intégré, dans son *Kitâb al-ishtiqâq al-kabîr* [Livre sur la théorie de la grande dérivation], les différents arrangements des lettres dans la langue arabe, en tentant d'associer à chaque combinaison trilitère un sens originel d'où dériveraient les sens de toutes les permutations possibles de la dite combinaison. En Andalus, c'est-à-dire la partie de la Péninsule ibérique qui a été gouvernée par des pouvoirs musulmans, az-Zubaydî (m. 989) a considéré, dans son *Mukhtasar* [L'abrégé], des dénombrements qui tiennent compte des contraintes liées à la prononciation et à l'usage.

On constate d'ailleurs, à la lecture d'un ouvrage du polygraphe Al-Suyûtî (m. 1505), que, même si les méthodes de calcul varient d'un auteur à l'autre, elles sont toutes soumises à des contraintes linguistiques (concernant la nature des lettres qui composent les mots et les règles de prononciation) qui n'ont peut-être pas favorisé le dégagement d'algorithmes généraux. De plus, ces méthodes ne sont pas exemptes d'erreurs, à la fois au niveau des résultats et au niveau des raisonnements qui les justifient. Ce qui tend à prouver qu'on ne disposait pas encore de formules ou de procédés pour obtenir les dénombrements cherchés.

C'est ce que confirme, par exemple le contenu de l'ouvrage du linguiste Ibn Durayd (m. 933). On y trouve, incomplètement exposées, deux méthodes de calcul de nature différente. La première consiste à disposer les lettres à combiner sur un disque fixe entouré de deux anneaux concentriques et à faire tourner les anneaux de manière à faire correspondre, à chaque fois des lettres différentes.

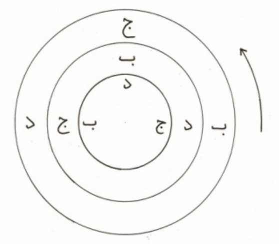

Figure 8: Disque d'Ibn Durayd

L'astrologue maghrébin al-Bûnî (m. 1205), et le célèbre mystique Ibn ^{c}Arabî (m. 1240) l'ont utilisé. On le retrouve même en Europe, en particulier dans des écrits théologiques du catalan Ramon Llull (m. 1316). La seconde méthode consiste à dénombrer, séparément, les mots sans répétition de lettres puis les autres. Et, dans le dénombrement des premiers, à distinguer entre les mots qui ne contiennent ni *w* (wâw) ni *y* (yâ) et les autres : Le calcul est correct pour les arrangements, 2 à 2 avec répétitions, des 28 lettres de l'alphabet arabe. Il est faux pour les arrangements de plus de deux lettres, l'erreur portant sur la valeur des combinaisons 3 à 3. Ibn Durayd semble utiliser en effet la formule $C_p^3 = p.C_p^2$, au lieu de $C_p^3 = \frac{p-3}{3}.C_p^2$. Il est étonnant que les auteurs successifs n'aient pas comparé ces résultats à ceux d'al-Khalîl Ibn Ahmad qui n'explicitent aucune méthode mais qui sont rigoureusement exacts.

D'après les informations dont nous disposons aujourd'hui, c'est au Maghreb que les préoccupations et les pratiques combinatoires ont débouché, trois siècles plus tard, sur l'élaboration d'un chapitre nouveau avec ses objets, ses outils et son domaine d'application. Mais, comme en Orient à la fin du VIII^{e} siècle, c'est une discipline non mathématique qui a préparé l'avènement de ce chapitre. Il s'agit de la linguistique. Cette activité n'était pas nouvelle au Maghreb. Mais, au cours des siècles qui nous intéressent, elle a bénéficié d'un réel dynamisme. En effet, on assiste, à partir du XII^{e} siècle, en particulier à Marrakech, la capitale de l'empire almohade, à un regain d'intérêt tel pour les différents chapitres de la linguistique que même des mathématiciens se sont consacrés à son étude. Ce n'est pas un hasard d'ailleurs que ce soit Ibn al-Bannâ (m. 1321), l'auteur de plusieurs ouvrages sur la langue arabe, qui a été un des auteurs dont les préoccupations combinatoires ont été les plus conséquentes.

A partir d'une simple lecture des textes qui nous sont parvenus, on voit que des problèmes sont posés et résolus en utilisant des formulations et des raisonnements à caractère combinatoire. On constate aussi qu'une terminologie, née des besoins de la linguistique et de la lexicographie, acquiert un statut mathématique et qu'un formulaire nouveau est établi pour devenir un instrument opérant sur des objets mathématiques. Cela dit, il faut avouer que dans l'état actuel des recherches sur ce sujet, nous n'avons pas suffisamment d'éléments pour cerner les différents aspects de cette activité depuis ses débuts. Nous allons donc nous contenter, ici, de présenter les résultats et les démarches connues en dégageant certaines caractéristiques de cette nouvelle pratique mathématique.

Une contribution du XII^{e} siècle

Dès la fin du XII^{e} siècle, des problèmes posés par la linguistique arabe de la fin du VIII^{e} ont été réexaminés mais, cette fois, dans le cadre d'un chapitre autonome de la science du calcul. C'est ce qu'a fait Ibn Mun^{c}im (m. 1228), un mathématicien de Marrakech, originaire de Dénia, une ville d'al-Andalus, dans son traité *Fiqh al-hisâb* [La science du calcul].

C'est dans la onzième section de son livre, intitulée "*Le dénombrement des mots qui sont tels que l'être humain ne peut s'exprimer que par l'un d'eux*", que ce mathématicien, spécialiste de géométrie et de théorie des nombres, expose ses démarches et ses résultats combinatoires. Le contenu de cette section est présenté par l'auteur comme une extension des résultats d'al-Khalîl Ibn Ahmad, et une généralisation de ses calculs qui avaient permis de déterminer les combinaisons sans répétitions des phonèmes arabes. A cet effet, Ibn Mun^{c}ie se propose d'abord de traiter le problème d'une manière générale, puis de faire suivre ses démonstrations d'exemples et de tableaux. La généralité dont parle l'auteur concerne l'établissement de formules mathématiques et de procédés en vue de dénombrer les mots de n'importe quelle longueur, dans n'importe quelle langue. Malgré tout, cette étude dépasse objectivement le cadre linguistique dans lequel elle a été formulée et réalisée, tant par la manière de poser les problèmes et de les relier l'un à l'autre, par les méthodes de raisonnement utilisées, que par les résultats établis.

L'auteur commence par établir, à partir d'un ensemble de couleurs de soie qui joue le rôle de modèle abstrait, une règle permettant de déterminer toutes les combinaisons possibles de *n* couleurs, *p* à *p*. Dans ce but, il construit, selon une démarche inductive, un tableau numérique dont il identifie les éléments avec les combinaisons cherchées. En faisant cela, il donne, pour la première fois à notre connaissance, selon une démarche strictement combinatoire, le fameux triangle arithmétique que des mathématiciens d'Asie Centrale avaient déjà construit, au XI^{e} siècle, dans un cadre algébrique, en vue de déterminer les coefficients du binôme.

Figure 9: Triangle arithmétique d'Ibn Munʿim

L'étude d'Ibn Mun´im se poursuit par l'établissement, selon des démarches combinatoires reposant sur l'induction, des formules relatives aux permutations, avec ou sans répétitions, d'un ensemble de lettres et celles donnant, par récurrence, le nombre de lectures possibles d'un même mot de n lettres, compte tenu de tous les signes de prononciation utilisés par une langue donnée. Il conclut cette première partie en établissant la formule des arrangements, sans répétitions, en adoptant une démarche analogue à la précédente et qui nécessite le recours à des tableaux de nombres. Comme pour la recherche des combinaisons simples, Ibn Mun'im fait fonctionner, une nouvelle fois, des ensembles d'objets (respectivement des lettres d'un alphabet, des couleurs de soie, des filaments de couleurs), comme des modèles abstraits en identifiant, à chaque fois, les objets étudiés aux éléments du modèle.

Les résultats établis par Ibn Munʾim

(1) Les combinaisons dans le triangle arithmétique :

$$C_n^p = C_{p-1}^{p-1} + ... + C_{n-1}^{p-1}$$

(2) Nombre de permutations d'un mot de n lettres distinctes :

$$P_n = 1.2.3....n$$

(3) Nombre de permutations d'un mot de n lettres dont p lettres sont répétées respectivement k_1, k_2, …, k_p fois :

$$P_n^k = \frac{P_n}{P_{k_1}.P_{k_2}...P_{k_p}}$$

(4) Nombre de prononciations d'un mot, compte tenu des voyelles :

$$S_n = 4S_{n-1} - 3S_{n-3}$$ ou bien $$S_n = 3S_{n-1} + 3S_{n-2}$$

La troisième partie de la onzième section renferme, en plus de quelques applications, une série de tableaux qui permettent de déterminer, de proche en proche, tous les éléments qui interviennent dans le dénombrement des mots qu'il est possible de prononcer, dans une langue donnée, c'est à dire les combinaisons, les permutations et les arrangements, avec ou sans répétitions.

Il nous reste à dire quelques mots sur les types de raisonnement utilisés par l'auteur pour établir ces résultats et sur leur statut par rapport aux outils et aux démarches mathématiques traditionnels. En ce qui concerne les justifications qui interviennent dans l'établissement des résultats précédents, l'analyse des démarches de l'auteur fait apparaître deux types de raisonnement que l'on pourrait qualifier, globalement, d'inductif et de combinatoire. Le premier, avec ses différentes variantes (et avec le sens qu'il a gardé jusqu'au XVII^e siècle), est un outil traditionnel dans les mathématiques arabes, avec son domaine privilégié, la théorie des nombres, et son statut particulier mais reconnu. Quant au second raisonnement, nous ne l'avons pas rencontré dans des écrits mathématiques antérieurs au *Fiqh al-hisâb*, il se distingue de l'induction par une démarche qui combine l'énumération et le dénombrement. L'utilisation de cette démarche pour établir des règles considérées comme générales apparaît comme une reconnaissance de fait de son caractère mathématique, sans que l'on puisse affirmer encore que cette reconnaissance a été explicitée par l'un ou l'autre des auteurs qui se sont occupés de combinatoire.

La combinatoire aux XIII^e –XIV^e siècles

Vers la fin du XIII^e siècle au plus tard, un nouveau pas est franchi dans l'activité combinatoire au Maghreb. Les formules exprimant le nombre de combinaisons et d'arrangements de n objets p à p sont données, mais avec de nouvelles démonstrations et dans le cadre d'une problématique classique, celle de la théorie des nombres. Cette contribution est exposée et revendiquée explicitement par Ibn al-Bannâ dans deux de ses ouvrages : Le *Tanbîh al-albâb*, que nous avons déjà évoqué et le *RAF al-hijâb ʿan wujûh aʾmâl al-hisâb* [Le lever du voile sur les formes des opérations du calcul].

Dans le problème n° 14 du premier texte, intitulé *Question tirée de la linguistique*, l'auteur énonce d'abord la règle arithmétique, déjà donnée par Ibn Mun^{c}im, qui permet le calcul des permutations d'un nombre quelconque de lettres de l'alphabet. Pui il donne, pour la première fois à notre connaissance, la formule arithmétique qui permet de calculer explicitement les combinaisons sans répétitions (C^p; $_n$, pour $n \geq 2$ et $2 \leq p \leq n$) :

$$C_n^p = \frac{n(n-1)...(n-p+1)}{p(p-1)...2.1}$$

Dans le second ouvrage, on retrouve les résultats que nous venons d'évoquer. Mais, cette fois, ils sont exposés dans un cadre classique plus général, celui de la théorie des nombres de la tradition néo pythagoricienne telle qu'elle est parvenue au Maghreb à travers la traduction arabe de *l'Introduction arithmétique* de Nicomaque de Gérase (II^e s.). En effet, les expressions combinatoires sont comparées à des sommes de suites finies d'entiers et à des éléments du tableau des nombres figurés.

Cette démarche arithmétique explicite nous autorise à dire qu'Ibn al-Bannâ a eu une perception nette de la liaison étroite, à la fois au niveau des résultats et des démonstrations, entre méthodes arithmétiques et méthodes combinatoires. On peut préciser, à ce sujet, que, au niveau des résultats, c'est le tableau des nombres polygones qui assure la liaison entre les deux disciplines. Mais, au niveau des démonstrations, ce sont les différentes méthodes d'induction qui justifient l'intégration, par ce mathématicien, des résultats combinatoires au vaste chapitre de la théorie des nombres.

Il faut enfin noter que, parties de préoccupations linguistiques, donc profondément culturelles, les démarches combinatoires reviennent à ces préoccupations en intervenant dans la résolution de problèmes non mathématiques. Le *Tanbîh al-albâb* d'Ibn al-Bannâ nous fournit un échantillon de ces problèmes : Enumération des différents cas d'héritage possibles lorsque les héritiers sont *n* garçons et *p* filles, énoncé de toutes les situations où l'ablution (qui précède toute prière musulmane), est nécessaire avec de l'eau, et de celles où elle est permise sans eau, dénombrement, selon les exigences du rite malékite, des prières à effectuer pour compenser l'oubli de certaines d'entre elles, dénombrement de toutes les lectures possibles d'une même phrase selon les règles de la grammaire arabe.

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Adolescent Learning and Secondary Mathematics

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In this paper I develop my thinking about how learning secondary mathematics can relate to the adolescent project of negotiating adulthood. All too often it does not, yet the same kinds of adolescent autonomous thinking which so often lead to disaffection and rejection, not only of mathematics but of school and life more generally, can be embedded and enhanced positively within the teaching and learning of mathematics. I suggest parallels, and some kinds of tasks which enhance the adolescent project through mathematics, and mathematics through adolescence.

Introduction

> *To confirm the deepest thing in our students is the educator's special privilege. It demands that we see in the failures of adolescence and its confusions, the possibility of something untangled, clear, directed* (Windle, 1988).

My personal rationale for taking this view lies in my work as head of a mathematics department during the late 80s and early 90s in a school which served a socio-economically disadvantaged area. In this school, academic achievement was generally low. Nevertheless, our mathematics results were usually the best of all subjects, sometimes ahead and sometimes just behind those achieved in creative arts, and nearly 100% of the cohort, including persistent absentees, achieved some kind of mathematics grade. I take little pride in this, however, because the teaching methods we used, while appealing to the students and to us at the time, did not enable students to do as well as those at nearby middle class schools. While students were learning *some* mathematics, we nurtured and depended on their powers of exploration, application, and their natural enquiry to achieve success. We were praised for this in some quarters, because we had developed ways of working on mathematics *in* school which were similar to the ways in which quantitative problems arise *out* of school. However, we did not enable the majority of students to contact the essentially abstract, structural, understandings which characterise the subject in its entirety. We did little to help them engage out of school with new ways of thinking. In Vygotsky's terms, we failed to support them in engaging with the scientific concepts which can only be deliberately taught (Kozulin, 1986, p. xxxiii). Since then my work has focused on how students with low achieving backgrounds can engage with mathematical thinking, without limiting them to merely being better at workaday thinking in mathematical contexts (Watson, 2006).

Adolescent Concerns

There is widespread agreement that adolescents are broadly concerned with the development of identity, belonging, being heard, being in charge, being supported, feeling powerful, understanding the world, and being able to argue in ways which make adults listen (Coleman & Hendry, 1990). Naming these as adolescent concerns assumes that Western psychological understandings of adolescence can be taken to be universal, which may not be the case due to the interaction of physiological and cultural factors, but I adopt this understanding for the purposes of this paper. Adolescents engage with these concerns through interaction with peers. This was regarded by Vygotsky as the 'leading activity' of adolescence (Elkonin, 1972), that is the activity which leads the course of development but is not necessarily the only influential activity. Nevertheless, as he saw it, peer interaction is the context in which adolescents work out their relationships with others, adults, the world and themselves. They do this by employing their new-to-them ability to engage in formal-logical thinking, so that they become capable of self-analysis, and analysis of other situations, as internalisations of social consciousness develops with peers (Karpov, 2003).

This new-to-them facility is attributed by Piaget to a biological development of the cortex, yet different societies appear to engender different kinds of world view, different forms of mature argument, and different kinds of abstraction. Vygotsky (1986) recognises the role of physiological maturity, and emphasises that the use of the maturing brain is influenced in how the notion of self is worked out by relations in the social world. Thus the biological ability to understand things in more complex, abstract, ways is not the most important influence on learning. Rather, it is how adolescents use adult behaviour and interaction as mediators of their main activity with peers that influences the development of identity and self-knowledge.

School classrooms are thus very important places for adolescents, because it is there that adult behaviour dominates, while the majority present are adolescents whose behaviour and interactions are constrained by adult goals and expectations. It is an easy step from this analysis to ask for more discussion, more work in groups, and more attention to students' ideas and their social world to produce more engaged learners.

'Cognitive bullying' is not too strong a term to describe the kind of teaching which ignores or negates the way a student thinks, imposes mental behaviour which feels unnatural and uncomfortable, undermines students' thoughtful efforts to make sense, causes stress, and is repeated over time, possibly with the backing of the system or institution. To be forced to revisit sites of earlier failure by, for example, doing fractions during grades 6,7,8 and 9, can, with this perspective, be seen as cognitive bullying which is at best marginally productive and at worst emotionally damaging. Students can become trapped into repeated failure with no way out except to adopt negative behaviour or to accept such treatment compliantly. However, compliance does not imply mental comfort or lack of anxiety. As well as repetitious failure, much mathematics teaching involves showing students how to manipulate and adapt abstract ideas – ways of thinking which are often in conflict with their intuitive notions. Even in the most therapeutic of classrooms this requires ways of seeing and paying attention which are contrary to what the student does naturally – against intuition. Thus for some students the mathematics classroom is a site where natural ways of thinking and knowing are constantly overridden by less obvious ideas. Without time to make personal adjustments, many students give up attempts to make personal sense of what they are offered, and instead rely on a disconnected collection of rules and methods. The term 'cognitive bullying' can therefore be taken to mean that students' own ways of thinking are constantly ignored or rejected, the mathematical experiences which generate fear and anxiety are constantly revisited for repetition, and they are expected to conform to methods and meanings which they do not understand.

This state of affairs reaches a climax in adolescence, when examinations become high-stakes, major curriculum topics become less amenable to concrete and diagrammatic representations, full understanding often depends on combining several concepts which, it is assumed, have been learnt earlier, and adolescents are developing a range of serious disruption habits. Those whose thinking never quite matches what the teacher expects, but who never have the space, support and time to explore why, can become disaffected at worst, and at best come to rely on algorithms. While all mathematics students and mathematicians rely on algorithmic knowledge sometimes, learners for whom that is the only option are dependent on the authority of the teacher, textbooks, websites and examiners for affirmation. Since a large part of the adolescent project is the development of autonomous identity, albeit in relation to other groups, something has to break this tension – and that can be a loss of self-esteem, rejection of the subject, or adoption of disruptive behaviour (Coleman & Hendry, 1990, pp.70, 155).

Enquiry Methods

Stoyanova (2007) reports on the evaluation of an extensive mathematics curriculum project in which students used tasks which encouraged enquiry, investigation, problem posing, and other features commonly associated with so-called ‘reform’ and enquiry methods. Results of analyzing test answers from 1600 students showed very few clear connections between aspects of the programme and mathematical achievement during adolescence. Most notably: problem-posing, checking by alternative methods, asking ‘what if...?’ questions, giving explanations, testing conjectures, checking answers for reasonableness, and splitting problems into subproblems were associated with higher achievement in year 10, while using general problem-solving strategies, making conjectures, and sharing strategies were not. Use of ‘real life’ contexts was negatively associated with achievement at this level. Other interesting aspects of her results include that teachers’ beliefs about learners’ ability to learn turned out to be very important, a finding repeated elsewhere (e.g. Watson & De Geest, 2005), and that explicit teaching of strategies was not found to be associated with achievement, either positively or negatively.

Realistic tasks can provide contexts for enquiry and often enquiry methods of teaching and learning are recommended for adolescent learning. Historically, mathematics has been inspired by observable phenomena, and mathematicians develop new ideas by exploring and enquiring into phenomena in mathematics and elsewhere. It is also possible to conjecture relationships from experience with examples, and thus get to know about general behaviour. But mathematics is not *only* an empirical subject at school level; indeed it is not *essentially* empirical. Its strength and power are in its abstractions, its reasoning, and its hypotheses about objects which only exist in the mathematical imagination. Enquiry alone cannot fully justify results and relationships, nor can decisions be validated by enquiry alone. Many secondary school concepts are beyond observable manifestations, and beyond everyday intuition. Indeed, those which cause most difficulty for learners and teachers are those which require rejection of intuitive sense and reconstruction of new ways of acting mathematically.

School mathematics as a human activity has at least two dimensions, that of horizontal and vertical mathematisations (Treffers, 1987).

> *In horizontal mathematization, the students come up with mathematical tools which can help to organize and solve a problem located in a real-life situation. Vertical mathematization is the process of reorganization within the mathematical system itself, like, for instance, finding shortcuts and discovering connections between concepts and strategies and then applying these discoveries* (Van den Heuvel-Panhuizen, 2007).

This shift from horizontal to vertical mathematisation has to be structured through careful task design; it does not happen automatically. A Vygotskian view would be that this kind of shift necessarily involves disruption of previous notions, challenges intuitive constructs, and replaces them with new ways of thinking appropriated by learners as tools for new kinds of action in new situations.

Grootenboer and Zevenbergen (2007) work within a social paradigm which emphasizes agency and identity, yet found that they had to go beyond Burton's analysis of professional mathematical activity to be informative about what actions learners had to make when working mathematically. In their study, learners had to identify patterns, construct generalizations, use examples to test hypotheses, and identify limits to make progress with a task. By naming these actions, they show how important it is that the intellectual demands of mathematics itself should be taken into account when thinking about teaching and learning. Many of the features of the programme evaluated by Stoyanova (2007) could be described as offering agency, but it was seen that only some of them led to improved mathematics learning. The effective features all engaged adolescents in exercising power in relation to new mathematical experiences, new forms of mathematical activity, and being asked to use these, express these, and to display authority in doing so. For example, year 10 level adolescents responded well to being given authority in aspects of mathematical work: checking answers, giving explanations, asking new questions, testing hypotheses, and problem-posing. These actions appeal to the adolescent concerns of being in charge, feeling powerful, understanding the world, and being able to argue in ways which make adults listen. They offer more than belonging by doing what everyone else is doing, or being heard merely through sharing what has already been done.

Shifts of Mathematical Action

Analyses of student errors in mathematics (e.g. Booth, 1981; Hart, 1982; Ryan & Williams, 2007) suggest that many students get stuck using 'child methods', intuitive notions, invented algorithms which depend on left-to-right reading, or misapplication of verbal tricks. When these methods do not produce the right answers to school questions, often at the start of secondary school, these could well be a contributory factor to rejection of the mathematics curriculum.

The contradictions between intuitive, spontaneous, understandings and the scientific concepts of secondary mathematics can be the beginning of the end of mathematical engagement for adolescents. If they cannot understand the subject by *seeing* what it does and how it works, but instead have to believe some higher abstract authority that they do not understand, then the subject holds nothing for them. This analysis has contributed to the belief that mathematics need not be taught to everyone and that many adolescents only need to become functionally numerate (Bramall & White, 2000). But this view misses the point. The authority of mathematics does not reside in teachers or textbook writers but in the ways in which minds work with mathematics itself (Freudenthal, 1973, p.147; Vergnaud, 1997). For this reason mathematics, like some of the creative arts, can be an arena in which the adolescent mind can have some control, can validate its own thinking, and can appeal to a constructed, personal, authority. But to do so in ways which are fully empowering has to take into account the new intellectual tools which simultaneously enable students to achieve in mathematics, and which develop further through mathematics (Stech, 2007). To understand this further I present key intellectual tools of the secondary curriculum as illustrations of what needs to be appropriated in order to engage with new kinds of mathematical understanding. Unsurprisingly, these turn out to be aspects which cause most difficulty and my argument is that it is likely that many teachers do not pay enough attention to these shifts as inherently difficult. Not only do these shifts represent epistemological obstacles, in Brousseau's terms, but they are also precisely those changes to new forms of action which constitute the scientific knowledge of mathematics – that which can only be learnt at school. It is inequitable to expect students to bring their everyday forms of reasoning to bear meaningfully on mathematical problems, when everyday forms do not enable these shifts to be made:

- Shift from looking for relationships, such as through pattern seeking, to seeing properties as defined by relationships;
- Shift from perceptual, kinaesthetic responses to mathematical objects to conceptual responses; and a related shift from intuitive to deductive reasoning;
- Shift from focusing on procedures to reflecting on the methods and results of procedures;
- Shift from discrete to continuous ways of seeing, defining, reasoning and reporting objects;
- Shifts from additive to multiplicative and exponential understandings of number;
- Shift from assumptions of linearity to analysing other forms of relationship;
- Shift from enumeration to non-linear measure and appreciating likelihood;
- Shift from knowing specific aspects of mathematics towards relationships and derivations between concepts.

A shift towards seeing abstract patterns and structures within a complex world is seen by many psychologists, not only those influenced by Piaget but also the Vygotskian school, as typical of adolescent development (Coleman & Hendry, 1990, p. 47; Halford, 1999). One of the ways in which these two schools of thought differ is that Piaget appears to be saying that this shift happens biogenetically, and new forms of learning follow. In mathematics, however, it is common at all levels of competent study to move fluently between concrete, diagrammatic and abstract approaches, and between examples and generalisations as appropriate, for the exploration being undertaken. Vygotsky's (1986) approach is that learners are capable of developing abstract ideas but need interaction with expert others to achieve this for themselves. He recognised that adolescence is a particularly appropriate time for conversations and scaffolding towards abstraction to take place, and that the biological and physical changes which occur at this age also relate to making sense of self in society and self in relation to ideas (p.107).

Shifts of Mathematical Action Compatible with Adolescence

Over three projects, IAMP, CMTP, and MkiTeR[1], I have observed many lessons with engaged adolescents in which a central feature appears to be the introduction and use of new intellectual tools which reflect new-to-them forms of mathematical action. For example, in one of the IAMP classrooms a teacher would sometimes be very explicit about shifts: '*You have been doing adding for years; we have to change to thinking about multiplication*'.

I shall now give some examples of tasks which generate and nurture mathematical identity in adolescence, while staying focused on the secondary mathematics curriculum as the locus of new forms of thinking. The point of showing these tasks is to demonstrate that, while exploratory and realistic learning environments have much to offer adolescents, it is also possible to structure short, tightly-focused, curriculum-led tasks in ways that lead directly to higher levels of engagement and also employ the social and emotional modes of working that are widely desired.

[1] Improving Mathematics Attainment (Esmee Fairbairn Grants 01-1415, 02-1424), Changes in Mathematics Teaching (Esmee Fairbairn Grant 05-1638) and Mathematics Knowledge in Teaching e-Research (John Fell Fund). The views expressed in this paper are mine and not those of the grantors.

Learner-Generated Examples

Students in a lesson were familiar with multiplying numbers and binomials by a grid method:

X	**20**	**7**
50	1000	350
9	180	63

X	**z**	**+3**
2z	$2z^2$	6z
-1	-z	-3

They had been introduced to numbers of the form $a \pm \sqrt{b}$. The teacher then asked them to choose pairs of values for *a* and *b*, and to use the grid method to multiply such numbers to try to get rational answers. Students worked together and began to explore. At the very least they practised multiplying irrationals of this form. Gradually, students chose to limit their explorations to focus on numbers like 2 and 3 and, by doing so, some realised that they did not need explicit numbers but something more structural which would 'get rid of' the roots through multiplication. Although during the lesson none found a way to do this, some carried on with their explorations over the next few days in their own time (Watson & Shipman, in press).

Tasks in which students gain technical practice while choosing their own examples, with the purpose of closing in on a particular property, relationship, or class of objects, can be adapted to most mathematical topics, and when the constraints in such tasks are incorporated into the goal, learners have to engage with new mathematical ideas.

Another and Another …

The following task-type also starts from learners' examples:

> *Ask students to give you examples of something they know fairly well; then keep asking for more and more until they are pushing up against the limits of what they know.*
>
> *e.g. Give me a number between zero and a half; and another; and another …*
>
> *Now give me one which is between zero and the smallest number you have given me; and another; … and another….*

Each student works on a personally generated patch, or in a place agreed by a pair or group. Teachers ensure there are available tools to aid the generation – in this case some kind of 'zooming-in' software, or mental imagery, would help.

This approach recognises learners' existing knowledge, and where they already draw distinctions; it then offers them opportunity to add more things to their personal example spaces, either because they have to make new examples in response to your prompts, or because they hear each other's ideas (Watson & Mason, 2006). Self esteem comes at first from the number of new examples generated, then from being able to describe them as a generality, and finally from being able to split them into distinct classes. This is not merely about 'sharing examples' but about adopting them and making claims about them; taking note of peers' contributions, mediated by an adult, and enhancing self-knowledge through this process.

Putting Exercise in its Place

If getting procedural answers to exercises in textbooks is the focus of students' mathematical work (whether that was what the teacher intended or not) then shifts can be made to use this as merely the generation of raw material for future reflection. Many adolescents have their mathematical identities tied up with feeling good when they finish such work quickly, neatly and more or less correctly; others reject such work by delaying starting it, working slowly, losing their books and so on; still more can produce good-looking work which shows little understanding, their self-esteem tied up with form rather than function. Restructuring their expectations is, however, easy to do if new kinds of goal are explicated which expect reflective engagement, rather than finishing, so that new mathematical identities can develop which are more in tune with the self-focus of early adolescence while requiring new forms of action, reflection on results and processes.

Examples of different ways to use exercises are:

Do as many of these as you need to learn three new things; make up examples to show these three new things.

At the end of this exercise you have to show the person next to you, with an example, what you learnt.

Before you start, predict the hardest and easiest questions and say why; when you finish, see if your prediction was correct; make up harder ones and easier ones.

When you were doing question N, did you have to think more about: method, negative signs, correct arithmetical facts, or what? Can you make up examples which show that you understand the method without getting tied up with negative signs and arithmetic?

Rules *versus* Tools

Student-centred approaches often depend on choice of method, and this, of course, celebrates autonomy. However, mathematics is characterised by, among many other things, variation in the efficiency and relevance of methods. For example, 'putting a zero on the end when multiplying by ten' and 'change side-change sign' are fine so long as you know when to do these – and mathematicians do not abandon these ways of seeing. Rather than it being a *rule* it becomes a *tool* to be used when appropriate. Adolescents often cannot see why they should be forced to abandon methods and behaviour which have served them well in the past (repeated addition for multiplication; guessing and checking 'missing numbers'; and so on) to adopt algorithms or algebraic manipulations. On the other hand, they often choose autonomously to abandon past behaviour in the service of new goals. One way to work on this is to give a range of inputs and show students that they can decide which of their methods works best in which situation, and why. This leads to identifying methods which work in the greatest range of cases, and the hardest cases. These 'supermethods' need to be rehearsed so that they are ready to use when necessary, and have the status of tools, rather than rules; empowering rather than oppressive. One teacher I have observed calls these 'bits of technology' to emphasise that appreciating their usefulness may be delayed.

Abstract Mathematics and Adolescence

In this paper, I have advanced the idea that ways can be devised to teach all adolescents the scientific conceptualizations, and methods of enquiry, which characterise hard mathematics. I suggest that these cohere with and enhance many features of adolescent development. Moreover this can be achieved without resorting to cognitive bullying which is counter-productive and alienating, because the epistemological changes of activity embedded in mathematics are similar to the ways in which adolescents learn to negotiate with themselves, authority, and the world. Agency and identity do not have to be denied, but neither does abstract mathematics have to get lost in the cause of relevance and personal investigation.

In all of the above task-types, students create input which affects the direction of the lesson and enhances the direction of their own learning. Classrooms in which these kinds of task are the norm provide recognition and value for the adolescent, a sense of place within a community, and a way to get to new places which can be glimpsed, but can only be experienced with help. To use the 'zone' metaphor – these tasks suggest that mathematical development, relevance, experience and conceptual understanding are all proximal zones, and that moves to more complex places can be scaffolded in communities by the way teachers set mathematical tasks.

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Working Groups

Groupes de travail

Mathematical Reasoning of Young Children

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Introduction

Young children's mathematics has been a focal point of research, curriculum initiatives and public policy in recent years. While mathematical reasoning is frequently given prominence, there is little consensus as to what mathematical reasoning is or the nature of young children's reasoning (English, 2004). Participants in this session were called upon to act as co-researchers, each bringing his or her philosophical orientation and history of experiences to video data of young children engaged in activity and to international examples of mathematics curriculum for preschool children. Through our discussions, no consensus was reached and many more questions were raised.

Young Children Engaged in Unstructured Activities

Descriptions of mathematical reasoning in the literature emphasize discursive features including conjectures, explanations, justification and argumentation (e.g., Stiff, 1999; Yackel & Hanna, 2003). The assumption posed at the beginning of this session was that that while language may be an indicator of reasoning, young children's verbal skills typically lag behind their ability to reason mathematically. If language is not used as the sole means to examine the nature of young children's reasoning, what other means might we use?

An embodied cognition perspective (Lakoff & Johnson, 1999; Lakoff & Núñez, 1997; Lakoff & Núñez, 2000) was presented as one lens through which to examine young children's mathematical reasoning in action. This perspective assumes that our (mathematical) concepts are necessarily shaped by our perceptual, cognitive and motor systems. Abstractions "work their way up" out of our embodied experience as a result of our recurring patterns of bodily activity, the movement of our bodies through space and time, the ways we manipulate objects and through our interactions with the environment and with others (Johnson, 1999). Further, reasoning is "mostly unconscious", "largely metaphorical and imaginative" and "emotionally engaged" (Lakoff & Johnson, 1999, p. 4). Although embodied cognition was presented as a lens, each participant brought his or her own perspective through which to engage in the discussion.

As a group we viewed two sets of video data in which children were engaged in unstructured activity. The first set was of preschool children (ages 4 and 5) at a block centre. As is typical of preschool settings there was no predetermined task posed by the teacher, nor was there a curriculum or set of specific learning objectives to be achieved. Children were free to come and go as they pleased and while the teacher was actively engaged in asking the children questions, she was clearly not ‘leading’ or ‘directing’ the activity in any way. Throughout four sessions many of the children seemed to focus on building tall towers. Their products became successively taller and more stable throughout their actions across the four sessions. While the teacher repeatedly asked the children to explain how to build stable structures and to explain why some structures were unstable, verbal responses were typically incomplete or answered based on alternative interpretations of the questions. Without verbal responses as indicators of reasoning, we discussed whether the children were indeed reasoning; whether that reasoning was mathematical; and what should we look and listen for in the children’s actions that demonstrated their reasoning.

The second video set included a three-year-old boy and mother playing with 2-D geometric shapes. In one segment, the boy built a series of “windows” (squares) and “doors” (rectangles) across the table. At one point the mother attempted to point out the repeating pattern that was emerging and where it broke down (“window, door, window, door, window, door, window ... *window*, door?”). The boy did not attend to the pattern and continued to build outwards. Although the emerging pattern was visible to the mother and to us as mathematics educators, we discussed questions very similar to those raised in the previous classroom videos.

Both sets of video data involved children engaged in unstructured activity with potentially mathematical 3-D and 2-D objects. Our conversation evolved around two related sets of questions identified below:

- Mathematics and mathematical reasoning:
  - Are the children’s activities mathematical?
  - In what ways are the children engaged in reasoning?
  - How is mathematical reasoning the same as or different from reasoning in general?
- Indicators of mathematical reasoning:
  - What might we look for and listen to that demonstrates reasoning?

1. Mathematics and Mathematical Reasoning

As a group we spent a considerable amount of time discussing whether the children’s activities were focused on mathematics, science, both or neither. The domain of mathematics is not clearly bounded for any of us. Historically, mathematics and science were not separate and we questioned the value of making a distinction, particularly for young children. Our discussions meandered from the philosophical to the practical.

As we viewed the videos we all saw mathematics in the children’s activities, but we also recognized that as mathematicians and mathematics educators we see mathematics everywhere—even when it is not apparent to others. When do the children’s activities become mathematical? Is it only when we see the mathematics in their actions, or do the children need to be cognizant of their own mathematical actions? Recognizing that we, as observers, are choosing to make distinctions we wondered what purpose it serves to distinguish between mathematics and science since it is only in schools that we separate content into isolated subjects. While we philosophically argued whether this should be the case, several participants raised practical perspectives.

To what extent do children need to be aware that they are doing mathematics? Many participants argued that preschool children did not need to be aware that they were doing mathematics. Much of their early learning activities are experienced holistically in that mathematics is integrated throughout their daily activities. However, many participants felt it was extremely important for teachers to be able to make distinctions and be explicit in the mathematical content of the children's activities. It is through these distinctions that the teacher can attend to and orient attention to important mathematical ideas that arise in the classroom. However, making distinctions necessarily results in boundaries placed on what is and what is not mathematics. The boundaries may in fact unknowingly privilege certain ways of knowing and doing mathematics at the expense of other ways of thinking. A bounded perspective, particularly in school mathematics, excludes some ways of thinking such that children from a very young age begin to assume that they can't do mathematics if their knowing and doing is outside typical boundaries. Young children come into school with open, questioning and experimental ways of thinking. Rather than capitalizing on young children's ways of knowing, we have a tendency, particularly in mathematics, to limit questioning and ways of reasoning. How we define mathematics and mathematical reasoning has an impact on what is valued in the classroom. Yet, if we define mathematics and mathematical reasoning broadly, what is there to distinguish mathematical reasoning from reasoning in general? That is, what *isn't* mathematical reasoning? What ways of knowing and doing underlie mathematical thinking (as opposed to other ways of knowing and doing)?

One way of distinguishing mathematical reasoning from reasoning in general was to focus on the content. Reasoning mathematically means reasoning about mathematical content. Although we returned to our previous discussion of what content is mathematical, the focus on reasoning with and about mathematical content provided a useful distinction from reasoning in general.

2. Indicators of Mathematical Reasoning

If language was used as the sole indicator of reasoning, then we would have to assume that little mathematical reasoning was occurring in both video data sets. The questions posed by the adults often went unanswered. When a child did answer, he or she often answered a different question than was posed or provided an incomplete answer in an attempt to guess what the teacher had in mind. The lack of substantial explanations, conjectures, predictions or arguments provided by the children led to an important focus in our discussion: "What evidence is there of mathematical reasoning?" Several responses were provided. First, we assumed reasoning was occurring because, in the first set, the children were successful at building successively taller towers. In the second set, the child's placement of shapes was clearly not haphazard. As we viewed the videos again we noticed several more features in the children's activities that indicated evidence of reasoning.

Since a task was not posed by the adults in any of the settings, we could assume that the children's play was unfocused and random, but it was not. Many of the children had set a deliberate task for themselves, although the task or problem was rarely articulated. By asking ourselves, "What task has the child set out for him or herself?", we could examine the actions and strategies used in the process of engaging in that task.

The children building with both blocks and polygons, made very deliberate choices of shapes. The choices were somewhat constrained by the materials close by and available, but it was clear that the children sought out particular attributes (e.g., rectangular prisms for block building; squares and rectangles in polygons) and ignored or replaced others that did not contribute to their self-defined task (e.g., half cylinders and hexagons).

After choosing objects, the children placed the blocks deliberately in their structures to either maintain balance and/or symmetry. If the placement was unsuccessful, we observed children either making minor adjustments to the positioning or discarding the object and selecting a different one. The environment also had a tremendous influence on the placement of materials. In one session of block building, the children's space was restricted so we see very similar structures arising that involve a narrow base and thin rectangular prisms placed vertically end to end. In the polygon video, we see the boy making a different choice of block to span a crack in the table. Through the deliberate choice, placement and readjustment of manipulatives, we saw generalized strategies emerging to address the self-generated tasks.

Through our discussion we noted the futility of many of the educator's questions to the children. Since the questions posed (e.g., "How can you make it stable?") did not appear to provide an opportunity for children to demonstrate reasoning, we wondered what types of teacher activities might. One possibility was to provide counter-examples to see whether the children noticed the features of their implicit strategies. For example, in the block building, almost all of the structures were symmetric and aligned along a single vertical axis. What if the children were provided with counter-examples that showed stability in alternative formats? What if the children's strategies were questioned by including blocks or placement of blocks that disturbed the stability? How would the children respond to these physical counter-examples?

Young Children Engaged in Structured Activities

In this portion of the working group, another two sets of video data were presented to participants. These data focused on young children engaged in more structured activities. While the focus on mathematical content was predetermined, the engagement of the children was spontaneous and exploratory. Different questions arose in the course of our discussion of the video data. Rather than questions related to what is mathematics and how might we examine reasoning, the discussion related to this video data focused on the mathematical reasoning young children are capable of. The first video allowed us to examine young children's ideas of multiplicative reasoning and proportional thinking based on halving and doubling relationships. The second video showed kindergarten children engaged in extending and finding rules for growth patterns. Both of these topics that the kindergarten children were engaged with—proportional reasoning and patterns for early algebra—are typically viewed as beyond the comprehension level of young children. Also, both topics are known to be challenging for school children when they are introduced in later grades. Despite these prior expectations, both videos show the children meaningfully engaged and successful in ways that may not have been anticipated.

In the first video, kindergarten children were working with representations of snakes drawn on rectangular strips that were placed in length sequence on the classroom floor. The snakes varied in length but not in any other dimensions, with lengths (heights) measured in half relations. In this activity, inspired by Inhelder and Piaget's 'eel task' for proportional reasoning (e.g., Nunes, Desli & Bell, 2003), students were given/shown a number of magic pellets (circular plastic discs) required for the tallest snake to perform a certain magic feat. The challenge was to determine a proportional number of magic pellets that a sequence of shorter snakes would need in order to perform the same magic feat. Specifically, in the video excerpt that we watched, the teacher began by placing four discs above the tallest snake, "Tallie", explaining that this was the number of magic pellets that Tallie required to fly to the top of the C.N. Tower. Then, pointing to a second snake, Shorty (half the height of Tallie), the teacher asked students to determine the number of pellets for this snake to accomplish the same feat. As a final challenge she introduced Tiny, (half the height of Shorty) and asked the same question. The students were successful in responding that the mid-sized snake would need two magic pellets, and the smallest snake would need one. What we then observed in the video, not anticipated in the planning of the lesson, was that the students spontaneously offered explanations of what would happen if the halving sequence were continued. First, they asserted that if an additional shorter snake were to be added (Teensey Weensey), that snake would require a half of a pellet to fly to the top of the tower. As the lesson progressed the children showed that they were able to take half of a half of a half and so on, using fractional language: "half of a half is a quarter, half of a quarter is an eighth, and half of that is a sixteenth". Some students even went as far as naming a thirty-second as half of a sixteenth. Watching the video, observers had the sense that the children recognized the infinite nature of the halving process. As one child asserted, "This would go on forever like a computer".

The second video of this set, like the first, involved students in a lesson that was designed by the teacher with specific mathematics goals at the core: in this case recognizing and generalizing rules for growing patterns that were based on $y = mx$ functions. In the video-taped lesson the students were shown the first three elements of a pattern composed of square tiles laid out in nx3-arrays (where n = the position number) with position cards placed beneath each element. Specifically, in the first position of the pattern three square tiles are laid in a row on the floor with edges touching; in the second position the first row is repeated with a second row added so that we have a 2x3-array; in the third position the pattern continues, resulting in a 3x3-array. The students were then given tiles and asked to make their own 4th position to follow from the first three, and then to describe how the squares would be placed in subsequent positions and to estimate how many tiles would be required to build further positions. Most of the students in the group were able to build/create further elements of the pattern. A few could go so far as to describe the tenth position by moving their fingers along an imaginary horizontal line, asserting that there would be three tiles along the bottom, and then, moving their finger vertically from the base line, they would say there would be 10 tiles up. Some students when prompted by the teacher were able to reason through and find the total number of square tiles in the 10th position.

Although kindergarten students typically engage in many kinds of patterning activities involving extensions and predictions and description of rules, it is unusual for such young students to reason about patterns that grow rather than repeat. Thus, in many ways, just as in the previous lesson on proportions and fractions, this lesson went well beyond what is typical for kindergarten children. Growing patterns is usually a topic occurring in later elementary years, but in this second video we see that aspects of reasoning with growing patterns are clearly accessible to kindergarten children.

The two videos of the structured lessons led to a great deal of discussion among the participants regarding the extent to which the students were actually grappling with and understanding the primary mathematical ideas that underpinned each of the two lessons: proportional reasoning in the first video and, in the second, generalizations of patterns with an underlying $y = mx$ structure.

In regards to the proportional reasoning task, although participants agreed that children could deal with halves, there was general agreement that halves are somehow intuitive in a way that, for example, thirds and tenths are not. Participants considered that young children can, in a sense, "feel" halves, but that thirds and tenths are less accessible. We recognized that there was an abstraction as children moved beyond concepts that they could somehow imagine, yet we wondered whether it was important to provide opportunities for children to experience portions of content (e.g., proportionality, infinity) at an age that it is accessible, rather than waiting to present the abstract content in its entirety.

With regard to the second lesson, participants acknowledged that pattern recognition—extension and rule generation—has a powerful influence on children's mathematical understanding. Participants commented on the ease with which children in the video could build, extend and describe simple growing patterns. However, the participants also expressed concern that children of this age are more attuned to, and ultimately interested in, repeating patterns. Thus, working with growing patterns not only is constraining for the students but also requires very strong teacher direction and, consequently, limits diverse responses and opportunities for children to explore beyond the boundaries of the lesson.

It is interesting to note here that in addition to the main videotaped lesson described above, the participants also watched a short excerpt from a previous lesson on patterns in which the children were asked to find the rule for a "times two" pattern. In that lesson, the teacher showed the children the first two positions: first a 1x2 rectangle of squares followed by a 2x2 rectangle (the second position). However, when the children were asked to make the third position of this pattern, rather than building a 2x3 rectangle with 6 squares, several students used 8 squares, building a 2x4-array, thus suggesting that the sequence 2, 4, 8 may be more intuitive or familiar to some students than the linear progression that was asked for. It was only with persistent teacher direction that these students were able to build the required pattern.

Viewing of the structured lessons led the participants to a renewed discussion of the significance of the unstructured activities that had been the subject of the first day of meetings. Participants compared some of the features of the structured and unstructured approaches. One topic emerging from this comparison centered on the extent of the children's control and ownership of mathematical activities. As was clear in the videos in which students were involved in unstructured activities, the children were fully engaged in self-directed activities of a mathematical nature. We know that more structure leads to fewer questions of whether children's activities are mathematical or not; unstructured tasks, on the other hand, allow students, including our preschoolers, opportunities for more diversity and creative thinking. Thus, although it was acknowledged that the videotapes of the structured lesson revealed that these lessons provided students with opportunities for engaging with significant mathematical ideas, there was a concern about the degree of teacher direction and the effect of structured mathematics lessons for students in the early years.

A second issue that played an important role in our discussions was the prioritising of numbers in the early years and the push to begin the formal school math before many children are ready and the effect of that practice. Here again the contrast between the two sets of videos—structured and unstructured—was revealing. In the cases of both videotapes of the structured lessons, while both used geometric representations, the snake lengths, and the tile arrays, both lessons had an important focus on numbers—how many magic pellets in the snake lesson, and how many tiles are needed to make the *n*th element of the pattern.

Preprimary Curricula

The final data set provided to participants included excerpts of pre-primary curriculum frameworks, including New Zealand's *Te Whãriki Early Childhood Curriculum* (Ministry of Education, 1996); England's *Statutory Framework for the Early Years Foundation Stage* (England Department for Education and Skills, 2008); Singapore's *Nurturing Early Learners: A Framework for a Kindergarten Curriculum* (Pre-school Education Unit Ministry of Education, 2000); and the province of Ontario's *Early Learning for Every Child Today: A Framework for Ontario's Early Childhood Settings* (Best Start Expert Panel on Early Learning, 2006).

There is an international trend to develop and promote pre-primary curriculum frameworks to provide guidance for educators. Stark differences in the curricula exist in relation to mathematics learning. The New Zealand curriculum integrates mathematics content throughout the document and is highly process-oriented, envisioning young children as explorers engaged in the development of "working theories". In contrast, England's recently mandated *Statutory Framework for the Early Years Foundation Stage* takes a highly prescriptive approach which identifies measurable outcomes for each child to attain by the end of their fifth year. Singapore's Kindergarten curriculum is brief, holistic, and describes mathematics content in minimal detail. Ontario's *Early Learning* framework appears to take a balanced approach between processes and content.

The contrast between the documents was startling. A number of questions were raised as to the appropriateness and value of a pre-primary curriculum. Some participants questioned whether a pre-primary curriculum that includes a significant amount of content was appropriate, particularly for three- and four-year-olds. Others indicated that children in families who do not have access to the curriculum and resources may be disadvantaged. Clearly the cultural context had implications for accessibility, equity and other resources.

We contrasted the importance for early mathematics learning with the successful promotion of literacy in Canada. Certainly, there is a void of information available to parents on ways of supporting mathematical thinking in the home. Children's literature books are sent home with children at birth. What resources could be given to parents to support early mathematics? We brainstormed a few suggestions such as blocks, games involving counting, strategy and logic, dominoes, snakes and ladders, puzzles, etc.

Unlike for literacy, there is not a strong community of mathematics educators in early childhood, particularly in Canada. Instead, we have left early childhood mathematics to psychologists. In Ontario's *Early Learning* framework we noticed that no mathematics educators were involved in the production of the document. What impact could and should mathematics educators have in the field of early learning in mathematics? What difference could we or should we make in the field in promoting a national perspective of mathematics learning in preschool?

Concluding Remarks

Throughout our discussion over the three days we raised many questions about early mathematics learning: What should be the focus of early learning in mathematics? What are the goals of early mathematics learning and experiences? Should it be objective driven or exploratory? Should mathematics in the early years focus on fostering an appreciation for mathematics or should early mathematics learning experiences be designed to prepare children for school mathematics? What experiences and what forms of interaction will nudge three- and four-year-olds onto paths of mathematical learning? In addition, we considered what forms of mathematical reasoning are important for young children. Should the emphasis of early mathematics be primarily oriented towards number and counting, or should the focus be more on spatial reasoning?

Given all that we know about how easily children become disenchanted and feel alienated from mathematics in later grades, what can we do in the early years to help students to be curious and feel empowered? Our discussions and ongoing questions emphasized the importance of mathematics educators attending to and participating in early learning initiatives in mathematics.

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Mathematics-in-and-for-Teaching (MifT): The Case of Algebra

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Introduction

The announced aims, goals, intentions and hopes for this working group were as follows: the collective and individual enrichment of appreciation of what mathematics is essential in and for teaching algebra, including what it is that could inform teacher choices as they construct tasks and interact with learners during mathematical activity arising from those tasks, and mathematical and epistemological obstacles whether inherent in algebra, or arising from different teaching approaches to algebra.

By means of the pre-conference website, participants were encouraged to do a little preparatory reading related to the Mathematics-in-and-for-Teaching (MifT) of Algebra theme of the working group. Some of the suggested readings included the following: Ball and Bass (2000), Baroody et al. (2004), Davis and Simmt (2006), Fosnot and Dolk (2002), Ma (1999), Núñez (2000), Simon (2006), and Simon and Tzur (2004).

But, as often happens in CMESG working groups, what transpires can be quite different from that which was originally planned. The MifT working group proved to be no exception to this general rule. This working group report attempts to capture some of the many dimensions that arose during our discussions. The first part below focuses on how a given problem, the Bus Problem, might be approached at different grade levels, as well as issues associated with students' generating multiple solution paths and with the teacher's organizing these for classroom discussion. This first part concludes with a discussion relating these two issues to questions on MifT, such as, '*What it means to know mathematics for teaching*' and *'How teachers might learn such mathematical knowledge'*. The second part of the report – by means of a second example, the Chewing Gum Problem – focuses on an individual teacher's orchestration of students' problem-solving activity so as to assist them in noticing a certain underlying form in their sequence of solving operations and to generalize that form. A summary discussion describing the features of this teacher's practice, features that illustrate the nature of the mathematics-in-and-for-teaching algebra awarenesses that this teacher had developed within herself, then follows. The report concludes with reflections that the participants shared during the three working group sessions, either orally or by means of their written conference notes.

Exploring Ways for Knowing and Learning Mathematics-in-and-for-Teaching Example 1: The Bus Problem

> *Bus Problem: There are 36 children on a bus. There are 8 more boys than girls. How many boys? How many girls? Show two different solutions.*
>
> *How might a teacher use this problem for teaching?*
> *What mathematics would students use to solve this problem?*

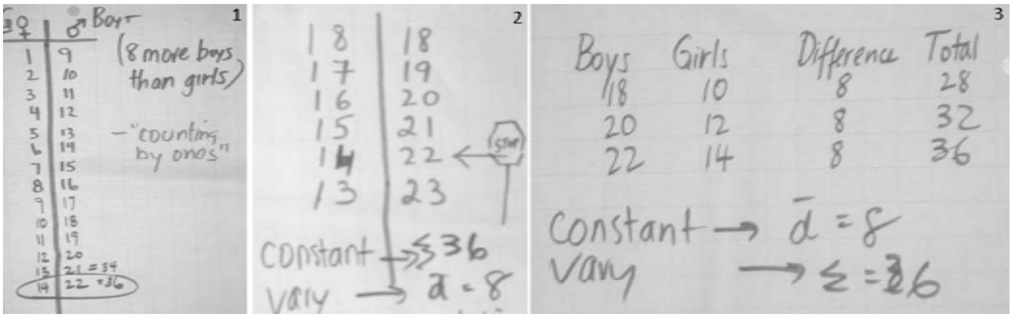
A grade 2 teacher might use this problem to provoke students to make a transition from counting strategies to addition and subtraction. A grade 5 or 6 teacher could use this problem to relate a "bar model" representation to symbolic notation. A grade 7 or 8 teacher might use this problem to teach students about representing linear relations using a diagram, table of values, and algebraic notation. A grade 9 or 10 teacher could use this problem to introduce students to systems of linear equations. This problem could be presented to students for solving in small groups, in pairs, or individually. Students could use counters, base ten blocks, an open number line, or a graphing calculator to develop and model solutions. When students have completed two different solutions, the teacher organizes a whole class discussion about the solutions. What mathematical criteria might a teacher use to decide which solutions to choose for class discussion? In what sequence might the solutions be shared? A teacher needs to consider numerous details when making decisions for planning and implementing a lesson, such as choosing problems and learning materials (e.g., manipulatives, technology), understanding the mathematics evident in student solutions, and organizing class discussion of the solutions for student learning.

Solving a Problem in Different Ways

In observing students and adult learners solve the Bus problem in different ways, often the first solution includes mathematical ideas and strategies that are most familiar to the learner, either because those ideas and strategies have been used recently or possibly because they have been well practiced over time. For example, a grade 1 teacher said, "*I used this counting strategy because this is the mathematics I have been doing with my kids*." While another teacher said, "*I just finished teaching my students to write algebraic expressions, so I see the boys and girls as x's and y's*." However, a greater potential for mathematical innovation and invention by the learner becomes possible when different ways of thinking about solving the problem are encouraged. Different solutions to the Bus problem, generated from educators (e.g., classroom teachers, graduate students, mathematics educators, and mathematicians), are provided below. The solutions to the Bus problem are offered as examples of the range of mathematical thinking provoked by the problem and as a context from which we can explore mathematics-in-and-for-teaching. Each set of solutions is described in terms of the mathematical strategy, model of representation, and mathematical relationship among the range of solutions.

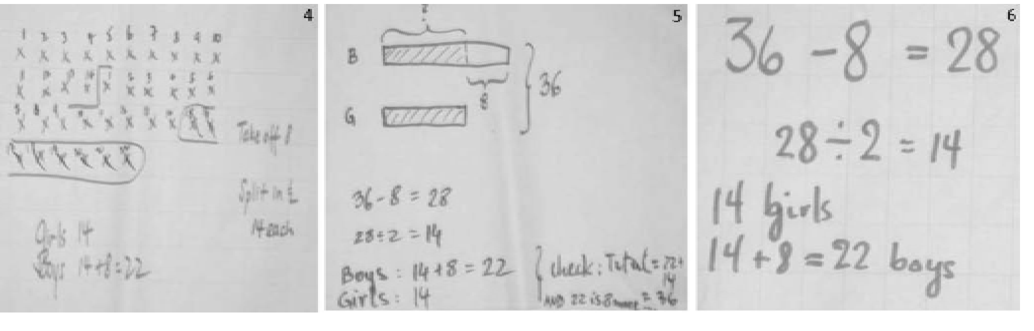
Solutions 1 to 3

For Solution 1, the focus was on keeping the difference of 8 between the number of boys and girls constant and on counting by ones (e.g., "*I started with 1 girl, then 9 boys because 9 – 1 = 8, then 2 girls, 10 boys, because 10 – 2 = 8, and continued until 14 girls and 22 boys because 22 – 14 = 8, and 22 + 14 = 36 which is the total number of students on the bus.*"). Solutions 2 and 3 also focused on the idea of either keeping the sum of boys and girls constant (36) and varying the difference between boys and girls (to 8) or keeping the difference of boys and girls constant (8) and systematically varying the sum of boys and girls (to 36).



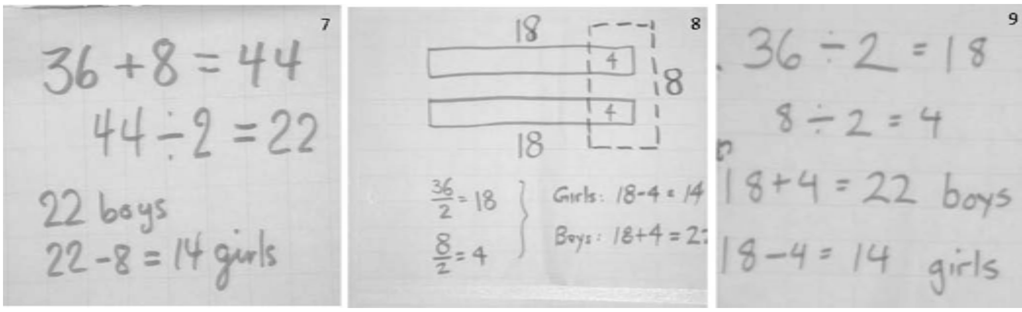
Solutions 4 to 6

In the next set of solutions, different models of representation are used to show the same mathematical strategy; that is, subtract the 8 boys from the total number of boys and girls, divide the difference of 28 into two equal groups of boys (14) and girls (14), then join the 8 boys to 14 boys for a sum of 22 boys. While solution 4 uses an array of 3 x 10 + 1 x 6, showing separation of 8 for the whole with 2 partitions of 14, solutions 5 and 6 describe the same strategy using a sequence of numerical equations. However, solution 5 shows evidence of reasoning about the sequence of operations, by providing a pictorial, "bar model" representation. When building a "bar model," information that is given, as well as information that is unknown, must be identified. In fact, identifying and solving for an unknown quantity is a key concept in algebra.



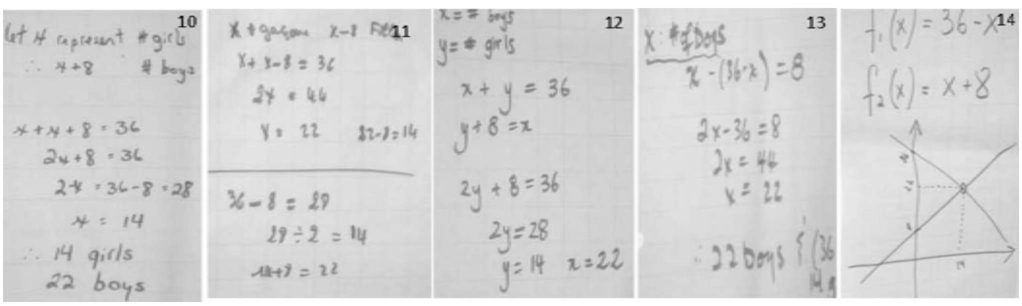
Solutions 7 to 9

How are solutions 7, 8 and 9 similar to and different from solutions 4 to 6? Solution 7 is similar to solutions 4 to 6, but the 8 boys are added to the whole, 36, then 2 equal groups of boys and girls (22) are made, followed by an adjustment of 8 to the girls (22-8). Solutions 8 and 9 focus on the idea of equal groups of boys and girls, like solutions 4 to 7, yet make the groups first, then adjust the number of boys and girls to be either + 4 or -4. Solution 8 represents the strategy using a "bar model," while solution 9 is a sequence of operations.



Solutions 10 to 14

Solutions 10 to 14 could be considered similar, as they are algebraic generalizations of the arithmetic solutions 1 to 9. For some solutions, the strategy of adding two different numbers that have a difference of 8 to get a sum of 36 is shown. Other solutions represent the concept of constant and variability; for example, the sum of two numbers is held constant at 36 while the differences between two numbers varies, until a unique solution of two numbers with a sum of 36 and a difference of 8 is achieved. In other solutions, a big idea of doing and undoing is illustrated (e.g., adding 8 and subtracting 8, multiplying by 2 and dividing by 2).



From our previous experiences, we've noticed that typical responses to the Bus problem from students in grades 2 to 5 include arithmetic solutions (like solutions 1 to 9). This observation makes sense given that the students' mathematics learning is focused on developing understanding, proficiency, and flexibility in manipulating numbers and operations. Yet for students in grades 6 to 9, both arithmetic and algebraic solutions (like solutions 1 to 14) are typical responses, as they are learning about the relationship between arithmetic calculations and algebraic generalizations. Take a moment to examine the mathematical strategies in algebraic solutions 10 to 14 and arithmetic solutions 1 to 9 to decide which arithmetic and algebraic solutions are similar.

Organizing Solutions for Discussion and Learning

It is significant mathematical work that a teacher does when preparing for and coordinating a class discussion about their solutions to a problem in order to make mathematical ideas and strategies explicit. Such work requires teachers and their students to make sense of the different strategies that they may not have seen before, discern the mathematical details inherent in the solution, see mathematical relationships between solutions, and notice the solutions with strategies that would work for all cases. Such sense making requires teachers to deconstruct their own mathematics knowledge to make visible mathematical ideas and strategies in multiple forms (e.g., concrete, semi-concrete, abstract). In doing so, teachers can anticipate the ways that students transform their mathematical knowing across models of representation, symbol systems, and problem solving contexts. In the example below, the working group produced a range of solutions, which were clustered by similar mathematical strategies, and they were organized to show an elaboration of arithmetic to algebraic thinking. For example, one cluster focused on the strategy of dividing the number of children into 2 equal groups, then adding 4 to boys and subtracting 4 from girls.

As the working group described and discussed each cluster of solutions, annotations of mathematical elaborations were made on and between the solutions. Sometimes, the mathematical intention of the author was understood differently by the working group, and solutions within one cluster were re-sorted to other clusters. Our organization of the solutions made different mathematical elements of the problem and its solutions visible for learning. Though our mathematical analyses focused on specific mathematical details in each solution, as Mason (2008) reminded us, it is also about "*trying to get a bigger picture, trying to see the wood instead of the individual trees*" (p. 5). Overall, this collectively-produced artifact or *bansho* displayed a mathematical landscape of the working group's thinking.

Designing Problems Beyond the Bus Problem

In choosing a task or problem, teachers contemplate several criteria: What is the mathematical potential of the task? What mathematics could students use to solve the problem? Does the problem have one or many solutions? What important ideas and processes are involved in the problem? What does the teacher have to know to use this task effectively with students? How might students get stuck and what would the teacher do? When teachers solve a problem in different ways, prior to using it with students, they are often able to answer many of the aforementioned questions.

Additional questions posed by the working group included: What choices were made (and could they have been different) concerning (re)presentation, notation, expressions of generality? What criteria are useful for making such choices? In considering these additional questions, the working group generated some variations on the Bus problem:

> Ribbon B is 5 m longer than ribbon A. Ribbon C is 6 m longer than ribbon B. There are 66 m of ribbon altogether. What are the lengths of ribbons A, B, and C?
>
> Children are on a bus. There are 8 more boys than girls. How many boys? How many girls?
>
> There are 36 children. There are more girls than boys. How many boys? How many girls?

> Given the sum and difference in whole numbers, what is possible? Fix the difference, change the total.

Such variations on the Bus problem included changing: the size of numbers, number relationships between known and unknown quantities, the problem context, the choice of variable (unknown quantity), and the number of variables in the problem.

Where's the Mathematics-in-and-for-Teaching?

What ways do teachers need to know mathematics for their teaching?

Ball and Bass (2003) distinguished several features of knowing mathematics for teaching, such as unpacking symbolic and abstract forms of mathematical ideas into their conceptual and constituent components, understanding and making connections across mathematical domains, helping students build links and coherence in their ever-changing mathematical knowledge, anticipating how mathematical ideas change and grow, and attending to the relevance of mathematical notation, use of terms, and representation for learners. These features include "*… knowledge of what is typically difficult for students, of representations that are most useful for teaching a specific idea or procedure, and of ways to develop a particular idea*" (Ball, 2000, p. 245). More specifically, within the context of coordinating class discussion of solutions to the Bus problem, the teacher's mathematical work also includes: representing mathematical ideas carefully by mapping out relationships between physical (operations, processes), graphical, and symbolic representations; interpreting and making mathematical and pedagogical judgments about students' questions, solutions, problems, and insights (both predictable and unusual); and being able to respond productively to students' mathematical questions and curiosities (Ball & Bass, 2003).

How might teachers learn such mathematical knowledge?

Given both the depth and scope of knowing mathematics-for-teaching, clearly, it cannot be simplified into a set of topics in a "How To Teach Mathematics" manual. Watson and Mason (2005) explained that "*mathematics is learned by becoming familiar with examples that manifest and illustrate mathematical ideas and by constructing generalizations from examples*" (p. 2). The same might be said for teachers learning mathematics-for-teaching. As students explain their mathematical thinking, a teacher is developing an understanding of the students' mathematical ideas and connections between their ideas, in order to make some generalizations about the students' collective and individual mathematical knowing. Such understandings and generalizations are generated in the moments of teaching and are used by a teacher over the course of a lesson.

Further, Davis and Simmt (2006) offer the notion that a focus on how concepts are presented and elaborated through the course of the K-to-12 curriculum is constructive for learning mathematics-for-teaching. Because the mathematical vocabularies, images, and algorithms used in schools impact the shaping of teacher and students' understandings, "… *it seems that teachers must be adept at 'translating' among available symbol systems and at recognizing when they are engaging in such translations*" (p. 316). Such adeptness is often cultivated during the practices of teaching.

Therefore, elaborating on the notions described above by Watson and Mason (2005) and Davis and Simmt (2006), we are hypothesizing that solving problems used for teaching and analyzing the mathematics inherent within and across different solutions to a problem are useful ways for exploring how teachers need to know and learn mathematics-in-and-for-teaching. Ball and Bass (2003) suggested that practice in solving the mathematical problems teachers face in their work would help them learn to use mathematics in the ways they will do so in practice. And the practice is likely to strengthen and deepen their understanding of the mathematical ideas. So, to begin our exploration, the Bus problem was offered. This problem has been offered to Grade 2 to 10 students, and their diverse solutions demonstrated a range of knowledge of arithmetic and algebraic concepts, algorithms, and strategies. Such a range prompts the teacher to recognize the mathematical ideas inherent in the solutions, make mathematical connections among the solutions, and then consider how the discussion of these solutions could be orchestrated for student learning.

By deconstructing the mathematics used to solve the Bus problem, we suggest that we were engaged in the study of mathematics-in-and-for-teaching. Comparing and analyzing the range of mathematics (e.g., concepts, procedures, strategies) inherent in our different solutions became the starting point for our beginning deliberations about what it is necessary to know, be aware of, and have come to mind, in terms of the relationship between arithmetic and algebra.

A Second Exploration of MifT: The Chewing Gum Problem

While the previous discussion focused to a large extent on a range of solutions that covered the gamut from arithmetic to algebraic approaches, the next example has a quite different focus. We now take a close look at the practice of one particular teacher who aimed at bringing her students beyond finding the solution to a word problem toward a realization of that which was *algebraic* about their work. This practice is contrasted with that of another teacher who, once her students had found the answer to the problem, was unsure how to move beyond this. This issue of moving beyond finding the answer to a mathematical problem in the algebra class might more properly be phrased: *When the question isn't the question and the answer isn't the answer, what then is the "real question"?*

The examples of teaching practice presented in this scenario are drawn from an analysis of the TIMSS-R video data of 8th grade algebra classes around the world (Stigler et al., 1999). The analysis, conducted by Margaret Smith (2004), compared the practice of two teachers (Teacher A and Teacher B) who gave similar problems to their grade 8 classes (see Figures 1 and 2), and illustrates how a teacher might promote algebraic reasoning by a focus on generalizable methods in the solving of word problems. Current algebra curricula tend to stress the importance of embedding algebra instruction in problem-solving situations. However, the algebraic component often gets lost. Even when the problem-solving activity culminates in the sharing of a multiplicity of alternate, sometimes quite ingenious, problem-solving approaches – and there is widespread agreement that it is vital for students to see alternate approaches to solving a problem and to learn from these – the sharing of solving approaches often becomes an end in itself. The problem-solving approaches are not used as a means to develop algebraic thinking in students. So, let us take a look at how this might be done, as per the description provided by Smith (2004).

> *A few days ago, Veronica and Caroline were both asked to the prom. That night, they went out to shop for dresses. As they were flipping through the racks, they each found the perfect dress. Both dresses were priced at $80. Neither of them had enough that night, but each went home and devised a savings plan to buy the dress. Veronica put $20 aside that night and has been putting aside an additional $5 a day since then. Caroline put aside $8 the day after she saw the dress and has put in the same amount every day since. Today, their friend Heather asks each girl how much she has saved for the dress. She says, "Wow! Caroline has more money saved." How many days has it been since Veronica and Caroline began saving?*

Figure 1: Problem given by Teacher A – the "Prom Dress" task

Teacher A gave her students the "Prom Dress" task because she thought it connected well with prior work done on writing equations and graphing lines and could be used to help students make a transition to systems of linear inequalities.

When she (Teacher A) noticed that most of her students were having some difficulty in getting started with the problem, she tried to help them by drawing three labelled columns on the chalkboard. While students continued to work on the problem, she encouraged those who had already finished to find another way to solve the problem. She later called upon various students to present their solutions. One student came forward and filled up the table of values on the chalkboard up to the seventh day, remarking that on the seventh day Caroline had more money saved than had Veronica. When the teacher asked if there were any other solutions, no one offered any and the "discussion" (and the lesson) came to a close. That is to say, Teacher A ended the lesson as soon as her class had found the solution to the problem.

Teacher B adopted a quite different approach in her classroom orchestration of the Chewing Gum problem (see Figure 2) – a problem similar to that of the Prom Dress task.

Ken and his younger brother enjoy chewing gum. One day, the boys go to the candy store and buy several packages of gum. Ken bought 18 ten-piece packages of gum, and his brother bought 24 five-piece packages of gum. Every day, each boy finishes one whole package of gum. One day, they looked at how much gum each boy had left. Ken noticed that his brother had more pieces of gum than he had. How many days has it been since the boys bought the gum?

Figure 2: Problem given by Teacher B – the "Chewing Gum" task

Teacher B introduced the problem as follows:

She first put the chewing gum task on the board. A student read the problem aloud.
Then she drew two rectangles on the board, one for Ken and the other for his younger brother.
The students counted out 18 circles and displayed them on Ken's rectangle to represent the 18 packages of gum, but counted them by tens to make clear the number of pieces of gum that Ken had started with. A similar process was followed for filling the brother's rectangle and announcing the number of his pieces of gum.
At that moment, the teacher asked students to try to solve the problem.

About halfway through the class period, she asked selected students to present their work to the class. The first group of students who came forward used a third rectangle and moved into it a 'package of gum' from Ken's rectangle and a package from the younger brother's rectangle, explaining that at the end of the first day, Ken had 170 pieces of gum and his brother had 115. They kept doing this until the younger brother had more pieces of gum left. The teacher then summarized their approach: "*You took one circle from each boy, counting down by tens for Ken and by fives for his brother until his brother had more gum. This is good, but it could take a long time when the numbers get bigger*. *Did anyone find an easier way than this?*"

Another group came forward and drew a table of values on the board with three columns labelled: Day, Ken, and Brother. The values they entered into this table showed that on the 13^{th} day, the younger brother had more gum (see Figure 3).

Day	Ken	Brother
1	170	115
2	160	110
3	150	105
4	140	100
5	130	95
6	120	90
7	110	85
8	100	80
9	90	75
10	80	70
11	70	65
12	60	60
13	50	55

Figure 3: This table was used by one group of students to illustrate their solution approach for the problem

But the teacher did not stop there. She continued with the following request: "*Now I wonder if any of you thought of a way to show how many pieces of gum each boy had every day. Many of you may not have thought of this way that we will do it, but that is okay, we will try it anyway. I would like you to add some columns to Group 2's table like this* (headers: Day, Equation, Ken, Equation, Brother) – see Figure 4 – *and think of an equation Group 2 might have used to find out how many pieces of gum each boy had. What would Day 1 look like?*"

Day	Equation	Ken	Equation	Brother
1		170		115
2		160		110
3		150		105
4		140		100
5		130		95
6		120		90
7		110		85
8		100		80
9		90		75
10		80		70
11		70		65
12		60		60
13		50		55

Figure 4**:** The table after the teacher had inserted two new columns for the "equations"

When one student from Group 2 responded that they took 10 away from 180 for Ken and 5 away from 120 for his younger brother, Teacher B filled this information in on the first line of the table (180 – 10 = 170; 120 – 5 = 115) and asked the students to continue working on the task of completing the table. When some students appeared to be confused, she asked them to stop and look at the two numbers for a given day and to decide what computation they needed to do to get each number.

As the students continued working, Teacher B asked two students to put their work on the board (see Figure 5).

Day	Student 1	Student 2	Ken	Student 1	Student 2	Brother
1	180-10=170	180-10=170	170	120-5=115	120-5=115	115
2	170-10=160	180-20=160	160	115-5=110	120-10=110	110
3	160-10=150	180-30=150	150	110-5=105	120-15=105	105
4	150-10=140	180-40=140	140	105-5=100	120-20=100	100
5	140-10=130	180-50=130	130	100-5=95	120-25=95	95
6	130-10=120	180-60=120	120	95-5=90	120-30=90	90
7	120-10=110	180-70=110	110	90-5=85	120-35=85	85
8	110-10=100	180-80=100	100	85-5=80	120-40=80	80
9	100-10=90	180-90=90	90	80-5=75	120-45=75	75
10	90-10=80	180-100=80	80	75-5=70	120-50=70	70
11	80-10=70	180-110=70	70	70-5=65	120-55=65	65
12	70-10=60	180-120=60	60	65-5=60	120-60=60	60
13	60-10=50	180-130=50	50	60-5=55	120-65=55	55

Figure 5: The work of two students at making explicit the methods used to compute the values in the KEN and BROTHER columns

The students of the class were then asked which of the equation-types (that of Student 1 or that of Student 2) would be more helpful if the number of days got really large. They decided that Student 2's method was more helpful because: "*All you need to know is how many days so you can multiply it by how many pieces of gum are in each package, ten or five*." As they then ran out of class time, the teacher concluded the lesson by asking the students to think about a more general way of writing the equation that would give them the number of pieces of gum each boy had on whatever day.

What Does This Example Suggest Regarding Mathematics-in-and-for-Teaching Algebra?

Zaslavsky and Leiken (2004) have pointed out that teachers engaging in learning activities for student mathematical learning can be an effective vehicle for their professional growth. The previous example of the Bus problem, and its ensuing discussion, provided an illustration of how teachers might develop knowledge of mathematics-for-teaching by engaging in the kind of activity referred to by Zaslavsky and Leiken. However, an additional factor to be taken into account in this conversation is one that has been emphasized by Mason (1998): it is, in fact, one's developing awareness in teaching practice that actually serves to constitute change in one's 'knowledge' of mathematics teaching and learning, as well as in – in some cases – one's knowledge of mathematics. It is in this sense that the phrase *mathematics-in-and-for-teaching* is being used; that is, one's knowledge of mathematics *for* teaching arises with*in* the act of teaching. As such awarenesses are personal constructs, an observer can only guess at the kinds of MifT awarenesses that may have developed within Teacher A and Teacher B during the Prom Dress and Chewing Gum scenarios respectively. The best that we can do here is describe features of Teacher B's practice that suggest the nature of the mathematics-in-and-for-teaching-algebra awarenesses that Teacher B had developed for herself, and that were, in turn, exemplified in her teaching approach.

The unfolding of the lesson in Teacher B's class suggests that, for this teacher, it was important for students to see sequences of solving operations in a general way – and for her to promote the sequences that could lead to efficient generalizations, thus prefiguring the use of algebraic equations for modeling and solving word-problem situations. One of the, often undisclosed to students, secrets of algebra is the need to be able to notice a certain underlying form in a given sequence of examples and to generalize that form, as well as the awareness that some generalizable forms are more efficient and easier to calculate with – and to represent in symbolic form – than others.

Another salient feature of Teacher B's teaching practice was the analysis and comparison of different solution methods. Teacher A's students presented only one solution method, allowing little room for developing mathematical connections across solution methods. Because Teacher B's students were encouraged to present more than one solution method, she was able to have them compare two different general forms. This practice has been reported in other research on algebra teaching, for example, the work of the researcher Jo Boaler and her collaborator the math teacher Cathy Humphreys (Boaler & Humphreys, 2005).

The students of Teacher B's class were encouraged to provide detailed explanations. When this was not spontaneous, Teacher B specifically probed students to give more detailed and connected explanations (Smith, 2004). Simply providing an answer was not acceptable in Teacher B's class: If students did not make connections, she asked questions that linked the pieces together. Teacher B also helped students to construct another solution method. She tried to get them to consider using a common variable as a first step to comparing the expressions. Teacher B worked very hard at getting the students to be more explicit about the calculation methods they had used to obtain their solutions. When she asked them to compare two particular methods, she felt that it was vital for them to reflect on which one might be easier to use in calculating a response for any number of days and to express their method in a general way. This example draws out the importance of the role of generalizing and generalization in teachers' knowledge of MifT for algebra.

The central role given by Teacher B to the process of generalizing a certain form within the sequence of solving operations was, however, only one of the components of MifT for algebra that was discussed, even if briefly, by working group participants. Other aspects of algebraic reasoning that it was felt teachers need to develop within their students include the key algebraic distinctions of Equality – Variation, Invariance – Variance, Unknown – Variable, and Doing – Undoing. The algebraic thread that involves the equality-equivalence-expressions-equations-functions dimension was also considered to be a crucial aspect. While issues of algebraic content, and how to dissect these so as to encourage and facilitate algebra learning, flowed quite easily in and out of the working group discussions. The "moves" of the teacher in deciding on the specific actions to be taken in the practice of teaching within a given class and in rendering the implicit questions of algebra more explicit was found to be much harder for working group participants to articulate.

Reflections

It had emerged in the previous two sessions that there is often a gap between what the teacher has in mind that the students will do, and what the students actually do. This raised questions about the role of mental imagery in imagining the students working so as to describe to them as precisely as possible what is intended, while accepting that there will always be interpretation. Indeed tasks which admit of variation and which encourage students to make significant mathematical choices have a greater chance of attracting student attention and sustaining relevant activity than tasks that do not.

As might be expected, questions such as "What is algebra?" and "What do we mean by knowledge?" kept arising during our work. An important variant is, "When are students doing algebra?", because, as many observed, you cannot always tell from student behaviour what the students are actually doing or thinking about. For example, they can be acting-as-if they understand by following templates. Put another way, 'thinking is only partially projected into behaviour', so there may be much more sophisticated thinking going on than is interpretable from observed behaviour. There is an analogy to the Turing test here: from student responses to probes, teachers are expected to determine whether the student understands (or what aspects are understood).

When we constructed a poster of aspects of algebra, it was noticed that of the 43 post-its, 8 more referred to concepts than to processes.

Observable behaviour involves the manipulation of objects, whether material, diagrammatic (iconic), or symbolic. There is a widespread faith that manipulation leads to learning. However, what is needed is closer attention to actions carried out on objects, and pedagogical interventions, so as to bring those actions and their effects, as Simon and his fellow researchers (Simon, Tzur, Heinz, & Kinzel, 2004) emphasize, to the surface. Promoting reflection is often referred to, but often there is little pedagogical substance behind such utterances.

Having stimulated activity in the form of the use of actions, preferably modified to deal with fresh challenges, it is vital that these modified actions become an extended, or even new, method for use again in the future.

It is essential therefore that teachers appreciate the intentions of tasks (more accurately, of the author of the tasks) and how pedagogical choices might influence the achievement of affordances. MifT includes therefore awareness of possible variation, and deep appreciation of the dimensions-of-possible-variation and their corresponding ranges-of-permissible-change that constitute the class of tasks which a given technique can be used to resolve. Behind this awareness is the awareness of pervasive mathematical themes such as doing and undoing (reversal), distinction between unknown and variable, and invariance in the midst of change.

The issue arose as to how conventions local (to a class) and global (in the mathematics community) are best handled: some people insist from the beginning that conventional notation and vocabulary is used, while others encourage students to bring their own thoughts to articulation and are content to work with local terms and definitions, at least for a time before introducing the more conventional versions.

Obstacles to pedagogical development include assumptions made by students themselves, influenced by assumptions made by guardians and teachers, and assumptions about how students learn. If developmental stages are presumed, then students whose thinking does not follow the norm can be seriously disadvantaged. Furthermore, implicit condescension, even if unintended, concerning what students are capable of is a severe limiter on what they actually do.

Things That People Reported Had Struck Them During the Sessions:

- The fundamental awarenesses which form the basis for our actions concerning arithmetic operations of +, –, x and ÷, not to say exponentiation and others.
- The presence of so many conventions about how these operations are ‘handled’.
- The tension between chaos and order, freedom and constraint, and the fragility of intuition in the presence of the pressure exerted to conform to socio-mathematical practices.
- Necessity of having pedagogical strategies come to mind when needed, and appreciating the likely consequences of using them, beyond ‘knowing about them’. Perhaps MifT is really about knowing the affordances of choices, and having those choices come to mind.
- “The question is not the question and the answer is not the answer”: the questions asked of students are prompts to action, activity, experience and learning from that experience, and are rarely the sole focus of attention when effective teaching is taking place; answers to questions are also not what the teaching and learning are actually about, but rather the search for effective approaches to classes of tasks.

- Crucial step in promoting multiple approaches and resolutions is linking them together, discussing effectiveness, efficiency and reconstructability. Many students are content to work mechanically; many of these crash; only a few discover the power and effectiveness of meaning. Yet at the same time, many people, especially women, report abandoning mathematics when the meaning became obscure and no one helped them rediscover it.
- Teaching by listening (a Brent Davis phrase) is occasioned by setting tasks with multiple approaches, and engaging students in seeking effective solutions rather than getting to an answer.
- It is easy to do easy things, but hard to sustain activity on hard things. The space between a student response and a teacher re-response is a locus for the difference between effective and ineffective teaching. In the sessions, we may have slipped past the core aspects of MifT algebra (harder to locate and fasten on). Perhaps need a psychoanalysis of mathematics … what is the mathematical Id that drives the mathematical Ego?
- The role of convention concerning arbitrary choices contrasts with the deductive necessity of structural aspects (see Wertheimer, 1945; Hewitt, 1999, 2001).
- As ever, the core question for mathematics educators involved in professional development (PD) – whether pre-service or in-service – is how to interpret the curriculum as it is enacted in the classroom. What are the essential ideas and concepts? How to avoid the production of lists, which tempts teachers into mechanically trying to enact them. The core PD question is whether participants can imagine themselves acting freshly as a result of the session(s), and whether they have built up a sufficient 'head of steam' to carry this sense of what is possible into action. But it is hard to be aware of everything, or even to remember to be aware of something particular that you wish to remember to be aware of!
- As Italo Calvino put it in *Mr. Palomar*, "It is only after you come to know the surface of things that you venture to see what is underneath; but the surface of things is inexhaustible".

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Mathematics and Human Alienation

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Initial Abstract

Webster's definition of 'alienation' is "a withdrawing or separation of a person or a person's affections from an object or position of former attachment." Note that this definition of 'alienation' implies a previous attachment. It seems that most of us have had experiences with mathematics that have precipitated alienation in some form: from our peers; from our previous attraction to some aspect of mathematics; or in general, from some aspect of ourselves as thinking/feeling human beings. On the other hand, we would not be part of CMESG/GCEDM if it were not for an existing affection and attachment to some aspect of mathematics, in addition to the CMESG community itself.

This working group will begin with an invitation for participants to share some aspect of the "story" of their interactions with mathematics. The purpose of this is to focus our discussions on a personal/grounded level. This working group will proceed to look at mathematics and alienation as they apply to students of mathematics, and to the relationship of the general public with mathematics. What is the source of this alienation? Is it unavoidable? What can we do about it?

Some aspects of alienation caused by mathematics are described in "*Alive Mathematics Reasoning*" the plenary talk that David Henderson presented at the 1996 Halifax CMESG/GCEDM meeting: http://www.math.cornell.edu/~dwh/papers/Halifax/talk.html

This Final Report is divided into four sections corresponding to the three days of the working group and followed by an appendix created by juxtaposing portions of reflections submitted by two members of the working group.

Day 1: Alienation from/by mathematics
Day 2: Drawn toward mathematics: The other side of alienation
Day 3: Connections to mathematics and the broader society

DAY 1: Alienation from/by Mathematics

Plan for Day 1

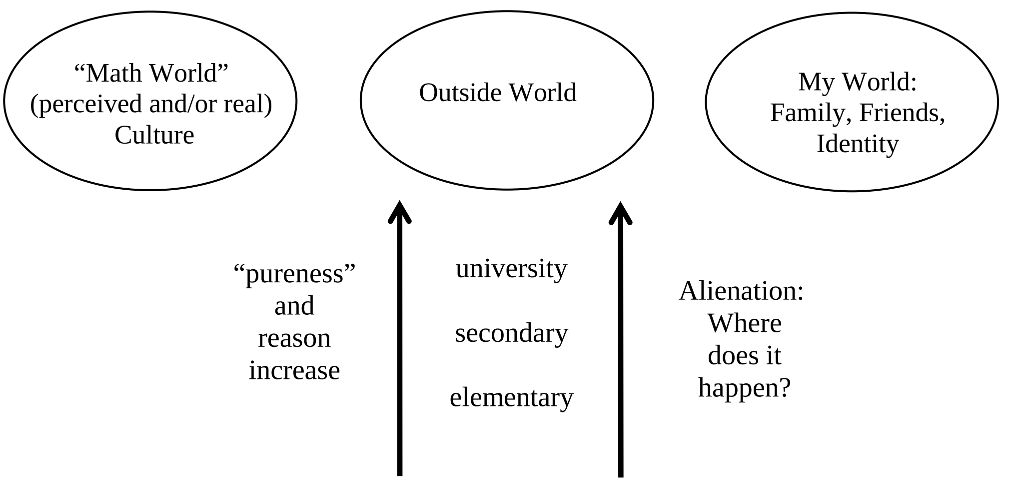
Each person relates 1 or 2 stories of instances from their life when they felt "alienated" by/from mathematics and how these experiences helped to shape their relationship with mathematics. "Mathematics" should be interpreted as broadly as possible.

After each one tells their story in small groups, then each small group reports back to the whole WG. The use of drawings, even drama, in the reporting is encouraged. The report is not meant to provide details of any one story (the stories are to be considered private and confidential within the small groups) but it should identify what the primary characteristics and nature of alienation are.

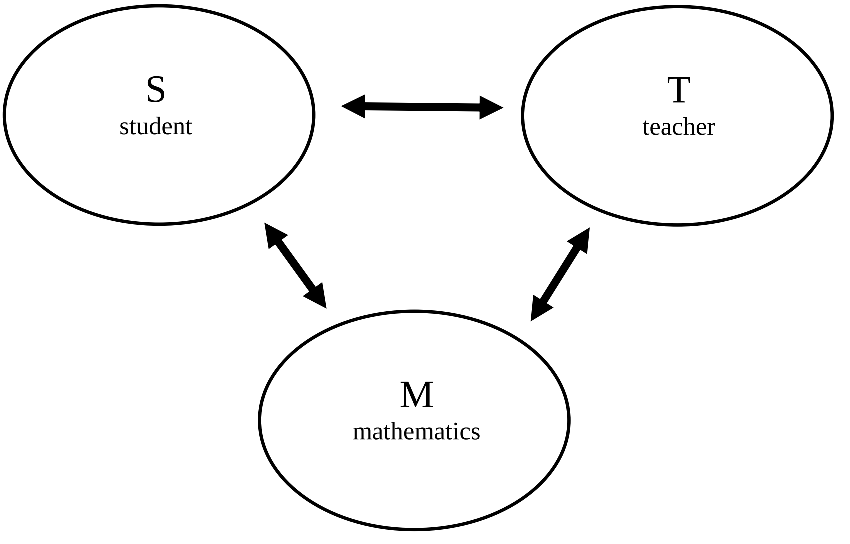
Then, reflecting back on the sharing of stories and on the characteristics which have emerged, could there be other stories participants would like to share, about themselves or others (as teacher, parent, friend) within the large group so that our picture is more fully developed?

"Poster" Reports from the Small Groups on the Nature of Alienation

Group 1: There are three cultures whose overlaps or perceived overlaps affect alienation.



Group 2: Alienation can happen when the relationships (both ways) between each vertex are not present, as is often the case. For example, often the student (S) cannot detect the relationship between mathematics (M) and the teacher (T), or feels no relationship with either T or M.



Group 3: Alienation often grows out of the imbalance of various dichotomies within mathematics. (This list of dichotomies was started the first day and added to in Days 2 and 3.)

applied	--	pure thought
messy	--	neat
co-operative	--	competitive
social	--	solitary
a language	--	not another language
easy	--	challenging
secure	--	unknown
teaching mathematics	--	discipline of mathematics
subjective	--	objective
ambiguous	--	always right or wrong
play	--	work we have to do
basis in meaning	--	formal basis

Generally speaking, people consider many of the items on the left side as 'feminine' aspects of math and many of the items on the right side as 'masculine' aspects of math.

Breadth of Mathematics-Induced Alienations

The study group was convinced that the varieties of mathematics-induced alienations are very wide. There are many barriers to a relationship with mathematics – some of these barriers are:

- socio-cultural assumptions about how we should learn math; for example, indigenous children in Australia can subitize much larger amounts, and so see no need to count, which works against them in school because school privileges the learning of counting
- failure of teachers to call on the mathematician inside every student
- the mythos of mathematics as cold, pure, perfect, Platonic
- math is the stuff "out there" that teachers merely transmit and students reproduce, but this removes the relationships out of math
- students are positioned and position themselves as either able or unable to do math; there is no middle ground, and there is little responsibility
- family patterns; for example, parent didn't do well in math, so it is not surprising when their child doesn't do well – this is an attitude that influences relationship with math
- math is about many things (cf. Bishop's framework) such as play, which is not evident in school math
- stereotypical images of mathematics and mathematicians; for example, math is not "sexy" and who would want to be a geeky mathematician?
- being good at math is like membership in an elite club; to enter, must want to, be asked to, and be able to enter the club; must be able to walk the walk and talk the talk
- the overemphasis on notation (e.g. algebra) and the delegitimatization of other representations (e.g., building a model by cutting, and so forth, is crafts, not math)
- narrow visions of the nature of math among both teachers and students; for example, what counts as proof; and math's role in/relationship with the world
- addiction to teaching/learning how and not why; right answers privileged above all else; technology can act as a barrier; for example, sin 37 means the answer was generated by the calculator
- math is a gatekeeper; hence it disenfranchises much more than subjects without the gatekeeper status of other subjects
- presentations of math in textbooks; for example, the accessible exposition being wrong (the formal is privileged); for example, "If this proof doesn't convince you, then there is something wrong with you."
- there is such a psychological phenomenon as *hating what we do best*

DAY 2: Drawn Toward Mathematics: The Other Side of Alienation

Plan for Day 2

Participants divide into different small groups to discuss: Everyone in the WG is now participating in mathematics, so they have been attracted or drawn into mathematics. What is the nature of this "attraction" or "drawing in" and what is the nature of the experienced ongoing tensions and mediations over time?

Each small group reports back to the whole WG. The report is not meant to provide details of any one story (the stories are to be considered private and confidential within the small groups) but it should identify what the primary characteristics and nature of alienation are.

Then, reflecting back, small groups comment on the characteristics that have emerged, so that our picture is more fully developed.

Big Ideas from the Discussions

People are drawn into mathematics by:

1. *Math as freedom, or lack of it.*
   Highly prescribed curriculum and rigid perceptions of math can be FREEING (well structured => one way to teach => easy to teach). Related to this, when students fail to learn it is their problem, which frees the teacher of responsibility (to know is to do well on a test => security on knowing how to do it). Math can give one a sense of independence.

Math as a way of knowing yourself, rather than as only a subject to know about.

This idea emerged from a story about a pre-service teacher who was good at math but was bored during the first course, which was a math content course. The pre-service teacher started gaining deeper insights about lower-level mathematics. Then, during the second course, which was about teaching math, she started learning about herself by doing math.

It is NATURAL for any human to feel "moments" of alienation or drawing in; what matters is the adaptations of each person over time, leading to a personal relationship with math that may or may not be defined/constrained by an alienation or drawing in narrative.

Math as challenge.

Many find a sense of gratification by getting answers, by discovering, and by solving a problem, even if it's a challenge. In a slightly different direction, there is the fun of solving problems and enjoying games (for example, Sudoku). Others are drawn to math because it is useful and can be applied to solve real world problems. And intertwined in all this is the sense that, in mathematics, one can't fake it – it's a sincere discipline

Good teachers can draw one in.

This is an antidote to alienation on both sides: Many of us were drawn to mathematics by good teachers who crossed our paths. Teaching mathematics draws many of us into mathematics. Good math teachers are rare because they could have ended up doing something else. Some are drawn to mathematics by the social/teaching parts of the discipline, even though others are drawn to mathematics precisely because they do not have to be social.

DAY 3: Connections to Mathematics and the Broader Society

Plan for Day 3

Divide into two small groups to discuss: What are the implications for us in our teaching, in our relationships within mathematics, and in our relationship with the broader society?

General discussion of the topic of our WG.

Main Themes and Questions

1. *Dichotomies*:
   We noted that alienation can be triggered by an imbalance on either side of each dichotomy listed under Day 1. It is important for us in our teaching roles and our communications with the public to keep this need for balance in mind. We do not want mathematics to be too challenging or too easy; but we do want to feel success (feeling

successful is necessary but not sufficient). We noted the dichotomy in the similar phrases: “I know it in my heart” (intuition) versus “I know it by heart” (memorization). One dichotomy was seen as particularly important: objective (for example, test scores, rigor, and so forth) and subjective (for example, subtle, culture stuff, intuition, and so forth).

2. *Everyone is a mathematician; but …*:
   - there are processes that deny/stop this from being perceived
   - there is a difference between insiders and outsiders. This was introduced through the story of a pre-service teacher who took a math methods course. She hated it and felt she couldn’t do well in math, and claimed that: “It is only those who can do math (or like math) who claim that everyone can appreciate math”. We, who are insiders, can end up in a circular argument, and never actually draw in the outsiders, because we try to draw in with what drew us in and fail to acknowledge that what didn’t alienate us may alienate others (the proverbial “everyone is different”).
   - gender issue: innate ability (male) versus hard work (female) – what about math does this?

3. *Social and cultural issues*:
   Culture impacts significantly on how students are able (or not) to engage with school mathematics. Students from particular social and cultural groups are more at risk of failing school mathematics, not due to ability but through the structuring practices of the field. Indigenous students, for example, come to school often being competent at subitising and with a strong spatial sense but these skills/dispositions are not part of the early years’ curriculum, so from an early age they are positioned by the practices as failing school mathematics. It is not that they do not have ability but that there is a mismatch between what the school values/rewards and what the learners bring to the learning environment.

4. *Identity in the world ... negotiating/navigating/forming that identity in conjunction with mathematics*:
   Several of our group members talked about mathematics as defining their personal identity at difficult life junctures.
   What is the educational goal? – It is based on leading to equity (not sameness).
   Early socializations of gender and math.

5. *Who owns the word “mathematics”*?:
   - Who defines it? What are the perceived views of math?
   - Does mathematics help us understand the world better? Every word in this question is more or less ambiguous, raises new questions, and prods for a debate no matter what the answer. Indeed, what is “mathematics” and who defines it? What does it mean to “understand”? And what “world” do we seek to understand “better” using mathematics – the natural world, the social realm, or our interior lives?
   - When we talk about “gaining mathematical insights” and “constructing mathematical arguments within constraints”, an unanswered question arises: What insights and arguments are mathematical?
   - The peculiar relationship between mathematics and its practitioners seems to have no equivalent in other disciplines. No other professionals are working in a similar crossfire of public emotionality: at the same time respected and reviled, appreciated and envied, needed and hated.

6. *Access to notation (language, discourse) of math* can *be a big part of alienation*:
   For example, an equal sign between two things that are qualitatively different but in some way equal such as with 31 + 11 = 6 x 7.

Language – access to math discourse – what about mathematics is important for this? Is it particular to the discipline?

Who owns the language of mathematics? For example, children from the working class (compared to middle/upper classes) do not have as much access to math discourse because it is not pointed out in everyday life in out-of-school life (for example, by parents), and because of this, they do not do as well in math upon entering school.

The language of instruction is likely to be middle-class. The further a student is from this desired form of language, the greater the chance of exclusion but also the greater need for explicit instruction around language. This means that when these students come into the learning situation, there is considerable potential for missing significant mathematical ideas. Explicit teaching of some of the language of mathematics becomes essential if students from some backgrounds are to gain entry into the discourse.

7. *Definition of a good teacher*:

Making sense of the sense that other people make.

Empathy is a necessary condition, both for reflexive self-understanding and for gaining the confidence to help others who might be alienated from mathematics.

Gender experiences: the observed differentiated responses by teachers based on gender of the student; and the common perceptions of female as teacher.

Conclusion

The working group effectively looked at the idea of alienation from various perspectives, as suggested through the themes identified in the report. Most people in the group have experienced alienation from mathematics on a personal level during their journey, though there were people who did not resonate with this personal experience. The alienation took surprising forms, or came at stages that most people alienated from mathematics could not identify. For example, the idea of being alienated from math during graduate studies in mathematics would seem foreign to most students and teachers of mathematics. Our own experiences of alienation are different than those of others who have not been drawn in, or possibly have been alienated for a dominant portion of their mathematical experiences. The collective discussion took place within a context where the discussants have generally been successful and enjoyed mathematics. At the same time, this group has observed alienation at many levels. This awareness of alienation from mathematics as a common experience (of students, peers, teachers) and a way of perceiving math for others in societal roles outside of formal education was critical to the context of our working group. This backdrop was present as we worked with what we know – our own experiences as mathematicians, teachers, students, and citizens – to move our own personal and collective experiences with the topic forward. The experience was enriching and the intention of the report has been to place some of the core ideas in a form that will provoke further thought and discussion on this issue.

Acknowledgments

The working group leaders wish to acknowledge the contributions of the participants in preparing the report. The ideas presented within the report are not individually attributed though they are drawn largely from the notes and discussions held in Sherbrooke. Gratitude is expressed to Paul Betts for his detailed notes. The notes were particularly valuable following the zealous efforts of custodians to clear any records in the form of chalkboard notes, flipcharts, and such from the room following the second day. Thanks to Robyn Zevenbergen for a contribution on cultural aspects of alienation that has been integrated into the report.

Appendix: An Imaginary Dialogue

As we were preparing this report, two members of the group agreed to share some of their personal story. We have combined their contributions in an imaginary dialogue, as there were striking similarities as well as contrasting features which we thought would better emerge this way. In what follows, Kate Mackrell and Vincent Martin are the contributors. We thank them.

Kate: *In high school, mathematics had been the pure, beautiful and simple structure which was a safe refuge from a chaotic family. The structures were obvious, all the problems in the textbook were easy and proofs were fun. And I got good marks. And then there was my problem. I loved the shape of the parabola – I'll never forget the day that this curved function emerged from the textbook. It was science fiction at its best. I was intrigued by the focus-directrix derivation, and I asked myself "what would happen if I used the parabola itself as a directrix?" I played with this problem for ages […] I can't remember why I stopped working on the problem (perhaps I felt I'd finished?), but I put the pieces of paper in my poetry file. I now see this as my first piece of real mathematics and I've continued to revisit it in different ways.*

Vincent: *Mon cheminement à l'école primaire m'a permis d'apprendre les bases des mathématiques. Je n'ai que de bons souvenirs mathématiques associés à cette époque de ma vie. Les choses y étaient simples: ma communauté d'apprenants n'était composée que d'une trentaine d'élèves, tout au plus, le curriculum enseigné était unique et chacun se trouvait confronté aux mêmes activités mathématiques. Mes performances d'apprenti-mathématicien y ont été aussi enrichissantes que remarquables.*

Kate: *But mathematics had also been the tricky problems of the mathematics contests. This was what being a mathematician was really all about – coming up with quick solutions to quite difficult problems. I knew I was not a mathematician when, as a result of the grade 11 mathematics contest, I, along with the other top 1% of girls, was invited to a week's seminar at the University of Waterloo. Only the top half percent of boys had been invited. There were a few girls in the top half percent, and I was not one of them, and hence was not a mathematician.*

Vincent: *Le passage au secondaire a entraîné bien des changements dans ma vie, y compris au plan de mon apprentissage des mathématiques. En effet, le nombre d'apprenants composant ma communauté a été multiplié par cinq ou six et plusieurs enseignants – plutôt qu'un seul comme au primaire – nous guidaient désormais dans notre apprentissage des mathématiques. […] Puis, pour les quatrième et cinquième secondaires, le curriculum s'est divisé en deux volets: les mathématiques et les mathématiques avancées. Pour des raisons plus ou moins obscures (peut-être simplement pour satisfaire mes parents), j'ai opté pour les mathématiques de haut niveau.*

Au fil du secondaire, mes résultats en mathématiques n'ont fait que décroitre et j'ai tranquillement quitté le peloton de tête. Peu à peu, les influences sociales ont su me faire croire qu'aimer les mathématiques et réussir dans l'apprentissage de ce domaine n'était pas valorisant. Rendu dans les classes de mathématiques avancées, vers la fin du secondaire, je considérais négativement les bons élèves et je tentais de me satisfaire de mes maigres résultats. Malgré tout, je n'étais vraisemblablement pas à l'aise avec moi-même et un tiraillement entre les influences des amis doués ou non en mathématiques s'est opéré pendant un bon moment.

Kate: *Then there was university. I'd expected that I could learn all the math there was to know by the end of university, and it was unsettling to discover just how much mathematics there was and that I had no chance of learning more than a very small fraction of it.*

And mathematics was hard, and increasingly not visual, and I just couldn't see the relevance of being a mathematician. It felt trivial compared with suffering with the poor in Africa or some such. So I put more time into pondering moral dilemmas than studying, and got the corresponding results. And again felt that I wasn't a mathematician – mathematics came easily to real mathematicians. […] By the time I came back to do my Master's I'd decided that I needed to settle down and do math properly. And properly meant diving into the algebra and seeing what emerged out the other end. […] Properly also meant doing what other people told me to do and not look for my own connections. So, I came out with high marks in the coursework (I'd done it properly), a thesis which demonstrated adequate understanding of a complicated area, and the total conviction that I was not a mathematician. That was the low point in my relationship with maths.

Vincent: *Après la fin de l'école secondaire, je suis allé au cégep et j'ai choisi un programme d'étude dans lequel j'ai été initié au calcul différentiel et intégral, ainsi qu'aux statistiques. Dans ce contexte, la valorisation d'apprentissages de qualités et des bons résultats a refait surface. C'est ainsi que j'ai eu l'impression d'œuvrer pendant deux ans à rattraper le retard que mon désengagement du secondaire avait entraîné.*

Kate: *The critical moment in my relationship with mathematics came in the middle of my PGCE year. (This is the UK equivalent to a B.Ed.). It was 1982, I had just turned 30, had come to terms with entering old age, and we had a session on "investigations".*

Start with square dotty paper and draw polygons with vertices only on the dots and with an area of 8 squares. How many dots can you get inside the polygons? On their borders? Explore.

My experience of investigations such as the dotty paper one […] totally changed my view of what mathematics, and mathematics teaching, should be about. Yes, I'd run across lots of problems before – but the point of the problem was to solve it as stated, either for a practical reason (which I could do) or to prove that you were clever (which I could do if I could see the "trick" to the problem, but not otherwise). Problems had never been a means to mathematical discovery. The problem […] had "easy" solutions, but then led to other problems – what if you change the area? Is there a relationship between the area, the dots on the inside and on the boundary? Why does this relationship occur?

All these questions, to me, are now at the core of mathematics, and I realize that this is the experience that was missing in my formal mathematics.

Vincent: *À l'université, j'ai fais un baccalauréat en enseignement au préscolaire et au primaire, durant lequel j'ai suivi différents cours de didactiques des mathématiques. Intrigué autant qu'attiré par ce domaine, j'ai choisi de creuser plus avant. Ainsi, après avoir réalisé une maîtrise en sciences de l'éducation en didactique des mathématiques, j'entame aujourd'hui un doctorat en éducation et il ne fait aucun doute que je porterai attention à un objet d'étude lié de près à la didactique des mathématiques.*

En somme, je contemple à rebours mon parcours d'apprenant des mathématiques en constatant qu'en dépit de mes résultats inégaux et parfois décevants, j'ai toujours cherché à rester en contact avec cette discipline. Mon rapport aux mathématiques s'est modifié à quelques reprises dans le passé et, aujourd'hui, je regrette de ne pas avoir su mettre davantage d'effort dans mon apprentissage des mathématiques.

Kate: *I now think of myself as a mathematician, as someone who enjoys playing with mathematics. What has been good for me along the way has been the beauty of mathematical structure, success, having fun (which often required thumbing my nose to the "proper"), making connections, solving problems and ultimately* *learning how to investigate within mathematics. The alienations have been anxiety about the hugeness of mathematics (which is now a relief – I'm not going to run out!), lack of connection to visual images, competition, lack of relevance, a belief that if mathematics is difficult then the learner is not clever, and a belief that mathematics is about algebraic manipulation and needs to be done" properly" (i.e. by following other people's rules).*

Communication and Mathematical Technology Use throughout the Post-secondary Curriculum

Utilisation de technologies dans l'enseignement mathématique postsecondaire

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Introduction

> *Although the computer hardware and software options have been present for decades, we have still not seen a major shift in pedagogy within our education systems such as was widely predicted . . . We need to dedicate perhaps 10% of our individual energy and working lives to the exploration of new ways of teaching—of reconceptualising how it is that we teach and students learn mathematics at all levels.*
>
> Seymour Papert (paraphrased)[1]

How is technology changing how we teach mathematics students and mathematics teachers, and how they learn? In this working group we focused on three themes: (1) new technologies for investigating and doing mathematics; (2) new communication technologies; and (3) implications. Below we elaborate on these themes based on our working group's discussion.

[1] Keynote Address, ICMI Study 17: Digital technologies in mathematics education—Rethinking the terrain, Hanoi (Vietnam), December 2006.

New technologies for doing mathematics

In the first day of our working group, our discussion was guided by these questions: What are the mathematical technologies available to us for doing and teaching mathematics, and how are we using them? Has the pattern of our use changed, e.g. what is the role of technology in modeling and simulation, and data management? Does our mathematical and pedagogical thinking change? Do such changes persist even when not using the technology tools? Do these technologies provide new opportunities for students to become more independent in their doing and learning of mathematics?

These questions concerned technologies such as Computer Algebra Systems (e.g., Maple, Mathematica, etc.), Statistical Analysis Systems (e.g., SAS, Minitab, etc.), Dynamic Geometry Software (e.g., Cabri, Geometer's SketchPad, etc.), Online Learning Objects such as Applets, Discrete Mathematics Systems (e.g. MatLab, Excel, etc.), and programming. Examples of technology uses at Brock University and Cégep de Rimouski were provided to participants to set off the discussion: use of Maple for rates of change, derivatives, simulation-modeling, and Euler's Method; use of an Applet for linear transformations; use of Minitab for the Central Limit Theorem; and use of programming for prime numbers. See (CMESG 2008 Working Group on Technology Wiki, n.d.) to access any of the files.

The following themes emerged from our discussion:

Challenges

One challenge is the students' knowledge of technology for doing mathematics. Students have been referred to as "digital natives" (Pensky, 2001) due to their immersion in digital environments and tools. However, we wondered whether this immersion leads to digital literacy and more specifically to a digital literacy that is of use in the mathematics classroom. We might for example consider ourselves as "automobile natives" but most of us only have a superficial knowledge of how automobiles work. Our knowledge is that of a consumer and not of a mechanic or a designer or a scientist, though the automobile has deeply modified our way of life: leisure, urbanism, etc. In a similar way, the young generation might not know how the digital technology works but all the same it is now an integral part of their culture. It is a zapping, multitasking and chattering generation, able to get a lot of information but juxtaposing it without logical structure, and who wants results as soon as possible (Piette, Pons, Giroux, & Millerand, 2001). We discussed the need to educate students to recognize appropriate use of technology in doing, including investigating, mathematics.

In this context, a second challenge is that of a systemic approach. How do we ensure that as students move from one course to the next that their technology skills are both valued and further developed, not only within a mathematics program, but also for cross-discipline programs or later in the profession? This is a difficult challenge since, even just within a core undergraduate mathematics program, very few systemic integrations of technology have taken place. Assude et al. (forthcoming) argue that

> *evolution and innovation in university mathematics education is a slow process. There is in fact a strong internationally uniform 'mathematics university department culture'. Traditionally, mathematicians view doing mathematics as an individual activity. There is a strong focus on proofs. Teaching is usually valued as secondary and way behind research, which may reinforce a common attitude towards teaching to copy one's own personal, traditionally abstract oriented, experience since one's success supports it, although the big majority of undergraduate mathematics students do not become mathematics academics.* (p.6)

A third challenge is deciding on which technology tools to use. Should we favour the use of open source software, for example? This points again to the need of systemic integration of technology that could guide us in selecting appropriate technological tools for longer-term impact on students. A final challenge is that it is difficult to suggest common solutions for different students and contexts, such as, mathematics majors, cégep science students, and prospective teachers. However, the reality might be of a heterogeneous group of students, e.g. at cégep, education with science students possibly heading to engineering, high-school science and mathematics teaching, sciences or health sciences; or for example a first-year university mathematics class for mathematics majors and prospective mathematics teachers. How can we find meaningful ways to integrate technology in classes that include such different students?

Technologies

We noted that technology is increasingly used by mathematicians and scientists to do mathematics (e.g., Maple). The use of technology offers students in mathematics opportunities to investigate new problems (rather than only show competency in established mathematics). For example, in the innovative core undergraduate *Mathematics Integrated with Computers and Applications* (MICA) program at Brock University (Ben-El-Mechaiekh et al., 2007), mathematics majors and prospective mathematics teachers learn to investigate self-stated conjectures and real world situations of their own choosing, by designing, implementing and using interactive computer environments (Muller et al., forthcoming). Examples of student original projects can be accessed from the website (MICA Student Project Website, n.d.). Buteau and Muller (2006) observe that through this mathematical activity, students have the opportunity to realize their creativity and to develop their intellectual independence and skills to communicate their understanding of mathematics in an exact way. Because the technology used may not document solution paths or may serve only as a computational and visualization support tool, such as MICA students' interactive computer environments, we noted the need to develop a writing culture for documenting mathematics with technology for learning (lab reports) and assessing (screen capture software, wink).

A 'Black Box'?

We discussed the recurrently raised issue that technology might be seen as a “black box”, where the process of doing mathematics remains hidden and not understood by students. We noted that any mathematical procedure (with or without technology) might be seen as a black box. We also noted that some of our mathematical concepts need to be treated as procedures in some contexts, for efficiency. For example, when differentiating a polynomial, we normally 'apply the rule' rather than using the first principles. We concluded that perhaps our focus should be on “openable (or white) boxes”, where we have opportunities and abilities for understanding mathematical processes.

Changes Due to Technology Use

Bill Higginson commented: “Don’t ask how I use it in my discipline; ask what is my discipline now that I use it”.

At Cégep de Rimouski, changes due to technology are very important. With the capacity of computer algebra systems (CAS; e.g. Maple) to easily compute and quickly draw precise graphs, it allows for a reduction in the curriculum of some technical aspects in numerous topics, such as the study of functions, geometry, etc. With its capacity to do simulations, it can support students' learning of many mathematical concepts. The use of CAS provides access to higher-level mathematics; for example at cégep level, it allows a deeper consideration about topics such as differential equations, multi-variable functions, linear algebra, etc. Technology makes it possible because long, arduous, and repetitive computations are no longer an obstacle. A systematic use of CAS in learning mathematics prompts students to develop an algorithmic process strategy that is transferable as a working process alongside the scientific method—what Cégep de Rimouski calls the simulation-modeling process (see *Modelisation.mw* or *Modelisation.pdf* in CMESG 2008 Working Group on Technology Wiki, n.d.). In fact, it is not the technology itself that changes the discipline, but the way we use it. Studies on use of technology in teaching show that if the technology is used for itself, it is generally a failure with students (Roy, 2005).

Another change with technology that was noted at Cégep de Rimouski is that it tends to fill the gap between students' perceptions of modern sciences and of the mathematics they learn, e.g., 18th century Calculus. Many disciplines use mathematics via technology to model and simulate their problems. Modeling and simulating has changed both the work in science and the role of mathematics in science. In fact mathematics is used more and more in disciplines like biology, finance, environmental sciences, etc. Cégep de Rimouski's position is that curriculum must pay attention to this reality.

The use of technology makes it easier to investigate mathematics, a potentiality that led, or at least strongly contributed, to the emerging Experimental Mathematics. Such a use can in particular play a role in strengthening (or wakening) students' engagement in mathematics. The technology use in teaching and learning can also provide more time and opportunities for more conceptual discussions. Finally, we noted that there is a necessity of developing a critical mind about technology, e.g. for the interpretation of results in modeling supported by technology computations and investigation.

New Communication Technologies

In the second day of our working group, we discussed these questions: What is new or different about Web 2.0 communication, and does it restructure and reorganize our thinking, as mathematics teachers and students? Do the read/write, collaborative affordances of wikis and other social software make a difference in mathematics and mathematics teacher education? When anyone with a video camera (or just a $20 webcam) can post a math video on Youtube, who is the teacher? the student? the textbook? the tutor? What if students were allowed unfettered access to the Internet? What would change in terms of curriculum, teaching, learning and assessment?

The following themes emerged from our discussion.

Internet

The Internet is increasingly used as a reference for information. It can also be a source of situations that can be analyzed mathematically, such as, the movement of a dancer in a Youtube video, or the stride rate of a runner in a race broadcast by an online news source. However, the content of the Internet is not stable (there is no guarantee that the site we referenced today will be the same tomorrow) and not always accurate. Students need to develop an ability to find and sort information and to be critical about information on the Internet.

What if your students have unfettered access to the Internet? There is then an enormous amount of available mathematics resources. While working with a laptop and Internet, a student has access within seconds to all kinds of courses, problems and solutions, encyclopaedias, fora, etc. The student also has access to all his/her work, and can communicate with other students, with the instructors, etc. This highly challenges the way we traditionally teach mathematics. It seems that we have more questions than answers. It challenges curriculum. And what about assessment? Instructors also have to adjust the class management to integrate the use of such communication technology. Students can ask their own questions and Google search them on the Internet, which then could lead to other ideas or concepts in addition to the initial quest. We might claim that traditional lectures become less important, whereas presenting explanations, developing working methods, prompting individual or team questions, and addressing information management become more important, and thus conclude that students become more autonomous and can manage more of their own learning. But do they? Or does it simply shift teaching issues to something new?

Collaboration

Web 2.0 tools (like wikis) offer collaborative affordances that potentially change how students learn mathematics and mathematics education in online settings. These changes may require shifts in student and instructor attitudes towards authorship, copyright, and forms of expression. Perhaps we need to take account of Vygotsky's (1978) view of knowledge as constructed in interactions with others. By "others" we can also refer to digital tools that permeate our new media culture. Levy (1997) suggests that technology itself is an actor in the collaborative process. Levy sees technology not simply as a tool used for human intentions, but rather as an integral component of the cognitive ecology that forms when humans collaborate in a technology immersive environment. Likewise, Borba & Villareal (2005) see humans-with-media as actors in the production of knowledge. They note that humans-with-media form a collective where new media also serve to disrupt and reorganize human thinking.

What if you taught an online mathematics or mathematics education course in a wiki setting? What are the issues, challenges and opportunities, for mathematics, for curriculum, for teachers, and for students? We suggest that the nature of the task makes a difference. Also, the nature of the online environment makes a difference: is it synchronous (using a platform like Elluminate) or asynchronous (using a wiki), for example. A lot also depends on the number of students in the class. Teacher education classes tend to be small in size whereas undergraduate mathematics courses tend to be quite large. Online collaboration takes on a very different meaning when there are 150 students in a course.

Online collaboration in a mathematics or mathematics teacher education setting also raises issues of what mathematical typesetting tools are available. It also raises issues of ownership. It is not easy to edit others' work or to have one's own work edited. Editing may easily be interpreted as a form of criticism, resulting in resistance to editing or being edited (ownership of ideas/work). Gadanidis, Hoogland & Hughes (in press) and Borba & Gadanidis (in press) note that in teacher education courses there is some resistance to using the read/write features of a wiki. This is not uncommon in initial uses of wikis (Grant, 2006). The resistance appears to be greatest in online graduate courses where students seem to have difficulty allowing themselves to edit the work of others. The dominant experience in graduate courses is that students write their papers in private and that only confidential suggestions for improvement come from the instructor. In addition to this tradition, there are also issues about ownership of ideas. When a student's paper or poem is edited by peers, is that paper or poem still the original student's work? Plagiarism and the scholarly need to acknowledge sources are especially important issues in graduate work. However, we perhaps need to make explicit in our graduate courses that peer editing is the norm in scholarly writing. For example, when a scholarly paper such as the present paper is submitted for publication, it is reviewed by peers, and their comments and suggestions, written in the margins, in summary statements, or in the case of electronic submissions sometimes in the text itself, come back to the authors. Such comments and suggestions are a learning experience for the author(s) and some are incorporated in the final version. It should be noted that the final version of the paper typically does not credit the reviewers (whose identity is kept confidential from the authors) and the original authors retain ownership of the final work. Also, a number of ideas emerge from the three cases for helping students feel more comfortable with the peer editing process in a wiki. These include, (1) maintaining a copy of the original work as well as the edited work, (2) giving students experiences with editing the work of someone who is not part of the course,

(3) using word play activities where students rearrange a jumbled poem or where students add nouns and verbs that have been removed from a poem or paragraph, and (4) using group assignments where 2-4 students submit a single piece of work created in a wiki.

Multimodality

Web 2.0 communications are also increasingly multimodal, with integrated use of such tools as drawing palettes, video capture, and concept mapping. Kress and van Leeuwen (2001) suggest that in a digital environment "meaning is made in many different ways, always, in the many different modes and media which are co-present in a communicational ensemble" (p. 111). Hughes (2006) suggests that the Web is also becoming a "performative medium". This is evident in the multimedia authoring tools used to create online content, such as Flash, which often use performance metaphors in their programming environment (like "stage", "actor", "scene" and "script"). The shift from text-based communication to multimodal communication is perhaps not simply a quantitative change. It is not just a case of having more communication modes. It might be seen as a qualitative shift, analogous to the change that occurred when we moved from an oral to a print culture. Our understanding of what this change implies is emergent and not fully conceptualized or articulated.

What if your online students could communicate with you and with other students using video, audio, drawings, and graphics? What are the issues, challenges and opportunities, for mathematics, for curriculum, for teachers, and for students?

Implications

In the last day of our working group, we discussed these questions, though some of them had indirectly been addressed during the first two days: What does the technological change mean for our pedagogical practices? Are there new paradigms emerging? Should there be new paradigms emerging? What about our curriculum? Is it changing? Should it change? If so, how? And, what are the implications for education research? Are we shining scholarly lights in the right places? What are our blind spots? With the progress of technological tools, does the place of mathematics in science and in society change?

The following themes emerged from our discussion:

Pedagogical practices

Studies show that integrating new technology without a change of pedagogy is one of the main factors of failure (Forget, 2005). For example, when implementing a course exploration of mathematical concepts with the use of technology, our role as instructor needs to shift from traditional lecturer to more of a facilitator (Muller et al., in press). Since most of us may have been educated through a 'lecturer-type' of teaching, it puts a demand on the instructors to review and challenge the pedagogical practices that can/could/should be adapted to the potentialities of mathematics instruction with technology. Integrating technology in teaching also brings demands on the instructors to need to know mathematics differently, different mathematics, and the selected technology. Thus, time investment by instructors is crucial for a successful integration of technologies.

It seems to us that when introducing technology in teaching mathematics there are three things to consider and the links between them: the technology available, the students and their culture, and the kind of mathematics they need and want. We have to find a balance and complementary links, and define what the role of a mathematics instructor becomes.

Curriculum

One important potentiality provided by the use of technology is to integrate more exploration and simulation in our pedagogy. However, this requires much time and thus some content may have to be dropped or adapted from the original curriculum. On the other hand, modeling requires a broader curriculum. In the end, mathematical investigations and modeling motivate students to become more active in their learning and doing mathematics, and to work harder to develop deeper understanding.

Place of Mathematics in Science and in Society

Traditionally, physics was the first choice application field for mathematical modeling. It is now shifting to biology. We noted however that this shift is not yet reflected in our textbooks. Mathematics is also being increasingly introduced in the arts; e.g., fractals in visual arts or computational music analysis. The modeling of complex biological systems relies highly, as do fractals in visual arts, on mathematics with use of technology.

Final Thoughts

In the words of Bill Higginson: "Ne me demandez pas comment j'utilise la technologie dans ma discipline, mais demandez-moi ce qu'est devenue ma discipline maintenant que je l'utilise" or, as originally stated: "Don’t ask how I use technology in my discipline; ask what is my discipline now that I use it."

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Appendix : Intégration de la technologie au Cégep de Rimouski

C'est la montée en puissance des ordinateurs personnels et le développement des logiciels de calcul symbolique qui ont incité le département de mathématiques du Cégep de Rimouski à intégrer vers 1994 l'informatique comme outil usuel dans tous les cours de mathématiques du programme pré-universitaire de Sciences de la nature. Ce choix a pu améliorer l'enseignement des mathématiques en revoyant le contenu des cours et en introduisant deux méthodes de résolution de problèmes.

D'abord, la puissance de calcul et les capacités graphiques de logiciels comme Mathematica ou Maple ont permis d'améliorer le contenu des cours. En effet, le temps nécessaire pour effectuer des calculs longs et répétitifs a pu être libéré et consacré à mieux comprendre des concepts comme la limite, les fonctions, etc. Ces concepts ont aussi pu être présentés à l'aide de simulations et de graphiques ce qui aurait été impossible sans ces logiciels. De plus, ces logiciels ont permis de traiter plus en profondeur des thèmes comme les équations différentielles, les fonctions à plusieurs variables, la géométrie dans l'espace, l'algèbre linéaire, etc. Ensuite, deux méthodes de résolutions de problèmes ont été introduites. D'une part, l'utilisation de Maple va de pair avec l'apprentissage d'une méthode algorithmique, car ce logiciel exige une rigueur d'écriture et une cohérence entre les lignes de commandes. Ainsi, l'utilisation de Maple corrige quelques lacunes générales des étudiants, comme résoudre par la recherche d'un exemple, par essai-erreur, etc.

D'autre part, l'utilisation de Maple rend aussi possible le développement d'une méthode de modélisation qui permet de traiter des problèmes de mathématiques appliquées, comme des problèmes de physique, de biologie ou des problèmes reliés à l'environnement. L'aspect central de cette méthode, et qui est seulement possible en utilisant l'ordinateur, est de comprendre un problème par des simulations numériques ou graphiques. Il est ainsi possible de suivre l'évolution du phénomène, d'identifier le rôle des paramètres ou les cas particuliers et aussi de déterminer les limites du modèle. L'interprétation des résultats par simulations permet alors de développer l'esprit critique. Avec l'ordinateur il ne s'agit plus de calculer une réponse numérique mais de comprendre le phénomène par le traitement informatique du modèle.

Toutefois cette démanche de modélisation soulève de nouveaux problèmes : la formation des enseignants, l'utilisation de l'interdisciplinarité, la mise à jour des programmes, etc. Par contre, cette démarche de modélisation-simulation rapproche l'enseignement des mathématiques de la façon dont elles sont utilisées aujourd'hui. Elle tend ainsi à combler le fossé entre la perception que les étudiants ont des sciences d'aujourd'hui où les mathématiques sont omniprésentes et les mathématiques enseignées, celle des XVII et XVIII siècle. Cette démarche tend enfin à répondre à l'éternelle question des étudiants : mais à quoi servent les mathématiques.

Cultures of Generality and their Associated Pedagogies

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Below is the three-paragraph description that was sent out ahead of the meeting

Mathematics educators, teachers and mathematicians do things in ways that try to convey some mathematical generality. When teaching, they do things that try to help students think about such generality. In this group, we will provide artifacts that attempt to carry or convey some generality from a range of cultures (of generality) for group members to interact with and work on. Our first intent is for participants to work towards a greater understanding of what generality is being expressed in and through them. Secondly, but of comparable importance, we wish to attend to the tacit or more explicit pedagogy at work in *how* the generality is being conveyed within the artifact.

As some historical examples of mathematical artifacts we might offer include excerpts from ancient mathematics texts including (translations of!) Babylonian problem texts, Greek geometric proofs, pages of Chinese mathematical texts and Vedic procedures.

By the term 'cultures of generality', we have in mind different historical forms in which mathematics has been presented (arithmetic, geometric, algebraic), as well as more recent mathematical and educational manifestations such as computer-based mathematics (dynamic) and tasks or problem-based genres of teaching, in addition to instances from cultures apparently outside mathematics (imagistic, poetic, aesthetic). We hope to explore connections among different forms of generality and their pedagogic conventions and possibilities.

Day 1

To make a start,
out of particulars
and make them general, rolling
up the sum, by defective means

(Williams, 1946/1985, p. 259)

We started by looking at a worked example (88 × 96 = 8448) of 'Vedic multiplication' (taken from George Joseph's book *The Crest of the Peacock*, pp. 244-245), based on the following array, which is claimed to exemplify the *sutra* 'all from nine and the last from ten' (see below for a little more discussion of what a *sutra* is).

84	48
88	12
96	4

Then we invited participants to work 'comparably', 'in the same way', on 64 x 92. The purpose of this task was to elicit ideas about generality in this example, which is so different from our techniques for multiplication but also from our ways of describing a multiplication method. What might this be an example of and how might we express it?

Possible things to notice include: the answer 8448 is a palindrome, so 84 could be the reverse of 48; 84 could be 88 – 4; 84 could be 96 – 12. How can we know which of these are specific and which are generic of a 'method'? Try another instance.

This led to an extensive conversation on a range of connected terms, including 'method' and the allied but interestingly distinguishable notions of sutra, technique, algorithm, strategy and heuristic. The level of generality and distance of 'sutra' from any given instance seems so far removed from the definiteness and prescription of an algorithm (the two arguably lying at opposite ends of a spectrum related to guiding mathematical practice in any specific instance).

Etymologically a sutra is a thread (cf. suture), from the Sanscrit word for 'sew' to which it relates, so sutra "suggests a number of topics strung together on a common thread of discourse" (www.diamond-sutra.com). (The German verb *nähen* also means "to sew", but literally means "to make nearer" i.e. "to nearen".) Sutras are often allusive and aphoristic (more akin to Zen Buddhist koans perhaps), a condensed or cryptic statement that often can be understood only through reflection and commentary, and as accretors of experience. They are deliberately condensed and intended to be memorised. Meditation requires time as an experience.

Following the break, we offered a sheet containing Babylonian examples of problems that to us look like quadratic equations solved by the technique of 'completing the square' (taken from Fauvel and Gray, 1987, p. 31). The Babylonian numeration system is sexagesimal floating point – the ';' is a historical interpolation suggesting where the sexagesimal point 'should' have been, had they used one.

1. I have added up the area and the side of my square: 0; 45. You write down 1, the coefficient. You break off half of 1. 0; 30 and 0; 30 you multiply: 0; 15. You add 0; 15 to 0; 45: 1. This is the square of 1. From 1 you subtract 0; 30, which you multiplied. 0; 30 is the side of the square.
2. I have subtracted the side of my square from the area: 14, 30. You write down 1, the coefficient. You break off half of 1. 0; 30 and 0, 30 you multiply. You add 0; 15 to

14, 30. Result 14, 30; 15. This is the square of 29; 30. You add 0; 30, which you multiplied, to 29; 30. Result 30, the side of the square.

3. (*actually number 7 on the tablet*) I have added up seven times the side of my square and eleven times the area: 6; 15. You write down 7 and 11. You multiply 6; 15 by 11: 1, 8; 45. You break off half of 7. 3; 30 and 3; 30 you multiply. 12; 15 you add to 1, 8; 45. Result 1, 21. This is the square of 9. You subtract 3; 30, which you multiplied, from 9. Result 5; 30. The reciprocal of 11 cannot be found. By what must I multiply 11 to obtain 5; 30? 0; 30, the side of the square is 0; 30.

There are some nice pedagogic points here: for example, there is a 'no-op' line (borrowing a computer term for a program statement that has no effect, other than to take time to execute, allowing the alignment of certain processes), the statement in the first question 'One is the square of one'. In this particular problem, there is no point to such a line in the specific solution. But it acts as a placeholder for the fact that at this point *in the general solution* a square root is to be extracted here. It gestures at the general.

In addition, there is a nice ambiguity in the first problem too, when the 'I' imperatively instructs the 'you' to 'write down 1, the coefficient'. There are, of course, two coefficients, both of which are 1 in this example. A possible, subtle pedagogical move may be at work here, suggesting that when there is a potential for ambiguity, deliberately invoke it in order to have students raise the question of 'which coefficient?' as a way of noticing there are two (which 'in general' will be different).

How too do we notice and attend to the 'sameness' across these various texts, where what stays the same is more salient than what differs, because it is in that growing awareness that a sense of 'the algorithm' comes.

There is a precept in medical school with regard to medical procedures ('see one, do one, teach one') that proved of relevance here. Even if it is see *m*, do *n*, teach *p*, what shifts of perception are required to move from *seeing* (what am I attending to?) to *doing* (what am I attending to?) to *teaching* (what am I attending to?).

No 'meta-text' explaining what is happening and directing my attention. There is no meta-language used either, except perhaps in the word translated as 'coefficient'.

Day 2

We began the second day by asking participants to write down what remained with them from Day 1, before having participants offer a few brief instances to the group. We then shifted our focus from the arithmetic considerations of the previous day to geometric ones, specifically the particularity/generality related to diagrams. We offered Archimedes's proof that the perimeter of a circumscribed polygon is greater than the perimeter of the circle (in Book I, Proposition 1 of his *On the Sphere and the Cylinder*). The proof, following Netz's (2004) translation, runs as follows:

> *For since BAΛ taken together is greater than the circumference BΛ. Similarly, ΔΓ, ΓB taken together are greater than ΔB, as well; and ΛK, KΘ taken together are greater than ΛΘ; and ZHΘ taken together is greater than ZΘ; and once more, ΔE, EZ taken together are greater than ΔZ; therefore the whole perimeter of the polygon is greater than the circumference of the circle. (p. 41)*

We asked the participants to draw the diagram they thought might accompany this proof text. In addition to the challenges that were encountered in "decoding" the text, and figuring out which polygon had been used, and how it had been labeled, we were interested in probing the issue that Archimedes offers an apparently general proof, yet visually provided a specific example of a polygon, namely, a pentagon (see Figure 1). We noted a large variance in the actual diagrams produced by participants, from triangles to decagons. We then discussed the extent to which the pentagon can be thought of as the most general of polygons, a claim that Netz advances in his book. The pentagon seems to have a sufficient amount of complexity and possibly irregularity (in contrast to the triangle or square) without being too difficult to work through (as a 17-gon, say, might be).

We discussed the interesting mismatch between the text (which makes a claim about any polygon) and the image (referred to in the text as a ‘polygon’), and the resulting tension between the particular and the general, both in terms of the mathematical validity of the proposition, and in terms of its pedagogical use (one can *see*, from the specific example of the pentagon, how the argument would follow for the hexagon). Indeed, Netz views diagrams being used by the ancient Greeks as sketches to show topological, rather than geometric relations – and, more importantly, to provide logical control of the proposition.

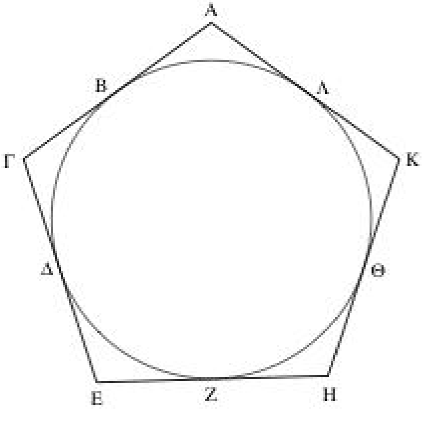

Figure 1. The diagram used by Archimedes in I.1 of *On the Sphere and the Cylinder*

We discussed at length the different issues at stake in moving from the particular to the general when diagrams are involved, the diagram being something that requires ‘setting down’ and, therefore, involves choices of size, orientation, etc. In contrast, one can quite easily use the single word ‘square’ to denote the entire range of possible sizes and orientations.

Following our work with Archimedes, we turned to other instances of the use of diagrams in other settings: ancient Japanese mathematics, contemporary mathematics, and diagrams from school textbooks. We analysed the different features of these diagrams that seem to evoke either particularity or generality. For example, comparing the three diagrams below, the left-most one evokes far greater particularity, with the addition of the indicated measurement of side BA. By contrast, the diagram for triangle GHI evokes much more generality, in that we have no idea of the scale, while still being a specific triangle. In the middle, triangle DEF gains particularity by means of the angle markers that denote it as being isosceles, but the presence of those symbols almost lend it greater generality in that, apart from any other constraints (measurements, size) the triangle appears almost as *any* isosceles triangle. IS GHI comparably *any* scalene triangle? If not, why not?

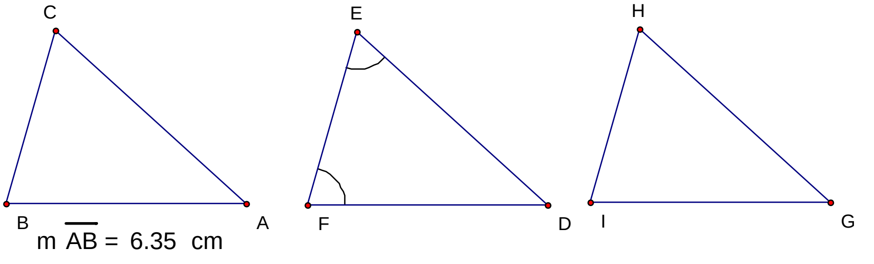

Figure 2: General and particular triangles

In the third part of the day, we turned to dynamic imagery, offering a selection of constructions created in *Sketchpad* for the participants to interact with. Our goal was to explore further the relation between the particular and the general given that the dynamics of the constructions allow users to create a much wider and varied set of specific examples, which may evoke for them a sense of generality.

In the first one, the user presses the Start button and the circle on the screen begins to move around, changing location and size. Eventually a point appears on the circle, and the range of variation of the circle changes, as it remains fixed to that point. Then a second point appears, again diminishing the range of variation of the circle, which now passes through the two fixed points. Finally, when the third point appears, the motion of the circle stops. It has been caught. The sketch illustrates the notion that three points completely specify a circle on the plane. However, it gets there by moving from the most general to the most particular of configurations.

The second sketch involves the area of a triangle. Here, users drag the vertex of a triangle along a line parallel to its base. Users may also drag the base of the triangle to investigate different initial configurations of triangles. This sketch illustrates the principle of shearing, and the constancy of the length of the height and of the base as the shape of the triangle changes (see Figure 3).

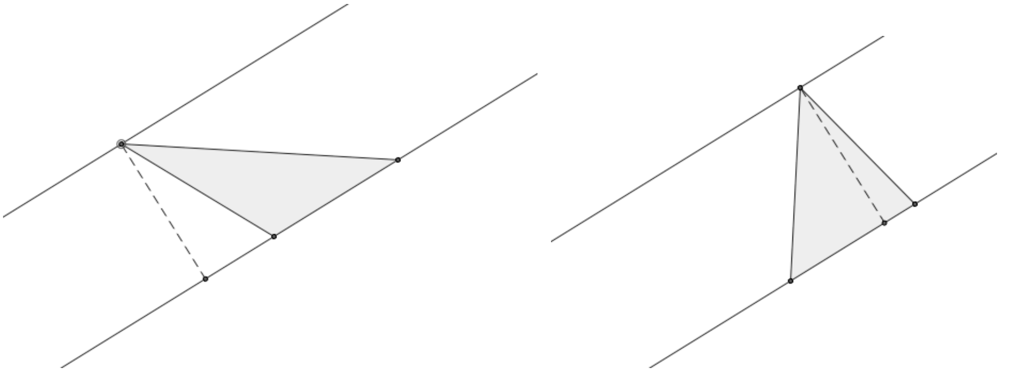

Figure 3: Shearing a triangle

The third sketch extends the second one, showing the rectangular product of the three pairs of bases and heights of a triangle. As users drag the vertices of the triangle, these rectangles change shape, not always looking as equal in area as we might expect (see Figure 4).

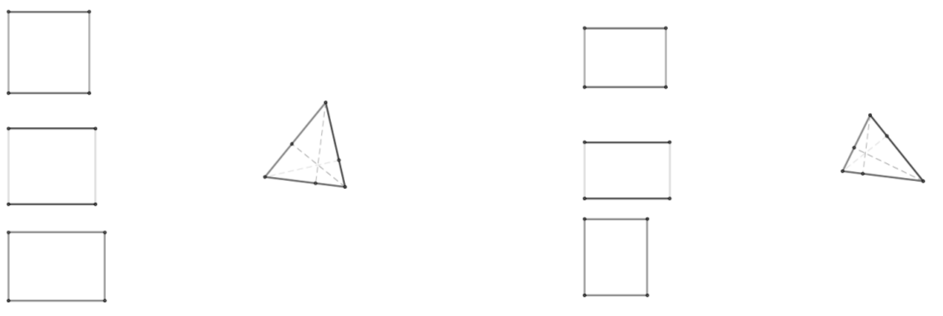

Figure 4: The rectangular areas associated with two different triangles

The fourth sketch relates to more arithmetic ideas. It contains a number line on which two points have been placed, as well as their product. Users can drag either of the two points (or both at once) and observe what happens to the product. Here, the specific values of the multiplier, multiplicand and product give way to more general, global aspects of the behaviour of multiplication: what happens as the multiplier decreases? As both multiplier and multiplicand increase? As they travel between 0 and 1? (See Figure 5.)

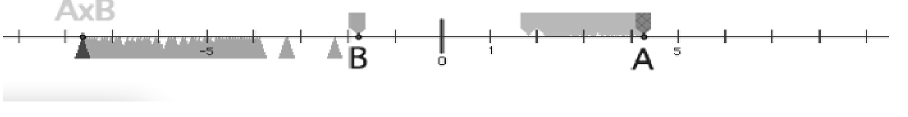

Figure 5: Using the dynamic number line for multiplication

Participants engaged enthusiastically with these constructions, on occasion discovering some generalities they had not seen before. During the closing reflections on our experiences for the day, we discussed the perceptual status of these dynamic diagrams: whether we could or should see them as many specific examples of the same thing, or as one thing continuously changing. These different possibilities would seem to change the relationship between the general and the particular that a sketch can evoke. We also commented on the extent to which attending to the behaviour of objects and relations allowed for ignoring the particular, and that this might have interesting pedagogical consequences.

The reader of the diagram infers its intent, the intention is carried by the semantics. I require my interpretation to make sense of a diagram. The syntax of symbols and diagrams of text all differ. Teachers value hand-made diagrams because they reveal what the student emphasizes by what is made accurate (see Candia Morgan's book *Writing Mathematically*).

Day 3

We began the final day by examining examples of images and generality from middle and secondary school textbooks related to the area of triangles, Pythagoras's theorem and the quadratic formula, in order to examine how contemporary pedagogic texts handled issues of generality and particularity, as well as the manner of invoking the use of diagrams. The text sources were several pages taken from Nelson *Mathematics* 11 (p. 50), Addison-Wesley *Minds on Math* 11 (pp. 319-320 and 481), College Preparatory Mathematics *Algebra*

Connections (p. 358) and *Foundations for Algebra* (p. 117), Chenelière *Interactions 8* (p. 252).

Some of them gave a worked example followed by a formula, some gave no worked example and just the formula and its algebraic generation, some had several examples and placed the generalised formula (here, the area of a triangle) in a box away from the main text, some focused on empirical derivations of formulas without giving any mathematical justification. There seemed to be a tension or struggle for the reader's attention. Are the particulars arbitrary, uninteresting or irrelevant? Are the diagrams added as 'extra'? How is the diagram tied to the text? What is the dialectic between them?

The two previous days focused discussion and attention rendered the pedagogic choices made by each set of authors/publishers quite apparent. We ended the meeting by asking participants to write down what had remained with them from the days before, and then had participants offer a few brief instances to the group.

One observation was given as a sutra, related to the experience of working on the Babylonian problems: *split the centre, re-establish the balance.*

Another concerned the issue of using the static in the dynamic: is there an oxymoronic quality in speaking of a 'dynamic' image? Is the default for 'image' static? This led to the question of whether too many examples (the 'cheapness' in some sense of generating examples in *Sketchpad*) distracts from or perhaps could even destroy a sense of generality. Is the particular or the general the more fragile? Is a static diagram something that can be played with? Various related continua or named dualities were offered: static/dynamic; discrete/continuous; specific/general. Are we looking for invariance across particulars? Is the static unremarkable? Or completely astonishing?

Is it all movement, flux? We explored this tension with regard to simple geometric objects.

- A line is an aggregation of points versus a point is called into being by the intersection of two (or more) lines.
- When I see a line, I see nothing but points versus when I see a point, I see nothing but intersecting lines.

We looked for parallels to the claim 'the concrete is the abstract made familiar by time': the particular is the general made singular in time? Do we experience particulars before generals (seriality versus gestalt)? What would constitute an act of the general? How might we synthesise particular and general?

The static diagram stays put, the text take me through. We construe the moving as animation, signs of life. It moves without moving itself: the figure is asleep, the diagram is dead. We kept coming back to the question of time in this context.

In her remarkable book *Wisdom and Metaphor,* Jan Zwicky (2003) remarks:

> *To realize it could be any right triangle, any square, is to experience the beauty of a mathematical truth. To grasp a geometrical truth is to grasp a gesture that is meaningful in an enormous array of contexts – in fact, all that are available to the spatial imagination.*
>
> *The experience of beauty is the experience of some form (or other) of relief from time. (p. 71 left)*

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Topic Sessions

Séances thématiques

Virtual Problem Solving Opportunities to Meet the Needs of the Net Generation: Knowledge Building, Knowledge Sharing and Being Part of the Community

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Introduction

Solving mathematical problems online is becoming one of the popular Internet supported learning opportunities for many schoolchildren. Our 5-year long experience of running a Problem of the Week project CASMI (Communauté d'Apprentissages Scientifiques et Mathématiques, www.umoncton.ca/casmi) is one of many examples of such opportunities that may potentially enrich students' mathematical experiences. At the CMESG-2007 Meeting, we shared our findings with members of Working Group A, *Outreach in Mathematics: Activities, Engagement and Reflection* (Freiman, 2007). During our 2008 topic session, we extended our reflection to a broader view of online mathematics learning that seems to catch the motivation of a new generation of students that is called the Net Generation.

The Net Generation is a relatively new concept in the field of educational studies. It designates a generation of young learners that has grown up with computers, the Internet and interactive multimedia tools. Using extraordinary abilities to adapt to all new tools that are constantly arriving on the market and turn them into specific social networks, they expand their learning space beyond the walls of the traditional classroom. Blogs, wikis, web- and pod-casting are just few of many examples of new ICT tools available for knowledge building, knowledge sharing and socialization. Are we, mathematics educators, ready ourselves to meet the learning needs of these students, to adjust our teaching to their learning styles and turn their natural interest and motivation in technology into an additional support for meaningful mathematics learning?

While an important body of research reveals potentially rich learning opportunities that are provided by technology, little is known about their effect on learning outcomes and how to integrate them in effective teaching practices. In our article, we will examine this issue in more detail. Our aim is to identify the characteristics of this new generation, its learning style and learning needs. We will then look at several examples of new mathematical activities initiated for and/or by these students using new Internet tools. We will also look at some data we collected within a CASMI project to analyze students' and teachers' perceptions of online problem solving opportunities to see how it may enhance mathematical learning. New paths of research and practice will conclude our analysis.

What is the Net Generation?

From a historical perspective, information and communication technology (ICT), the use of computers, multimedia and internet, is a relatively recent tool of teaching and learning. However, researchers have remarked that there is already a new generation of learners that is growing up digital (Tapscott, 1998). This generation is living in a new, networked, visually rich, digitally constructed communication and information world in which everybody has democratic access to a variety of resources that are the same for a 13-year-old ordinary youngster as for an adult multimillionaire, once both of them have an access to the Internet.

Furthermore, the Net Generation’s way to talk is also different. According to Roos (2007), to someone who is over 30-years-old, talking means speaking to a colleague on the phone or chatting face-to-face with a friend. To those born after 1982 – also known as the Net Generation – the word "talk" takes on a slightly different meaning. Instead of talking on the phone using their voice, they would rather use a 10-digit cell phone text message, do an e-mail exchange, and initiate an instant message conversation or a message board discussion. The Net Generation has grown up in a world full of communication gadgets and software; they don't even see these tools as technology.

How Do They Learn?

From a point of view of learning, the Net Generation is often characterized by authors as an autodidactic generation. These students do not need mountains of the step by step manuals for ‘dummies’ but rather short visual explanations. The *Homo Zapiens* (another term used to designate a Net Generation representative) focuses on activity not on technology. While doing schoolwork, she does several tasks simultaneously while shifting her attention from one project to another. Not surprisingly, the way these students learn is different as they prefer receiving information quickly, are adept at processing information rapidly, prefer multi-tasking and non-linear access to information, have a low tolerance for lectures, prefer active rather than passive learning, and rely heavily on communication technologies to access information and to carry out social and professional interactions (Veen & Vrakking, 2006).

According to some available data, a Net Generation student is a strong visual learner, but usually a weaker textual learner. While she puts high priority on speed, the result sometimes lacks depth and critical thinking. The ‘Net-Gener’ does not limit her actions by only downloading information; being rather an experiential learner, she learns by discovering and by doing, thus creating new information (like making movies instead of downloading movies). This type of learner is an excellent collaborator and likes to work in groups (Veen & Vrakking, 2006; Gokhale, 2007).

Pletka (2007) also emphasizes the role of multitasking fast-paced visually oriented environments in which the Net generation student would adapt and discover information where it can be accessed randomly in associative contexts rather than in step-by-step linear ways. She uses a variety of technical skills and competencies to personalize the digital world for her needs.

Are We Ready to Teach Them?

Several authors point out the generation gap as they talk about teachers’ readiness to meet the needs of the Net generation. Among them, Prensky (2001) analyses a dichotomy between *digital natives* versus *digital immigrants*. According to him, *digital natives* "speak the language" of technology fluently and spontaneously. They navigate the virtual and physical world seamlessly. *Digital immigrants* may share some of the characteristics of the Net Generation – preference for e-mail, Google and buying tickets online – but they'll always speak "with an accent". They would not expose their daily thoughts and emotions on a blog. Neither, would they write a report, instant message to six friends and watch TV at the same time.

What impact does this difference between a digitally native (student) and a digital immigrant (teacher) have on mathematics teaching and learning? More precisely, how do we teach mathematics to a digital native? Do we still use books? Do we use paper and pencil? Do we use explanation and ask them to take notes? Do we guide them? Moreover, do they actually need us? Maybe we just leave them on their own? To answer those questions, we need more research evidence and more practical results. For the moment, we will briefly analyze some examples of Internet tools and environments used by the Net Generation to see what kind of mathematical activities they may generate.

Web 2.0 Tools and Their Use in Mathematics

Solomon and Schrum (2007) use the year 2000 as a turning point in the development of a new Internet based technology called Web 2.0. They begin their timeline with year 2000 when the number of web sites reached 20,000,000. The year 2001 was marked by the creation of *Wikipedia*, the first online encyclopaedia written by everyone who wanted to contribute to the creation of the shared knowledge. In 2003, the site *iTunes* allowed creating and sharing musical fragments. In 2004, the Internet bookstore *Amazon.com* allowed buying books entirely online. In 2005, the video sharing site *Youtube.com* appeared, allowing producing and sharing short video sequences. The authors state that by the year 2005, the Internet had grown more in one year than in all the years before 2000, reaching 1,000,000,000 sites by 2006.

The result of this tremendous growth of internet-based environments and the educational resources generated by them is a transformation of e-learning itself. According to O'Hear (2006), the traditional approach to e-learning was based on the use of a Virtual Learning Environment (VLE) which tended to be structured around courses, timetables, and testing. That is an approach that is too often driven by the needs of the institution rather than the individual learner. In contrast, the approach used by *e-learning 2.0* (a term introduced by Stephen Downes) is 'small pieces, loosely joined', as it combines the use of discrete but complementary tools and web services – such as blogs, wikis, and other social software – to support the creation of ad-hoc learning communities. Let us look at several features of these tools as we analyze a few examples of mathematical opportunities they create.

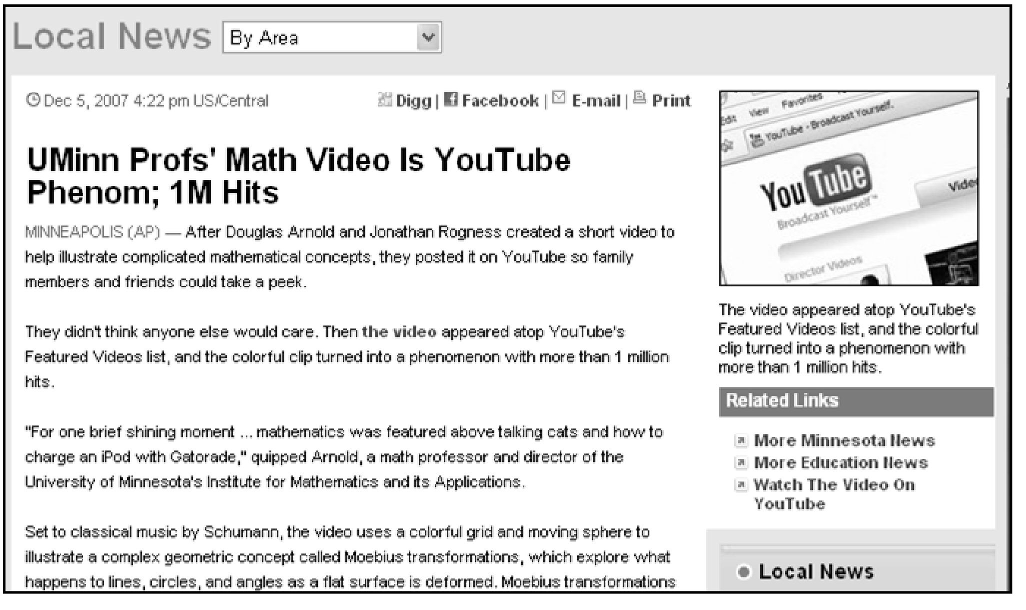
Wiki is an Internet tool allowing a collective writing of different texts as well as sharing a variety of information. Everybody can eventually be a contributor to the creation of a web site on a certain topic (or several topics, as it is in the case of the Wikipedia, www.wikipedia.org/). Let us look at an example of a wiki related to mathematics: http://cmath.wetpaint.com/ This site, about combinatorial and recreational mathematics, invites everyone to contribute to its development. The menu item *EasyEdit* on the home page allows writing and modification with simple word processing functions. Another function, *EditTags*, helps authors to create keywords, facilitating searches of the useful information by other members. The page can be e-mailed to everyone using the *E-Mail* function, thus informing others about changes made on the site. The special menu makes the wiki a genuine virtual community of learners: people can join the community, share news, photos, or participate in the discussion Forum. Each contribution is rated and the top 5 contributors are listed. It motivates participants to be more active in a co-creation of knowledge.

Podcasts can be used to audio-share mathematical knowledge among a larger auditorium than one with people sitting in a traditional classroom. It can be used as a method of delivering mathematical lectures online as well as for the promotion of mathematics as is done on the

Mathematical Moments of the American Mathematical Association site: (http://www.ams.org/mathmoments/browsemoments.html?cat=all)

Mathematical Moments features a series of posters to help the public discover the World of Mathematics. It contains ready-to-print PDF files on many different topics in science, nature, technology, and human culture, some of which have been translated into other languages. There are also many podcast interviews with experts in the field. Playing a Media Player file (MP3), everybody can listen to the podcast, download its PDF text version, as well as browse related resources.

Video-casting opportunities are provided by multiple Internet sites, allowing the creation and sharing of video sequences produced by the users. For example, an article published in one local newspaper informs the readers about one university professor who put a 2-minute video about a *Moebius strip* on the *youtube.com* site. The sequence was viewed by more than 1 million users within 2 weeks, a phenomenon unthinkable without the technology – no one university professor can reach that many students during a whole career (see Figure 1).



Local News By Area

Dec 5, 2007 4:22 pm US/Central

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UMinn Profs' Math Video Is YouTube Phenom; 1M Hits

MINNEAPOLIS (AP) — After Douglas Arnold and Jonathan Rogness created a short video to help illustrate complicated mathematical concepts, they posted it on YouTube so family members and friends could take a peek.

They didn't think anyone else would care. Then the video appeared atop YouTube's Featured Videos list, and the colorful clip turned into a phenomenon with more than 1 million hits.

"For one brief shining moment ... mathematics was featured above talking cats and how to charge an iPod with Gatorade," quipped Arnold, a math professor and director of the University of Minnesota's Institute for Mathematics and its Applications.

Set to classical music by Schumann, the video uses a colorful grid and moving sphere to illustrate a complex geometric concept called Moebius transformations, which explore what happens to lines, circles, and angles as a flat surface is deformed. Moebius transformations

The video appeared atop YouTube's Featured Videos list, and the colorful clip turned into a phenomenon with more than 1 million hits.

Related Links

More Minnesota News

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Watch The Video On YouTube

Local News

Figure 1: Local newspaper informs readers about a mathematical video on youtube.com

The environment offers not only an opportunity to view the video, but also to assess it (using a 5-star system) and to share it with others, as well as publish a comment. Among the comments found on the site, one comes from a 13-year-old user who writes that '*the answer you are trying to prove is wrong*' and tries to give her own explanation saying, however, that she '*hasn't figured it out yet, but that's my guess.*' This comment illustrates how democratic access to mathematical ideas is provided by the technology to everyone disregarding the school or ability level.

Another interesting comment comes from a Net generation student who says that the video explained in a simple way something that 99% of teachers wouldn't be able to do in two hours. This comment illustrates one of the above-mentioned characteristics of net generation students who look for short and simple explanations.

Photo-sharing is yet another form of creating and sharing knowledge, available on several dynamic sites with photo galleries like Flickr (www.flickr.com – see Figure 2).

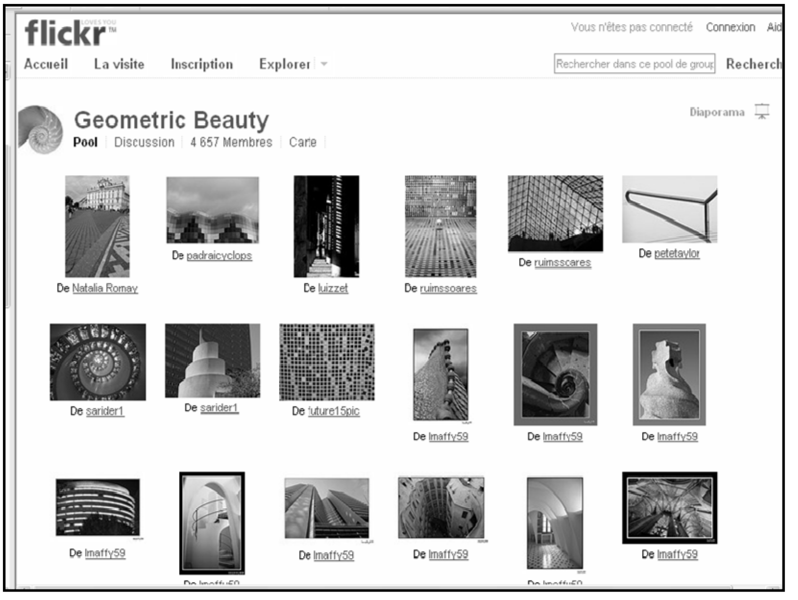

Figure 2: Pictures of Geometric Beauty on Flickr

Regrouped by categories that can be found by an easy-to-use search engine, the photos can be published and discussed by the members of a community, as for example, the community that discusses *geometric beauty* which numbers almost 5000 members. Each photo is provided with a kind of ID card that documents useful information such as the date of its publication, the author's (or publisher's) username, as well as the list of all other categories to which the photo belongs, the date when the photo was taken, and how many other users added it to their albums.

<u>*Discussion forums*</u> allow building online communities that talk to each other by posting questions and giving answers. This collective work may enable a student who is struggling with mathematical homework to address other people and ask for help, as illustrated by the following example from the Math Forum site (mathforum.org). The message posted by one user says that '*after having asked a teacher and having read a book*', she '*still had a feeling'* that she needed more explanation, so she appealed to the whole virtual community asking for help. The discussion on some questions can take the form of multiple exchanges between members.

<u>*Blogs*</u> may provide multiple educational opportunities as they are built by means of easy-to-use software that removes the technical barriers to writing and publishing online. The 'journal' format encourages students to keep a record of their thinking over time facilitating critical feedback by letting readers add comments – which could be from teachers, peers or a wider audience. Students may use blogs for different purposes: to provide a personal space online, to pose questions, publish work in progress, and link to and comment on other web sources. For example, the following record found in a blog written by a student contains her comments on the web site CASMI. The student writes (in French, see Figure 3) that '*CASMI is a useful site; two mathematics teachers are using the site for teaching and learning many things. I suggest you try the link below to explore the site'*. The empty dialogue windows below the message prompt all readers to write a comment on this message.

Figure 3: Message in a school blog written by a Net Gen student

The learning model that can be extracted from our examples features three major educational trends related to the web 2.0 technology: knowledge building/co-constructing, knowledge sharing, and socialization by interaction with other people.

Virtual Problem Solving Community

The Web 2.0 tools examined in previous sections provide us with several examples of knowledge building and knowledge sharing. According to Jonassen et al. (2008), social software which is at the heart of Web 2.0, *'enables people to unite to collaborate through computer-mediated communication and to form online communities'* (p. 101). In such a virtual community, potentially rich learning may emerge. In the description of the key-elements of a Community-Based Online Learning model, Palloff and Pratt (2007) put emphasis on people who are communicating and interacting while being actually 'present' online. The purpose of such communication and interaction is being established by mutually negotiated guidelines based on practical considerations (including privacy, security and ethical norms). The process of learning is reflective, transformative and constructivist. According to the authors, only a combination of those three equally important elements, namely, people, purpose and process, creates optimal learning conditions for reaching such outcomes as co-created knowledge, meaning, reflection, transformation, increased self-direction, and finally, reinforcement of presence.

In mathematics, several virtual communities have proved their vitality and rich learning potential. In the *Math Forum*, for example, the culture (or norms for interacting) has been developed across services. According to Renninger and Shumar (2004), it includes '*(a) assuming that participants are using the site to learn or figure something out, (b) accepting at face value what a person says about both their interest and understanding, and (c) using an inquiry approach of questioning, exploring, and modeling in order to enable the participant to understand'* (p. 184). In order to ensure interactivity, such online services as *Ask Dr. Math* and *Problem of the Week* have been created. They are monitored by a team of specially trained volunteers that provide each participant with some personally adapted clues, making her work through the problem instead of giving an answer right away. Moreover, the authors insist on the importance of combining interactivity with a rich mathematical content that *'has depth and breadth'* (Renninger & Shumar, 2004, p. 185).

Another example of a virtual mathematical community is *L'agora de Pythagore* which is built up around philosophical mathematical questions that are being discussed online by 10- to 15-year-old schoolchildren. Its common goal is posted on the home page. Working within a group that treats some particular mathematical question (like '*Does chance exist*?'), each participant is invited to contribute, producing her own ideas and/or reacting to ideas of the others. The rules of interaction are also clearly announced (like '*each participant has to sign the post with her real name*' or 't*he person involved in the discussion must react within 7 days*'). In the discussion, the student has a particular role of not only consumer of a particular knowledge as established fact but rather one of actor, so creator of a new knowledge that is becoming shared and discussed with other community members (Pallascio, 2003).

Our experience with developing and implementating a virtual community on mathematical problem solving, CASMI (www.umoncton.ca/casmi), has also shown that mathematical reasoning and communication can be fostered by means of rich and complex problems that have been posted online since October, 2006 (Freiman & Lirette-Pitre, 2009). Daily statistical monitoring by *E-STAT* (http://persos.estat.com/) between September 2007 and August 2008 shows that more than 300 000 pages have been viewed during more than 30 000 visits from about 30 000 visitors. About 150 problems split into 4 levels of difficulty have been posted online through this period and more than 20 000 solutions have been received electronically from more than 8000 K-12 students as well as more than 1500 teachers and pre-service teachers. The community has constantly been growing since that time. Each solution sent through the web site is being individually analyzed by mentors (university students) and kept in a member's e-portfolio. Small communities can be organized by teachers and their students, so the access to the portfolios can be shared. Moreover, the members can use the whole database of all problems accompanied by a general comment and examples of the most interesting solutions. Some problems are available for discussion through the discussion Forum. Each member can propose a new problem that can be shared with other members.

The community generates other activities through different partnerships with other communities (like MATHENPOCHE, http://mathenpoche.sesamath.net/) or organisations (like the *Canadian Mathematical Society* that helped to organize *The 1st Virtual Mathematical Marathon* in 2008). Two collaborative projects with local schools on robotics and mathematical giftedness allowed creating new problems with students or developing learning scenarios (including virtual ones) with MINDSTORM Lego Robots (http://cahm.elg.ca/archives/robomatic/).

The following example of electronically submitted solutions to a mathematical problem illustrates several features that are not often found in paper-and-pencil mathematical solutions but are rather typical for a participative web communication. The first solution addresses the CASMI team ('*Bonjour à toute équipe*') and finishes with thank you words ('*Merci à toute l'équipe au revoir*') and is accompanied by 'smilies', emoticons expressing emotions (see Figure 4).

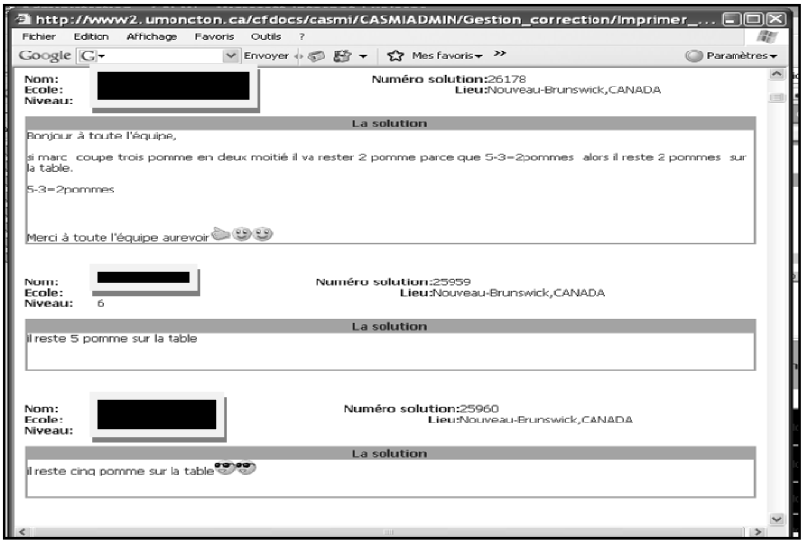

Figure 4: Electronically submitted solutions

From the interviews we conducted with students and teachers (Freiman, 2008), we have learnt about several characteristics that make problem solving virtual communities potentially rich for building up solid mathematical knowledge and skills.

Both students and teachers underline 'challenge' as an important component of a good mathematical problem (Student - St.) '*Problem has to be a problem*'; (St.) '*Sometimes problems are easy to solve, sometimes they are very difficult, you have to think a lot in order to solve them*'; (Teacher -T.) '*Good variety of mathematical content; complex enough to challenge the students'*. The participants argue that the participation in such online projects increases students' motivation to learn mathematics: (T.) '*No need to motivate students to solve problems*'; (St.) '*Solving problems online is much more motivating than having them from the textbooks of 70-s'*. The creativity has been also mentioned in teachers' responses: (T.) '*I was pleased getting with one student who was telling me a very original way to solve a problem*'; (T.) '*Students like discussing CAMI problems and different ways of solving them, as well as differentiation*'; (T.) '*Giving a choice to the student';* (T.) '*Each student can move at her own pace through the problem solving process'.*

For both teachers and students, the feed-back given by mentors is important for supporting learning: (St.) '*It is excellent that someone analyses our solutions and gives us a feed-back*'; (St.) '*The comments I get help me to become a better problem solver*'; (T.) '*Students are very proud when they get a positive comment and like to share it with them: It is like they have won a cup in a competition*'. Also, according to the teachers, participation in the project helps students to go beyond the curriculum: (T.) '*One of my students has to use a Pythagorean Theorem that we didn't study before; she found it on the web and presented it to the rest of the group*'.

Our findings lead us to several conclusions about overall excitement of teachers and students about the site as a source of rich and challenging mathematical problems that allows bringing informal elements into the classroom routine, ensuring appropriate depth and breadth of the curriculum, and at the same time allows going beyond the curriculum. It lets us suggest that virtual problem solving communities may provide an important learning resource for the NET-Generation, leading to their better involvement in mathematical activities and thus increasing the potential of the virtual community as community of learners.

These conclusions are compatible with those made by other authors who see a positive impact of participative educational environments supported by Web 2.0 technology:

1. Computer supported collaborative learning environments based on model-eliciting problems do provide a rich context for mathematical knowledge building discourse (Nason & Woodruf, 2004) with a variety of rich online resources (Renninger & Shumar, 2004).
2. Interactivity and communication about mathematical problems are the key advantages of technology in fostering making connections, engaging into questioning and finding solutions, as well as working with challenging problems (Renninger & Shumar, 2004), and helping students to gain a wide appreciation of mathematics (Jones & Simons, 1999).
3. Communication and discussion is seen as help for in-depth explorations (Pallascio, 2003) that complements other new opportunities for learning by means of tools that reinforce cognitive development (Rotigel & Fello, 2004; Depover et al., 2007).
4. Altogether, online collaboration fosters the generation of new knowledge, initiative, creativity, and critical thinking (Pallof & Pratt, 2007).

Overall, an active participation in virtual mathematical experiences may help to turn the natural motivation and interest into meaningful mathematics learning, increase opportunity for enrichment and collaboration, thus supporting an emergence of a new learning culture. However, more empirical evidence is needed to evaluate the outcome of virtual learning from (meta)-cognitive perspectives, affective perspectives, and social perspectives.

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Towards the 2009 Canadian Mathematics Education Forum

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France Caron, *Université de Montréal*
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In preparation of the fourth meeting of the Canadian Mathematics Education Forum (to be held in Vancouver, April 30 to May 3, 2009), a session was held at the 2008 CMESG meeting to present the objectives of the CMEF 2009, its organizing themes, and the various working groups that had been set up. This session was also an opportunity for participants to contribute further ideas or suggestions for the activities and working groups of the forum.

The Purpose of the Forum

The Canadian Mathematics Education Forum (CMEF) has met three times: in 1995 (Quebec City), in 2003 (Montreal, cosponsored by CMS and CRM), and in 2005 (Toronto, cosponsored by CMS and Fields). During the 2005 forum, the question arose whether we should continue with these meetings. Since we have the semi-annual CMS Education sessions and the annual CMESG meeting, what is the purpose of the CMEF? After extended discussions and a close look at the successes and recommendations of the three CMEFs, it was decided that the Forum should continue, to maintain valuable collaboration between school teachers, university/college mathematicians and math educators, and that it move to a 4-year cycle. For the 2009 CMEF, we proposed that such collaboration should start much earlier than the Forum itself, based on expressed needs and existing initiatives.

The Theme of the Forum

Almost every province is struggling with mathematics curriculum reform. It is generally felt that "getting it right" is a long-term process that requires sustained work, continuous partnership with teachers from design to implementation and adjustments, consultation with mathematicians and mathematics educators, support for teacher professional learning, access to rich resources, manageability and coherence of assessment policies and practices.

"Curriculum" in its many dimensions emerged as a recurring theme in the previous two CMEFs. With a view to address some of the main concerns and challenges that were expressed there, it was decided to have the 2009 Forum focus on *the ways in which resources and assessment define, inform and mould curriculum.* This objective requires the participation and collaboration of people involved at the many relevant constituencies: the school systems, teachers at all levels, coordinators, school boards, colleges and universities, mathematics and statistics departments, faculties of education, Ministries of Education, parent groups, and business and industry.

CMEF 2009 is cosponsored by the Canadian Mathematical Society (CMS) and the Pacific Institute for the Mathematical Sciences (PIMS).

Working Groups

A primary purpose of the Forum is the development or sharing of concrete materials and resources (booklets, modules, examples, web-pages) to support mathematics education at all levels. A second objective is to engage in a national discussion on the important role of assessment in teaching and learning mathematics.

To facilitate this, a call went out in early 2007 for project proposals. In response, more than 40 proposals were received, some already well developed and ongoing, and others at the idea or design stage. With the objective of promoting collaboration of participants with like interests and complementary perspectives, these projects were grouped into a total of thirteen working groups for the 2009 Forum. A list of the working groups and the contributing proposals is found at the end of this report.

The 2008 CMESG Topic Session

A brief presentation was followed by extensive questions and discussion, focused for the most part on the question of what activities could be expected at the meeting itself, how the participants would be chosen, and what sources of funding might be available. Other questions dealt with the operation of the working groups over the following year. How was this to be facilitated, particularly for those groups which were spread out geographically?

Program. The current draft of the program was presented to the participants of the topic session. We began with the presentation of the two invited plenary speakers, and mentioned the possibility of an additional Public Lecture in conjunction with the meeting of *Changing the Culture*, a 1-day conference sponsored by PIMS that has met annually at SFU Vancouver since 1998. We solicited ideas for topics of panels and discussion groups, some of which could run in parallel. Building on a much appreciated feature of the CMESG meetings, we expressed our intention of providing opportunity for *ad hoc* sessions. We described the format for the working groups, which will form a major component of the program, receiving around six hours of the schedule. Working group meetings will also be attended by delegates who are not part of any group, but who will have an opportunity to contribute, or simply to get a sense of what’s happening in that area. During these meetings the working groups would have a chance to consolidate and/or validate their work to date and discuss possible next steps.

Participants. There should be around 200 participants from various levels of the education system. The working groups will be invited in December to submit a list of names of those who wish to attend the forum as part of the work of the group. Prior to that, it will have been made possible, through the Forum website, for people interested in joining an existing group to indicate their interest to the contact person for that group. Other delegates will be invited to ensure strong participation of teachers and fair representation of all levels, regions, and linguistic communities. To this end, it was suggested by some participants of the session to contact the French immersion schools in the Vancouver area.

Working together. Various mechanisms have been put in place (mainly on the web or through e-mail) within the different working groups for setting up their own collaborative environment. Some meetings (either physical, video, or audio) have been held to help launch the work.

It was announced that the CMEF website, hosted by the CMS, would have a link to an active working group site at Queen's which would provide greater flexibility for updates: http://www.cms.math.ca/Events/CMEF2009/

Working Groups

- Assessing for Problem Solving Development
- Mathematics for Elementary Teaching
- Rethinking Assessment
- Online Learning
- Transition to University
- Mathematical Modeling and Science
- Philosophy of Mathematics
- Aboriginal Ways of Knowing Mathematics
- Problem Solving in Secondary Mathematics
- Problem Solving in Elementary Mathematics
- Significant Statistics
- Early Childhood Geometry
- Textbook Design

Les flocons de neige au service des mathématiques

Marie-Pier Morin
Université de Sherbrooke

Introduction

Dans le cadre de notre travail comme professeure à la formation des futurs maîtres du primaire, nous enseignons la didactique de la géométrie à l'aide du logiciel de géométrie dynamique Cabri-géomètre. Nous souhaitons ainsi amener les futurs enseignants vers une meilleure conceptualisation et une meilleure compréhension de la géométrie. De façon plus spécifique, nous abordons notamment le thème des quadrilatères et celui des transformations géométriques à partir de ce logiciel. Étant donné que les futurs enseignants sont appelés à exploiter les technologies de l'information et de la communication (TIC) dans leur enseignement, l'accueil réservé à ce logiciel est très positif. À titre d'exemple, après avoir travaillé l'inclusion des quadrilatères avec les futurs enseignants, plus de la moitié d'entre eux (N=43) ont affirmé que l'utilisation de Cabri-géomètre leur a apporté un éclairage nouveau sur l'étude des quadrilatères, et les trois quarts pensent que l'utilisation de ce logiciel pourrait favoriser une meilleure compréhension des concepts de base en géométrie. Toutefois, les futurs enseignants sont souvent réfractaires face à l'utilisation d'un logiciel de géométrie dynamique avec les élèves du primaire. Leurs propres difficultés en mathématiques peuvent expliquer ces réticences. En effet, pour 28 % des futurs enseignants interrogés, il n'est pas possible d'utiliser ce type de logiciel en classe : *« Si moi-même j'ai eu des difficultés, je crois que les élèves auraient aussi de la difficulté ».*

De façon à montrer aux futurs enseignants qu'il est possible d'utiliser Cabri-géomètre à l'école primaire, nous travaillons depuis cinq ans à l'élaboration et l'expérimentation de séquences d'enseignement-apprentissage avec des élèves du 3^e^ cycle du primaire (11-12 ans). Ce travail est réalisé en collaboration avec Carole Morelli, conseillère pédagogique à la Commission scolaire des Hauts-Cantons et Jocelyne Lambert, conseillère pédagogique à la Commission scolaire Marie-Victorin. Au plan organisationnel, le travail se fait en équipes, soit dans le laboratoire informatique de l'école, soit directement en classe, avec des ordinateurs portables. Cette contribution a pour but de présenter une partie du travail accompli depuis 2003 avec les élèves du primaire. Contrairement au travail que nous réalisons avec les futurs enseignants, cette démarche ne s'inscrit pas dans le cadre d'une recherche et est davantage réalisée à titre exploratoire.

Cabri-géomètre à l'école primaire

Activités de prise en main

Lorsque nous utilisons Cabri-géomètre avec les élèves du primaire, étant donné qu'ils n'ont généralement jamais utilisé de logiciel de géométrie dynamique, les deux premières leçons sont des activités de prise en main. Dans le cadre de ces activités, nous visons l'utilisation de certains outils de base tel le point, le segment, la droite, la droite parallèle, la droite perpendiculaire, le cercle et le polygone régulier, à partir de la construction d'un bonhomme allumettes (figure 1) et d'un train (figure 2). Pour que les élèves saisissent bien comment utiliser ces outils, ils doivent suivre des étapes de construction prédéfinies, lesquelles ont entre autres comme but de leur faire prendre conscience que les constructions ne sont pas des dessins. Effectivement, une construction conserve ses propriétés géométriques lors de sa manipulation avec un logiciel de géométrie dynamique, car elle a été réalisée à partir d'objets géométriques élémentaires choisis avec soin. Un dessin quant à lui repose sur des éléments perceptifs qui ne résistent pas au déplacement dans le logiciel. Ainsi, par exemple, ces étapes montrent l'importance de construire les wagons du train à partir de droites parallèles et perpendiculaires, de façon à ce que les rectangles formant les wagons conservent leurs propriétés lors d'un déplacement. Comme en témoignent les productions obtenues, ces deux activités sont généralement très bien réussies par les élèves. Ces derniers sont très habiles avec le logiciel et démontrent une grande dextérité dans la manipulation des différents outils.

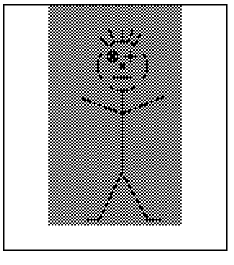


Figure 1

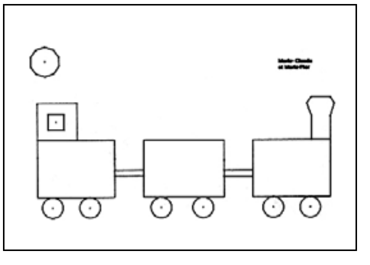
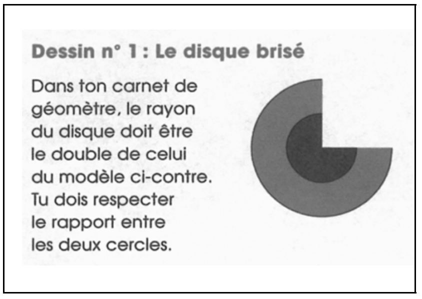


Figure 2

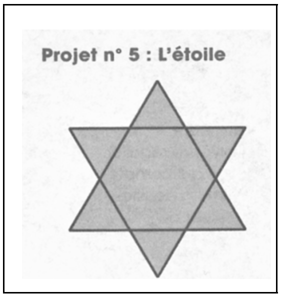
Si nous étions titulaire d'une classe du primaire, nous laisserions certainement plus de latitude aux élèves lors de cette prise en main, et ce, de façon à les laisser découvrir par eux-mêmes le fonctionnement de Cabri-géomètre. Toutefois, étant donné le temps qui nous est souvent imparti, nous devons nous contraindre à faire une prise en main plus dirigée.

Académie des géomètres

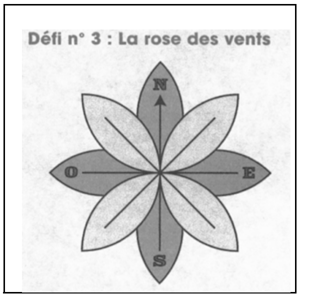
Après les activités de prise en main, pour faire ressortir la pertinence de travailler avec un logiciel de géométrie dynamique, nous laissons momentanément le travail à l'ordinateur et nous proposons une séquence d'enseignement-apprentissage élaborée par Lyons et Lyons (2004), l'*Académie des géomètres*. Par la réalisation de constructions géométriques faites uniquement à partir d'un compas et d'une règle non graduée, cette situation permet aux élèves de refaire les découvertes des géomètres de l'Antiquité. Par exemple, ils doivent découvrir comment faire un angle droit, un triangle équilatéral ou encore comment diviser un segment en quatre parties égales. Par la suite, toujours avec les mêmes instruments, ils doivent réaliser des constructions plus complexes dont, entre autres, un disque brisé, une étoile ou une rose des vents (figures 3, 4 et 5). Dépendamment du nombre de constructions faites par les élèves, cette séquence a une durée de deux à trois leçons.



(Lyons et Lyons, 2004, p. 143)
Figure 3



(p. 144)
Figure 4



(p. 145)
Figure 5

Quand les élèves montrent qu'ils comprennent bien comment utiliser la règle non graduée et le compas pour réaliser leurs figures, nous leur permettons d'utiliser l'ordinateur. À ce moment, ils découvrent l'intérêt de travailler avec Cabri-géomètre. En effet, si les constructions ne sont pas plus faciles à réaliser, parce que tout le travail conceptuel derrière les constructions reste le même, les constructions se font plus rapidement parce que les manipulations sont réalisées à l'ordinateur. Ainsi, une erreur de construction se voit plus facilement corrigée.

Séquence d'enseignement-apprentissage sur les flocons de neige

Une fois que les élèves ont saisi la pertinence de faire de la géométrie à l'aide d'un logiciel de géométrie dynamique, nous leur proposons une séquence d'enseignement-apprentissage dans laquelle ils sont appelés à utiliser Cabri-géomètre. L'activité que nous avons choisie de présenter ici s'est déroulée dans deux classes de 3^{e} cycle de l'école Louis St-Laurent à Compton. Cette séquence avait pour thème « Les flocons de neige ». Par cette activité, nous voulions utiliser les TIC non seulement pour travailler des notions mathématiques, mais aussi pour donner la possibilité aux élèves d'explorer un phénomène complexe et d'établir un lien entre les mathématiques et la nature.

<u>Activité 1 : Création d'un flocon de neige (1)</u>

Après avoir fait l'observation de flocons de neige dans la nature et à partir de photos de flocons agrandis (Libbrecht, 2004), nous avons amené les élèves à faire ressortir certaines des particularités du flocon, dont le nombre de branches toujours égal à six, la symétrie de chacune de ces branches, la symétrie du flocon même et le caractère unique de chacun. Par la suite, nous avons élaboré, en groupe, une démarche possible pour construire un flocon de neige et nous avons construit un flocon avec eux à l'ordinateur. Ce flocon de neige s'est fait à partir de la rotation d'un triangle équilatéral modifié (figure 6). Concrètement, nous avons fait un triangle équilatéral et nous en avons modifié un côté. Ensuite, en utilisant l'outil polygone, nous avons repassé sur tous les points de la nouvelle figure de façon à identifier le polygone créé. Enfin, nous avons fait faire des rotations de 60 ° à ce polygone, jusqu'à la création d'un flocon complet. Suite à cette construction collective, en équipes de deux, les élèves ont réalisé leur propre flocon de neige.

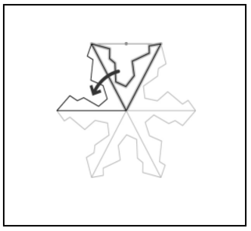


Figure 6

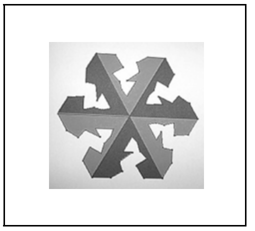


Figure 7

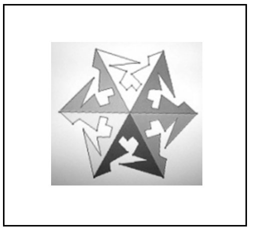


Figure 8

Activité 2 : Création d'un flocon de neige (2)

La deuxième séance de cette séquence a débuté par l'observation des flocons réalisés lors de la première activité. Nous avons alors interrogé les élèves quant à la conformité de ces flocons au regard des particularités énoncées au cours précédent. Ils ont alors pris conscience que les flocons construits n'étaient pas symétriques (figures 7 et 8). En conséquence, nous les avons questionnés sur une façon possible d'élaborer un flocon de neige parfaitement symétrique et nous en avons construit un avec eux à l'ordinateur. Brièvement, pour faire un tel flocon, nous avons séparé le triangle initial en deux parties égales à l'aide d'un axe de symétrie. Nous avons par la suite modifié la moitié du côté sur lequel passe l'axe de symétrie et nous avons fait la réflexion de cette moitié sur l'autre moitié. Pour terminer, tout comme le flocon élaboré à l'activité 1, nous avons identifié le nouveau polygone ainsi formé et nous lui avons fait faire des rotations de 60 º, jusqu'à la formation d'un flocon complet (figure 9). Enfin, les élèves ont construit leur flocon en équipes de deux (figures 10 et 11).

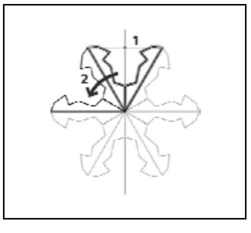


Figure 9

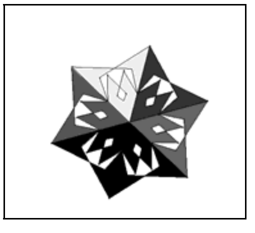


Figure 10

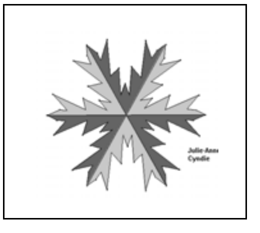


Figure 11

Nous aurions pu, dès la première séance, faire remarquer que les flocons n'étaient pas symétriques. Toutefois, comme c'était le programme de construction sur lequel les élèves s'étaient entendus, nous voulions les laisser aller au bout de leur démarche collective. Suite à l'évaluation du niveau de compréhension des élèves, nous pensons même que ce choix a fait en sorte que les apprentissages qui ont découlés de cette erreur étaient plus solidement ancrés.

Activité 3 : Création d'un dallage

À la dernière séance, nous avons proposé la construction d'un dallage de flocons de neige. Pour commencer, nous avons étudié les particularités du dallage à l'aide d'un dallage d'oiseaux (figure 12). Nous avons ainsi fait ressortir qu'il s'agit du recouvrement d'un plan à l'aide de polygones, sans espace libre ni superposition entre les polygones. Par la suite, en groupe, nous avons fait l'étude d'une démarche possible pour élaborer un dallage à partir de translations d'un flocon (figure 13) et nous avons réalisé un dallage. Nous avons choisi de fournir le flocon initial aux élèves parce que le but visé n'était pas qu'ils sachent faire un flocon qui peut s'imbriquer mais plutôt qu'ils réalisent un dallage. En effet, si nous n'avions pas donné le flocon initial, beaucoup de temps aurait dû être consacré à sa création et nous aurions ainsi eu moins de temps pour le dallage. Enfin, en équipes de deux, les élèves ont réalisé leur propre dallage.

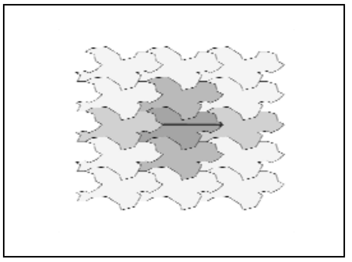


Figure 12

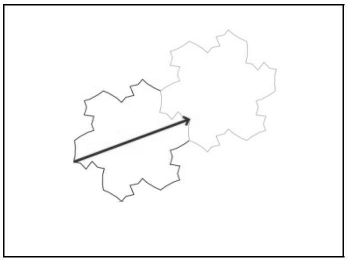


Figure 13

Il a été intéressant de constater qu'une fois la construction complétée, les élèves ont personnalisé leur dallage en y ajoutant de la couleur et en le transformant. En effet, à leur grande surprise, ils ont réalisé qu'en modifiant leur flocon initial, tous les autres flocons du dallage subissaient la transformation (figures 14 et 15) ! Il s'agit là d'un avantage considérable des logiciels de géométrie dynamique par rapport aux logiciels de dessin.

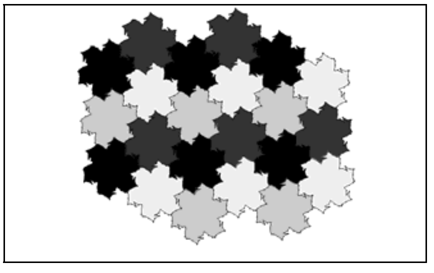


Figure 14

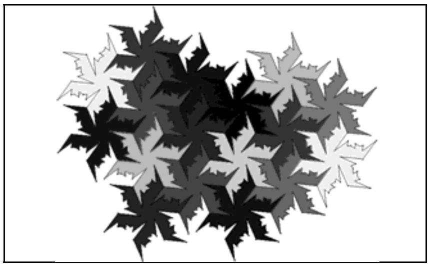


Figure 15

Rédaction des démarches de construction des élèves

Au cours des trois activités sur les flocons de neige, nous avons demandé aux élèves d'écrire leurs démarches de construction sur une feuille, parallèlement au travail à l'écran. Le but que nous poursuivions était double. D'une part, nous souhaitions soutenir le travail des élèves en les amenant à prendre conscience de leurs démarches. D'autre part, faire noter les démarches de construction était pour nous une façon de travailler au développement de la compétence disciplinaire qui consiste à *Communiquer à l'aide du langage mathématique* du Programme de formation de l'école québécoise (Gouvernement du Québec, 2001). Lors de l'étude des démarches, quatre niveaux ont pu être observés :

Niveau 1: Description négligée, très incomplète ou absente ;
Niveau 2: Description incomplète, inexacte ou ne permettant pas de comprendre la démarche de construction ;
Niveau 3: Description complète, mais comportant soit des oublis, des inexactitudes ou des imprécisions ;
Niveau 4: Description très complète, précise et détaillée.

Si nous avons pu remarquer ces quatre niveaux dans les productions des élèves, la majorité se situait aux niveaux 2 et 3. Nous pouvons penser qu'ils ont moins bien réussi cette partie parce qu'ils n'étaient pas habitués à cette façon de faire. Nous pouvons ainsi inférer qu'à la longue, cette production pourrait s'améliorer. Toutefois, ces niveaux de production peuvent aussi traduire un certain désintérêt pour cette partie que nous considérons importante afin d'éviter que les élèves ne se perdent dans des constructions non réfléchies. Effectivement, écrire les démarches de construction oblige à s'arrêter et à penser aux constructions avant de les réaliser.

Évaluation du dispositif

Après la séquence d'enseignement-apprentissage sur les flocons, nous avons fait un retour sur les apprentissages réalisés. Nous avons alors recueilli les commentaires des élèves et des enseignantes quant à cette expérience. Ce retour a montré une réelle appréciation de la séquence.

En effet, d'une part, lorsque nous avons questionné les élèves quant au travail réalisé avec Cabri-géomètre, plus des trois quarts (N=42) se sont dit très satisfaits, ce qui montre que même s'ils ont moins aimé certains aspects, ils sont contents de leur travail. Aussi, 92 % des élèves interrogés se sont dit très intéressés à utiliser ce logiciel de nouveau, ce qui se traduit par une vraie réussite pour nous. Ainsi, si cette expérimentation avait comme but d'intégrer les TIC dans l'enseignement des mathématiques au primaire, les résultats montrent qu'il est non seulement possible de le faire, mais que les principaux intéressés sont prêts à répéter l'expérience.

D'autre part, une rencontre avec les enseignantes a montré qu'elles sont satisfaites du dispositif de formation mis en place. Elles ont apprécié la séquence d'enseignement-apprentissage sur les flocons pour la motivation qu'elle a suscitée chez les élèves, et ce, malgré le fait que ces derniers étaient beaucoup moins enthousiastes à l'idée d'écrire leurs démarches de construction. Selon elles, ils n'aiment pas écrire et n'affectionnaient pas cette partie. Peut-être pour tenter d'atténuer le manque d'investissement à ce niveau, elles ont soulevé la proximité physique des élèves au laboratoire informatique, qui peut avoir rendu le travail d'écriture plus difficile.

Aussi, les enseignantes disent avoir apprécié la séquence d'enseignement et plus précisément Cabri-géomètre pour son accessibilité aux élèves en difficulté. Selon elles, la séquence élaborée était appropriée pour tous les jeunes. À cet effet, nous avons pu remarquer qu'il est facile de proposer une différenciation pédagogique pour l'enseignement de la géométrie avec ce logiciel. Tandis que ceux qui présentent des difficultés réalisent les tâches à leur rythme, il est possible de proposer des défis aux autres élèves, qui vont facilement au-delà des exigences demandées. Enfin, les enseignantes ont souligné que travailler avec Cabri-géomètre permet d'utiliser le langage mathématique avec plus de rigueur. Pour illustrer leur propos, elles ont pris l'exemple de la rotation. Lorsque les élèves effectuent la rotation d'un triangle avec Cabri-géomètre, ils doivent préciser autour de quel centre et selon quel angle faire la rotation. Ainsi, ils sont appelés à utiliser un vocabulaire qu'ils n'utilisent pas nécessairement lorsqu'ils effectuent une rotation de façon technique sur papier.

Conclusion

À la lumière de ces expériences, bien que nous soyons satisfaite du travail réalisé, nous pensons que certains changements sont souhaitables pour une exploitation plus profitable des TIC à l'école primaire. Ainsi, comme nous l'avons mentionné plus haut, nous sommes consciente que le fait d'utiliser des activités dirigées peut réduire l'activité mathématique des élèves. Dans des conditions idéales, il nous semblerait important de leur laisser plus de latitude, de façon à les laisser explorer plus librement les fonctionnalités de Cabri-géomètre. Toutefois, nous devons composer avec l'horaire des enseignants qui nous reçoivent dans leur classe, ce qui réduit souvent passablement le temps alloué.

Aussi, nous pensons que les activités avec le logiciel devraient être mieux intégrées aux activités de classe, afin que les élèves voient un meilleur arrimage entre les deux. Selon nous, la formation des enseignants et des futurs enseignants à l'utilisation des outils informatiques est un incontournable si nous souhaitons que les TIC soient mieux intégrées en classe. À cet effet, nous projetons, dans les années qui viennent, accompagner des enseignants en exercice afin de les aider à intégrer Cabri-géomètre dans leur enseignement de la géométrie au primaire et au secondaire. Ce choix est d'autant plus justifié que de façon générale, dans les pays industrialisés, on constate une sous-utilisation des ressources numériques dans les pratiques quotidiennes de la majeure partie des enseignants du préscolaire, du primaire et du secondaire (European Commission, 2006; Hennessy, Ruthven et Brindley, 2005). Il devient ainsi très pertinent d'amener des enseignants à intégrer un didacticiel puissant au regard du soutien à l'apprentissage disciplinaire de la géométrie dans leur enseignement. Ce propos vient appuyer une des recommandations faites par Kahane (2002) dans son rapport sur l'enseignement des mathématiques à l'effet qu'il faudrait faire plus de place aux TIC pour l'enseignement de cette discipline scolaire et, plus précisément, pour soutenir celui de la géométrie. Selon lui, l'utilisation des outils informatiques tels que les logiciels de géométrie dynamique peut favoriser le processus d'abstraction et aider l'élève dans ses apprentissages en géométrie puisqu'ils ouvrent un champ de possibilités qu'il est impossible d'obtenir dans le cadre d'un travail effectué uniquement sur papier.

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New PhD Reports

Présentations de thèses de doctorat

Innovative Methodology: Pre-Service Secondary Mathematics Teacher Knowledge

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The primary purpose of this study was to investigate the pre-service secondary mathematics teachers' knowledge in the context of logarithms and logarithmic functions. Particularly, it targeted the subject matter content knowledge and pedagogical content knowledge of pre-service teachers. Concurrent with those efforts, the study also focused on the development of the research methodology for the purpose of the collection and analysis of data.

The following attempts formed the cornerstones of the present study. Firstly, the study provided an account of pre-service secondary school teachers' subject matter content knowledge and pedagogical content knowledge of logarithms and logarithmic functions. Secondly, it explored how and why pre-service teachers envision applying their subject matter content and pedagogical content knowledge of logarithms and logarithmic functions in designed simulated activities. And finally, it described the relationships between pre-service secondary school mathematics teachers' subject matter content and pedagogical content knowledge.

In summary, there were two research questions addressed in this study:

1. What is pre-service secondary mathematics teachers' knowledge of logarithms and logarithmic functions? What is the relationship between their subject matter content knowledge and pedagogical content knowledge?
2. What do the designed tasks reveal about the nature of teachers' knowledge? To what extent are these tasks effective and useful as data collection tools for research in mathematics education?

For the purpose of this report, I decided to focus on one of the tasks employed for instructional and research purposes in this study.

Research Site and Context

The reported research took place during the secondary mathematics methods course, *Designs for Learning Secondary Mathematics*, offered by the Faculty of Education at Simon Fraser University. The duration of this course was 13 weeks, with meetings once a week for four hours.

When designing the tasks used in this study, I found myself in a dual-role position: as an instructor and as a researcher. As an instructor of the secondary mathematics methods course, I hoped to create engaging activities and rich learning environments where pre-service teachers would come into contact as closely as possible with the real life situations of a mathematics teacher. These tasks would incorporate an implicit review of the mathematical content, while explicitly focusing on pedagogical implications. As a mathematics education researcher, I tried to construct methodologies that would reveal valuable insights about pre-service teachers' mathematical and pedagogical content knowledge, in particular related to logarithms and logarithmic functions.

The data consisted of the accumulated participants' responses gathered from their completion of the two tasks: peer-interviews conducted, transcribed and analyzed by participants; and written responses in the form of *Math Play* scenarios. Both tasks were employed as ongoing learning activities during the methods course.

Participants of this study were 6 pre-service secondary mathematics teachers in their final term of studies before certification.

The task, called the *Math Play,* was assigned during the eighth meeting of the course. This session was focused on assessment for understanding. One of the topics discussed in the class was errors in mathematics classrooms. Pre-service teachers were working on the activity of assessing different "student made" erroneous solutions, and analyzed the sources of the mistakes that had occurred. As a follow up of such explorations, participants were asked to individually complete the *Math Play* task. Once again, several content choices were given to students. Six students focused on logarithms and only those were analyzed for this study. A five-week period was provided for the completion of this task.

The Math Play

All the participants were prompted by the same task. Each pre-service teacher was to analyse the following erroneous situation that exposed a student's misunderstanding:

> *Act 1, Scene2:*
>
> *There is a conversation between a teacher and a student (there are 30 students in a class):*
>
> > *T: Why do you say that* $\log_3 7$ *is less than* $\log_5 7$*?*
> >
> > *S: Because 3 is less than 5.*

Pre-service teachers were asked to diagnose the student's misunderstanding, formulate a plan for remediation of the misunderstanding, and write out the balance of the interaction(s) in the form of a *math play*. The use of the word "diagnosis" in this situation, meant that teachers were to establish how, when, and why the misunderstanding could possibly occur. For this task, they were to write Act 1, Scene 1. From this, I anticipated an accumulation of some reliable assumptions about the participants' knowledge.

To confirm or refute my assumptions, I analysed Act 2, Scene 1 of the math play. In this remediation part of the activity, the pre-service teachers were to create a teachable moment, when they would orchestrate the events, tasks, and conversations, to lead the student out of the problematic situation. It was expected that pre-service teachers would help the imagined student realise the nature of his/her mistake, and in some way verify that the student corrected the mistake and demonstrated understanding of the provided explanation. The teachers' methods of such verification would expose their knowledge of mathematics and pedagogy that would allow me to confirm my initial evaluation, or on the contrary reject it, or perhaps simply modify it.

For example, in Act 1, Scene 1, a pre-service teacher would write that the misconception is a result of a student's misunderstanding of logarithmic notation. Then, in Act 2, Scene 1, the teacher would have to focus on the student's knowledge of logarithmic notation. There needs to be a consistency between the two acts. From the instruments used for remediation I learned about the extent of a pre-service teacher's subject matter and pedagogical knowledge. The absence of such consistency could be considered as an indication of insufficiencies in a pre-service teacher's knowledge.

Yet again, data sources associated with this task included the following writings:

- Pre-service teachers' diagnose(s) of the students misconception;

Through these particular data, I investigated how well pre-service teachers can assess a student's learning from a subject matter perspective. What exactly about the logarithms did the student not understand?

- Their personal encounter on where and how a given situation could take place, in the form of Act 1, Scene 1;

In these materials, I was looking for the pre-service teacher's ability to situate the provided episode, Act 1, Scene 2, in a sequence of educational events.

- The remediation part, where the pre-service teachers had to organize a situation to guide student's learning, in the form of Act 2, Scene 1.

This was the most important part of the data collected in this task. Here, I was looking for the evidence of pre-service teachers working on the "fixing" of the student's misconception. Pre-service teachers' abilities to deal successfully with the given misconception served as an indication of their teaching proficiency. On the contrary, pre-service teachers' limited knowledge of the mathematical content and pedagogy resulted in their attempt to "re-teach" the concept.

Theoretical Considerations

> *"…research methodology is not merely a matter of choosing methods and research design, … methodology is about the underlying basis for the choices that are being made…"* (Goodchild & English, 2003, p.xii)

In contemporary mathematics education, one encounters different ideas, methodologies, and various approaches to investigate research questions. For example, clinical interviews and questionnaires are the most commonly used instruments for collecting data. Some others are: journaling (Liljedahl, (in press); Flückiger, 2005), error activities (Borasi, 1996), technology based tasks (Dubinsky, 1991; Weber, 2002), and example generation tasks (Bogomonly, 2006; Rowland, Thwaites & Huckstep, 2003; Zazkis & Laikin, 2007). A detailed account on a variety of research methods can be found in Goodchild & English (2003). The research in mathematics education confirms that different methodologies and approaches allow for the creation of situations that enable researchers to collect more diverse data.

In the following, I present the reader with the discussions of the research ideas from an existing body of educational research that proved to be valuable in designing, understanding, and analyzing the described research task.

Error Activities

Practice shows that errors as a source of learning have become recognized in certain areas of mathematics education research. Students' mistakes receive constant attention from educators. In many studies they are collected, classified, and explored in terms of their roots, etc. The most recent area of attention to errors recognized in the studies, focused on the integration of errors into teaching practices for the purpose of creating inquiry learning activities. I believed that this particular area of the research was important to the study; therefore, it was elaborated in greater detail.

One of the most comprehensive works on this topic was done by Raffaella Borasi, and was presented in *Reconceiving Mathematics Instruction: A Focus on Errors* (1996). Her entire study centered on mathematical errors. She believed that they play an important role in learning and teaching mathematics. Her beliefs were based on a detailed analysis of several teaching experiments involving *error activities*. According to Borasi (1996), *error activities* are "*instructional activities designed so as to capitalize on the potential of 'errors' to initiate and support inquiry*" (p.30). Taking the constructivist approach of learner-based inquiry, she used errors as an opportunity "*…to generate doubt and questions that, in turn, can lead to valuable explorations and learning*" (p. 285). Borasi identified several types of error that could be successfully employed in *error activities*. The great majority of errors used in the case studies were students' mathematical mistakes. These errors helped to generate conflicts that, in turn, exposed and challenged the students' limited knowledge about mathematics.

In addition, this study showed that various error activities offer diverse learning opportunities. Among such opportunities were the prospects of experiencing constructive doubt and conflict regarding mathematical issues, pursuing mathematical exploration, engagements in challenging mathematical problem solving, experiencing the need for justification of the mathematical work, and taking an initiative and an ownership in the learning of mathematics (Borasi, 1996).

The most important objective of Borasi's (1996) study is to regard mathematics instruction as supporting students' own inquiries, by "*using errors as springboards for inquiry*" (p. 143) that contributes to students' mathematical learning and growth in more than one way. In her research, e*rror activities* were identified as alternatives to traditional methods of mathematics instruction. Borasi's ideas contributed to the design of my research methodology. I extended and adapted them to the learners of the teaching of mathematics, and employed them for the greater benefit of mathematics education research. For instance, Borasi concluded that error activities were the type of instructional activities that would initiate and support learner-based inquiry, thereby exposing and challenging the students' limited mathematical knowledge. Even though *error activities* offer a certain freedom to the learner in terms of exploring the possibilities of fixing mathematical mistakes (which is beneficial for the students enrolled in a mathematical content course), a greater space for imagination is necessary when moving beyond correcting. To make imaginative students' cognition tangible to the pre-service teachers, the exploration into *when*, *why* and *how* such errors could occur should take place. In the *Math Play*, pre-service teachers have to deal with a misconception developed by an imagined student. This misconception is presented as a student's erroneous answer to a posed mathematical problem. The pre-service teachers were asked to analyze the possible sources of this misconception, and accordingly situate them in the sequence of learning events, such as a lesson plan which contained a pre-existing dialogue between an imaginative student and a teacher. In addition to the diagnostic stage, pre-service teachers had to create a remediation part. The use of the erroneous examples in which pre-service teachers first considered student's conceptions and developed explanations, responses and remediation, proved to be a valuable activity for future teachers. For the researcher, this activity revealed both types of teacher's knowledge: subject matter and pedagogical.

Role-Playing

The multidisciplinary studies of aspects of the real world in the physical and social sciences over the past century had lead to the articulation of important new conceptual perspectives and methodologies that are of value to both researchers as well as professionals in these fields. The simulation of real life problems has become one of the popular teaching methodologies in many subject areas. There were several studies that reported on the effectiveness and importance of simulation activities in language education, science education, and in education in general (Blatner, 1995, 2002). For the purpose of this study, the following discussion will be focused on findings that view role-playing as a less technologically elaborate form of simulations activities, where participants personify somebody else for a particular reason.

According to Blatner (2002), role-playing is a good inquiry approach. It possesses two distinct properties: it transforms the content from information into experience, and it exposes how the person would act when placed in another person's situation (it could be either imagined or the act of pretending). Blatner claims that role-playing is an effective method for developing the ability to think about the ways one thinks: metacognition. It is also shown that role-playing is a powerful teaching methodology. This methodology helps students to understand the nature of education. Even though the research on role-playing was conducted with drama students, it seems that this approach can provide the pre-service secondary mathematics teachers with an opportunity, which was lacking in the previously mentioned error activity, to experience the understanding of the subject matter from someone else's perspective and position.

The role-playing approach was employed in the *Math Play* task. The *Math Play* was designed as an activity in which the pre-service teachers were to play roles of a classroom mathematics teacher and a student, simultaneously. In this setting, the pre-service teachers were to experience the metacognitive aspects that role-playing has to offer. The pre-service teachers would orchestrate the entire interaction for themselves and by themselves. The only constraint that remained would be the particular mathematical content. Though role-playing alone has great metacognitive potential, a greater effectiveness could be reached when used specifically in a combination of other approaches practiced in mathematics education research.

The *Math Play* is a self-exploratory activity, which focuses on the metacognitive aspects of mathematics teaching and learning. It is an example of an error activity that capitalizes on the imagined student's misconception. The pre-service teachers' investigations into the sources of such misconception led to valuable explorations into teaching and learning of a particular mathematical domain, logarithms and logarithmic functions. In turn, such explorations shed light on pre-service teachers' content knowledge. The detailed description of the research tasks is provided in the dissertation, but here the main intent was to focus on the nature of one research task and discuss its multifaceted potentials.

Excerpt from the Data: Teachers' Explanation(s) of the Possible Reasons Why the Given Error Occurred

This particular lens allowed me to measure the depth of the participants' subject matter knowledge. It is an open-ended task, as it provides the participants with the freedom to elaborate not only on the issues related to the knowledge of logarithms, but also on those of pedagogy. This task presents participants with an opportunity to reflect on possible pedagogical inconsistencies that turn out to be fatal to student understanding.

The richness of pre-service teachers' knowledge is evident from the variety of possible explanations as to why the particular mathematical error could occur. Participants' imagination, subject matter knowledge and understanding of envisioned students' performances are main factors that contribute to the assessment of teachers' preparedness to teach. While four out of six pre-service teachers provided only one explanation of why a given misunderstanding took place, two participants, Mike and Greg, presented multiple reasons.

The summary of sources for a student's misconception provided by all six pre-service teachers is presented in the following table.

Table 1: Sources of the Student's Misconception (from the *Math Play*)

Name	Root(s) of misconception		
Greg	Missing meaning change into exponential form	Student's attitude	Misunderstand multiplication
Natalia	Misunderstanding of change of base law, common logarithm and definition		
Kurt	Definition of logarithm		
Mike	Misunderstanding how logs are related to the exponents, $\log_5 7$ misread as 5^7 and $\log_3 7$ misread as 3^7	"misconceive solution $\log_5 7 = y_1$ and $\log_3 7 = y_2$ to comparison"	Relating logarithms to fractions $\log_3 7$ misread as 3/7 and $\log_5 7$ misread as 5/7
Nora	Change of base law		
Kal	Missing meaning – change into exponential form		

The data provided in the table presents the synopsis of the participants' responses. The three columns to the right of the pre-service teachers' names represent the information about the sources of possible student's misconceptions. Teachers indicated that these particular misconceptions led student to a faulty response to the given task. As is shown in the table, all six participants were concerned with mathematical content. For them incomplete or absent mathematical knowledge is the reason for the student's inability to respond correctly. There was only one pre-service teacher, Greg, who expressed his concerns not only with the content, but also with the student's attitude. For Greg, the student's personality could be a possible contributor to the mistaken answer.

Concluding Remarks

It was found that the subject matter knowledge of pre-service secondary mathematics teachers within the content domain of logarithms and logarithmic functions is very limited. It was also found that the pedagogical content knowledge teachers hold is related to their subject matter knowledge (subject matter content knowledge).

When focusing on the concepts of logarithms and logarithmic functions, a systematic approach was used in analyzing pre-service teachers' understanding of these particular topics. Teachers' pedagogical knowledge (at the times they are considered to be inseparable) was explored through the lens of the teachers' prepared questions and examples, and used in the designed instructional activities.

The analysis suggested that pre-service secondary mathematics teachers lack subject matter knowledge. It was especially evident in the participants' responses in the *Math Play* task. Even though participants were able to identify and utilize the possible difficulties in the teaching and learning of logarithms and logarithmic functions, their limited content knowledge prevented them from pursuing their inquiry further.

The participants' responses to the *Math Play* task also showed that most participants were able to compensate for their limited content matter knowledge with workable level pedagogical knowledge. This allowed them to situate learning around the mathematics they were comfortable with.

The effectiveness of the research methodology developed and used in my study is measured through the research possibilities it offered. This activity allowed me to investigate pre-service teachers' knowledge from many different sources that yielded very diverse information about the participants' knowledge. The detailed account of the collected materials can be found in chapter 4 of my dissertation. The *Math Play* proved to be a valuable data collection tool and was used for the purposes of analysis.

The *Math Play* created meaningful pedagogical experiences for both the participating pre-service teachers and me. Firstly, I found that this particular activity required pre-service teachers to review the mathematical content at hand. Secondly, it provided pre-service teachers with an opportunity to experience the complexity of real teaching situations that required knowledge of mathematics, pedagogy, and the students' learning. Pre-service teachers who participated in and completed these tasks exposed the deeper insights about their subject matter and pedagogical content knowledge regarding logarithms and logarithmic functions.

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From Mathematics Learner to Mathematics Teacher: Preservice Teachers' Growth of Understanding of Teaching and Learning Mathematics

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Introduction

Most preservice teachers enter their teacher education with predetermined ideas of what it means to teach mathematics and with predetermined ideas about how they think students learn mathematics (Parjares, 1992; Kreber, 2002; Roth & Tobin, 2002). Much of their understanding of what it means to teach and to learn is based on their own observations and experiences as students in school, and more recently as students at the university.

While there is little doubt that a strong background in theory of mathematics is essential to understanding and explaining mathematical concepts, it can be argued that a course in theoretical calculus has little bearing on how to explain the intricacies of addition and subtraction of fractions. Thus, the university mathematics courses the preservice teachers take in their undergraduate program, although essential for understanding the mathematical content, often do not prepare them for the reality of the activity of teaching mathematics in the high school classroom. Teaching involves understanding how students learn and teacher education can be thought of as an intermediary step to integrate theory of subject content with theory of learning.

This research considered the integration of theory with practice as well as the development of preservice teachers' understanding of what it means to teach and what it means to learn mathematics. It was also concerned with determining a method by which this change could be analyzed, and as such, addressed the question: Can theory applied in one setting be effectively re-addressed and applied in a different setting?

Outline of the Study

The Pirie-Kieren Dynamical Theory for the Growth of Mathematical Understanding

The study focused on the Pirie-Kieren Dynamical Theory for the Growth of Mathematical Understanding (the P-K Theory) to determine if it could be used to enhance preservice teachers' understanding of teaching and of learning and also to determine if the same theory could be used as an illustrative framework with which to analyze their growth of understanding.

Briefly, the P-K Theory looks at the developing understanding of mathematics as "*a whole, dynamic, leveled but non-linear, transcendentally recursive process*" (Pirie & Kieren, 1991, p. 1). A model used to discuss the developing understanding consists of a set of eight nested circles, implying eight layers/levels of understanding (Primitive Knowing, Image Making, Image Having, Property Noticing, Formalizing, Observing, Structuring and Inventising) (see Figure 1).

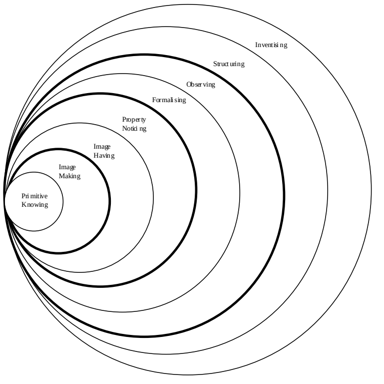

Figure 1: Model for the P-K Theory

As students develop mathematical understanding, they are likely to proceed outward, stopping and working at different levels in order to integrate their understanding. The darker circles in the model indicate “don’t need boundaries”. When a student crosses one of these, he/she need not refer back to the previous level because the level of understanding developed allows him/her to work at the new level until he/she comes to an epistemological obstacle, at which point it will be necessary to “fold back” to an inner level of understanding in order to re-examine and possibly re-define that understanding before being able to move out again.

Collecting the data

The teacher education program at the University of British Columbia provided an opportunity to consider the questions outlined above. In the Secondary Mathematics Integrated Program (SMIP), an intensive program in which prospective secondary mathematics teachers were enrolled, three of their courses (Mathematics Methods, Principles of Teaching and Communications) were integrated and taught by the same instructor with a focus on how these aspects of teaching related specifically to mathematics teaching.

At the beginning of the program, background information was obtained on the prospective teachers and they completed a questionnaire modeled on a survey used by Hart (2002), to determine their initial beliefs about what it means to teach and how they think students learn mathematics. They were then presented with the P-K Theory as one way to consider the manner by which students come to an understanding of mathematical concepts. Throughout the program, they revisited the theory and were asked to consider it in their discussions of student activities and of their own activities. The purpose of these re-visitations was to determine if the theory enhanced their understanding of teaching and learning.

Original data was collected using video, and all the SMIP classes were videotaped for later reference. The intrusiveness that may be felt by having a video camera present was alleviated through the processes of explaining the reasons for collecting the data, assuring confidentiality of information and identification, and of forming good rapport with the participants as outlined by Marland (1984). As well, videoing took place every day and the SMIP students seemed to become oblivious to the presence of the camera as Bottoroff (1994), and Cudmore and Pirie (1996) indicated usually happens when taping takes place over a period of time. Lastly, with the presence of so many new technological devices, people have become more accustomed to being videotaped.

In the process of data collection, depending on the task at hand, the video focused on the class as a whole or on smaller groups of students involved in discussions or activities. To obtain more specific information on the preservice teachers' opinions on what it means to teach mathematics and how students develop mathematical understanding, all were videoed in self-selected groups of two to three while discussing their beliefs about teaching and learning. As well, copies of their class assignments were collected and co-related to the class discussions.

The next stage of original data collection involved video taping four selected students during their practicum (four to five classes each) to determine if and how their understanding of how students learn mathematics and what it means to teach mathematics carried through to practice. These individuals, referred to as Sophia, Lance, Ellie and Wayne, later observed (and were videotaped while doing so) one of the videos of him/herself teaching. While observing the tape, and after, they commented on their reactions to their presentations and on their understanding of the learning that took place.

The final piece of data collection took place during a social evening in which the preservice teachers once again completed the questionnaire given them at the beginning of the data collection so this could be cross-referenced with their initial statements to determine what changes took place. This social event was also videoed to obtain some final views of the preservice teachers. In total, there was approximately 240 hours of video material.

Analysis of Data

The analysis of the data involved two major steps: 1. Determining what the preservice teachers thought it meant to teach mathematics and what they thought about how students learn mathematics. This information would be gleaned from their discussions and statements through their teacher education program. And, 2. Determining how their understanding evolved during the SMIP program. This second step in the analysis involved developing a means by which to determine the growth of understanding of what it means to teach mathematics and of how students learn mathematics. This was the aspect of analysis that considered if theory developed to analyze the developing understanding in one area could be used as a framework by which to analyze the developing understanding in a different area. That is, it directly addressed the issue: Can the P-K Theory be effectively transposed to analyze the growth of understanding of what it means to teach and of understanding how students learn mathematics?

Modifying The P-K Theory to fit a new context

Since one purpose of this study was to determine if the P-K Theory could be used as an illustrative framework by which to analyze growth of understanding of teaching and learning mathematics, it was essential to consider the theory in this new light. Each level of developing mathematical understanding was observed through two new lenses – one which would critically examine how that level would look if one were considering an individual's growth of what it means to teach mathematics and one which would examine what it would look like if it were considering an individual's growth of understanding of how others (students) learn mathematics. Through the process of developing these definitions and of determining exemplars for the different levels, close contact was kept with Dr. Susan Pirie (one author of the P-K Theory) so that she was able to determine if the new definitions and exemplars maintained the integrity of those of the original theory.

While developing the definitions and exemplars applicable to developing understanding of teaching and of how students learn mathematics, it became apparent that at the Observing level of understanding, the two concepts – what it means to teach and how students learn mathematics – had to merge. That is, at this level, one <u>consciously</u> had to realize that how one teaches affects how others learn, and one <u>consciously</u> had to work to align one's definitions of teaching and learning. Thus, in order to map the growth of understanding of these concepts, a new model needed to be developed to take this into account (see Figure 2).

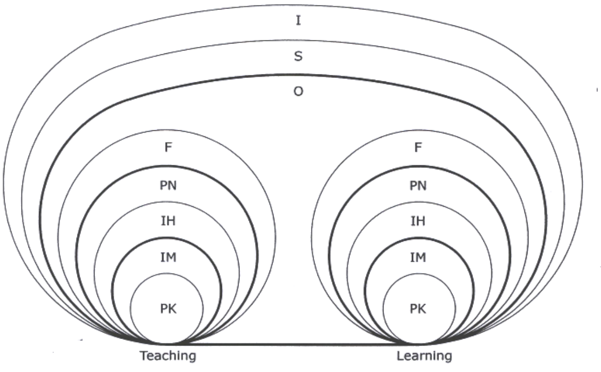

Figure 2: Borgen's Modified Dual Model for the Mapping of the Growth of Understanding of Teaching and Learning

Working with the data

Testing the theory with respect to developing understanding of teaching and learning involved determining the prospective teachers' understanding of what it means to teach mathematics and how students develop an understanding of mathematics. This involved analyzing the activities of the SMIP students. Analysis of the video tapes began only at the end of the final social gathering by reviewing the video tapes involved.

Initial viewing involved about forty hours of tapes chosen at random. This random viewing provided for an opportunity for re-immersion into the data and to observe activities over time. This was an important aspect of the analysis as by this time it had been over nine months since the initial taping began and, while some opinions had been formed, it was essential to see these, not as a single activity, but to determine how they fit into the picture as a whole. The videos were then viewed in order from beginning to end. "Post-field notes" were written. Post-field notes are detailed notes written while observing the video tapes. They contained specific quotes and activities of all SMIP students, noting tape number, time on it, and who was involved. In order to observe the subtleties of behavior and the actions that often spoke more clearly than the words uttered, the tapes often had to be viewed several times over. While the post-field notes included information on all the students in the class, they focused mostly on interactions involving Sophia, Lance, Ellie and Wayne. The notes were cross-referenced with the original questionnaires and the background data obtained at the beginning of the program, class assignments throughout the program, and the final questionnaire given at the end of the practicum.

A means of describing the individual preservice teachers and to detail their activities was needed. Portraiture was chosen for this because it seeks "*to record and interpret the perspectives and the experience of the people [being studied], documenting their voices and their visions*" (Lawrence-Lightfoot & Hoffman Davis, 1997, p. xv). In order to determine the growth of understanding of the preservice teachers, it was essential to have a "picture" of their activities throughout their teacher education. A picture, or portrait, of each of the four prospective teachers had emerged during the viewing and post-field note making. As these portraits emerged, they vividly expressed the individuality of each prospective teacher. So that the reader would be able to visualize the person as a real-life individual, not drawn out of proportion, but one with an individual personality and style, they were written as much as possible using the words of the individual. Each portrait could be read as an individual development of understanding of teaching and learning. Once the portraits were written, each was shared with the individual it represented to check the accuracy of the portrayal (Lawrence-Lightfoot & Hoffmann Davis, 1997).

Analysis of the portraits

The four individual portraits developed through analysis of the interactions of the SMIP students were used as the data which detailed the understandings of Sophia, Lance, Ellie and Wayne, in order to determine their growth of understanding of what it means to teach mathematics and of how students learn mathematics, and to determine if the P-K Theory offered an illustrative framework by which to analyze this development. Incidents (activities and/or statements) within the portraits were identified and numbered. They were then compared to and categorized according to the modified definitions and exemplars of the eight levels of developing understanding as outlined for the modified P-K Theory on developing understanding of what it means to teach mathematics and how students learn mathematics. A charting of growth was done, using different colors to map the different aspects – blue for teaching and red for learning. The initial charting of incidents used a linear model (see Figure 3 for the charting of Ellie's portrait as an example) on which each incident was numbered as in the portrait. This charting was then mapped on to the revised, combined model (see Figure 4 for Ellie's mapping as an example).

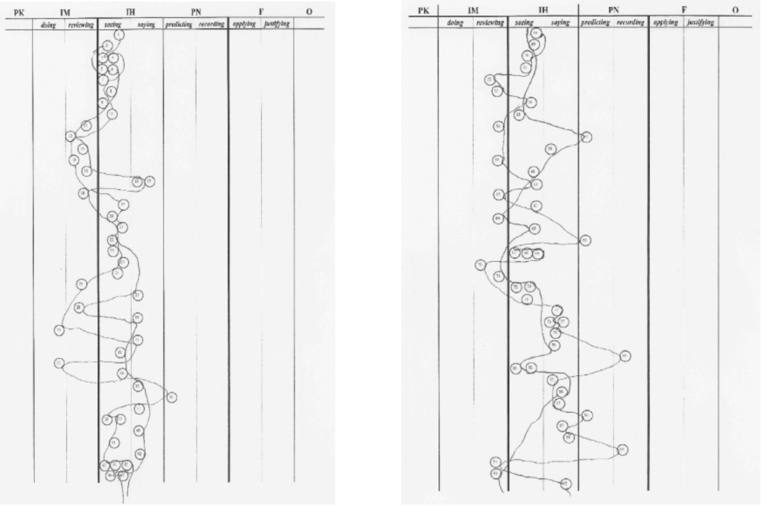

Figure 3: Charting of Ellie's Growth

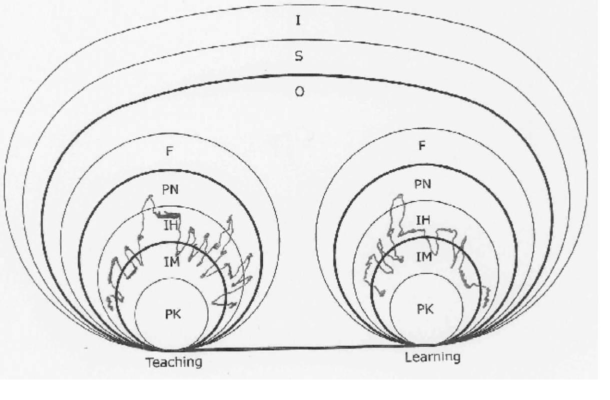

Figure 4: The Mapping of Ellie's Growth
(mapping starts at the outer side of each circle)

Conclusions

One purpose of this study was to determine if it was possible to use the Pirie-Kieren Theory for the Growth of Mathematical Understanding as an illustrative framework by which to analyze the growth of preservice teachers' understanding of teaching and learning mathematics. That is: Could the theory be effectively used to describe growth of understanding in an area different from the learning of mathematics? The ability to identify the learning activities of Sophia, Lance, Ellie and Wayne and to classify them according to the modified version of the Pirie-Kieren Theory indicates clearly that theory developed in one area can be revised/modified and used in a different, albeit related area, providing special attention is given to the re-defining of levels and the determining of exemplars. It remains to be determined if the theory can be used to discuss the developing understanding in different fields of study, such as the learning of history and the integration of historical events into one's understanding.

Related to the fact that the theory could be modified to be used in a new situation is that, examination of the data and the mappings indicates that, as with the use of the mappings to consider the growth of mathematical understanding in which each individual student develops his/her own individual learning schemata, fundamentally different profiles were revealed for the developing understanding of what it means to teach and to learn mathematics for the four preservice teacher. These learning profiles seemed to be related to the individual's initial understanding of what it means to teach and of how students learn mathematics and possibly more so to their willingness to accept new ideas and to try new methods of teaching. Only one of the four preservice teachers, Sophia, came to the point of consciously seeing a relationship between the teaching that took place and the developing understanding of mathematics. Thus, she was the only one to reach the Observing level. The others, although noting a relationship at times, did not do so consciously. This outcome would seem to be expected as these were preservice teachers at the beginning of their career.

The answer to the second question addressed in this study is embedded in the data and in a follow-up of the preservice teachers, now mostly practicing teachers, involved in the study. It was clear throughout the SMIP program that the P-K Theory provided the preservice teachers with a languaging with which to discuss and think about teaching and learning. While it is possible that another theory would have provided the same opportunity it is apparent that the theory and the language of the revised model gave them a specific way to discuss the growth of their own understanding of what it means to learn and to teach, not merely what teaching and learning entails. Providing a common language with respect to discussing teaching and learning of mathematics is imperative in a teacher education program. Communication is an important aspect of mathematical learning (NCTM, 2000) and of learning to teach mathematics.

Throughout the SMIP program, a collegiality and professionality among the preservice teachers developed. This has been maintained since the time of the study, with most of them keeping in professional contact, sharing ideas, and, in essence, developing a mentoring relationship. They have developed into reflective practitioners as evidenced through the fact that they are presenting at conferences, organizing workshops, have become leaders in the form of mathematics department heads of high schools, and have travelled to different countries to work in the area of mathematics education. Having a common language through the use of the Pirie-Kieren Dynamical Theory for the Growth of Mathematical Understanding with which to discuss teaching and learning, and having had the opportunity to develop as a cohesive group through the Secondary Mathematics Integrated Program seems to have been an integral part of the development of these teachers' understanding of what it means to teach and to learn mathematics.

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Tracking the T.I.P.S. Mathematics Document: Curriculum Negotiation and Professional Development Models

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This research study featured a case study of 64 participants involved in the planning, writing, and implementation of an Ontario teacher resource document for middle school mathematics, namely, *Targeted Implementation and Planning Supports* (TIPS) (Consortium of Ontario School Boards, 2003). The first part of the study looks at the entire process of curriculum negotiation, drawing upon the previous work/models of MacDonald and Walker (1976), and Pitman (1981). It seeks to understand the factors influencing the development of the resource document in terms of participant perceptions and influences. The second part of the study contrasts three particular professional development models that were used to implement the TIPS resource in Ontario, a decision that was left up to the local boards and coordinators. Although the three board-based PD models share certain elements, a certain primary focus within each of these three contexts emerged from the data analysis. These three models are discussed and compared in order to highlight perceived advantages/disadvantages from the different professional development approaches, and considerations for various stakeholder groups within education are also shared.

Research Study Purpose

The processes of resource development and implementation surrounding the TIPS project were susceptible, as in any other comparable situation, to the complex dynamics of curricular negotiation (MacDonald & Walker, 1976; Pitman, 1981). Research that sheds light on specific patterns or themes emerging from an analysis of the perceptions held by multi-level participants throughout this type of process has the potential to increase general understanding of multiple aspects of such reform efforts. The purpose of this participatory case study, then, was primarily that of seeking to better understand the curriculum negotiation process, as it related to the unfolding of the TIPS resource document, and to ultimately build emergent theory and related conceptual models that would serve as possible reference points for those involved in such processes. The following list of key questions regarding curriculum negotiation and professional development guided the research case study:

Curriculum Negotiation

- To what extent, or in what particular ways, did the messages within the TIPS project undergo "necessary distortion" throughout development and implementation?
- How do the dynamics surrounding issues of flexibility/accountability relate to, or bear influence upon, the "gap between images" found in the MacDonald/Walker/Pitman models?

- How might future attempts to strategize and implement new curricular initiatives be influenced by these deeper understandings surrounding gap analysis related to the curriculum negotiation model?

Professional Development

- To what extent did the professional development experienced with the delivery of the TIPS document meet, or not meet, the needs of classroom teachers?
- What form(s) of professional development was perceived by the various multi-level participants as being most desirable in terms of teacher support and long-term effectiveness?
- What were perceived as specific benefits and limitations of the various configurations of "Pilot Team" approaches as experienced by those in the school boards studied?

Theoretical Framework

The three related research areas, which most directly influenced my work, were those of education reform, curriculum negotiation, and professional development for educators. I will restrict the following sections to dealing only with the second and third of these areas for sake of brevity.

Curriculum Negotiation

The phrase curriculum negotiation was used throughout this study to denote a complex process, which involves the development, mediation and implementation of a given curricular product. This process is affected by many factors such as timing, flexibility, accountability, system scale, existing beliefs and attitudes among participants, support and training mechanisms, funding, and communication. In *Changing the Curriculum*, MacDonald and Walker (1976) drew upon the work of Havelock (1971) and his three change models (Social Interaction; Research Development and Diffusion; and Problem-solving); Schon (1971) and his three diffusion models (Centre-periphery, Proliferation of Centres, and Shifting Centres);

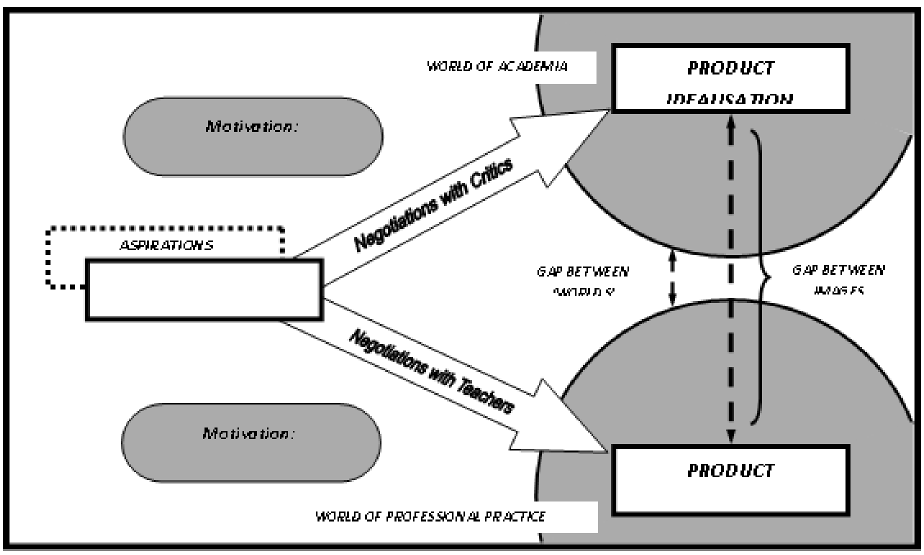

Figure 1. Curriculum Negotiation Model (MacDonald & Walker, 1976)

and House's (1974) theory of diffusion in urban societies. They traced the post-war history of curriculum design and dissemination in the United States, beginning with the "triggering" event of the Soviet's launching of Sputnik 1, moving through the US academic/military alliance and its influence on education policy, and ending with the major reforms of the 1970s. The "dissemination" movement is shown to have failed to live up to its designers' expectations, primarily because of its overlooking of the human element at the consumer reception stage of dissemination, and how the entire process of transmitting an innovation is deeply affected by beliefs of, and prior attitudes held by, those individuals or groups receiving the product. MacDonald and Walker further indicated that failure of the desired 'fidelity' within the dissemination process was not primarily connected to poor communication, but rather to content and reception. MacDonald and Walker, therefore, developed their own *Curriculum Negotiation* model (see Figure 1), which attempted to more accurately capture both the *physical* and *psychological* realities.

Of particular interest in their conceptualization is the noted "gap between images/worlds", clearly demonstrating that the idealised product, in the minds of the developers, is often far-removed from the actual product that is implemented by teachers in school classrooms.

Pitman (1979), in analyzing the MacDonald and Walker model, maintained that it provided for "*significant differences to exist between the implemented form and idealisation of an innovation and for such difference to be explained in terms other than simply those involving teacher ignorance, teacher conservatism, and lack of sufficient contact.*" However, Pitman criticized the model on several points: (i) that it assumed that teachers are the ultimate target of curriculum innovation, as opposed to students; (ii) that it failed to account for the role of various mediators in the process; and (iii) that it did not acknowledge that teachers/mediators, not unlike developers, follow different motivations when negotiating with critics and when negotiating with students through classroom interactions. Based on these observations, and with a view to preparing for his own study focusing on the negotiation of a new Science Integration curriculum in Hong Kong, Pitman developed an extended version (see Figure 2).

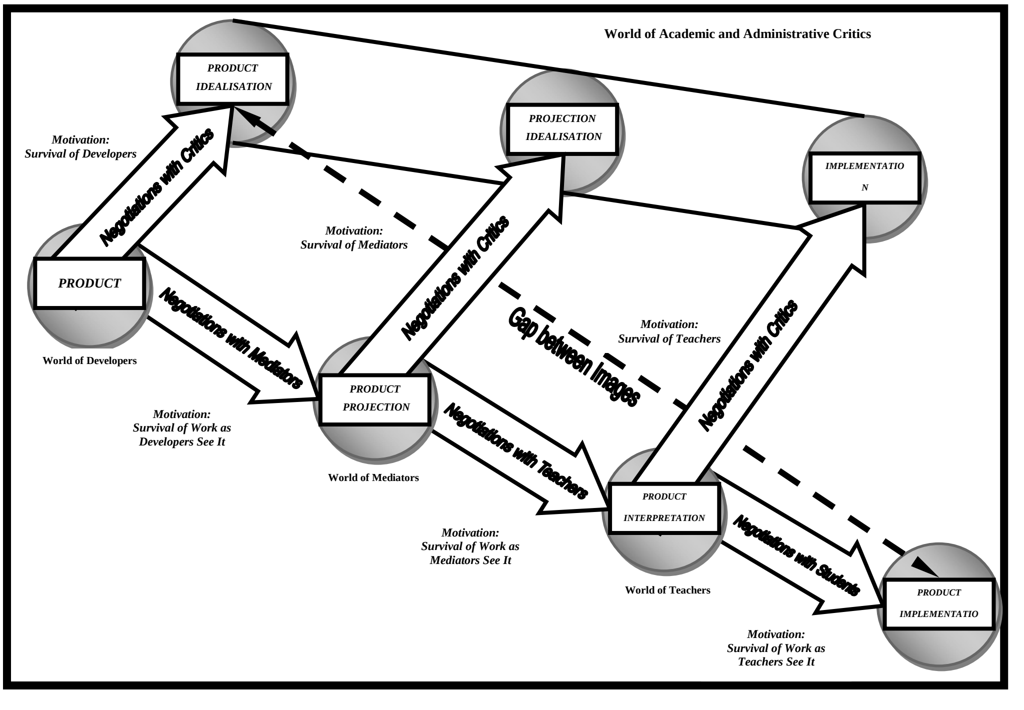

Figure 2. Curriculum Negotiation Extended Model (Pitman, 1981)

Pitman also adopted Berlyne's (1965) conflict resolution categories, namely, disequalization, conciliation, swamping, and rejection, as a way of structuring his quantitative methodology (i.e., a series of questionnaires developed specifically for his research project, and which, through factor analysis, revealed certain patterns in the negotiation process). Among Pitman's conclusions, perhaps the most interesting is the fact that the prolonged influence of the mediators on teachers was shown to reduce the likelihood of conciliation or swamping from occurring. He further explains the ramifications of this particular fact, "*Such an effect is to be regarded as desirable if one is to argue that contact with mediators is likely to sensitise teachers to any real changes involved in implementing the innovation in a form in which the implementation/idealisation gap is minimized*" (1979). Drawing upon this conceptual framework and using Pitman's extended model as a theoretical point of departure, the researcher has visually represented the *TIPS Development and Implementation* processes in a subsequent iteration of the Curriculum Negotiation Model (see Figure 3).

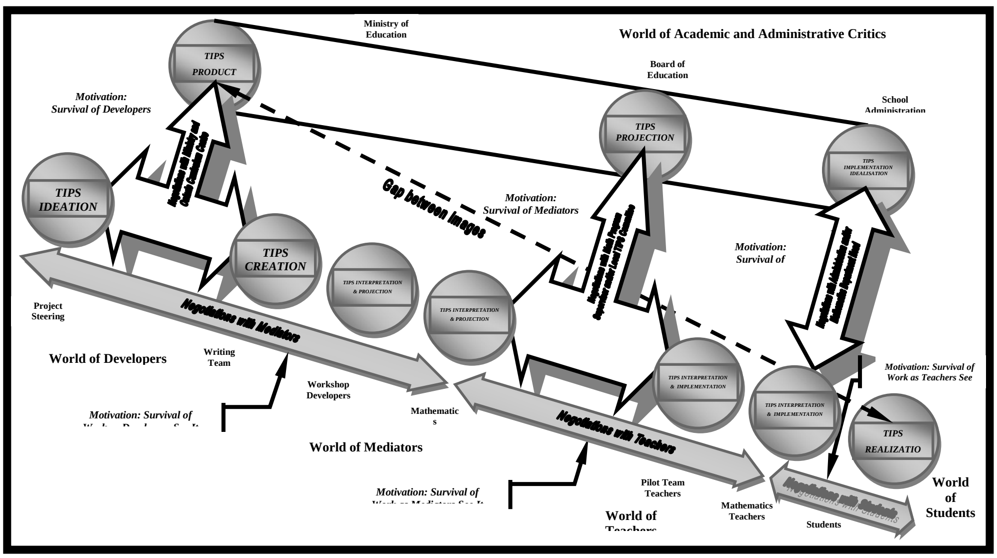

Figure 3. TIPS Development and Implementation Model (Jarvis, 2006)

Several drafts of this model were modified in an attempt to more accurately capture the cyclical nature of the negotiation processes found within this specific curricular context—the model being fitted to the reality of the complex negotiation experience, and not vice versa. Perhaps of most significance in the ongoing refinement of the model were the two dialectic spirals, or triple-pointed arrows, found to exist at both the developmental (i.e., TIPS ideation/creation) and projection (TIPS professional development) stages of the curriculum negotiation process.

Professional Development

Notwithstanding the broader definition of professional development which may be argued to include any form of professional learning undertaken by an individual or group both inside and outside of formal practice and/or training (e.g., personal reading, travel), the term was somewhat more tightly defined within this study as that which encompasses all of the activities and resources that are planned and provided by either the Ministry of Education, local District School Boards, or school-based groups in an attempt to further the professional growth and competencies of educators. Professional development for teachers forms a critical piece in the overall educational reform agenda and, more specifically, the curriculum negotiation process. Quality professional development is generally depicted by researchers as featuring a well-researched and presented rationale, the provision of adequate time and resources, and long-term support mechanisms (see, for example, Elmore, 2005). All of these aspects, when combined effectively, attempt to alter the fundamental pedagogical beliefs of teachers, and thereby effect, in the words of Fullan (2003), "second-order change." Although substantial international research has been conducted surrounding the area of professional development for teachers (Darling-Hammond & McLaughlin, 1995; Earl et al., 2000; Guskey, 2002), and, more specifically, for mathematics educators (Loucks-Horsley & Matsumoto, 1999; Snead, 1998), there has been relatively little research conducted regarding the professional development of Ontario mathematics educators, especially at the upper elementary and secondary school levels (Muise, 2003; Ross et al., 2004; Suurtamm et al., 2004). This doctoral research project was also undertaken to address this apparent paucity, and to thereby add to the growing body of such research in Ontario, Canada.

Summary of Findings

To summarize some of the key findings of this doctoral research study, the following list of seven observations, or concluding statements, has been formulated based on the emergent and consolidated themes, and in relation to the theoretical models which were analyzed in detail:

1. TIPS was interpreted as a bridge to reform-oriented practice including the use of such strategies as manipulatives, cooperative group work, technology, and increased student communication. These strategies, particularly the use of manipulatives, were shown to be more favorably adopted by elementary school teachers than by their secondary counterparts. TIPS was also viewed as a bridge to innovative, cross-strand curricular planning, whereby teachers were encouraged to cluster expectations around "big ideas" in their planning and teaching. TIPS was perceived as a bridge to increased teacher communication, particularly as a potential and effective springboard for cross-panel professional development activities.
2. All four of Berlyne's (1965) conceptual conflict responses (i.e., disequalization, conciliation, suppression, and swamping) were shown to be evident amongst the study participants. Perhaps most interesting of these was the *systemic swamping* of the original TIPS vision—that of "TIPS as planning template" as opposed to "TIPS as complete course of study," as expressed by members of the Steering Committee—by teachers, administrators, and coordinators in the field. This response was ostensibly due to a combination of the perceived high quality of the completed sections of the resource, and to the lack of time, resources, and capable writers (i.e., those who felt competent enough with the mathematics content to be able to complete such detailed and reform-oriented lessons) in the local boards. The TIPS Steering Committee listened to this field-based input and, with extended funding and support from OME, ultimately re-assembled an even larger Writing Team to complete the expanded *TIPS4RM* resource in summer 2005.
3. The "gap between images" featured on the MacDonald/Walker and Pitman *Curriculum Negotiation Models* can be negated (i.e., ignored at one's peril), reacted to (i.e., addressed once the implemented results become known), or proactively negotiated. By involving Writing Team members directly in the shaping and writing of the TIPS resource, Evgren arguably narrowed the "gap" and indeed used it to her advantage by proposing non-prescriptive parameters for her Steering Committee and Writing Team at the onset, and by providing the necessary flexibility and ongoing support to all those involved.
4. Based on the interview data and field observations, particularly with regard to the leadership and actions of Evgren and Valery while working with their respective teacher groups, the *Parametric Creativity Model* was introduced as an analytic generalization and strategy. This negotiation model featured five components:

position parameters and determine the indeterminate; assemble selectively and trust the team; communicate and facilitate; magnify, modify, and mollify; and, reflect, regroup, and recreate.

5. In analyzing the different professional development models employed in the three selected District School Boards for the implementation of the TIPS resource, each was given a label based on a predominant characteristic of the model used: B1 was referred to as a *Participatory Development Model*; B2 as a *Differentiated Development Model*; and B3 as a *Sustained Development Model*. The use of a Pilot Team group for testing and presenting new resources to peers; the insistence on full representation of targeted grade panel teachers for introductory sessions; the facilitation of teacher choice regarding PD options; and the provision of ongoing communication, workshops, and support (i.e., Summer Institute days followed by school year days with the same group of teachers) all emerge as positive characteristics of the combined professional development models examined in the study. Less effective were the following characteristics: short “rough-cut” video documentaries of classroom practice; differentiated sign-up, insofar as there existed a somewhat unclear communication of sign-up protocol and options; and lesson development within a limited time frame and an unfamiliar setting. These findings would seem to coincide with those of Darling-Hammond and McLaughlin (1995) in which they describe characteristics of an “ideal” professional development model based on the results of a large meta-analysis study.
6. Beyond the experiences of the teachers within the three case study District School Boards, it was shown that the members of the *TIPS Writing Team*, composed of mathematics teachers, coordinators, and Faculty of Education researchers, arguably experienced the greatest form of sustained professional development which involved many of the above-mentioned elements such as voice, choice, and ongoing teacher support.
7. Qualitative methods such as original “self-interviewing” (conducted by colleague), follow-up reflective interviews with key participants, full disclosure of workshop/interview transcripts via digital CD-ROM appendices, and the use of *Atlas.ti* software for organizational purposes were presented as effective for such forms of case study research.

Web-Based Reporting

The final chapter of the thesis dealt with overall findings and considerations for various stakeholder groups within education including policymakers, curriculum leaders, mathematics coordinators, school administrators, mathematics teachers, and BEd faculty. The online doctoral research menu (available at: www.nipissingu.ca/faculty/danj/PHD/HOME.htm) features the full thesis (bookmarked PDF), interview transcripts, data matrices with emergent themes (PDF files of all relevant quotations), and the interactive “Parametric Creativity” model. This digital presentation of data and findings (see Figure 4) demonstrates how the Internet can offer researchers new opportunities for sharing research results and resources.

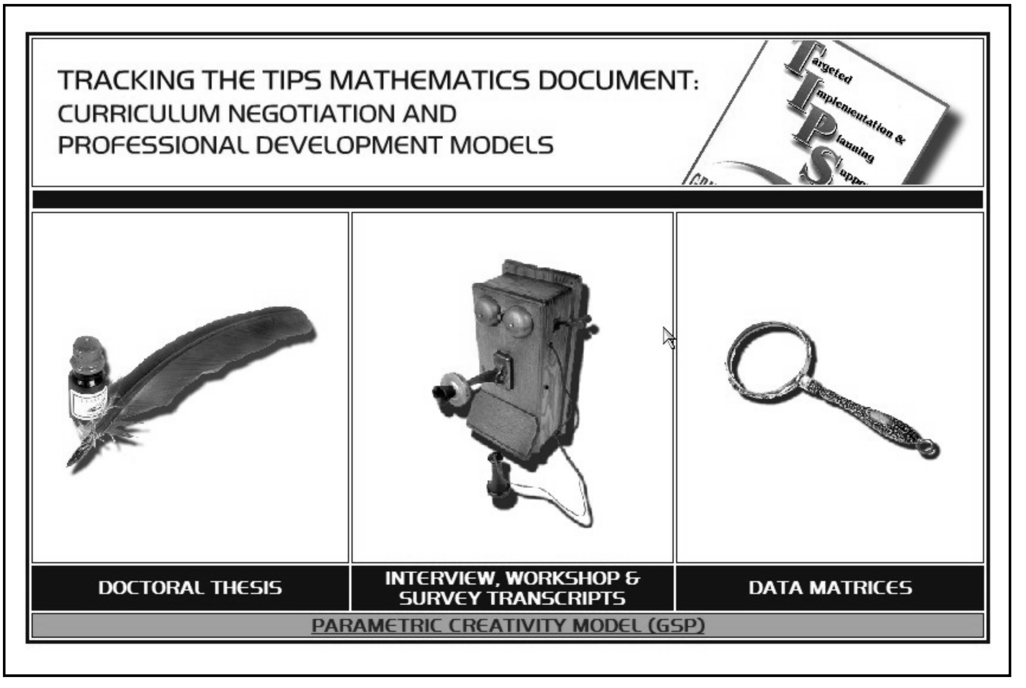

Figure 4. Online Research Menu with Thesis, Transcripts, Data Matrices, and PC Model

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Investigating ‘Epistemologically Correct’ Experiences of Mathematical Learning
Expériences d’apprentissage mathématique « épistémologiquement correctes » : une investigation

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Introduction

In her 2004 report on mathematicians as learners, Burton cited Wenger:

> *Educational processes based … on actual participation are effective in fostering learning not just because they are better pedagogical ideas, but more fundamentally because they are ‘epistemologically correct’, so to speak. There is a match between knowing and learning, between the nature of competence and the process by which it is acquired, shared, and extended* (Wenger, 1998).

The project that this report presents focuses on the application of this perspective to a mathematics course for elementary student teachers. To this end, a framework describing ‘epistemologically correct’ mathematics learning experiences was outlined. Based on this framework, a description of the type of tasks that could be considered to lead to such experiences was developed, and an undergraduate course that encapsulates these intentions was designed and implemented. This implementation was then evaluated in terms of the intervention’s success at providing ‘epistemologically correct’ experiences (as defined) of mathematics and in terms of changes in the affective responses of the participants. The report focuses on the first two points: the development of a framework for developing teaching approaches that provide the appropriate experiences, and the resulting teaching approach. The thesis itself also included the evaluation of the approach in terms of its intention, and of its influence on the participants’ affect.

Integrating a Historical Perspective on Epistemology

In order to define what could be considered ‘epistemologically correct’ experiences of mathematics learning, I investigated the literature on the philosophy of mathematics, with a particular look at the changing perspectives throughout history. This survey revealed a movement from what was mostly empirical observation, to views that were more and more rationalist, with a peak in the early twentieth century, at which time the philosophical intention was to completely explain all of known mathematics using only a system of pre-established rules (logic and set theory), thereby eliminating the empirical ontology altogether. This intention is well known to have failed due to the extreme nature of the underlying view. Lakatos (1976) provided a more balanced, integrated view of the development of mathematical ideas, suggesting that the perspectives are not incompatible, and can be integrated.

Though a widely held contemporary perspective is that mathematics is socially constructed, in practical terms, in the classroom, empirical observation and the idea of Universal Truth are very much alive. To reconcile the various views, as Lakatos has done, it is possible to develop a pluralistic, non-homogenous framework of the epistemology of mathematics. This framework integrates empirical observation, logical reasoning and social norms and consists of four types of mathematical notions[1]:

- Observational notions, which are the result of basic empirical observation, without explanatory content. The epistemological source of these notions lies in personal sensory experience with a phenomenon.
- Conventional notions, which are not the result of empirical observation or logical derivations from more basic or fundamental notions. They have been chosen by the experts or imposed by simple enculturation (Pimm, 1995) as convenient for the task, are accepted socially, remain unquestioned, and are treated as monolithic.
- Applicational notions, which are the product or application of some form of mathematical reasoning upon the previous two categories, and can therefore be explained and traced back to this reasoning.
- Theorisational notions, which make possible the reasoning that itself produces the applicational notions and promotes their adaptability.

The four types of mathematical notions are all part of the wider body of work that constitutes what is today accepted as mathematics. The typology of a specific notion, however, is not necessarily fixed. It is possible to treat a notion as conventional, even though it was derived empirically or logically. This is often done when the specific notion is left unquestioned and simply *used* to solve a problem, even if the user knows it as applicational or even theorisational. Any mathematical notion, therefore, can be 'treated as' conventional, or observational, even though it is also potentially applicational, for the purpose of the task at hand. For this reason, a framework describing the way of knowing these notions must be superposed onto this typology.

The simplest types of notions, conventional and observational, tend to be perceived as monolithic and left unquestioned. If known, they can be replicated when needed. Familiarity with them can be expressed as 'knowing-what' or 'knowing-how', in the case of a process or algorithm.

Applicational notions can be known that way too, but familiarity can also be expressed as 'knowing-why' when the underlying reasoning is familiar enough to bear examination.

Finally, theorisational notions can be treated like conventions, or reasons for other notions, but they can also be expressed using 'knowing-when', that is, as framing domains of application, or ways of reasoning about, etc. An individual 'knows when' to apply the notion.

The table below shows a summary of the three ways of knowing mathematical notions, and several of their properties, including the 'thinking register'.

[1] *I interpret mathematical knowledge as being about mathematical objects and of relationships between them. In addition, these relationships can themselves be regarded as objects of higher-level relationships, as in the example of mathematical logic, which uses relational statements as objects. At any given moment in a mathematical situation, therefore, mathematical statements are treated either as statements of relationship, or the object of relationships with one, several or a whole class of other mathematical objects. For this reason, and to lighten the flow of the text, I use the term (mathematical) notions to include both (mathematical) objects and (mathematical) relationships, unless the distinction is significant* (Knoll, 2007).

Table 1: Levels of Knowing Mathematical Notions

Level	**Knowing-what/how**	**Knowing-why**	**Knowing-when**
Register	Low level	Mid-level	High level
Manifestation	Recalling a fact or performing a process	Monitoring a process	Abstracting from a process
Engagement	Passive	Active	Critico-Creative
Properties	Replicative	Transferable	Constructive
Perception	Determined by 'external Authority To be memorised	Instance of reasoning behind a notion	Rigour in mathematical structure. Can be used to (re-)construct

A mathematical learning experience, to be 'epistemologically correct', must incorporate all these levels. Tasks that promote this experience, therefore, need to present the engaged participant with an opportunity to use and develop knowledge about notions at all the levels. I define such tasks in the next section, and call them 'Mathematical Enquiry'.

Classifying Mathematical Tasks

In order to examine a mathematical task in terms of the levels of the preceding typology, it is necessary to evaluate its potential for *knowing-when* as well as *-why* and *-what/how*. To facilitate this evaluation, I propose a 2-dimensional classification system based largely on the literature on problem solving.

Routine versus Non-Routine Problems

Firstly, a task can be classified according to the level of engagement that is required to perform it successfully. This classification has been expressed as the difference between a problem, defined as a "*situation where the [participant] cannot at once decide what rule to apply or how it applies*", and an exercise defined as "*a situation in which this is at once obvious*" (Passmore, 1967; see also Zeitz, 1990). This distinction is similar to that made by Polya (1957) between routine and non-routine problems. An important aspect of this classification is that it is not simply a condition of the task itself, but of the participant's familiarity with it. A specific situation can evoke a routine exercise for a more knowledgeable individual, or great puzzlement for a less knowing one. This focus on the participants' readiness to perform the task is well developed in Goldin's (1982) framework, which focuses more on this condition and develops a 5-fold distinction. In his framework, a task can belong to any one of the following categories:

1. The subject '*knows the answer*' or is already at the goal when the task is posed.
2. The subject does not '*know the answer,*' but '*possesses a correct procedure*' for arriving at it
3. Same as 2, but the subject is unable to describe the procedure in advance of carrying it out.
4. Same as 3, but the subject '*does not know for sure*' (cannot state with certainty) that he or she possesses the procedure until after the problem has been attempted.
5. The subject does not possess a procedure for arriving at the answer (pp. 95-96)

In this categorization, the participant's possible relationship with the task is much more differentiated than in the previous versions, and the obstacle can be of various natures. At Goldin's level 2, for example, although the participant knows what procedure to carry out, suggesting that he only needs '*knowing-what/how*', s/he cannot explain it "*in advance of carrying it out*". '*Knowing-when*' is not required. In 1992, Mason advanced a different definition:

> *I take the word problem to refer to a person's state of being in question, and problem solving to refer to seeking to resolve or reformulate unstructured questions for which no specific technique comes readily to mind* (p. 17, footnote).

The key to Mason's statement is the student's state of being when performing a task, her/his *engagement* with the problem. This condition of problem solving is neither a feature of the problem, nor of the solver, at the start of the solving process: it is a property of the relationship between the two. It thereby connects, through the level of engagement, to the level of knowing required, as shown in the table.

In summary, mathematical tasks can be classified according to how routine they are for a specific individual. Mathematical enquiry, if it is to require all the levels of knowing described above, needs to be as non-routine a task as possible, so as not to engage solely the lower levels.

Problems Requiring Mathematics to be Solved, versus Problems of a Mathematical Nature

A second distinction can be made, between problems that require the application of mathematics, and problems that are mathematical by their very nature, that is, those which, if solved, produce new mathematics, at least for the solver. This distinction is akin to that made by Polya (1957), between "*problems to find*" and "*problems to prove*". According to his description:

> *The aim of a "problem to find" is to find a certain object, the unknown of the problem. […] We may seek all sorts of unknowns; we may try to find, to obtain, to acquire, to produce, or to construct all imaginable kinds of objects. […] The principal parts of a "problem to find" are the unknown, the data and the condition* (p. 154-55).

This corresponds to the category of problems "*that require mathematics to be solved*". In contrast, the resolution of problems that are mathematical in nature elicits, on the part of the solver, the development and creation of a mathematical system. The distinction could be considered one of focus, between the search for a solution to a concrete, specific application, or in general to a class of situations, with the corollary that the boundaries of the domain of applicability need to be defined. If the focus is on a specific application, the solution is the goal-state and work can stop when this is found. If generalisability is sought, the problem can be pursued further by examining similar cases, classifying them, working on defining boundaries, special cases, etc. This suggests the use of knowing-when. Problems of a Mathematical Nature are therefore good candidates for mathematical enquiry.

Defining the Criteria for Mathematical Enquiry

The classifications described above form the basis for the conditions for mathematical enquiry to take place. They do not, however, pose specific criteria for the design of such tasks. Such a framework can, however, be reified into a series of criteria, as follows. In 2003, Grenier and Payan's developed a framework designed to determine the conditions for engagement, on the part of school children, in 'professional research situations'. For them, it was essential that:

> *En situation de recherche, le chercheur peut, et doit, pour faire évoluer sa question, choisir lui-même le cadre de résolution, modifier les règles ou en changer,*

> *s'autoriser à redéfinir les objets ou à modifier la question posée. Il peut momentanément s'attaquer à une autre question si cela lui semble nécessaire.*[2]

Researchers have ownership of their enquiries in a way that allows them to redefine, modify or even temporarily abandon them in favour of another. To replicate this kind of context requires the development of criteria for the design of a teaching approach that is intended to provide experience with mathematical enquiry. These criteria are delineated below.

Criterion 1: A Novel Starting Point

> *Une SRC s'inscrit dans une problématique de recherche professionnelle. Elle doit être proche de questions non résolues. Nous faisons l'hypothèse que cette proximité à des questions non résolues - non seulement pour les élèves, pour l'ensemble de la classe, mais aussi pour l'enseignant, les chercheurs - va être déterminante pour le rapport que vont avoir les élèves avec la situation.*[3]

A research situation, to be acceptable according to their criteria, needs to be unsolved as far as the whole community of mathematics is concerned, in order to ensure full, 'professional' engagement. This is not always practical, and in any case, as Mason (1978) explains:

> *The question by itself cannot replace the process leading to its articulation, so the student is not in the same state as the originator.* (p. 45)

In this project, rather than making use of unanswered mathematical problems, the participants developed their own starting point. In the larger sense, a task can be called one of mathematical enquiry if the starting point is new to the (local) participant(s). If they create it, it is new to all the participants, because the originator holds the meaning of the problem.

Criterion 2: An Open-Ended Process

> *Plusieurs stratégies d'avancée dans la recherche et plusieurs développements sont possibles, aussi bien du point de vue de l'activité (construction, preuve, calcul) que du point de vue des notions mathématiques.*[4]

According to Grenier and Payan, a mathematical task, to emulate professional activities, must not prescribe a specific method of resolution. The process must remain the choice of the student-participant. This is also true of mathematical enquiry.

Criterion 3: An Open-Ended Goal-State

> *Une question résolue renvoie très souvent une nouvelle question. La situation n'a pas de « fin ». Il n'y a que des critères de fin locaux.*[5]

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[2] In a research situation, for the problem to evolve, the researcher can and should determine the domain of applicability of his questions, modify or replace the rules under which s/he operates, allow her/himself to redefine the objects of the problem or indeed the problem itself, focus temporarily on a different question if it seems necessary. (my translation)

[3] An RSC [Research Situation for the Classroom] is framed by a professional research question. It must be connected to problems which are unsolved in the canon. We make the hypothesis that the fact that the problem is unsolved, not only for the pupils, but for the instructors and for the participating professionals, is key to the rapport which the pupils will develop with the situation. (ibid)

[4] Several investigation approaches and developments are possible, both from the point of view of the activity (construction, proof, calculation), and from the point of view of the mathematical knowledge. (ibid)

[5] An answered question often leads to a new question. The situation has no 'goal-state'. There are only criteria of local resolution. (ibid)

As discussed previously in the description of problems that are mathematical in nature, a task, to be considered mathematical enquiry, must have an open-ended goal state: any solution will lead to new questions, of generalisability, of domains of applicability, of special cases…

Criterion 4: An Atmosphere of Security

> *La question initiale est facile d'accès : la question est « facile » à comprendre. Pour que la question soit facilement identifiable par l'élève, le problème doit se situer hors des mathématiques formalisées et c'est la situation elle-même qui doit « amener » l'élève à l'intérieur des mathématiques. Des stratégies initiales existent, sans que soient indispensables des prérequis spécifiques. De préférence, les connaissances scolaires nécessaires sont les plus élémentaires et les plus réduites possibles.*[6]

The uncertain nature of mathematical enquiry needs to be mitigated, for participants to feel safe in engaging actively, by the social context of the experience. In particular, in the classroom context, a participant will engage in an activity if it is deemed feasible, under the didactic contract and based on the knowledge available. This is an essential criterion for the implementation of mathematical enquiry in the classroom as it is a condition for engagement.

Criterion 5: The Experience of Mathematical Enquiry

In addition to the above-mentioned criteria, mathematical enquiry, to be authentic, needs to be experienced in a way that is analogous to the experience of professional mathematicians. Descriptions of such experiences often are expressed in terms of stages, as in Hadamard's scheme (see Liljedahl, 2004), which includes preparation (initiation), incubation, illumination and verification, or Mason's (1992) energy states. A common feature of these descriptions seems to be the importance of giving the experience time to unfold. Burton (2004), for example, argues:

> *The strategy of a student […] of having time and space to retreat, reflect, research, is not only appropriate to the unsolved problems of research mathematicians. Students undertaking a mathematical challenge also need to have room to manoeuvre, to work together, to consult people or books, to think.*

Time, therefore, needs to be given for risky avenues to be investigated, questions to be re-formulated, ideas to be incubated and results to be examined and re-examined.

Additional constraints were derived, in this project, from the research methodology and the context within which the study took place. The data collection consisted of both point-in-time (questionnaires, recorded class discussion) and continuous events (journals), promoting participants' reflections on their experiences both before and during the intervention, thereby transforming those experiences. The programmatic context of the course also influenced both its mathematics curriculum and the type of participants.

Chronology of the Implementation

The teaching approach resulting from this development was implemented in the context of an undergraduate mathematics course taken by students registered in an integrated bachelor of education programme in a Midwestern state university. The mathematics course forms one of several that are required of elementary student teachers, and is traditionally given by the

[6]The initial question is easily accessible: the question is "easy" to understand. For the question to be easy to identify, the problem must be situated outside of formalised mathematics, and must pull in the pupil. Initial strategies exist, without requiring specific pre-requisite knowledge. Preferably, this required knowledge is made minimal. (ibid)

mathematics department. The participants consisted of 33 women and 4 men, ranging from less than 21 years old (16) to over 25 (3). 24 were registered in the integrated education programme, 4 in the Masters' for teacher certification, and 4 as undecided. 4 were in their second year of undergraduate studies, 15 in their third and 12 in their last. Classes met twice a week, and were broken down into five distinct phases.

Phase 0 consisted of the first class and focused mainly on the collection of pre-intervention data gathering, using a questionnaire and recorded whole-class discussion, and the students' introduction to the course syllabus. During this class, expectations were disclosed, by explaining that the student would engage in open-ended mathematical investigation. Journals were also distributed in the second class.

Phase 1, which consisted of 10 classes, saw the participants engage in 'mini-projects', that is, small-scale tasks designed to model the type of activities that they would be required to engage in during the main project. These projects consisted of 'research situations', not 'well-formed questions', which, with some guidance, got the participants used to the pace. This phase was considered to provide the participants with a 'ramping up' of engagement. A typical instruction during this phase was as follows:

> *Once you have got a pattern, check [that] it works for different examples, then ask [yourself] why it may always be true. Do the simple cases very thoroughly.* (Blackboard)

Phase 2 lasted for 7 classes and consisted entirely of student-led activities, based on student-identified starting points, development and goal-states. Interactions with the instructor-team were kept very hands-off and consisted largely of responses to questions with questions: the enquiry was the students', and so the answers should be theirs. An example of such interaction, reported by the main instructor was as follows:

> *Rob [pseudonym] asked about his 'number of manipulations'. 'Is this one or two?' he asked. I [wanted to ask] him to define it his way, but he completed the sentence [for me]* (Dr Zachary, personal journal, p. 38).

Many of the interactions between the instructor team and the students demonstrated willingness on the latter's part to engage critico-creatively. The research reports reflected this.

Phase 3, which also consisted of 7 classes, brought the class back to a more regular way of working. This phase was mainly implemented in order to 'catch up with the curriculum' so that the required content was addressed.

Finally, Phase 4 consisted in summative assessments and closing data collection. The students presented orally the results of their enquiries, an exam was administered that covered the content of Phase 3, the journals were collected one last time, and the participants responded to the post-intervention questionnaire. This phase lasted 4 classes, including the exam. The analysis of the journals revealed a strong engagement, on the part of the majority of participants, in 'mathematical enquiry'. The thesis elaborates on these results and on the affective responses.

Discussion

The project as a whole consisted not only of the development of the framework and derived teaching approach described here, but of an analysis of the authenticity of the experience in terms of the exemplar that is 'professional practice', and of the participants' affective responses and changes therein. Briefly, the experience seemed to be authentic for a majority of the participants a great deal of the time, and the responses, as predicted by the literature, showed significant changes in the participants' beliefs about mathematics rather than in their attitudes, known to be more stable.

One of the limitations of this teaching approach that stood out, as much from the practical considerations at hand as through the participants' own responses, was the scale of the time investment required. This limitation is not easily done away with, however; it

> *… is so fundamental to the practice that it is constitutive of one of the five design criteria of the approach. As such, it changes from a characteristic of the practice to a condition for its authenticity, and therefore the question becomes not one of adjustment, but of the worthwhile nature of the practice as a whole* (Knoll, 2007).

Another issue, directly linked to criterion 1 regarding the participants' ownership of the starting point, arose in the case of students who, within the parameters of the intervention, could elect to opt out of the experience by choosing a mathematical topic and task with which they were so familiar and thereby reduced the practice to an exercise. This appeared to happen only with one student. Other limitations of the practice include the difficulty in assessing activities whose results are uncertain.

In the case in point, the practice was applied to the preparation of future teachers. The claim is that for teachers to be able to speak with authority about mathematics as a discipline, it is important for them to have experienced it in all its aspects, including its creation. The questions remain whether this practice promotes such authority and whether the practice is applicable to the teaching of mathematics in schools itself.

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Communication in Mathematics: A Discourse Analysis of Peer Collaborations

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Introduction

Significant emphasis has been placed on the benefits of peer discourse in learning mathematics (Expert Panel on Student Success in Ontario, 2004; National Council of Teachers of Mathematics/NCTM, 2000). Specific ways of communicating *in* and *about* mathematics have been mandated; that is, students are expected to use mathematical discourse to explain their thinking, even to one another (Ontario Ministry of Education and Training/OMET, 2005). At the same time, great importance has been placed on collaborative learning amongst peers (see for example, Gillies & Ashman, 2003; Jacob, 1999; Kagan, 1994; Slavin, 1995; Vermette, 1998), where students spend their time predominantly talking to one another rather than the teacher. In response to these emphases on peer discourse in collaborative settings my two central research questions are: (1) What is the nature of peer communication in mathematics? and, (2) What is its relationship to learning and knowing?

Theoretical Framework

The theoretical framework informing this research is a sociocultural perspective. The sociocultural perspective, as I conceive it, consists of the following key constructs: (1) knowledge is a re-construction (Goldin, 2003; Lerman, 1996, 1998a, 1998b, 2001; Lerman & Zevenbergen, 2004; Sfard, 2000), (2) learners construct knowledge about themselves and about mathematical objects from interactions with people, activity, and cultural artefacts (Vygotsky, 1962, 1978), (3) individuals are acculturated into their social roles through their ability to access linguistic codes within social settings (Bernstein, 1971, 1990, 2000), and (4) individuals through social and cultural transmission enter into discourse with particular *capitals* (Bourdieu, 1991) or *status characteristics* (Berger, Cohen, & Zelditch, 1972) which affect their ability to participate (Brown, Collins, & Duguid, 1989; Lave & Wenger, 1991) or to learn.

The sociocultural perspective recognizes that coming to have knowledge or *access* to the meaning making cannot be assumed and is neither arbitrary nor neutral. Lerman suggests that a student being able to communicate becomes a reflection of them *becoming* mathematical. However, as Alrø and Skovsmose (1998) point out "*communication in mathematics education assumes a particular character*" (p. 42) which unless rendered accessible to students can result in "*an amputated discussion of purpose*" (p. 42). Adler (1998) says that "*teaching mathematics successfully to all in school is a complex task. It includes: enabling epistemic access*" (p. 24). A student demonstrating what they know through communication is not in itself problematic; rather, the problem arises from false assumptions of taken-as-shared codes, or common linguistic capitals within the classroom and, more specific to this research, within peer collaborations.

Literature Review

Adler (1998) describes from her study of small group settings in a multilingual classroom that when students work in small groups, the inner dynamics of the group often reflects a small class setting. However, as Mason in Sfard et al. (1998) points out, in the absence of a more expert other, communication alone in small groups may be insufficient in the development of mathematical meaning.

Barnes and Todd's (1978) seminal research with secondary students on peer collaborations suggests that knowledge is a negotiable commodity between students *only* when students are fully and willingly engaged in the learning task. Of particular importance is the observation that Barnes and Todd make regarding peer collaborations outside the visible range of the teacher. Barnes and Todd's describe instances where peer collaborations occasionally resulted in verbal attacks on each other, thus preventing epistemic access (Adler, 1998).

Cohen (1994) offers an important meta-analysis of existing research on conditions for productive small groups. In this piece of work she conceptualizes "*conditions under which use of small groups in classrooms can be productive*" (p. 1). Although productivity is often taken to imply academic achievement, according to Cohen, productivity is the "*occurrence of equal-status interactions within the small group*" (p. 3). This view of productivity contrasts the idea that stronger students somehow take responsibility for weaker students within the small group setting. Cohen makes clear that *cooperative* in this analysis implies only to those learning opportunities that cannot be accomplished individually by a learner; that is, there is a need to collaborate with either another or a peer in order to progress in the problem solving.

Johnson and Johnson (1989, 1992, 1994) identify five basic elements of effective group work or peer collaborations. The authors suggest that in order for peer collaborations to be effective, the following elements must be present: (1) individual accountability, (2) social and academic appraisal of the groups' efforts (i.e., *process* as defined by Johnson and Johnson), (3) collaborative skill, (4) face to face interactions, and (5) positive interdependence. Positive interdependence implies a willingness on the part of the students to accept accountability for one another's learning. As Johnson and Johnson suggest, one or more elements may be missing, with the exception of positive interdependence, for group work or peer collaborations to be effective. Finally, also pointed out by Johnson and Johnson, these elements must be both taught and included in teachers' pedagogical choices.

Methodology

To examine peer communication, adopting a complex view of discourse as action, words, and gestures (Lerman, 2001), I used *video study methodology* paired with *interactional sociolinguistics*. The following assumptions frame my conceptualization of video study methodology: (1) *what* is said to another (actual words used) and *how* one makes sense of what is said is shaped by *how* things are said; (2) actions, with or without words, can also project meaning – intentional or otherwise; (3) individuals can come to know something about themselves and others through interactions; and (4) participants' interpretations/perspectives on their video participation are central if the research is intended to be transformative in some way. By engaging participants through an analysis of their own roles within videos, the participants become "*co-constructors of knowledge*" (p. 236).

Paired with video study methodology was *interactional sociolinguistics.* Schiffrin (1994) says that interactional sociolinguistics is not a discourse analysis tradition that proposes to explain *how* discourse conveys intention. Nor does interactional sociolinguistics attempt to elucidate underlying motives of discourse; rather, it uses discourse as a way of developing knowledge about how meaning is being conveyed within the particular social cultural interaction and context. An important implication of considering discourse as a product of context is that motivation, goals, or intentions are *not* points of consideration in the analysis; rather, sense making occurs through an analysis of how things are said (i.e., explaining the behaviour) and what is said.

This research was conducted in an eighth grade classroom. The school itself is located in a suburban setting of a moderate sized city. The overall socioeconomic status of the school is average to high, and the school has placed in the 90^{th} percentile of overall provincial testing in its school district, as reported by administration and the teacher. There were 34 students in total in this eighth grade class, 19 boys and 15 girls. The students were all either 13 or 14 years of age. The students were randomly assigned to groups by the teacher for peer collaborations, with a heterogeneous composition of mixed ability and mixed gender. I conducted this research in the classroom of a male teacher with eleven years of intermediate teaching experience.

My data collection took place during the 2005 to 2006 school years. The following data sources were collected for this research: (1) in-class structured field notes spanning over 7 months (59 days of 70 minute mathematics lessons, (2) 19 student interviews of approximately one hour in length, transcribed, (3) one teacher interview, transcribed, (3) 38 hours of video taped peer collaborations, (4) transcriptions of the videotaped peer collaborations (approximately 35 pages for every hour of footage), (4) observational notes from the video transcriptions, (5) student artefacts, (6) sociometric questionnaires, (7) student response journals at the conclusion of the final task, and (8) focus group session transcriptions.

There were seven stages to the data analysis: (1) multiple viewings of the videos, (2) defining ‘episodes of communication’, (3) coding types of mathematical talk (Pirie,1998), (4) coding tacit function of the communications (Mercer, 1996, p. 368), (5) coding episodes according to level 1, 2, 3, or 4 of communication using the current grade 1-8 Achievement chart for communication in mathematics, (6) identifying and coding the ‘receiver’ of the episode of discourse, and (7) theory building.

Findings and Discussion[1]

At the onset of this research I anticipated that I would be examining how words and actions used in mathematical communications among peers facilitated learning and knowing. My findings and conclusions rather center on the social spaces of peer collaborations. The social spaces create the initial access points for students to engage in communication. Therefore, limited access *to* the social spaces or limited mobility *within* the social spaces prevents communication and thus inhibits learning and knowing (Kotsopoulos, 2008).

My findings suggest that students may create only limited opportunities to develop mathematical discourse from one another, despite notable pedagogical intention on behalf of a teacher, and despite the task. Limited opportunities exist to develop mathematical discourse from one another because of the nature of the peer group work. Therefore, the nature of the group work has a specific relationship to learning and knowing in that some students are

[1] For a full account of this research, please see my dissertation which is available online through Dissertations Complete (Kotsopoulos, 2007a).

potentially excluded from the discourse practices of the group. That is, certain individuals fail to gain even legitimate peripheral participation (Brown et al., 1989; Lave & Wenger, 1991) in the discussions, and therefore only exist beyond the peripheries of discourse, and thus beyond the peripheries of learning.

Particularly excluded from meaningful discourse in this research were those students perceived as ‘low achievers’. Two key factors were determined that contribute to the findings: (1) the impact of *situated identities* (identities specific to a context but not necessarily relevant, accurate, or internally consistent with the individual), and (2) how the groups functioned, despite pedagogical intention (i.e., no real expert other, foreman-like leadership, etc.). The situated identities, coupled with the ways in which the groups interacted, resulted in particular patterns of discourse (Kotsopoulos, 2007b). These patterns of discourse had exacting outcomes for students – both those identified as low achievers and other students, who *may* have been low achieving, but having as Berger, Cohen, and Zelditch (1972) suggest more desirable specific status characteristics. Despite efforts throughout the school year to build strategies for working together, peer discourse was largely exclusionary.

Students come to group work with preconceived ideas about their peers in the areas of leadership, mathematics, and group work. I refer to these identities as situated, in that the identities do not necessarily hold merit either within the group work or perhaps even in other settings. In addition, consistently observed throughout all the videos was that the groups had particular ways of functioning. In each group there was a *foreman*, who delegated tasks to the *mathematical labourers*. The foremen took no responsibility as the ‘expert other’ in the group and as such, support for the mathematical labourers that were challenged in their learning (i.e., low achievers or those that struggled), was not automatically forthcoming. Even in instances where students who struggled talked aloud, no support was forthcoming.

The situated identities, coupled with the ways in which the groups functioned, create specific discourse patterns amongst the group members. That is, little mathematical discourse is undertaken with certain members of the group. Moreover, participation on the part of certain members of the group based upon situated identities is extremely limited as was the case with the low achievers. Low achievers, in particular, had little opportunity to engage in discourse and thus little opportunity to benefit from peer discourse. This being said the quality of the discourse in general did not appear to support learning for other members as well.

As a result of the peer collaborations, some students experienced *productive silencing*, which occurs when attempts by students to become legitimate participants (Brown et al., 1989; Lave & Wenger, 1991) in the group, are met with resistance. In contrast, other students experienced *productive positioning* – permitting individuals to exist within a particular social setting *with* limited responsibilities to the collective *and* despite the wider needs of the collective (Kotsopoulos, 2006). A student’s lack of meaningful participation is not challenged by members of the group or is overtly or covertly excused. Therefore, there is no form of resistance on behalf of the student or the other members of the group with respect to their participation. One mechanism that seemed to disrupt productive silencing and productive positioning is video modeling, whereby students themselves view video data.

Educational Implications

The findings from this research have unanticipated wider-reaching potential in that my findings have multi-disciplinary implications. An overarching finding of this work is that the social spaces of peer discourse are much more potent than might be anticipated. As an experienced classroom teacher and a researcher, who was present during the data collection, I was often astonished by what I viewed in the videos. The video data were disturbing. Watching the subjugation of some of the students in the videos was difficult.

Given the sound and extensive pedagogical strategies undertaken by this classroom teacher, this research highlights the complexities associated with teaching students *how to* communicate mathematically. Structuring *into* teaching and learning social transparency through video modeling may assist in teaching communication. Video modeling did seem to have a positive effect on the overall ethos within the groups. Video modeling uncovered the subversive mechanisms within the group that allowed for some to benefit from productive positioning, while others were subject to productive silencing. Following the viewing of the videos, there seemed to be somewhat of a movement towards a collective of learners rather than a collection of learners. At the very least the issues of productive silencing and productive positioning were made explicit as a result of the video modeling.

Although there is wide-reaching consensus on the benefits of peer collaborations in the learning of mathematics, a critical review of this body of research may be needed. *Were data that supported peer collaborations gathered through transcriptions alone? Were students interviewed? What was missed if no video data was included in the analysis? What were the ages of the participants? Which schools were researched? What pedagogy was considered?* These are my remaining ponderings regarding the endorsement of peer collaborations in the learning of mathematics. I also now remain leery of data consisting of transcriptions alone. Such research is missing the social context of learning, which I argue is most important and readily available through video data.

In addition, careful contemplation needs to occur with respect to evaluative measures of communication practices within curriculum documents. Do evaluating peer communications evaluate something *other than* mathematical knowledge (i.e., legitimate participation, existing status characteristics or capitals, etc.)? If the video data was used to evaluate the communication of mathematical knowledge (to a variety of audiences) for the students in this research, few, if any, would have reached a level 1 (i.e., "*communicates for different audiences and purposes with* limited *[my emphasis] effectiveness*" (Ontario Ministry of Education and Training/OMET, 2005, p. 23)).

According to the current curriculum guidelines (Ontario Ministry of Education and Training/OMET, 2005), students are required to be able to communicate with different audiences (e.g., peers) for different purposes (p. 23). The findings from this research have important implications with respect to some evaluative aspects of communication of mathematics as a category of achievement (as seen in the Province of Ontario). Given that mathematical communication seen in this research is, at best, level 2 within the achievement chart, the expectation and importance of students communicating effectively about mathematics to one another seems untenable and certainly questionable in terms of the evaluative validity.

As educators, we may need to interrogate our assumptions more closely regarding peer collaborations, given some of the findings from this research. Certainly, learning to interact with one another in ways that values all participants is a worthy goal of peer collaborations in of itself. However, *can this be evaluated*? *Can we as educators gain knowledge of how a student communicates to audiences other than ourselves*? Furthermore, *what other sorts of knowledge (i.e., social, cultural, racial, etc.) are being communicated within those interactions that cannot be mediated by the teacher or pedagogical intention*? *What lasting effect does this other knowledge that is being communicated in such a setting, have on students' perceptions of themselves as mathematical doers?*

Conclusions

My sense from this research is that learning to communicate about mathematics (or any other subject for that matter) can only occur after students learn to communicate with one another in ways that legitimize and validate all students and not just select few. Only after this sort of communication is achieved can other forms of knowledge *through* communication occur. As shown in this research, video modeling, as a pedagogical tool, may have some potential in this respect.

Admittedly, the outcomes from this research are surprising for me. Often in my viewing of the videos I wondered how difficult it must be for some students (particularly the low achievers) to come to group work knowing fully the forthcoming challenges.

I began this research questioning how students come to develop mathematical discourse from one another. My conclusions and my findings center hardly on mathematical discourse but rather on the way in which we, as humans, interact *and* intersect with one another. As such, the findings from this research contribute to current understandings in sociologies of education, as well as sociologies of mathematics education.

Acknowledgment

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Ad Hoc Sessions

Séances ad hoc

The Nature of Mathematics Students' Questions

Gregory Belostotski
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At the brief ad hoc conversation at the 2008 CMESG meeting in Sherbrooke, I invited participants to consider the counter-intuitive nature of secondary mathematics students asking or not asking questions of their teacher. Specifically, I outlined what I view as a puzzling event that repeats itself in most mathematics classrooms. The interplay of social pressures, motivations and interest, student-teacher relationships, and lesson structures provides a very viscous medium in which questions often do not emerge. However, when a student does ask the teacher a question about mathematics, that question transcends all the possible negative pressures and thus is a significant, yet not very well understood event in learning and teaching of mathematics.

At the time of the meeting, my thoughts were pre-occupied with the locating of relevant literature, as well as situating my study within appropriate theoretical, philosophical and practical frameworks. Below, I give personal highlights of the conversation and provide a brief outline of the direction that my research has taken since the meeting.

One of the key ideas for me that has emerged from the conversation is to consider the type of questions that are asked as a form of help-seeking and the type of questions students ask to pursue deeper sense of knowing. I was immediately drawn to the notion of help-seeking as that was precisely the focus of my attention. In fact, the term 'help-seeking' has proved to be very fertile in searching for relevant literature.

Recently, as I continued to build the framework for my research, I decided to take a more inclusive approach to mathematics students question asking. In particular, I have recently considered the possibility of including the environment such as space and time for asking questions as an important aspect of research into the nature of student questions. The issues of space and time allow for both help-seeking and meaning-forming types of questioning.

Other ideas that have surfaced from the conversation are less tangible and therefore difficult to describe. The audience consisted of one faculty mathematics educator, and several mathematics education students and researchers with a variety of backgrounds including expertise in mathematics teacher training and mathematics learning. Several people took turns talking about their own research experiences, research sites, practicality of classroom observations, and conducting of interviews. As I continue to outline my research proposal, I frequently refer to various suggestions and shared experiences during that and several other conversations.

Since the ad hoc presentation, my research has become more defined. I now wish to consider the *social*, the *discursive*, the *experiential*, and the *temporal* perspectives of secondary mathematics student questions. Here, social perspectives refer to interactions between the individual student and the community of the classroom including the teacher. I plan to research the discursive perspective in an attempt to describe the conversational space for asking questions. This space will include issues of politeness and student-teacher dialogue. Selecting secondary mathematics students as the focus of my study suggests the importance of their past experiences (experiential) with question asking. Finally, temporal perspective refers to the importance of time in question forming.

I am grateful to CMESG for this first opportunity to talk about my research during the ad hoc session. In the end, I have found this ad hoc a rewarding opportunity to talk about my ideas, practice voicing my interests, and listen to helpful suggestions from well-meaning perfect strangers.

Disrupting (Gifted) Teenagers' Mathematical Identity with Epistemological Messiness

Paul Betts, *University of Winnipeg*
Laura McMaster, *Miles Macdonell Collegiate*

Mathematics is commonly perceived as a universal and unquestioned body of knowledge. On the other hand, philosophers have debated the epistemological status of mathematics (e.g., Davis & Hersh, 1981; Ernest, 1998; Lakatos, 1976; Lakoff & Núñez, 2000; Byers, 2007). It is our belief that students can be provided with opportunities to perceive mathematics as more than the tidy and undisputable collection of facts prevalent in school math classrooms (Betts, 2007).

We developed a "messy" conception of the nature of mathematics, based on ideas drawn from the authors listed above, where both the products of mathematics (e.g., theorems, definitions) and processes for making truth claims (e.g., logic) are not considered unquestionable, and so there is room to critique even the most obvious (e.g., 2+2=4). We established two broad philosophical "camps" (with two positions for each) concerning the nature of mathematics, namely (1) Absolutism - Platonism (e.g., Erdös) and Formalism (e.g., Hilbert) and (2) Humanism - Proofs and Refutations (e.g., Lakatos) and Embodiment (Lakoff and Núñez). We then developed a series of activities which developed and illustrated the ideas from each camp, and exposed students to the potential messiness of mathematics.

Our goal was to use messiness to expand (gifted) high school students' conceptions of the nature of mathematics. Not surprisingly, given many years of exposure to an absolutist vision of mathematics, the students we worked with struggled to make sense of mathematics as messy. In this Ad Hoc, we looked closely at how three students (Dorothy, Mary and John, all enrolled in the "Theory of Knowledge" course within the International Baccalaureate Program) adapted to the disruptions triggered by a messy rendering of the nature of mathematics.

Dorothy's adaptation to her discomfort with messiness involved the use of behaviours that covered her discomfort, such as laughing and changing the subject. Mary seemed to deliberately present herself as tentative and unsure, while always maintaining an emotional comfort with Platonism. John, on the other hand, was continually curious, questioning, challenging and skeptical; for him it was a fascinating mental game to play, but at a purely cerebral level only. In each case, we saw the students navigating the disruption of their mathematical experiences by compartmentalizing ideas, emotions, and/or experiences. They kept philosophy of mathematics separate from their experiences of school math; Dorothy and Mary at an emotional level, while John did so at a cognitive level.

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Problem Solving Strategies in the Primary Grades

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Ball, Bass, Sleep, and Thames (2005) suggest that teachers must understand the content in the curriculum to be able to make sense of student errors, know the nature of the learner, and have the ability to structure and sequence learning. This project explores a problem solving approach to teaching mathematics to primary students (grades 1, 2 and 3). Our research question guiding our work is: How do we best structure and sequence learning to help students become better problem solvers? The goals are: (1) to develop problem solving skills in students, (2) to increase an awareness of the efficiency of various problem solving methods in particular contexts for students, and (3) to develop pedagogy around problem solving as a teaching method.

The teachers in this study will develop a series of 9 Smartboard technology lessons that are intended to develop the following 9 problem solving strategies: guess and check, draw a picture, use a calculator, use a number sentence (formulas), find a pattern, work backwards, make a table, and make or use models/ manipulatives. As the lessons progress, students will also be asked to identify, through another chart, which methods they view as most efficient. The tracking charts will be analyzed both throughout and at the end of the series of lessons to determine (a) whether students had an increase in the methods they used, and (b) their perceptions of efficiency of methods - recognizing that some responses may suggest more advanced abilities in mathematics (i.e., using multiplication versus repeated addition).

The teachers in this project are working towards improving our teaching of mathematics and attempting to engage in our own mathematical work. We hope to have a better understanding of problem solving ourselves as a result of this project. We hope to learn how to use new manipulatives to deepen our own mental frameworks as well as help support student learning. The teachers involved feel confident in the first three and have identified the final point as an area of need.

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Recruitment and Retention of Mathematics Students

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Marcella Fioroni, *York University*
Miroslav Lovric, *McMaster University*

The theme of Survey Team 1 work at the recent ICME Conference (ICME 11, Monterrey, Mexico, July 2008), was 'Recruitment, entrance and retention of students to university mathematics studies in different countries'. Chaired by Derek Holton (New Zealand), the team took on the task to survey and review factors that may be co-responsible for present decreasing trends in studying mathematics worldwide, including those that determine students' choice of studies, entrance conditions, and the ways in which universities respond to student needs and preferences when shaping their programs, undertaking teaching and supervision, and conducting examinations.

The three of us have been working on a report on the situation at Canadian universities. We started by looking at the data that we obtained from Dr. Eric Muller: graduation numbers in mathematics and statistics at all levels (undergraduate and graduate) from Statistics Canada, as well as written responses to the short 5-question survey that was mailed out to all members of the Canadian Mathematics Society. We take this opportunity to thank Dr. Eric Muller, not just for sharing all this data with us – his knowledge, discussions, and insights helped us a great deal.

Because we identified a number of additional issues that were not part of the CMS survey, we decided to create our own instrument and administer it to the same constituency. To the original list of 5 questions we added 11 new ones, in order to collect qualitative data that would allow us to understand the issues better, as well as to create a more comprehensive image of activities within mathematics departments at Canadian universities. Discouraged by the low response rate to the original survey given by the CMS, we decided (initially) not to email our survey to potential respondents. Instead, we felt that a more personal approach might be more effective. Besides approaching CMESG conference participants in person, we decided to organize an ad hoc session in order to advertise the survey, and to collect as many responses as possible. Furthermore, we wanted to generate discussion and information on what is happening with student enrolments in undergraduate mathematics courses in Canadian universities. Are student numbers really falling, as seems to be the case in some places? If they are, what can be done about it? If in other places enrolments are increasing, what is happening there for this to be the case?

Ad Hoc discussants touched upon several themes, such as issues with the CEGEP in Quebec, recruitment of female students, and the situation at Queen's university. We identified some interesting points (for example, retiring CEGEP instructors being replaced by young graduate students who are deterred from their studies by teaching jobs) that will not appear in our report but are valuable nonetheless, especially to inform future research. As well, we managed to persuade a few more people to reply to our survey.

Our report, 'Recruitment and retention of mathematics students in Canadian universities', will be published in early 2009 in the *International Journal of Mathematical Education in Science and Technology*.

Experiences of Female Undergraduate Mathematics Students

Jennifer Hall
University of Ottawa

In this ad hoc session, I discussed findings from my master's thesis research, which investigated the high school and university mathematics experiences of women currently in upper years of undergraduate mathematics degree programs. Through individual semi-structured interviews about the women's personal characteristics, families, peers, and experiences in the formal education system, I explored the supports available to them and the challenges they faced.

Two of the early findings from my research were somewhat surprising to me, and I thus shared them in the ad hoc session with the hope of gaining new insight from other conference participants. The first finding was that nearly all the participants in my study had a second major besides mathematics, had begun their university studies in another field, or were planning to obtain a second degree in a field outside mathematics upon graduation. The second finding was that most of the participants strongly voiced a separation between themselves and the 'other' women in mathematics. The participants found the 'other' women in mathematics to be very shy, singularly focused on mathematics, and not very helpful or friendly, and the participants found that they did not fit in with these women.

Discussions in the ad hoc session were quite varied. For instance, one participant in the session questioned whether my requirement of the participants having received all their education in Canada may have limited my participant pool to women with certain characteristics. Another participant shared that most of the female mathematics students at his university also have a second major. The conversation included a discussion about recent research (e.g., Lubinski & Benbow, 2006; Pinker, 2008) that found that mathematically-talented women are self-selecting away from the field due to preferences for fields associated with people or organic materials. Furthermore, Lubinski and Benbow (2006) found that, due to their wider range of skills, women have more choices for fields of study and careers, so thus opted out of the 'hard' sciences whereas men did not have that option due to their narrower range of skills. One participant questioned whether the low enrolment of women in mathematics may be linked to an evolutionary attraction to human beings and relationships. The discussion in the ad hoc session was helpful in providing me with new insight into my research findings.

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Isn't All Teaching Scaffolding:
What Do *You* Scaffold For?

Eva Knoll, *Mount Saint Vincent University*
Mary Jane Harkins, *Mount Saint Vincent University*

In the context of education, scaffolding can be conceptualized as the contribution of the 'other' (often the teacher), when a learner is operating within Vygotsky's ZPD (zone of proximal development). Verenikina (2003) notes that Mercer and Fisher (1992) further specify that:

> *the teaching and learning event […] be followed by evidence of the learners having achieved some greater level of independent competence as a result of the scaffolding experience.*

If this independent competence is to be developed in the learner, the scaffolding needs to be pitched at the right level, and directed to the right aspect of the learning situation for the intended competence to develop. This aspect of the scaffolding intervention is connected to the targeted learning, that is, the additional competence that the learner should take away from the event. This suggests that, for the scaffolding to be effective, its character depends on several factors. Firstly, it depends on the learner's existing knowledge in that the scaffolding is intended to supplement this existing condition. Secondly, it is connects to the learning that the teacher targets, and the nature of this learning. Thirdly, it depends on the socio-cultural context of the event.

In Knoll (2007), the author suggests that the intervention involved in the scaffolding event can be characterized using multiple sets of parameters; for example, the intervention can be positioned on a continuum between pro-active (i.e., planned in advance), and reactive (ad hoc). In addition, it can be aimed at the development of different types of 'independent competencies', including fact recall, routine algorithms, and problem solving competencies.

In order to enhance the scaffolding practice of teachers in the mathematics classroom, we aim to develop a concept map of characterizations of scaffolding practices. The proposed research will begin with a survey of the relevant literature, beginning with Wood, Bruner and Ross (1976), and continue with a series of interviews and dialogues with mathematics educators.

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Translating Messages from Curriculum Statements into Classroom Practice: Communication in Grade 9 Applied Mathematics

Jill Lazarus
Queen's University

The topic of this ad hoc session was the qualitative case study research that I conducted for my Master's thesis. The thesis describes images of communication in the mathematics curriculum. In the Ontario Mathematics Curriculum, communication is one of seven mathematical process expectations that should be addressed in instruction (Ontario Ministry of Education, 2005). It is also one of four categories of knowledge and skills that should be assessed and evaluated. My thesis describes how two teachers translate communication messages from curriculum statements into classroom practice in the Foundations of Mathematics, Grade 9, Applied (MFM1P) course.

Teachers across Ontario indicate on provincial surveys that they support communication in their Grade 9 Applied mathematics classrooms (EQAO, 2007). The teachers who participated in this research would also indicate their support. These two cases illustrate different images, or meanings, associated with this mathematical process. The two cases in this study are complementary. One teacher has a strong background in mathematics and has been heavily involved in mathematics curriculum development and research in Ontario. Her images of communication are heavily tied to those articulated in the curriculum. In her classroom, communication is important for constructing knowledge and learning. The other teacher has a strong background in Special Education and teaches Applied-level courses because of her ability to help students who struggle. Her images are based more heavily on her professional experiences in the classroom and in Special Education. Her reasons for encouraging communication are more practical. In her classroom, communication is important for motivational and management purposes.

During the ad hoc session, participants discussed the research and my interpretations of the findings from the two cases. At the time of the conference, I was in the process of writing the thesis. The feedback that I received during this session was valuable for helping me decide how to present my findings in the thesis.

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High School Students and Mathematics Homework

Ralph T. Mason
University of Manitoba

In this ad hoc session, I shared examples of data taken from an ongoing five-year longitudinal study called *Trajectories* (Mason & McFeetors, 2007). About 100 students in three high schools are participating twice a year in on-line surveys and personal interviews that focus on the experiences of students as they choose courses in mathematics and science and as they choose how to pursue success in those courses. This session addressed directly the range of approaches of academically-oriented students to their mathematics homework.

Grade 9 Mathematics in Manitoba is a one-size-fits-all course, and academically oriented students in our study report that they emerge from their grade nine mathematics without having needed to learn by doing homework. Because high school mathematics is organized into four streams of courses beginning in grade 10, academically oriented students report a very different context in the Pre-Calculus or Honors Mathematics streams. In this ad hoc session, participants looked at one to two pages selected from individual students' interview transcripts. Here are some snippets of students' statements about their mathematics homework.

> *I:* *You mentioned in your survey that homework is more important in grade ten than last year. Can you tell me a little more?*
>
> *Rina:* *Yeah because just if you don't do any homework you won't understand it. Because I have to go over my notes like 5 times before I finally understand them. And just to even like try a question, it's way more like important to even do one question then last year. I just kind of did the notes and I kind of understood it. Sometimes I didn't have to do like any practice questions at all. But now you'll really get confused and lost if you don't do any. So yeah.*

> *I:* *If you already understand it why do you take it home, why do you even do it?*
>
> *Jeremy:* *Practice. So it stays in my head.*
>
> *I:* *Practice makes it stay in your head? How does that happen?*
>
> *Jeremy:* *Just doing it over and over again makes it stay in my head. Yeah.*

> *Zoe:* *I don't like doing math homework. It's boring. It's a waste of energy and then I get frustrated when I don't get it so I just give up 2 seconds after looking at it.*

The participants in the ad hoc session clearly enjoyed the direct access that the transcript selections provided to the voices and experiences of the students. They were struck by the differences among the students' strategic approaches to their homework. Some participants recognized similarities to students they had taught, or even themselves as mathematics students. The ad hoc session closed with discussion about whether effective homework strategies should be taught in mathematics.

In the interviews, students were asked if they ever received advice on how to do homework effectively. The students portrayed their grade 10 mathematics classes as a strategic desert.

They are exhorted to do their homework, and warned of dire consequences if they do not. And those consequences often came to pass, to some degree, for many students. But they report getting no advice on *how* to do their homework so that their efforts result in success, whether they interpret success as good marks or as learning the content fully. None of the interviewed students, drawn from 30 different classrooms of grade 10 mathematics, reported that their teachers noticed or asked about *how* or *why* they approached their homework as they did. Students recognized that they needed to do their homework to be effective, but many students were finding that working *hard* wasn't enough. They want to know how to work *well,* and did not receive guidance in doing so.

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Certainty and Ambiguity: Prolegomena to an Undergraduate Course in Writing and Mathematics

Mircea Pitici
Cornell University

My *ad hoc* presentation is anticipatory. I am preparing to teach a Writing in Mathematics seminar, part of the university-wide program of First-Year Writing Seminars overseen by the Cornell Knight Institute for Writing in the Disciplines. After teaching various mathematics courses, each time following a strict syllabus modeled after a mandatory curriculum, I have my first opportunity to design entirely an undergraduate course and to teach it according to my plans. I will explore, with undergraduates interested in learning mathematics (and, perhaps, interested in fundamental mathematical research), a corpus of literature usually neglected by mathematics professionals and educators.

Whether academics acknowledge it or not, their disciplines are engaged in a battle for "attention as a scarce resource" (Lanham, 2006, p. xi). In this competition, mathematics—a subject demanding the mobilization of abundant cognitive resources—is often shortchanged. Part of the problem resides with the mathematics educators, who put too little effort into bridging the methodological gap between their discipline and other learning domains. Reuben Hersh (1989) observed long ago that most mathematics teachers and professional mathematicians are incompetent writers. That unflattering remark, coming from someone who can hardly be suspected of not sympathizing with mathematics educators, described a situation with antecedents going back to C. P. Snow's (1959) distinction between "word people" and "number people". Yet Hersh's observation coincided with the momentous start of an effort to close the gap between humanistic education and mathematics learning. Things *are* changing in some places. The writing-to-learn-math movement is almost two decades old now and has resulted in a large body of literature, of uneven quality.

In my seminar I will guide the students through the resources currently available. We will explore critically the worthiness of the arguments made in the readings; we will consider aspects of style and substance; we will inquire into the elements that contribute the most to effective expository writing on mathematics; we will opine on what merits emulating and what does not. Then we will follow up by applying what we learn in discussions to the course assignments.

My seminar will be animated by the conviction that writing well in and about mathematics increases the chances that mathematics literacy (or *numeracy*, as the remarkable historian of mathematics education in America Patricia Cline Cohen (1982) called it) will improve. Mathematics helps humans represent simple or complex phenomena, model and refine processes, exercise mental skills. Due to the diversity in this array of qualities, most people are intimidated by mathematics. The first role of the good mathematics writer is to challenge successfully the misperceptions that make so many students anxious when learning mathematics.

Good mathematics writing and good writing *about* mathematics are difficult. They require more than the understanding of the subject discussed; they require the talent to attract and persuade readers not necessarily familiar with the technical aspects of mathematics. A few authors excel in writing well on mathematical topics, whether they popularize or they address to a learned readership. Others sparkle in occasional contributions to collective volumes, in singular monographs, in professional journals, in conference talks, or in reply to polemics. For most people, including some renowned mathematicians (see Dieudonne (1973) for an elitist *credo*), higher mathematics is (and should remain) so abstruse as to appeal only to an elite group of individuals, who supposedly were endowed with the rare ability to comprehend it. I am teaching the Writing in Mathematics seminar with the conviction that it is in the interest of our students and of the society at large to broaden as much as possible mathematical understanding.

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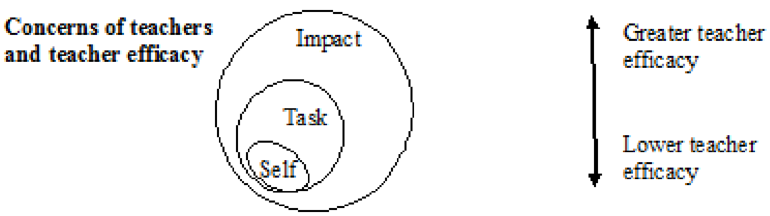
The Complexity of Teacher Concern, Orientation, and Efficacy in Preservice Programs

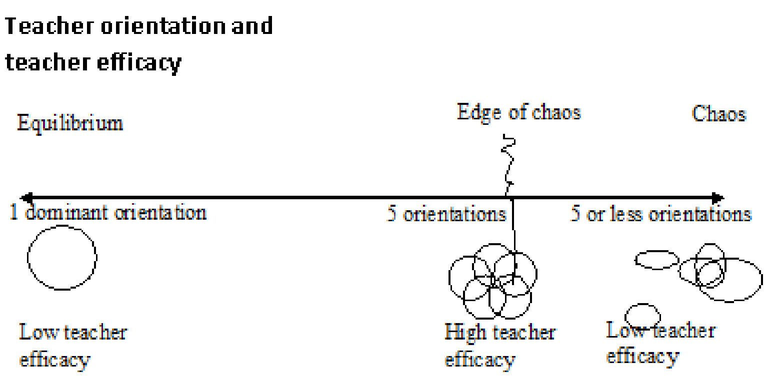
Jamie Pyper
University of Western Ontario

The purpose of the Ad Hoc is to share a conceptualization of mathematics teacher preparation and with collegial input, discuss and elaborate on such a conceptual model. Conceptual frameworks from complexity science, teacher concern, teacher orientation, and teacher efficacy integrate to create an emerging conception of preservice teachers and their teacher preparation.

Complexity science metaphors are used to articulate the dynamics and experiences of education (teaching and learning); embededness, nestedness, emergence, reduction, self-organization, reiteration, and critical mass for an 'edge of chaos' that facilitates a complex learning 'system' (Biesta, 2008; Davis & Simmt, 2003; Davis & Upitis, 2004; Rasmussen, 2008; St. Julien, 2008; Waldrop, 1992). These metaphors provide a basis for integrating the three stages of teacher concern – self, task, impact (Fuller & Bown, 1975); the five orientations one may hold on views and goals about teaching and learning and teacher preparation – academic, technological, practical, personal, critical/social (Feimen-Nemser, 1990); and teacher efficacy (Bandura, 1986; Tschannen-Moran & Woolfolk Hoy, 2001).

The following diagrams provide initial graphics to visualize the above frameworks as components of a larger conceptualization of mathematics teacher preparation.





Four specific questions were posed in the Ad Hoc session: What changes to the graphic models do you suggest? What conceptual frameworks would you contribute to this model?

What contributions to teacher preparation do you think of? Where do one's 'beliefs' fit in, and what kind of beliefs are they? An informative and extensive conversation developed resulting in possible conceptual enhancements.

References available from jpyper@uwo.ca

Enabling In-Service Teacher Professional Development Through Action Research

Natasa Sirotic, *Simon Fraser University*
Susan Oesterle, *Simon Fraser University*

Opportunities for professional development for teachers are limited to attending sessions at conferences and taking courses. These can be helpful, but often the theory they encounter is too far removed from day-to-day practice. It is sometimes difficult even for practical ideas to be integrated effectively, and the often narrow focus can fail to take into account the complexities of the classroom.

We are interested in exploring the idea of bringing in-service into the classroom with the goals of: shifting teacher focus from “how to cover the material” to “how to develop mathematical thinking”, creating communities of practice within schools where teachers can support each other in on-going teacher development, and helping teachers begin to see themselves as researchers and participants in a life-long process. Japanese Lesson Study offers one possible model for how this can be done.

What can a researcher do to facilitate these goals? How can one foster a culture accepting of critique (which seems essential for progress)? As an observer in the classroom, what lenses are useful in filtering our observations?

Panel

Table ronde

Rupture and Coherence in Advocacy in Public Policy

Mary Cameron, *Memorial University of Newfoundland and Labrador*
Peter Liljedahl, *Simon Fraser University*
Frédéric Gourdeau, *Université Laval*
Walter Whitely, *York University*
Florence Glanfield, *University of Alberta, moderator*

Forethoughts

In the last few years, the perceived successes or failure of mathematics education has been a centre of attention in the media. Radio, television, newspapers – we have probably all heard comments or read articles which pleased us and others which made us less than happy. How do we react to this? What might we do as individuals or as a collective? What can we learn from our various experiences engaging with public policy? Should we be more proactive, individually or collectively? Are there some parts of the country where we feel that the voice of mathematics educators is better heard? What role does CMSEG play, or what role might it play in this respect?

Dans les dernières années, les succès et les échecs perçus étaient placés au centre de l'attention médiatique. La radio, la télévision et les journaux ont publié des articles et de commentaires dont certains nous plaisent et d'autres nous déçoivent. Comment réagit-on à cette situation? Que pouvons-nous faire en tant qu'individu ou comme collectivité? Que retient-on de nos expériences avec la politique publique? Devrions-nous être plus proactifs, individuellement et collectivement? Y a-t-il des parties de notre pays où la voix des didacticiens et didacticiennes des mathématiques sont mieux entendues? Quel est le rôle du GCEDM présentement et comment peut-il changer?

You'll notice that in the next few pages, Mary's words provide a context for this panel discussion – it was Mary's questions in the fall of 2006 and spring of 2007 that prompted this panel discussion at the 2008 Annual Meeting. As you read through the next few pages you will see that there are multiple opinions within our community as to whether or not the community as a whole, or individuals, should become engaged in public policy advocacy.

Mary Cameron

In July 2006 I joined Memorial University's Faculty of Education as an assistant professor in primary/elementary mathematics education. I was an outsider to the province and the educational community of Newfoundland and Labrador but looked forward to the challenge of learning how to make my way in a new faculty at a new university in a new province. My excitement began to wane six months later when a loud message about the perceived failure of mathematics education began to resonate through local media. Letters to the editor, news-talk radio, and television interviewers reported that computational skills were not being taught in schools. Problem solving became the brunt of most of the criticism, with the main thrust of arguments suggesting that the 'new math' curriculum, with its focus on problem solving, was to blame. I was more than familiar with the 'math wars' but what surprised me was the one-sided thrust of the message in the media and the strong impact that the thirty-second sound bytes were having on the general public. Standing in line for coffee at Tim Hortons did not leave anyone exempt from experiencing the public disruption created by the local media regarding how allegedly terrible the teaching of mathematics had become. The voices in the media ran counter to my own understanding about the teaching of mathematics to children and to the growing body of research on teaching mathematics to children.

I communicated to the provincial ministry that I belonged to a national body of mathematicians and mathematics educators (CMESG) and perhaps the national body might be able to provide some leadership to the province. I turned to the executive of CMESG for advice. The executive informed me that they could not speak as a body representing the collective. Individual members of CMESG could lend support (some members did give generously, for which I am grateful) but as one voice, CMESG could not engage in public policy. Rather than enter the debate in the media, and feeling rather isolated, I chose to respond to the situation in the one way I felt would be most helpful and that was through my university teaching. Without question, teaching is a political act. I focused on helping pre-service teachers come to understand the heart of the debate and to make sense of the tensions they were witnessing in staff rooms and coffee shops. However, I was still left wondering about the role of CMESG with regards to supporting its members who are supporting the aims of CMESG. Does CMESG have a responsibility to support the voices of mathematics educators in areas of the country where they are often silenced? If so, how might CMESG support its members? What is the role of CMESG when it comes to advocacy, and in particular when it comes to the politics of math education? Indeed, mathematics education remains a powerful topic in the media as we have all witnessed. My hope is that CMESG can continue to evolve in ways that allow for many viewpoints, but also provide a community of support for members who strive to promote the four aims of CMESG. It was a pleasure and an honor to be a part of this pan-Canadian perspective on rupture and dissonance in advocacy in public policy.

Peter Liljedahl

Mathematics education in the media: who is reporting and who is seen as the expert?

In order to better address these questions I felt it was important that we, as a community, have a better understanding of exactly whose voices are represented in the media. Such an understanding would better equip us to make ourselves heard.

Focusing on print media accessible to the general public – newspapers and magazines – I performed a content analysis of 42 articles. In doing this I found that, in essence, there were only six types of articles represented in the sample. There were *editorial* articles written by columnists (n=6), articles written by *investigative reporters* (n=8), articles reporting results from *institutes or research centres* (n=6), articles reporting research performed by *mathematics education researchers* (n=2), articles reporting research performed by *other researchers* (n=19), and *letters to the editor* (n=1)[1]. See Table 1 for a summary of these results.

[1] Although there was only one *letter to the editor* published I also found internet postings for 20 letters to the editor that were not published.

Table 1: Various types of articles in the sample

TYPES OF ARTICLES	n	%
editorial	6	14.3
investigative reporter	8	19.0
institute/centre researcher	6	14.3
mathematics education researcher	2	4.8
other researcher	19	45.2
letter to the editor	1	2.3
TOTAL	42	100

Within these articles there were a number of voices represented as experts. These voices are summarized in Table 2.

Table 2: Various voices seen as experts in the articles

VOICES	n	%
mathematician	8	9.4
teacher	10	11.8
parent	8	9.4
mathematics education professor	5	5.8
administrator	5	5.8
district mathematics specialist	4	4.7
student	2	2.3
psychology professor/researcher	12	14.1
neuroscience professor/researcher	2	2.3
other professor/researcher	12	14.1
institute/centre researcher/spokesperson	5	5.8
school board member	1	1.2
mathematics teacher organization spokesperson	4	4.7
other	7	8.2
TOTAL	85	100

Most troubling in these figures is the lack of representation from the mathematics education community. Our voices were instrumental in the authoring of only 4.8% of the articles and made up only 5.8% of the population of experts. For all intents and purposes we are **silent in** the media. Even more troubling is the possibility that we are **silenced by** the media. In either case, the implications are less than comforting. As a community, we are not seen as being agents in the object of our study. Clearly, this needs to change.

Frédéric Gourdeau

Le milieu de l'enseignement a le dos large. Du décrochage au taux de suicide, tout est la faute du système d'éducation. La situation est évidemment beaucoup plus complexe. Lorsqu'on lit dans les journaux des critiques qui paraissent simplistes, ou qu'on entend des arguments qui invoquent le gros bon sens pour revenir à un enseignement simple et routinier, il y a de bonnes raisons de vouloir réagir.

Au Québec, les multiples réformes de l'enseignement des mathématiques, qu'elles soient spécifiques aux mathématiques ou partie prenante d'une réforme tout azimut, ont eues et ont encore leurs ardents défenseurs et leurs pourfendeurs. Ailleurs au Canada, la situation n'est pas différente. Elle est sans doute même exacerbée à cause des nos voisins du Sud, chez lesquels le mouvement *Back to basics* a fait couler beaucoup d'encre.

In these reforms, what role should or could CMESG play? As an organization, what should we promote? As I was preparing for this panel, I remembered the words of Mogens Niss, ex Secretary General of ICMI, at the 2005 Canadian Forum on Mathematics Education. Let me briefly put this in context. At the Forum, we were considering the possible ways in which a Canadian organization of mathematics educators (CMS, CMESG, or something new) could decide to act. There are varied models internationally and we wanted Mogens to share some of his reflections, what he had seen work best, and what pitfalls to avoid. In his plenary talk, I remember him addressing this issue and though I cannot recall his exact words, my recollection is that he strongly advised us to stay away from specific public advocacy as an organization.

There are at least two good reasons for this. The first reason is the nature of our community. We can be a network, put people in contact and generally play an active role in this respect but, even in our small community, we are not all in agreement as to what should be done concretely: this is probably a massive understatement, as even if there might be some broad consensus on some general statements about mathematics education, getting it down to program and classroom work is bound to get very different answers within our community.

If we could agree, should we then go ahead as a community? I would argue that we should not. Even the best proposed reform in education may not go according to plan. There are many actors involved in education and for proposed changes to yield good results, one needs to have long term engagement, a lot of dedication and possibly some luck! If it does not work out, or even if it simply does not seem to work out, the credibility of any group which has been identified as a key proponent of the reform may suffer badly. The lack of influence which ensues may have long lasting effects. This, as I recall, was something which partly happened in Denmark.

Is that to say that CMESG/GCEDM does nothing and that the mathematical education community idly stands by the side as programs are being rewritten? I would not say so. Working with ministry officials, being part of smaller groups reacting or advising on reform is something which has been done, and which could possibly be done even more. There have been some (and probably many) initiatives which have been nurtured by the CMESG community over the years, if not initiated within it. In Quebec, mathematicians like myself, and mathematics educators, have been invited to sit on advisory committees close to the actual elaboration of programs. This was probably not done in the most effective way, but it was done.

Au Québec toujours, la dernière réforme (*la* réforme) de l'éducation va de l'avant. Malgré plusieurs ratés et des évaluations extrêmement variées (pour ne pas dire carrément contradictoires), la réforme tient en bonne partie le cap. Cette réforme, que je qualifierais d'idéologique, s'est faite à partir de grands et nobles principes, mais aussi en écartant ceux qui ne pensaient pas de *la* bonne manière. Parmi ceux là, les disciplinaires (didacticiens inclus) ont été maintes fois exclus des rencontres et décisions importantes, et leurs analyses et recherches n'ont pas été sollicitées. On trouvera bien quelques exceptions, je sais, mais cette réforme a relégué le contenu à un rôle de soutien, tout au plus, au développement des compétences. Le discours officiel quant à l'importance du contenu disciplinaire a d'ailleurs été tout-à-fait erratique, et les multiples contradictions que l'on peut relever au fil des ans illustrent bien son rôle accessoire pour les bonzes de la réforme.

Lors de l'élaboration de cette réforme en mathématiques, l'étroite collaboration de quelques personnes en mathématiques et en didactique des mathématiques avec quelques fonctionnaires attachés au contenu a permis de nuancer quelques dérives franchement ridicules. Au royaume de l'absolutisme, on pourrait presque parler de subversion! Je crois que ce travail de collaboration, qui s'était développé au fil des ans, a eu une influence très positive en mathématiques.

Finally, let me say a few words on the Canadian political landscape. Education is a provincial responsibility. It is intimately linked to language and culture. It is closely linked to history and to the education systems in place. It is not disjoint from television, music, art, and I could go on, and to the influences these have on our youth and our society, both as a reflection of who we are and as a maker of who we are. While reflections and discussions at CMESG/GCEDM have played and will continue to play a crucial role in our understanding of mathematics education, the implementation of these understandings needs to respect our differences. In my opinion, the implicit respect and collaboration which is a key feature of our community is our greatest asset, and we should strive to preserve it.

Walter Whiteley

My comments are based on my background of some prior advocacy experiences in Ontario. If you are interested in advocacy work then I must caution you, you must be prepared to work for a decade or more to see the impacts of your work.

Research does not count!

Overwhelming experience shows that 'research results' are not a basis for policy makers and even less a basis for the public response. Their stories and their 'common sense' matters far more.

It is hard even to make research a basis for altering teaching practices at any level (including in universities, even in mathematics departments or in faculties of education). There is simply no impact (see Burkhardt and Schoenfeld and paper of Beggs, Davis and …). This was the conclusion of my graduate class (several members in this audience participated in this).

An example: Strategies to avoid the math wars. Success to date – but requires constant work.

I share a message from my general social advocacy work. It is important to cultivate allies – carefully, and think hard, issue by issue, who are your allies? We need allies across the country to resist some of the pressures sweeping North America.

We have to work hard to keep the mathematics community from turning on the mathematics education community. For example, which conversations do you have with engineers, physicists, pure mathematicians? What is the difference between a check list of topics vs. discussion of abilities and processes?

This takes time, but results in much better outcomes. It is terribly important to plan for multi-year campaigns.

Note on the press

In general, the press does not have an interest in thoughtful analysis. During the Ontario curriculum revisions, I along with Peter Taylor, were associated with a position which was the consensus of mathematics departments in Ontario (with the exception of Waterloo and Downtown University of Toronto): don't do calculus in high school. It is crowding more important things (like processes, reasoning, … making meaning, …) in high schools. I was interviewed at least four times and explained the context. The reporters thanked me for helping them make sense of some of the debate – but I was never quoted! That was O.K. – but interesting. If the CMESG does take (occasional) positions, then hopefully it will be thoughtful enough to share that fate – make sense of events, but not be quoted!

Some other examples of 'advocacy' around CMESG

Several years ago, a working group created a Manifesto on 'unstuffing the curriculum'. It was a good working group, and I have been able to use the one page Manifesto to give clear context for positions being taken (with colleagues, with the ministry, with my classes, ….). This was advocacy – but was not endorsed directly by CMESG or CMS. I have also used other working group reports to back up proposals, such as development of new programs for pre-service teachers. (Come to my CMS talk in 8 days in Montreal.) So working group reports can be (should be?) advocacy!

YES, BE advocates

I think there are several areas where we could have a substantial discussion and create other position papers which will help us do our work outside of CMESG/GCEDM. They probably would not be 'big in the press' but they could draw on research findings and give credence. I offer three possible examples:

<u>Example 1: What is a good process for curriculum revision?</u>

We'd need to identify our allies (all the classroom teachers, many of the textbook publishers), and who we'd need to battle (people at higher levels in the Ministry as they would lose some of their control, and it could become ministry bashing; or lobby groups with specific goals, e.g. back to basics). But an open process would be harder to influence. There is a risk that in open and fair process, some of our pet projects (e.g. more geometry!!) could lose out.

Sample: There is currently a three year curriculum project in Ontario. People from three Mathematics Education communities are involved: the provincial classroom teacher organization, the provincial school board mathematics coordinators organization, and the Fields Math Ed Forum (includes college and university faculty and grad students). We plan to work for three years in advance. The support from research and from individual stories would help us in our work. It's taken us 18 months to narrow down a topic.

<u>Example 2: What is the role of mathematics (and math beyond number) in pre-school/early childhood?</u>

In this example our allies could include parents/equity seeking groups, psychologists (we'd need to be careful), early childhood educators, and private (and some public) day care/pre-school operators. However, there are problems with groups such as the College of Early Childhood Educators and private (and some public) day care/pre-school operators as their goals/missions may not align with the position that this example is advocating.

<u>Example 3: What is the role of specialists in Math Ed in the middle years (Grades 5-8)?</u>

In this example our allies could be mathematics teachers at the next level, many parents, potentially many teachers in the middle years (some would be relieved depending on how this idea is implemented), and potentially science educators (particularly in small schools if the statement 'mathematics and science education' was in the proposal).

Who would the 'battle' be with in this example? Possibly many people in Faculties of Education; possibly school administrators (don't know), elementary teachers unions (don't know…but could shift into teacher bashing), Ministries of Education/College of Teachers, school boards (there would be a shortage and might be pressure to pay better!).

The press might be neutral or even positive. The early childhood statement would be an easier sell, but one would need stories, not data. Our position, or our newsworthy 'story,' in examples 2 and 3 is that we need better prepared AND more motivated students in mathematics/science as citizens and parents. Equity might also be our story but we would have to be careful as it could end up with the story of being more competitive and generating more people who lose.

Criteria for involvement

If the CMESG/GCEDM did become involved in public policy advocacy then it should be in areas where there is dominant outcome from research (rather than the middle of a debate). We'd have to figure out who might be our allies and whether or not we need to work with them to develop the next generation of advocacy documents and strategies. Perhaps it might be good enough for CMESG/GCEDM to just live as working group reports – but reports written to include advocacy documents and even strategies. Advocacy needs people to work for multiple years.

Afterthoughts – Florence Glanfield

The panelists were asked to respond to the following questions (you will notice that I've changed 'we' to 'mathematics educators'): How do mathematics educators react to the media's attention to the perceived successes or failure of mathematics education? What might mathematics educators do as individuals or as a collective? What can mathematics educators learn from our various experiences engaging with public policy? Should mathematics educators be more proactive, individually or collectively? Are there some parts of the country where the voice of mathematics educators is better heard? What role does CMSEG/GCEDM play, or what role might it play in this respect?

Peter's comments remind us of the positioning of mathematics education researchers in the media and lead us to wonder why mathematics education researchers are silenced in the media. Frédéric's comments help us to consider an international and Canadian perspective of this topic, arguing that our CMESG/GCEDM community as a whole should not be engaged in advocacy in public policy. Walter shares examples of advocacy experiences and possible advocacy positions. Walter also suggests that the community as a whole could be seen as advocates through our working group reports, and urges individuals to become public policy advocates.

In reflecting on these thoughts, further questions cross my mind: In what way(s) do mathematics educators believe that public policy is reflected in classrooms and in classroom practices? What might, or might not, be different for children learning mathematics in classrooms if mathematics educators are (or are not) involved in public policy advocacy? I also wonder what role a community of mathematics educators might play in helping the public (and then in a sense public policy) come to understand that mathematics and the study of mathematics are not just for the elite. The work of Tom Archibald (2008) of Simon Fraser University enters my thoughts here. Tom's work in the history of mathematics has offered me insight into the present-day mathematics curriculum that is privileged in schools and school systems across Canada. Tom's work helped me to see the historical nature of the way in which mathematics has come to been seen as a study only for the 'elite' or 'privileged' – thereby suggesting to 'all others' that there is no room for them in mathematics (and, by association in mathematics education). At the same time organizations such as the National Council of Teachers of Mathematics (who are actively involved in public policy advocacy) and other organizations (such as Ministries of Education) are promoting the notion that 'Mathematics is for ALL.' In what way(s) do mathematics educators come to acknowledge this historical perspective (of which many mathematics educators and members of the public are a product) as we move into attempting to begin to 'restory' mathematics and the study of mathematics in a different way – a way that suggests that 'Mathematics is for ALL?'

At the same time, I go back to Mary's questions, "However, I was still left wondering about the role of CMESG with regards to supporting its members who are supporting the aims of CMESG. Does CMESG have a responsibility to support the voices of mathematics educators in areas of the country where they are often silenced? If so, how might CMESG support its members?" As current president of CMESG/GCEDM, I offer an answer to Mary and all members who are looking for support in response to public policy initiatives – whether the initiatives are being debated in the media or in other venues such as public forums:

The individuals that form the organization known as the CMESG/GCEDM bring a breadth of experiences to the collective. As one comes to learn about the individuals within the CMESG/GCEDM one comes to know about the rich diversity of the individuals that contribute to the strength of the collective. When one wonders about the ways in which they 'might' or 'might not' engage in these public discussions around policy and mathematics education, any member is invited to submit these queries and encourage dialogue on the list serve. The list serve provides our membership with access to individuals who've engaged in public policy advocacy and discussions in various ways throughout their careers.

I've come to consider the organization called the CMESG/GCEDM as a mathematical community or a collective learning system (Davis and Simmt, 2003) over the course of our interactions together. I see our mathematical community known as the CMESG/GCEDM coming together in a variety of ways "(i) to study the theories and practices of the teaching of mathematics; (ii) to promote research in mathematics education; (iii) to exchange ideas and information about all aspects of mathematics education in Canada; and (iv) to disseminate the results of its work" (CMESG/GCEDM Constitution, 2008). I believe that as one engages in interactions within the CMESG/GCEDM community one comes to develop an understanding of the way(s) in which the individuals themselves might or might not be engaged in public policy advocacy.

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Appendices

Annexes

APPENDIX A / ANNEXE A

Working Groups at Each Annual Meeting / Groupes de travail des rencontres annuelles

1977 *Queen's University, Kingston, Ontario*

- Teacher education programmes
- Undergraduate mathematics programmes and prospective teachers
- Research and mathematics education
- Learning and teaching mathematics

1978 *Queen's University, Kingston, Ontario*

- Mathematics courses for prospective elementary teachers
- Mathematization
- Research in mathematics education

1979 *Queen's University, Kingston, Ontario*

- Ratio and proportion: a study of a mathematical concept
- Minicalculators in the mathematics classroom
- Is there a mathematical method?
- Topics suitable for mathematics courses for elementary teachers

1980 *Université Laval, Québec, Québec*

- The teaching of calculus and analysis
- Applications of mathematics for high school students
- Geometry in the elementary and junior high school curriculum
- The diagnosis and remediation of common mathematical errors

1981 *University of Alberta, Edmonton, Alberta*

- Research and the classroom
- Computer education for teachers
- Issues in the teaching of calculus
- Revitalising mathematics in teacher education courses

1982 *Queen's University, Kingston, Ontario*

- The influence of computer science on undergraduate mathematics education
- Applications of research in mathematics education to teacher training programmes
- Problem solving in the curriculum

1983 *University of British Columbia, Vancouver, British Columbia*

- Developing statistical thinking
- Training in diagnosis and remediation of teachers
- Mathematics and language
- The influence of computer science on the mathematics curriculum

1984 *University of Waterloo, Waterloo, Ontario*

- Logo and the mathematics curriculum
- The impact of research and technology on school algebra
- Epistemology and mathematics
- Visual thinking in mathematics

1985 *Université Laval, Québec, Québec*

- Lessons from research about students' errors
- Logo activities for the high school
- Impact of symbolic manipulation software on the teaching of calculus

1986 *Memorial University of Newfoundland, St. John's, Newfoundland*

- The role of feelings in mathematics
- The problem of rigour in mathematics teaching
- Microcomputers in teacher education
- The role of microcomputers in developing statistical thinking

1987 *Queen's University, Kingston, Ontario*

- Methods courses for secondary teacher education
- The problem of formal reasoning in undergraduate programmes
- Small group work in the mathematics classroom

1988 *University of Manitoba, Winnipeg, Manitoba*

- Teacher education: what could it be?
- Natural learning and mathematics
- Using software for geometrical investigations
- A study of the remedial teaching of mathematics

1989 *Brock University, St. Catharines, Ontario*

- Using computers to investigate work with teachers
- Computers in the undergraduate mathematics curriculum
- Natural language and mathematical language
- Research strategies for pupils' conceptions in mathematics

1990 *Simon Fraser University, Vancouver, British Columbia*

- Reading and writing in the mathematics classroom
- The NCTM "Standards" and Canadian reality
- Explanatory models of children's mathematics
- Chaos and fractal geometry for high school students

1991 *University of New Brunswick, Fredericton, New Brunswick*

- Fractal geometry in the curriculum
- Socio-cultural aspects of mathematics
- Technology and understanding mathematics
- Constructivism: implications for teacher education in mathematics

1992 *ICME–7, Université Laval, Québec, Québec*

1993 *York University, Toronto, Ontario*

- Research in undergraduate teaching and learning of mathematics
- New ideas in assessment
- Computers in the classroom: mathematical and social implications
- Gender and mathematics
- Training pre-service teachers for creating mathematical communities in the classroom

1994 *University of Regina, Regina, Saskatchewan*

- Theories of mathematics education
- Pre-service mathematics teachers as purposeful learners: issues of enculturation
- Popularizing mathematics

1995 *University of Western Ontario, London, Ontario*

- Autonomy and authority in the design and conduct of learning activity
- Expanding the conversation: trying to talk about what our theories don't talk about
- Factors affecting the transition from high school to university mathematics
- Geometric proofs and knowledge without axioms

1996 *Mount Saint Vincent University, Halifax, Nova Scotia*

- Teacher education: challenges, opportunities and innovations
- Formation à l'enseignement des mathématiques au secondaire: nouvelles perspectives et défis
- What is dynamic algebra?
- The role of proof in post-secondary education

1997 *Lakehead University, Thunder Bay, Ontario*

- Awareness and expression of generality in teaching mathematics
- Communicating mathematics
- The crisis in school mathematics content

1998 *University of British Columbia, Vancouver, British Columbia*

- Assessing mathematical thinking
- From theory to observational data (and back again)
- Bringing Ethnomathematics into the classroom in a meaningful way
- Mathematical software for the undergraduate curriculum

1999 *Brock University, St. Catharines, Ontario*

- Information technology and mathematics education: What's out there and how can we use it?
- Applied mathematics in the secondary school curriculum
- Elementary mathematics
- Teaching practices and teacher education

2000 *Université du Québec à Montréal, Montréal, Québec*

- Des cours de mathématiques pour les futurs enseignants et enseignantes du primaire/Mathematics courses for prospective elementary teachers
- Crafting an algebraic mind: Intersections from history and the contemporary mathematics classroom
- Mathematics education et didactique des mathématiques : y a-t-il une raison pour vivre des vies séparées?/Mathematics education et didactique des mathématiques: Is there a reason for living separate lives?
- Teachers, technologies, and productive pedagogy

2001 *University of Alberta, Edmonton, Alberta*

- Considering how linear algebra is taught and learned
- Children's proving
- Inservice mathematics teacher education
- Where is the mathematics?

2002 *Queen's University, Kingston, Ontario*

- Mathematics and the arts
- Philosophy for children on mathematics
- The arithmetic/algebra interface: Implications for primary and secondary mathematics / Articulation arithmétique/algèbre: Implications pour l'enseignement des mathématiques au primaire et au secondaire
- Mathematics, the written and the drawn
- Des cours de mathémathiques pour les futurs (et actuels) maîtres au secondaire / Types and characteristics desired of courses in mathematics programs for future (and in-service) teachers

2003 *Acadia University, Wolfville, Nova Scotia*

- L'histoire des mathématiques en tant que levier pédagogique au primaire et au secondaire / The history of mathematics as a pedagogic tool in Grades K–12
- Teacher research: An empowering practice?
- Images of undergraduate mathematics
- A mathematics curriculum manifesto

2004 *Univerité Laval, Québec, Québec*

- Learner generated examples as space for mathematical learning
- Transition to university mathematics
- Integrating applications and modeling in secondary and post secondary mathematics
- Elementary teacher education - Defining the crucial experiences
- A critical look at the language and practice of mathematics education technology

2005 *University of Ottawa, Ottawa, Ontario*

- Mathematics, education, society, and peace
- Learning mathematics in the early years (pre-K – 3)
- Dicrete mathematics in secondary school curriculum
- Socio-cultural dimensions of mathematics learning

2006 *University of Calgary, Alberta*

- Secondary mathematics teacher development
- Developing links between statistical and probabilistic thinking in school mathematics education
- Developing trust and respect when working with teachers of mathematics
- The body, the sense, and mathematics learning

2007 *University of New Brunswick, New Brunswick*

- Outreach in mathematics – Activities, engagement, & reflection
- Geometry, space, and technology: challenges for teachers and students
- The design and implementation of learning situations
- The multifaceted role of feedback in the teaching and learning of mathematics

2008 *Université de Sherbrooke, Sherbrooke*

- Mathematical reasoning of young children
- Mathematics-in-and-for-teaching (MifT): the case of algebra
- Mathematics and human alienation
- Communication and mathematical technology use throughout the post-secondary curriculum / Utilisation de technologies dans l'enseignement mathématique postsecondaire
- title xx

APPENDIX B / ANNEXE B

Plenary Lectures at Each Annual Meeting / Conférences plénières des rencontres annuelles

1977	A.J. COLEMAN	The objectives of mathematics education
	C. GAULIN	Innovations in teacher education programmes
	T.E. KIEREN	The state of research in mathematics education
1978	G.R. RISING	The mathematician's contribution to curriculum development
	A.I. WEINZWEIG	The mathematician's contribution to pedagogy
1979	J. AGASSI	The Lakatosian revolution
	J.A. EASLEY	Formal and informal research methods and the cultural status of school mathematics
1980	C. GATTEGNO	Reflections on forty years of thinking about the teaching of mathematics
	D. HAWKINS	Understanding understanding mathematics
1981	K. IVERSON	Mathematics and computers
	J. KILPATRICK	The reasonable effectiveness of research in mathematics education
1982	P.J. DAVIS	Towards a philosophy of computation
	G. VERGNAUD	Cognitive and developmental psychology and research in mathematics education
1983	S.I. BROWN	The nature of problem generation and the mathematics curriculum
	P.J. HILTON	The nature of mathematics today and implications for mathematics teaching

1984	A.J. BISHOP	The social construction of meaning: A significant development for mathematics education?
	L. HENKIN	Linguistic aspects of mathematics and mathematics instruction
1985	H. BAUERSFELD	Contributions to a fundamental theory of mathematics learning and teaching
	H.O. POLLAK	On the relation between the applications of mathematics and the teaching of mathematics
1986	R. FINNEY	Professional applications of undergraduate mathematics
	A.H. SCHOENFELD	Confessions of an accidental theorist
1987	P. NESHER	Formulating instructional theory: the role of students' misconceptions
	H.S. WILF	The calculator with a college education
1988	C. KEITEL	Mathematics education and technology
	L.A. STEEN	All one system
1989	N. BALACHEFF	Teaching mathematical proof: The relevance and complexity of a social approach
	D. SCHATTSNEIDER	Geometry is alive and well
1990	U. D'AMBROSIO	Values in mathematics education
	A. SIERPINSKA	On understanding mathematics
1991	J .J. KAPUT	Mathematics and technology: Multiple visions of multiple futures
	C. LABORDE	Approches théoriques et méthodologiques des recherches françaises en didactique des mathématiques
1992	ICME-7	
1993	G.G. JOSEPH	What is a square root? A study of geometrical representation in different mathematical traditions
	J CONFREY	Forging a revised theory of intellectual development: Piaget, Vygotsky and beyond
1994	A. SFARD	Understanding = Doing + Seeing ?
	K. DEVLIN	Mathematics for the twenty-first century
1995	M. ARTIGUE	The role of epistemological analysis in a didactic approach to the phenomenon of mathematics learning and teaching
	K. MILLETT	Teaching and making certain it counts
1996	C. HOYLES	Beyond the classroom: The curriculum as a key factor in students' approaches to proof
	D. HENDERSON	Alive mathematical reasoning

1997	R. BORASSI	What does it really mean to teach mathematics through inquiry?
	P. TAYLOR	The high school math curriculum
	T. KIEREN	Triple embodiment: Studies of mathematical understanding-in-interaction in my work and in the work of CMESG/GCEDM
1998	J. MASON	Structure of attention in teaching mathematics
	K. HEINRICH	Communicating mathematics or mathematics storytelling
1999	J. BORWEIN	The impact of technology on the doing of mathematics
	W. WHITELEY	The decline and rise of geometry in 20th century North America
	W. LANGFORD	Industrial mathematics for the 21st century
	J. ADLER	Learning to understand mathematics teacher development and change: Researching resource availability and use in the context of formalised INSET in South Africa
	B. BARTON	An archaeology of mathematical concepts: Sifting languages for mathematical meanings
2000	G. LABELLE	Manipulating combinatorial structures
	M. B. BUSSI	The theoretical dimension of mathematics: A challenge for didacticians
2001	O. SKOVSMOSE	Mathematics in action: A challenge for social theorising
	C. ROUSSEAU	Mathematics, a living discipline within science and technology
2002	D. BALL & H. BASS	Toward a practice-based theory of mathematical knowledge for teaching
	J. BORWEIN	The experimental mathematician: The pleasure of discovery and the role of proof
2003	T. ARCHIBALD	Using history of mathematics in the classroom: Prospects and problems
	A. SIERPINSKA	Research in mathematics education through a keyhole
2004	C. MARGOLINAS	La situation du professeur et les connaissances en jeu au cours de l'activité mathématique en classe
	N. BOULEAU	La personnalité d'Evariste Galois: le contexte psychologique d'un goût prononcé pour les mathématique abstraites
2005	S. LERMAN	Learning as developing identity in the mathematics classroom
	J. TAYLOR	Soap bubbles and crystals
2006	B. JAWORSKI	Developmental research in mathematics teaching and learning: Developing learning communities based on inquiry and design
	E. DOOLITTLE	Mathematics as medicine

2007	R. NÚÑEZ	Understanding abstraction in mathematics education: Meaning, language, gesture, and the human brain
	T. C. STEVENS	Mathematics departments, new faculty, and the future of collegiate mathematics
2008	A. DJEBBAR	Title
	A. WATSON	Adolescent learning and secondary mathematics

APPENDIX C / ANNEXE C

Proceedings of Annual Meetings / Actes des rencontres annuelles

Past proceedings of CMESG/GCEDM annual meetings have been deposited in the ERIC documentation system with call numbers as follows:

Proceedings of the 1980 Annual Meeting	ED 204120
Proceedings of the 1981 Annual Meeting	ED 234988
Proceedings of the 1982 Annual Meeting	ED 234989
Proceedings of the 1983 Annual Meeting	ED 243653
Proceedings of the 1984 Annual Meeting	ED 257640
Proceedings of the 1985 Annual Meeting	ED 277573
Proceedings of the 1986 Annual Meeting	ED 297966
Proceedings of the 1987 Annual Meeting	ED 295842
Proceedings of the 1988 Annual Meeting	ED 306259
Proceedings of the 1989 Annual Meeting	ED 319606
Proceedings of the 1990 Annual Meeting	ED 344746
Proceedings of the 1991 Annual Meeting	ED 350161
Proceedings of the 1993 Annual Meeting	ED 407243
Proceedings of the 1994 Annual Meeting	ED 407242

Proceedings of the 1995 Annual Meeting	ED 407241
Proceedings of the 1996 Annual Meeting	ED 425054
Proceedings of the 1997 Annual Meeting	ED 423116
Proceedings of the 1998 Annual Meeting	ED 431624
Proceedings of the 1999 Annual Meeting	ED 445894
Proceedings of the 2000 Annual Meeting	ED 472094
Proceedings of the 2001 Annual Meeting	ED 472091
Proceedings of the 2002 Annual Meeting	*submitted*
Proceedings of the 2003 Annual Meeting	*submitted*
Proceedings of the 2004 Annual Meeting	*submitted*
Proceedings of the 2005 Annual Meeting	*submitted*
Proceedings of the 2006 Annual Meeting	*submitted*
Proceedings of the 2007 Annual Meeting	*submitted*
Proceedings of the 2008 Annual Meeting	*submitted*

Note

There was no Annual Meeting in 1992 because Canada hosted the Seventh International Conference on Mathematical Education that year.