CANADIAN MATHEMATICS EDUCATION STUDY GROUP

GROUPE CANADIEN D'ÉTUDE EN DIDACTIQUE DES MATHÉMATIQUES

PROCEEDINGS / ACTES
2014 ANNUAL MEETING / RENCONTRE ANNUELLE 2014

CMESG GCEDM

University of Alberta
May 30 – June 3, 2014

EDITED BY:
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in memory of Doug Franks and Sandy Dawson

dedicated mathematics educators and dear friends

they will be missed

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PROCEEDINGS OF THE 2014 ANNUAL MEETING OF THE CANADIAN MATHEMATICS EDUCATION STUDY GROUP / ACTES DE LA RENCONTRE ANNUELLE 2014 DU GROUPE CANADIEN D’ÉTUDE EN DIDACTIQUE DES MATHÉMATIQUES

38th Annual Meeting
University of Alberta
May 30 – June 3, 2014

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INTRODUCTION

Olive Chapman – President, CMESG/GCEDM
University of Calgary

The 38th annual meeting of the Canadian Mathematics Education Study Group/Groupe Canadien d'étude en didactique des mathématiques [CMESG/GCEDM 2014] was another memorable learning and social event!

Our hosts at the University of Alberta made sure we were well fed, entertained, and accommodated. The excursion to Fort Edmonton Park (which allowed some of us to also enjoy a nature hike to the venue) with dinner at the rustic Egge's Barn and the conference dinner at the Faculty Club were beyond my expectations. Thanks to our colleagues, faculty members, Drs. Yvette d'Entremont, Florence Glanfield, Julie Long, Lynn McGarvey, and Elaine Simmt for their thoughtful planning and hosting of the meeting. Also thanks to the other members of the organizing team for their valuable contribution to the planning and smooth running of the meeting: the sessional instructors, Robert Bechtel, Janelle McFeetors, and Carrie Watt and the undergraduate and graduate students, Shelley Barton, Priscila Dias Correa, Trina Ertman, Behnaz Herbst, Dakota Jesse, Lixin Luo, Billie Dawn McDonald, Marina Spreen, Jayne Powell, and Christine Wiebe Buchanan. Finally thanks for the financial support of the Faculty of Education conference fund, the Faculty of Education Centre for Mathematics Science Technology Education, the Dean of the Faculty of Education, the Departments of Elementary and Secondary Education – Faculty of Education, and the Faculté St Jean.

I also acknowledge the CMESG/GCEDM executive for organizing another stimulating program with topics relevant to our membership of mathematicians, mathematics teacher educators and mathematics education researchers. On behalf of the executive, thanks to the two plenary speakers, Dr. Dave Hewitt for engaging us in the economic use of time and effort in mathematics classrooms and Dr. Nilima Nigam for the meaningful examples of mathematical problem-solving as applied to real problems from industry and the non-profit sectors. Thanks to Dr. Tom Kieren whose Elder Talk offered insights of the various ways interaction affects mathematics knowing and of mathematics knowing-in-action in mathematics classrooms. Thanks also to the leaders of the five Working Groups; the presenters of the three Topic Sessions; the ten new PhDs; the Ad Hoc and Math Gallery Walk presenters; the presenters of the Panel for tackling the issue of what we have not been hearing about PISA in the reporting and interpretation of the results for Canada; and all the participants for making the 2014 meeting a stimulating and worthwhile experience.

This "Proceedings" of the meeting offers the opportunity for readers to learn about some of the mathematics education research and interests of our community. The variety of topics covered from the early grades to post-secondary mathematics education will definitely provide a meaningful way for participants to further reflect on and build on their experiences at the meeting and for others to share in and be inspired by the work of the mathematics education community in Canada.

Down from Grande Prairie
a contingent of thirteen
descended upon

CMESG (GCEDM)
meeting here in Edmonton
changing the climate

by being present
thus enriching discussion.
Transforming the place

in multiple ways
affecting interactions
expanding circles

changing perceptions
nudging boundaries moving
forth inclusiveness.

More teachers welcomed
whether elementary
or secondary.

Adding dimensions,
understandings, perspectives
widening the lens.

Viewing with fresh eyes.
Ears hearing differently.
Voicing new questions.

Verses expressing
gratitude today for your
participation.

John McLoughlin
June 2, 2014

Horaire

Vendredi 30 mai	Samedi 31 mai	Dimanche 1 juin	Lundi 2 juin	Mardi 3 juin
	8:00 – 8:45 **Inscription**			
	8:45 – 10:15 **Groupes de travail**	8:45 – 10:15 **Groupes de travail**	8:45 – 10:15 **Groupes de travail**	8:45 – 10:15 **Panel**
	10:15 – 10:45 Pause café	10:15 – 10:45 Pause café	10:15 – 10:45 Pause café	10:15 – 10:45 Pause café
	10:45 – 12:15 **Groupes de travail**	10:45 – 12:15 **Groupes de travail**	10:45 – 12:15 **Groupes de travail**	10:45 – 11:30 **La parole aux anciens**
				11:45 – 12:30 **Séance de clôture**
	12:30 – 13:30 Dîner	12:30 – 13:30 Dîner	12:30 – 13:30 Dîner	
	13:30 – 14:00 **Café dessert et Gallérie mathématique**	13:30 – 14:00 **Séances *ad hoc***	13:30 – 14:00 **Petits groupes**	
14:30 – 18:45 **Inscription** 15:30 – 16:20 **Amis de FLM**	14:10 – 14:40 **Petits groupes**	14:10 – 14:40 **N^{lles} thèses (3)**	14:10 – 15:10 **Plénière 2 : Séance questions**	
	14:50 – 15:50 **Plénière 1 : Séance questions**	14:50 – 15:50 **Plénière 2**	15:20 – 16:20 **Séances thématiques**	
	15:50 – 16:10 Pause café	16:30 – 21:00 **Excursion** *Edmonton Architectural Tour & Fort Edmonton Park* **Souper**	16:20 – 16:40 Pause café	
	16:10 – 16:40 **N^{lles} thèses (1)**			
17:00 – 18:30 **Souper**	16:45 – 17:15 **N^{lles} thèses (2)**		16:50 – 18:15 **Assemblée générale annuelle**	
18:45 – 19:30 **Séance d'ouverture**	**Souper libre**		19:00 – ? **Souper** *Faculty Club*	
19:30 – 20:30 **Plénière 1**				
20:30 – 23:30 **Réception**				

Schedule

Friday May 30	Saturday May 31	Sunday June 1	Monday June 2	Tuesday June 3
	8:00 – 8:45 **Registration**			
	8:45 – 10:15 **Working Groups**	8:45 – 10:15 **Working Groups**	8:45 – 10:15 **Working Groups**	8:45 – 10:15 **Panel**
	10:15 – 10:45 Coffee Break	10:15 – 10:45 Coffee Break	10:15 – 10:45 Coffee Break	10:15 – 10:45 Coffee Break
	10:45 – 12:15 **Working Groups**	10:45 – 12:15 **Working Groups**	10:45 – 12:15 **Working Groups**	10:45 – 11:30 **Elder Talk**
				11:45 – 12:30 **Closing Session**
	12:30 – 13:30 Lunch	12:30 – 13:30 Lunch	12:30 – 13:30 Lunch	
	13:30 – 14:00 **Dessert, Coffee & Mathematics Gallery**	13:30 – 14:00 ***Ad Hoc* Sessions**	13:30 – 14:00 **Small Groups**	
	14:10 – 14:40 **Small Groups**	14:10 – 14:40 **New PhDs (3)**	14:10 – 15:10 **Plenary 2 Discussion**	
14:30 – 18:45 **Registration** 15:30 – 16:20 **Friends of FLM**	14:50 – 15:50 **Plenary 1 Discussion**	14:50 – 15:50 **Plenary 2**	15:20 – 16:20 **Topic Sessions**	
	15:50 – 16:10 Coffee Break	16:30 – 21:00 **Excursion** *Edmonton Architectural Tour & Fort Edmonton Park* **Dinner**		
	16:10 – 16:40 **New PhDs (1)**		16:20 – 16:40 Coffee Break	
17:00 – 18:30 **Dinner**	16:45 – 17:15 **New PhDs (2)**		16:50 – 18:15 **Annual General Meeting**	
18:45 – 19:30 **Opening Session**	**Dinner on your own**		19:00 – ? **Dinner** *Faculty Club*	
19:30 – 20:30 **Plenary 1**				
20:30 – 23:30 **Reception**				

Plenary Lectures

Conférences plénières

THE ECONOMIC USE OF TIME AND EFFORT IN THE TEACHING AND LEARNING OF MATHEMATICS

Dave Hewitt
Loughborough University, (previously) University of Birmingham
United Kingdom

INTRODUCTION

I start with two statements:

1. The learning of very young children before they enter school is impressive.
2. The learning of those same children, later on when they are in secondary school, is less impressive.

With respect to the first, newly born children cannot walk, speak in their first language, control their bowel movements, feed themselves, throw and catch things,…, etc. The list goes on.

For the second, I look at the mathematics curriculum at the end of primary school and compare this with the end of high school and the difference does not seem so profound. Of course, there are many other subjects as well, but overall I find myself far more impressed with the learning which takes place in a child's first few years (see Hewitt, 2009, for how observation of this has helped me reflect upon my practice).

I also note that:

1. Children are not taught formally how to do the things they achieve in their pre-school years.
2. The students in secondary school are formally taught in their subject lessons in school.

These statements raise the issue of how we are asking students to work in school and how this relates to the way they worked as younger children before entering school.

During my talk I showed two videos, each available on *YouTube*, which use Cuisenaire rods to help teach the addition of fractions. The first is a clip[1] which lasts for 5 minutes 48 seconds where there is a lot of verbal explanation. It uses the rods to explain how to work out $\frac{2}{3}+\frac{5}{6}$. The second[2] lasts for 1 minute 32 seconds and has no spoken words as it works on $\frac{1}{2}+\frac{1}{3}$. What explanation there is comes mainly from the way the rods are arranged and some

[1] http://www.youtube.com/watch?v=QuJayqMsXE0 [Accessed 3rd November 2014]
[2] http://www.youtube.com/watch?v=1_E_SrpyPvU&list=UUOE7NqEwBhF-bhN7Sh77_Ag [Accessed 3rd November 2014]

pointing gestures. The dynamics in what might be learned through watching each of these is complex but an oversimplified impression I have is that with the first I am listening to explanations and with the second I am trying to generate explanations. These are examples of two quite different ways in which a learner is being asked to work. There is a temptation here to offer a constructivist perspective, however, in both cases students have to construct their own knowledge, from a radial constructivist viewpoint (von Glasersfeld, 1987). So, for me the language of construction does not help me to work on the difference I experience when viewing these two videos. Instead I turn to what Gattegno (1971) calls *powers of the mind*.

For each *power of the mind* I will offer an activity to try to help a reader gain a sense of that power from within. Mason (1987) makes reference to the *Rig Veda* which talks of two birds, one eating the sweet fruit whilst the other looks on without eating. In this spirit, I ask for you to engage in each activity, and to also observe yourself whilst doing so. There are nine powers I will introduce within three separate headings: Guiding; Working with 'material'; and Holding information.

POWERS OF THE MIND

I would like to start by asking you to remember the word *pimolitel*.

The powers of the mind are exactly that. This means, since we all have minds, that we all have these powers of the mind. As such, there is nothing profound about these activities. They are designed just to help get in touch with those things which are ordinary and which we use moment by moment, every day of our lives.

GUIDING

Activity 1:

Read the following and do as it says:

- If you can read this please put your left hand on your head.
- If you can read this the please use your right hand to point to your nose.
- If you can read this then say "I am sorry but I cannot read this."

As you try to read the above note the effort which you are putting into your eyes and the straining involved. The wanting to read is an act of will, and the *Will* places energy to where it is needed in order to try to do what you want to do. As Dewey (1975, p. 8) remarked "The exercise of will is manifest in the direction of attention". Of course, you will only experience this presence of the *Will* if you really tried to read the above. Instead you may have taken one look and decided either not to engage in the activity or started engaging and then quickly decided that the text was too small and so not bothered trying to read it. In such circumstances your *Will* did not place energy into the act of reading and you are unlikely to have noticed anything. If you go back and try to read each line, then note how the energy placed in your sight increases. You may also note the moving of your head forwards. The *Will* is the first *power of the mind* and one which controls the placement of energy within your internal system. It is at the heart of everything we do. As such it is also an indicator of the nature of all the *powers of the mind*; they are within us, no matter what gender, race, socio-economic class or disability we may have. It is an attribute of the mind (Gattegno, 1971) no matter what our circumstances.

A consequence of the *Will* is that energy is channelled somewhere in particular and this results in some things being stressed whilst others, as a consequence, are ignored. Stressing and ignoring is the result of an act of the *Will*.

Activity 2:

Take out a pen and a pencil and hold them both in one hand.

Close your eyes and keep them closed.

Drop both the pen and the pencil so that they fall on the floor.

With your eyes remaining closed, bend down and pick up the pen, not the pencil.

As you bent down, there is an issue about how much you have to bend. This will be judged partly from a bodily ‘memory’ of bending down many times in the past but also from the sense of touch you experience when your hand touches the floor. You may well have found that you bent down a certain amount but your hand had not touched the floor yet and so more bending was needed. Alternatively, you may have bent down too much and found that your hand ‘hit’ the floor and so you came back up a little so that your body was in a position such that the height of your shoulder above the floor was just a little less than the length of your arm. So the amount of bending was informed by the sensations gained from your hand being in contact with the floor. Following this, you may have moved your hand over an area until you felt it touch something. You may then have moved your hand so that your fingertips could come in contact with the object and you had to decide whether this was the shape of something which might be a pen and judge whether it was the pen or the pencil. If that tactile sensation did not ‘feel right’ you would have let go and continued moving your hand along the floor. It might have been the case that your body position needed to change as only a certain area of the floor could be covered from one body position and the pen may have bounced further away. Eventually, you felt that the touch sensations from your fingers were consistent with that of feeling a pen, rather than a pencil, and that was when you picked it up with a degree of confidence that you had the pen in your hand.

During this activity, it is possible to gain a sense of how your body position changed so that the height of your shoulder ‘felt right’ in order for you to explore an area of the floor with your hand. You then made use of the tactile sensations in your fingertips until those sensations were consistent with what you would expect from feeling a pen. Your actions are guided by what feels right and consistent with your expectations. This *sense of truth* is a power of the mind which guides your actions.

So there are two powers of the mind which are concerned with guiding:

- Will
- A sense of truth

WORKING WITH 'MATERIAL'

Let me first address the word 'material' (Hewitt, 1997). A common usage of this word relates to the substance with which we might work in order to create something. For example, curtains are made from material, or the materials with which a shelf is made might include wood, metal brackets and screws. Materials are the things with which we work in order to produce or make something. In a similar way, I can work with ideas in order to produce something, which may or may not be physical. For example, within the sphere of history there are scripts which contain written comments and ideas which a historian may use to argue for a particular perspective upon someone's life or about a series of events which happened in the past. Although the scripts may be physical, it is the ideas and information which come from the written texts which are the real 'material' with which such a historian may work. A mathematician works with certain ideas, theorems and images in order to create a line or argument which can result in a proof. The material with which they work are the awarnesses they have of certain mathematical properties and relationships. A politician may work with statistical information on a particular issue and data on popular opinion about that issue, in order to offer an argument for why a particular policy should be adopted. All of these use ideas, information, images from our senses, etc. We work with those things in order to make our actions and decisions. It is these things with which we work, that I describe as *material*.

Activity 3:

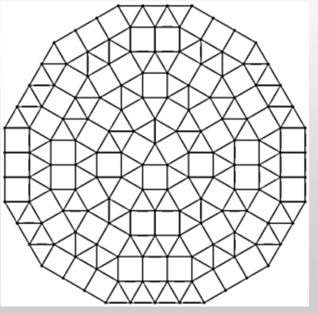
Write down a 'sum' equalling -12 which involves all of these operations:

- Add
- Subtract
- Square
- Cube root

I have provided some material within the box above and you have provided other material from your own knowledge and awareness. However, you have not been told how to use that material in order to succeed with the challenge. You have tried this and tried that, perhaps finding that a little adjustment needs to be made to your initial ideas. You may have used an awareness that certain numbers might be helpful within your calculation, such as cube numbers, and you have made decisions about what to try out through the knowledge and awareness you have and worked within the constraints stated within the task. You had to come up with numbers and ideas of how you might get -12 whilst meeting the constraints stated. Another power of the mind is *creativity*. I am not talking here about exceptional creative talent, but about the everyday ability to generate ideas and ways to proceed with the material at hand given certain constraints. Indeed, the constraints are part of the material with which one works. As such constraints are a necessary aspect of creativity. Whether someone else gets the same expression or not does not change the fact that someone has been creative in order to produce their expression. Creativity, in this sense, has nothing to do with the uniqueness of the final product. The creation by two people of the 'same' final expression, will inevitably have involved unique ways in which each person used their creativity to arrive at what looks the same in terms of an expression on paper. The final articulation can never reflect all that has been involved in producing it.

Activity 4:

Look at this and say something that you can see.



This image is one for which there is no common name. As such it is difficult to express the whole in words. Instead it is likely that you are driven to attend to the parts which make the whole. There are many things which can be noticed, of which a few are:

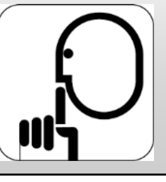
- There are squares/triangles.
- There is a triangle in the centre… or is it a square at the centre?
- If I move out horizontally from the centre, then I see a collection of one, three and then five triangles.
- I see squares with a triangle on each side (but not every square has this).
- I see a triangle with a square on each side (are there more like this?).

Another power of the mind is that of *extraction*. We can extract parts from a whole. Indeed to do otherwise would make our lives almost impossible. There is so much potential within our field of vision alone that to act in any way will require stressing part of what is available. Those people who suffer from a little deafness and wear a hearing aid talk about finding it difficult to hear in crowded, noisy, situations. It is not because they cannot hear; it is because the hearing aid magnifies all the sounds and is not discriminating. What we use in hearing something is not only the volume level but the ability to stress one particular set of sounds over all the sound waves which enter our ears. Indeed, there are times when you might not have heard something because your attention was elsewhere. The sound may well have been loud enough, it was because your *Will* directed your attention elsewhere. To *hear* is not about sounds being loud enough but about an ability to stress and ignore.

To attend to something has a consequence that some things are stressed whilst others are ignored. This can result in us becoming aware of something in particular. This is the power of *extraction* and something which we all possess. We can, do, and must, extract parts from the whole.

Activity 5:

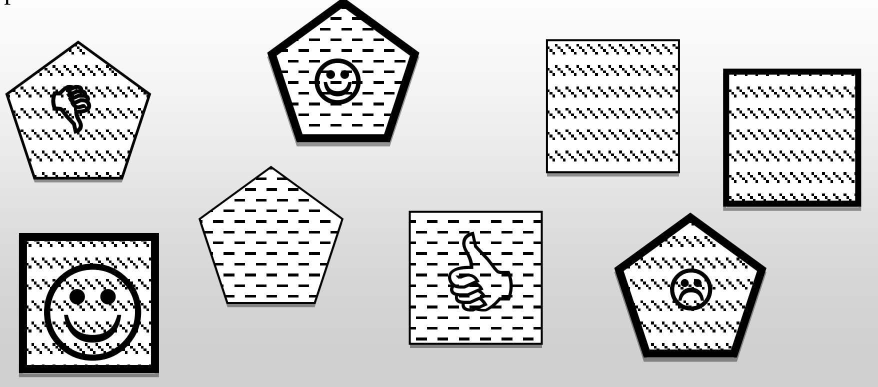
(a) Say the following out loud: 1, 2, 3, 4, 5, 6, 7, 8, 9
(b) What does this sign mean?



The symbols 1, 2, 3, etc. are just squiggles on paper. You have associated a sound with each particular squiggle. There is nothing within the squiggle ‘2’ which means you have to say *two*. Indeed, you may well have said *deux*. For someone new to this squiggle, there is nothing about it which someone can ‘work out’ as to how the squiggle is to be said. It is about associating a sound with a squiggle. Likewise the sign is representing a gesture of a finger being placed vertically by the lips, a gesture which we associate with being quiet or not talking. However, there is no reason why such a gesture must mean this. We make associations with signs happening at the same time as the context in which they appear. *Association* is another power of the mind. We have been exposed to two pieces of material – the squiggle ‘2’ and the word ‘two’ – and we are able to associate one with the other.

The next activity I offer is slightly different to the one I used in the talk. This is because the original activity made use of time in the way which cannot be done within just this text.

Activity 6: What sort of thumb or face should be in each of the three remaining shapes?



I have not told you any rules behind which symbol appears within which shapes, except saying that it will be a thumb or a face. Instead I offer examples and a question which implies that there are particular symbols which *should* be in the remaining three shapes. You are likely to have looked at what is the same and what is different in the given shapes, such as there are a number of squares, there are different shadings and some have a bold boundary whilst others do not. Each of these awarenesses come from extracting some things from the whole. I suggest you began to see whether you can associate a face, for example, with some attributes of the shapes. When is there a face and when is there a thumb? You also need to consider the range of different possibilities, there are thumbs and faces, but what kind of thumb and what kind of face? Some are larger than others, some have thumb up and a smiley face, and others have thumb down and a sad face. Within all these variations I suggest that you were looking for what attributes within the shapes are associated with which of these variations. A consistent association can then lead to a sense of spotting rules and from there you might apply those rules to the three remaining shapes. *Abstraction* of patterns and rules from examples is another power of the mind and, as with all powers of the mind, is something which we use on a daily basis.

We have not run our lives by only those things which we are told to do. I can say this as it is not possible for other people to tell us everything that is involved in speaking our first language or knowing how to manage our way around a city we have not visited before. The shear variety of what is involved in such activities means that there is simply too much for us to be told everything. The amount we are told is minute in comparison to the amount we have worked out for ourselves. In our first language, we looked for patterns and rules in how the language seemed to behave and we applied those patterns to similar situations. An example in English is how verbs tend to change when shifting from present to past tense. Instead of having to be told how each separate verb behaves, we apply an observed pattern of adding *–ed* on the end. The evidence is seen every day with children saying sentences such as "*I goed to the park yesterday*." They have not been told to say this and so it comes from their ability to abstract a rule from the examples they have heard and apply the perceived rule with other verbs (Ginsburg, 1977). Most often, this results in them saying the verbs correctly, but they then learn that there are exceptions, and these do have to be learned on a more individual basis. Abstraction allows us to deal with new situations based upon what we have noticed and learned from the experiences we have had up to this point in time.

The powers of the mind which relate to working with material are:

- Creativity
- Extraction
- Association
- Abstraction

These are ways in which we work with material to select, link and take forward what we have noticed, into new situations.

HOLDING INFORMATION

As well as working with material, we also need to retain information which is of significance to us.

Activity 7:

Several pages ago, I asked you to remember a word.

Without looking back, say that word and write it down.

I have asked people to do some something equivalent to this on many occasions. On each occasion, only about half the people managed to write down the word correctly spelt. This is just one word and, within a relatively short space of time, so many failed to remember it correctly. We all have the power of *memory*, but sometimes as educators, we do not acknowledge the inefficiency of this power. I do not know whether you, the reader, did try to remember this word when I asked you to do so several pages back. If so, I suggest that you spent a certain amount of energy trying to memorise it at the time of reading. You may have used association to try to link it with another word or words that you knew already. Those that were successful at remembering the word reported shifting attention back to that word on several occasions whilst engaged in the rest of the tasks and talk (or text, in the case of you reading this now).

Remembering and forgetting exist alongside each other. To remember successfully requires significant effort, both at the time of being asked to memorise, and also at intervals thereafter. That is why *practice* has played such a significant role in many classrooms, because it is memory which is called upon so often. There *are* times when we need to memorise but as

Gattegno (1986, p.126) said "memory should be relegated to a limited area in education – that it should be used only for that which we cannot invent".

The next activity is again slightly adapted from that which I gave in the talk.

Activity 8:

Consider somewhere you have visited in the last month which is not a place you spend a lot of time on a frequent basis.

Imagine that place now and say out loud two things about the surroundings there.

When you visited that place you did not try to memorise the surroundings just in case you were going to be asked about it when you came to read this article. You have not spent your time in between time to check whether you still remembered it. Indeed, this is something quite different to memorisation. Here, you put no effort at all whilst you were at that particular place and you have not needed to do so since either. It is only now, when asked to recall some things about the surroundings, that you have used a little energy to do so. Another way we hold information is through *imagery* and this is very different in nature to memory.

Activity 9:

- Get a pen or pencil.
- Hold out your hand, palm upwards and place the pen so that the nib is pointing away from you.
- Throw it from one hand to another so that it rotates 360º with nib of the pen ending up pointing in the same direction as it did at the start.
- Do this again, going back to the original hand.
- Continue.
- Now stop. Do not throw it again.

Whether you were successful with this is not important. I am about to ask you a question but before doing so I would like you to put any free hand you have (you can continue to hold this book with one or both hands) on your lap, palm down. Now, without moving your hands, what provided the twist of the pen: fingers, wrist, or something else?

Without the freedom to repeat the physical movements, you are likely to use your imagery in trying to answer this question. What is of significance is that I suggest you were not able to answer this question immediately without recall to some imagery to try to run through doing this movement again. Yet, at the time of doing the activity, I conjecture that you did manage to make the pen twist and so within you, at some level, you knew what to do. Yet now, when asked about it, the answer is not immediately available despite you doing this only a matter of seconds previously. We hold a lot of information at a deeper level than that of which we are consciously aware. Here the information is of a functional nature, it is available as and when we need it with no or little cost in terms of energy. Knowing how to walk, or scratching an itch, are other examples. We have not always been able to do these things, so they are learned activities. I suggest that there was a time in our lives when we were very conscious of what was involved with such things. However, now we are so skilled that we often do not even notice what we are doing. The same can be said about counting. Watching young children learn the complexities of counting (and counting *is* a complex activity) can help us realise how much we can do this with such little attention; so much so that we might find it difficult to answer the question "how did you manage to count those objects?" other than to say "I just did". The ability to make some things seemingly automatic frees us to give our attention to new things and learn more. Wood (1988) expressed the significance of automaticity:

> *developing 'automaticity' means that the child no longer has to consciously* attend *to the practised elements of her task activity. 'Automated' actions may be performed without the need for constant monitoring or awareness. As some aspect of the developing skill is automated, the learner is left free to pay attention to some other aspect of the task at hand.* (p. 175)

So, the powers of the mind are:

Guiding:

- Will
- A sense of truth

Working with material:

- Creativity
- Extraction
- Association
- Abstraction

Holding information:

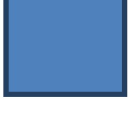
- Memory
- Imagery
- Automaticity

It seems to me that common practice in many mathematics classrooms means that of all the powers which can be called upon, it is memory which is called upon most. Yet, as demonstrated by my little activity, it is one which requires significant energy and is often accompanied with forgetting. If learning is then based upon memory, and something has been forgotten, then it is hard for that to become known again. So, the task for us is to consider ways in which we can call upon more of the available powers and restrict memory to its rightful place. To do so I will consider four frameworks:

- Arbitrary and necessary
- Practise through progress
- Subordination
- Direct Access

ARBITRARY AND NECESSARY

I have discussed this framework in greater detail elsewhere (Hewitt, 1999). The basis of the framework is one of viewing each part of the mathematics curriculum and asking the question, "*Is it possible for someone to come to know this for sure without being informed of it?*" If the answer is no, then that aspect is arbitrary; if yes, then it is necessary. For example, what is the name of the shape below?



I cannot stare at this shape and know for sure it is called *a square*. If I am in a French speaking area then it is not called a square, it is called *un carré*. In other languages it has a different name. There is nothing about the shape which means a particular collection of sounds, in the form of a word, must be associated with it. Consequently, if I am trying to learn the name of this shape then I will not know for sure what it is called within a certain language unless I am informed. Of course, I can also invent a name. However, if we all did this then we would find that we are unlikely to have come to the same decision. So, for agreement to happen, for new learners to use the same names as other people do, then they need to be informed. So names are arbitrary and I use arbitrary as this describes how it might feel for a learner. As a learner might ask, *why is it called that?* A question for which there is no mathematical reason.

As well as names, conventions are also arbitrary, in the sense that I am using this word. Why is the *x*-coordinate written before the *y*-coordinate when a position on a graph is described? Why is there a comma in-between the two? And why are brackets involved? I class all socially agreed conventions as arbitrary. No matter how often I might turn round or stare at a circle, I cannot know for sure that there are 360 degrees in whole turn. Why 360? Why not 100? There are historical reasons, as the Babylonians were working in base 60 and there were mathematical conveniences with having a number with many factors. However, this is still about *choice*; there is nothing that means a learner would know it *has* to be 360 and could not be anything else. Indeed, for other mathematical reasons, radians are preferred. Even though there are reasons, it is still not necessary, it is only convenient. Thus I still class this as an arbitrary aspect of the curriculum.

Not everything on the curriculum is arbitrary. For example, in Euclidean geometry all triangles tessellate. This is not a matter of choice, it is something which can be worked out and argued that it *must* be true. Given the conventions (arbitrary) regarding names, symbols and number system (base 10), then 3+4=7. This is something which everyone will agree upon. It is the necessary where mathematics lies. It is here where things must be how they are. It is not a matter of choice, instead it is a matter of justification and proof. It is with the necessary that the question *why?* is appropriate, unlike with the arbitrary.

The arbitrary is about acceptance and memory as there is no mathematical reason why something is how it is. Without reasons, a learner is left only with memory. The necessary is about questioning and awareness as here there are always reasons. The fact that there are reasons means that a learner can use and educate their awareness to come to know these things. This dichotomy has significant implications for the teacher as well as a learner. To teach something which is arbitrary involves assisting memory, whereas I suggest teaching something which is necessary involves the very different task of educating awareness.

With respect to the arbitrary, I know as a teacher that I need to inform my students of what is arbitrary. More detail can be found elsewhere (Hewitt, 2001) but there are many ways in which I can go about telling someone something. This is, in itself, something worthy of careful consideration. The *when* and *how* to tell is important. I will offer one example here relating to the order in which a coordinate is written. One way is to say that you write the *x*-coordinate first and then the *y*-coordinate and leave it there. A minor addition would make use of the fact that *x* comes before *y* in the alphabet to help them know which comes first. This small addition does, at least, try to make use of the power of association in their attempt to memorise this.

Practise of the arbitrary is important since it is about memorisation, and students need to be helped in their attempts to memorise. One example of a way to practise the convention of co-ordinates is for two pairs of students to play a game on a coordinate grid. The game is secondary to the way in which the game is played. The rules of playing the game involve each pair having one person who decides which coordinate they should go for next (in whatever game it is they are playing) and *saying out loud* (or *writing down*) the coordinate, whilst the other person has to *listen to* (or *read*) this and put a cross at that coordinate (the first person not being allowed to say "no not there", etc.). This means that each person is practising the convention of which comes first, one saying/writing and the other listening/reading. The game itself could be one of many. An example would be taking turns between the teams to put a cross in their colour on a coordinate grid in order to have four crosses of their colour which would lie on the corners of a square. One point for each square created. The size of the grid could be decided and whether it involved all positive numbers in the coordinates or a mix of positive and negative numbers.

The arbitrary is the rightful place to call upon memory and a teacher's role is to acknowledge this and assist students in their task of memorising.

The necessary can be known without students being informed. As such, I suggest there is a very different job to be done. Figure 1 represents two aspects to consider, the awareness of students and the desired mathematical property or relationship which you might want them to come to know.

Figure 1

The first challenge of a teacher is to design or choose and activity which can (a) be meaningfully engaged with the awareness the students already possess and (b) engagement with that activity can lead to an awareness of the desired properties or relationships (see Figure 2).

Figure 2

However, the choice of the activity is not the end of a teacher's role, of course. What is important is the way in which that teacher works with the students whilst the students are working on the activity. This may involve a series of questions which help challenge and focus students on particular aspects whilst they are working (see Figure 3).

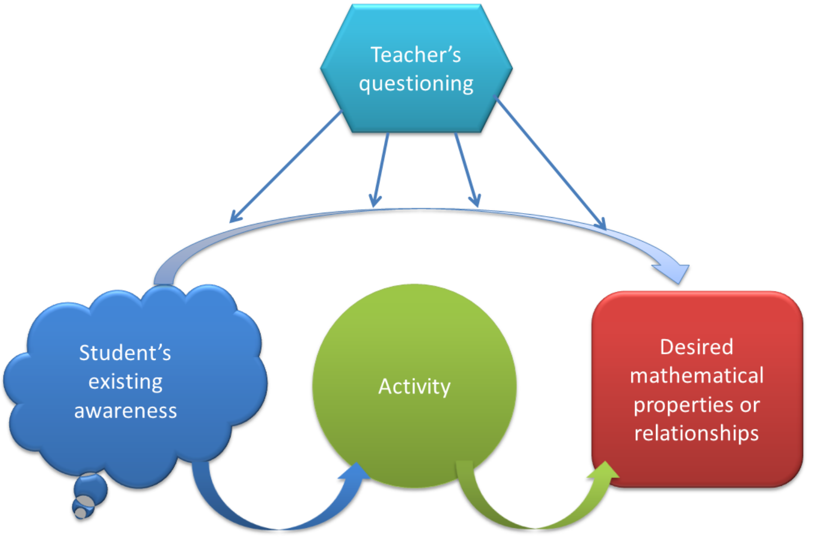


Figure 3

An example of such an activity might be where students are given the information that 100% is $360 and asked what other percentages they could work out. A gradual development of something like Figure 4 can come from students starting to say they can work out 50% by halving, then 25% by halving that; 10% is often stated quite quickly and this can lead to other percentages, such as 5% and 20%. Then someone might realise that if they know 10% and 20%, they can also work out 30%. This can go on for some time until 1% is known and someone realises that they can then find any percentage at all from the 1%. This can lead to an awareness of how any percentage can be found.

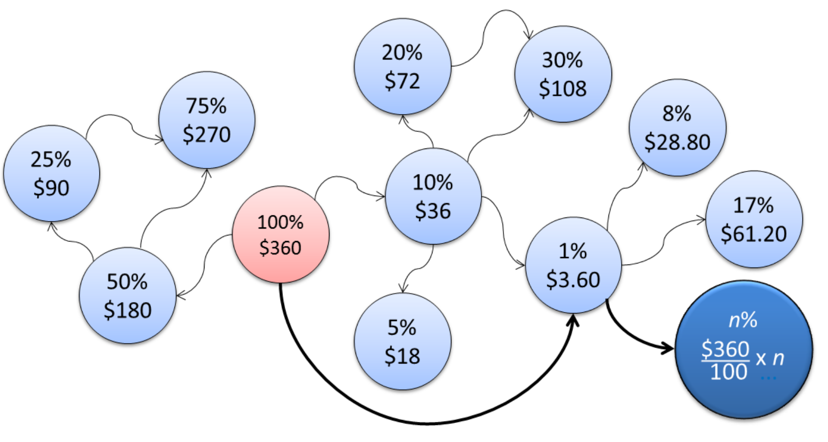


Figure 4

Such an activity does not call upon students being told a rule, which they then have to memorise. Instead they have to use a variety of powers, such as will, a sense of truth, creativity and abstraction. Memory is kept in its rightful place and not called upon explicitly.

PRACTICE THROUGH PROGRESS

I will only briefly discuss this aspect. However, the nature of practice is important for successful learning. Firstly, I will make use of the two topics mentioned above: coordinates and percentages. The suggested activity above for practising the convention of saying and writing coordinates gives the possibility of becoming aware of something new. It is sometimes a while into such an activity before one team realises that squares do not have to have horizontal or vertical sides. In fact, going for such obvious placements of your team's crosses are more likely to be thwarted by the other team as they are more obvious. Consequently, students are coming to learn about different orientations of a square and how they can be sure whether any four particular crosses of the same colour are positioned at the corners of a 'squiffy' square. The practice of coordinate conventions does not just result on students standing still. Instead they can continue to progress in terms of educating their awareness whilst practising a particular convention. This contracts to a traditional form of practice in the form of an exercise, where they have to plot given coordinates, or write down the coordinates of given points (and are never doing anything with those answers). In such a traditional exercise the best that can be hoped for is that someone does not 'go backwards'.

With respect to percentages, rather than doing a traditional exercise with questions such as *find 40% of 260*, a challenge such as that in Figure 5 will have students carrying out a lot of practice of find percentages but also this practice is carried out with a purpose in order to succeed with the given challenge. Finding more examples which have one or two steps to make a 50% increase overall can result in educating their awareness in how percentages behave and that attention needs to be placed on what a percentage is *of*, as much as the numerical value of the percentage.

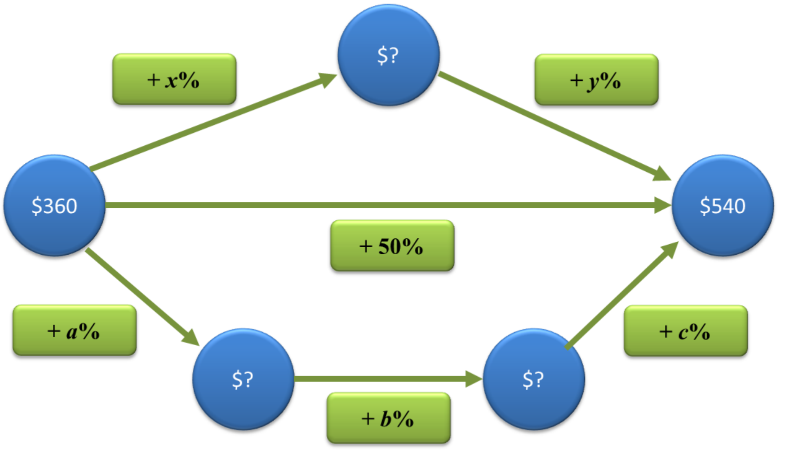


Figure 5

SUBORDINATION

Subordination has some of the features of *practise through progress* but with significant shifts in some of the components. With practise through progress, something has already been learned and it is a matter of finding an activity which practises what is already known whilst simultaneously allowing opportunities to progress in other areas. Subordination turns this on its head, by having the activity clearly understood whilst the thing which the activity practises is *not* known to the student. The activity calls upon the practice of something of which the students do not yet know. So the learning takes place with what is being practised rather than what comes out from the activity itself. So, as a teacher I will appear, as far as the students are concerned, to be interested in the outcome of the activity. However, my real agenda is not that at all, but whether what is required to be practised by the activity has been acquired or not.

The example I offer is based upon the computer program *Grid Algebra* [3]. This is based upon a grid of numbers in multiplication tables (see Figure 6).

There are many activities and features of the software, but here I concentrate mainly on just one. Initially students become familiar with the structure of the grid through several activities built into the software. Then it is revealed that any number can be picked up and dragged to a cell either horizontally or vertically (see Figure 7). When such movements are made, the software shows the notational consequence of such a movement. So, the number 3 is dragged one cell to the right (addition), and this results in 3+1 appearing in the cell which previous had shown 4. The 3+1 then becomes an object in its own right and can be dragged down (from the one times table down to the six times table: multiplication) to show 6(3+1), which in turn is dragged to the left (subtraction) to show 6(3+1)-12. The 'peeled back corners' in certain cells indicate that there is more than one expression in those cells. For example, the number 4 is still in the cell which now has 3+1 showing and can be seen again by clicking on the peeled back corner.

1	1	2	3	4	5
2	2	4	6	8	10
3	3	6	9	12	15
4	4	8	12	16	20
5	5	10	15	20	25
6	6	12	18	24	30

Figure 6

[3] Grid Algebra is available from the Association of Teachers of Mathematics (ATM): http://www.atm.org.uk/Shop/Primary-Education/Software-Media/Grid-Algebra---Single-User-Licence/sof071

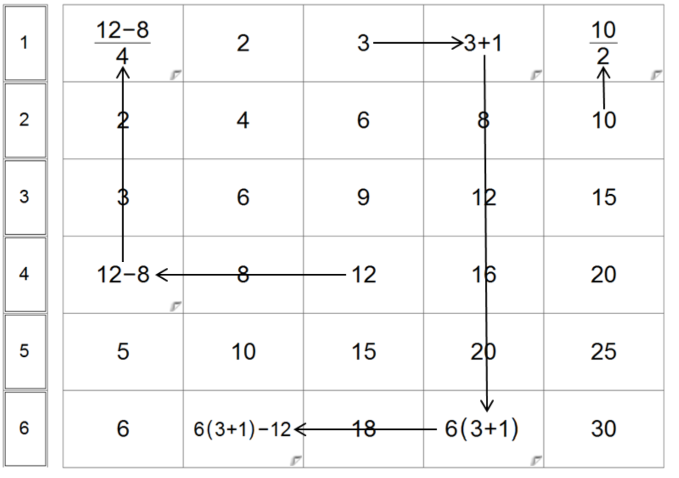


Figure 7

A key activity can then be set up where the grid is empty apart from one number and the result of that number being dragged around the grid with all intermediate expressions rubbed out. Thus, only the number and the final expression can be seen (such as in Figure 8).

The task for the students is to re-create that journey by picking up the 17 and dragging it to various positions until they produce the expression $\frac{6(17+2)-18}{2}+12$. The formal notation of this expression might be something of which students are not familiar and they may not know about order of operations either. Yet these are the things which need to be used in order to carry out the activity. This is a situation which involves subordination. The students are familiar with the idea of physical journeys and so can understand the nature of the challenge. They know they have to make movements on the grid and this is something they can do (whether or not the movements are correct!). Yet this activity requires practice of interpreting formal notation and knowing order of operations – something, let us assume, they do not know about. So the desired learning is in what is required to be practised rather than the result of the activity.

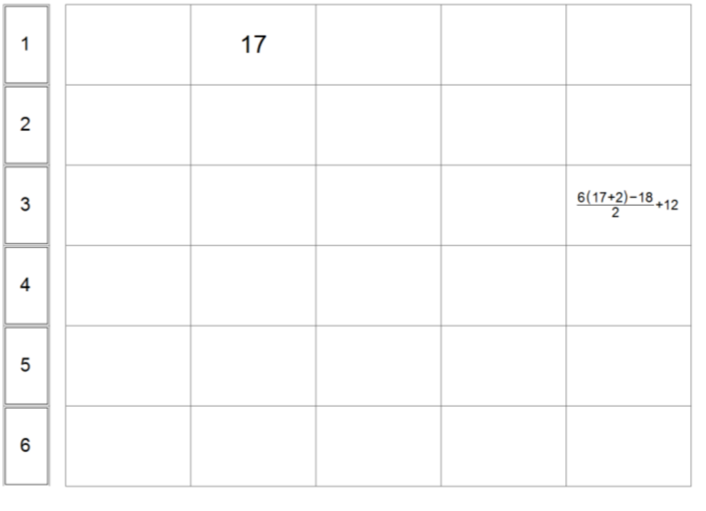


Figure 8

A key factor with subordination is that someone needs to be able to see the consequences of their actions and be able to understand those in relation to their success or otherwise in achieving the challenge. So, in the case of the *Grid Algebra* activity, students will make a decision to move 17 somewhere and the software will feed back the consequence of their movement in the form of notation. The students can then see whether this notation is beginning to build up to the desired final expression $\frac{6(17+2)-18}{2}+12$. Suppose, for example, they started off with the correct order of operations by adding two (moving to the right) and then multiplying by six (moving down). However, after they then subtracted 18 (moved to the left), they felt that they should add 12 next and then divide by two, they would find the software would show $\frac{6(17+2)-18+12}{2}$ which looks different from the target expression. Hence, they can tell that they have done something wrong simply because it does not look the same. They would also end up in a different place (see Figure 9).

Thus, the feedback is understandable in terms of the achievement or otherwise of the task and so they can become aware from this feedback whether they have interpreted the notation correctly or not and make adjustments accordingly. My experience working with students is that it does not take long for them to learn how the four operations are written in formal notation and the order of operations. Furthermore, they become very fluent with this quite quickly (Hewitt, 2012).

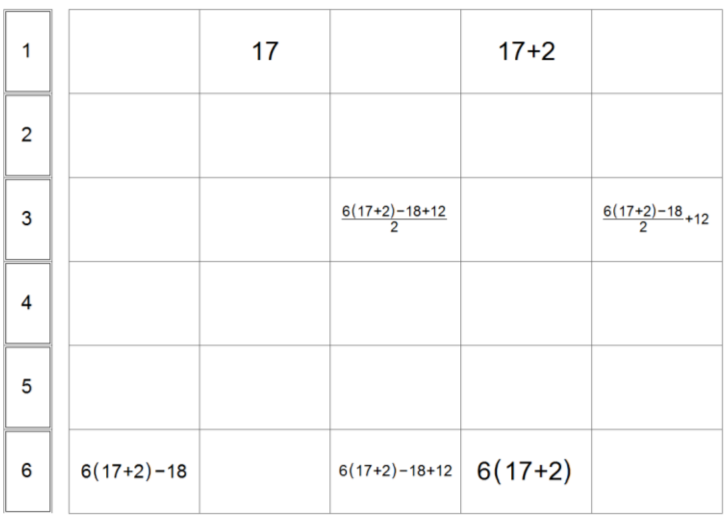


Figure 9

When something has become fluent, we hardly know we are doing what we do. This includes a whole range of knowings and skills, such as walking, counting, spelling of many words and, for many of us, correct use of algebraic notation. Every person has a long list of things which have become automated and little or no attention is given to these things. Instead attention is placed on some other goal for which one or more of these are required. Subordination attempts to mimic this relative imbalance of where attention is placed. As a teacher I focus students' attention on the goal, rather than the means of achieving that goal. So, in the *Grid Algebra* example above, the goal is to re-produce a particular expression through movements on the grid. The means of how to achieve that – being able to interpret formal notation and know order of operations – is not explicitly mentioned and certainly not 'taught' beforehand. The learning comes from noticing the effect of movements on the 'look' of the expression generated compared with the target expression. Plenty of incorrect movements are made initially, but after many of these tasks, students become very adept at interpreting notation and can begin to articulate the 'rules' of what the notation means in terms of operations and order of operations. The fact that their attention is deliberately placed on the goal rather than the means, allows the means to be learned in a way more akin to much of their early pre-school learning, where little was explicitly explained and yet they had to find the means to achieve what they wanted to achieve.

DIRECT ACCESS

The phrase 'direct access' comes from Laurinda Brown with whom I spent many an evening discussing our classrooms when we were both teaching in Bristol in the UK. Too often the mathematics curriculum is structured in a way where small steps are made and one piece of the curriculum is built upon another which is, itself, built upon another, which, in turn, is built upon another, etc… Figure 10 gives a sense of a typical situation where, in order to learn Y, you need to know C, which in turn requires you to know B, which is built upon A. The problem with this is two-fold: it takes time to come to know A, B and C; and by that time, there is a chance that at least one of A, B or C have been 'forgotten'. So trying to teach Y becomes a problem.

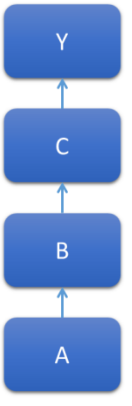


Figure 10

Instead, a pedagogic challenge is to analyse Y to find its fundamental essence and structure, and consider what is the least which needs to be used to engage in a meaningful way with Y (see Figure 11).

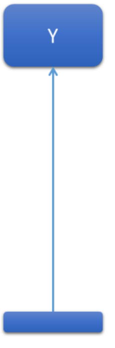


Figure 11

This has connections with Bruner's (1960, p. 52) statement that "any subject can be taught to any child in some honest form". This statement becomes possible if we do not call upon a list of previous learning. Instead, the powers of the mind can be called upon since we all possess these. So the challenge is to use as little prior learning as possible in order to engage in the mathematical essence of what Y is really about. What can be used is the expectation that a student will engage in a way where their powers of the mind are being utilised. An example is the image in Figure 12, of a dot moving round a circle.

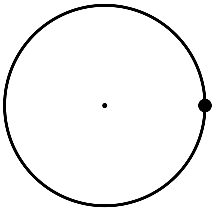


Figure 12

I will briefly give a sense of how I might work with a class. The dot starts as in Figure 12 and rotates anti-clockwise. I ask a class to say "now" when the dot is at its highest position and I tell them that at this point the height of the dot is one. I do likewise with the place where it is lowest and tell them the height is negative one. I then ask them to say "now" when the height is zero. We establish that this gets said twice within one revolution. I introduce the radius and the angle the radius has turned through from its start position (see Figure 13).

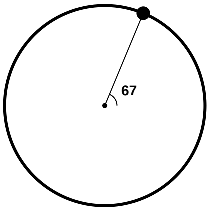


Figure 13

The students articulate that the angle says 90 when the height is one, 270 when it is negative one, and 0 and 180 when it is zero height. A discussion often arises as to whether one of those occurrences of zero height is at 0 or 360 and I suggest it is both and ask what happens after the 360 as the point continues turning. We establish a sequence of 0, 180, 360, 540, 720…. I rotate my hand many times round the circle quickly and then move another 90 degrees. I begin to introduce some notation and the awareness that a height of one can be obtained from lots of 360 degrees followed by another 90 degrees is written as

$$\text{height}(360n + 90) = 1$$

This, later on, becomes:

$$\sin(360n + 90) = 1$$

I will not go into detail but I have a way of addressing the issue of which angle produces a height of 0.5 and the following is established:

$$\text{If } \sin x = 0.5 \text{ then } x = 360n + 30 \text{ or } x = 360n + 180 - 30$$

The notation comes only as a form of expressing the awareness they have already revealed. Whether the values n can take is expressed as $\forall n \in Z$ or in words is not of importance. However, sometimes I find students can be excited by a bit of 'weird' notation being used as a way of expressing an awareness as long as they are feeling quite comfortable with that awareness.

A different stressing of the dot leads to establishing cosine and tangent and further work leads to working on equations such as $\cos x = \sin x$. At some point the graphs of these functions are produced (see Figure 14).

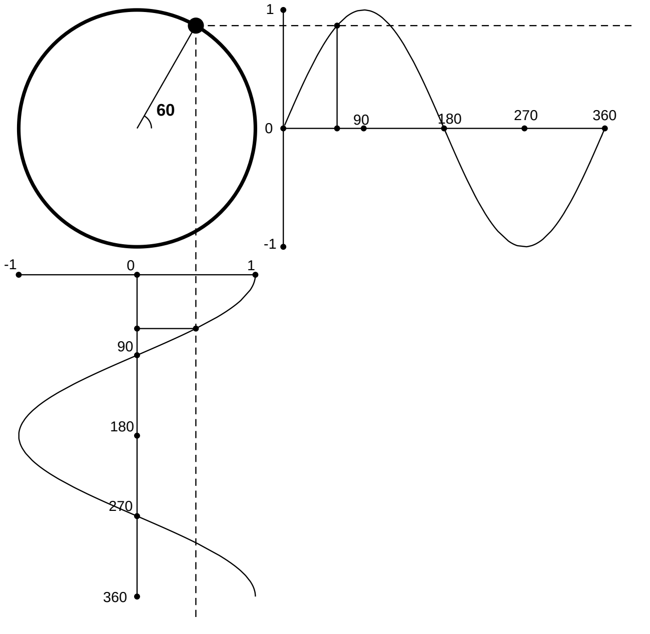


Figure 14

This image is quite a well-known one. The significance is in the way of working with students so that they are not being asked to remember anything particular from their previous mathematics lessons. Instead they are just being asked to watch, observe and comment upon what they see. They will use most if not all of the powers of the mind listed earlier. The labelling and notation is something that I will look after as teacher. So, I provide only that which is arbitrary – the notation. The students provide that which is necessary – the properties and relationships. I simply label any awareness which becomes established. Gradually the students begin to adopt the notation as part of the way they communicate any further awareness they have. As little previous knowledge is called upon, such work can take place with relatively young students. I have worked with 11-12 years olds in a mixed attainment class in such ways and it may be that this is just as possible with younger students. However, the aim is not to play a game of seeing how young students can be to engage with this particular idea but to note that it is possible to work on a topic such as finding general solutions to some trigonometric equations without the need to have a long series of previous mathematics work to prepare them for this. Note also that this image is not something to be remembered (as in memorising) but is something which is recalled. The significance of recalling rather than remembering is that it is not necessary to ask students to memorise this image; instead the process of working with the image on activities over time means that this is an image which can be recalled in the same way as you recalled a place in Activity 8 above.

CONCLUDING REMARKS

The framework of arbitrary and necessary offers a way to distinguish between the areas of the mathematics curriculum which necessitate the use of memory and those areas where the use of memory is best avoided. Most significantly the aspects of the curriculum which are necessary are where the real mathematics lies. This is the area where the other powers of the mind can be called upon so that awareness is educated rather than the inappropriate and inefficient use of memorisation. Whether practising an arbitrary name/convention, or the use of a necessary property/relationship, there are ways to practise which can call upon a range of powers of the mind where progress is made alongside the desired practice; and the practice is seen as purposeful.

Subordination offers a way new things can be met for the first time as the vehicle through which a certain challenge can be achieved. The notion of immediately having to use and practise something which is unfamiliar may seem strange, yet I argue that this is what we all did as young children in learning language in order to say what we wished to communicate, and in developing the skills of walking in order to get to objects we find desirable. Subordination mirrors the behaviour of when something is already automatised. In such circumstances, we do not place our attention on what it is we are using; instead we place our attention on the effect its use has upon a desired goal. I suggest that this form of practice not only helps what is used to be learned relatively quickly, but also drives it into becoming automatised.

The notion of direct access takes away the need to have remembered earlier mathematics content; instead a carefully designed activity utilises the powers of the mind which can result in educating awareness, which can take the form of items on the mathematics curriculum through direction of attention and carefully timed notating and formalising.

A final note is that I have not specifically focused on the use of one power of the mind over others. This is due to the fact that they do not come singly. For example, association and imagery are frequently used when someone is trying to memorise. In fact, it is impossible to stop any of these powers being at work. The argument I am making is about the degree to which each is stressed when working with students. In some classrooms, students can come to know that it is memory which is stressed over other powers and come to expect this within mathematics lessons. As a consequence, when a greater use of the other powers is suddenly expected, students can respond in a way which makes it appear that they do not have them! All students do, of course, have all of these powers, and use them on a daily basis. However, there can be a culture established where the guiding powers of *will* and a *sense of truth* can result in a student directing energy into seeking what it is that needs to be memorised (since this is the norm). Something different to this can mean their *sense of truth* provides a feeling that this is not what mathematics lessons are about; and this can result in a lack of engagement. Shifting the culture is required, which means that a student's *will* and *sense of truth* are aligned with the expectation that the full range of powers of the mind are utilised rather than seeking only to memorise. To change that culture is a teacher's responsibility so that students can learn more, faster and in a deeper way.

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MATHEMATICS IN INDUSTRY, MATHEMATICS IN THE CLASSROOM: ANALOGY AND METAPHOR

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> *The skill set of graduates encompasses technical depth in a relevant discipline, breadth of knowledge across the mathematical and computational sciences, interest in and experience with the scientific or business focus of the employer, enthusiasm for varied challenges, the flexibility and communications skills required to work in an interdisciplinary team, the discipline to meet time constraints, and a sense for a reasonable solution.* (Society for Industrial and Applied Mathematics, 2012, p. 28)

> *There is a hidden curriculum of cognitive and verbal skills, of abstraction and interpretation and of some very quantitative skills (estimation, bounding, determining approximate and qualitative solutions) that form a core of the syllabus equal in importance to the core that is represented by the table of contents of the textbook. Students profit from being made explicitly aware of these course goals.* (Pemantle, in press)

> *A metaphor is useful only for transforming what happens, enriching it in some way. It never tells you what actually happened, how it happened, or why it happened. A fleeting thought might be compared to a ship on the horizon, but surely it's saying something that a shop on the horizon is never compared to a fleeting thought? ... If metaphors increase our understanding, they do so only because they take us back to a familiar vantage, which is to say that a metaphor cannot bring anything nearer. Everything new is on the rim of our view, in the darkness, below the horizon, so that nothing new is visible but in the light of what we know.* (Haider Rahman, 2014, p. 290)

INTRODUCTION

It was an honour to be invited to the CMESG 2014 meeting, and I'd like to thank the organizers as well as the conference participants for their hospitality and patience. My own research interests lie in the mathematical modelling of natural systems, and the subsequent theoretical and numerical analysis of such systems. In other words: I claim neither training nor research expertise in the *pedagogy* of mathematics. My presentation concerned my own personal experiences and thinking in the classroom; the opinions I expressed in my lecture and in this article are representative of nothing more than my own story. The willingness of the audience members, each with considerable training in education, to listen to such a story is laudable.

In the first part of the lecture, I attempted to describe what a good mathematical problem in industry may be, and what skills may be needed to solve it. I tried to abstract these skills away from their specific contexts, and described three attempts to *teach* them.

In the second part of the lecture, I reflected on the current public discourse around the teaching of mathematics, particularly with the intent of *training a skilled workforce*. I shared my personal confusion about the often contradictory, but passionately held, beliefs around a *crisis* in the training of STEM (science, technology, engineering and mathematics) students.

In terms of structure, I discussed somewhat controversial questions during the lecture.

1. Are problems in industry worthy of our mathematical attention?
2. Should we tell our students about mathematical problems in industry? How?
3. Are the mathematical abilities one acquires at university worthy of industry's attention?
4. Is it worth trying to teach industrially-relevant mathematics?
5. Perhaps most controversially, why are we teaching mathematics to non-mathematicians?

In this article, I'll focus on the content of the first part of the lecture.

MATHEMATICS IN INDUSTRY?

The fact that the quantitative sciences form the underpinning of much of our current society—ranging from banking, internet commerce, the design of highways, scheduling of the logistics of delivering healthcare to remote places—is neither novel nor surprising. We know, with varying degrees of expertise, that science and technology are intimately connected with most facets of our lives, and that mathematics plays a key role.

We *know* this fact about the central importance of mathematics in our lives, we *teach* this to our students, yet most of us do not actually *use* a lot of mathematics directly in our daily routines. Mathematics is ubiquitous, and yet somehow hidden from us in its explicit use. We swipe a credit card, but don't actually know or daily think of the principles of encryption which made the transaction secure. We board a flight, but don't explicitly solve an optimization problem to see if there's a reason we board in the order we do. We're *consumers* of mathematics, and particularly in how it is used in industry. We use mathematical ideas in the form of black-boxes (Damlamian, Rodrigues, & Strässer, 2013). This extends even to companies and industries which routinely employ mathematical tools while solving problems, but may not *recognize* that they are doing so.

I think this disconnect—we are told mathematics is important, yet most of us never actually see more advanced topics explicitly in use—is interesting. It is also personally frustrating to me.

Students in a typical mathematics class may learn advanced concepts. They may see beautiful results. So intoxicating, so pristine is the sheer beauty of mathematics that the relation of these ideas to problem solving in industry seems almost like a sullying of these ideas. They may see an 'application' or a 'word problem', but these too frequently seem arid and contrived. Without a prior (or concurrent) exposure to the application, this is not surprising. Thinking about the convergence of series is interesting. An example about pensions and compounded interest is less interesting, especially if the student has never heard about pensions or compound interest elsewhere.

I've been told by a stellar mathematics student that she did these word problems because she had to, but what she really loved was proving theorems. Because the mathematics in industry is so hidden, she wasn't aware that some of the most interesting mathematics is motivated by questions arising there.

Students of other disciplines, on the other hand, see advanced mathematical concepts, and are not impressed by their beauty. They'd like to see how this idea is relevant to their core interest in some other field, otherwise it seems a waste of time. It would be ideal if students in, say, first-year biology learned mathematical concepts relevant to what they were learning in their biology class. Unfortunately, the profusion of other fields and the very real constraints of universities and colleges makes it nigh-impossible to achieve this integration of curricula. There are constraints of funding, there is politics, there is institutional history. The student doesn't see these constraints. The student only sees a mathematics class that he/she does not want to be in, doesn't see the relevance of, and (if we are honest) is scared will decimate his/her odds of a high GPA.

I've been told by a stellar engineering student that he hated the mathematics classes he took, particularly numerical analysis, but that what he really loved was the finite element analysis of mechanical structures. He saw this in his engineering classes, using commercial software. When I told him that the analysis of the finite element method was, in fact, a major part of numerical analysis, he told me to "*get out of here*!" Once again, because the mathematics is hidden from the user of these commercial software packages, he wasn't aware that mathematics was key to much modern engineering design.

ARE PROBLEMS IN INDUSTRY WORTHY OF OUR MATHEMATICAL ATTENTION?

I think we owe it to our students to answer this question. Of course, I personally believe the answer is a resounding yes. It is worth defining what I mean by a good mathematical problem in industry. First, it should possess the feature that its mathematical reformulation is close to the original problem, that is, it satisfies

$$\left\| Question_{mathematical} - Problem_{original} \right\| < \epsilon$$

for some small $\epsilon > 0$. The question, "*Will eating this candy right now make me sick, yes or no?*" does not possess this feature, since a half-ways decent mathematical formulation would be a question about expected outcomes and distributions, with probabilities of getting sick being the answer. But this was not what the question asked. In a related sense, a good mathematical problem in industry leads to models which are verifiable. We should arrive at a model with descriptive as well as predictive powers.

A mathematical problem in industry may require new mathematics to be developed. Or, it may need very simple, existing mathematical tools. So, if we start from the premise that we need to teach students about mathematics in industry, we should alert them to these possibilities. They will sometimes need to learn new mathematical concepts to solve a given problem. Occasionally they may use basic arithmetic. What is important is for a student to recognize *how to formulate a mathematical question*, and then *use the most appropriate mathematical tool to solve it.*

These are somewhat nebulous ideas. I would like to believe this is what I teach, but in truth I teach calculus or analysis or partial differential equations, and only convey these ideas along the way. There is no class that I've taught called 'how to be mathematically open-minded' or 'how to be mathematically curious about your surroundings'. When I teach differential equations, I focus on the concepts, and then show a lot of instances where students need to use differential equations to solve a problem. With a few exceptions, I have not walked into a classroom, given students a problem and said: "*Go figure out what mathematics you need to solve this, and if you don't know it, go learn it.*" But this is, I think, how they will frequently encounter problems in industry—not in some neatly packaged form where the mathematical tools required are obvious.

At this point it is useful to distinguish between:

- well-defined mathematical problems of industrial relevance, where the company understands the need for mathematics; and
- poorly-defined problems of industrial relevance, where mathematics may be useful.

The first set of problems is nice to have. Someone confronted with a well-defined mathematical problem with industrial relevance in a supportive managerial environment should, with a solid mathematical training, be able to make good progress. Unfortunately, as described above, I think many problems in industry fall into the second category. A company may not even recognize that a mathematical solution is what is required. Indeed, a quick look at any jobs listing will reveal lots and lots of job descriptions like *supply chain expert* or *business analytics for enterprise software*, but very few for *mathematics*. Yet, embedded in the former job descriptions are lots of (poorly defined) problems where mathematics could contribute a lot. What skills are needed for problem solving in this setting? Are we teaching these?

WHAT MATHEMATICAL SKILLS SHOULD WE BE TEACHING STUDENTS TO PREPARE THEM FOR CAREERS IN INDUSTRY, AND HOW SHOULD THEY LEARN THESE SKILLS?

In the previous section, I've tried to describe two broad classes of mathematical problems in industry. What should a student learn at university in order to be successful at solving these? The Society for Industrial and Applied Mathematics (SIAM) has released two very thoughtful reports (*Mathematics in Industry*) in 1996 and 2012. The ICME/ICIAM study on *Educational Interfaces Between Mathematics and Industry* (Damlamian et al., 2013) collected pedagogical experiences of colleagues from a host of countries on precisely the interface between mathematics, education and industry. The details of how and what to teach varies considerably, but one can think like a mathematician and try to abstract some key ideas.

What I've extracted from these reports, and numerous interactions with colleagues in the academe as well as industry, is the following: in order for a student to successfully use mathematics in industry, they need disciplinary content, interdisciplinary breadth and *soft skills*.

Obviously, a student first needs to see mathematical concepts. These can vary depending on the core discipline, but if one is going to solve mathematical problems in industry, one must know some mathematical concepts. These we are good at identifying, and these we teach. We may decide to emphasize some concepts (for example, limiting processes in Calculus) over others (for example, discrete probability), but we can revisit these choices. What else should students be learning in our classes?

As the first epigram from the beginning of this article suggests, the Society for Industrial and Applied Mathematics (a very large international scholarly society for mathematicians) believes that in addition to core technical knowledge, students must also develop intellectual breadth (including exposure to another discipline), flexibility, and the ability to interact, communicate and collaborate with others (SIAM, 1996, 2012).

Why are flexibility and breadth of knowledge so important? I think it is because they allow a mathematically-trained student to draw fruitful analogies between a new problem and a problem they may have seen in a different incarnation, in a different setting. The problems may be drawn from disparate fields, but a successful student of mathematics ought to be able to identify if the underlying mathematical principles needed to study the problems may, in fact, be the same. An excellent example comes to mind when thinking about how long it takes to bake a cake, and how long it takes an accidental ink-drop to colour water in a glass. The framing and the setting of these problems is rather different, but the mathematical models describing them end up being very similar indeed. So if one has seen the heat-flow problem solved mathematically, and if one understands the features of the model and its solution, *and* one understands a bit about how ink might mix with water, then one can fruitfully use mathematical insights from the former problem to understand the latter. In such an instance, a successful analogy between situations has been drawn. The mathematical model, then, plays the role of the metaphor expressing the analogy.

For this enterprise to be successful, one needs to have seen a lot of mathematical concepts, and one needs to be prepared to use these in unanticipated situations. I like to tell students that likely no one will give them a 5-by-5 matrix to invert for money, but that if they know how matrix inversion works, they can help mechanical engineers design cars.

In the second epigram of this article, abstraction and synthesis are highlighted as critical skills. I think this is again because the ability to abstract the essence from a given problem, and then synthesize these ideas with other concepts one knows, is important for drawing fruitful analogies.

If I assume that all three—disciplinary content, interdisciplinary breadth and *soft skills*—are important, and that a central aspect of successful problem solving is the ability to draw fruitful analogies, how do I go about translating this into pedagogical practice?

MATHEMATICAL PROBLEM SOLVING IN TEAMS

When appropriate and when resources allow, I have found open-ended mathematical problem solving in teams to be an interesting pedagogical tool. Depending on format, open-ended problem solving in teams teaches/uses:

- Disciplinary content
- Interdisciplinary breadth
- Soft skills
- Ability to draw fruitful analogies

I think such activities are valuable as training. If possible, students should be given poorly formulated problems, and organized into teams which comprise of students from different disciplines, and where they take their time to formulate a mathematical problem, learn the tools they need, and then solve the problem. As I write this, colleagues in a university setting will recognize that I'm talking of a luxury. I have not attempted this in large classes with no assistants where the curriculum is specified; I have not attempted this in classes which are required as part of accredited programs (where the content is non-negotiable). In the settings where I have tried such team-based approaches, I have *sometimes* found that students see:

- few problems in industry are even formulated as clean word problems;
- they don't know what mathematical concepts they will need during their lives;
- intellectual flexibility, humility and willingness to learn are key;
- they need to defend and critique their mathematical ideas with honesty;
- there are no medals awarded for a complex solution, if a simple one suffices;
- sometimes there is no 'correct' answer;
- (non)-mathematical communication is useful.

I find that working in teams to arrive at a mathematical solution to a non-mathematical problem reinforces the need for mastery of the 'hidden curriculum' (in Pemantle's formulation).

I'll now describe, as promised, three different attempts to teach skills which are useful in mathematical problem solving in industry.

EXAMPLE 1: PROBLEM SOLVING IN GROUP TEAM PROJECTS

I attempted to train 3rd-year undergraduate students within an *existing* course in Elementary Numerical Analysis (Math 317), at McGill (between '01 – '07). This also coincided with my attempts to apply the principles of *reflective teaching,* and I am grateful to Prof. Lynn McAlpine and Dr. Denis Berthiaume, Faculty of Education, McGill, for their help and advice during this (McAlpine, Weston, Berthiaume, Fairbank-Roch, & Owen, 2004).

The curriculum of the course was specified. It was a co-requisite in an accredited course (in Mechanical Engineering), and I had no flexibility in terms of drastic changes to content or assessment. However, I had the students work in groups on a semester-long project. The class was a 3rd-year Numerical Analysis class, and the audience consisted of students in mechanical engineering, math, physics and computing science. I had anywhere from 80 to 120 students. Over and above the group project, the students had five assignments (including coding projects), two midterms and one final exam. I had two teaching assistants, who also helped with the marking. The introduction of these group projects required me to schedule six additional office hours (over and above the usual) per week for this course.

The groups were self-selected with 4-5 students with complementary expertise. The goal for the group projects was to show students:

- the relevance of mathematical concepts learned to problem of *their* interest;
- concepts beyond those taught in class;
- how to write modular scientific codes;
- how to work in teams;
- how to communicate their results.

The students were asked to find an interesting mathematical problem with a time-dependent partial differential equation that they wanted to solve. They were asked to set up a precise mathematical model, and then discretize it using the method of lines. Along the way, each team was required to write an algorithm to use a Newton iteration for a system of nonlinear equations, embed it as a part of an implicit solver for ODE, and then finally combine it with a finite difference discretization in space to yield a solver for their evolution problem. They were asked to relate their findings back to their original problem, and post their work on a publically-viewable Wiki. There were three 'check points' during term where I monitored their progress.

I think the students learned to work in an interdisciplinary team. They certainly learned concepts beyond the curriculum, in context; they learned the details of their specific team's open-ended problem. They had public exposure of work as an incentive. It was hopefully fun and rewarding for them as well, but extremely time-intensive. I am not sure it would work well for students who have other work or family obligations.

I learned from this experience. Groups were initially over-ambitious in the problems they wanted to solve, picking projects that were simply too complex for the mathematical and science/engineering background they possessed. Some wanted to study heat flow in a jet turbine, some wanted to model entire financial markets, and some wanted to understand the buckling of a car under impact. Working with them to find a more tractable problem which was still interesting took a lot of time, and (I hate to admit it) involved a certain disappointment among the students. Groups learned that communicating and sharing code is difficult in heterogeneous teams... and yet this is something they will be expected to do most of the time in industry. I also learned that group projects were a lot of work for students and instructors. It was fun and rewarding for me, but I cannot recommend this strategy without reservation. I also learned that I had neither the training nor the patience to properly resolve the inevitable issues which arose between personalities in groups. But these issues are very important.

Most importantly, however, I learned that perhaps students at this stage did not have enough disciplinary information to fruitfully draw analogies and identify metaphors. Too many things were simultaneously new—the application (problem), the mathematics, the computational skills—and while the students learned quickly, I believe this strategy would work better with students with more experience in either the mathematics or in the application area.

EXAMPLE 2: PROBLEM SOLVING IN A COURSE

I decided to structure an entire course around problem solving, where the problems and applications would become the focus and the mathematical/computational science ideas would be taught as the need arose. This is a mathematics-by-case-study approach. I also tried to separate the *soft-skills* aspect from the *mathematical training* aspect.

The class was a 3rd-year ‘special topics’ class, consisting of around 20 mathematics, physics and computer science students. This group possessed more in terms of mathematical background than the group in the previous example. However, they had less exposure to problems arising in technological or engineering applications.

The class was structured around five case studies. There were five assignments, one midterm and one final. My role as instructor was to provide case studies and help students with new concepts.

The intent was for mathematical and computational concepts to be introduced in the context of applications. These applications and the related concepts were:

- Mathematical epidemiology, ODE, ODE solvers;
- Walking on coals, non-dimensionalization, scaling, asymptotics;
- Mystery chord in a Beatles’ song, Fourier analysis, FFT;
- Cost of annuities, probability, Monte-Carlo methods.

A fifth case study was picked by the students.

The students certainly worked on several interdisciplinary problems. They learned how to write a mathematical formulation of these problems, and also how to identify when their existing mathematical tools would need to be augmented by new tools. They learned the details of five open-ended problems, and got a broad exposure to modeling and simulation techniques.

However, the lack of familiar structure was a challenge for students. Once again, I found that lack of familiarity with both the application and the mathematics made the enterprise challenging for the students—they had to learn not only about asymptotics, but also about how heat transfer works, what the thermal conductivity of skin is, etc. Moreover, I’m not sure

I was successful in conveying the mathematical ideas in a manner which could be abstracted away from the specific example. In this approach, students learned concepts in the (narrow) context of a given application. I had no clear idea of how to test whether they'd learned how to transfer these concepts to a different application entirely. And, once again, this course was a lot of work for students and instructor.

So, returning to Pemantle's quote, while I made the course goals explicit and exposed the *hidden curriculum*, I'm not certain how to assess whether it was pedagogically successful. I do not have a clear idea how to assess the effects of this class on their long-term mathematical habits or their ability to make mathematical analogies and connections. Ideally, I would have an opportunity to evaluate their abilities in these areas before, during, and a few months after the course. If the intent of the class was, explicitly, to train students in problem-solving strategies of the kind they'd need in industry, then it should be possible to design long-term assessment strategies to measure the success of the pedagogical ideas I tried. If I ever teach such a class again, I plan to get expert help on this.

EXAMPLE 3: MATHEMATICAL MODELLING GRADUATE CAMPS

There is a long tradition of mathematical problem-solving study groups for industry, intense workshops where mathematicians and graduate students spend a few days trying to model and analyse problems in industry. I'd like to share my experience with running a training camp for students in preparation for such a study group.

The structure of an Oxford-style study group is as follows. Prior to the event, the number of problems and the number of participants (M, N natural numbers) are known. The duration (T + 2) days of the event is also known.

- On day 1, people from industry/non-profits present M problems.
- N mathematicians/students self-select into M teams.
- Teams work on problems for T + 0.5 days.
- The work is fast-paced, and can be intellectually intense.
- On day T + 2, teams present their solutions.
- Solutions usually = mathematical models, some analysis and simulation.

Typically, $M = 5$, $N = 40$, $T = 3$ (so, duration of event = 5 days). Usually, the graduate student participants have the opportunity to gain experience in a training camp, held the week before the main event. The format is similar, except the problems are presented by mathematicians, and the teams comprise only of students.

I was involved as a mentor as part of such a training camp at Oxford in April 2014. The problem I assigned was the so-called *Airbus 380 Problem*. I knew that this was a mathematically challenging, industrially important problem with no 'perfect' solution.

The set-up of the Oxford Grad Modelling Camp was as follows: There were several teams of graduate students, each assigned a mentor who presented a problem. I had a team of six graduate students of mathematics with diverse backgrounds. They had 3.5 days to study the problem, propose model(s), and prepare a presentation. My role as mentor was to guide, but not direct.

The stated goal for the students was to design a boarding protocol for the Airbus 380.

The protocol:

- was for an Airbus 380 in economy-only configuration;
- must be efficient: reduce total boarding time for the flight;
- must be robust to out-of-order boarding.

I gave them the total number of seats in the plane, and the configuration of the seats. It makes sense that mathematics may be useful to help design such a protocol, but how? What is even a good mathematical question to ask? Based on this very loosely formulated question, they had to construct a mathematical model, and then use it to give me a protocol.

Very quickly, the team found important resources (papers). The students in my team built a computational tool for timing and visualizing boarding strategies; investigated optimization algorithms—integer programming, genetic algorithms; located actual physical parameters in the problem; and examined combinatorial questions

An impressive amount of work got done, but the team tended to split into sub-groups to pursue different ideas. The tendency to work alone, and locate the ‘perfect’ answer, was in evidence at the beginning!

My observations based on this, and similar camps in the past, are that it is:

- good to let the team make their own mistakes;
- important to have ‘group summary’ meetings 4-5 times a day, to force team members to explain their thoughts and progress;
- key to not explicitly favour one approach over the other;
- helpful to keep reminding the students of the *problem at hand*, not the *problem we wished we had*;
- strategic to distinguish between useful and unproductive frustrations.

I believe the students became aware of the need to recognize analogies with problems in other contexts. This was critical given the short duration of the workshop. They learned to model this specific problem in deterministic, stochastic, and computational ways—this represented a very large number of new mathematical concepts they had to learn on the spot. They learned to locate relevant information, and discard irrelevant information. They worked with a team of peers under time pressure, and learned to accept that there won’t be a single, ‘perfect’, answer to every problem in industry.

REFLECTIONS

Returning now to the questions I raise in the introduction, I have some (personal) answers.

1. Are problems in industry worthy of our mathematical attention?—Yes.
2. Should we tell our students about mathematical problems in industry? How?—Maybe/I don’t know the best way.
3. Are the mathematical abilities one acquires at university worthy of industry’s attention?—Maybe.
4. Is it worth trying to teach industrially-relevant mathematics?—I don’t know.
5. Why are we teaching mathematics to non-mathematicians?—We should be honest about the answer, whatever it may be.

The first two questions I have attempted to address in previous sections. Even the third—Are the mathematical abilities one acquires at university worthy of industry’s attention?—has been partially addressed. We have seen that it is not enough for our students to acquire disciplinary expertise. They need to work in an interdisciplinary setting, get comfortable with poorly-formulated open-ended problems and learn *soft skills* such as teamwork and communication.

I think success in solving mathematical problems in industry follows from the ability to draw fruitful analogies and identifying mathematical commonalities. I am not sure we are universally able to help our students learn these skills with the very real constraints at a university. As I've tried to describe, my attempts to convey some of these skills, some of this 'hidden curriculum', required a very serious commitment of time and resources. These challenges facing us at universities—larger student bodies, more educationally-diverse backgrounds and aspirations, constrained resources, poorly-defined yet very real pressures to 'teach industry-ready skills', institutional traditions and rivalries—play as significant a role in what happens in our classrooms as our thoughts on pedagogy. It is all well and good for me to suggest team problem solving or a flipped classroom as a good pedagogical idea, but for you to adopt it, you must look around you and determine if it would even begin to make sense in your context. This isn't the math-education/mathematics dialectic. This is the can-this-be-done-with-what-I-have question.

After the CMESG meeting, I had the opportunity to design and teach a mathematics class for first-year students with strong Calculus backgrounds, who are interested in physics. The content of the class was structured to support a physics class the students would be taking at the same time. Our lecture schedules were coordinated, as were some of our homework assignments. We explicitly drew connections between topics in the mathematics and the physics classes. The students enjoyed it a great deal—there was no question about the relevance of the mathematics they were seeing, nor of the formalism of the physics—and the instructors learned a lot, too. We tried very hard to demonstrate how we approach problems in our disciplines, including the importance of drawing connections, trying a range of quantitative and qualitative approaches and genuinely collaborating with peers. As a consequence, this cohort of students saw material at a much more sophisticated level in both their mathematics and physics classes.

Teaching these courses was a privilege and a joy. I was able to try a lot of the pedagogical strategies described above, and some new ones. Some worked, and the feedback from the students was invaluable. They were truly active learners, which, after all, is what we hope of our students at university. However, since both classes were very challenging and structured in an unfamiliar way, the initial enrolment was low. We operate in a system with constraints, and so, despite our collective best intentions pedagogically and the demonstrable success of the course (in terms of student learning), the future of this pair of courses is unclear.

This last example serves to remind us that while we may want to train our students for successful careers in industry, where the drawing of connections (analogies) will be rather important, at the university our pedagogical focus is on the disciplinary tools (the metaphors). As mathematicians, we recognize the need to make explicit the hidden curriculum, and to show 'mathematics in action'. Indeed, we know we need to help our students acquire both technical depth and a simultaneous breadth in using those tools in novel situations. Many of us seek to explicitly integrate problem solving into our curricula. But we do this within the framework of the institutions we teach in. At least I have to temper my expectations and pedagogical aspirations with this in mind, in the light of what I know.

In conclusion: yes, I think there is very interesting mathematics in industry, and we do our students a disservice by not showing them instances of it. I'm not sure how best to do this, but at all levels of mathematical instruction we should be honest about the importance and role of abstraction and logical argumentation. These are the traits which will be useful for them going forward, in addition to learning how to communicate, be intellectually flexible and work in professional environments.

Most of all, I think when we teach mathematics, we should remember that it is fun for us. Maybe some of our students will be convinced it is fun for them as well.

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Elder Talk

La parole aux anciens

INTER-ACTION AND MATHEMATICS KNOWING – AN ELDER'S MEMOIRE

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PROLOGUE

Fifteen years ago, in my discussing with Les Steffe his study of various aspects of children's construction of mathematical ideas, Les made an interesting distinction between two kinds of concepts: *children's mathematics* and the *mathematics of children*. By the latter, Les points to the various observed knowing actions that children take as they work on mathematical tasks—in his case, mainly dealing with whole numbers and the fractional numbers of arithmetic. But Les sees as his project to develop a *children's mathematics* that is formed by selecting and abstracting, from observations of such a body of mathematical actions, key action/action sequences and the relationships between them, which form schemes of knowing a particular set of inter-related mathematical concepts in a particular domain. To what Brent Davis (1996) calls *the formal aspect of the body of mathematics*, I think Les wishes to contribute a category that, at a formal level, portrays necessary action sets, sequences and relationships among them that would characterize, say, the meaning of knowing whole numbers or fractional numbers by children. As noted above, this mathematics comes out of, or is an abstraction from, the body of careful observation of and describing of and cataloging of actions that children take in doing what can be observed as mathematics in very clever— frequently computer-based—action settings (*mathematics of children*). Les, as well as being a great theorist, has with his colleagues and students amassed one of the greatest collections of such data. Les has written about this multi-faceted project of many years extensively (e.g. Steffe & Olive, 2010) and his work, though different from mine, continues to stimulate my thinking. I will use these two concepts of child/student related mathematics in my own way in what follows.

From my point of view, one can see a wonderful example of an abstract portrayal by Merlyn Behr, Gershon Harel, Tom Post and Dick Lesh of a form of *children's mathematics* related to fractional numbers (related to each of the sub-constructs that I developed in 1976), especially in their 1992 paper. In this very dense paper that uses systems of representations of fractional objects and systems of representations of (mental-physical) mathematical actions on them that are seen as necessary for a child—to use one very small example—to see, that ¾ seen as three ¼ units is equivalent to ¾ seen as a 3-unit divided by 4 is equivalent to a ¾ unit…. While Steffe's idea of *children's mathematics* tends to focus on the order of schemes or mechanisms that he sees as a sufficient condition for a particular mathematical idea complex—e.g. adding whole numbers, the Behr et al. (1992) ideas with respect to the different constructs of rational numbers try to provide a theoretical picture for each of a wide variety of fractional number tasks (showing equivalence, adding, multiplying, ordering…) within each construct (e.g. fractions as measures or operators…).

While I have worked with both Steffe and the four *Rational Number Project* researcher-writers above, what you will see below are examples of the *mathematics of children* occurring within one fractional number lesson. Yet I see this mathematics knowing-in-action as related to the *children's mathematics*, particularly as seen in the work of Behr et al. cited above. I have talked with Merlyn Behr many times, before his untimely death, about his beautiful representation systems. I have seen many examples of instances of his abstractions in my observations of children in action (see Kieren & Simmt, 2002), particularly examples of abstractions such as *unit-forming*, *unit-comparing* and *unit-transforming*. One can also observe such abstractions-in-action in the actions and inter-actions shown in the children's work later in this paper, actions that can be related to Behr et al. (1992) ideas.

Because of my interest in the work of Maturana and Varela (1980, 1987) for the last 30 years, contrasted with the *children's mathematics* work pointed to above, I have looked at *mathematics of children* as it arises in inter-action with aspects of the environment and with others in it, including the teacher. I have sought to think about the implications of this for mathematics knowing and teaching using the ideas garnered from these observations of students from age 5 to 17. Following ideas from Maturana and Varela's book, *Tree of Knowledge* (1987), I, with colleagues Elaine Simmt and Joyce Mgombelo (Kieren, Simmt, & Mgombelo, 1997), have come to see knowing as occurring in the praxis of living in an environment with others. In this living, *individuals bring forth a world of SIGNificance with others that an observer sees as involving mathematics*. Varela, Thompson, and Rosch (1991) would argue that in inter-acting with an environment and others in it, the knowing action on the part of the knower can be observed to co-emerge with the environment. It is almost a cliché to say that the environment affects the individual knower and knowing, but what is interesting here is that the knowers in action and inter-action necessarily affect and change the environment and in particular the cognitive domain. These enactivist ideas can be observed in action as the children as described in my memoire below are seen bringing forth a world in the flow of their praxis of living in inter-action in a particular setting.

PART I: AN EXAMPLE OF MATHEMATICS IN INTER-ACTION

VIGNETTE 1—DOING MATHEMATICS DIFFERENTLY IN INTER-ACTION

Ja and her partner Ru sit at side-by-side desks at the back of the classroom. They are two of 20 students, mostly girls in a Grade 4 Spanish bilingual classroom. They are working on a task, using a particular "Fraction Kit" where they are asked to find at least three examples of ways to make ¾ of a pizza without using fourth-pieces; they are also asked to sketch their work and write addition sentences representing it. They are allowed, if they wish, to make their sketches and write their expressions on the board as well. Before we follow their work, we'll consider the setting in which it occurred.

I have done mathematics lessons of various kinds with this class over the last five years, starting even in pre-school, so I am no stranger. In fact I am known as *Abuelo* Tom to this group. The children have done a short unit on fractional numbers in Grade 3 and have in the Grade 4 year done initial work on decimal fractions, mainly using tenths and hundredths, previously.

The lesson from which the three vignettes in this paper are drawn will not come as a surprise to those of you who know me.

At the front of the class is a large sign:

ABUELO TOM'S TIENDA DE LA PIZZA
LA CASA DE "1 CENT PIZZA"

Each student has a kit in an envelope containing two ones-units worth of each of these fractional number amounts: ones, halves, fourths, fifths, tenths, twentieths, twenty-fifths, fiftieths, and hundredths. These fractional pieces, each fraction piece being a different colour, are based on 'one' being a square 20 cm $\times$ 20 cm; with the others being assorted rectangles of appropriate sizes, ending with 2 cm $\times$ 2cm-squares for hundredths.

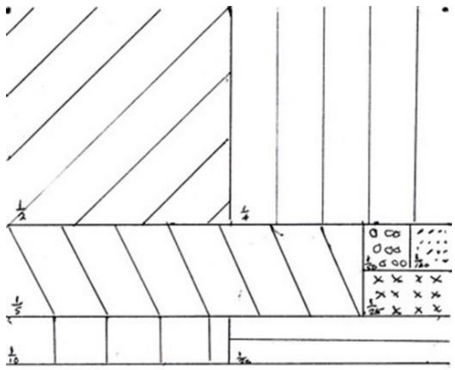

Figure 1. Pieces in the fraction kit (stacked on the ones unit).

The amounts are not labeled on the pieces. It is important to note that one can't judge the relationship between piece sizes for many comparisons either by simple looking or simple coverings—for example, comparing the fifth and the fourth pieces cannot be done by looking, although one can easily see that the tenth is one half of a fifth piece by laying a tenth piece on a one-fifth piece (and even this relationship is not obvious if one centres the tenth piece on the fifth piece). This feature of the kit is there, in part, to promote student 'reasoning' as to relationships between amounts shown by fractional numbers, rather than relying simply on 'looks like' perceiving. (This distinction is related to the difference between the first two vanHiele levels.) The children had not worked with all of these fractional numbers in combination before. Nor had they worked with this kit before. Although the children made use of all of the different pieces in dealing with tasks, the activities described below deal mainly with fractional parts of 1 whose relative 'size' is greater than or equal to 1/20.

The first task, that proved very easy for the class with nearly everyone offering ideas, is:

> Given that the large square is one, what is the fractional number for each of the other pieces of 'pizza'?

So each of the coloured pieces now has a fractional number associated with it, based on the largest piece being 1. (See the first column in Figure 2 below.) You will notice right from the start both the ratio relation aspect of rational numbers and rational numbers as being "muchnesses"—quotients or measures, in particular—come into play. From previous work in Grade 3, these students are at least informally aware of the actions needed to name at least some of the pieces in relationship to the whole or 1 with the correct fractional number (e.g. ½ and ¼). And as we started naming the smaller pieces using fractional numbers, they easily saw how tenths could be created from fifths, as well as how both fiftieths and hundredths could be created using a twenty-fifth as a unit—$\frac{1}{2}$ of $\frac{1}{25}$ is $\frac{1}{50}$; $\frac{1}{2}$ of $\frac{1}{50}$ is $\frac{1}{100}$ and $\frac{1}{2}$ of $\frac{1}{2}$ of $\frac{1}{25}$ is $\frac{1}{100}$. Notice the connection to Behr et al (1992) ideas of unit-forming and unit-transforming noted above. Also note, just in this task itself, the relationship of forming fractional pieces from a whole in various ways, as well as the possibility that children would see one fiftieth both as $\frac{1}{2}$ of a twenty-fifth and as $\frac{1}{50}$ of 1. Examples of Piaget, Inhelder and Szeminska's (1960) critical ideas on fractional thinking from their book *The Child's Conception of Geometry* include: forming fractional parts of the whole of the form $1/n$; given such a fraction, use it to form a fraction of that amount $1/np$; and from $1/np$ recreate $1/n$; as well as 1 (where n and p are natural numbers).

The second task is to give the cost for each piece, given that the largest is priced at \$1 or 100 cents. This task was a little harder, but most of the class, if not all members in it, soon worked out that $\frac{1}{2}$ is 50 cents; one-fourth is 25 cents; $\frac{1}{10}$ is 10 cents—so $\frac{1}{5}$, which is twice as big, is 20 cents. Some children notice that you can just divide 100 by n to get the cost of a $1/n$ size pizza. Of course, this problem setting is an artificial one; nevertheless, several children also argued that there should be an added "cutting cost", especially for the smaller fractional parts.

When task two was finished the children were asked to write the costs in "dollars" form (e.g. 50 cents as \$0.50). This left us with what is shown in Figure 2 below.

Fractions	Cost (cents)	Cost (\$)	
1	100	1.00	
$\frac{1}{2}$	50	0.50	
$\frac{1}{4}$	25	0.25	
$\frac{1}{5}$	20	0.20	
$\frac{1}{10}$	10	0.10	
$\frac{1}{20}$	5	0.05	
$\frac{1}{25}$	4	0.04	
$\frac{1}{50}$	2	0.02	
$\frac{1}{100}$	*1*	*0.01*	*The One Cent Pizza*

Figure 2. Various designations related to fractions and fractional numbers.

Now the children were asked to do the following: Find several ways to make an amount of pizza equal to one half using the kit and write a mathematical sentence to describe this situation—e.g. $\frac{2}{4} = \frac{1}{2}$; $\frac{1}{4} + \frac{5}{20} = \frac{1}{2}$. As noted above, students were urged to draw pictures of their solutions and write their mathematical sentences next to them on the white board. Of course once a couple of students had done this there was a parade to the board. Some of this work is shown in Figure 3 below. (Notice the variety of responses, including one using *subtractive* methods.)

Although this was not a research study, nor did I have another person observing groups of children (making observations of their mathematical actions) during the lesson, I did watch the three pairs of students discussed here more closely than I did the others, and what is described here is my reconstruction from things I noted down in each case. The purpose is to provide the readers with situations to stimulate discussing roles that inter-action has in the mathematics-knowing actions of these children.

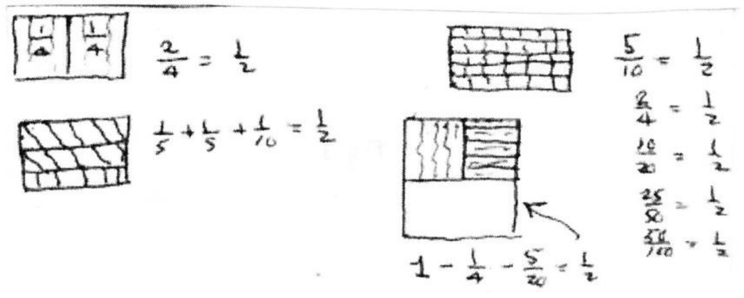

Figure 3. Reports of making one half a pizza.

Getting back to Ja and Ru and their work, the class then was asked to work on the following:

> An order comes in for ¾ of a pizza but there are no fourth pieces available. Make up ¾ of a pizza without using fourth pieces in at least 3 ways. Write mathematical sentences to describe your work.

When I came to their desks Ru was putting fractional pieces together to make three fourths and talking aloud as she did so: "*Yes, I think one half plus 2 one tenths will do it, because these parts here will just make it fit.*"

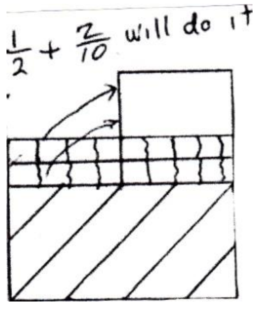

Figure 4. Ru's actions/expressions in making ¾.

Ja immediately replied, "*That can't be right. Three fourths should cost 75 cents but your amount costs just 50 cents plus 20 cents.*" Ru, now talking to Ja, "*I see. My two parts won't cover the rest of the fourth. I need another little piece; oh yea, a twentieth, so* $\frac{1}{2} + \frac{2}{10} + \frac{1}{20} =$ *¾.*" Ja replied, "*Yup 50 + 20 + 5* [for the twentieth] *is 75 cents*." Then aloud, but to herself, appearing to be looking up at the chart on the board, "*Oh this is just like decimals.*"

Following that remark Ja says, "*Let's do something else. We'll do five fourths instead*." Ru is puzzled and looks at her. Ja says: "*You know* $\frac{5}{4}$*; that's like 1 and ¼ pizzas. You make* $\frac{5}{4}$ *of a pizza and I'll write down the decimal for each piece you use and check if mine adds up to 1.25.*" Ru puts out 5 one-fourth pieces. Ja says, "*That's too easy. It's 5 times 0.25. You know .25; .50 ; .75; 1.00; 1.25*." "*OK,*" says Ru and puts out 2 halves and 2 tenths as Ja writes 0.50 + 0.50 + 0.20 in column form. Ru, not attending to Ja, says, "*Wait. That's not enough. I need another half of a tenth—oh, a twentieth piece*." Now Ja adds a 0.05 to her column and says: "*You're right. That adds to 1.25*."

PART II: SOME THEORETICAL NOTES

There was more to learn in watching these two, but I looked up and saw Ro and Jo doing something that looked interesting, so I moved on to them. I'll turn to Ro and Jo in Vignette 2 below. But before that I'd like to talk about the point of view I'm taking as I think about the roles inter-action plays in this classroom and in mathematics learning more generally. I want to highlight again the definition of *knowing* derived from Maturana and Varela (1987). In knowing acts a person *brings forth, with others, a world of SIGNificance in a sphere of behavioural possibilities (of the knowing actors)*. Notice that this definition sees knowing as living, and in fact Maturana and Varela, in the same book, *identify knowing with doing, with living and with learning* (or necessarily adjusting one's behaviour to new circumstances). Thus I view mathematics learning as part of the process of living which, as noted earlier in this paper, necessarily involves inter-action with the environment and others in it. I think one can use this picture of knowing in looking at the classroom I was in and part of. Clearly one can observe the children *bringing forth a world of significance* for them:

- Notice that, although fully cognizant of the artificiality of the 'pizza' task situations, children enter into them vigorously and bring in their everyday knowledge—e.g. physically comparing sizes of the pieces but also mentally dividing them to form new 'units'.
- They are clearly motivated by being able to show others their work (on the white board).
- They watch each other and notice elements of what others are doing. They look to others and records of their work on the white board for different or interesting-to-them ways of responding to tasks.
- Clearly they observe that this form of living involves mathematics. Notice Ru using her knowledge of the relationship between $^1/_{10}$ and $^1/_{20}$, or Ja realizing that money expressed in dollars was just what she already knew as "decimals". In fact, her persistent use of decimals and Ru's accepting her comments in this form suggest that they were inter-acting effectively using two fractional 'dialects'.

Thus we see the knowing *with others* as a significant contributor to the actions, the expressions and the mathematics one can observe the children—and me too, as I watch and listen to them—engaged in. Although the two girls above are acting differently, they are recursively coordinating their actions together and bringing forth a micro world which I, and they too—after all this is mathematics class—observe as mathematical. Notice also that the cognitive domain is changed by their actions and inter-actions. Mathematics as they see it and act upon it and communicate about it changes. Although it is beyond the scope of this memoire to discuss it in detail, this action of the two girls above involves and is done in language and making distinctions in that language. That is the import of highlighting the world of *SIGNificance* that they are bringing forth and acting in.

To look more explicitly at inter-action and its qualities, I now turn to a model that Elaine Simmt developed to study inter-action within parent-child pairs as they worked together on variable entry mathematical tasks (Simmt, 2000). She and I have used it in many circumstances, including observing Grade 3 children working on fractional tasks (Kieren & Simmt, 2002) and practicing teachers developing various approaches to thinking about the mathematics of integers (Kieren & Simmt, 2009). See Figure 5 below for a simple form of this model.

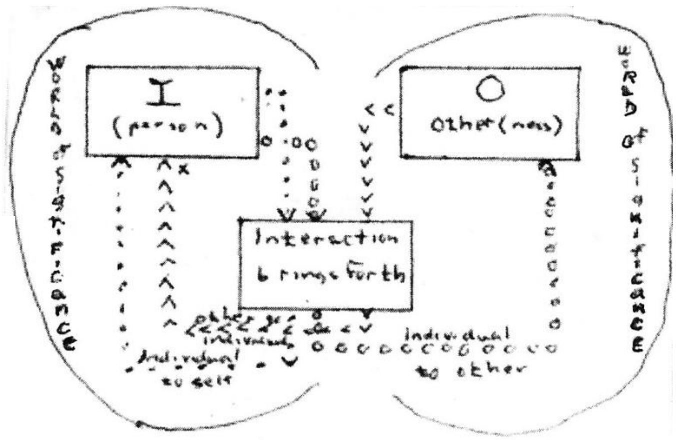

Figure 5. Bringing forth a world in inter-action.

This figure has many features as well as an accompanying set of ideas. The three principle boxes in the figure are labeled. I is for the individual person and cognitive actions associated with the individual. The second box is labeled O (otherness) and can be thought of as the other persons in the environment and their related actions, as well as material in the environment. The third box is labeled and highlights the *bringing forth in inter-action* to indicate the inter-actions between I and O or I and I (an individual with his or herself). There are three denoted pathways that highlight three different inter-action forms. The first is shown by >>>>> and relates to actions of the other (or otherness) related to the action of the individual. The second indicated by ***** depicts the inter-action of the individual with her- or himself. The third pathway labeled ooooo indicates the way in which actions of the individual potentially impact the otherness and others in it. This interaction necessarily occurs, according to Maturana and Varela (1987), within the sphere of the behavioural possibilities of the individual(s) involved. That sphere is necessarily changing over the course of the inter-actions occurring as the persons in and with the environment bring forth worlds of SIGNificance together. This domain is necessarily a social one, featuring many simultaneously occurring and changing *structural couplings* (Maturana & Varela, 1987) between individuals, as well as persons and the other elements of the environment. Actions and inter-actions in this social domain can be seen to affect the cognitive domain in which they exist as well. Not to over-simplify this, these actions/interactions change what is meant by mathematics in the environment. Thus mathematical knowing is observed as a dynamical phenomenon based on inter-actions even though, at the individual level, actions are determined by the *structures* of the individuals (in the Maturanian sense of the living out of their biological organizations and their lived histories of actions in environments). The children are not automatons giving pre-programmed responses to the challenges of the environment, but are in fact acting as autopoietic beings, continually changing their capacities to respond to the environment and others in it and in so doing also changing the environment.

A vivid example of this is the remark by Ja, "*This* [these problems and what I am doing] *is just like decimals*." Although this change is subtle, it might be the occasion for her changing the task they were working on and how she deliberately announced the roles for both her and Ru to now assume. Thus at once there are dynamical mathematical understandings occurring for individuals (see the work of Pirie and Kieren (e.g. 1994) on dynamical understanding or John Mason's (e.g. 2002) work on noticing) as well as the inter-active knowing that seems pertinent in considering the *mathematics of children* portrayed in this paper. As with other work I have done (Kieren & Simmt, 2002, for example), the examples in all three vignettes show the fractional mathematics of these children exhibiting interesting characteristics including:

- a central focus on unit fractions as nouns;
- unit transformations in action;
- adjectival use of fractional language—the 2 in $\frac{2}{10}$ being an adjective modifying $\frac{1}{10}$, but also as part of the expression for a two-tenths unit or fractional number entity;
- and flexibility in using systems of mathematical symbolism to portray and compare mathematical products and reasoning. This latter capacity can be observed as a tool in dynamical personal understanding and also a means by which inter-action and its product, *Worlds of SIGNificance,* are facilitated.

Before returning to the classroom featured in this memoire, I would like to discuss in more detail the paths (>>>>>;*****; ooooo) noted above:

- Pathway >>>>> might point to the inter-action between elements in a teacher lesson in action (in our case, the prompts related to the fractions kit), other student's actions (say Ja's suggestion to change tasks), and other elements of the environment (the material on the whiteboard) with the individual I. Just outside the I box, right where both the >>>>> and the ***** paths enter it, there is the letter X. The X points to the phenomenon of *occasioning*: Suppose a person's(s') actions or particular elements of the environment (in the otherness, O) are noticed by an observer to be taken up by I and transformed by her/him for her/his use. When this occurs one can say the inter-action with O *occasioned* the actions of I.

- The pathway ***** points to the inter-action between I and her/his records of, memories of, reflections upon, formation of re-presentations of, etc., of previous actions. Von Glasersfeld's 1995 book is replete with many examples of such an inter-action path related to all of his various forms of re-presentations, especially empirical re-presentations done by a person in constructing and reconstructing his or her schemes of acting. While students in these vignettes are working in interactive pairs, each student can be seen in their actions to exemplify this form of reflexive pathway in their actions.

- And finally, reciprocally with >>>>>, the pathway ooooo implies that the actions of I have an impact on and change persons or the observed nature of objects in the otherness. This phenomenon is at the heart of Varela et al.'s (1991) maintaining that *the individual and the environment, and for our purposes I and O, assuming they are structurally coupled* (see Maturana & Varela, 1987), *are co-determined through inter-action*.

Some of the effects of inter-action, especially those from O to I (>>>>>) and I to O (ooooo) are observable in the episode involving Ru and Ja above. One can clearly observe the effects of Ja's "decimal" computations on Ru's appraisal of her work, but also to reiterate a point made differently above, the effect of the table in Figure 2 above on Ja's relating and using the relationship between dollar representations and decimal fractions to regularize her own activities, but also to change the environment in which she and Ru are working.

"ETHICS" AND KNOWING ACTIONS

From the point of view of the mathematics knowing in the classroom, this set of pathways of inter-action carries with it ethical implications. Elaborating initial thinking done on this by Varela et al. (1991), Kieren and Simmt (2009) have noted three forms such ethical actions might take: *provisional*, *attentional*; and *occasional ethics*. Such ethics are not seen in formal rules of conduct, but are observed in action and are thought by Maturana (2005) and von Foerster (2003) to contribute to the social fabric that allows, for example, these pairs of students to make places beside themselves for each other, and thus allow interaction to continue effectively.

The first, *provisional ethics*, is usually thought to impact the ways in which a teacher prepares and, on the fly, acts to provide students with tasks, materials, mathematical insights, and help. In the vignettes, the teacher (I) exercised such ethics:

- in designing and providing the kits, designing tasks that allowed for variable entry into them (Simmt, 2000); and
- in encouraging students to exhibit their work for others.

But the students also exhibit such an ethic in action when they, through their actions, consciously act to maintain their working relationship with another or with the whole class. This was obvious in the inter-actions of Ja and Ru where especially Ja acted in such a way as to include Ru's actions in her thinking in action, thus *valuing* Ru's actions, but also providing tools—use of decimal equivalents of fractional numbers to portray and evaluate their mathematical actions. On reflection, while the teacher (I) provided and encouraged an avenue, the collection of student work on the board, the teacher (I) missed the opportunity to *provide* deliberate opportunities for students to comment upon or use the ideas represented on this white board display. In an electronic age, such a provisional ethic would prompt the teacher to ask the students to make effective use of a smart board, were one available.

Attentional ethics are observable (or not, by their absence) when both students but also teachers *notice* (to use a concept of John Mason) the actions of others and take them into account in their own knowing. Clearly in Vignette 1 above, both Ru and Ja indicate by their continuing actions and interactions that they are attentively taking account of one another's thought/actions as well as of the materials and tasks provided by the teacher even when, perhaps especially when, one sees Ja modifying the task at hand.

The third way of making a place beside yourself for the others actions/ideas, etc., is to act with an *occasional* sensibility. That is, as pointed out by Varela et al. (1991), acting in this manner implies that one, whether a teacher or a student, is aware that anything he or she does may act to occasion another to act as well. Thus responsibility for one's actions, even having bright ideas, carries with it an awareness of the effects of one's actions on others and the environment for learning.

I will now turn our attention to two other vignettes with an eye to observing the concepts above used as tools to point to ways in which inter-action affects mathematics knowing and vice-versa as part of living—mathematics knowing of participants affects the inter-action.

PART III: MORE EXAMPLES

VIGNETTE 2—INTER-ACTION WITH ONESELF?

> [In this brief episode we see Ro principally in inter-action with herself and her work. Clearly this episode can be observed to show the structure-determined nature of her actions, affected by her lived history with the mathematics of fractions but also by her lived history as a person—she likes to do things differently from others. But what might distinguish this from a 'constructivist analysis' is noticing the way that she is inter-acting with the environment. She is not simply being triggered by it and making empirical abstractions from her actions on it (to use von Glasersfeld's (1995) term); her actions co-emerge with the fraction kit materials which can be observed to change

> their nature too. Further, her thought/actions are tempered by the attention of others and by the work of her partner, seemingly working on 'the same' task in a very different manner. Thus the phrase, *bringing forth a world of SIGNificance with others*, resonates with me as I observe her work, as individual as it seems.]

Ro and Jo sit side by side, a couple of metres from Ja and Ru. They typically sit with one another in this classroom and are friends. They are working on the task of making $\frac{3}{4}$ without using fourth pieces. Jo is working on finding ways to do this using $\frac{1}{2}$ as one of the elements. Ro watches her for a moment and then says aloud but to herself, "*I'm going to use fifths*," and starts by laying out a half piece and a fourth piece and then laying three fifths pieces on top of them. Of course they do not 'cover' three fourths, but overlap as shown in Figure 6.

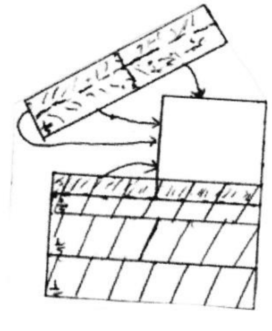

Figure 6. Expressing ¾ using fifths.

Ro ponders what she sees and then moves her finger from the fifth piece that is 'hanging over' and moves her finger onto the fourth piece as shown in Figure 6. Now she lays another fifth piece above the three she has already placed and runs her finger horizontally across the fifth in this position and then vertically through the middle of this piece as well. I watch her move her finger from the fifth piece to the fourth piece several times. Noticing that I am watching (notice the effect of attention), she says, "*I think I've got it using fifths*," and writes $\frac{3\frac{3}{4}}{5} = \frac{3}{4}$. I am puzzled and ask her to explain how that works.

She then re-enacts what she had done tentatively before rather quickly, first by laying down the half and fourth pieces, laying three fifths on top of them. She says, "*See, the bottom half of the fifth covers up the half piece. You see that half of the upper half of this fifth* [piece] *covers part of the fourth and so you 'move' the other half of the half of the fifth piece onto the fourth too. But that covers less than half of the fourth. So I took another fifth piece. I saw I could use fourths of it to cover the rest of the fourth. By doing this… I could see that three fourths of that fifth piece would completely cover* $\frac{3}{4}$. *So* (*three and three fourths*) *fifths equals three fourths.*"

By this time Jo had covered $\frac{3}{4}$ with one half piece and five twentieth pieces and had drawn its picture on the white board with the equality next to it: $\frac{1}{2}+\frac{5}{20}=\frac{3}{4}$. (See Figure 7.) Ro noticed that Jo had been watching her explanation to me. Ro then looked at what Jo did and said to both Jo and me, "*If I actually cut up the fifth pieces like I said, mine would be like hers. So* $\frac{3\frac{3}{4}}{5} = \frac{1}{2}+\frac{5}{20}$." Notice here the impact of Jo's work on this statement. If Ro was thinking, "*Jo's is just another way of doing* $\frac{3}{4}$*, so mine and hers must be equal*," then she would not have constructed the explanation showing a piecewise comparison between the two pieces of work as she did here.

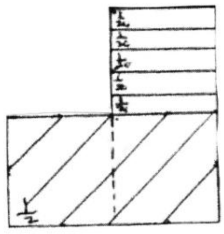

Figure 7. Jo's idea for ¾.

Looking back at this vignette, the focus is principally on an observed inter-action of Ro with herself—and of course the environment, including her use of the kit provided. In problem-posing terms, Ro posed a unique (in the class) problem or task for herself and accomplished it to her satisfaction. Still one could observe that her actions co-emerged with the materials provided in the kit—that is, for Ro, the pieces in the kit became consciously dividable in ways useful to her. While her oral explanations with her actions seemed to be a conversation with herself, still my presence and attention appeared to occasion her to a more careful explanation. Finally, while it appears that Ro paid little attention in this vignette to what Jo was doing (and vice versa), it is clear that Ro took the opportunity at the end to tell in what way her result was equivalent (and even like) Jo's last result. Thus, in this vignette one sees several effects of inter-action on knowing.

One might say that the work portrayed on the board by others in the class occasioned Ro, as she deliberately tried to do something different (a not unusual behaviour for her). But more importantly, notice that even in contrast to Ru's initial work in Vignette 1, Ro's inter-actions with the materials (fifth pieces in particular) suggests that she sees these materials as quantities and units that are transformable (not like fixed puzzle pieces). Thus while her mathematical thought/actions are changed by using the materials, in that inter-action the materials 'change' for her (and others in the class) as well, as we will see in Vignette 3 below. In other words, because Ro used the materials as she did, it allows for the possibility for others to do so (and entails an occasional ethic), just as showing one's work on the board did for many in the class—if you act, others may attend to your actions and be influenced by them.

VIGNETTE 3—OCCASIONED TO EXTEND AND GENERALIZE ONE'S THINKING

As a good example of this occasioning, at this point Ro's classmates Ka and Er hear Ro talking to me and Ka called across, "*What are you doing?*" Ro says, "*I made ¾ using fifths; it's 3 and ¾ fifths!*"

Ka and her partner Er return their attention to their own work. And as I move over to observe them, Ka says to him, "*Fifths huh? Oh, I see like* $2\frac{1}{2}/5$ *makes* ½" (an expression of which is already on the board, although I doubt she has noticed it—perhaps another lapse in my provisional ethics). She then enacts this idea with the pieces, laying three fifth pieces on top of a ½-piece with the third fifth extending beyond one half—she points to the half of the fifth she imagines is covering the ½-piece, by 'drawing' an imaginary line across it. Er looks puzzled but then accepts what Ka is saying. He continues, "*If* $2\frac{1}{2}$ *fifths is* $\frac{1}{2}$ *then* $4\frac{1}{2}$ *fifths is one* [pause]. *No wait that can't be right—*$\frac{5}{5}$ *is 1. So* $4\frac{1}{2}$ *fifths is less than that*." Ka replies, "*Ya,* $\frac{4}{5}$ *is 20, 40,…, 80 cents, and a* $\frac{1}{2}$ *a fifth more is 10 cents…so 90 cents, so* $4\frac{1}{2}$ *fifths is* $\frac{9}{10}$." (Notice the almost casual use of both money languaging and fractions in her reasoning.)

I notice that Ka and/or Er have several ways of 'making three fourths' on the board, with interesting sentences describing them; e.g. $\frac{1}{2}+\left(\frac{1}{25}+\frac{1}{100}\right)+\frac{1}{20}+\left(\frac{1}{25}+\frac{1}{100}\right)+\frac{1}{20}+\left(\frac{1}{25}+\frac{1}{100}\right)=\frac{3}{4}$ (Brackets are inserted here and in the other unusual fractional expressions for clarity for the reader—the children did not regularly use them.) So I am not surprised when Ka (likely occasioned by her brief encounter with Ro's work and the previous short conversation with Er) says [out loud but seemingly to herself], "*I'm going to try to make all these fractions* [meaning not clear] *using fifths*." Turning to Er and me she says, "*We know* $\frac{3\frac{3}{4}}{5}$ *equals* $\frac{3}{4}$*. I'm going to try* $\frac{1}{4}$ *first.*" She lays a fifth piece across a $\frac{1}{4}$-piece and points to the half of the fifth piece covering the fourth and says "*The other part of the fifth goes here*," pointing above the part of the fifth piece on the $\frac{1}{4}$-piece. "*So we need part of another fifth piece. Oh I see,* $\frac{1}{4}$ *of a fifth. Cool!*" Now she writes and says too "$\frac{1\frac{1}{4}}{5}=\frac{1}{4}$". (See Figure 8.)

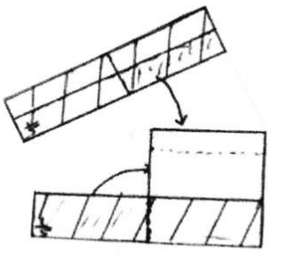

Figure 8. Making ¼ using fifths.

Ka now says, "*Ro showed us that* $\frac{3\frac{3}{4}}{5}=\frac{3}{4}$". "*And you did* $\frac{2\frac{1}{2}}{5}=\frac{1}{2}$," Er adds, "*but then* $\frac{2\frac{2}{4}}{5}=\frac{2}{4}$." "*Hey, a pattern!*" Ka replies. "$\frac{1\frac{1}{4}}{5}=\frac{1}{4}$; $\frac{2\frac{2}{4}}{5}=\frac{2}{4}$; $\frac{3\frac{3}{4}}{5}=\frac{3}{4}$" Er adds, "So $\frac{4\frac{4}{4}}{5}$ *must be* $\frac{4}{4}$ *—is that right?*" "*Yup,* $\frac{4}{5}+\frac{1}{5}=1$*, so* $\frac{4}{4}$," replies Ka, "*because we can write* $\frac{\frac{4}{4}}{5}=\frac{1}{5}$." "*OK, let's make a table*," says Er, and they construct with some discussion the following:

Fourths	Fifths
$\frac{1}{4}$	$\frac{1\frac{1}{4}}{5}$
$\frac{2}{4}$	$\frac{2\frac{2}{4}}{5}$
$\frac{3}{4}$	$\frac{3\frac{3}{4}}{5}$
$\frac{4}{4}$	$\frac{4\frac{4}{4}}{5}$
$\frac{5}{4}$	$\frac{5\frac{5}{4}}{5}$

This latter was done by checking back to $1/4 = 1\frac{1}{4}/5$; and adding 4 and $4/4$ fifths and 1 and ¼ fifths to get 5 and $5/4$ fifths. This was followed by the extension of the chart to include the row:

$6/4$	$6\frac{6}{4}/5$

(still to be tested by Ra and Er)

Notice the various uses of fractional language both standard and not standard in developing this table. I observe this episode, among other things, as a nice example of *bringing forth a world of SIGNificance.*

At this point the class came quickly to an end with my suggesting that they show some of this work to the whole class tomorrow—when I would not be there. This ending was a missed opportunity. I might have asked what they thought $27/4$ would be in fifths (a prompt for a concrete generalization of their pattern). From there I might have asked this pair and involved the whole class as well, to figure out, using their answer to my previous question, what you would have to add in fifths to find $28/4$ in fifths (actions here even for these 9-year olds would give them a chance to think/act *inductively* (in the mathematical sense). *This thought illustrates how the actions of children impact on and occasion the didactical mathematical thinking of the teacher. This thought also reflects another important reason for the teacher him/herself to employ an attentional ethic in her/his work.*

PART IV: EVERYTHING SAID IS SAID BY AN OBSERVER – EVEN BY AN ELDER

I hope the vignettes above provide you with a setting for observing roles of inter-action on knowing. This inter-action occurred in many forms and was directed in many ways. For example, the fraction kit and the initial tasks of naming its pieces and associating these pieces of 'pizza' with monetary values occurred through the children's actions with the kit itself, always affected and effected by their own lived histories of work on fractions and even work on previous such kits. Similarly, because of the imaginary environment and their lived knowledge of money values less than one, most students individually, but also as a collective entity, realized the parallel roles of partitioning to get fractions of 1, that is, $(1/n)$; and division of 100 by n to get related money values for each piece. Thus each child's lived history and the group's history of working with me on fractions in the past (in this case over a year ago) can be observed to affect her/his actions and their collective actions in this new environment.

One can and should ask in what ways are the actions in this episode *mathematical*? While there are answers to that question throughout the paper and in the paragraphs immediately above and below, let me be explicit here. In a variety of ways, actions on the objects have a mathematical character. One can observe many examples of reasoning with objects themselves, especially in Ro, Ra and Er, that for me at least show the children thinking of the objects as having 'mathematical properties' rather than being game or puzzle pieces only. In many ways, these children relate the fractional numbers they use as re-presenting quotients and measures. In making the expressions they use, the children can be observed as making mathematical models both of and for mathematical actions—further they can be observed to think of needing to alter the mathematics in practical use (considering the cost of cutting small fractional pieces of pizza). They can be observed as using two notational systems in describing the same mathematical situations, showing the equivalence at the level of the expressions and making judgments for choosing among representational systems (especially

Ja). Further, one can observe children thinking in terms of patterns, and especially in noticing and extending patterns. I find this observable, especially in Ro's showing herself and others the relationship between $\frac{3\frac{3}{4}}{5}$ and $\frac{1}{2}+\frac{5}{20}$; in Ja's seeing dollar work as one form of more general decimal work; and of course in the explicit work of Ra and Er in pattern making. So I believe the case can be made for these children and the whole class bringing forth a world of SIGNificance that an observer takes to be mathematics or mathematical. The work of these students took place under circumstances that remind one of the Lynn Gordon Calvert (2001) book on conversations in the practice of mathematics. In that book she relates the mathematical thinking illustrated in various conversations among school students working in pairs on mathematical tasks to some of Polya's (1954) work on plausible reasoning and mathematics knowing. From the ideas in her book, I find that one might observe among the children whose work is sampled from above, what I, following Gordon Calvert, notice as being a *conversarial form of mathematics*.

Turning back to actions and inter-actions, making records, both personally and especially as a class, further provided support for such inter-action. In this, children could individually interact with the work of the collective. For example, the class chart with the pieces, fraction names, money values in cents, and money values in dollars, was used by Ja in Vignette 1 to see the role of money values in checking complex actions on fraction pieces and occasioning her to see/re-member that the money values associated with fractions written in dollar form were "just decimals". (So perhaps she thought, "*I can use decimals to work on these problems*."). As I have noted many times in my work across many age groups and many mathematical topics, the act of creating a public record of one's work acts in two ways. Clearly it is an expression of the child making it that inserts them into the collective and is satisfying and world making in that sense, an act of inserting their reasoning into the discourse of the class ("*I belong to this conversation*"). But the collective records in general occasion continuing actions by children, but also frequently provide them with models that they imitate. My many years of working in such environments convinces me that this imitation is in no way simply copying, but is an expression of "*I can do that* [act] *too*" and frequently involves (personal) changes in the pattern/expression/reasoning form being imitated. This latter is vividly seen in Ra's and Er's taking up of Ro's example. And as noted in Part II, these actions and inter-actions are replete with ethical considerations, and supported by both the children and the teacher taking (in situ) what can be observed as ethical forms of action—*provisional, attentional and occasional*.

This discussion of the various vignettes reminds me of the fact that that the knower and the environment are co-emerging and changing one another. This is vividly seen in Vignette 2 where Ro's actions change the nature of the fraction pieces for her, but also allow her to think of and show for herself that it is legitimate, in her and later the class' sense, to think of quantities expressed in the form $\frac{3\frac{3}{4}}{5}$. Notice that this use of fractional notation was picked up and used by Ka and Er as well. It is my experience (Kieren & Simmt, 2002) that it becomes (in almost an unspoken manner) a collective understanding that fractions as quantities are such that the $1/n$ and then the denominator acts as the *noun* while the numerator acts as *adjective* in fractional forms of notation. In many classes, at different grade levels, while students used expressions like $\frac{\frac{1}{2}}{2}$ = ¼ often in their work, they never, except later when studying division of fractions, write a fractional amount using a fraction in the denominator. In other words, the key role of $1/n$ fractional numbers seems to be part of the collective understanding of the classes I have worked with—much like $1/n$ *exists for all n not 0* is typically the first assumption related to the axioms of rational numbers as a quotient field. Thus the *mathematics of children* in inter-action with one another can be seen to be related to *big 'M' Mathematics* (see the discussion above) and certainly to any form of *children's mathematics*.

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Panel

BASIC FACTS ABOUT PISA 2012

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INTRODUCTION

After the release of the Programme for International Assessment (PISA) results in December, 2013, the *Globe and Mail* headline read, “Canada’s fall in math-education ranking sets off alarm bells” (Alphonso, 2013). This article and dozens of others over the following months sparked a national debate over the state of mathematics education in Canada (see Chernoff, n.d.). Although PISA is a problem-solving test for 15-year-olds, newspaper articles and public response, including parents, teachers, and mathematicians, blamed the results on the perceived removal of ‘basic facts’ from the elementary curriculum and pointed to‘discovery math’ pedagogical approaches as the direct cause (McGarvey & McFeetors, accepted).

Petitions were launched in British Columbia, Ontario, and Alberta (Houle, 2013; Murray, 2013; Tran-Davies, 2013). The Alberta petition, initiated by a parent of an elementary school student, gained substantial traction with more than 16,000 supporters signing the petition and received nation-wide media coverage. The petition letter to the Alberta Minister of Education speaks to many of the public perceptions about mathematics education in Canada. Tran-Davies’ (2013) opening statement reads: “The news regarding the students’ failing math grades since the institution of the new strategy-based curriculum is extremely disturbing, alarming and unsettling,” and that the PISA results “confirm that the system has clearly failed the first wave of children subjected to their grand experiment.” She claims that without mastery of computation algorithms, memorizing times tables, and “vertical additions”, students will be unable to problem solve or think critically; they will be “repulsed by math” and doors will be closed to them. In conclusion, Tran-Davies asks the Minister to “once again go back to embracing the basics.” Similar sentiments were part of the dominant discourse in the headlines for months:

- “Education fad may have harmed a generation of Albertans” (“Education fad”, 2014).
- “Provinces stick with discovery math despite back to basics push” (Morrow, 2014).
- “Canada’s math woes are adding up” (Wente, 2014).
- “Alberta’s touchy-feely math curriculum is appalling” (Gunter, 2014).

In this paper, I examine issues related to what PISA results were reported, how they were reported and what was ignored in the media.

RANKING, SCORES, AND DISTRIBUTION

PISA results are most frequently reported as a ranking of average scores. In 2012, Canadian students fell out of the top 10 ranking for the first time in PISA history. Canadians ranked seventh in 2003, when mathematics was the focus subject, to thirteenth place in 2012. The use of rankings as a relevant form of comparison is occasionally questioned, but what media did not report is that there was a 37% increase in the number of countries/economies participating from 2003 to 2012, including the entry of first, second, and fourth 2012-ranked Shanghai, Singapore, and Chinese Taipei. Also, although Canada ranked thirteenth, only nine countries/economies scored statistically above Canada and only two were non-Asian countries (Liechtenstein and Switzerland).

In addition to the ranking, the declining Canadian score on the test was also frequently reported. The average Canadian student dropped 14 points from 532 in 2003 to 518 in 2012. The quantitative meaning of the PISA scores is not discussed in the media. The 14 points represents a statistically significant change, but it is not a substantial drop. If we ignore critiques about what the PISA test is and what it presumes to measure, what do these scores represent? As a standardized test, the average score is set at 500 points (the Organisation for Economic Co-operation and Development (OECD) average was 494) with a standard deviation of 100 points, and 40 points is estimated as being one year of study. Assuming that the tests in 2003 and 2012 are comparable, Canada's 15-year-olds are, on average, about a third of a year behind where they were in 2003. The tendency to report the results of the 21,000 participating students as a single ranking or single score ignores the distribution around the scores. Do we have significantly more students failing now compared to a decade ago, thus pulling down the average score?

While not defined as 'failing', Level 2 questions are described as "the baseline level of mathematical proficiency that is required to participate fully in modern society" (Brochu, Deussing, Houme, & Chuy, 2013, p. 24). In Canada, the percentage of students scoring below Level 2 did increase over the past decade. In 2003, 10% of Canadian students fell below the acceptable level of proficiency. In 2012, this rose to almost 14%. There are 4% more students 'failing'. Yet, 86% of students are at or above the minimal level of proficiency. Not mentioned in the media was that there are fewer Canadian students in the failing category than the other non-Asian countries ranking higher than Canada.

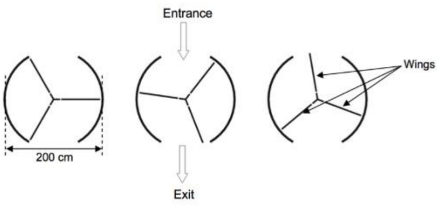
Examining the distribution of scores further, there was little change in the percentage of students scoring in the middling levels of 2, 3 and 4. In all four PISA tests since 2003, nearly 70% of Canadian students scored at these levels. Of concern and not mentioned in media is that our largest change in distribution is in the category of "top performers" (Levels 5 and 6). In 2003 over 20% scored in that zone, but in 2012 this percentage shrank to 16% of the student sample. If the economic development of a country is to rely on the top performers (another assumption), then it is somewhat surprising that the drop in numbers in the upper levels was not reported.

UNDERSTANDING THE BASICS

What 'basic math' is needed for successful performance on the PISA test? The *Revolving Door* (Figure 1), one of the released items from PISA 2012, consists of three questions at levels 3, 4 and 6 (OECD, 2013a).

Revolving Door

A revolving door includes three wings which rotate within a circular-shaped space. The inside diameter of this space is 2 metres (200 centimetres). The three door wings divide the space into three equal sectors. The plan below shows the door wings in three different positions viewed from the top.



Level 3 *(Difficulty 512): Shape and Space*
What is the size in degrees of the angle formed by two door wings?

Level 4 *(Difficulty 561): Quantity*
The door makes 4 complete rotations in a minute. There is room for a maximum of two people in each of the three door sectors. What is the maximum number of people that can enter the building through the door in 30 minutes?

Level 6 *(Difficulty 840): Shape and Space*
The two door openings (the dotted arcs in the diagram) are the same size. If these openings are too wide the revolving wings cannot provide a sealed space and air could then flow freely between the entrance and the exit, causing unwanted heat loss or gain. This is shown in the diagram opposite. What is the maximum arc length in centimetres (cm) that each door opening can have, so that air never flows freely between the entrance and the exit?

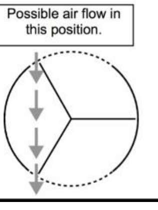


Figure 1. *Revolving Door* PISA 2012 Released Item (Adapted from OECD, 2013a).

Despite media attention to the assumed poor computational skills of students, it was rarely reported that the 15-year-olds were allowed calculators for the PISA test. As revealed by the *Revolving Door* questions, the types of questions asked had minimal computational requirements. The computational elements of the three questions are as follows:

Level 3: $360 \div 3$

Level 4: $4 \times 3 \times 2 \times 30$

Level 6: $(1/6)(200\pi)$

The questions are challenging due to the geometric and proportional reasoning needed to determine what computation was necessary—not the computation itself. In fact, the content category Canadians were marginally weaker in was not quantity or change and relationship, but space and shape. A curriculum that offers stronger geometric reasoning and visualization processes and more opportunities to develop novel strategies to unfamiliar problems is possibly more justified than one emphasizing mastery of computation. Another basic set of skills essential for successful test taking as evident in the *Revolving Door* is reading and interpreting information.

While mathematics was the feature of most headlines, scores in reading and science also declined (see Figure 2). Attribution of cause is assumed to be the general erosion of the quality in the Canadian education system. Yet, there was little connection to other news reports printed around the same time:

- "Canada's immigration numbers peaking for the seventh consecutive year: 2012 statistics" (Radia, 2013).

- "ESL students in the majority at more than 60 schools in Metro Vancouver" (Skelton, 2014).
- "Aboriginal populations surge in Canada, StatsCan says" ("Aboriginal populations", 2013).
- "Are there more students with special needs, or are we better identifying them?" (Sherlock, 2014).

The face of Canadian classrooms has changed considerably over the past decade. At present, one in five people in Canada are foreign born with far fewer new immigrants entering Canada with English or French as a first language. There have been changes to policies related to inclusive education and increasing numbers of students with special needs are now integrated into classrooms with little increase in supports for students with cognitive and physical difficulties.

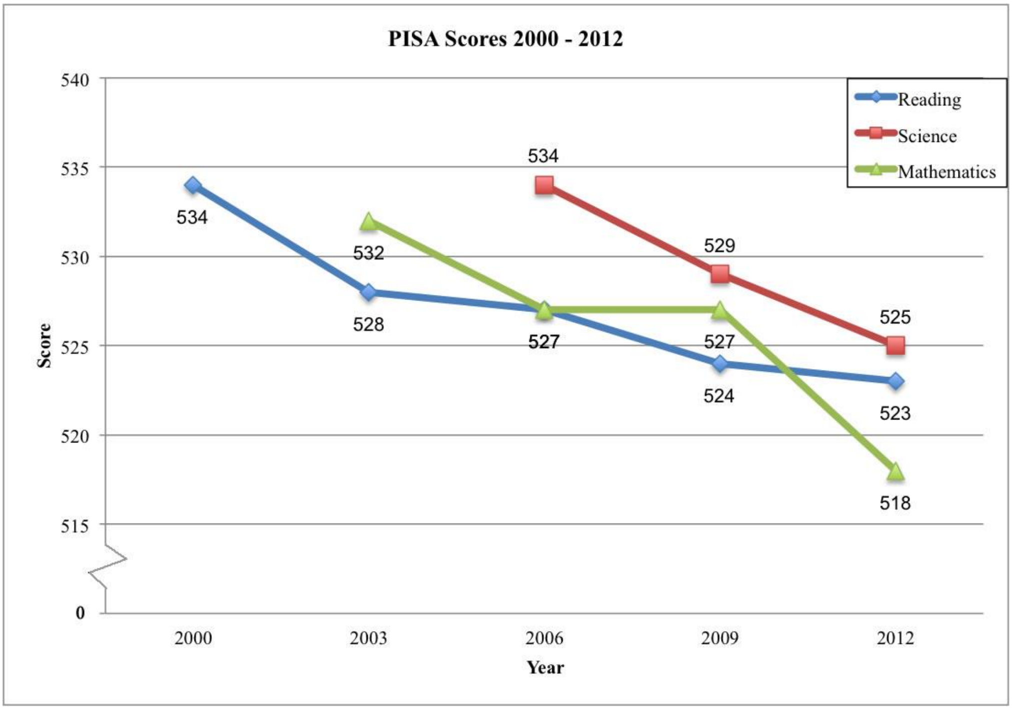


Figure 2. Downward trend in PISA scores across all subjects.

In addition to the changing demographic, one of the basics that OECD suggests as being a factor of student performance is being at school, on time, all the time (OECD, 2013b). Yet, 44% of Canadian students reported having arrived late to school at least once in the two weeks prior to the test. Also, about 25% of Canadian students reported skipping some classes and 22% reported skipping at least one full day of school in the two weeks leading up to the test. Canadian students ranked far below the OECD average in self-reported categories for arriving late to school and missing class, particularly in comparison to students from other top ranked countries. The score-point difference associated with these results was 23 – 37 points lower than students who were not late and did not skip. A return to the basics of computation in elementary school to ameliorate perceived issues with PISA scores seems unlikely to address these larger social circumstances.

THE GOOD OLD DAYS

While the media and public response scorned the mathematics skills of today's Canadian students with hundreds of headlines, little media attention was paid to the adult version of the PISA—the PIAAC. The results for the 2012 Programme for the International Assessment of Adult Competencies were released a couple months prior to the PISA with only a handful of headlines in the news such as the following:

- "Canadian adults match global peers in reading, fall short in math: OECD test" ("Canadian adults", 2013).
- "Canada's math, science lag bad for economy, report says ("Canada's math", 2013).
- "Does Canada have the skills to pay the bills?" (Yakabuski, 2013).

The PIAAC is a test of literacy, numeracy, and problem solving in technology rich environments for 16 to 65 year olds. The average Canadian adult's numeracy score is "significantly below the average" of the 24 countries participating (OECD, 2013c). The results show that 25 to 34 year olds have the highest skill level, but performance declines within the 35 and older age groups. The results are interesting given that the criticism waged by older adults on the state of mathematics education is from the demographic who scored poorly compared to their international counterparts.

Our own CMESG panel presentation was not immune from public critique. The *Edmonton Journal* fairly reported the general theme of the panel with the headline, "Slipping math scores don't equal a crisis, say math conference panel" (Sands, 2014a; 2014b). But the public responses to the article glorify basic facts from curriculum past:

> *Our kids do not learn the basics anymore. I read through my sons* [sic] *grade 3 math lessons and was appalled at the method he uses for basic addition and subtraction methods. It took me some time to see where he was getting his answers. Although they were correct answers he could have saved time with the old methods. We are creating lazy minds with the methods that are taught today.* (Gitersos Nazarali in Sands, 2014a)
>
> *What a bunch of garbage! They are not teaching our kids the basics. There is no memorization of the times tables. Ask a kid what 6x8 is and it will take them five minutes to come with an answer which may or may not be correct. It is a constant frustration to see them decline in ability and understanding.* (Bolduc Czyz in Sands, 2014a)

Nowhere in the newspaper article does it refer to Grade 3 computation facts. Yet, these two emotional critiques are representative of thousands of others posted in response to Canadian newspaper articles targeting the elementary mathematics school curriculum as the primary reason for the decline in PISA scores. The disconnect between the PISA results and the public response as indicated in the responses above speaks to another set of unresolved issues: Parents feel angry, disempowered, and frustrated because they don't understand the mathematics that is being expected of their children and they don't know how to help. No media report or set of results will mitigate these feelings. Until we are able to communicate with and re-engage parents and the public in the mathematics of today's children and youth, "The war over math is [and will continue to be] distracting and futile" (Anderssen, 2014).

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WHAT HAVE WE NOT BEEN HEARING ABOUT PISA?

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There is a lot to be said about PISA, most of which most people do not hear, and there are also many things about PISA we do not know, such as the extent to which education systems worldwide are becoming more alike as a result of a common assessment regime. Researching some of the things we do not know is the focus of a research group I belong to (http://conedu.uni-bremen.de/en/con-edu/). In May 2014 we organised an international inter-disciplinary workshop on the theme "PISA: More than just a survey?" at which I learned many things I did not know about PISA, some of which I will share here.

First, I would like to share something you already know. What most people hear about PISA is not complete. We have all read newspaper articles that give incomplete or inaccurate information about PISA results. But you might like to know (or be horrified to know) how widespread this is. Figazzolo (2009) reports that:

> *Out of about 12,000 articles published at worldwide level between December 2007 and October 2008:*
>
> - *around 40% make a simple reference to PISA 2006, without further explanation*
> - *around 29% quote PISA 2006 rankings, again with no further explanation*
> - *around 27-28% use PISA as a reference to advocate for reforms (it is the case, particularly, in Mexico)*
> - *around 1-2% blame teachers for the results*
> - *around 2-3% give technical explanations on PISA rankings.* (p. 23)

Figazzolo goes on to note "almost 30% of the articles make a reference to PISA results in order to advocate for reforms without even analyzing what these results mean" (p. 23). You may have observed a similar phenomenon in Canada's newspapers.

As Figazzolo's mention of Mexico reminds us, "shock" at PISA results occurs in many places worldwide. This can occur because the scores are lower than expected, or are declining over time, or when they reveal wide gaps between high scoring students and low scoring students, especially when these are related to socio-economic factors.

In Germany, the PISA shock came with the publication of the 2000 PISA results. Germans had considered their school system one of the best in the world, which makes sense if you believe that a good education system is a foundation for a good economy. Germany has a very strong economy, and so one would expect it to have a strong education system. But the 2000 PISA results revealed that Germany's scores were below average, and that the German education system did an especially bad job of educating students from low socio-economic backgrounds.

One effect of this shock is visible in the amount of educational research related to PISA published in German after 2000 (see Figure 1). *FIS-Bildung*, the main German language index of educational research, shows a huge leap in publications referring to PISA, while *ERIC*, the main North American language index of educational research, shows no such shock effect.

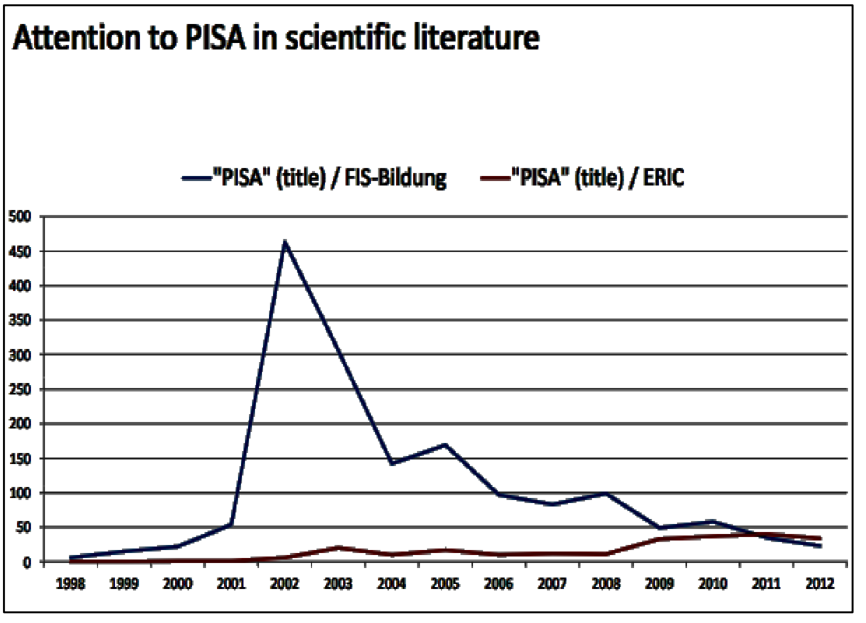

Figure 1. The PISA shock in Germany (from Bellmann, 2014).

Figazzolo also found that newspaper articles advocate for reform based on PISA results, a phenomenon that occurs also in Canada. And these calls for reform have effects. PISA changes education policy. In about 50% of the participating countries, reforms of schools and education systems were initiated in response to PISA (Martens, 2014). This is not surprising, because PISA intends to change policy. If you have a look at the table of contents of *PISA 2012 Results in Focus* (OECD, 2013b), you will find that every section ends with a chapter called “What this means for policy and practice”, in which specific policy recommendations are made by the OECD related to the PISA scores. Of course, this does not mean that the policy changes that are occurring are those advocated by the OECD. The OECD does not, for example, advocate more attention to basic facts, but this is a reform being called for in reaction to Canada’s latest PISA results.

Another effect of PISA shock in Germany was a change in the nature of faculties of education at universities, which might be of interest to CMESG members, if only for selfish reasons. Bellmann (2014) reports that in Germany between 2003 and 2010 there was a significant expansion in the number of educational research professorships in Germany. One hundred seven positions were advertised, specifically for empirical research. And all the newly appointed researchers after 2001 who focus on PISA were appointed to positions in education faculties. So PISA became an entry ticket to a career as a professor in education. You might think this is a good thing. Education needs more professors. But where did these new professors who focus on empirical research related to PISA come from? Bellmann reports that all of them did their doctorates in psychology.

I’d like to turn my attention at this point away from effects of PISA shock, and to look instead at *mathematical literacy*. One thing that is supposed to make PISA different from TIMSS, and most large scale mathematics assessment, is a focus on *mathematical literacy*. The definition has changed over the years, but here is what *mathematical literacy* is supposed to mean:

> Mathematical literacy is defined in PISA as: *the capacity to identify, to understand, and to engage in mathematics and make well-founded judgements about the role that mathematics plays, as needed for an individual's current and future private life, occupational life, social life with peers and relatives, and life as a constructive, concerned, and reflective citizen.* (Schleicher, 1999, p. 50)
>
> The mathematical literacy definition for OECD/PISA is: *... an individual's capacity to identify and understand the role that mathematics plays in the world, to make well-founded judgements and to use and engage with mathematics in ways that meet the needs of that individual's life as a constructive, concerned and reflective citizen.* (PISA, 2003, p. 24)
>
> PISA defines mathematical literacy as: *Mathematical literacy is an individual's capacity to identify and understand the role that mathematics plays in the world, to make well-founded judgements and to use and engage with mathematics in ways that meet the needs of that individual's life as a constructive, concerned and reflective citizen.* (Cresswell & Vayssettes, 2006, p. 72)
>
> PISA defines mathematical literacy as: *... an individual's capacity to identify and understand the role that mathematics plays in the world, to make well-founded judgments and to use and engage with mathematics in ways that meet the needs of that individual's life as a constructive, concerned and reflective citizen.* (OECD, 2010, p. 84)
>
> *Mathematical literacy is an individual's capacity to formulate, employ, and interpret mathematics in a variety of contexts. It includes reasoning mathematically and using mathematical concepts, procedures, facts and tools to describe, explain and predict phenomena. It assists individuals to recognise the role that mathematics plays in the world and to make the well-founded judgments and decisions needed by constructive, engaged and reflective citizens.* (OECD, 2013a, p. 25)

PISA is supposed to assess *mathematical literacy* but one thing I had not heard about, is that it does not. At the "PISA: More than just a survey?" Workshop, Stephen Lerman drew my attention to an article by Kanes, Morgan, & Tsatsaroni (2014) which discusses what PISA assesses. They analyse one item in detail (see Figure 2).

There are three questions related to this item. The third one is:

> 44.3. Mandy and Niels discussed which country (or region) had the largest increase of CO_2 emissions. Each came up with a different conclusion based on the diagram. Give two possible 'correct' answers to this question and explain how you can obtain each of these answers.

The scoring rubric for this question is:

> Full credit: Response identifies both mathematical approaches (the largest absolute increase and the largest relative increase) and names the USA and Australia.
>
> Partial credit: Response identifies or refers to both the largest absolute increase and the largest relative increase but the countries are not identified or the wrong countries are named.

Kanes et al. (2014) comment:

> *...in part 44.3, in spite of the apparent focus on a "discussion", the scoring rubric makes it clear that only "mathematical" approaches that draw on the information provided in the item may be considered valid. Other possible lines of argument, such as drawing on knowledge or evidence exterior to the question (which could certainly be considered valid in some "real-world" discussions), are excluded.* [...] *allocation*

> *of the full score depends on the student recognising the hidden assumptions embedded in the recontextualisation of mathematical knowledge in this assessment task, and which contradict the apparent valuing of real-world knowledge and the explicit privileging of communication over performance.* (pp. 157-158)

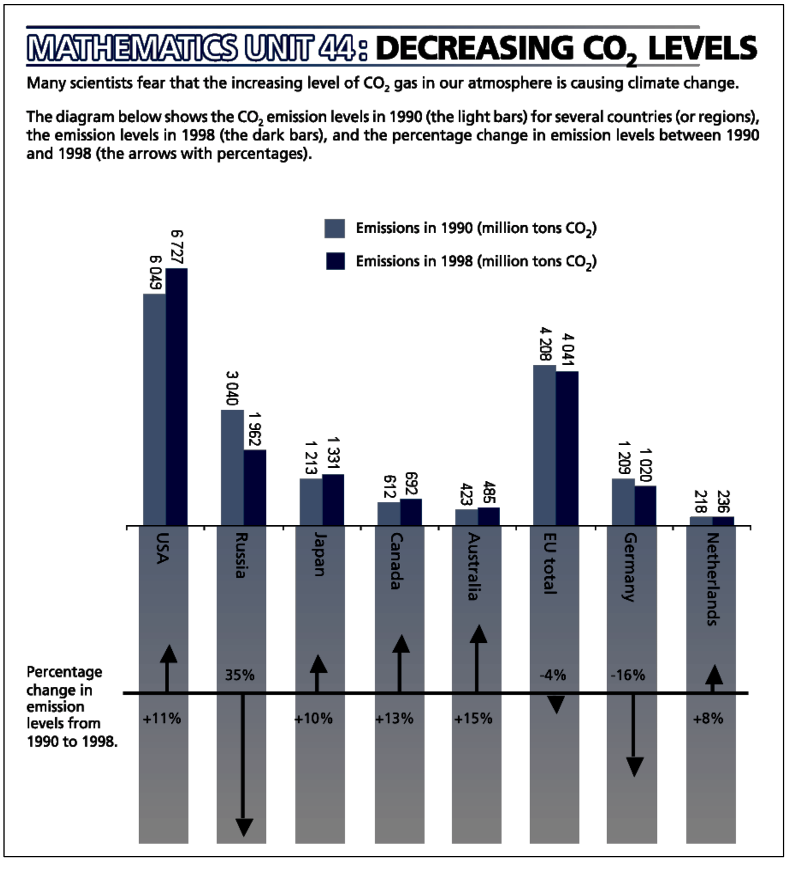

Figure 2. PISA sample item analysed in Kanes, Morgan, & Tsatsaroni (2014).

In other words, the item does not assess mathematical literacy, recognising "the role that mathematics plays in the world" and the ability to "make the well-founded judgments and decisions needed by constructive, engaged and reflective citizens" (OECD, 2013, p. 25). Instead it assesses whether students recognise the role that mathematics plays in testing, "the hidden assumptions embedded in the recontextualisation of mathematical knowledge in this assessment task" and penalises those who would found their judgements on "knowledge or evidence exterior to the question".

Mathematical literacy as defined by PISA, may be a worthy goal. However, to improve PISA scores the goal should be instead focussed on the kind of pseudo-real contexts and hidden criteria that PISA tasks employ. Germany, prior to 2000, had no standards for mathematics at the federal level. Individual states set their own curricula. Following the PISA shock, Germany very quickly established federal standards. A central competence in those standards is 'modelling' which, based on the example tasks usually cited, refers to solving tasks with pseudo-real contexts and hidden criteria, like PISA tasks.

Ironically, this focus on 'modelling' may make the gap between high-scoring, socially advantaged students and low-scoring, socially disadvantaged students even wider in Germany. There is research that suggests that pseudo-real contexts in mathematical tasks make them more difficult for socially disadvantaged students than for their socially advantaged peers (Cooper & Dunne, 2000; Lubienski, 2000). If this is the case, and schools change their measure of mathematical competence to ability to solve such tasks, socially disadvantaged students may perform even more poorly than before. And if Canada wishes to preserve its relatively narrow gap between the high performers and the low performers, extreme caution must be taken when "teaching to the test" by making more use of such tasks.

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THE PERFORMANCE OF QUÉBEC STUDENTS ON PISA'S MATHEMATICS ASSESSMENT: WHAT'S GOING ON IN QUÉBEC?

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INTRODUCTION

The recently released results of Canadian students on the 2012 PISA Mathematics Assessment created a wave of shock across much of Canada. Results varied from one province to the next and many people were disappointed by the performance of students in their home province. With the release of the PISA results, many began asking questions in order to better comprehend the drop in performance of Canadian students, as well as why the results of Québec students were better than those of students from elsewhere in the country. In this paper, I will attempt to shed light on the variation in Canadian students' results on the PISA assessment by responding briefly to a complex question: What is responsible for Québec students' performance? To answer this question, I will look at differences in teaching and learning, not at students. In particular, I will address how Québec students are taught and how they learn differently. Additionally, I will describe Québec teacher preparation programs, as well as the structure of the Québec education system. I will then show how research in *didactique des mathématiques* influences teacher preparation programs, initial teacher training, and professional development.

TEACHER PREPARATION PROGRAMS

In Québec, all teacher preparation programs are four years at the undergraduate level (B.Ed.) and two years at the master's level (MATL). Québec universities seek to support pre-service teachers in developing 12 professional competencies. Each university program offers at least 700 hours of field experience (*stage*). Additionally, all programs are aligned with an '*Approche programme*', whereby each course taken by pre-service teachers builds upon others, as well as the field experience. Upon completion of the program, an official teaching accreditation is delivered to graduates by the Ministère de l'Éducation, du Loisir et du Sport.

Teaching programs for elementary school teachers result in the receipt of a B.Ed. degree. Depending on the university, a pre-service teacher preparing to teach elementary school could take between two and five Mathematics and/or Mathematics Education courses. As such, Québec teachers receive from 78 to 245 class hours in Mathematics Education, whereas pre-service teachers in other provinces can have as little as 39 hours. Needless to say, this is a huge difference.

Teaching programs for secondary school teachers also result in the receipt of a B.Ed. degree. Depending on the university, a pre-service teacher in a secondary school teacher education program might have between two and eight Mathematics Education courses in addition to a number of pure Mathematics courses that are also required. A two-year program in secondary school education is also offered at the graduate level (Master's in Teaching and Learning); this professional graduate program leads to a teaching accreditation from the provincial government.

The Mathematics Education courses offered to pre-service teachers at both the elementary and secondary levels have a strong focus on *didactique des mathématiques*. That is, prospective teachers do not merely focus on a bunch of methods, but rather focus on how to create the best conditions for each student to learn mathematics.

EDUCATION SYSTEM

The structure of the Québec education system is quite different from other provincial education systems. For instance, upper middle school grades (Grades 7 and 8) are part of high school and are therefore taught by secondary school teachers, not elementary school teachers. There are no junior high schools in Québec. Additionally, secondary school ends at grade 11; there is no grade 12 in Québec. Instead, Québec students attend Cégep, which offers 2-year pre-university programs or 3-year technical programs. Upon graduating from Cégep, students are qualified to attend a university program.

Québec	Most other provinces in Canada
Pré-maternelle 4 ans Maternelle 5 ans	Kindergarten
Élémentaire Grades 1 to 6	Elementary Grades 1 to 6
Secondaire 1 & 2	Middle School Grades 7 & 8
Secondaire 3, 4, & 5	High School Grades 9 to 11
Cégep Programmes pré-universitaires de 2 ans Programmes techniques de 3 ans	Grade 12
Université	University

Table 1. Education systems in Québec.

Québec's education system also aims to provide equitable access to necessary learning resources to all students. DeBlois, Lapointe, and Rousseau (2007) sought to examine the equity of the Canadian education system by looking at the variation in student test-scores on past PISA assessments taken by Canadian students in each province. In order to accomplish this task, these authors used data from the 2000 PISA literacy exam and 2003 PISA numeracy exam. As a result of their efforts, Deblois et al. were able to identify that variation in student test-scores, particularly in the area of literacy, was larger between Canadian schools located outside of Québec. The smaller achievement gap observed between schools within Québec suggests that equal access to necessary learning resources for students may be less common in other provinces. It appears as though an education system in which students are provided equal access to necessary resources has a greater chance of producing elevated test-scores on assessments like those of PISA. Reducing inequities observed would require working within individual schools rather than merely focusing on comparisons of schools to one another.

RESEARCH IN *DIDACTIQUE DES MATHÉMATIQUES*

For many decades, Québec researchers' work has focused on the French *didactique*, especially the *théorie des situations didactiques* of Guy Brousseau (Brousseau, 1986a, 1986b, 1997, 1998; Brousseau, Brousseau, & Warfield, 2002). The focus of this theory is the learning environment created by the schoolteacher, which is referred to as the *milieu*. The different interactions within this milieu, it is argued, create opportunities to learn mathematics. According to the theory, when students take charge of a task and are empowered in their learning, rich learning opportunities result. This phenomenon is referred to as *devolution of the task* and implies that students take an active role in their learning. But students typically expect that their teacher will provide them information regarding what to do. Moreover, a teacher also typically has their own set of expectations of students by which students may be envisioned as more passive than active. These implicit expectations held by both students and the teacher comprise the *didactical contract*. In the devolution process, the didactical contract tends to be broken, and the implicit expectations are supplanted.

Another important idea put forth by Brousseau's theory regards the status of knowledge. According to the theory, the knowledge that students hold individually is referred to as *connaissance* (C-knowledge) (Warfield & Brousseau, 2007). The learning process, it is argued, involves having students transform their individual *connaissances* into socially shared knowledge within a community of mathematicians. This socially shared knowledge is referred to as *savoir* (S-knowledge). Learning tasks given to students should allow them to use their *connaissances* to develop socially shared knowledge or *savoir*.

HAVING STUDENTS USE THEIR C-KNOWLEDGE TO ACQUIRE S-KNOWLEDGE

Each of the items from the 2012 PISA assessment consisted of contextualized tasks. Below is an example retrieved from the 2012 international assessment (OECD, 2013, p. 92):

TEST QUESTIONS (LEVEL 5)
CLIMBING MOUNT FUJI

Mount Fuji is a famous dormant volcano in Japan.

QUESTION

The Gotemba walking trail up Mount Fuji is about 9 kilometres (km) long. Walkers need to return from the 18 km walk by 8 pm.

Toshi estimates that he can walk up the mountain at 1.5 kilometres per hour on average, and down at twice that speed. These speeds take into account meal breaks and rest times.

Using Toshi's estimated speeds, what is the latest time he can begin his walk so that he can return by 8 pm?

Figure 1

The context presented in this task provides students a point of entry to the task (Savard, 2008, 2011). It is important to discuss the contextual features in any given problem, as doing so gives students access to a task (Jackson, Shahan, Gibbons, & Cobb, 2012). If such discussion takes place, it is then possible for students to use their *connaissance* (C-knowledge) to create a mathematical model in order to solve the problem under consideration. Doing so, however, requires that students leave their sociocultural context to get into the specific mathematical context of the problem. Using their mathematical *connaissance* (C-knowledge) to solve the task might lead students to develop their *savoir* knowledge (S-knowledge). This transition typically occurs at the end of a lesson when the teacher discusses their *connaissance* (C-knowledge) with students and highlights links between this C-knowledge and students' *savoir* knowledge (S-knowledge).

DeBlois (2006) studied elementary school teachers' interpretations of their students' mathematical work and the influence of such interpretations on the teachers' later choice of interventions in class. Analysis of six seminars between Deblois and 21 teachers from the same Québec school revealed the elements that contributed to this interpretative process. This analysis called attention to five *milieux* of which teachers were aware as they interpreted their students' work: 1) conformity to what was taught; 2) familiarity of the students with the task; 3) students' understanding; 4) the features of the task; 5) *savoirs* (S-knowledge) from the curriculum. The manner in which teachers planned to intervene with students was found to depend on which of these five specific *milieux* they were most sensitive to. By attending to different *milieux*, the nature of teachers' interventions also changed, resulting in greater attention being paid to students' mobilization of *connaissance* (C-knowledge) than the discrepancies between students' mistakes and *savoir* (S-knowledge) from the curriculum. As a result of this work, a better understanding of teachers' attribution of causes to students' mistakes and of their preferred interventions with students who experience learning difficulties was developed. This work also highlighted what teachers pay attention to when supporting students in their learning.

Are Québec teachers more sensitive to students' *connaissance* (C-knowledge) than other teachers in Canada? More research is needed to respond to this complex question.

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PISA REPORTING: NOT SAYING WHAT PISA IS AND DOES

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The question posed to our panel was, "*What have we not been hearing about the PISA?*" I take this to be a question about what the public is not hearing—not a question about what we mathematics educators have not been hearing. However, not all mathematics educators are paying attention to PISA and its effects. I was not paying much attention to it until the press started asking me to comment on PISA.

My concern is that public popular press discourse about PISA focuses on the results without thought or concern about the instrument of testing. This suggests a belief that a measurement is empirical fact and values-free, like other mathematics. I argue, however, that mathematics is values-laden and measurement in particular reflects values. PISA results are an example of values-laden measurement. And because people do not think of it as values-laden, its impacts are all the more subtle and all the more powerful.

In my prepared comments for the panel discussion, I talked about what PISA is and how I have engaged public discourse about it when invited to do so. Interestingly enough, the Organisation for Economic Co-operation and Development (OECD), which has commissioned the Programme for International Student Assessment (PISA), begins its overview of the 2012 results with the same question with which I began my panel discussion comments – *What is PISA?* Here is the gist of their answer to this question.

> *PISA assesses the extent to which 15-year-old students have acquired key knowledge and skills that are essential for full participation in modern societies. The assessment, which focuses on reading, mathematics, science and problem-solving, does not just ascertain whether students can reproduce what they have learned; it also examines how well they can extrapolate from what they have learned and apply that knowledge in unfamiliar settings, both in and outside of school.* (OECD, 2013, p. 2)

In this paper, I (re)present my panel discussion comments but first contextualize them with some thoughts on measurement.

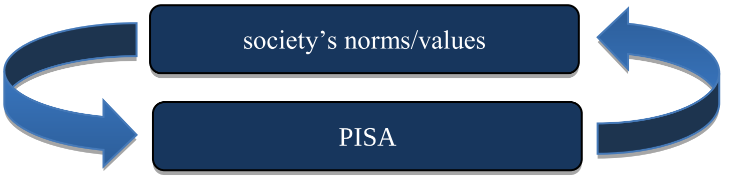
MEASUREMENT / ASSESSMENT

The heart of measurement is comparison. A comparison may be *relative* or *normative*. For example, I might measure my height by standing back-to-back with David Reid. Witnesses of this relative measurement would be able to say I am taller or shorter than David Reid. Or I might measure my height in relation to some norm, like a unit of measure—I am 184 cm tall. Measurement indexes values because any act of measurement requires a choice about what attribute warrants measurement, and because the process requires a choice of referent (either comparative or normative). CMESG participants probably don't care who is taller between David and me, nor about my height in centimetres or inches. Thus height measurements do not appear in our discourse. By contrast, our community may well care about the length of time we speak, which is easily quantifiable. Our community also cares about what we say, which is not easily quantifiable. I know the difficulty of quantifying certain qualities of people's discourse, partially through my experience doing so in research with Beth Herbel-Eisenmann.

Knowledge and skills (the mandate of PISA) are as or more difficult to quantify than the qualities of what people say. Unlike the things people say, to which we have some access, we don't have direct access to anyone's knowledge nor to their skills. It isn't even clear what knowledge or skill is. Making things more challenging, we have to decide which knowledge and which skills are the most important to measure, and we have to come up with a means for measurement—trying for relative or normative measurement. PISA positions its work as comparison (and thus a relative measurement), and the popular press complies. However, I think we could argue that the PISA instruments are normative. The OECD and our popular press report on comparison among jurisdictions, but these are based on normative results gleaned from analysis of students' performance on particular questions, chosen as norms.

What's more, not only do measurements index or reflect values, they also change values. Quantum physicists refer to the 'observer effect'—when they measure something they change its behaviour. Early speech act theorists were excited about a similar effect in discourse. Austin (1975) noted how declarative statements (statements of fact) do more than describe the world; they change the world. This idea is taken as obvious in more advanced linguistic theory. Indeed, when I told my mother about this as part of my response to her question about what "all this post-modern stuff is about," she said it's pretty obvious and even her father knew that apparent statements of fact actually change things. I'm just trying to make clear that it is well accepted that a 'presentation of fact' is never benign. Such presentations change the world. Thus, I am shocked that the press and public accept PISA reporting as objective statements of fact. Perhaps this acquiescence is a sign that the public wants to take as undisputable fact the claims based on PISA results.

In fact, PISA is both a reflection of certain social values and a driving force for change in those values. Like any discourse move it does both at the same time.



WHAT IS PISA?

At this time, I will not argue the basic OECD claim of what PISA is, except to remind us that its reference to "key knowledge" indexes a political agenda in that some people decided what knowledge is key. Furthermore, the OECD's reference to knowledge and skills "essential for full participation in modern societies" is yet another political choice about what "full participation" entails. Also, unlike the OECD, I prefer to refer to the PISA results in plural form because there were many different tests. The test cannot possibly be the same in every language. Furthermore, any one of these versions would test differently in different contexts, and the variety of versions exacerbates this effect.

One way to think about the range of possible ways to explore how 15-year-olds "extrapolate from what they have learned and apply that knowledge in unfamiliar settings, both in and outside of school" (OECD, 2013, p. 2) is to ask what PISA is/does and what it is not and does not do. PISA's comparison study discourse reflects and sustains a public wish for quantitative results and out-doing one's neighbours. An alternative would be to allocate the significant temporal, creative, and monetary resources used on PISA to understand better how we can focus on developing children's understanding. For example, PISA engages countless educators in writing test items that should be relatively culturally-neutral. My experience of claims for cultural neutrality indicate that they tend to make dominant cultures normative. So, for example, the PISA questions about buying a car would be more familiar to wealthy North American children than others. Of course there is little surprise that the OECD would foreground economically developed contexts, and try to make car ownership normative.

I also suggest that reporting on PISA should be taken as part of what PISA is. Yes, PISA is a test, but it is also educators and policy-makers constructing a test, countless children being coerced into doing these tests, the rhetoric published by the OECD about selected results from the tests, and the co-opting of these results by public press and lobby groups to further certain agendas. This all distracts from other possible foci of attention in the classroom and in public discourse about education. I suggest that the impact is more directly impacting the public discourse, but that this discourse, in turn, makes huge impacts on what happens in the classroom. To illustrate the power of this impact, I will list a couple of worthwhile endeavours eclipsed by the focus on comparative performance. First, I ask myself how education would be different if public discourse focused on stories of powerful learning in diverse contexts and discussion about how the practices in those contexts might be applicable in local contexts. Second, I ask myself what would happen if the public were not distracted from considering significant disparities within Canada's education system—particularly the significant underfunding of First Nations schools.

INTERACTING WITH THE PRESS AND OTHER STAKEHOLDERS

As noted above, I have been drawn into discourse about PISA by media requests for commentary. I have also been asked to speak with community gatherings about education. My reflections about what PISA is, as described above, are central to the messages I try to convey in these settings.

Most prominently, I draw attention to the comparison aspect. I clarify the perception that we are performing badly in PISA comparisons because we are, according to the OECD reporting, doing very well. My colleagues on the panel point to the statistics to demonstrate how Canada is doing well in comparison.

To complement this, I have also done a search for "Canada" in the summary document (OECD, 2013), and found Canada appearing thrice, not counting its appearance in lists of country data. The first item identified under the heading "Excellence through Equity: Giving Every Student the Chance to Succeed" states, "Australia, Canada, Estonia, Finland, Hong Kong-China, Japan, Korea, Liechtenstein and Macao-China combine high levels of performance with equity in education opportunities as assessed in PISA 2012" (OECD, 2013 p. 12). Later, in the first sentence after the heading "The PISA results of several countries demonstrate that high average performance and equity are not mutually exclusive", the OECD reports that "Australia, Canada, Estonia, Finland, Hong Kong-China, Japan, Korea, Liechtenstein and Macao-China show above-OECD-average mean performance and a weak relationship between socio-economic status and student performance" (OECD, 2013, p. 14).

Finally, and I will return to this one shortly, Canada is praised for teaching for understanding, beyond procedural knowledge:

> *Canada is more successful in this regard: 60% of students in Canada reported that their teachers often present problems for which there is no immediately obvious way of arriving at a solution, and 66% reported that their teachers often present them with problems that require them to think for an extended time. Education systems could and should do more to promote students' ability to work towards long-term goals.* (OECD, 2013, p. 22)

In my comments to the press, I do, however, assuage the public's taste for scandal and tragedy. After pointing to evidence that Canada is doing well in comparison, with emphasis on the word 'comparison', I say that if asked about how we are doing in our mathematics teaching, I would not say we are doing well. Perhaps we are doing well in teaching procedural mathematics (which I refer to as 'knowing how'), and doing better than other countries in teaching children how these procedures work and how they can be used to solve problems ('knowing why'), but we are not addressing other imperatives. Our mathematics education system is not preparing people to 'know when' to use mathematics and when mathematics is not the most appropriate tool. And, most importantly, our system is not generating a society that 'knows to' use mathematics to critique and question claims made by others. We have a society that lacks mathematical agency.

This in-ability, or perhaps in-activity, is clear in the way the public fails to look at the PISA results themselves to test the claims made by lobbyists with certain agendas. Our public fails to do the mathematics necessary to challenge popular claims drawn inappropriately from PISA reports. I have read in many newspaper articles and have heard in many phone-in discussions, claims that PISA results tell us we need to get back to the basics in teaching mathematics. Shockingly, PISA results do not do that at all. They tell us we are doing well. What's more, they promote teaching for understanding and praise us for teaching this way. So I challenge the PISA finding. We cannot be doing well, if our public fails to do straightforward mathematics to question what lobbyists are telling us.

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Working Groups

Groupes de travail

MATHEMATICAL HABITS OF MIND

MODES DE PENSÉE MATHÉMATIQUES

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INTRODUCTION

In 1996, Cuoco, Goldenberg and Mark raised a question that continues to be appropriate today: “Given the uncertain needs of the next generation of high school graduates, how do we decide what mathematics to teach?” In their article, they go on to question the appropriateness of a content driven curriculum, proposing instead that it be organised around “Mathematical Habits of Mind”.

> *A curriculum organised around habits of mind tries to close the gap between what the users and makers of mathematics do and what they say. ...[It] lets students in on the process of creating, inventing, conjecturing and experimenting ... It is a curriculum that encourages false starts, calculations, experiments, and special cases.* (p. 376)

Since that time, many others have tried to define and elaborate what is encompassed by the notion of *mathematical habits of mind* (see Lim & Selden, 2009). Broadly speaking, they can be thought of as productive approaches and ways of looking at problems (and the world) that are typical of practising mathematicians—potentially including Mason, Burton, & Stacey's (2010) "natural powers and processes" (p. 8): specializing and generalizing, conjecturing and convincing, imagining and expressing, stressing and ignoring, classifying and characterizing.

Over the last decade, advisory committee recommendations (US-based) seem to be increasingly emphasising the importance of preparing students to approach problems and look at the world the way that mathematicians do. This is captured in the *NCTM Principles and Standards* (2000), which advocates students acquiring "habits of persistence and curiosity" and observes that "[p]eople who reason and think analytically tend to note patterns, structure, or regularities in both real-world and mathematical situations." In *Adding it Up* (2001), Kilpatrick, Swafford, and Findell describe a "productive [mathematical] disposition" as "a habitual inclination to see mathematics as sensible, useful, and worthwhile, coupled with a belief in diligence and one's own efficacy" (p. 116). These ideas have begun to find their way into the Canadian mathematics curriculum: they are implicit in the WNCP (2006) K–12 curriculum and have now become explicit in the new Draft BC Mathematics K–9 curriculum (BC Ministry of Education, 2013), released late in 2013, which describes its major change as "a focus on developing mathematical habits of mind and encouraging students to wonder how mathematicians think and work".

This has implications for how mathematics will be taught in schools and will impact how we prepare teachers. The Conference Board of the Mathematical Sciences (2012), clearly addresses this, recommending: "All courses and professional development experiences for mathematics teachers should develop the habits of mind of a mathematical thinker and problem-solver, such as reasoning and explaining, modeling, seeing structure, and generalizing" (p. 19).

Mason et al. (2010) stress that "it is vital to educate one's awareness by engaging oneself in mathematical tasks which bring important mathematical awarenesses to the surface, so that they can inform future action" (p. xii). In this working group, we plan to take this advice to heart as we seek to come to a collective understanding on what might constitute mathematical habits of mind and to consider how we might foster/nurture these in our preservice teachers.

DAY 1

Day 1 began with introductions followed by a presentation by Susan to set the context for the Working Group discussions of mathematical habits of mind.

WHY ARE WE TALKING ABOUT MATHEMATICAL HABITS OF MIND?

Susan began by answering this question on a personal level, describing how she had come to be co-leading this working group. It all started with another working group, a session that she was co-leading with Lynn Hart, Susan Swars (both from Georgia State University) and Ann Kajander (from Lakehead University) at the PMENA 2012 Conference in Atlanta. The theme of the session was "What Preservice Elementary Teachers of Mathematics Need to Know". During that working group, participants (including Mary) brain-stormed to come up with a list of important things that preservice teachers should learn in their preservice content course(s). From this discussion, *mathematical habits of mind* arose as a theme. The group resolved to write a book aimed at instructors of mathematics content courses for preservice teachers, and Susan and Mary found themselves in a subgroup assigned the task of writing a chapter on mathematical habits of mind. (Work on the book continues. The group met again at PMENA 2013 in Kalamazoo and planned to meet a third time at PME 2014 in Vancouver.)

As part of gathering information for the book, Frédéric was approached to provide comments from a mathematician's perspective. In order to seek more input and further the discussion, Mary, Frédéric and Susan decided to run an *ad hoc* session about mathematical habits of mind at CMESG in 2013. The session was very well-attended, so much so that it was decided to try to organize a Working Group with this theme for 2014.

BACKGROUND

For Susan, her interest in mathematical habits of mind had arisen from the PMENA working group discussions in 2012. At the time, she had no idea that the topic would soon be so relevant to mathematics education in BC. Coincidentally, in the Fall of 2013, the British Columbia Ministry of Education released its preliminary draft of its intended curriculum revisions, and front and centre in the self-described essence of the changes to the mathematics curriculum was the phrase "mathematical habits of mind". Susan provided a brief summary of some of the literature on mathematical habits of mind, going back to Cuoco et al. (cited above). She touched on a few others who had made contributions in this area, including Harel (2007, 2008), Selden and Selden (2005), Bass (2008), Leikin (2007), and Lim (2008, 2009), but noted especially John Mason et al.'s *Thinking Mathematically* (2010), whose original version (1982) significantly predated Cuoco et al. (1996). Mason's work coaches readers on how to think mathematically, describing effective strategies of *specialising* and *generalising* and how to cope with being *stuck*.

She also summarised the significant advisory group recommendations (described in the introduction, above), emphasising the report of the Conference Board of the Mathematical Sciences (2012) which stresses the importance of developing mathematical habits of mind in mathematics teachers. She noted that working with teachers would be a primary focus of this working group.

The influence of the literature that addresses mathematical habits of mind on curriculum seems to continue to grow. They appear implicitly in some of the "Mathematical Processes" outlined in the Western & Northern Canada Protocol (2006) curriculum. Under the process of problem solving it describes: "Creating an environment where students openly look for and engage in finding a variety of strategies for solving problems empowers students to explore alternatives and develops confident, cognitive, mathematical risk takers" (p. 8); under reasoning it states: "High-order questions challenge students to think and develop a sense of wonder about mathematics" (p. 8). British Columbia's new draft mathematics curriculum (BC Ministry of Education, 2013) is much more explicit. Not only does it declare "a focus on developing mathematical habits of mind", its self-described essence is "helping students appreciate a uniquely mathematical perspective: how embodying mathematical ways of thinking and acting changes how one interprets the world around them".

This new focus on mathematical habits of mind puts new pressures on teachers, particularly on those whose only experiences have been in content-driven courses. Whether or not one agrees with the direction that this curriculum is taking, for better or worse it has reinforced that there is more to teaching mathematics than transmitting the content. The challenge we face as those who support teachers is to better understand what mathematical habits of mind might mean, and to find ways to both build them in preservice teachers and to prepare teachers to foster them in their own students.

WHAT ARE MATHEMATICAL HABITS OF MIND?

Susan offered a preliminary definition and invited participants to respond:

> In their broadest interpretation, mathematical habits of mind are…
>
> thinking about mathematics (and the world) the way that mathematicians do.

Some suggestions from the group about what this might mean included: they look at it in a unique way, they look for patterns, generalise, they look for multiple solutions, they are excited and intrigued by mathematical tasks. There was also a strong assertion that there is no specific approach.

Susan proposed a thought experiment, asking participants to consider the following scenario and to think about how they would respond personally, and how a mathematician might respond.

<u>The Groupon Scenario</u>

You and a friend have just shared a lovely lunch at a local bistro. You chose this restaurant on this day because your friend had purchased a "Groupon". She had paid $15 for a $30 credit toward the cost of the lunch. It is now time to pay the bill, which comes to $27 after the discount, including the taxes, but not the tip. What do you do?

Participants shared a variety of responses to the scenario, including various strategies for determining how much each person should pay. A question was raised as to whether this was a mathematical problem or a social problem. Some would have simply paid the full tab and let the friend pay next time. Someone else noted that a mathematician's response might be to consider all options within the scenario and then choose the optimal one, while another observed that his mind was drawn to wondering about the business implications of selling groupons. Interestingly, some found that they responded to the scenario as mathematics educators, wondering how this problem could be used/adapted/extended in their classroom teaching. What was clear was that any given situation can be seen through a variety of lenses, including mathematical, social, economic and educational.

Next, Susan offered a tighter definition of mathematical habits of mind:

> We demonstrate MHoM when we habitually choose actions and strategies, pose questions and display attitudes that are PRODUCTIVE in a mathematical context.
>
> They help us understand the math, solve problems and maybe even help us create mathematics.

She spoke briefly about habits, noting that they are often automatic, can be learned and unlearned (not always easily), and that they can be good or bad. She invited the group to brainstorm 'bad' habits that students of mathematics sometimes display. Responses included:

- Stopping when they are 'done' – anti-joining
- Asking immediately: "What formula do you use?"
- Mimicking the teacher, peers: "Show me an example and I'll do it."
- Bringing haste to situations – believing that all questions can be solved quickly
- Glossing over and looking for numbers – not engaging in the problem
- Memorizing – trying to guess or recall what is expected
- Giving up

It was noted that these are often developed from experiences with mathematics and mathematics teaching. They are survival techniques. One participant even questioned the appropriateness of using the word 'bad' to describe these habits as they are often effective in school mathematics.

Some 'bad habits' of mathematics teachers were also identified, including:

- Requiring the answer to "go in the box"

- Setting up work for efficiency of marking
- Never making a mistake while solving a problem in front of students

Discussion then moved to consideration of ‘good’ mathematical habits of mind. Note that soon after, the word ‘good’ was replaced by ‘productive’. Susan invited participants to work in small groups to consider what habits of mind they would like to foster in their mathematics students, including preservice teachers. She suggested that they try to think about these under four categories: attitudes (encompassing the affective aspects and emotional responses), actions (high-level, often physical), strategies (more specific, mathematical approaches) and questions (what should they be asking?).

A sample of results appears in Table 1, below. Participants’ original work appears in Figures 1 and 2. We note that there were cases where it was unclear where to place a particular attribute or behaviour. For example, the predisposition towards reflection could be seen as an attitude, while the act of reflecting might be an action. The distinction between actions and strategies was particularly ambiguous.

Attitudes • Persistence • Open-mindedness • Confidence • Willingness to take risks • Willingness to tinker • Curiosity • Humbleness • Wonder, interest • Inventivity • Reflection • Being okay with the messy • Celebration • Patience • Self-efficacy, self confidence • Thoroughness, attention to detail • Interest in others approaches • Flexibility • Laziness: value efficiency	Actions • Read the question • Self-regulate, monitor • Recognize the solution • Write things down • Know when to abandon an approach • Seek help • Know where to look for resources and when • Know how to get out of a rut • Analyze • Use tools/objects/manipulatives • Collaborate • Take a break
Strategies • Start with a simpler or similar problem • Draw a picture • Act it out, model it • Make the problem more specific • Talk it out • Borrow from others • Be creative, inventive, trust intuition • Let it percolate • Keep a record of strategies tried • Develop a way to communicate so that others can understand • Use algebra • Work backwards • Draw on past experience • Use Google	Questions • Where am I and where am I going? • What is relevant, what is not? • Have I done a similar problem? • How can I generalize this? • Is there another way? • Is there another solution? • What are my resources? • Why? • What if? • What stays the same? What changes? • What else can I try? • Can I simplify the problem?

Table 1

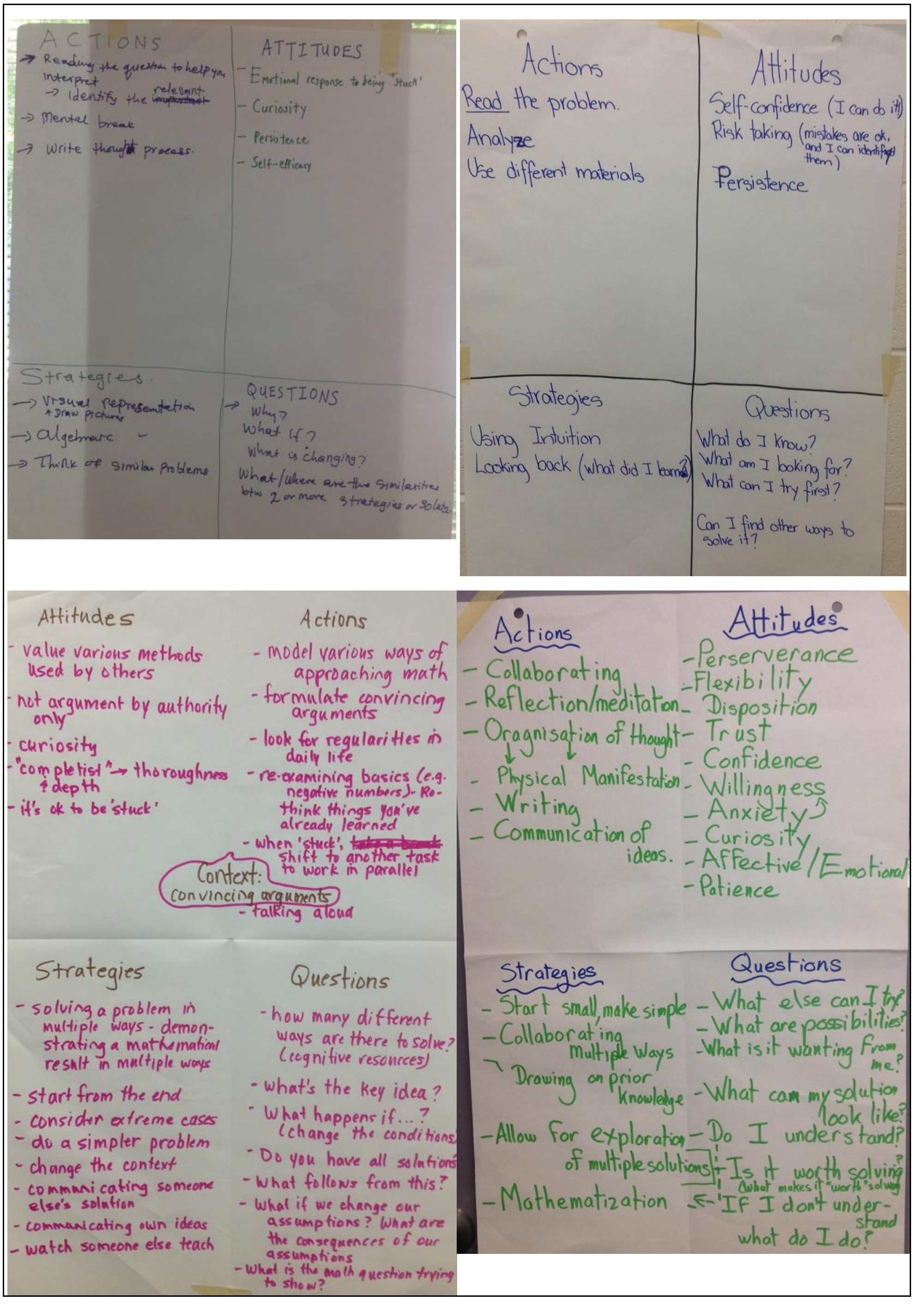

ACTIONS
→ Reading the question to help you interpret
→ Identify the relevant
→ Mental break
→ Write thought process.
ATTITUDES
- Emotional response to being "stuck"
- Curiosity
- Persistence
- Self-efficacy
Strategies
→ Visual representation * Draw pictures
→ algebraic
→ Think of similar problems
QUESTIONS
→ Why?
What if?
What is changing?
What/where are the similarities btw 2 or more strategies or
Actions
Read the problem.
Analyze
Use different materials
Attitudes
Self-confidence (I can do it!)
Risk taking (mistakes are ok, and I can identify them)
Persistence
Strategies
Using Intuition
Looking back (what did I learn?)
Questions
What do I know?
What am I looking for?
What can I try first?
Can I find other ways to solve it?
Attitudes
- value various methods used by others
- not argument by authority only
- curiosity
- "completist" → thoroughness & depth
- it's ok to be 'stuck'
Actions
- model various ways of approaching math
- formulate convincing arguments
- look for regularities in daily life
- re-examining basics (e.g. negative numbers) - Re-think things you've already learned
- when 'stuck', shift to another task to work in parallel
Context: convincing arguments
- talking aloud
Strategies
- solving a problem in multiple ways - demonstrating a mathematical result in multiple ways
- start from the end
- consider extreme cases
- do a simpler problem
- change the context
- communicating someone else's solution
- communicating own ideas
- watch someone else teach
Questions
- how many different ways are there to solve? (cognitive resources)
- what's the key idea?
- What happens if...? (change the conditions)
- Do you have all solutions
- What follows from this?
- What if we change our assumptions? What are the consequences of our assumptions
- What is the math question trying to show?
Actions
- Collaborating
- Reflection/meditation
- Organisation of thought
- Physical Manifestation
- Writing
- Communication of ideas.
Attitudes
- Perserverance
- Flexibility
- Disposition
- Trust
- Confidence
- Willingness
- Anxiety
- Curiosity
- Affective/Emotional
- Patience
Strategies
- Start small, make simple
- Collaborating
Multiple Ways
Drawing on prior knowledge
- Allow for exploration of multiple solutions
- Mathematization
Questions
- What else can I try?
- What are possibilities?
- What is it wanting from me?
- What can my solution look like?
- Do I understand?
- Is it worth solving? (what makes it "worth" solving)
- 'If I don't understand what do I do?'

Figure 1

Figure 2

Participants took time to look at others' results, posted up on the walls around the room. Full group discussion of what we noticed and what surprised us followed and a number of interesting ideas arose:

- It was noted that many of us wanted students to have the confidence to try.
- It is a habit of mathematicians to be lazy (in a good way) to be efficient; they pack as much as possible into notation. It is often the role of the teacher to unpack.
- A distinction was raised between habits of learners of mathematics versus habits of producers of mathematics.
- There was discussion about intuition, which one group had listed as a strategy. Is it an attitude? Ultimately it leads to actions.
- The importance of play was emphasised: it implies a certain perspective, an approach to problems that reduces anxiety.

There was no intent to develop a concise definition of mathematical habits of mind, nor to come to consensus on what this notion might include. Rather, the objective was to raise awareness of the range of things (beyond mathematics content) we might/could/should be communicating/teaching/sharing with our students, particularly with prospective teachers, in our mathematics classes. The session for the day concluded with a promise to spend part of the next session engaging in tasks to further help us consider what could be encompassed by mathematical habits of mind, and an invitation to think about what types of activities might help develop such habits in prospective elementary teachers.

DAY 2

Day 2 began with Frédéric leading us in a warm-up activity called "Chicks and Mother Hen".

The main aim of the first half of Day 2 activities was to allow participants to notice their own mathematical habits of mind as they engaged in solving problems. Five problems were given (see below) and participants were invited to work in groups of any size on any of the problems they liked. While they worked they were asked to consider the following:

1. What mathematical habits of mind do you notice/observe in yourself as you approach these problems?
2. What was missed on the sheets created the previous day? Go back and add/edit/emphasise as seems appropriate. What seems especially important?
3. How might a preservice teacher respond to these or similar problems?
4. Could an activity like this support developing mathematical habits of mind in preservice teachers? Or help preservice teachers value mathematical habits of mind?

THE PROBLEMS

Problem 1: The Diagonal of a Rectangle

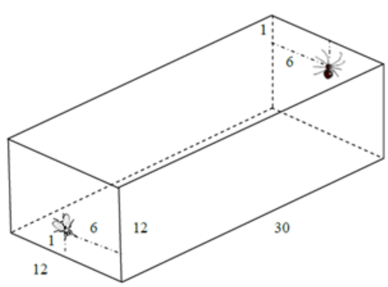
A homework assignment asked students to draw the diagonal of a rectangle. Erin used tracing paper to draw the diagonal. However, when she came to class, she only had the tracing paper with the diagonal she had drawn. Draw a possible rectangle for which Erin drew the diagonal.

Problem 2: The Locker Problem

In a college hallway there are 1000 lockers, numbered 1 to 1000. At the start of this story all of the lockers are closed. The first student enters and opens all of the lockers. The second student comes along and closes every second locker. The third student goes to every third locker and closes the open ones and opens the closed ones. The fourth student repeats the process with every fourth locker. This goes on until the thousandth student changes the state of the thousandth locker. Which lockers are still open?

Problem 3: The Spider and the Fly

Suppose a spider and a fly are on opposite walls of a rectangular room, as shown in the figure. The spider wants to 'visit' the fly, and assuming that the spider must travel on the surfaces of the room, what is the shortest path to the fly? Be careful! The shortest distance is less than 42!



Problem 4: Knights of the Round Table

King Arthur has a daughter and wants her to marry one of his knights. He will choose which knight by asking them to all sit down at the round table. He will say to the first knight, "You live." He'll say to the second knight, "You Die" and kills him, and lets the third one live and fourth one die and so forth around and around the round table until there's only one left. Where should a knight sit to be sure that he is

> the one to marry the king's daughter? (This will depend on how many knights there are—can you find a formula for the knight to use?)

<u>Problem 5: Magic Trick</u>

> Twenty or so two-colour coins (yellow on one side, red on the other) are placed on a table by the magician. One person is asked to do the following, while the others watch and the magician looks away: turn coins one at a time (so that the colour exposed on top changes), turning as many coins and as many times as you wish, simply saying out loud "I turn" each time you do. Then, cover one coin of your choice. The magician can then look, and will guess if the covered coin top face is yellow or red.

DISCUSSION OF THE PROBLEMS

After we had worked on the problems for about an hour, the group came together to comment on the experience and address the discussion prompts.

Observations of personal mathematical habits of mind included the following comments:

- Being skeptical about a possible solution pushed other group members toward more precise justification. The group noticed degrees of justification, citing Mason's "convince yourself, convince a friend, convince an enemy".
- Prior knowledge, including mathematical facts, played a big part. An important question is: "What do you already know?" If you don't have the needed information, you need to know what to ask.
- Sometimes there can be a clash of intuitions.
- Some felt curiosity, persistence, a desire to go further…frustration, a need to change strategies.
- Multiple approaches to the same problem added to depth of understanding. The diagonal problem was tackled using geometric facts/theorems, paper-folding, algebra and computer software.
- The Knights problem inspired individuals to start with small numbers and look for patterns. One group took a 'divide and conquer' approach, dividing up work among themselves, making predictions and checking.
- Many talked about persistence. Students often give up easily, but some observed in themselves that even if they temporarily switched to another task, they went back to the original problem to try again. It was noted that you need to have a certain level of trust/self-confidence to go back. There is often something to be gained from taking a break and walking away from a problem for a while. Minds can continue to work on a problem, even when engaged in other activities. It was noted that often fixed time-frames in school settings don't allow for this experience. Students may need to have opportunities to pause and come back to a problem.

Discussion of preservice teachers' possible responses to these questions brought to light the importance of creating a safe space for exploring. Anxiety can inhibit thinking. Even some of the participants in the group shared having some initial anxiety entering into the problem-solving activity, feeling insecure about their own mathematical abilities as compared to others at their table. Having a choice of problems to work on, and being able to choose who to work with and where to work allowed participants (and would allow prospective teachers) to feel safer. Explicitly discussing the phenomenon of being *stuck* as Mason (2010) does, particularly the emotions involved and the strategies for moving forward anyway, would be worthwhile. Just as important is modeling being stuck. This means being willing to allow ourselves to get stuck in front of our students. Having successful experiences of risk-taking can empower teachers to take risks in their teaching and to provide experiences for their students to do this as well.

Choice of problems would also help with engagement, allowing preservice teachers to choose what they found to be more interesting.

Following the break, we moved to a new topic in order to move beyond problem solving as a vehicle for mathematical habits of mind. Mary gave a brief presentation on Carol Dweck's (2006, 2008) research on *mindsets* and how mindsets relate to teaching and learning in general and to mathematics teaching and learning in particular. Dweck's work has become more widely known in mathematics education circles through the work of Jo Boaler (2013). Boaler has recent publications citing the latest scientific research from neuroscience on brain plasticity and the incredible potential the brain has to grow and change. This research fits well with Dweck's work on mindsets and achievement. It is through Boaler's work that Mary began to pay attention to mindsets and to consider the strong relationship of mindsets to mathematical habits of mind.

In 2006, Dweck, a well-known Stanford University psychologist who has been researching success and achievement for years, published *Mindsets: The new psychology of success*. Dweck's premise is that it is not just our abilities that lead to success, but the mindset we possess about our abilities. Her research has found that there are two mindsets: a *fixed* mindset and a *growt*h mindset. She found that in the United States, 40% of students have a fixed mindset, 40% have a growth mindset, and 20% have a mix. People with a *fixed* mindset think of their abilities as being set in stone. This gives rise to the belief in math geniuses and people who believe learners are either good at math or not good at math. Effort does not come into play when considering ability. Intelligence is viewed as static. Dweck's research shows there is a desire on the part of people with a fixed mindset to look smart and avoid situations that may expose their perceived lack of ability. They may avoid challenges, give up easily without much effort, ignore useful feedback, and feel threatened by the success of others. On the other hand, Dweck argues that people with a *growth* mindset think of their abilities as developing with effort and experience. Such people think of intelligence as something that can be enhanced through effort, tend to face setbacks as a chance for learning by embracing challenges as part of the process, learn from feedback, and become inspired by the success of others. In 2008, Dweck prepared a paper for the Carnegie Corporation of New York-Institute for Advanced Study Commission on Mathematics and Science Education called *Mindsets and Math/Science Achievement*. In this paper, she argues that students who viewed their math/science ability as fixed are at a clear disadvantage when compared to learners who believe their math/science abilities can be enhanced. Her paper provides research evidence toward the following:

> *Mindsets can predict math/science achievement over time; Mindsets can contribute to math/science achievement discrepancies for women and minorities; Interventions that change mindsets can boost achievement and reduce achievement discrepancies; Educators play an important role in shaping students' mindsets.* (p. 2)

As a group we discussed Dweck's four recommendations that make up the second half of the paper: 1) ways in which educators can convey a growth mindset to students; 2) ways that educators can be taught a growth mindset and how to communicate it to students; 3) ways to convey to females and minority students that past underachievement has its roots in *environmental* rather than *genetic* factors and can be overcome by enhanced support from their educational environment and by personal commitment to learning; and 4) ways to incorporate the growth mindset message into high stakes tests and into the teaching lessons that surround them. We concluded the day by discussing the implications of these research findings and recommendations on teacher education for both practicing and preservice teachers.

DAY 3

Our final day started with a brief presentation by Frédéric of a paper by Hyman Bass (2011), *A Vignette of Doing Mathematics: A Meta-cognitive Tour of the Production of Some Elementary Mathematics*. In this paper, Bass presents a personal description of the doing of mathematics in a general context, followed by a description of actual mathematical work done with a problem from elementary arithmetic (division of a cake).

The description of the doing of mathematics by mathematicians presented was close to a number of ideas that had already been explored in the working group. The description is as follows (where the "we" refers to "we, mathematicians"):

We…

- Question
- Explore
- Represent, possibly in more than one way.
- Look for structure, which may lead to conjectures or new questions.
- Consult, if we're stuck: others, the literature, the web.
- Connect with other mathematics, through our research, reflection, analogies.
- Seek proof, to prove or disprove our conjectures. Often this proceeds by breaking the task into smaller pieces, for example by formulating, or proving, related, hopefully more accessible, conjectures, and showing that the main conjecture could be deduced from those.
- Can be opportunistic, letting the mathematics guide us if we see inviting trails.
- Prove, writing a finished exposition of the proof (if one is found), using illuminating representations of the main ideas, meeting standards of mathematical rigor, and crafted to be accessible to the mathematical expertise of an intended audience.
- Analyze proofs, which even if they are conceived of as a means to an end, are a product worthy of note and study, since the theorem typically distills only a small part of what the proof contains.
- Use our sense of aesthetics and taste, associated with words like elegance, precision, lucidity, coherence, unity.

The participants of the working group were then invited to work in small groups to consider activities which they have used in their teaching, or have seen others use, and reflect on which mathematical habits of mind were explored or solicited in these activities. They were also asked to try to bring to the fore which features of the activities were important in this respect.

Participants chose to be divided into five small groups which focused on either the elementary (2 groups), secondary (2 groups) or tertiary (1 group) level. After a time for discussion which lasted until the break, we came together for a large group discussion, based on short reports from each of the groups. The discussion was free-flowing and rich, and we present some of the ideas which were discussed.

ELEMENTARY

Both elementary groups brought up specific actions teachers can take to promote development of mathematical habits of mind (MHoM) in their students. In particular:

- Teachers can (and should) demonstrate MHoM. They can model learning: struggling, sense-making, and acknowledging emotions. They can also acknowledge the courage to ask questions and ask for help.

- They can model some specific habits of mind and be explicit about it: for instance, generalizing and iterative thinking (revision).
- Teachers can make thinking visible, flexible, and public. The use of prompts that foster expression of and development of MHoM is of curricular importance. Annette and Michelle described their use of vertical surfaces in the classroom, creating and using as much of it as possible, for instance using a white shower curtain and Crayola washable markers. Giving students a way to make their thinking visible allows for mathematical conversations to occur.
- Teachers can foster a reflection about where the mathematics is located when groups are working together, doing activities that question the location of the 'mind' in our understanding of MHoM. When working in a group, an idea which is proposed does not belong to a person—it belongs to the group. Talking about an idea and discussing it is not about saying that the idea is right or wrong, but rather discussing so that the group gets somewhere. One needs to be bold in challenging the mathematics (and not challenging the person).

One group observed that a key feature of activities that enable MHoM is that they build relationships.

While one of the elementary level groups noted that anxiety could hinder or hold back developing mathematical habits of mind, another participant noted that with preservice teachers, their anxiety can be an opening for discussing them. This raised an interesting issue about different types of anxiety and a possible distinction between elementary and secondary preservice teachers: elementary preservice teachers may be more anxious about the content and discussion of MHoM can draw their attention to other important aspects of learning mathematics; secondary preservice teachers may be more confident about content, but may be anxious about letting go of what has worked for them in the past. In either case, awareness of student anxiety is a factor that instructors need to keep in mind in choosing activities to develop MHoM.

It was also observed that there should be an awareness that children come to school with some of these MHoM, and that teaching is about exploiting them, developing them, not installing them or educating it into them.

How do we change? Beyond experiencing methods in a teacher education program, it may be necessary to see it work in someone else's classroom, with real students. So it may not be enough to model it in the teacher education classroom, as future teachers may need to see their peers model it, in context.

SECONDARY

The secondary groups also identified features of activities for fostering MHoM. There should be a narrative, and elements of play, curiosity, and reflection. They should allow for questions about what is possible and what is relevant. One stresses some aspects, ignores others. There is an engagement in mathematical ideas. Choosing problems that are sufficiently complex, so that the solvers are discouraged from resorting to simply counting, but look for underlying structure is also effective.

Some particular activities that had been tried were described, including The Peanut Butter Jar Conundrum, presented to calculus students. They are asked to redesign the jar to avoid the conundrum…*There is peanut butter in the bottom of the jar but the knife is not long enough to get the remaining peanut butter out without getting your hand dirty*.

It was also argued that there ought to be mathematics for mathematics sake, and that MHoM can be developed through pure mathematics rather than subordinating them to other activities or applications, which may distract from the main idea.

Like in the earlier discussion arising from the elementary group discussions, participants stressed that promoting MHoM includes making decisions explicit when teaching. In secondary school mathematics, what we do (when teaching) is very fluid, we know it is not always like this, but our students do not necessarily know that. Other aspects which were discussed included the following:

- Having students notice MHoM in peers is significant.
- These opportunities allow students to become aware of their own habits and then, if they have the will and motivation, they may be able to change them.
- Seeing the same mathematics across or through different contexts is empowering. However, students may not be aware of the context to which the teacher is referring and it is important to make it accessible.
- Using history, with evidence of struggle, persistence and eventual success can be useful.

Participants referred to several potentially useful frameworks including: Polya's heuristics; Mason's mathematical thinking and processes; and Ann Watson's (2006) adolescence and math.

One line of argument, which was presented for all levels of teaching, is that the activity has nothing to do with it—it is the way we work which matters. What we might stress and what we ignore. For instance, going back to some examples presented, you (the teacher) never carry out any arithmetic but point out what arithmetic you would carry out: it is a decision to focus on the structure rather than the eventual answer. We can bring out MHoM in almost any mathematical activity we do, through what we choose to focus on, although perhaps some activities are more suited to bringing out particular habits of mind.

TERTIARY

Bringing in the group which focused on the tertiary level into the discussion at this stage enabled us to add some new dimensions to our discussion. The following mathematical habits of mind were stressed:

- Precision—in communication, in argumentation, in reading, in definition, in inference
- We would like for our students to concentrate and immerse themselves in one thing—to find 'flow'.
- Ability to use knowledge in different contexts
- Distinguish between relevant and irrelevant
- Self-regulation was also mentioned

One task was presented by Nilima Nigam. She has her students read poetry and newspapers. Students put a box around what they think is mathematics. She wants to shape them to think precisely about the world they inhabit. It is important to have them not disassociate from the rest of the world and their experiences. A participant noted that this is important at all levels: we want students to learn to live mathematically. Even teachers at the primary level can talk to children about looking at the world with 'mathematical glasses' on.

The tertiary group also observed that there is a need to re-form the mathematical habits of mind of many students. They enter remedial type courses with a certain state of mind, because their strategies have worked for them until this point. If you are constantly told "do this, do that", you lose your agency. It is essential to give tasks that allow choice, freedom, exploration, and allow them to make decisions. Our students are the product of systems that have focused on content.

CONCLUSION

At the end, we asked participants to tell us what they had learned or what ideas had been brought out for them during our working group sessions. We present some of their (edited) comments here, organized around some of the key themes that arose:

THE NATURE OF MATHEMATICAL HABITS OF MIND

- Habits of mind: inspire, reveal, develop, encourage, grow, etc.
- MHoM form an overarching phenomenon that permeates throughout mathematics education.
- MHoM are available to us all, and their power can and should be enhanced for all students individually and collectively, as a contribution to their lives and to our society.
- Understanding may occur 'in the moment' and may be temporary or fleeting, however crystal clear it feels. Mathematical Habits of Mind may help understanding be visible, natural, authentic (and personal), and transformative, … into a sense of permanence of 'knowing-to'.
- In a broad sense, maybe we can ground MHoM in metacognition as a theoretical frame.
- *Je n'avais jamais vu les maths (ens./app.) sous l'angle des MHoM. Je sens que ma participation à ce groupe m'a aidée à mettre des mots sur ce que je conçois être "faire des maths". Ces MHoM forment ce qu'est l'activité mathématique. La résolution de tâches mathématiques en gardant en tête quels sont nos propres modes de pensée mathématiques aide à réaliser comment se produit mon activité mathématique et comment je fais des maths, ce qui peut être très utile à exploiter avec les étudiants de tout niveau scolaire.*
- I have been thinking mainly of two things. First, framing MHoM as mathematizing the world—we have touched on this briefly in the working group. Second, that the larger contextual element is ontological—our being and becoming in the world. That part of it is that we all have the possibility of being mathematically in the world.
- Everything is habits…perhaps. A very useful way of thinking about teaching and learning mathematics. Exposing implicit habits, or at least looking for them will be easier…perhaps…given the experience of the last few days.

WHAT TEACHERS NEED TO KNOW ABOUT MATHEMATICAL HABITS OF MIND

- The 'need' for creating a community/environment that allows, fosters, stimulates the making explicit, bringing out often innate and implicit characteristics of mathematical habits of mind to be more explicit and valued.
- Mathematical habits of the mind are not to be formally 'taught' as if they were part of a curriculum list to be 'ticked off'; they are already within us all. The task of a teacher is to use stressing and ignoring to nurture and develop them and to help students see the power which can come from their regular use.
- There is a need to provide students with rich experiences and encourage them to notice MHoM themselves.
- The importance of 'making the implicit explicit'.
- To teach is to unpack your experiences as a learner of mathematics.

- As teachers, we have the freedom to choose how to demonstrate our MHoM to students, but students don't necessarily have the choice of the MHoM they receive from us. So...we need to be aware of the MHoM we demonstrate, and in order to demonstrate valuable/productive MHoM, we need to have these MHoM, then be aware of them, and understand them. Then we can truly live them and influence students by modelling these MHoM. Definitely not an easy process. But very worthwhile.
- Focusing on creativity prompts us to think about MHoM.
- Educators in K-12 and in post-secondary can foster these habits in their students.

WHAT MATHEMATICS LEARNERS NEED TO KNOW ABOUT MATHEMATICAL HABITS OF MIND

- "Work with your capacity to develop your MHoM. You are not working to cope with your deficiencies." "If it is truly a problem, being stuck is part of it."

WHAT ARE THE CHALLENGES?

- Perhaps the biggest challenge is supporting teachers in recognizing and valuing MHoM in children.
- We (post-secondary) tend to concentrate too much on teaching the content and then complain that our students do not develop certain ways of thinking and/or writing, expressing themselves, observing, rationalizing.

MORE QUESTIONS

- I am left with more questions…I like that. The essential question: Is almost everything a potential MHoM and further, productive or not depending upon context?
- How are MHoM different from/same as HoM? Is this a human endeavour in multiple contexts? I'm provoked and challenged by ontological issues and will take more seriously how I can explicitly point my students toward recognizing and naming MHoM.

As Working Group leaders, our three days of discussion and activities left us energized and inspired, more in tune with the nuances of mathematical habits of mind, and appreciating the challenges of and potential for explicitly addressing mathematical habits of mind in order to improve our preparation of preservice teachers. We would like to express our deep appreciation for the contributions of all who participated. They shared insights and experiences and engaged in respectful, constructive dialogue, enriched by their varied backgrounds and interests to the benefit of all present.

[Note: References follow the French version.]

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INTRODUCTION

En 1966, Cuoco, Goldenberg et Mark ont soulevé une question qui est toujours d'actualité : « Puisque qu'on ne connait pas avec certitude ce que seront les besoins des générations de diplômés à venir, comment décider quelles mathématiques enseigner? » Dans leur article, les auteurs critiquent la pertinence d'un curriculum basé sur des objectifs de contenu et proposent d'articuler le curriculum autour de « Modes de pensée mathématiques[1]».

> *Un curriculum dont l'organisation est structurée par les modes de pensée mathématiques vise à combler le fossé qui existe entre ce que les utilisateurs et les créateurs de mathématiques font et ce qu'ils disent. … [Il] permet aux étudiants de prendre part aux processus de création, d'invention, d'émission de conjectures et d'expérimentation … C'est un curriculum qui encourage les faux départs, les essais numériques, les expériences, et les cas particuliers.* (p. 376)

Depuis, plusieurs chercheurs ont essayé de définir et d'expliquer ce que sont les « modes de pensée mathématiques » (voir Lim et Selden, 2009). Grossièrement, on peut les concevoir comme des approches et des manières d'aborder des problèmes (ainsi que la vie en général) qui sont typiques des mathématiciens—incluant ainsi les « capacités et processus naturels » (p. 8) de Mason, Burton et Stacey (2010) : spécialiser et généraliser, émettre des conjectures et convaincre, imaginer et exprimer, porter attention et ignorer, classifier et caractériser.

Dans la dernière décennie, des comités consultatifs aux États-Unis ont émis des recommandations qui semblent accorder une plus grande importance à ce que les étudiants puissent attaquer les problèmes à la manière des mathématiciens. Cet aspect était déjà important dans les *NCTM Principles and Standards* (2000), lesquels mettaient l'emphase sur l'acquisition par les étudiants « d'habitudes de persévérance et de curiosité » et soulignaient que « les personnes qui raisonnent logiquement et de manière analytique sont enclines à remarquer les patrons, la structure et les régularités dans les situations de la vie courante et en mathématiques ». Dans *Adding it Up* (2001), Kilpatrick, Swafford et Findell décrivent une « disposition mathématique productive » comme une « propension naturelle à voir les mathématiques comme raisonnables, utiles et pertinentes, à croire à l'importance du travail et de la persévérance, et à avoir confiance en ses capacités personnelles » (p. 116). Ces idées se fraient un chemin dans les curriculums mathématiques au Canada : elles sont implicites dans le curriculum K-12 du *Protocole de l'Ouest et du Nord Canadiens* (WNCP, 2006) et sont explicites dans le projet de Colombie-Britannique (K-12), publié fin 2013, et qui décrit comme changement principal « l'attention portée au développement de modes de pensée mathématiques et l'incitation à ce que les étudiants cherchent à comprendre comment les mathématiciens réfléchissent et travaillent » (BC Ministry of Education, 2013).

Ces changements influencent la manière dont seront enseignées les mathématiques à l'école et aura donc un impact sur la formation des enseignants. Le Conference Board of the Mathematical Sciences (2012) adresse ces changements directement, recommandant : « Tous les cours et toutes les activités de perfectionnement pour les enseignants de mathématiques devraient développer les modes de pensée mathématiques et de résolution de problèmes mathématiques, tels que l'analyse et l'explication, la modélisation, le détection de structure, et la généralisation » (p. 19).

Mason et al. (2010) soulignent qu' « il est vital d'éduquer sa propre conscience en s'investissant dans des tâches mathématiques qui permettent de révéler des aspects, des réalités mathématiques de sortes que ceux-ci puissent informer nos actions à venir » (p. xii). Dans ce groupe de travail, nous avons pris cette suggestion à cœur en cherchant à parvenir à une meilleure compréhension collective de ce que sont les modes de pensée mathématiques, et en considérant comment aider à les développer chez les enseignants en formation initiale.

[1] L'expression anglaise « Mathematical Habits of Mind » est traduite par « Modes de pensée mathématiques ».

PREMIÈRE JOURNÉE

Nous avons débuté avec une présentation de chacun des membres du groupe de travail. Puis, Susan a présenté le contexte dans lequel se situait le thème du groupe de travail.

POURQUOI DISCUTER DES MODES DE PENSÉE MATHÉMATIQUES?

Susan a tout d'abord expliqué comment elle en était venue à être l'une des co-animatrices de ce groupe de travail. Le tout avait débuté lors d'un autre groupe de travail, qu'elle animait avec Lynn Hart, Susan Swars (toutes deux de la Georgia State University) et Ann Kajander (de l'Université Lakehead), dans le cadre de la PMENA 2012 à Atlanta. Le thème de travail était « Qu'est-ce les étudiants en formation initiale à l'enseignement des mathématiques au primaire ont besoin de savoir ». Les participants (dont Mary faisait partie) y ont fait un remue-méninge afin d'établir une liste d'éléments importantes pour les enseignants en formation initiale. Les modes de pensée mathématiques sont ressortis de cette discussion. Le groupe a alors décidé d'écrire un livre destiné aux enseignants des cours de mathématiques pour les programmes de formation initiale, et Susan et Mary se sont retrouvées dans l'équipe chargée d'écrire un chapitre sur les modes de pensée mathématiques. (Le travail sur le livre continue. Le groupe s'est réuni à nouveau à PMENA 2013 à Kalamazoo, et doit poursuivre son travail à PME 2014 à Vancouver.)

Dans le cadre de la collecte d'informations pour le livre, Frédéric a été approché pour contribuer en tant que mathématicien, en commentant sur ce qu'il trouvait important pour les futurs enseignants au primaire. Afin de favoriser une discussion plus large et d'approfondir les réflexions à ce sujet, Mary, Frédéric et Susan ont par la suite tenu une séance *ad hoc* portant sur modes de pensée mathématiques à la rencontre annuelle du GCEDM de 2013. La séance a connu un franc succès et il a été décidé de soumettre une proposition pour tenir un groupe de travail sur ce thème cette année.

CONTEXTE

L'intérêt de Susan pour les modes de pensée mathématiques a véritablement surgi des discussions du groupe de travail PMENA en 2012. À l'époque, elle n'avait aucune idée de l'importance que ce thème allait prendre pour l'enseignement des mathématiques en Colombie-Britannique. Parallèlement, le Ministère de l'éducation de la Colombie-Britannique a publié à l'automne 2013 son avant-projet de révisions du curriculum : y figurent comme thème central les *modes de pensée mathématiques*, formant ce qui est décrit comme étant au cœur des changements proposés.

Susan a présenté un bref résumé de la littérature sur les modes de pensée mathématiques, à partir de Cuoco et al. (précité). Elle a abordé quelques autres auteurs ayant contribué au domaine, incluant Harel (2007, 2008), Selden et Selden (2005), Bass (2008), Leikin (2007), et Lim (2008, 2009), soulignant particulièrement le livre *Thinking Mathematically* de John Mason et al. (2010) – dont la traduction française est *L'esprit mathématique* – et dont la version originale (1982) précède significativement Cuoco et al. (1996). Le livre de Mason amène les lecteurs à développer leur esprit mathématique, notamment en décrivant des stratégies efficaces pour *spécialiser* et *généraliser*, et pour savoir comment agir face à *un blocage*.

Elle a également résumé les recommandations importantes de comités consultatifs (décrits dans l'introduction ci-dessus), mettant l'accent sur le rapport du Conference Board of the Mathematical Sciences (2012), qui souligne l'importance de développer les modes de pensée mathématiques chez les enseignants de mathématiques. Elle a noté que le travail avec les enseignants serait un des axes de travail les plus importants de ce groupe de travail.

L'influence de la littérature portant sur les modes de pensée mathématiques sur le curriculum semble continuer de croître. Ils apparaissent implicitement dans certains des « processus mathématiques » énoncés dans le curriculum du Protocole de l'Ouest et du Nord Canadiens (2006). On retrouve, dans le processus de résolution de problèmes : « un environnement dans lequel les élèves se sentent libres de rechercher ouvertement différentes stratégies contribue au fondement de leur confiance en eux-mêmes et les encourage à prendre des risques » (p. 8). Dans la rubrique portant sur le raisonnement, on trouve : « Le défi relié aux questions d'un niveau plus élevé incite les élèves à penser et à développer leur curiosité devant les mathématiques » (p. 8). Le projet de nouveau programme de mathématiques de la Colombie-Britannique (BC Ministry of Education, 2013) est beaucoup plus explicite. Non seulement on y déclare « un accent sur le développement des modes de pensée mathématiques » mais on indique, dans la description de l'essence même du nouveau curriculum, qu'il vise à « aider les élèves à apprécier la perspective unique offerte par les mathématiques : comment l'utilisation des modes de pensée et d'action mathématiques modifient l'interprétation du monde dans lequel on vit ».

Cette emphase nouvelle sur les modes de pensée mathématiques ajoute de la pression aux enseignants, particulièrement sur ceux dont la formation mathématique est constituée de cours axés uniquement sur le contenu. Que l'on soit ou non d'accord avec la direction prise par ces programmes, celle-ci renforce l'importance de faire plus que transmettre du contenu dans les cours de mathématiques. En tant que personnes qui soutiennent les enseignants, le défi que nous devons relever est de mieux comprendre ce que sont les modes de pensée mathématiques, de trouver des moyens de favoriser leur développement chez les futurs enseignants, et de les préparer à favoriser leur développement chez leurs étudiants.

QUE SONT LES MODES DE PENSÉE MATHÉMATIQUES?

Susan a offert une définition préliminaire et a invité les participants à répondre :

> Dans leur interprétation la plus large, les modes de pensée mathématiques sont ...
> de penser aux mathématiques (et au monde) à la manière des mathématiciens.

Parmi les suggestions émises, on note : les mathématiciens portent un regard différent ; ils cherchent des patrons, généralisent ; ils cherchent des solutions multiples ; ils sont stimulés et intrigués par des tâches mathématiques. On affirme aussi avec conviction qu'il n'y a pas une approche unique.

Susan a par la suite proposé une expérience, demandant aux participants d'envisager le scénario suivant et de réfléchir à la façon dont ils répondraient personnellement, et comment un mathématicien pourrait répondre.

<u>Le scénario du « Groupon »</u>

Vous et un ami venez de partager un délicieux dîner dans un bistrot local. Vous avez choisi ce restaurant parce que votre ami avait acheté un « Groupon ». Elle avait payé 15 \$ pour obtenir un crédit de 30 \$ pour couvrir le coût du repas. Il est maintenant temps de payer la facture, qui est de 27 \$ après le rabais, y compris les taxes, mais sans le pourboire. Que faites-vous ?

Les participants ont suggéré plusieurs possibilités, dont diverses stratégies pour déterminer combien chaque personne devrait payer. On a demandé s'il s'agissait d'un problème mathématique ou social. Certains auraient tout simplement payé la facture totale et laissé l'ami payer la prochaine fois. Une personne a fait remarquer que la réponse d'un mathématicien pourrait être d'envisager toutes les options, puis de choisir la solution optimale, tandis qu'un autre a fait observer qu'elle avait été amenée à s'interroger sur les implications commerciales des « groupons ». Fait intéressant, certains ont répondu au scénario en tant qu'enseignant de mathématiques, se demandant comment ce problème pourrait être utilisé, adapté et développé dans leur classe. Dans les discussions, il était clair que toute situation peut être vue à travers différents filtres : mathématique, social, économique, éducatif, etc.

Puis, Susan a offert une définition plus précise des modes de pensée mathématiques :

> Nous démontrons des modes de pensée mathématiques lorsque nous choisissons habituellement des actions et des stratégies, posons des questions et démontrons des attitudes qui sont PRODUCTIVES dans un contexte mathématique.
>
> Ces modes de pensée nous aident à comprendre les mathématiques, à résoudre des problèmes, et peuvent même nous aider à créer des mathématiques.

Elle a parlé brièvement des habitudes[2] notant qu'elles sont souvent machinales, qu'elles peuvent être apprises et désapprises (ce qui n'est pas toujours facile), et qu'elles peuvent être bonnes ou mauvaises. Elle a invité le groupe à réfléchir aux « mauvaises habitudes » que les étudiants des mathématiques affichent parfois. Les réponses incluaient

- Arrêter dès qu'ils ont terminés, se désengager
- Demander immédiatement quelle formule utiliser
- Imiter les enseignants, les pairs : « Montre-moi un exemple et je vais le faire. »
- Aborder les situations comme si tout devait avoir une réponse rapide
- Lire superficiellement à la recherche d'information numérique – ne pas véritablement s'engager dans la résolution du problème
- Mémoriser, essayer de deviner ou de se rappeler de ce qui est attendu
- Abandonner

On a noté que ces habitudes étaient souvent issues des expériences antérieures avec les mathématiques et leur enseignement. Ce sont des techniques de survie. Un participant a questionné la justesse du qualificatif « mauvaise » pour décrire ces habitudes puisqu'elles sont souvent efficaces en mathématiques scolaires.

On a aussi abordé certaines « mauvaises habitudes » des enseignants de mathématiques :

- Exiger que la réponse soit écrite dans la petite boîte à cet effet.
- Concevoir les travaux en fonction de la facilité de correction.
- Ne jamais faire d'erreur lors de la résolution d'un problème devant la classe.

La discussion s'est ensuite portée sur les « bonnes » habitudes, rapidement décrites comme « productives » plutôt que « bonnes ». Susan a invité les participants à travailler en petits groupes pour réfléchir aux modes de pensés qu'ils aimeraient favoriser chez leurs étudiants en mathématiques, y compris les futurs enseignants. Elle a suggéré de les diviser en quatre catégories : attitudes (englobant les aspects affectifs et émotionnels), actions (de haut niveau, souvent physiques), stratégies (approches plus spécifiques, mathématiques) et questions (lesquelles poser).

Un échantillon des réponses figure dans le tableau 1, ci-dessous, alors que les écrits originaux des participants sont à la figure 1 et 2. Il y avait des cas où il était difficile de savoir où placer un attribut ou comportement particulier. Par exemple, la prédisposition à la réflexion peut être

[2] Note de traduction: l'expression anglaise *Mathematical Habits of Mind*, traduite par *Modes de pensée mathématiques*, contient le mot « habit », lequel est traduit ici par *habitude*.

considérée comme une attitude, alors que l'acte de réflexion peut être vu comme une action. La distinction entre actions et stratégies était particulièrement ambiguë.

Attitudes	Actions
• Persistance • Ouverture d'esprit • Confiance • Volonté de prendre des risques • Volonté de bricoler • Curiosité • Humilité • Émerveillement, intérêt • Ingéniosité • Réflexive • Accepter une certaine désorganisation • Célébration • Patience • Efficacité, confiance en soi • Rigueur, souci du détail • Intérêt pour d'autres approches • Flexibilité • Une certaine paresse qui pousse à l'efficacité	• Lire la question • S'autoréguler, se surveiller • Reconnaître une solution • Écrire • Savoir quand abandonner une approche • Demander de l'aide • Savoir quand et où chercher des ressources • Savoir comment sortir d'une routine • Analyser • Utiliser des outils, des objets, du matériel de manipulation • Collaborer • Prendre une pause
Stratégies	**Questions**
• Débuter par un problème plus simple ou similaire • Faire un dessin • Modéliser, représenter par une action • Rendre le problème plus spécifique • En parler • Emprunter aux idées des autres • Être créatif, inventif, faire confiance à l'intuition • Laisser le problème macérer • Garder une trace des stratégies utilisées • Chercher à communiquer afin que d'autres puissent comprendre • Utiliser l'algèbre • Travailler à rebours • S'appuyer sur l'expérience passée • Utiliser Google	• Où suis-je et où est-ce que je veux aller ? • Qu'est-ce qui est pertinent ? Qu'est-ce qui ne l'est pas ? • Ais-je fait un problème similaire ? • Comment généraliser ? • Peut-on aborder d'une autre manière ? • Peut-on trouver une autre solution ? • Quelles sont mes ressources ? • Pourquoi ? • Qu'est-ce qui se passe si…? • Qu'est-ce qui reste identique, et qu'est-ce qui change ? • Quoi d'autre essayer ? • Peut-on simplifier le problème ?

Tableau 1

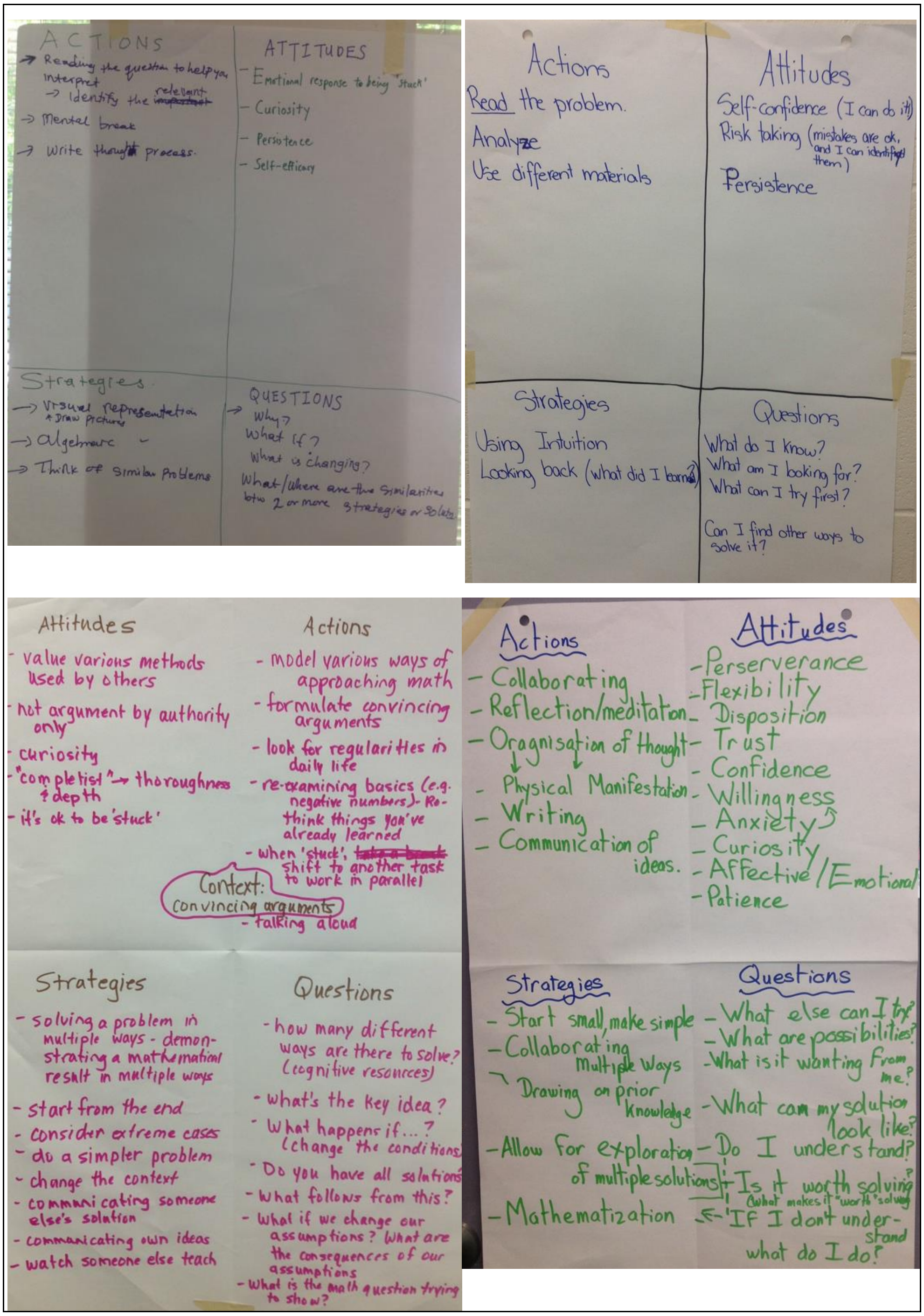

ACTIONS
→ Reading the question to help you interpret
→ Identify the relevant
→ Mental break
→ Write thought process.
ATTITUDES
- Emotional response to being "stuck"
- Curiosity
- Persistence
- Self-efficacy
Strategies
→ Visual representation
* Draw pictures
→ algebraic
→ Think of similar problems
QUESTIONS
→ Why?
What if?
What is changing?
What/where are the similarities btw 2 or more strategies or solutions
Actions
Read the problem.
Analyze
Use different materials
Attitudes
Self-confidence (I can do it)
Risk taking (mistakes are ok, and I can identify them)
Persistence
Strategies
Using Intuition
Looking back (what did I learn)
Questions
What do I know?
What am I looking for?
What can I try first?
Can I find other ways to solve it?
Attitudes
- value various methods used by others
- not argument by authority only
- curiosity
- "completist" → thoroughness & depth
- it's ok to be 'stuck'
Actions
- model various ways of approaching math
- formulate convincing arguments
- look for regularities in daily life
- re-examining basics (e.g. negative numbers) - Re-think things you've already learned
- when 'stuck', shift to another task to work in parallel
Context:
convincing arguments
- talking aloud
Strategies
- solving a problem in multiple ways - demonstrating a mathematical result in multiple ways
- start from the end
- consider extreme cases
- do a simpler problem
- change the context
- communicating someone else's solution
- communicating own ideas
- watch someone else teach
Questions
- how many different ways are there to solve? (cognitive resources)
- what's the key idea?
- What happens if...? (change the conditions)
- Do you have all solutions
- What follows from this?
- What if we change our assumptions? What are the consequences of our assumptions
- What is the math question trying to show?
Actions
- Collaborating
- Reflection/meditation
- Organisation of thought
- Physical Manifestation
- Writing
- Communication of ideas.
Attitudes
- Perserverance
- Flexibility
- Disposition
- Trust
- Confidence
- Willingness
- Anxiety
- Curiosity
- Affective/Emotional
- Patience
Strategies
- Start small, make simple
- Collaborating
Multiple Ways
Drawing on prior knowledge
- Allow for exploration of multiple solutions
- Mathematization
Questions
- What else can I try?
- What are possibilities?
- What is it wanting from me?
- What can my solution look like?
- Do I understand?
- Is it worth solving? (what makes it "worth" solving)
- If I don't understand what do I do?

Figure 1

Figure 2

Après avoir pris le temps de consulter les réponses des autres groupes, affichées sur les murs, Une discussion plénière portant sur ce qui se dégageait et sur les éléments surprenants a permis l'émergence d'idées intéressantes :

- Pour plusieurs, il est important que les étudiants aient assez confiance pour oser essayer.
- Un aspect du mode de pensée des mathématiciens est d'être paresseux (dans un bon sens) pour être efficace. La notation est compacte. L'enseignant doit souvent faire le travail inverse, et décompacter le tout.
- On fait une distinction entre les modes de pensée des apprenants des mathématiques et ceux des créateurs de mathématiques.
- Il a été question de l'intuition, qu'un groupe avait répertoriée comme stratégie. Est-ce une attitude? Ultimement, elle mène à des actions.
- L'importance du jeu a été soulignée : elle implique une certaine perspective, une approche des problèmes qui réduit l'anxiété.

Il n'y avait aucune intention d'élaborer une définition concise de ce que l'on entend par modes de pensée mathématiques, ni de parvenir à un consensus sur ce que cette notion pourrait comprendre. Au contraire, l'objectif était de sensibiliser à la gamme de ce que nous pourrions ou devrions communiquer, enseigner ou partager avec nos étudiants, en particulier avec les futurs enseignants, dans nos classes de mathématiques. La journée s'est terminée avec la promesse de s'engager, lors de notre deuxième séance, dans des tâches qui pourraient nous aider à mieux cerner ce que pourrait comprendre cette notion, et on a invité les participants à réfléchir aux types d'activités pouvant aider à développer ces habitudes chez les futurs enseignants du primaire.

JOUR 2

Après une activité de mise en train animée par Frédéric, nous avons débuté nos travaux. L'objectif principal de la première partie de la séance était de permettre aux participants de remarquer leurs propres modes de pensée mathématiques en résolution de problèmes. Cinq problèmes ont été présentés (voir ci-dessous) et les participants ont été invités à travailler en groupes de tailles quelconque sur un des problèmes, à leur choix. On les a invités à considérer les éléments suivants tout en travaillant :

1. Quels modes de pensée mathématiques observez-vous en vous-même alors que vous abordez ces problèmes ?
2. Qu'est-ce qui manque aux listes produites lors de la première séance ? Ajoutez, modifiez ou soulignez au besoin. Qu'est-ce qui est particulièrement important ?
3. Comment un enseignant en formation initiale répondrait-il à ce problème ou à un problème semblable ?
4. Est-ce qu'une activité du genre de celle que vous faites présentement peut appuyer le développement de modes de pensée mathématiques chez les enseignants en formation initiale, ou les aider à leur accorder de l'importance?

LES PROBLÈMES

Problème 1 : La diagonale d'un rectangle

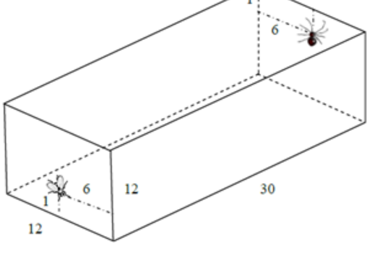
> Comme devoir à la maison, on a demandé aux élèves de dessiner la diagonale d'un rectangle. Erin utilisé du papier calque pour tracer la diagonale. Cependant, quand elle est venue à la classe, elle n'avait que le papier calque avec la diagonale qu'elle avait tracée. Dessinez un rectangle pour lequel la diagonale tracée par Erin est possible.

Problème 2 : Le problème des casiers

> Dans un couloir d'un collège, il y a 1000 casiers, numérotés de 1 à 1000. Au début, tous les casiers sont fermés. Le premier élève entre et ouvre tous les casiers. Le deuxième élève entre et ferme un casier sur deux, en débutant par le deuxième. Puis, le troisième étudiant entre et ferme ou ouvre les casiers, de 3 en 3, fermant un casier s'il est ouvert, et l'ouvrant s'il est fermé. Le quatrième étudiant répète le processus, en allant aux casiers de 4 en 4. Cela continue jusqu'au millième étudiant, qui ne change l'état que du casier 1000. Quels casiers sont encore ouverts ?

Problème 3 : L'araignée et la mouche

> Une araignée et une mouche sont sur les parois opposées d'une chambre rectangulaire, tel que sur la figure. L'araignée veut atteindre la mouche. En supposant que l'araignée doit se déplacer sur les surfaces de la pièce, quel est son plus court chemin jusqu'à la mouche? Attention! Le chemin le plus court mesure moins de 42!



Problème 4 : Chevaliers de la Table Ronde

> Le roi Arthur a une fille et veut lui faire épouser un de ses chevaliers. Il le choisira en leur demandant de s'asseoir à la table ronde. Il dira au premier chevalier, « Vous vivez. » Il dira au deuxième chevalier, « Vous mourrez » et le tuera, et laissera vivre le troisième, tuera le quatrième, et ainsi de suite autour de la table ronde jusqu'à ce il y ait seulement un chevalier demeurant en vie. Où doit s'asseoir un chevalier pour être celui qui épousera la fille du roi ? (Cela dépendra du nombre de chevaliers. Pouvez-vous trouver une formule que le chevalier pourra utiliser ?)

Problème 5 : Tour de magie

> Une vingtaine de pièces de deux couleurs (jaune d'un côté, rouge de l'autre) sont placées sur une table par le magicien. On demande à une personne de faire ce qui suit, pendant que les autres regardent et que le magicien est retourné. Tourner une pièce à la fois (pour que la couleur exposée change), autant de fois qu'il le souhaite, en disant clairement « je tourne » à chaque fois qu'il tourne une pièce (sans dire laquelle ou sa couleur). Puis, couvrir une pièce de son choix. Le magicien peut alors regarder. Il dévoile, sans se tromper, si la face supérieure de la pièce couverte est jaune ou rouge. Comment fait-il ?

DISCUSSION DES PROBLÈMES

Après avoir travaillé sur les problèmes pendant environ une heure, le groupe s'est réuni pour formuler des observations sur l'expérience et répondre aux questions proposées.

Les éléments suivants sont ressortis lors de la discussion portant sur les modes de pensée mathématiques personnels :

- Être sceptique quant à une solution possible a amené d'autres membres du groupe vers une justification plus précise. Le groupe a remarqué divers degrés de justification, citant Mason : vous convaincre, convaincre un ami, convaincre un ennemi.
- Les connaissances antérieures, notamment mathématiques, ont joué un grand rôle. Il est important de se demander ce que l'on sait déjà. Si on manque d'information, il faut savoir quoi demander.
- Parfois, il peut y avoir des intuitions conflictuelles.
- Certains ont ressenti de la curiosité, le goût de persévérer, le désir d'aller plus loin, et aussi de la frustration, la nécessité de changer de stratégie.
- Des approches multiples ont enrichi la compréhension. Le problème de la diagonale a été abordé en utilisant géométrie, pliage, algèbre et logiciels.
- Pour le problème des chevaliers, certains ont procédé par induction : essais de petits nombres, recherche d'un patron. D'autres ont travaillé en groupe, se divisant les cas à vérifier, émettant des hypothèses et les vérifiant.
- Plusieurs personnes parlent de persistance. Les élèves abandonnent souvent facilement, alors qu'ici, si certains participants décrochaient par moments, ils s'attaquaient à nouveau au problème par la suite. On remarque que cela prend une certaine confiance en soi, en ses chances de réussite, pour revenir au problème. En fait, prendre une pause et revenir au problème est souvent bénéfique. Le travail intellectuel se poursuit même lorsqu'on est engagé dans d'autres activités. L'horaire scolaire, souvent compartimenté de manière rigide, ne permet pas facilement aux élèves de prendre du recul et de revenir au problème. Des telles opportunités sont importantes.

Pour les enseignants en formation initiale, il apparaît important de créer un climat qui permet d'explorer en toute confiance. L'anxiété peut inhiber la réflexion. Des membres du groupe ont expliqué avoir ressenti une certaine anxiété au début de l'activité, une insécurité quant à leurs aptitudes en mathématiques relativement aux autres personnes de leur groupe. Le fait de pouvoir choisir quel problème aborder, avec qui et où travailler, leur a permis (et permettrait aux enseignants en formation initiale) de se sentir plus sécures. Discuter explicitement le « blocage », comme Mason (2010) le fait, et en particulier les émotions en jeu et les stratégies pour aller de l'avant, serait utile. Il est aussi important de démontrer ce qu'est un blocage, et il faut donc accepter d'être bloqué devant nos étudiants. Un enseignant qui a vécu des expériences positives en prenant des risques dans la résolution de problèmes pourra davantage prendre des risques dans son enseignement et offrir la chance à ses élèves de faire de même.

Pouvoir choisir les problèmes peut aussi aider à susciter l'intérêt et l'engagement des futurs enseignants, et leur permet de choisir ce qu'ils ont trouvé plus intéressant.

Après la pause, nous avons abordé une nouvelle thématique, pour aller au-delà de résolution de problèmes. Mary a donné une brève présentation de la recherche de Carol Dweck (2006; 2008) portant sur *l'état d'esprit*[3] et sur ses liens avec l'enseignement et l'apprentissage en général, et avec l'enseignement et l'apprentissage des mathématiques en particulier. Les travaux de Dweck ont été davantage diffusés dans le domaine de l'enseignement des mathématiques grâce aux travaux de Jo Boaler (2013). Dans des écrits récents, Boaler s'appuie sur des recherches de pointe en neuroscience portant sur la plasticité du cerveau, et qui indiquent que le cerveau peut croître et se modifier de manière importante. Ces recherches sont en concordance avec le travail de Dweck portant sur les liens entre l'état d'esprit et la réussite. C'est par les travaux de Boaler que Mary a été amenée à porter une attention aux liens forts entre l'état d'esprit et les modes de pensée mathématiques.

C'est en 2006 que Dweck, une psychologue bien connue de l'Université Stanford dont les travaux de recherche portaient sur le succès et la réussite, a publié le livre *Mindsets : The new psychology of success*. La prémisse de Dweck est que nos capacités ne sont pas seules responsables de notre réussite, et que c'est davantage notre état d'esprit à propos de ces capacités qui est déterminant. Dans ses recherches, elle a identifié deux états d'esprit : un état d'esprit fixe (*fixed mindset*) et un état d'esprit orienté vers le développement de soi (*growth mindset*, que l'on désignera par *état d'esprit en développement* dans la suite). Selon ses travaux, aux États-Unis, 40% des étudiants ont un état d'esprit fixe, 40% ont un état d'esprit en développement, et 20% ont un profil mixte.

Les gens ayant un état d'esprit fixe croient que leurs capacités sont innées et ne peuvent changer. Ils croient davantage au génie mathématique, et croit que les apprenants sont de deux types : ceux qui ont la bosse des maths et ceux qui sont nuls en maths. L'effort ne fait pas vraiment partie de l'équation. L'intelligence est statique. Les travaux de Dweck montrent que les personnes ayant un état d'esprit fixe veulent paraître en contrôle, intelligentes, et évitent les situations qui risquent de les montrer comme étant moins capables. Ils peuvent éviter les défis, abandonner facilement sans faire d'effort, ignorer des critiques utiles, et se sentent menacés par le succès des autres.

D'autre part, Dweck fait valoir que les personnes ayant un état d'esprit en développement pensent que leurs capacités se développent grâce à leurs efforts et à l'expérience acquise. Ces gens considèrent que l'intelligence peut se développer grâce à l'effort, ont tendance à considérer les revers comme une opportunité pour apprendre et les défis à relever comme partie intégrante du processus, apprennent de la rétroaction, et sont inspirés par le succès des autres.

[3] L'expression anglaise *mindset* est traduite par *état d'esprit*.

En 2008, Dweck a préparé un article pour la Carnegie Corporation of New York-Institute for Advanced Study Commission on Mathematics and Science Education intitulé *Mindsets and Math/Science Achievement*. Dans cet article, elle soutient que les étudiants qui considèrent que leurs capacités en mathématiques ou en sciences est fixe sont nettement désavantagés par rapport à ceux qui les considèrent comme pouvant être améliorées. Elle présente des résultats de recherche qui appuient la conclusion suivante :

> *Les états d'esprit peuvent prédire l'évolution du rendement en mathématiques et en sciences au fil du temps; les états d'esprit peuvent contribuer aux écarts de rendement en mathématiques et sciences pour les femmes et les minorités; les interventions qui modifient l'état d'esprit peuvent améliorer le rendement et réduire les écarts; les éducateurs jouent un rôle important dans la formation de l'état d'esprit des étudiants.* (p. 2)

Nous avons discuté des quatre recommandations qui constituent la seconde moitié de l'article de Dweck : 1) comment les éducateurs peuvent transmettre un état d'esprit en développement aux étudiants ; 2) comment enseigner aux éducateurs un état d'esprit en développement et les manières de le communiquer aux étudiants ; 3) comment faire comprendre aux femmes et aux membres de minorités que les sous-performances passées sont issues de l'environnement et non de facteurs génétiques, et peuvent être surmontées avec un appui accru du milieu éducatif et par un engagement personnel envers son propre apprentissage ; et 4) comment intégrer la compréhension de l'importance de l'état d'esprit dans les évaluations dont les enjeux sont importants ainsi que dans l'enseignement préparant à ces évaluations. Nous avons conclu la journée en discutant des implications des résultats de ces recherches et des recommandations qui sont formulées pour la formation des enseignants.

JOUR 3

Nous avons débuté notre dernière journée avec une brève présentation de Frédéric d'un article de Hyman Bass (2011), *A Vignette of Doing Mathematics : A Meta-cognitive Tour of the Production of Some Elementary Mathematics*. Dans cet article, Bass présente une description personnelle de ce que veut dire « faire des maths » dans un contexte général, suivie d'une description de travail mathématique réel sur un problème d'arithmétique élémentaire (division d'un gâteau).

Cette description du travail mathématique tel qu'il est fait par des mathématiciens touche certaines des idées explorées dans le groupe de travail. Cette description (où le « nous » réfère à « nous, mathématiciens ») se divise ainsi :

Nous ...

- Questionnons
- Explorons
- Représentons, éventuellement de plus d'une manière.
- Cherchons de la structure, ce qui peut conduire à des conjectures ou de nouvelles questions.
- Consultons, si nous sommes bloqués : les autres, la littérature, le web.
- Établissons des liens avec d'autres mathématiques, grâce à nos recherches, à notre réflexion, à des analogies.
- Cherchons une preuve, pour confirmer ou infirmer nos conjectures. Pour ce faire, on subdivise souvent la tâche, par exemple en formulant ou en démontrant des résultats ou conjectures liés, que l'on espère plus accessibles, et en montrant que l'on peut en déduire la conjecture principale.

- Sommes parfois opportunistes, laissant les mathématiques nous guider si nous voyons des pistes intéressantes.
- Démontrons, en rédigeant la preuve (si on y arrive), en essayant de bien présenter les principales idées tout en respectant les normes de rigueur mathématique, et en ciblant un auditoire spécifique.
- Analysons les preuves, qui contiennent généralement davantage que le résultat prouvé.
- Utilisons notre sens esthétique et nos goûts ; on pense à l'élégance, à la précision, à la clarté, à la cohérence, à la cohésion.

Les participants ont ensuite été invités à travailler en petits groupes pour considérer des activités qu'eux ou d'autres ont utilisées en enseignement, les modes de pensée mathématiques touchés ou utilisés dans celles-ci, et les caractéristiques des activités qui le permettaient.

Les participants se sont réunis en cinq groupes, selon le niveau d'enseignement : élémentaire (2 groupes), secondaire (2 groupes) et tertiaire (1 groupe). La discussion nous a mené jusqu'à la pause du matin, après laquelle nous nous sommes engagés dans une discussion en plénière, prenant comme appui de brefs rapports de chacun des groupes. C'était une discussion vivante et fluide, dont nous présentons certaines idées.

PRIMAIRE

Les deux groupes ont présenté des actions spécifiques les enseignants peuvent prendre pour promouvoir le développement des modes de pensée mathématiques chez leurs élèves. En particulier :

- Les enseignants peuvent (et devraient) être des modèles en utilisant eux-mêmes des modes de pensée mathématiques. Ils peuvent jouer le rôle d'un apprenant : trébucher, faire des efforts pour comprendre, reconnaître les émotions en jeu. Ils peuvent reconnaître le courage requis pour poser des questions et demander de l'aide.
- Ils peuvent modéliser certaines habitudes spécifiques et être explicites à ce sujet : par exemple, la généralisation et la pensée itérative (procédant par révisions successives de ce qu'on a obtenu).
- Les enseignants peuvent rendre la réflexion visible, flexible et publique. L'utilisation d'outils qui favorisent cette expression et le développement de modes de pensée mathématiques est cruciale. Annette et Michelle ont décrit leur utilisation des surfaces verticales dans la classe, qu'elles utilisent au maximum, par exemple en utilisant un rideau de douche blanc et des marqueurs lavables Crayola. Donner aux élèves un moyen de rendre visible leur pensée permet aux conversations mathématiques de se produire.
- Les enseignants peuvent favoriser une réflexion sur la localisation des mathématiques lorsque l'on travaille en groupe, en faisant des activités qui amènent à réfléchir au travail en équipe et aux modes de pensée en présence. Dans un groupe, une idée proposée n'appartient pas à une personne, mais bien au groupe. Discuter de cette idée ne revient pas à dire si elle est bonne ou mauvaise, mais plutôt à discuter en vue de mener le groupe vers la réussite. Il faut oser remettre les mathématiques en question (sans que cela ne soit une remise en question de la personne).

Un groupe mentionne qu'une caractéristique clé des activités qui promeuvent des modes de pensée mathématiques est de favoriser les relations entre les individus.

Alors que l'un des groupes de niveau élémentaire a noté que l'anxiété pourrait entraver le développement de modes de pensée mathématiques, un autre participant a noté que l'anxiété ressentie par les futurs enseignants pouvait être une opportunité pour en discuter. Cela a amené à se questionner sur les différents types d'anxiété, notamment chez les futurs enseignants du primaire et du secondaire : ceux du primaire peuvent ressentir davantage d'anxiété quant au contenu, et une prise en compte des modes de pensée mathématiques peut attirer leur attention sur d'autres aspects importants de l'apprentissage des mathématiques; ceux du secondaire peuvent se sentir davantage en maîtrise du contenu, et ressentir de l'anxiété face à la proposition de ne pas faire ce qui a fonctionné pour eux dans le passé. Dans les deux cas, il faut se préoccuper de l'anxiété des apprenants dans le choix des activités visant à développer des modes de pensée mathématiques.

On a également remarqué que les enfants possèdent déjà certains modes de pensée mathématiques, et que l'enseignement doit en favoriser le développement et non pas chercher à les implanter.

Comment changeons-nous? Il peut être nécessaire de voir ce genre de travail dans la réalité d'une classe, et pas uniquement dans le cadre d'un programme de formation des enseignants. Il n'est possiblement pas suffisant de modéliser cela dans un cours, mais bien de voir des pairs le faire dans un contexte réel.

SECONDAIRE

Les groupes considérant le niveau secondaire ont également identifié les caractéristiques d'activités favorisant le développement de modes de pensée mathématiques. Il devrait y avoir une trame narrative et des éléments de jeu, de curiosité et de réflexion. Elles devraient permettre de se demander ce qui est possible, de se questionner sur ce qui est pertinent : ignorer certains aspects, en prioriser d'autres. On doit s'engager dans les idées mathématiques. On peut choisir des problèmes dont le niveau de complexité rend une solution par simple calcul trop longue ou complexe, et pour lesquels on peut rechercher une structure sous-jacente.

Parmi les activités spécifiques discutées en groupe, on mentionne le problème du pot de beurre d'arachides, présenté dans un cours de calcul différentiel et intégral. *Comment repenser la forme d'un pot de beurre d'arachides pour éviter de se retrouver avec un couteau trop court et des mains salies lorsque l'on veut vider le pot ?*

On a également fait valoir qu'il devait y avoir des mathématiques en tant que telles, et que les modes de pensée mathématiques pouvaient se développer en faisant des mathématiques sans les subordonner à des activités ou des applications, qui peuvent détourner de l'idée principale poursuivie.

Comme pour les groupes portants sur le primaire, les participants ont souligné que mettre de l'avant les modes de pensée mathématiques impliquait d'exprimer clairement ses décisions en enseignant. Au secondaire, l'enseignement est fluide, alors que souvent le travail mathématique ne l'est pas : les élèves ne le savent peut-être pas. Parmi les autres points de discussion, on note les suivants.

- Il est important que les étudiants remarquent les modes de pensée mathématiques de leurs pairs.
- Ce genre de travail permet aux étudiants de prendre conscience de leurs propres habitudes. S'ils sont motivés et ont de la détermination, ils peuvent être en mesure de les changer.

- Bien que le fait de voir des mathématiques dans différents contextes soit utile, il est important que ces contextes soient accessibles aux étudiants.
- L'histoire, dans laquelle on peut voir les efforts, la persévérance et la réussite, peut être utile.

Les participants ont évoqué plusieurs cadres de réflexion potentiellement utiles dont : l'heuristique de Polya, l'esprit mathématique de Mason, et l'adolescence et les maths de Ann Watson (2006).

Une ligne d'argumentation, pour tous les ordres d'enseignement, est que ce n'est pas tant l'activité en elle-même qui importe, mais la manière dont on travaille ; ce qu'on souligne et ce que l'on passe sous silence. Par exemple, un enseignant peut ne pas effectuer le calcul arithmétique, mais souligner quel calcul il ferait : c'est une décision, celle de se concentrer sur la structure au lieu de la réponse. Nous pouvons faire ressortir les modes de pensée mathématique dans presque n'importe quelle activité mathématique, bien que certaines activités soient mieux adaptées pour faire ressortir certains modes de pensée en particulier.

TERTIAIRE

La venue du groupe ayant travaillé sur l'enseignement supérieur dans la discussion a permis d'ajouter de nouveaux éléments à celle-ci. Les modes de pensée suivants ont été soulignés.

- Précision – dans la communication, l'argumentation, la lecture, les définitions, les déductions.
- Nous tenons à ce que nos étudiants puissent se concentrer et s'engager à fonds, avec énergie et en ayant un but, une direction.
- Aptitude à utiliser les connaissances dans différents contextes.
- Distinguer ce qui est pertinent et ce qui ne l'est pas.
- S'autoréguler.

Une activité a été présentée par Nilima Nigam. Elle demande à ses étudiants de lire de la poésie et des journaux. Elle veut les amener à élargir leur vision des maths, à les sortir du cadre dans lequel ils ont enfermé les maths. Elle veut les amener à penser précisément au monde qu'ils habitent. Il est important qu'ils ne se dissocient pas du reste du monde et de leurs expériences personnelles. Un participant souligne que c'est important à tous les niveaux : nous voulons que les élèves apprennent à vivre mathématiquement. Même les enseignants au niveau primaire peuvent parler aux enfants de regarder le monde avec des « lunettes mathématiques ».

On remarque aussi qu'il est nécessaire de réformer les habitudes intellectuelles de plusieurs étudiants au niveau tertiaire. Ils entrent dans les cours compensateurs avec un certain état d'esprit, leurs stratégies ayant fonctionné jusque-là. Si on vous a toujours dit quoi faire, vous avez perdu votre autonomie. Il est essentiel de proposer des activités qui permettent des choix, de l'exploration, et permettent de prendre des décisions. Nos étudiants sont issus de systèmes ayant mis l'accent sur le contenu.

CONCLUSION

En conclusion de notre groupe de travail, nous avons demandé aux participants ce qu'ils avaient appris ou ce qui ressortait pour eux à la suite de nos discussions. Nous présentons certains de leurs commentaires, partiellement édités et regroupés par thèmes.

LES MODES DE PENSÉE MATHÉMATIQUES

- Inspirer, découvrir, développer, encourager, croître, etc.
- Les modes de pensée mathématiques sont de nature globale et se manifestent partout en didactique des mathématiques.
- Les modes de pensée mathématiques sont accessibles à tous, et leur développement peut et doit être améliorée pour tous les élèves, individuellement et collectivement, contribuant ainsi à leurs vies et à notre société.
- La compréhension peut être fugace même si elle semble limpide. Les modes de pensée mathématique peuvent aider à rendre la compréhension visible, naturelle, authentique, et transformatrice : la permanence du « savoir-faire ».
- Dans un sens large, peut-être qu'ils s'insèrent dans le cadre théorique de la métacognition.
- Je n'avais jamais vu les maths (enseignement ou application) sous l'angle des modes de pensée mathématiques. Je sens que ma participation à ce groupe m'a aidée à mettre des mots sur ce que je conçois être « faire des maths ». Les modes de pensée mathématiques forment ce qu'est l'activité mathématique. La résolution de tâches mathématiques en gardant en tête quels sont nos propres modes de pensée mathématiques aide à réaliser comment se produit mon activité mathématique et comment je fais des maths, ce qui peut être très utile à exploiter avec les étudiants de tout niveau scolaire.
- J'ai surtout pensé à deux choses. Tout d'abord, décrire les modes de pensée mathématique comme étant de mathématiser le monde, ce que nous avons abordé brièvement dans le groupe de travail. Deuxièmement, que le contexte global est ontologique : être et devenir dans le monde. En fait, ici, c'est que nous avons tous la possibilité d'être mathématiquement dans le monde.
- Tout est mode de pensée... peut-être. Une manière très utile de penser à l'enseignement et à l'apprentissage des mathématiques. Exposer les modes de pensée implicites, ou à tout le moins être à leur recherche sera plus facile... peut-être... compte tenu de l'expérience de ces derniers jours.

CE QUE LES ENSEIGNANTS ONT BESOIN DE SAVOIR SUR LES MODES DE PENSÉE MATHÉMATIQUES

- Le besoin de créer une communauté ou un environnement qui permette et favorise l'expression de ce qui amène à prendre des décisions, faisant ainsi ressortir des caractéristiques des modes de pensée, souvent innées et implicites, et rende ainsi les modes de pensée mathématiques plus explicites et davantage valorisés.
- Les modes de pensée mathématiques ne sont pas à enseigner comme s'il s'agissait d'un item à cocher dans une liste ; ils sont déjà en chacun de nous. Le rôle d'un enseignant est de mettre l'emphase sur certains aspects, d'en ignorer d'autres, afin de soutenir le développement des modes de pensée mathématiques chez les élèves et de les amener à apprécier les bénéfices découlant de leur usage régulier.
- Il est nécessaire de fournir aux étudiants des activités riches et de les encourager à reconnaître leurs modes de pensée mathématiques.
- L'importance de rendre explicite ce qui est implicite.
- Enseigner, c'est décompresser vos expériences en tant qu'apprenant des mathématiques.
- En tant qu'enseignants, nous avons la liberté de choisir comment démontrer nos modes de pensée mathématiques, mais les étudiants n'ont pas nécessairement le choix de ce qu'ils reçoivent de nous. Donc ... nous devons être conscients des modes de pensée mathématiques que nous démontrons, et afin de démontrer leur grande utilité, nous avons besoin de les posséder nous-mêmes, d'en être conscient et de les comprendre. Ensuite, nous pouvons vraiment les vivre et influencer les étudiants en

tant que modèle. Ce n'est certainement pas un processus facile, mais il en vaut la peine.
- Mettre l'accent sur la créativité nous incite à penser aux modes de pensée mathématiques.
- Les enseignants de tous les niveaux peuvent favoriser ces habitudes chez leurs élèves.

CE QUE LES APPRENANTS ONT BESOIN DE SAVOIR SUR LES MODES DE PENSÉE MATHÉMATIQUES

- Travaillez à développer vos modes de pensée mathématiques. Vous ne travaillez pas pour faire face à vos lacunes. Si c'est un « problème », être bloqué fait partie du processus normal.

QUELS SONT LES ENJEUX?

- Le plus grand défi est de soutenir les enseignants dans leur capacité à reconnaître et valoriser les modes de pensée mathématiques des enfants.
- Nous (au post-secondaire) avons tendance à trop nous concentrer sur l'enseignement du contenu et à déplorer le fait que nos étudiants ne développent pas comme on le souhaite une certaine manière de penser, d'écrire, de s'exprimer, d'analyser ou de réfléchir.

D'AUTRES QUESTIONS

- Je repars avec davantage de questions… ce qui me plait. La question essentielle : est-ce que presque tout est un mode de pensée mathématique potentiel, qui est productif ou pas selon le contexte?
- Qu'est-ce qui différencie les modes de pensée mathématiques des modes de pensée en général? Est-ce une même entreprise humaine, qui varie selon les contextes? Je suis interpellé par des questions ontologiques et je vais accorder plus d'importance à la manière dont je peux aider mes étudiants à reconnaître les modes de pensée mathématiques.

Comme facilitateurs du groupe de travail, nous sommes ressortis de nos trois jours de discussion avec énergie et inspiration, davantage en contact avec les multiples nuances recélées dans le concept de modes de pensée mathématiques, et pleinement conscients des défis et du potentiel d'un travail explicite sur les modes de pensée mathématiques avec les futurs enseignants. Nous tenons à exprimer notre profonde gratitude pour la contribution de chacun des membres du groupe. Tous ont partagé idées et expériences, et se sont engagés dans des conversations constructives et respectueuses, enrichies par la variété de leurs contextes, de leurs bagages et de leurs intérêts, pour le bénéfice de chacun.

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FORMATIVE ASSESSMENT IN MATHEMATICS: DEVELOPING UNDERSTANDINGS, SHARING PRACTICE, AND CONFRONTING DILEMMAS

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INTRODUCTION

Formative assessment has been shown to be a strong lever in improving student achievement, particularly for students who are struggling (Black & Wiliam, 1998). Yet, interpretations of what formative assessment is vary greatly (Shepard, 2005). In this working group we collectively explored the meanings and understandings of formative assessment. Through group sharing of research and classroom artifacts (video, audio, records of assessment evidence, transcripts, blogs) we considered a range of formative assessment practices used to elicit, record, and respond to students' mathematical thinking. Issues, challenges and dilemmas that arise in the use of formative assessment were discussed, as well as suggestions to address these challenges.

The working group was guided by the following questions and we addressed these through group sharing of both research and practice.

- What meanings are given to formative assessment?
- In what ways do formative assessment practices best support student learning?
- What does formative assessment look like? With different students? At different grades?
- What dilemmas do teachers face in incorporating formative assessment?
- How can formative assessment be supported?

While these questions were suggested to guide the group, the discussion often took different directions based on the interests of the group.

WHAT MEANINGS ARE GIVEN TO FORMATIVE ASSESSMENT?

Opening graffiti wall activities provided space for participants to share their understandings of formative assessment as well as their goals in participating in the formative assessment working group.

Participants' goals were varied and spanned from practical to epistemological. Figure 1 provides part of the graffiti wall where participants posted their goals in joining this working group. Goals included:

- Going deeper in how epistemology influences instructional and assessment practices;
- To reconcile formative assessment with reporting demands of 'the system';
- Learning more about questions and concerns of teachers with respect to formative assessment;
- Understanding different conceptions of assessment across Canada;
- How to use formative assessment effectively in classrooms;
- How to gather and use the information gathered through formative assessment.

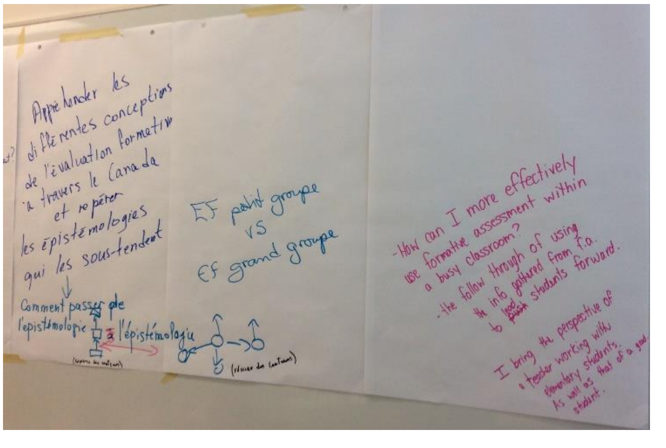


Figure 1. Responses to: "Why did you want to be part of the Formative Assessment Group? What do you hope to gain from being in the group?"

Participants also provided descriptions of formative assessment (see Figure 2) that included: evidence of student learning that is used to inform teaching; evaluation *au service de l'apprentissage*; a dialogue between teachers, students, and learners to foster learning; and giving and receiving feedback.

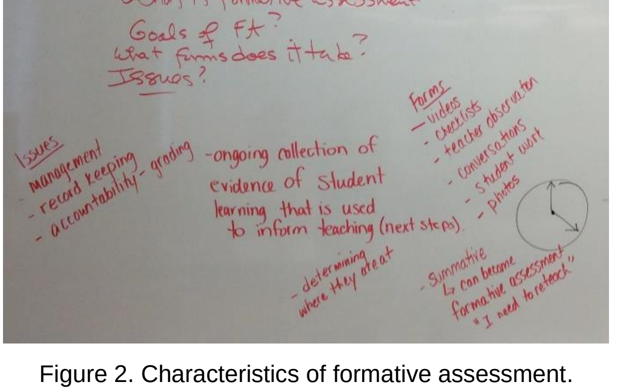


Figure 2. Characteristics of formative assessment.

Participants also posted several purposes of formative assessment: to promote learning and adapt teaching, such as determining where students are in their learning and *vise à guider l'apprentissage et à reconnaître le rôle formateur de l'erreur*. They reported on a variety of forms that formative assessment could take: video recordings, checklists, teacher observations, conversations, student work, photographs, etc.

Participants reported several challenges with enacting formative assessment, many of which were closely aligned with their goals in participating in this working group: finding ways of record keeping; determining grades and being comfortable with accountability for those grades; finding time to effectively gather assessment evidence and provide meaningful feedback; communicating new assessment methods with students, parents, and other teachers; and working with evidence of student learning that is qualitative to determine a numeric grade for a report card.

To continue the conversation about definitions and descriptions of formative assessment, participants worked in small groups and were given a range of definitions of formative assessment from a variety of sources:

- "An assessment is formative to the extent that information from the assessment is fed back within the system and actually used to improve the performance of the system in some way" (Wiliam & Leahy, 2007, p. 31).
- It appears to be widely accepted that Michael Scriven was the first to use the term *formative*, to describe evaluation processes that "have a role in the on-going improvement of the curriculum" (Scriven, 1967, p. 41). He also pointed out that evaluation

  > *may serve to enable administrators to decide whether the entire finished curriculum, refined by use of the evaluation process in its first role, represents a sufficiently significant advance on the available alternatives to justify the expense of adoption by a school system.* (pp. 41-42)

  suggesting "the terms 'formative' and 'summative' evaluation to qualify evaluation in these roles (p. 43)" (as cited in Wiliam, 2014, p. 2).
- *Since the purpose of formative assessment is to guide decisions about how to help learning, the next steps for pupils cannot be planned in detail until the evidence is collected and interpreted. Although teachers' experience will enable them to be prepared for the kinds of ideas and skills that they are likely to find, the implementation of formative assessment allows for decisions to be taken on the basis of how pupils respond to their learning activities*. (Harlen, 2007, p. 123)
- *Formative assessment is defined as assessment carried out during the instructional process for the purpose of improving teaching or learning. [...] What makes formative assessment formative is that it is immediately used to make adjustments so as to form new learning*. (Shepard, 2008, p. 281)

Through discussion of these definitions and their own research and experiences each small group created a working definition of formative assessment. The working definitions were posted (see Figure 3) and discussed. Many of the groups focused on formative assessment as an ongoing process. They saw formative assessment as part of a dynamic system of interactions of the teacher, students, and curriculum (or mathematical ideas).

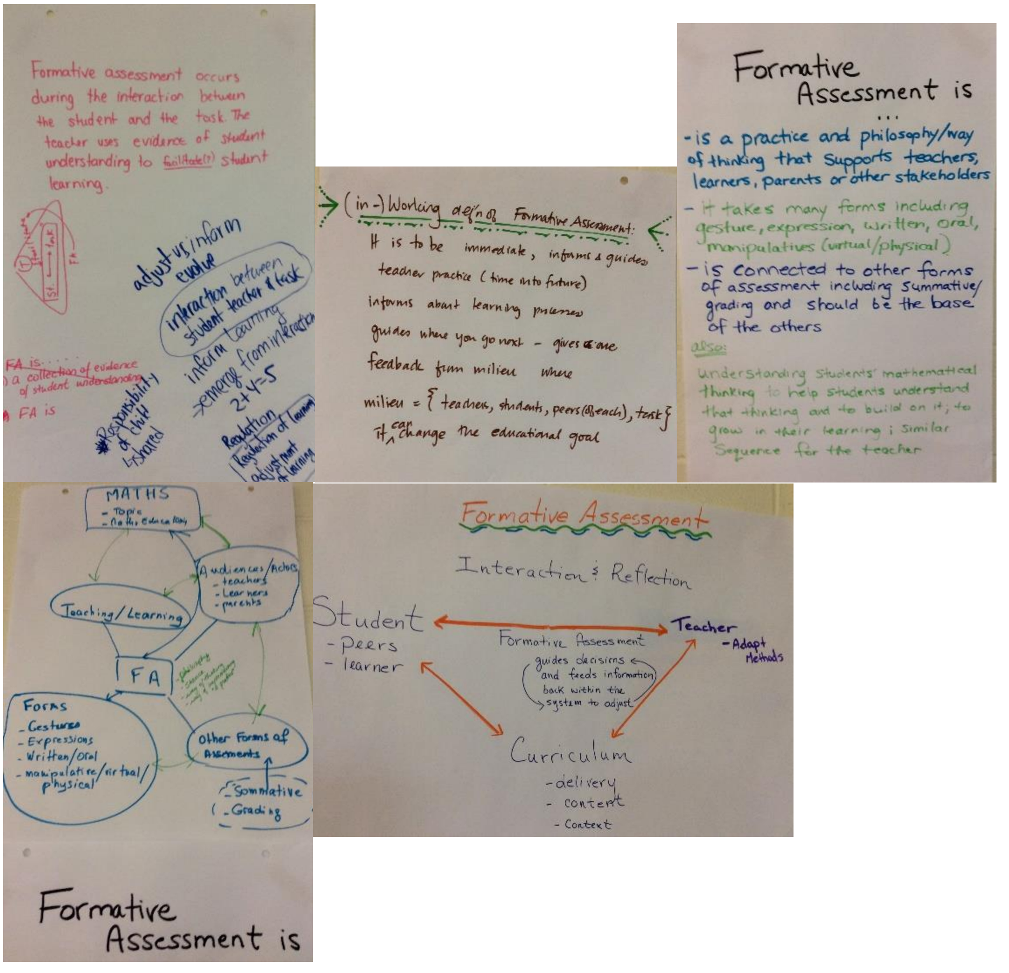


Figure 3. Working definitions of formative assessment.

WHAT DOES FORMATIVE ASSESSMENT LOOK LIKE IN PRACTICE?

Our focus then shifted to sharing and examining formative assessment practices. We began by discussing formative assessment as attention to listening and responding to students' mathematical thinking. Davis (1997) distinguishes between three types of listening: (1) evaluative listening which seeks to hear a 'correct answer', (2) interpretive listening which seeks to 'make sense' of what the other person is saying, and (3) hermeneutic listening which values deep understanding rather than convergence. We engaged in the practice of listening to student thinking by working in pairs and viewing short video clips taken from the work of Carpenter, Franke, and Levi (2003) of students explaining their thinking. Each clip provides an opportunity to listen to upper elementary students explaining their thinking as they solve an algebraic problem. The viewing of the videos highlighted the importance of listening to student thinking and being aware of the multiple ways that problems can be solved. Our discussion emphasized that written evidence of student thinking does not suffice if educators are interested in understanding the ways that students think about and solve problems. The explanations that students gave provided much more information about student thinking. Discussion ensued about how these opportunities to listen to student thinking might occur in mathematics classrooms.

Question posing and providing prompts that elicit student thinking are also important components of formative assessment. The working group discussed a case study of a teacher who used rich problem solving with Grade 8 students. As students worked in pairs on the problem, the teacher circulated and elicited and supported student thinking. We examined the following transcript and discussed the types of questions that the teacher used to probe student thinking and to make it visible.

Teacher: Ok, so can you explain this to me?

Kareem: Well, we thought that 4 frogs and 5 fairy godmothers were of the same strength, so we divided 4 by 5, for a rate of what it takes for one frog to beat a fairy godmother.

Teacher: What made you think of unit rates?

Kareem: Uh, because we wanted [*inaudible comment*] a number.

Teacher: What did you find out when you did that? [...] How many frogs were there?

Kareem: Four.

Teacher: And how many fairy godmothers were there?

Kareem: Five.

Teacher: So, if you…you said you divided 4 frogs by 5 fairy godmothers…

Kareem: Ya. […] So, we got 0.8, because that's how much…

Brian: That's how many fairies there are…

Kareem: So, basically, one frog could beat a fairy because the frog is stronger. One whole frog is equal to….Or not. One fairy godmother is equal to…

Teacher: Ok, why does that make sense?

Kareem: Well, they tied when it was 4 frogs, but it was 5 fairy godmothers. It was equal. So, we wanted to find out what it was for one frog. So, we divided 4 by 5 to figure out what was the 4 out of 5 [*inaudible*] to find out what could beat one fairy godmother.

Participants described the different purposes of the questions. For instance, "Ok, so can you explain this to me?" helps to make the thinking visible. "What made you think of unit rates?" helps to provide appropriate terminology to what the students are describing. "Ok, why does that make sense?" seemed like a question that could be used by a teacher in many circumstances to help students explain their thinking in more depth, or to help them connect their thinking to other experiences. We discussed different frameworks for categorizing questions (e.g. Boaler & Brodie, 2004) and discussed the categories of questions that elicit student thinking.

To further our discussion of formative assessment practices, several participants shared their work, which included research on formative assessment and formative assessment practices in classrooms. Several group members discussed how they used a variety of iPad applications such as one that created an interactive whiteboard space and audio recording of students working on a problem. In this way, the recording captured the students thinking as they described what they were doing, as well as the recording of their written solution (as it happened). Several members of the group (e.g. Teresa Raslyk, Doris Duret, Anna Taplin, Sandy Bakos) are classroom teachers or have worked with classroom teachers who have used such applications with their students. Many of these teachers incorporate these recordings in an electronic portfolio for each student that might contain images of written work, audio recordings on conversations and conferencing with the student, video recordings of the student explaining their thinking, or other evidence of students' mathematical thinking. Manon Leblanc shared formative assessment strategies that she has observed in a Grade 3 classroom that encourages students to self-assess. Steven Pileggi presented his use of quizzes as formative feedback in teaching an adult algebra class. Martha Koch shared her research on formative assessment that indicated that teachers are incorporating formative assessment in their practice to varying degrees. The research demonstrated teacher practices that included not-for-grades quizzes, such as clicker quizzes and collaborative open-book quizzes. She also shared examples of teachers using summative assessments in formative ways. This led to a discussion about the idea that it is not the assessment tool that is distinctly formative or summative but the way it is used that is summative or formative.

WHAT DILEMMAS ARE POSED WHEN INCORPORATING FORMATIVE ASSESSMENT?

Dilemmas and issues regarding formative assessment arose throughout the three days that the working group met. The dilemmas and challenges included:

- Helping parents understand new approaches to assessment;
- Incorporating formative assessment in very large classes (60+);
- Finding time for appropriate formative assessment and time to cover the curriculum;
- Dealing with institutional constraints and policies;
- Knowing what to look for, what questions to ask;
- Relationship between teachers' mathematical understanding and recognition of next steps for student thinking;
- Keeping records of observations and other formative assessment evidence of student learning;
- Deciding which data to use and how to use it to best support learning;
- Developing students' ability for self-assessment and determining actions for improvement;
- Letting go and including the student in the assessment process;
- Developing prospective teachers' expertise in formative assessment;
- Respecting differentiation and inclusivity in the assessment process;
- Relationship between formative and summative assessment;
- How to use formative assessment effectively within a busy classroom.

In our discussion of the dilemmas and challenges that are faced we looked at the framework developed by Windschitl (2002) that has been applied to assessment dilemmas (Suurtamm & Koch, 2014). This framework talks about four types of dilemmas: conceptual, pedagogical, cultural, and political. The following table outlines the dilemma types:

Dilemmas	Explanation
Conceptual	Grappling with current thinking in assessment and mathematics teaching and learning; considering the 'why' of assessment
Pedagogical	Grappling with the creation of assessment tasks, strategies, and tools; dealing with the 'how to' of assessment
Cultural	Focus on changes in classroom and school culture with regard to assessment practice; often arise when new assessment practices threaten existing cultural practices
Political	Dealing with school, district, or provincial policies on classroom and large-scale assessment that may or may not align with teachers' assessment thinking and practices

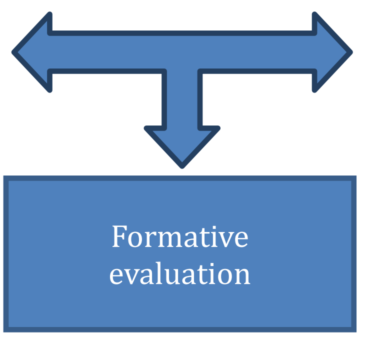
We considered how some of the dilemmas that the group discussed might align with some of the dilemma types. For instance, understanding the relationship between formative and summative assessment might be considered a conceptual dilemma. Developing strategies for keeping records of observations might be considered a pedagogical dilemma. Helping parents understand new assessment practices could be considered a cultural dilemma. Political dilemmas might include such things as grappling with institutional and administrative constraints. The advantage of thinking about the different dilemma types is that each dilemma type might be addressed in a different way. For instance, cultural dilemmas can best be supported through increased communication and recognizing that shifting perspectives and beliefs takes time. Pedagogical dilemmas, on the other hand, are often dealing with 'how to' issues and might be able to be addressed by sharing resources or through a workshop. However, it is also important to recognize that the dilemma types are interwoven. In other words, one cannot prescribe policy without understanding that it might have pedagogical and cultural dilemmas.

WHAT UNDERSTANDINGS DID WE TAKE AWAY?

At the end of our three days together, participants summarized a variety of ideas. One group of participants developed a diagram to illustrate their understandings of formative assessment (see Figure 4).

Others expressed that formative assessment takes into account the interactions of students, tasks and mathematics. Formative assessment provides evidence of student understanding through a variety of strategies of questioning, listening, and responding or providing feedback. Some participants described formative assessment as similar to providing an opportunity for 'readjustment' or 'redirection'. Ideally formative assessment helps student learn to readjust their actions and understandings and to confirm their learning. Formative assessment takes a variety of different forms. It can be formal or informal, planned or spontaneous. It can include conversations with the student, observations and questioning, quizzes, peer feedback, or self-assessment. Formative assessment is a dynamic process that helps to provide feedback to the teacher but also can increase students' awareness of their own learning, their next steps.

- L'enseignant établit une relation entre les réponses et les programmes
- Souci de l'amélioration de la performance
- Niveaux pour apprécier la progression de façon linéaire
- Enseignant a le statut d'expert en mathématiques
- Conceptions de l'apprentissage des mathématiques est linéaire



- Curieux de ce qui se passe dans la «boîte noire»
- Souci de l'apprentissage
- Dialogue pour susciter des prises de conscience (success/understanding)
- Critères pour appréhender les réseaux de connaissances
- Développer un sentiment de pouvoir chez l'élève
- Enseignant a le statut d'expert en mathématiques et dans le processus d'apprentissage
- Redéfinit ce qu'est enseigner comme un partage de la responsabilité
- Conception de l'apprentissage des math en réseaux

Figure 4. Understanding of Formative Assessment
(by Leblanc, Savard, Braconne-Michoux et Duret).

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TEXTER MATHÉMATIQUE

TEXTING MATHEMATICS

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This working group report is based on text and notes contributed by Richard Barwell, Jean-François Maheux and the participants.

TEXTER MATHÉMATIQUE?

What does it mean to read and write mathematics? How do we read and write mathematics? What can we say about how it is done? We understand 'reading' and 'writing' broadly to include not only the interpretation and production of mathematical texts as conventionally understood (historic or contemporary, in textbooks or students' work, etc.), but also the ephemeral texts of spoken words and gestures, the visual texts of moving or still images and, ultimately, any of the semiotic 'traces' we come across in doing mathematics.

D'une certaine manière, le langage et les mathématiques nous précèdent : nous naissons dans un monde où les mots et les idées des autres pénètrent tout ce ne faisons, y compris des mathématiques. Ainsi, nous lisons et écrivons des mathématiques à travers les mots des autres, dans un langage ayant une histoire, comme le note David Wheeler dans le rapport du Groupe de travail Mathematics and Language *de 1983* (Wheeler, 1983). *Ce besoin et cette histoire font en sorte que tous les textes mathématiques sont liés d'une manière ou d'une autre, à travers le temps et l'espace. La géométrie d'aujourd'hui, par exemple, porte les traces d'Euclide et de Descartes. Et dans n'importe quelle classe de mathématiques, les échanges s'inscrivent dans le temps, chaque conversation étant marquée par des références, des citations, des échos de discussions antérieures, tant au sein de la classe que dans les* *mathématiques comme discipline. Notre idée pour ce groupe de travail était d'explorer les liens entre les textes produits quand on fait des mathématiques.*

For this working group, we therefore considered exploring links and connections in/between the texts that arise in doing mathematics. *Nous pensions à la fois observer des produits déjà existants de l'activité mathématique (d'autrui), mais surtout mettre nos participants en position de produire eux-mêmes des textes que nous pourrions ensuite examiner ensemble, à la recherche de liens, de relations, de traces d'autres textes, etc.* The main components of the working group thus included:

- Working on mathematics problems offered by the group leaders. Participants were invited to bring and use mobile devices to record their work. The group leaders also collected some footage of participants working.
- Examination of extracts of recordings to explore the links and connections that emerged within and between texts.
- Discussion of theoretical prompts on the topic of intertextuality, as well as historical examples.

POINTS DE DEPART

Participants were offered several problems to work on. Various materials were provided, including flipchart paper, a whiteboard, pens, etc. We asked participants to bring electronic devices and several groups worked partly or entirely on tablets or laptops. During this initial activity, we asked each group to record their working in some way. Some used their phones or tablets to film themselves, while one group used recording software to capture their voices and what they were doing on the screen simultaneously. Jean-François also filmed some of the work of each group.

Parmi les problèmes proposés aux participants, voici les énoncés qui ont retenus leur attention :

1. How many different triangles can you make on a circular pegboard that has nine pegs? Or any number of pegs? (Source: NRICH (n.d.) – see http://nrich.maths.org/2852/note for a version of the problem as well as an interactive java applet for working on it).
2. Make some maths with Figure 1 (from Dudeney's (n.d.) *Amusements in Mathematics*). The image was projected onto a screen.

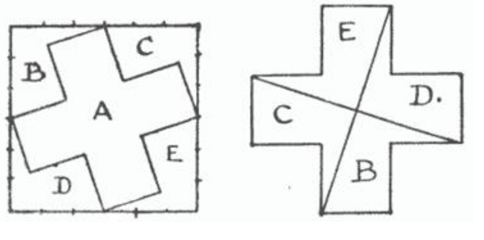


Figure 1. The images proposed for Problem 2.

3. The Pericut problem: it concerns a semicircle which has a point, Q, on the diameter, and smaller semicircles constructed on the diameter touching Q and the endpoints of the diameter. Given a point, P, on the circumference of the initial semicircle, a ray is constructed through Q. The problem is to explore what happens to the lengths of the two parts of the perimeter formed by the ray through PQ. Figure 2 shows one

participant's construction of the diagram. (Source: NRICH (n.d.) – see http://nrich.maths.org/276 for an online interactive version of the problem.)

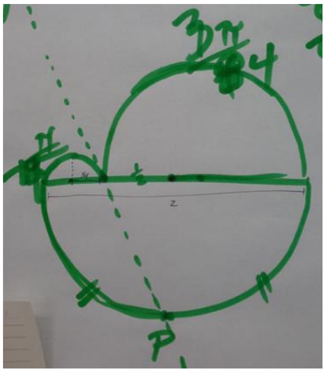


Figure 2. One group's sketch of the Pericut problem.

All three problems prompted rich mathematical activity. For example, a group consisting of Barbara and Lixee worked on Problem 3. Their sketch is shown in Figure 2. Here is Barbara's account of some of their work:

> What Lixee and I explored with the Pericut problem: Initially we had a conversation using the printed problem as a focal point. We talked and pointed as to what we thought would happen when the dot on the bottom moved around the bottom semi-circle. We also wondered what would happen if the two top circles were the same. We made our own sketches and then predicted. Since it is hard to have the accuracy we wanted, we turned to the technology to explore further our conjectures. We confirmed out initial hunches. Next we wondered if the dot joining the two upper small circles would move up and down the intersecting line. We also wondered if the circles on top were exactly the same would the same results be generated. Using the technology, our own drawings and the problem, we went back and forth between.

David, Elaine, Yasmine and Lyla worked on Problem 2. Here is Yasmine's account of what they did:

1. Selection criterion: least likely to require dynamic texting (i.e., software-based texts) to be solved. We selected the upright cross/tilted cross in squares.
2. Not having a particular 'word text' question, we decided to see what we could see/solve using the drawings and the figures.
3. Having our problem (text?) far up on the smart board and not having our perceived useful texting tools (e.g., graphing paper and scissors), we found it difficult to re-draw the figures on our papers. We tried to make graphing paper, to fold paper and to draw approximate figures.
4. Drawing additional lines and texts on an 'approximation of the figures' helped us see the relation between the area of the whole square and the cross in it. The number 18 came up frequently! (See Figure 3)
5. Not being sold on some properties of the figures (i.e. the connection between the "1/6 of the line" point and the "1/2 of the line" point of the square side) and not being able to draw the shapes nicely, moved us to use a more dynamic texting environment.… This, in turn, helped us see other things that we couldn't see as easily as on the computer screen! … (i.e., one side of the tilted cross really connected the 1/6 point to 1/2 point of the square side! Or at two points the area of the cross is half the area of the whole square and at one point the area of the cross is zero).

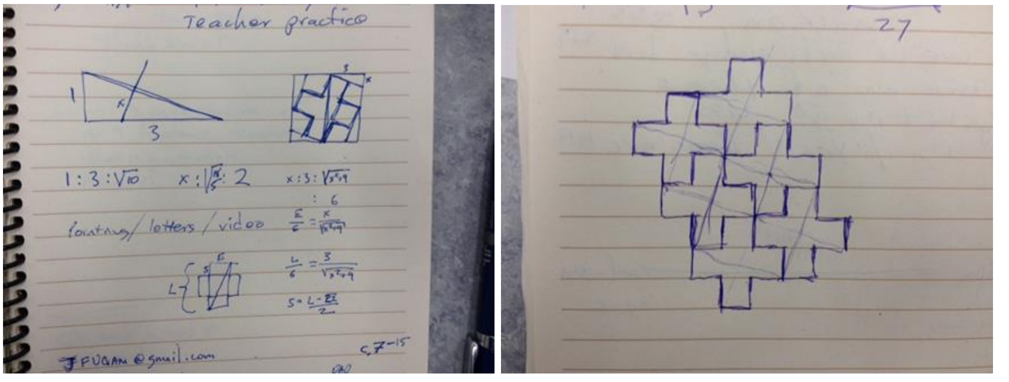


Figure 3. Texts produced by one group while working on Problem 2.

LAYER 1: COLLECTIVE ANALYSIS OF A TEXT

Jean-François and Richard selected a short extract from a video-recording of one of the groups (Petra, Billie-Dawn and Carmen) working on Problem 1 at a whiteboard. *Ce problème a été choisi par plusieurs équipes, et travaillé par l'une d'elle directement sur le tableau blanc. Cette équipe a chercher à identifier les différents polygones qu'il est possible d'obtenir en reliant des points distribués régulièrement sur un cercle. La mise en commun du travail de chaque équipe autour des traces laissées par l'équipe ayant travaillé au tableau a été particulièrement intéressante.* Figure 4 shows the whiteboard as it was at the end of the session.

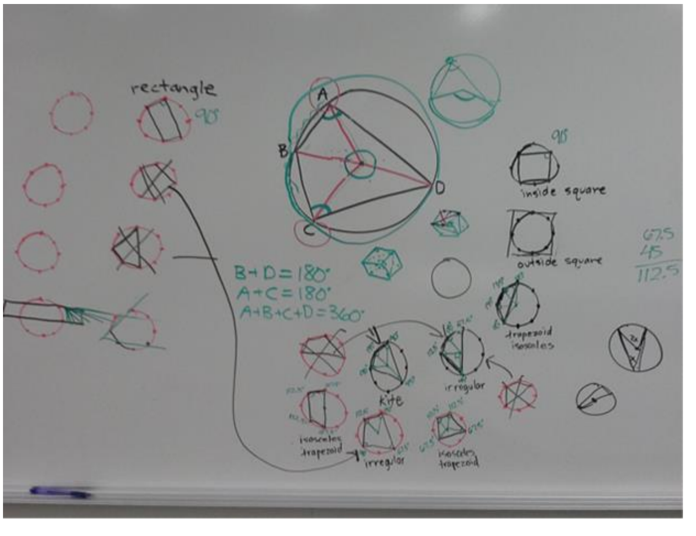


Figure 4. Petra, Billie-Dawn and Carmen's work on the circular geoboard problem.

The discussion of the video clip of a group working on Problem 1 was rich and detailed. We began by observing and describing what we could see. Here is Petra's account of the group's work:

> Our group (Billie Dawn, Carmen, Petra) worked on the quadrilateral problem. We chose to work on the whiteboard. Carmen and Billie Dawn drew replicas of the circle with 8 evenly distributed points along the circumference. Carmen became the scribe and we discussed together how to find all possible quadrilaterals using the diagrams we had. We soon found a method that worked and so we started recording

> the quadrilaterals in the little circles we had. One circle for every quadrilateral. Mid-way through we started to label the quadrilaterals with *square*, *rectangle*, *isosceles trapezoid*, etc. Working together went really well. We came together fairly quickly, understanding each other readily. Most of our discussion was based on the diagrams we had in front of us. We even commented afterwards, that there seemed to be a common language among each other that was only disrupted twice: once when the method seen by Petra and Billie Dawn was unclear to Carmen, the scribe, but that was resolved fairly quickly; and then a second time, when we were stuck trying to convince ourselves that we found all possible quadrilaterals. Once the problem was solved, we even started to extend it by posing some extra questions. The only time we labelled the points around the circle, was when we proceeded to prove one of our claims. Otherwise, our mathematical discourse did not seem ridden with symbols and formalism. Having had the opportunity to view a taping of our work, it seemed like the whiteboard was an important 'fourth person' in our group that had an equal part in our mathematical journey.

Gradually, as we reconstructed the sequence of the group's work, more detail emerged. In particular, participants in the plenary viewing of the video began to attend to the detail of how different diagrams were constructed and how different inscriptions were related through the participants' interaction. A key point of interest was the role of the whiteboard as a point of reference, and the associated role of gestures for pointing at inscriptions on the whiteboard. For example, participants viewing the video became interested in a moment where Petra is pointing at two places on the whiteboard at once, one with each hand. This moment seemed to be significant in the development of the group's thinking and Petra's pointing served to highlight a connection between the two parts of the board and so to create new meaning. This discussion left Petra with the following questions:

> How much interplay is there between the different texts (written, spoken, visual) in an individual, in the group?
>
> What happens at the moment of discovery/understanding? What type of text was prevalent or necessary? Or perhaps rather, what about the text facilitated/elucidated the discovery/understanding?

The discussion of this extract prompted participants to reflect on their own texting during their initial work on one of the problems. These reflections in turn served to generate questions and issues to be explored in subsequent sessions.

In their reflections on these various activities, participants' observations including the following:

- There's lots of redundant text—this reflects a division of labour, but sometimes we swapped roles.
- "I needed to think with the numbers."
- There were different ways of producing text—forming and reforming, arrowing and associating, 'informing', reforming not abstracting or composing or decomposing. Visualising geostrips. Rewriting and redrawing. Naming and renaming.
- Redrawing is linked to finding relationships; reorganising to find a nice layout.
- Systematisation—different entries into a coherent predictable future. Language of 'discerning' or 'making'.
- We were always going clockwise. Hopping. Limited to 360…could go more. We needed a notation of movement.
- We noticed notation-switching.
- On the board: immediately had to redraw the large circle. Used a lot of visual text. Had to come up with common language. Tightening up the language to exclude some cases 'outside squares'. Rereading the problem multiple times. Labelling came after

a few quadrilaterals. The interaction often flowed really well….but then our own descriptions were not so smooth. Discussion was really tied to these 8 dots.

- Diagrams became our talking point. We all had access to the diagrams.
- We needed to add a layer of colour. We redrew bits and pieces.
- Do children get to redraw multiple times?
- At the end we were all drawing at the same time.
- I couldn't just watch someone draw without a pen in my hand.
- There was a movement between a focus on our own thing and sometimes a focus on others' ideas. We all had our own scraps of paper. Sometimes there was a diagram in the middle of the table. So we were alternating between functioning as a group and individual activity. How did we establish that reconnection—how do I make things visible to other people when it's so obvious to me?

SOME QUOTATIONS ABOUT TEXT / QUELQUES CITATIONS À PROPOS DE TEXTES

At this point, to further enrich the discussions, our participants were invited to send us quotations from the literature that they felt spoke to the issues we had been discussing. These quotations were compiled and circulated to participants. Some time was allocated to discussing the thoughts prompted by these quotations. Here is a selection:

> *To language is to interact structurally. Language takes place in the domain of relations between organisms in the recursion of consensual coordinations of consensual coordinations of actions, but at the same time language takes place through structural interactions in the domain of the bodyhoods of the languaging organisms. In other words, although languaging takes place in the social domain as a dance of recursive relations of coordinations of actions, interactions in language as structural interactions are orthogonal to that domain, and as such trigger in the bodyhoods of the participants structural changes that change as much the physiological background (emotional standing) on which they continue their languaging, as the course that this physiological change follows. The result is that the social coordinations of actions that constitute languaging, as elements of a domain of recursive operation in structural coupling, become part of the medium in which the participant living systems conserve organization and adaptation through the structural changes that they undergo contingent to their participation in that domain. ... As the body changes, languaging changes; and as languaging changes, the body changes. Here resides the power of words. Words are nodes in coordinations of actions in languaging and as such they arise through structural interactions between bodyhoods; it is through this interplay of coordinations of actions and changes of bodyhood that the world that we bring forth in languaging becomes part of the domain in which our ontogenic and phylogenic structural drifts take place.* (Maturana, 1988, section 9.5)

> *Every word is directed toward an answer and cannot escape the profound influence of the answering word that it anticipates…. Forming itself in an atmosphere of the already spoken, the word is at the same time determined by that which has not yet been said but which is needed and in fact anticipated by the answering word. Such is the situation in any living dialogue.* (Bakhtin, 1981, p. 280)

> *But if thought corrupts language, language can also corrupt thought.* (Orwell, 1946, paragraph 20)

> *Now, my co-mates and brothers in exile,*
> *Hath not old custom made this life more sweet*
> *Than that of painted pomp? Are not these woods*

> *More free from peril than the envious court?*
> *Here feel we but the penalty of Adam,*
> *The seasons' difference, as the icy fang*
> *And churlish chiding of the winter's wind,*
> *Which, when it bites and blows upon my body,*
> *Even till I shrink with cold, I smile and say*
> *'This is no flattery: these are counsellors*
> *That feelingly persuade me what I am.'*
>
> *Sweet are the uses of adversity,*
> *Which, like the toad, ugly and venomous,*
> *Wears yet a precious jewel in his head;*
> *And this our life, exempt from public haunt,*
> *Finds tongues in trees, books in the running brooks,*
> *Sermons in stones, and good in every thing.*
> *I would not change it.* (Shakespeare, 1599, *As You Like It*, 2.1.548-565)

> *Comprendre [...] est attraper le geste et pouvoir continuer.* (Cavaillès, 1938/1981, p. 178)

> *The limits of my language means the limits of my world.* (Wittgenstein, 1921/1922, p. 150)

> *Je pense en fait avec la plume. Car ma tête bien souvent ne sait rien de ce que ma main écrit.* (Wittgenstein, 1956/2002, p. 27)

> *Many examples in this book have to do with failing to see, and the final chapter explores one of the deepest strata of vision, the complicity between blindness and sight. Recent medical experiments have shown that a great deal of vision is unconscious: we are blind to certain things and blind to our blindness. Those twin blindnesses are necessary for ordinary seeing: we need to be continuously partially blind in order to see. In the end, blindnesses are the constant companions of seeing and even the very condition of seeing itself.* (Elkins, 1996, p. 13)

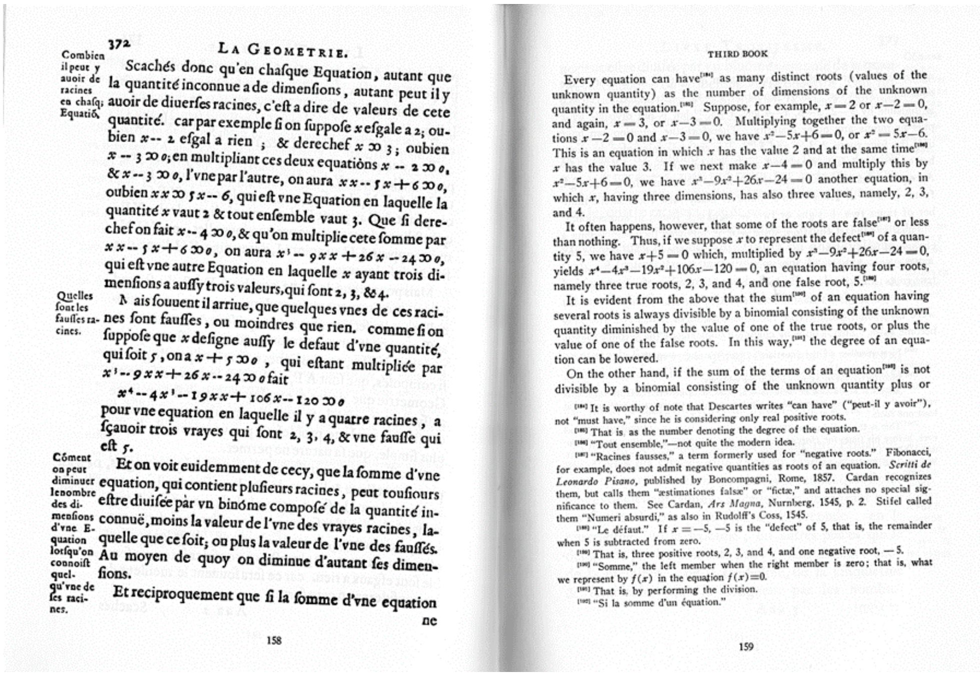
> *A blind spot is an absence whose invisibility is itself invisible…there are objects in every scene that we don't see – both psychologically and also physiologically.* (Elkins, 1996, p. 219)

Ceci a été le prétexte à une discussion autour de l'activité mathématique et de l'observation de celle-ci en termes de texte.

HISTORICAL TEXTS

Participants were also offered two historical texts and invited to read and comment on them. *L'un de ces documents était composé de quelques pages du troisième livre de* La geométrie *de Descartes, présentant de part et d'autre le texte en ancien français, et une traduction en anglais moderne de chacun des paragraphes. Il fut intéressant de constater combien les traductions (linguistique, mathématique) transformaient le texte, ajoutant parfois à l'écriture mathématique, par exemple, non seulement au niveau symbolique, mais aussi au niveau sémantique. D'autre part, le travail même de lire un texte mathématique (ancien) est apparu comme assez exigeant… sollicitant bel et bien une activité mathématique de la part du lecteur.*

Figure 5 shows one of the pages of Descartes' *Geometry* and its translation, where it is easy to see the transformation of mathematical texts through the ages, how this plays in our reading of old texts. But we can also see, reading the proposed translation, how re-writing mathematics is a special kind of mathematical work, which in the end does not eliminate the necessity for the reader to mathematically engage with the text. A translation here not only goes across centuries, but also from a language (French) to another (English). One way or another, (the same?) mathematical ideas live through those very different texts. Or come to life through our reading thereof.



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Combien il peut y auoir de racines en chasq; Equatiõ.

Scachés donc qu'en chasque Equation, autant que la quantité inconnue a de dimensions, autant peut il y auoir de diuerses racines, c'est a dire de valeurs de cete quantité. car par exemple si on suppose x esgale a 2; oubien $x - 2$ esgal a rien; & derechef $x \propto 3$; oubien $x - 3 \propto 0$; en multipliant ces deux equations $x - 2 \propto 0$, & $x - 3 \propto 0$, l'vne par l'autre, on aura $xx - 5x + 6 \propto 0$, oubien $xx \propto 5x - 6$, qui est vne Equation en laquelle la quantité x vaut 2 & tout ensemble vaut 3. Que si derechef on fait $x - 4 \propto 0$, & qu'on multiplie cete somme par $xx - 5x + 6 \propto 0$, on aura $x^3 - 9xx + 26x - 24 \propto 0$, qui est vne autre Equation en laquelle x ayant trois dimensions a aussy trois valeurs, qui sont 2, 3, & 4.

Quelles sont les fausses racines.

Mais souuent il arriue, que quelques vnes de ces racines sont fausses, ou moindres que rien. comme si on suppose que x designe aussy le defaut d'vne quantité, qui soit 5, on a $x + 5 \propto 0$, qui estant multipliée par $x^3 - 9xx + 26x - 24 \propto 0$ fait

$$x^4 - 4x^3 - 19xx + 106x - 120 \propto 0$$

pour vne equation en laquelle il y a quatre racines, a sçauoir trois vrayes qui sont 2, 3, 4, & vne fausse qui est 5.

Cõment on peut diminuer le nombre des dimensions d'vne Equation lorsqu'on connoist quelqu'vne de ses racines.

Et on voit euidemment de cecy, que la somme d'vne equation, qui contient plusieurs racines, peut tousiours estre diuisée par vn binóme composé de la quantité inconnuë, moins la valeur de l'vne des vrayes racines, laquelle que ce soit; ou plus la valeur de l'vne des fausses. Au moyen de quoy on diminue d'autant ses dimensions.

Et reciproquement que si la somme d'vne equation ne

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THIRD BOOK

Every equation can have[184] as many distinct roots (values of the unknown quantity) as the number of dimensions of the unknown quantity in the equation.[185] Suppose, for example, $x=2$ or $x-2=0$, and again, $x=3$, or $x-3=0$. Multiplying together the two equations $x-2=0$ and $x-3=0$, we have $x^2-5x+6=0$, or $x^2=5x-6$. This is an equation in which x has the value 2 and at the same time[186] x has the value 3. If we next make $x-4=0$ and multiply this by $x^2-5x+6=0$, we have $x^3-9x^2+26x-24=0$ another equation, in which x, having three dimensions, has also three values, namely, 2, 3, and 4.

It often happens, however, that some of the roots are false[187] or less than nothing. Thus, if we suppose x to represent the defect[188] of a quantity 5, we have $x+5=0$ which, multiplied by $x^3-9x^2+26x-24=0$, yields $x^4-4x^3-19x^2+106x-120=0$, an equation having four roots, namely three true roots, 2, 3, and 4, and one false root, 5.[189]

It is evident from the above that the sum[190] of an equation having several roots is always divisible by a binomial consisting of the unknown quantity diminished by the value of one of the true roots, or plus the value of one of the false roots. In this way,[191] the degree of an equation can be lowered.

On the other hand, if the sum of the terms of an equation[192] is not divisible by a binomial consisting of the unknown quantity plus or

[184] It is worthy of note that Descartes writes "can have" ("peut-il y avoir"), not "must have," since he is considering only real positive roots.

[185] That is, as the number denoting the degree of the equation.

[186] "Tout ensemble,"—not quite the modern idea.

[187] "Racines fausses," a term formerly used for "negative roots." Fibonacci, for example, does not admit negative quantities as roots of an equation. *Scritti de Leonardo Pisano*, published by Boncompagni, Rome, 1857. Cardan recognizes them, but calls them "æstimationes falsæ" or "fictæ," and attaches no special significance to them. See Cardan, *Ars Magna*, Nurnberg, 1545, p. 2. Stifel called them "Numeri absurdi," as also in Rudolff's Coss, 1545.

[188] "Le défaut." If $x=-5$, -5 is the "defect" of 5, that is, the remainder when 5 is subtracted from zero.

[189] That is, three positive roots, 2, 3, and 4, and one negative root, -5.

[190] "Somme," the left member when the right member is zero; that is, what we represent by $f(x)$ in the equation $f(x)=0$.

[191] That is, by performing the division.

[192] "Si la somme d'un équation."

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Figure 5. Une des pages de la *Géométrie* de Descartes.

LAYERS 2 AND 3: SMALL GROUP ANALYSIS OF TEXTS

Each group was asked to select a short extract from their recording. On the second day, each group received the selected extract of one other group and was invited to analyse it. They were also asked to record their analysis; using whatever electronic devices they might have to hand. *Le partage par chaque groupe du court extrait qu'ils avaient choisis dans le travail enregistré la veille a été l'occasion de poser un second regard sur l'activité mathématique à travers ces traces… en plus de fournir un nouveau texte à enregistrer: celui de l'interprétation et de la discussion de travail des autres. Voici les commentaires de Petra à ce propos :*

> Our group (Elaine, Lyla, Petra) analyzed the clip from a pair of participants that had worked on the quadrilaterals-in-circle problem. This pair labelled the quadrilaterals by the distance of points between edges. For example, the only possible square in the 8-point-circle is called 2222. The only possible rectangle is called 1313. The three minute segment was about a discussion the pair had on the possible permutations of 1124 and the meaning of the position of each number. From what we observed, they tried to visualize the permutation. The recording only included the voices of the pair and the drawings and writings but not the bodies, especially the hands that did the drawing, pointing and writing. It was interesting to observe

> that the discourse often gave clues to who did the writing and drawing: “from here to there” or at mark 16:03 and 16:45 (I didn't transcribe what they said there and only marked the time). Nonetheless, a question that stood out for our group is “Whose is the hand that draws? And does it matter?” There was also a strong need by both participants to underline, to circle, to label, to redraw, to segment the quadrilateral from the circle, which seemed to help them unravel the meaning of the position of each number in the permutation.

On the final day, new groups were formed and each group received an extract from a recording of another group analysing the extracts of one of the first groups working on one of the problems.

FINAL REFLECTIONS

At the end of the three days, we discussed new insights and awarenesses that had arisen during the activities of the working group, including the following points:

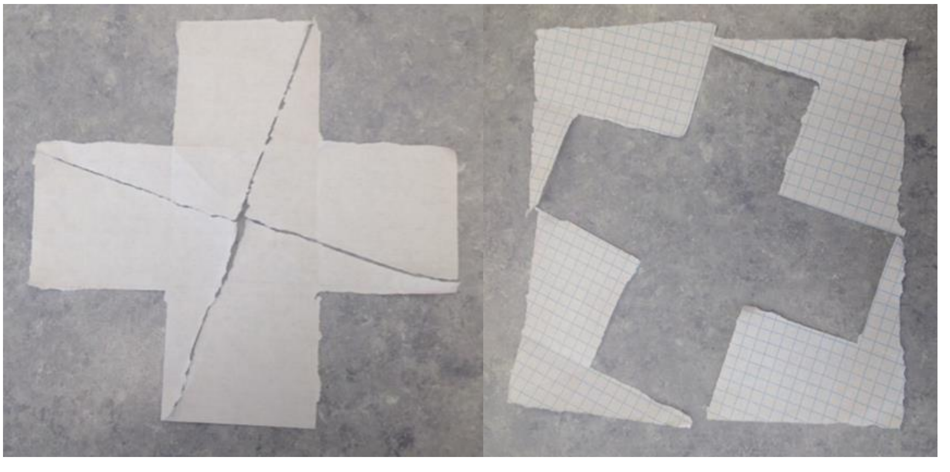
- I noticed the different ways in which individuals approach the same problem: the importance of the wording of the problem in the understanding of what the problem asks/requires.
- It was interesting to see similar ideas emerging in different groups and also to see different symbols/keywords emerging within group discussions as people converge on agreed meaning.
- *Text* means different things to different people. We are each approaching the conversation from a different direction.
- The medium of the text (software, pen and paper, ruler and graph paper) is part of the text.
- I was struck by the differences between the texts produced by others vs. texts produced by ourselves: it’s interesting to see how we produce different texts when working with the same problem.
- We seem to have or to gravitate to particular favourite textual forms…to lean on these more heavily than others…to return to them during times of least confidence. They anchor us ‘better’.
- I was taken by the action, motion, creation, etc. The mathematics was active, in motion, alive. I do love the verbs of it all.
- I became more aware of the interplay between fixed (given) diagrams, my own diagramming, dynamic representations, labelling, oral discussion and gesturing as we communicate.
- How broadly the term *text* can be interpreted: the conversations we are having are very similar to conversations I have had around complex systems thinking.
- It was sometimes more interesting to look at how text inter-acts with non-textual interventions than to look only at text.
- Visual text is just as alive as written or spoken text. It engages, it informs, it questions, it invents, …
- Having/not having different tools changed the text(s) produced.
- I am interested in the affordances that emerge with the ‘technology’ we have for our texting.

Participants’ were also asked for their questions at the end of the three days:

- How are the movements of the hand that produce text like those that produce diagrams?
- Do we communicate equally among different forms of text? When do we use one more than the other?

- What texts do we desire most as mathematical people? Why and how do certain texts satisfy our mathematical desires? As human beings, are there other kinds of desires that take different forms from our mathematical desires?
- What do we expect to see in a 'perfect' text?
- What do tools do in texting?
- What can our 'text' reveal about our understanding or the ways in which we perceive a problem?
- What are the classroom implications of this topic?
- How does an external artefact mediate languaging between people?
- When we produce a mathematical text, how do we include our thoughts and present them as valuable as the results we come up with?
- How do these favoured familiar textual modes come to be? How do they emerge? Where do we feel safest and why?
- If we structure group work in our classrooms with giving students specific roles, do we limit/deter their engagement in the task? I ask this because [during our problem solving] everyone needed to be active, write, draw etc.
- How is what we are focusing on in text different from when we are focusing on the emergence of a complex system?

Finally, the report of the working group to the closing session of the conference was in the form of a short video, which can be viewed at:
http://docsmaheuxjf.uqam.ca/Publications/CMESGvideoReport/ or
http://tiny.cc/CMESGvideoReport



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COMPLEX DYNAMICAL SYSTEMS

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INTRODUCTION

> *Complex systems are systems that comprise many interacting parts with the ability to generate a new quality of collective behavior through self-organization, e.g. the spontaneous formation of temporal, spatial or functional structures. […] This recognition, that the collective behavior of the whole system cannot be simply inferred from the understanding of the behavior of the individual components, has led to many new concepts and sophisticated mathematical and modeling tools for application to many scientific, engineering, and societal issues that can be adequately described only in terms of complexity and complex systems.* (Meyers, 2011, p. v)

> [Un système complexe est un ensemble constitué d'un grand nombre d'entités en interaction, ayant la capacité de générer un nouveau type de comportement collectif à travers une auto-organisation, incluant la formation spontanée de structures temporelles, spatiales ou fonctionnelles. […] La reconnaissance du fait que le comportement du système ne peut être simplement inféré de la connaissance des comportements des composantes individuelles a mené à plusieurs nouveaux concepts et outils sophistiqués de modélisation mathématique pouvant contribuer à l'étude de nombreuses problématiques scientifiques, sociales et technologiques, qui ne peuvent être décrites adéquatement qu'en termes de complexité et de systèmes complexes. (Translation of Meyers, 2011, p. v)]

Having 'cleared' the issue of defining complex systems with the inclusion of this workable definition, we made the choice of centering the working group on experiential learning. Through activities, simulations and games, the participants of this working group explored some models, concepts and approaches associated with the mathematics of complex dynamical systems and later analyzed their potential for inclusion in secondary and post-secondary mathematics.

CELLULAR AUTOMATA

We chose cellular automata as the entry point for our deliberations. Conceived in the 1940s, and popularized in 1970 through John Conway's (n.d.) *Game of Life*, a cellular automaton is an array of cells in which the state or behaviour of each cell in one generation of the array is determined by rules that involve the states of neighbouring cells in the previous generation. We started with the following game:

A STANDING GAME

Look around you and notice your 8 neighbours. At our signal, please do the following:

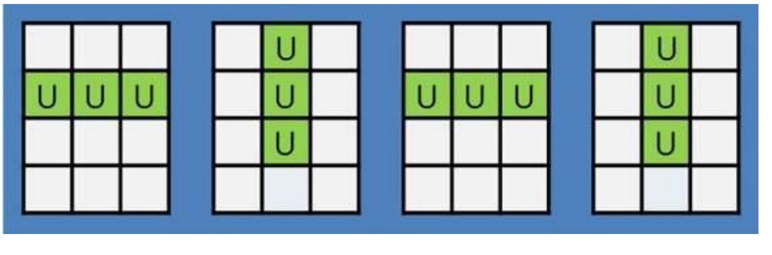
- If you are sitting and 3 of your neighbours are standing then STAND UP.
- If you are standing and 0 or 1 or 4 or more of your neighbours are standing, then SIT DOWN.
- If you are standing and 2 or 3 of your neighbours are also standing then remain standing.

Regardez autour de vous et observez vos 8 voisins. À notre signal, veuillez faire ce qui suit :

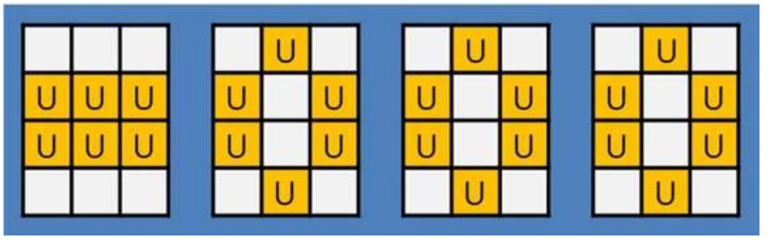
- Si vous êtes assis et que 3 de vos voisins sont debout, LEVEZ-VOUS.
- Si vous êtes debout et que 0 ou 1 ou 4 ou plus de vos voisins sont debout, ASSEYEZ-VOUS.
- Si vous êtes debout et que 2 ou 3 de vos voisins sont également debout, alors restez debout.

After extending these rules to include the participants who do not have eight neighbours, we tested different sets of initial conditions with a 3 × 4 grid of people and considered mathematical notions that emerged through the iterative process.

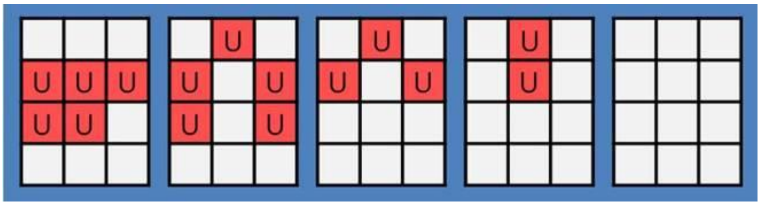
1. We started with three in a row standing and then played the game: we saw a cyclic solution, or periodic behaviour, develop rapidly.



2. We started with a 2 by 3 rectangle of people standing and then played the game: we saw the system stabilize to a non-trivial solution in one generation.



3. We started with a 2 by 3 rectangle with only one person sitting and then played the game: we saw the system evolve to the trivial solution in 4 generations.

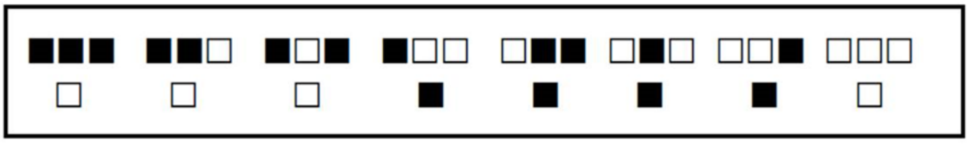


The rules for this game are exactly those used by Conway in his *Game of Life* (see http://www.bitstorm.org/gameoflife/). Of course, our physical setting offers constraints not present in an automated version of the game. Nonetheless we have a very accessible introduction to the importance of initial conditions on the evolution of a system under a given set of rules. As presented, this game only affords a limited view of the variety of possible evolutions of the system, and does not address more complicated states such as cyclic states of higher degree or those containing patterns that translate across the grid.

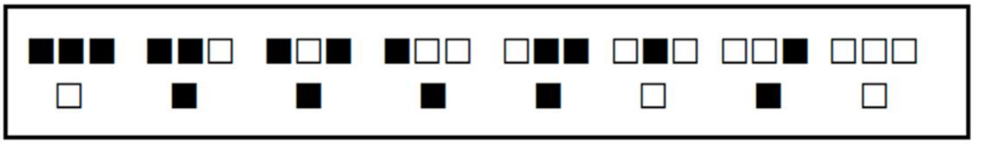
Participants focused their attention on their immediate neighbours and noted the difficulty of envisioning and appreciating the whole system while being within the system. This emphasizes the notion that “that the collective behavior of the whole system cannot be simply inferred from the understanding of the behavior of the individual components” (Meyers, 2011, p. v).

This activity does not readily offer a record of the evolution of the system across time. Indeed neither does the *Game of Life* except that ‘time’ can pass quickly in the simulations. So we moved to a pen and paper activity with one-dimensional cellular automata that offered a higher-level view of how systems might evolve across time.

Here the array of cells is a row rather than a grid. The state of each cell in one generation is determined by the states of the three cells above it in the previous generation according to rules expressed as per the diagrams below.



Rule 30

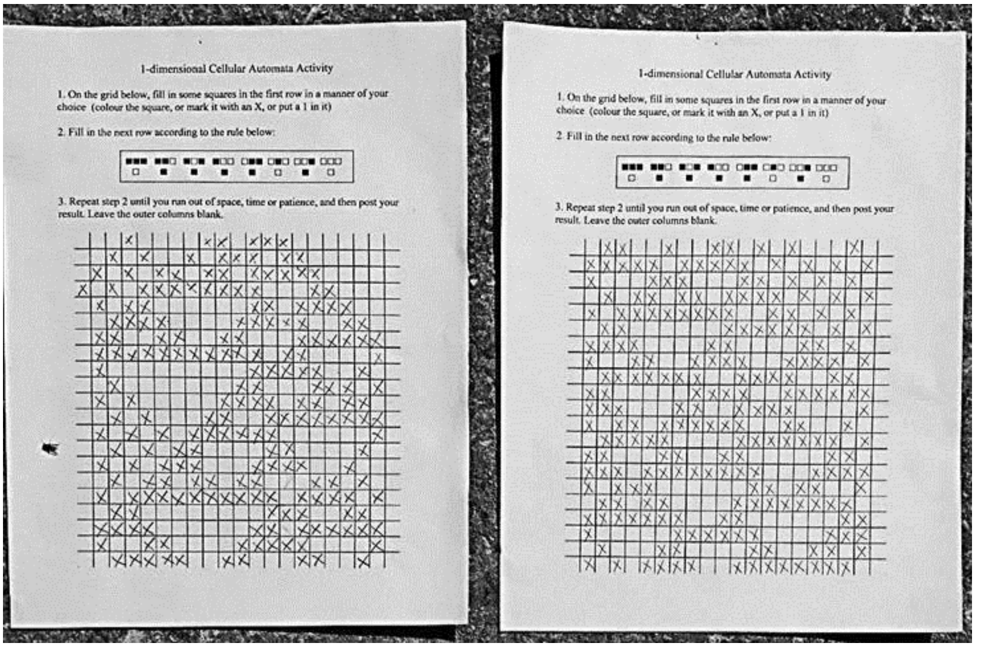


Rule 122

It is somewhat fun to note that the rules are named according the base 2 representation of numbers. Notice that with □ and ■ representing 0 and 1 respectively, the top row from right to left shows base 2 representation of the numbers 0 through 7. These are the exponents for the place values in base 2 representation. Rule 30 appears as it does because

$$30 = 16 + 8 + 4 + 2 = 2^4 + 2^3 + 2^2 + 2^1 = 11110_{\text{BASE 2}}.$$

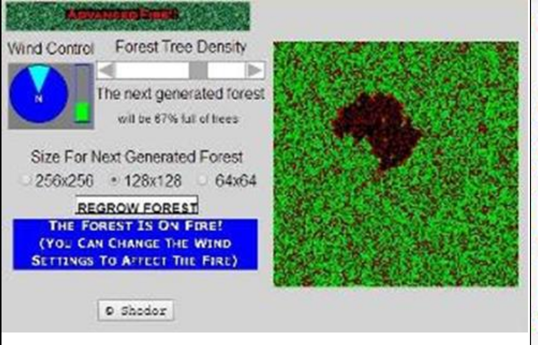
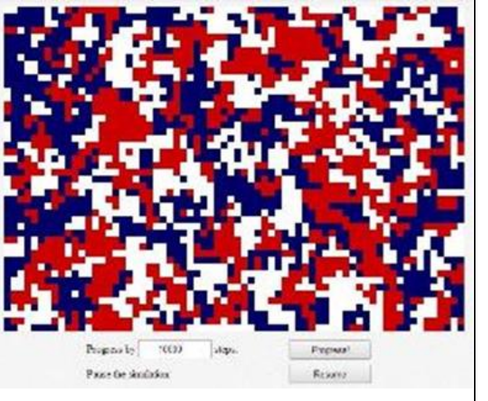
Rule 122 leads to the emergence of beautiful Sierpinsky-like triangles as displayed below.



Through repeated application of the rules, participants got a better feel for them. However, as expected, this did not lead to being able to generate the next line by looking globally at the preceding ones—such is the nature of complex dynamical systems.

The somewhat tedious character of the task and the non-negligible risk of error, prompted many participants to feel a growing need to code the process rather than do it manually. And so we proceeded to do just that with *Excel* files.

We followed with different applications of cellular automata as models of complex dynamical systems, with simulators available on the web.

Fire Propagation (renamed as *Game of Death* by a participant)	**Voting Dynamics**
	
http://shodor.org/interactivate-java/activities/ABetterFire/	http://math.berkeley.edu/~bgillesp/apps/voter/

Playing with such simulators allows for the identification of key variables for some of these dynamical systems. For example with fire propagation, participants observed that, in some situations, a high density of trees in a forest plays a much greater role in propagating a fire than does a strong wind.

The introduction of more than two possible states (for example having three parties to choose from in voting dynamics, or considering four sub-populations—susceptible, infected, recovered, and dead—within an infectious disease model) raises the level of complexity and consequently the unpredictability of the behaviour of a system. This took us to consider the complexity emerging in ecological systems.

MODELLING ECOLOGICAL SYSTEMS

Inspired by a newspaper article (Redfern, 2013), we looked at a complex ecosystem involving four species.

In the 1990s, wolves were re-introduced into Yellowstone National Park. A couple of decades later, the grizzly bear population in the park increased significantly. Why did that happen? Researchers hypothesized that a chain of interactions was at play. As wolves prey on elk, they decrease their population. All kinds of berries, previously consumed by elk (which also destroy berry shrubs), are now able to recover, thus providing an abundant source of food for bears, especially in fall, when they prepare for hibernation.

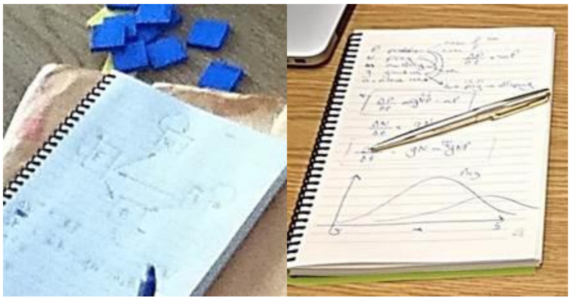
How can we verify researchers' claims about the chain of events that led to the increase in the bear population? What will happen with the wolf, deer and berries populations in the long run, in particular if the deer population declines to very small numbers? We decided to approach these questions by creating a simulation, and running it for an appropriate amount of time. Working in groups, the participants were asked to design the rules of a game (named the *Yellowstone Game*), which:

- would faithfully reproduce the interactions between the four species;
- would lead to the same outcome (i.e., an increase in grizzly bear population);
- would allow for a prediction of the future of the four populations;

- would use whatever resources participants felt like using. We had brought as possible resources: game boards, tiles of different colours, spinners, and dice (biased and unbiased).

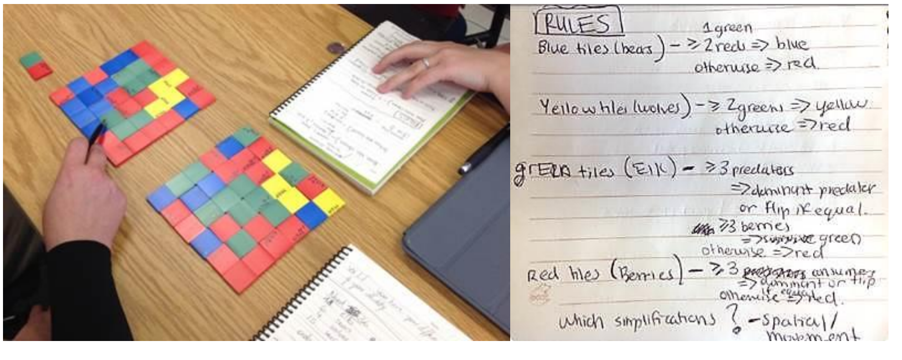
Coming up with the rules for the Yellowstone Game was one of the most creative moments of the working group, with different approaches taken by the participants and the consequent emergence of various models. We saw:

- Compartmental models and differential equations, which track the size of each population, similar to the way we analyze predator-prey interactions:



- Groups of animals, modelled by piles of coloured square tiles, with dynamic rules governing the changes in the sizes of the piles:

- Integration of space with cell automata, represented with coloured square tiles, where, for instance, a cell can change from being elk dominated to bear dominated, depending on availability of food (and, again, governed by the rules defined by the participants):



Integration of movement (dynamics is a key feature of animals!) was achieved with rules such as:

- If a bear has not eaten in 10 turns, it disappears.
- Every 10^{th} encounter between bear and wolf, bear is replaced by wolf. Otherwise, they bounce off.
- If a bear encounters berries, the bear sticks beside the berries for one turn and then moves on. The bear gets power points. Every 10 spaces, the bear drops a berry plant that will grow the next season.

The coding of such rules would lead to an *agent-based model*, a computational model used to investigate the ways in which interactions of agents (i.e. individuals or groups of individuals) affect the behaviour of a system as a whole.

If there is one lesson we learned, it is that we liked to play! Unfortunately, we could not spend as much time as necessary to fully develop the models and to run them as many times as needed in order to arrive at reliable predictions for the future of the populations.

The agent-based version of the game was inspired by the exploration we had done of an agent-based simulator, aimed at reproducing the movement and behaviour of each individual of two species (in a predator-prey relationship) within a given bounded region. The simulation, found at http://demonstrations.wolfram.com/PredatorPreyEcosystemARealTimeAgentBasedSimulation/, allows a user to change parameters (mobility and growth rates of each species, and the time a predator can survive without catching any prey).

This exploration complements the usual approach (found in first-year math textbooks) which consists of tracking the sizes (population numbers) of the two species. In particular: one considers the number of predators $f(t)$ and the number of prey $r(t)$ as functions of time and builds a system of differential equations based on two biological facts. First, the relative rate of change f'/f of the predator population is an increasing function of the prey population. In a simplest form, modelling the increase using a linear function, we obtain

$$\frac{f'}{f} = ar - b$$

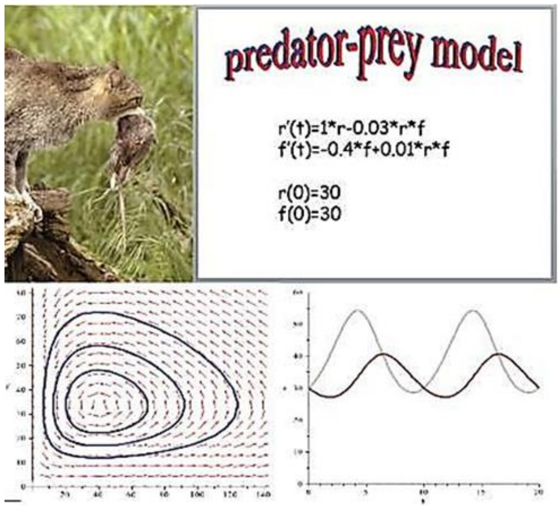
where a, $b > 0$ (the minus sign accounts for the fact that when there is no prey, the number of predators will decrease; a is the consumption of prey rate, and $-b$ is the per capita reproduction rate of predators). Second, the relative rate of change r'/r of prey population is a decreasing function of the predator population size. Using a linear function to echo this fact, we obtain

$$\frac{r'}{r} = -cf + d$$

where (when there are no predators, prey population will increase; $-c$ is the predation rate, and d is the per capita growth rate of prey). This system, usually written in the form

$$f' = -bf + arf \qquad r' = dr - crf$$

is called the predator-prey model.



This predator-prey model is then studied using various techniques, such as phase plane analysis and discretization (i.e., translating the two equations into their discrete versions and using computers to extract the behaviour of the two species). In any case, the most important feature of the model is the emergence of a pattern—periodicity (identified, in the case of a phase plane, by observing that the orbits are closed).

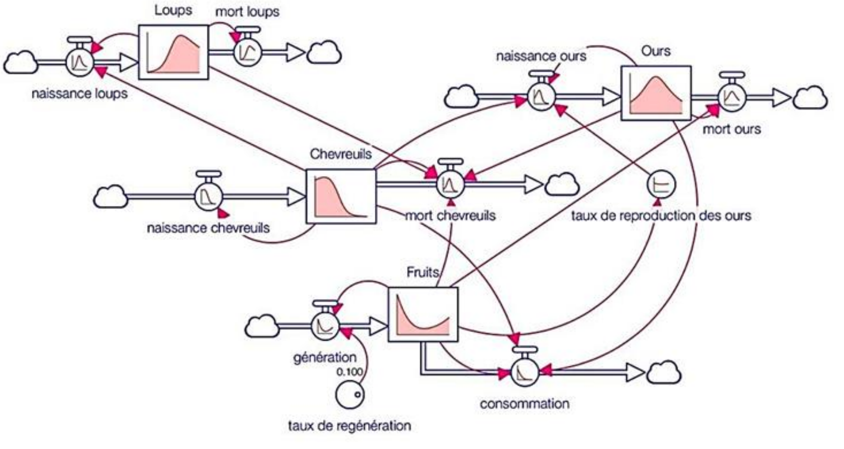
Algebraic, geometric and even numerical techniques available to us in the case of two species turn out to be more challenging when we consider more complex situations, where we need to model with more species, with delays in the interactions, with discrete events, with a greater consideration for space and its own complexity. In the sequence of modelling approaches adopted by our participants, we observed that a reduction in the presence of 'sophisticated' mathematics techniques was accompanied with a greater reliance on manipulation, automation and simulation to capture the complexity of the interactions. These powerful techniques have a place in the mathematical toolbox that allows us to study complex dynamical systems.

Since it is very likely that complex systems involve nonlinearity, the emergence of chaotic behaviour poses a whole new set of challenges. Further mathematical analysis is thus needed to give true insights into the structural properties of a system. For this, we need qualitative analysis of differential equations, phase plane techniques, and geometric analysis of trajectories, to mention a few. These tools and strategies not only increase our understanding of the complexity, but also provide a control structure to validate the results of the simulations.

MODELLING COMPLEX DYNAMICAL SYSTEMS WITH TECHNOLOGY

Dynamic modelling and simulation tools such as *Stella* and *StarLogo* have been around for quite some time as means to study the evolution of complex systems. Although they are regularly updated with more features and friendlier interface, they have had relatively low penetration in schools. Encouraged by Kaput and Roschelle (1999; republished in 2013) who have expressed the belief that "with time and effort, innovations in computational representations will make democratic access to systems dynamics possible" (2013, p. 24), we decided to take yet another look at the potential of some of these tools for studying complex dynamical systems and for learning mathematics. And in fact, not only can these simulation tools help appreciate the complexity of the phenomena which emerge from a collection of mutually interacting elements, and predict their long-term behaviour, but they also put users (and therefore students) in the driver's seat, by enabling them to define these interactions.

By exploring the features of the program *Stella*, the group came to appreciate how its focus on modelling interactions as inflows and outflows of reservoirs containing aggregate quantities could make it a useful tool to support the development of skills for modelling dynamical systems. The ease of building and defining the relations between system components, through graphical description and limited use of symbols, and the flexibility for parameterizing the simulation should warrant consideration in secondary and undergraduate mathematics.



Moreover, by aiming at describing change on a discrete time scale, one develops a sense of what a recurrence relation means, and how, when applied on a progressively smaller time scale, such relations develop into differential equations. These features can also be put to use in studying calculus, for instance to gain a deeper understanding of functions by making them emerge from the simulation of how they grow over time.

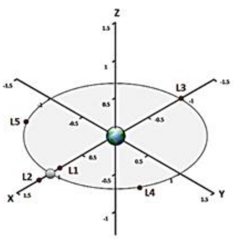
As the simulations run by *Stella* and similar programs make use of basic numerical integration techniques, there was an initial feeling among participants that the proper use of such software would have to rely on the knowledge of calculus and numerical methods. However, as one participant pointed out, maybe this is how some of us initially felt about dynamic geometry software (DGS) and the necessary geometry knowledge to use them, until experiments with younger students showed that well designed exploration activities done with DGS could contribute to developing new intuition and motivate the learning of concepts, properties and proofs. Consequently, we could envision a similar approach for developing intuition with respect to functions and calculus through the study of dynamical systems with modelling and using simulation tools.

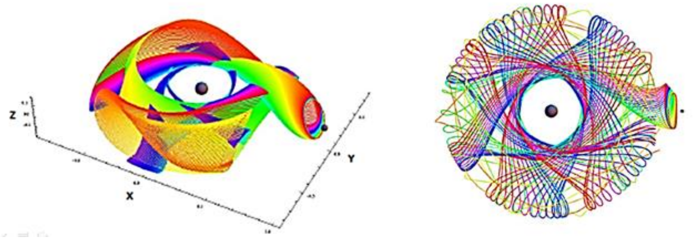
Ultimately, the black box associated with numerical integration is well worth opening at the undergraduate level. Not only does it expand substantially the class of problems that one can tackle, it represents, in its simplest form (Euler's method), a generalization of Riemann sums. Once that connection is made, it is not difficult to build from a *Stella*-like model an *Excel* file that will perform equivalent computations.

For space modelling and movement of a population within that space, agent-based simulation software like *StarLogo* can provide interesting insight. By modelling and observing the dynamics at the level of the individual creatures, rather than at the aggregate level of population densities, students can appreciate the dynamics of the interactions without having to think in terms of differential equations. Although this can be an advantage for introducing complex dynamical system modelling at the school level, the integration of such an agent-based approach in today's mathematics curriculum still represents a challenge. But it could well be an interesting entry into programming, which constitutes in itself a very rich toolbox to tackle complexity.

As an example of what can be achieved when one uses programming to solve a system of differential equations that models a complex dynamical system, Laura Broley shared her experience of programming to study unstable manifolds of halo orbits that characterize the trajectory of a satellite in interaction with the earth and the moon.

$$x'' = 2y' + x - \frac{(1-\mu)(x+\mu)}{r_1^3} - \frac{\mu(x-1+\mu)}{r_2^3}$$
$$y'' = -2x' + y - \frac{(1-\mu)y}{r_1^3} - \frac{\mu y}{r_2^3}$$
$$z'' = -\frac{(1-\mu)z}{r_1^3} - \frac{\mu z}{r_2^3}$$





She emphasized the value of programming in getting a deeper understanding of the model and in controlling the various simulations and making sense of their output.

CONCLUSIONS

The variety of approaches to analyzing complex dynamical systems, as well as the abundance of tools (programming, ready-made software, online resources, and so on) give us confidence that we can—to a point—improve our understanding of dynamics of complex systems. Furthermore, by running a mathematical model, we can calculate the future of a complex system and this, in many ways, constitutes our best answer to predicting how it will change over time.

Teaching mathematics constitutes a complex system as well, so it makes sense to think about its evolution over time. But as far as content goes, is there a place for complex systems in mathematics curricula?

Perhaps the best reason for the affirmative answer lies in the benefits that studying complex systems will bring. By learning to look at the dynamics of a given complex situation through different mathematical lenses, including but not restricted to differential equations, our mathematical experiences can become richer. We think geometrically, we think analytically, we think numerically, we build computer programs, we simulate, we build models or adjust existing ones. Studying complex dynamical systems is an excellent way to improve essential mathematical activities for our students: logical thinking, quantitative and qualitative reasoning, and engaging with true, authentic and significant applications representative of the global challenges the next generations will have to address.

Teaching complex systems might not require a major shift in undergraduate math curricula. Instead, we need to refocus certain courses, such as calculus and differential equations, to emphasize some topics which are traditionally on the back burner (such as Euler's method, qualitative analysis of ODEs, or numerical methods for solving systems of ODEs). Perhaps the largest change could be the introduction of one or two courses which would cover non-traditional mathematical topics, such as programming, creating simulations, using existing software such as *Stella*, applets, and so on, for problem solving. Such curricular shifts might also better serve other disciplines by offering a mathematical lens with which to view such issues as sustainable development, global health, social equity and countless issues in biology.

Integrating complex dynamical systems in K-12 curricula might prove much more difficult. Beyond the mathematical and curricular reflection needed to determine what exactly can or should be taught to support complex systems, there might be other challenges such as appropriate support for teachers. Nonetheless, our activities offer some ideas for broaching this challenging topic and introducing key concepts such as rate of change, iteration, stability, and periodicity. Simulations and computer programming could be used more widely to investigate ideas in mathematics, and perhaps be introduced as early as elementary education. Engaging students in studying and/or investigating relevant and significant problems with these tools would enrich their mathematical experiences, provide them with an opportunity to develop teamwork, modelling, programming and other problem-solving skills, and equip them with insights to, and means of understanding, complex challenges our dynamic world faces.

WEB RESOURCES

- Wolfram Simulators http://www.wolfram.com/cdf-player/
- Cellular automata:
  - Discrete models (to be linked with *Game of Life* and the *Standing Up Game*):
    - *Game of Life*: http://www.bitstorm.org/gameoflife/
    - Fire propagation:
      http://www.shodor.org/interactivate/activities/ABetterFire/
      http://demonstrations.wolfram.com/ForestFireSimulationUsingARandomizedHexagonalAutomaton/
    - Opinion propagation and vote:
      http://math.berkeley.edu/~bgillesp/apps/voter/
      http://demonstrations.wolfram.com/VoterModel/
- A first model of prey-predator:
  - http://demonstrations.wolfram.com/PredatorPreyEcosystemARealTimeAgentBasedSimulation/
- Yellowstone:
  - BBC report: http://www.bbc.com/news/science-environment-23495074/
  - Video: http://www.youtube.com/watch?v=GeF25GJleA8
  - Other source: http://oregonstate.edu/ua/ncs/archives/2013/jul/bears-and-berries-return-wolves-aids-grizzly-bears-yellowstone
  - Maps: http://www.yellowstone-natl-park.com/ywstone.htm
  - Topographic map: http://yellowstone.net/maps/yellowstone-topo/
- Stella trial version:
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ROLE-PLAYING AND SCRIPT-WRITING IN MATHEMATICS EDUCATION: PRACTICE AND RESEARCH

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INTRODUCTION

Role-play involves staging a problematic situation with characters taking roles. It may be used to fulfill various objectives such as therapeutic objectives, personal and professional training objectives, or may be used as a pedagogical method. Despite various reports on the benefits of this method, its use in Mathematics Education is underdeveloped. As such, the goal of our working group was to examine the affordances of role-playing in mathematics education in various settings. We also considered the affordances of script-writing, which we consider as imagined (rather than enacted) role-playing.

To achieve this goal the participants engaged in the following activities:

- They examined various scenarios in which role-play is enacted or imagined. Of course, 'active examination' involved engagement in role-playing and reflection on different roles.
- They analyzed various plays composed by students and discussed how those can be used in research.
- They designed scenarios or prompts for settings of their choice.
- They considered limitations and advantages of both methods, in comparison with each other, and in relation to traditional methods used in mathematics education in general and teacher education in particular.

In what follows we introduce each method and provide a summary of the particular activities that were presented, implemented and discussed as we worked in the group.

ROLE-PLAY: DEFINITIONS, ADVANTAGES, LIMITATIONS AND TWO SOLUTIONS

Role-playing is an unscripted "dramatic technique that encourages participants to improvise behaviors that illustrate expected actions of persons involved in defined situations" (Lowenstein, 2007, p. 173). In other words, role-playing is "an 'as-if' experiment in which the subject is asked to behave as if he [or she] were a particular person in a particular situation" (Aronson & Carlsmith, 1968, p. 26). Role-playing is used as an effective pedagogical strategy in a variety of fields, including limited use in teacher education, which is our main focus here.

In considering role-play in teacher education, Van Ments (1983) described it as experiencing a problem under unfamiliar constraints, as a result of which one's own ideas emerge and one's understanding increases. In this sense, role-playing can also be seen as role-training. It is aimed at increasing teachers' awareness of various aspects of their actual work. Despite the known advantages, role-playing in teacher education is underdeveloped. While some authors advocate for this method and report on its implementation, this is most often done in the form of self-reports and anecdotal evidence of participants' experiences.

Kenworthy (1973) described a method in which one participant takes on a teacher role while others take on the roles of various students (e.g., a slow student, a gifted student, a disturbing student). He considered this type of role-playing to be "one of the most profitable, provocative and productive methods in the education of social studies teachers" (p. 243). He claimed that engagement in role-playing activities helped participants anticipate difficulties they might encounter in their classrooms and as such gain security in their successful experiences should they face similar situations on the job.

More recently, in Palmer's (2006) study, pre-service teachers took on the roles of children as their professor modelled science teaching. It was reported that teachers' self-efficacy increased and they were more open to the idea of implementing role-playing in their teaching. In Howes and Cruz (2009) research students in an elementary science methods class were invited to assume roles of scientists and take part in an *Oprah Show* interview. In addition to learning about contributions of different scientists, this activity sharpened the prospective teachers' understanding of what science is and what image of science they wish to convey to their students.

Despite the recognized advantages, time and participation logistics are a significant limitation of role-playing. If we intend to engage our students in role-playing during class time, only a few will be active players and the remainder will serve as an audience.

How can one give all students the opportunity to participate in the role-playing scenario? This question troubled the leaders of the working group, Caroline and Rina, and they came up with two very different solutions. Caroline found a creative way to achieve participation of all students in her class. Rina turned to imagined (rather than enacted) role-playing, that is, writing a script for a dialogue between characters. We attend to both methods below.

ROLE-PLAY: IMPLEMENTATION

Since the beginning of the 2000s, role-playing has been used by Caroline and her colleagues in methods courses for prospective elementary school teachers at UQAM. In their setting, a student takes the part of a teacher while another acts as student, and they improvise in an "informed way" (Maheux & Lajoie, 2011) around a mathematical task, a student's question or production, the use of teaching material, and so on (Lajoie, 2010; Lajoie & Maheux, 2013; Lajoie & Pallascio, 2001a; 2001b; Marchand, Adihou, Lajoie, Maheux, & Bisson, 2012).

Their undergraduate course *Didactique de l'arithmétique au primaire* is designed around ten different role-plays on various topics including numeration, operations and algorithms, fractions and decimal numbers.

Each role-play is organized into four moments:

- First, the theme on which students will need to improvise is introduced (*introduction time*).
- Second, students have about 30 minutes to prepare in small groups (*preparation time*). In the preparation time, students reflect on what might happen between the teacher and the student, not knowing beforehand who will play the role of a pupil or that of a teacher. To prepare themselves, they generally draw on an article they've read at home, and the analysis of pupils' productions related to the topic. Based on what is observed in the preparation time, the instructor picks groups to go and play in front of the class, assigning them their roles (teacher or student). The groups get to choose which member is going to play.
- Third comes the play itself (*play time*). Since each group prepared separately, the students really have to improvise, and the goal is, for the 'teacher' to work from the 'pupil's' perspective (having him/her explain a solution, use a manipulative, reformulate, exemplify, and so on) in order to move his/her mathematical thinking forward. There is no script, and no specifically predefined end-point either.
- Finally, there is a whole classroom discussion (*discussion time*) in which the instructor engages with the students on the variety of possible interventions, and how they relate to one's intention and to what comes from the pupils. It allows the instructor to reflect with the class on what happened, what might have been done, what could be done next, and so on.

In the working group, we presented participants with a variety of tasks for role-playing (see Example 1 below). We asked the whole group to choose one in order to experiment with role-play together. Problem 1, in Example 1, was chosen (see below). Participants, in teams of three or more, had some time to prepare for the play (preparation time). A first set of two volunteers coming from two different teams came to play in front of the group (play time). Then, there was a discussion (discussion time) very similar to one that could take place with pre-service elementary school teachers. Afterward, a second set of two volunteers, coming from two other teams came to play. The choice was made by the group to use exactly the same task. Finally, our discussion evolved around what can be achieved by students role-playing, what can be learned about students while they are role-playing, how the approach might be adapted depending on context and intentions, etc.

EXAMPLE 1: UNEXPECTED SOLUTIONS TO WORD PROBLEMS

For this role-play, one designated teacher and one designated pupil will interact. The teacher will investigate the pupil's solution and, if appropriate, will lead the pupil to correctly solve the problem. **The teacher will be asked (as much as possible) to concentrate on interventions proceeding from the pupil's reasoning.**

<u>Problem 1 (Lajoie & Pallascio, 2001b):</u>

> Among the following mixes, which will taste of orange juice the most? Will it be mix #1 (1 glass of orange juice and 2 glasses of water) or mix #2 (2 glasses of orange juice and 4 glasses of water)?
>
> **Solution:** "In mix #1, I have one additional glass of water. In mix #2, I have two additional glasses of water. Mix #1 will be the tastiest orange mix. "

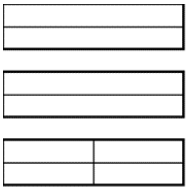
Problem 2 (Lajoie & Pallascio, 2001b):

> Which mix will we obtain if we pour the two mixes in a large container?
>
> **Solution:** "1/3 + 2/6 = 2/6 +2/6 = 4/6"

Problem 3 (Maheux & Lajoie, 2011):

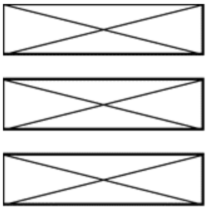
> We must share three chocolate bars between four persons. What will each person get?

Solution A:



Answer: 1 and ¼

Solution B:



Answer: 3/12

SCRIPT-WRITING

As mentioned above, script-writing is implemented in order to give all students the opportunity to participate in the role-playing scenario. We consider script-writing, that is, writing a script for a dialogue between characters, to be imagined (rather than enacted) role-playing.

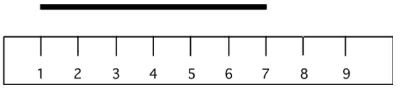
The use of script-writing as an instructional tool has been implemented in prior mathematics education research. For example, Gholamazad (2007) developed the *proof as dialogue* method. Prospective elementary school teachers participating in her study were asked to clarify statements of a given proof in elementary number theory by creating a dialogue, where one character had difficulty understanding the proof and another attempted to explain each claim. This method was amended and extended by Koichu and Zazkis (2013) and Zazkis (2014) in their work with prospective secondary school teachers. In both studies the participants had to identify problematic issues in the presented proofs and clarify those in the form of a dialogue, referred to as a *proof-script*. These scripts revealed participants' personal understandings of the mathematical concepts involved in the proofs as well as what they perceived as potential difficulties for their imagined students.

Additionally, the *lesson play* method was developed and used in teacher education where participants were asked to write a script for an imaginary interaction between a teacher-character and student-character(s) (Sinclair & Zazkis, 2011; Zazkis, Liljedahl, & Sinclair, 2009; Zazkis & Sinclair, 2013; Zazkis, Sinclair, & Liljedahl, 2009, 2013). *Lesson play* was juxtaposed with the traditional *lesson plan* and how the former may account for the deficiencies of the latter was outlined. The method was advocated as an effective tool in preparing for instruction, as a diagnostic tool for teacher educators, and as a window for researchers to studying a variety of issues in didactics and pedagogy (Zazkis, Sinclair, & Liljedahl, 2013).

In the working group we presented participants with a variety of prompts that were used in prior work, asked them to choose one and create a script. Then those scripts were shared with the group and discussion evolved around what can be achieved by a learner who uses this tool and what can be learned about students using this tool.

PROMPTS USED WITH ELEMENTARY SCHOOL TEACHERS

The prompts in Table 1 are taken from Zazkis, Sinclair, and Liljedahl (2013).

Ch. 4	Students in your class were asked to measure the length of different objects. The teacher collected their responses. **Teacher**: Johnny, how long is the stick that you measured **Johnny**: It is … seven. **Teacher**: Seven what? **Johnny**: Seven centimeters. **Teacher**: Can you show me how you measured? (Johnny places the stick next to the ruler as shown below)  **Teacher**:…
Ch. 5	There is a conversation between the teacher and a student. There are 20-25 other students in the room. **Teacher**: Why do you say that 462 is divisible by 4? **Student**: Because the sum of the digits is divisible by 4.
Ch. 5	There is a conversation between the teacher and a student. There are 20-25 other students in the room. **Teacher**: Why do you say that 354 is divisible by 4? **Student**: Because the sum of the digits is divisible by 4.
Ch. 5	There is a conversation between the teacher and a student. There are 20-25 other students in the room. **Teacher**: Why do you say that 354 is divisible by 4? **Student**: Because …
Ch. 6	There is a conversation between the teacher and a student. There are 20-25 other students in the room. **Teacher**: Why do you say 91 is prime? **Student**: Because it is not on our times tables.
Ch. 6	There is a conversation between the teacher and a student. There are 20-25 other students in the room. **Teacher**: Why do you say 143 is prime? **Johnny**: Because 2, 3, 4, 5, 6, 7, 8 and 9 don't go into it.
Ch. 6	There is a conversation between the teacher and a student. There are 20-25 other students in the room. **Teacher**: Why do you say 37 is prime? **Johnny**: Because 2, 3, 4, 5, 6, 7, 8 and 9 don't go into it.
Ch. 7	There are 20-25 students in the classroom. They are working on the following problem: A toy train has 100 cars. The first car is red, the second is blue, the third is yellow, the fourth is red, the fifth is blue and sixth is yellow and so on. (a) What is the colour of the 80th car? (b) What is the number of the last blue car? The teacher is moving through the room observing how the students are progressing. S/he stops and points at one student's work. **Teacher**: Why is the 80th car red? **Student**: Because the 4th car is red, and 80 is a multiple of 4.

Ch. 7	There are 20-25 students in the classroom. They are working on the following problem: A toy train has 100 cars. The first car is red, the second is blue, the third is yellow, the fourth is red, the fifth is blue and sixth is yellow and so on. (a) What is the colour of the 80th car? (b) What is the number of the last blue car? The teacher is moving through the room observing how the students are progressing. S/he stops and points at one student's work. **Teacher**: Why is the 80th car red? **Student**: Because the 10th car is red. So, the 20th car, the 30th car, the 40th car, and so on, will be red.
Ch. 8	There are 20-25 students working on problems comparing pairs of proper fractions. As you move around the class you overhear one student telling another student of a strategy that he has discovered. **Sam**: This is easy. Just look how close the top number is to the bottom number. The fraction that is closest is biggest. **Jennifer**: Does this always work? **Sam**: It has for all the questions we've done so far.
Ch. 9	There are 20-25 students in the classroom. They are working on the following problem: Once upon a time there were two melon farmers; John and Bill. John's farm was 200m by 600m and Bill's farm was 100m by 700m. Who grew the most melons? The teacher sees that the student has written: *They both grew the same amount.*

Table 1. Prompts from Zazkis, Sinclair, & Liljedahl (2013).

The prompts in Table 2 are taken from Zazkis and Zazkis (2014b).

TASK 1 Bonnie and Clyde are discussing numbers and their factors. Bonnie claims that the larger a number gets, the more factors it will have. Clyde disagrees. Write a script for a conversation between these two characters that includes their exchange of arguments as both sides are convinced they are right. Consider what examples and what experiences could have led Bonnie to this conclusion. Consider why Clyde would disagree. Consider what arguments and what examples they both use to convince each other and what each one of them finds convincing. Annotate your script, analysing the arguments of your characters and their examples.
TASK 2 Tom and Jerry are discussing rational and irrational numbers. Tom claims that 23/43 is an irrational number, because his calculator shows 0.53488372 when 23 is divided by 43, and there is no repeating pattern of digits. Jerry disagrees. Write a script for a conversation between these two characters that includes their exchange of arguments as both sides are convinced they are right. Consider what examples and what experiences could have led Tom to his conclusion. Consider why Jerry would disagree. Consider what arguments and what examples they both use to convince each other and what each one of them finds convincing. Annotate your script, analysing the arguments of your characters and their examples.

Table 2. Prompts from Zazkis & Zazkis (2014b).

The following task is taken from Zazkis and Nejad (in press). A short version will appear in the 2014 PME 38 proceedings.

Continue either of these conversations:

(a) **Conversation with a colleague:** It is 8:15 in the morning and you are busy preparing for your classes. A colleague comes to your room and says something like: "*Listen, I know you are doing your Master's and all. But have you thought about what this is doing for the kids?*"

(b) **Conversation with a school principal:** It is the lunch break and you are invited to the principal's office. (You noticed that the principal walked by your class earlier that day). The principal says: "*I asked you to come in here because I want to give you a little guidance*."

Table 3. Prompts from Zazkis & Nejad (in press).

PROMPTS USED WITH SECONDARY SCHOOL TEACHERS

The following is taken from Zazkis and Zazkis (2014a). A short version appears in the RUME 2014 proceedings, available at
http://timsdataserver.goodwin.drexel.edu/RUME-2014/rume17_submission_31.pdf

Consider the following proof of the Pythagorean theorem

Draw a square ABCD in which the length of the side is $a + b$.

Connect points KLMN.

The area of ABCD is $(a + b)^2$

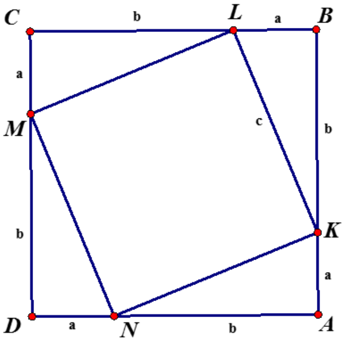
However, this area can also be calculated as composed of the square KLMN and 4 triangles, that is,

$$4 \times ½(ab) + c^2 = (a + b)^2$$

$$2ab + c^2 = a^2 + 2ab + b^2$$

$$c^2 = a^2 + b^2$$

QED



Imagine that you are working with a high school student and testing his/her understanding of different aspects of this proof.

What would you ask? What would s/he answer if her understanding is incomplete? How would you guide this student towards enhanced understanding? Identify several issues in this proof that may not be completely understood by a student and consider how you could address such difficulties. In your submission:

- Write a paragraph on what you believe could be a "problematic point" (or several points) in the understanding of the theorem/statement or its proof for a learner.
- Write a scripted dialogue between teacher and student that shows how the hypothetical problematic points you highlighted in part (a) could be worked out (THIS IS THE MAIN PART OF THE TASK).
- Add a commentary to several lines in the dialogue that you created, explaining your choices of questions and answers.

Table 4. Prompt from Zazkis & Zazkis (2014a).

The next is from Zazkis (2014).

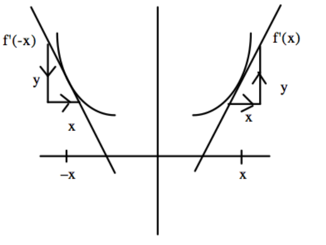
Theorem: The derivative of an even function is an odd function.

Consider the following proof of this theorem:

If $f(x)$ is an even function it is symmetric over the y-axis.

So the slope at any point x is the opposite of the slope at $(-x)$

In other words, $f'(-x) = -f'(x)$, which means the derivative of the function is odd.



Imagine that you are working with a student and testing his/her understanding of different aspects of this proof.

What would you ask? What would s/he answer if her understanding is incomplete? How would you guide this student towards enhanced understanding? Identify several issues in this proof that may not be completely understood by a student and consider how you could address such difficulties. In your submission:

- Write a paragraph on what you believe could be a "problematic point" (or several points) in the understanding of the theorem/statement or its proof for a learner.
- Write a scripted dialogue between teacher and student that shows how the hypothetical problematic points you highlighted in part (a) could be worked out (THIS IS THE MAIN PART OF THE TASK).
- Add a commentary to several lines in the dialogue that you created, explaining your choices of questions and answers.

Table 5. Prompt from Zazkis (2014).

The following is from Koichu and Zazkis (2013). It involves a proof of Fermat's Little Theorem (adapted from *Wikipedia*).

Theorem:

For prime number p and natural number a, such that GCF$(a, p) = 1$

$$a^p \equiv a \pmod{p}$$

Proof:

$0, 1, 2, \ldots, (p-1)$ is a list of all possible remainders in division by p.

When these numbers are multiplied by a, we get $0, a, 2a, 3a, \ldots, (p-1)a$. When the numbers are reduced modulo p we get a rearrangement of the original list.

Therefore, if we multiply together the numbers in each list (omitting zero), the results must be congruent modulo p:

$$a \times 2a \times 3a \times \ldots \times (p-1)a \equiv 1 \times 2 \times 3 \times \ldots \times (p-1) \pmod{p}$$

Collecting together the a terms yields

$$a^{p-1}(p-1)! \equiv (p-1)! \pmod{p}$$

Dividing both sides of this equation by $(p-1)!$ we get

$$a^{p-1} \equiv 1 \pmod{p} \text{ or } a^p \equiv a \pmod{p}\text{, QED.}$$

Create a dialogue that introduces and explains the above theorem and its proof. Highlight the problematic points in the proof with questions and answers. In your submission:

- Describe the characters in your dialogue.
- Write a paragraph on what you believe is a "problematic point" (or several points) in the understanding of the theorem/statement or its proof for a learner.

- Write a dialogue that shows how you address this hypothetical problem (THIS IS THE MAIN PART OF THE TASK)
- You may add a commentary to several lines in the dialogue that you created, explaining your choices of questions and answers, in connection to the characters, which may not be obvious for the reader.

Table 6. Prompt from Koichu & Zazkis (2013).

ROLE-PLAY AND SCRIPT-WRITING IN RESEARCH

Of course, our work with teachers inevitably leads to research questions. A part of the third day was devoted for discussion on how role-playing and script-writing can be used in research. The following ideas were presented by participants and elaborated upon:

- focus on understanding of particular mathematical concepts by students and teachers;
- focus on pedagogical approaches in dealing with particular student errors;
- focus on teacher's attention to the language of mathematics;
- focus on teachers' scaffolding students;
- focus on participants' progress over a period of time;
- focus on feedback: what do prospective teachers notice when observing colleagues' role-play.

Overall, we believe that the participants were exposed to valuable tools that enriched their experiences and presented a variety of ideas for future implementation. The work of the working group ended with expressed intentions of the participants to implement some variation on role-playing (either enacted or imagined) in their settings.

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Topic Sessions

Séances thématiques

MATHEMATICAL PROBLEM-SOLVING PROFICIENCY AND KNOWLEDGE FOR TEACHING

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INTRODUCTION

Mathematics teachers' knowledge for teaching mathematics with deep understanding has been a focus of research in mathematics education in recent years. One aspect of this knowledge that requires attention, given its importance to doing and learning mathematics, is problem-solving knowledge for teaching. This importance was highlighted by the National Council of Teachers of Mathematics [NCTM] (NCTM, 1980) with their declaration that problem solving should be the "focus of school mathematics in the 1980's" and the subsequent emphasis on problem solving as a standard for school curriculum (NCTM, 1989; 2000). As Kilpatrick, Swafford, and Findell (2001) explained,

> *We believe problem solving is vital because it calls on all strands of [mathematical] proficiencies, thus increasing the chances of students integrating them. ... Problem solving should be the site in which all of the strands of mathematics proficiency converge. It should provide opportunities for students to weave together the strands of proficiency and for teachers to assess students' performance on all of the strands.* (p. 421)

However, in spite of the emphasis the literature places on problem solving and ways to teach it, and while mathematics curricula have embraced it, problem solving continues to be a challenge for students and teachers in the mathematics classroom, which raises concerns about the knowledge teachers hold of it. This paper of the topic session highlights the nature of some of these concerns and some aspects of what constitutes mathematical problem-solving knowledge for teaching (MPSKT) based on the research literature. The focus is on the knowledge teachers ought to hold to help students to become proficient in problem solving.

PROBLEM-SOLVING PROFICIENCY

Problem solving could mean different things to teachers depending on their experiences with it as learners and their understanding of it. For example, they could correlate it with solving routine word problems or rote exercises, a view that will not support student's development of proficiency in genuine problem solving. Theoretically, genuine problem solving means "engaging in a task for which the solution method is not known in advance" (NCTM, 2000, p. 52). It is "a form of cognitive processing you engage in when faced with a problem and do not have an obvious method of solution" (Mayer & Wittrock, 2006, p. 287). MPSKT as discussed here is based on this perspective of problem solving and is related to teaching for the development of problem-solving proficiency.

Problem-solving proficiency is being used here to represent what is necessary for one to learn and do genuine problem solving successfully. This is similar to Kilpatrick et al.'s (2001) use of mathematical proficiency. As they explained: "we have chosen mathematical proficiency to capture what we believe is necessary for anyone to learn mathematics successfully" (p. 116). For example, Schoenfeld (1985) identified four factors as necessary for successful problem solving: appropriate resources, heuristic strategies, metacognitive control, and appropriate beliefs. This suggests that problem-solving proficiency requires one to be equipped with and competently use these factors. For mathematical proficiency, Kilpatrick et al. (2001) identified five components in defining it: conceptual understanding; procedural fluency; strategic competence; adaptive reasoning; and productive disposition. Table 1 shows a possible relationship between these components and the factors of problem-solving proficiency. In this relationship, conceptual understanding and procedural fluency embody the type of knowledge and skills that are the resources required for effective problem solving; strategic competence involves ability to formulate, represent, and solve mathematical problems; productive disposition includes beliefs; and adaptive reasoning includes capacity of logical thought and reflection.

Problem-solving proficiency	Mathematical proficiency
Resources	Conceptual understanding and procedural fluency
Strategies	Strategic competence
Metacognitive control	Adaptive reasoning
Beliefs	Productive disposition

Table 1. Connecting problem-solving and mathematical proficiency.

This relationship has important implications for teaching and learning problem solving in the mathematics classroom. For example, as Kilpatrick et al. (2001) noted, "the components of mathematical proficiency are not independent, they represent different aspects of a complex whole" (p. 116). They explained:

> *The most important observation we make here, one stressed throughout this report, is that the five strands are interwoven and interdependent in the development of proficiency in mathematics. […] Mathematical proficiency is not a one dimensional trait, and it cannot be achieved by focusing on just one or two of these strands. […] Helping children acquire mathematical proficiency calls for instructional programs that address all its strands.* (p. 116)

Thus, like mathematical proficiency, problem-solving proficiency is not a one-dimensional concept and cannot be achieved by focusing on just one or two of the factors that define it. Helping children to develop problem-solving proficiency will require instructional practices that address all of the factors in an interrelated way. In order to accomplish this, teachers will have to know not only how to solve problems but hold a deep understanding of other factors that are associated with the development of proficiency in problem solving.

CONCERNS WITH MATHEMATICS TEACHERS' PROBLEM SOLVING

Studies involving mathematics teachers' problem-solving ability have raised issues of teachers' knowledge of problem solving. For example, Schmidt and Bednarz (1995) and van Dooren, Verschaffel, and Onghena (2003) found that prospective teachers demonstrated a lack of flexibility in their choice of problem-solving approaches. Similarly, Taplin (1996) found that her participants preferred to work with a narrow range of strategies. They tended to select a method and not change from that, implying inflexibility in their choice or management of problem-solving strategies. Leung (1994) analyzed problem-posing processes of prospective elementary teachers with differing levels of mathematics knowledge and found that those with low mathematics knowledge posed problems that might not be solved mathematically and the mathematics problems posed were not necessarily related in structure. Chapman (1999) found that practicing elementary teachers believed that they should stick with a solution even if not productive. They were surprised at the different ways of thinking about and solving a problem based on each of the participants' solutions of the same problem. Chapman (2005) found prospective secondary mathematics teachers tended to made sense of problem solving in a linear way and equated it to solving algorithmic word problems.

Teachers, then, are not likely to be equipped with the kind of knowledge they should hold for problem-solving proficiency based on their experiences as learners of mathematics in traditional mathematics classrooms. While this could be changing due to efforts of reformed teacher education experiences/programs, there is still much to learn about the knowledge teachers should hold and how to facilitate their development of it. Recent studies, directly or indirectly, have investigated instructional practices in helping prospective teachers to grow in particular aspects of their knowledge of problem solving for teaching (e.g., Ebby, 2000; Lee, 2005; Roddick, Becker, & Pence, 2000; Szydlik, Szydlik, & Benson, 2003).

While these studies attend only to particular factors of problem-solving proficiency, they imply that simply reading about problem solving and/or solving problems is not enough to develop proficiency about and for teaching problem solving. They also imply what the researchers consider to be important for teachers to learn about problem solving, which contributes to our understanding of what ought to be considered as key components of MPSKT.

SOME KEY FEATURES OF MPSKT

While there are different perspectives of mathematics knowledge for teaching (e.g., Rowland & Ruthven, 2011), the pioneering work of Ball and colleagues has provided an important example of special ways in which one must know mathematical procedures and representations to teach mathematics meaningfully and effectively (e.g., Ball, Thames, & Phelps, 2008; Hill & Ball, 2009; Hill et al., 2008; Thames & Ball, 2010). They suggest that general mathematical ability does not fully account for the knowledge and skills needed for effective mathematics teaching. A special type of knowledge is needed by teachers that is not needed in other professional settings and the conceptual demands of teaching mathematics are different than the mathematical understandings needed by mathematicians. This suggests that a general problem-solving ability, while necessary, is not sufficient for teaching mathematical problem solving. Instead, a special type of knowledge is needed for effective teaching of mathematical problem solving. This knowledge, MPSKT, can be implied from what research suggests to be necessary for one to learn and do problem solving successfully. Based on a review of published theories and studies on mathematical problem solving, some of the key components of it are highlighted here.

There is a large body of research on mathematical problem solving that goes back to the early 1900s. It has addressed a variety of issues regarding the nature, learning, and teaching of mathematical problem solving. Works, such as, Garofalo and Lester (1985); Kilpatrick et al. (2001); Mason, Burton, and Stacey (1982, 2010); NCTM (1989, 1991, 2000); Polya (1957); and Schoenfeld (1985, 1992) that address the nature of mathematical problem solving and mathematical proficiency, provide insights on components of MPSKT that deal with problem-solving content knowledge. These works also address learners and learning of mathematical problem solving, which, in addition to studies on mathematics teachers' knowledge, thinking, learning and teaching of mathematical problem solving (e.g., Arbaugh & Brown, 2004; Carpenter, Fennema, Peterson, & Carey, 1988; Chapman, 1999, 2005, 2009a, 2009b; Ebby, 2000; Lee, 2005; Leung, 1994; Roddick et al., 2000; Schmidt & Bednarz, 1995; Szydlik et al., 2003; Taplin, 1996; van Dooren et al., 2003) provide insights on components of MPSKT that deal with pedagogical problem-solving knowledge. Table 2 summarizes some of these key components of MPSKT.

MPSKT	
Problem-solving content knowledge	Pedagogical problem-solving knowledge
Nature of problems Nature of problem solving Nature of problem solving proficiency	Students as problem solvers Instructional practices for problem solving Problem solving in the curriculum

Table 2. Some key components of MPSKT.

Table 3 provides brief descriptions of these types of knowledge that constitute MPSKT.

MPSKT	Description
Knowledge of mathematical problem-solving proficiency	Understanding of what is needed for successful problem solving and being proficient in problem solving
Knowledge of mathematical problems	Conceptual understanding of meaningful/worthwhile mathematical problems/tasks
Knowledge of mathematical problem solving	Conceptual understanding of mathematical problem solving as mathematical thinking and process (stages problem solvers often pass through in the process of reaching a solution)
Knowledge of students as mathematical problem solvers	Understanding of students as problem solvers, for example, what constitutes productive beliefs and dispositions toward problem solving
Knowledge of instructional practices for problem solving	Understanding of instructional practices for problem solving, including instructional techniques for strategies, metacognition, use of technology, and assessment
Knowledge of problem solving in the curriculum	Understanding of curriculum expectations in relation to problem-solving proficiency

Table 3. Theoretical components of MPSKT.

Teachers should have conceptual and procedural knowledge of mathematical problem solving. This includes understanding the stages problem solvers often pass through in the process of reaching a solution, for example, models of problem solving such as those of Mason, Burton and Stacey (1982, 2010), Polya (1957), and Schoenfeld (1985). Related to this, teachers should have knowledge of the nature of problems. Based on a study on practicing teachers' conceptions of contextual problems, Chapman (2009b) identified conceptions of contextual problems (both routine and non-routine) that teachers could hold that have the potential to limit or enhance how problem solving is perceived, experienced, and learnt by their students. A teacher can hold several of these conceptions, but the combination and depth involved were found to be important in relation to teaching. The findings indicated that for meaningful teaching with contextual problems, teachers should hold knowledge that includes: depth in a humanistic and utilitarian view of contextual problems; understanding of the relationship between contextual problems, students, and teacher; understanding of the role of problem context in problem solving or modeling and learning; and understanding of problem-solving thinking or process.

The literature also suggests that teachers need to understand students as problem solvers, for example, what constitutes productive beliefs and dispositions toward problem solving; what one knows, can do, and is disposed to do; and adequate level of difficulty of the problems assigned. They should have knowledge of skills students need to be competent technological problem solvers and how to evaluate students' problem-solving process and progress. They need to understand instructional practices for problem solving, including instructional techniques for strategies and metacognition. They must have strategic competence in order to face the challenges of mathematical problem solving during instruction. They must perceive the implications of students' different approaches, whether they may be fruitful and, if not, what might make them so. They must decide when and how to intervene—when to give help and how to give assistance that supports students' success while ensuring that they retain ownership of their solution strategies; what to do when students are stuck or are pursuing a non-productive approach or spending a lot of time with it; and what to look for. Teachers will sometimes be in the position of not knowing the solution, thus needing to know how to work well without knowing all.

CONCLUSIONS

Teachers need to hold knowledge of mathematical problem solving for themselves as problem solvers and to help students to become better problem solvers. Thus, a teacher's knowledge *of* and *for* teaching for problem-solving proficiency must be broader than competence in problem solving, that is, it requires more than how to solve mathematical problems. This paper highlighted some key components of the MPSKT a mathematics teacher should hold. However, it does not address ways of knowing that are important for this knowledge to be held in a meaningful and useful way for the classroom. How this knowledge is held by the teacher is important in terms of whether or not it is usable in meaningful and effective ways in supporting problem-solving proficiency in his or her teaching. In addition, it is also not only knowledge of each of the components in isolation but the connections among them that contributes to MPSKT. They are interdependent, which adds to the complexity of MPSKT and how to engage teachers in learning it. Understanding this interdependence may be important for teachers to hold this knowledge so that it is usable in a meaningful and effective way in supporting problem-solving proficiency in their teaching.

This discussion of MPSKT can be built on to provide a framework of key knowledge mathematics teachers ought to hold to inform practice-based investigation of it and to frame learning experiences to help teachers to understand the nature of this knowledge. It can also be built on to offer opportunities to prospective teachers to help them to be prepared to teach problem solving meaningfully by exploring these conceptions in terms of their nature and possibilities.

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INTERACTIONS DE LA CLASSE : TENSIONS ENTRE COMPRÉHENSION ET DIFFICULTÉS À APPRENDRE LES MATHÉMATIQUES

INTERACTIONS IN THE CLASSROOM: TENSIONS BETWEEN INTERPRETATIONS AND DIFFICULTIES LEARNING MATHEMATICS

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(English translation follows.)

Nos recherches sur l'interprétation à donner aux activités cognitives des élèves en mathématiques (DeBlois, 2003, 2014a) et sur la transformation de la sensibilité des enseignants à l'égard des erreurs de leurs élèves (DeBlois, 2006, 2009) montrent l'émergence de tensions dans les interactions de la classe (DeBlois, 2014b). Nous avons posé l'hypothèse selon laquelle la variété d'erreurs des élèves en mathématiques prend son origine dans les interactions de la classe (DeBlois, 2008, 2012) et le contrat didactique qu'elles laissent émerger. Nous avons ensuite choisi d'entrer dans l'étude de ces difficultés lorsque des comportements d'évitement, d'anxiété et d'agitation surgissent. Nos résultats montrent que le développement de règles et d'habitudes ont pu contribuer à l'émergence de tensions. Par exemple, des associations entre des mots et des opérations, plutôt que les relations entre les données, pourraient prendre leur origine dans une tension entre le temps d'apprentissage et le temps d'enseignement. Il devient ainsi important de cerner les rôles joués par les différents acteurs vis-à-vis du sens des savoirs et de la valeur qui leur sont attribués.

LE CONTEXTE

Au plan social, nous entendons parler de plus en plus des difficultés comportementales dans les classes ordinaires. C'est ainsi que Massé, Desbiens, et Lanaris (2006) reconnaissent que le taux de prévalence des troubles du comportement chez les élèves serait à la hausse. Selon Royer (2009), les causes de ces troubles seraient souvent liées à l'environnement physique et social des élèves. Ce type d'analyse conduit le plus souvent à adapter l'environnement dans lequel baigne l'élève d'un point de vue institutionnel, physique, social ou affectif (DeBlois & Lamothe, 2005). Par exemple, adapter l'environnement physique de l'élève pourra conduire à diminuer la quantité de matériel mis à sa disposition ou encore à espacer les tables de travail alors qu'une adaptation de l'environnement social pourra se manifester en attribuant des récompenses pour certains comportements attendus. Différentes stratégies comportementales proactives sont aussi proposées : réduire la durée de la tâche, découper le contenu en parties ou alterner entre des activités calmes et des activités plus actives, entre le travail individuel et de groupe, entre des activités d'écoute et des activités participatives (Massé & Couture, 2012). Nous avons cherché à répondre à la question : Peut-on faire autrement?

Nos recherches nous ont conduit à observer des relations entre l'interprétation des enseignants et le choix des interventions privilégiées (DeBlois, 2006, 2009). Nous avons pu constater que lorsque les enseignants étudient l'écart entre les résultats d'un élève et le résultat attendu, pour comprendre l'origine de l'erreur, ils attribuent l'erreur à un problème d'attention de ces derniers alors que s'ils considèrent l'erreur comme une extension d'un apprentissage antérieur, ils essaient de créer un conflit cognitif afin que les élèves reconnaissent la différence entre les situations. En outre, nous avons pu remarquer que lorsque les enseignants s'intéressent aux interactions que les élèves entretiennent avec la tâche, ils souhaitent comprendre leurs représentations à l'égard de la tâche, à cerner d'où elles viennent et comment elles ont été construites. Ainsi, interpréter les solutions erronées de ces deux élèves de 8 ans de façon différente conduit à intervenir autrement.

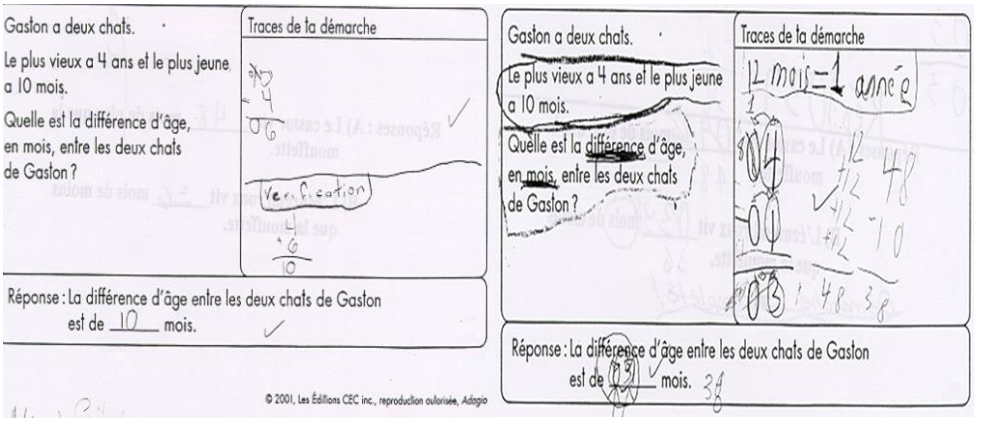

Figure 1

En effet, une interprétation des deux erreurs qui se détache des difficultés d'attention permet de considérer l'apprentissage antérieur de ces élèves à l'égard des opérations d'addition et de soustraction avec des nombres naturels. Les hypothèses posées par les enseignants concernent ces apprentissages. L'élève de la production de gauche réalise une soustraction en portant une attention au nombre le plus grand (10) et le plus petit (4) au détriment du sens de ces nombres. L'élève de la production de droite interprète le nombre qui représente la plus longue durée (ans) et la plus courte durée (mois) en « adaptant » le nombre 10 (01) afin de rendre possible la soustraction. Ces erreurs manifestent des attentes que ces élèves entretiennent à l'égard de la tâche. Il nous a donc semblé important d'étudier ces dernières en posant comme hypothèse que ces dernières pourraient contribuer à la désorganisation ou à l'évitement devant les situations chez certains élèves. Toutefois, ces attentes font partie des activités cognitives des élèves. Il est donc nécessaire de les situer pour mieux les étudier.

LE CADRE THÉORIQUE RETENU

Afin de cerner les activités cognitives des élèves, nous nous sommes appuyés sur la théorie de l'abstraction réfléchissante de Piaget (1977). À partir de cette théorie, nous avons réalisé des recherches sur la numération et les problèmes ayant une structure additive (DeBlois, 1996; 1997a, 1997b). Ainsi, à l'occasion de la recherche portant sur la numération, nous avons pu observer comment une alternance entre les représentations que les élèves se donnent de la tâche, leurs procédures de comptage et certaines prises de conscience à l'égard des caractéristiques de la numération contribuaient à leur apprentissage. Par exemple, l'attention des élèves qui ont illustré une quantité avec un matériel, comme des enveloppes opaques et des jetons, a alterné entre les jetons (le contenu de l'enveloppe) et les groupements d'objets (l'enveloppe comme contenant) avant de prendre conscience des relations d'équivalence, notamment entre 1 dizaine et 10 unités, puis 10 dizaines et 1 centaine. Ces relations d'équivalence ont ensuite contribué à utiliser un comptage par 100, par 10 ou par 1 des différents groupements illustrant un nombre. Toutefois, lorsque les élèves accordent peu de confiance au comptage, ils visent à arriver au résultat « prévu », délaissant le comptage comme moyen pour obtenir un résultat. Nous avons donc été en mesure de reconnaître que certaines attentes s'insèrent ainsi dans le processus d'apprentissage des élèves.

Nos recherches sur les problèmes ayant une structure additive (Vergnaud, 1981) ont permis de reconnaître à nouveau l'importance des représentations que les élèves se donnent de la situation, de leurs procédures, de leurs prises de conscience et de leurs attentes. Par exemple, un problème de complément d'un ensemble[1] exige des élèves de considérer la relation logico-mathématique de l'inclusion des sous-ensembles dans un ensemble. Toutefois, pour que cette compréhension se développe, il semble qu'il soit nécessaire que les élèves se donnent une compréhension des nombres comme les représentants d'une quantité pour comparer les moments de l'histoire (avant – après), puis chacun des sous-ensembles ou des nombres, pour enfin prendre conscience de la quantité manquante permettant de retrouver le total (DeBlois, 1997a). Bien que la manipulation de matériel représentant la situation semble contribuer à cette réorganisation, elle n'est pas suffisante. En effet, la manipulation permet une illustration des nombres, ce qui attire l'attention des élèves sur l'organisation en centaine, en dizaine et en unités au détriment de la relation logico-mathématique entre les données (inclusion des sous-ensembles dans un ensemble). Cette prise de conscience semble nécessaire à une évolution des procédures des élèves.

Nous avons observé un cheminement semblable devant des problèmes de comparaison d'ensembles[2] (DeBlois, 1997b). Une évaluation qualitative (peu vs. beaucoup) conduit d'abord les élèves à établir une correspondance terme à terme entre les éléments des deux ensembles pour prendre conscience de la « différence » entre les ensembles. Cette prise de conscience leur permet ensuite de compter ce qui reste, ou ce qui manque, pour ensuite construire la relation d'implication (si… alors). L'élève peut alors reconnaître que s'il a 5 crayons de couleurs de moins que son enseignante, alors son enseignante a 5 crayons de plus que lui. Ces travaux ont conduit à définir un modèle d'interprétation des activités cognitives des élèves qui invite à porter une attention aux représentations mentales de leur élève, à leurs procédures, aux prises de conscience et à leurs attentes (Figure 2).

Ce modèle permet de débuter le questionnement à partir de la production de l'élève plutôt qu'à partir de la situation proposée. Afin de cerner les attentes entretenues par les élèves à l'égard d'une situation, le concept de contrat didactique (Brousseau, 1988) est nécessaire. Ce type de contrat permet de cerner la façon dont les élèves pensent que les connaissances

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[1] Tu as un panier de 118 fruits. Tu sais que tu as 37 pommes. Les autres fruits sont des kiwis. Combien as-tu de kiwis?

[2] Tu as 8 crayons de couleurs. Tu as 5 crayons de couleurs de moins que ton enseignante. Combien de crayons ton enseignante a-t-elle?

s'articulent dans une situation, le rôle qu'ils croient devoir jouer[3] (Dencuff, 2010) et le rôle auquel ils s'attendent de l'enseignant. Ainsi, le contenu mathématique des situations est constitutif du contrat didactique, ce qui le distingue du contrat pédagogique. Le contrat didactique est contextualisé à la situation alors que le contrat pédagogique est générique à la vie scolaire de l'élève. En distinguant le contrat didactique du contrat pédagogique, il devient possible de distinguer la confrontation, entre l'enseignant et l'élève qui se désorganise devant une situation, et le conflit cognitif issu d'une rupture de contrat didactique[4]. Alors que la confrontation situe l'élève dans le rôle de « l'enfant », un acteur social qui remet en question les règles sociales; le conflit cognitif situe ce même élève dans le rôle de « l'apprenti » qui se trouve face à des « connaissances » qui ne fonctionnent plus, connaissances traitées à la manière de règles[5] ou d'habitudes[6] (DeBlois & Larivière, 2012). Un engagement dans le rôle de « l'apprenti » lui permettrait de s'émanciper des règles et des habitudes élaborées pour entrer dans les savoirs mathématiques.

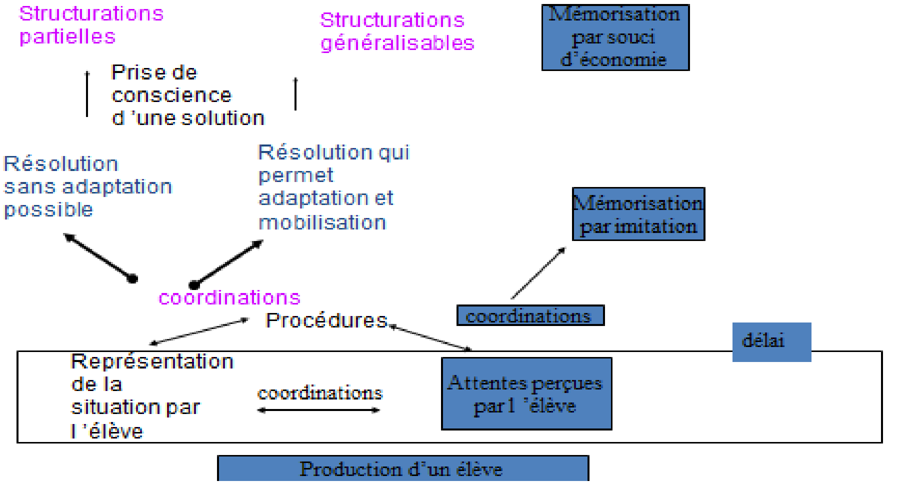

Figure 2. Modèle d'interprétation des activités cognitives des élèves (DeBlois, 2003) pour interpréter le processus d'Alex.

Toutefois, Brousseau (1988) rappelle que certains « effets de contrat didactique » peuvent réduire les apprentissages des élèves. Il a d'ailleurs documenté un certain nombre d'effets de contrat dont l'effet Topaze qui consiste à donner tellement d'indices à l'élève pour qu'il réussisse que son activité cognitive est réduite de plus en plus. Mary (2003) ajoute comment les paroles d'introduction d'une enseignante peuvent être interprétées par l'élève comme étant des indications liées à la situation proposée, créant ainsi un effet de contrat didactique. Pour répondre aux attentes de l'enseignant telles qu'il les interprète, l'élève cherche et organise son travail à partir de ce qu'il sait. Les attentes des élèves s'organisent à partir de règles et /ou d'habitudes élaborées souvent à l'insu des intervenants. La mise à l'épreuve de notre

[3] Inspiré par les travaux de Dencuff (2010), nous définissons l'enfant comme un acteur social qui développe des habitudes sociales et familiales; l'élève comme un acteur scolaire qui cherche à se conformer aux règles scolaires et aux routines de la classe et l'apprenti comme un acteur qui s'engage dans une situation qui risque de transformer ses connaissances.
[4] Une rupture de contrat didactique offre une opportunité d'apprentissage. En effet, la rupture de contrat didactique survient au moment où une connaissance ne fonctionne plus. Par exemple, pour comprendre que $0 - 3 = -3$ l'élève doit délaisser la conception selon laquelle 0 signifie « rien », pour élargir sa conception selon laquelle 0 correspond à un point d'origine sur la droite numérique.
[5] Une connaissance élaborée par l'élève sur la base de ses observations personnelles.
[6] Une connaissance élaborée par l'élève sur la base des routines et des règles de la classe.

hypothèse permet d'interpréter la désorganisation, l'anxiété ou l'évitement de la tâche comme une manifestation des attentes de l'élève.

Il ne s'agit pas de reconnaitre que les élèves en difficultés de comportements créent des règles différentes des autres élèves, mais bien que certaines règles sont issues du contrat didactique expliqueraient autrement les comportements inappropriées de certains élèves, notamment ceux souvent identifiés en difficultés comportementales. Ainsi, tous les élèves ayant créés ces règles ne développent pas de difficultés comportementales mais tous les élèves rencontrés et ayant manifestés des comportements d'évitement avaient créé ce type de règles. Poser la question des contenus sur lesquels travaillait l'élève au moment où son comportement est devenu inacceptable permet de porter une attention particulière au « discours intérieur » de l'élève qui se désorganise. Ce discours fait intervenir les habitudes et les règles qui se sont développées durant l'ensemble de sa scolarité. Quelles attentes les élèves ont-ils lorsqu'ils font des mathématiques? Ces attentes prennent leur origine dans quelles règles ou dans quelles habitudes? Ces règles ou ces habitudes peuvent-elles expliquer les comportements d'évitement et les difficultés comportementales observées?

UNE MISE À L'ÉPREUVE DE NOTRE HYPOTHÈSE

Les trois années d'expérimentation ont permis de réaliser au total 46 médiations[7] avec des élèves de 6-7 ans (2011), de 8-9 ans (2012) et de 11-12 ans (2013). Le contenu des médiations, d'une durée de 10 à 15 minutes, portait sur les contenus vus par l'ensemble de la classe en présence de l'enseignante et étaient enregistrées au moyen d'une caméra flip. Une anticipation des erreurs des élèves, par une analyse préalable des contenus, a été réalisée. Des questions ouvertes ont été prévues. Le choix des élèves qui ont participé à une médiation a été motivé par des réactions d'anxiété[8], d'évitement[9] ou encore d'agitation[10] chez ces élèves dans leur classe. Ces médiations ont d'ailleurs eu lieu en classe en présence de l'enseignante.

L'étude des médiations a permis de repérer les attentes entretenues par les élèves à l'égard d'une grande variété de situations mathématiques : problèmes à résoudre portant sur les nombres naturels, sur les fractions, sur les statistiques, en géométrie et en probabilités. Il a ainsi été possible de constater d'abord la présence de rupture de contrat didactique, confirmant ainsi notre hypothèse de recherche.

GERMAIN, UN ÉLÈVE DE 9 ANS

Germain pousse des soupirs en disant que sa solution ne fonctionne pas. Cette agitation, avec anxiété, motive le choix de l'élève pour réaliser une médiation. Placé devant la tâche demandant de colorier ¾ des framboises illustrées (Figure 3), il associe le dénominateur à un partage égal entre amis pour réaliser ce partage sur la base de sa connaissance des facteurs de 12 (6 + 6… des ensembles de 3). Toutefois, pour illustrer le numérateur, il réfère à ses expériences sur la fraction « partie d'un tout » en expliquant : « Ils disent d'en colorier 1 dans chaque…». Il utilise une connaissance familière pour identifier le numérateur mais il « sent » que cela ne fonctionne pas, ce qui provoque son agitation. Il manifeste la présence d'une

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[7] Nous avons préféré le concept de médiation à celui d'entrevue puisque nous ne pouvions prévoir avec qui nous aurions à travailler ni sur quel contenu. Des questions générales ont été planifiées, mais des adaptations importantes étaient nécessaires, et des erreurs possibles ont été anticipées compte tenu de l'état d'avancement des recherches.

[8] L'élève se parle, à voix haute, pour exprimer qu'il ne comprend pas ou qu'il ne sait pas quoi faire.

[9] L'élève regarde autour de lui sans commencer la tâche, se couche la tête sur le bureau ou est dans « la lune ».

[10] L'élève prend son crayon et cherche à attirer l'attention de son voisin d'en face. Il pousse sur un panier de plastique qui les sépare.

rupture de contrat didactique. Devant cette rupture, Germain semble adopter la posture de l'enfant.

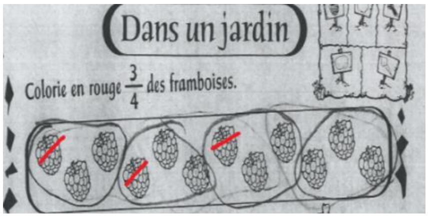

Figure 3

ALEX, UN ÉLÈVE DE 7 ANS

Nous avons aussi pu repérer d'autres phénomènes à l'origine des réactions des élèves. Ainsi, à l'instar des travaux de Mary (2013), la présence d'effets de contrat didactique a provoqué une réaction d'évitement, sans qu'il n'y ait nécessairement anxiété. Par exemple, lorsque nous débutons la médiation, Alex, un élève de 7 ans, joue avec ses crayons et discute avec ses voisins de table. Il a dessiné des cercles pour illustrer les nombres de la tâche. Toutefois, il a trouvé la solution 8 à la soustraction 8 – 2 (DeBlois & Larivière, 2012). Cette erreur découle d'une attention portée à la méthode de travail (surligner les mots importants) et aux nombres (qu'il souligne en jaune), délaissant la relation de retrait du problème. Il s'agit donc d'un effet de contrat didactique. Il explique déterminer l'opération à effectuer en fonction de l'ordre de présentation des nombres de l'énoncé. Ainsi, il choisit de soustraire parce que 8 est plus grand que 2 et qu'il est placé avant le 2 dans le problème. L'attention portée à la méthode de travail et à l'ordre de présentation des nombres, plutôt qu'à la relation logico-mathématique de retrait, manifeste de sa posture d'élève plutôt que d'apprenti. Cette posture ne suscite pas une validation de sa solution.

ALBERT, UN ÉLÈVE DE 11 ANS

Enfin, nous avons pu constater que certaines réactions d'élèves émergeaient d'une extension de ses connaissances, sans rupture de contrat didactique. Par exemple, connaissant le volume de la partie émergeante d'un iceberg (587 m^3) Albert est invité à trouver 100% du volume de cet iceberg. Il a écrit 90/100 sur sa feuille, reconnaissant ainsi que 90% correspond à 90/100, une écriture familière à une activité antérieure. Puis, il fixe le mur, joue avec son crayon, soupire et parle à voix haute. Cette réaction d'évitement, sans anxiété, conduit à réaliser une médiation. À cette occasion, l'élève ne parvient pas à interpréter la mesure 587 m^3 qui correspond à 10% du volume de l'iceberg, une manifestation d'une rupture de contrat didactique puisqu'il ne peut associer 10% au grand nombre 587. Il désigne l'iceberg complet avec son crayon lorsqu'il est invité à montrer ce que représente 90% de ce dernier. Parvenu à illustrer 90%, grâce à une figure carrée qu'il a quadrillée, puis les dix 10% composant le total, l'élève utilise la multiplication. Toutefois, si l'utilisation de cette procédure familière permet à Albert de trouver le volume total de l'iceberg, aucune interprétation du sens du nombre 587 m^3 (10%), qui correspond à ce qui manque à 90% du volume immergé pour obtenir 100%, n'apparaît. L'attention est portée sur l'algorithme de la multiplication, plutôt qu'à la relation logico-mathématique du complément d'un total exprimée en pourcentage, une manifestation de son rôle d'élève. Cette posture ne suscite pas une validation de sa solution.

Le tableau 1 permet d'observer que les ruptures de contrat didactique jouent un rôle dans les réactions d'évitement, d'anxiété et d'agitation des élèves rencontrés. Toutefois, d'autres phénomènes, comme les effets de contrat didactique ou l'extension d'une connaissance (DeBlois & De Cotret, 2005), sont à considérer.

Phénomènes à l'origine de la médiation	Effet de contrat didactique	Extension d'une connaissance	Rupture de contrat didactique
1er cycle primaire (/15) Classe ordinaire	2	9	5
2e cycle primaire (/16) Classe ordinaire	9	1	6
3e cycle primaire (/15) Classe spécialisée	4	6	5
Total (47)	15	16	16

Tableau 1. Phénomènes à l'origine de la médiation.

Ainsi, le développement de méthodes de travail qui mettent l'accent sur une démarche ou sur une conformité à l'égard des prescriptions des enseignants ont pu contribuer à l'émergence d'habitudes qui réduisent l'apprentissage. Une tension entre l'aide à l'apprentissage, prévue dès la planification et visant la collectivité comme l'illustre le cas d'Alex, et l'aide à l'élève comme individu ne permet pas à Alex d'entrer dans le rôle de l'apprenti pour valider sa solution. De plus, favoriser les associations, plutôt que les relations, semble contribuer comme l'illustre le cas d'Albert qui identifie 100% du volume d'un iceberg, sans avoir pu reconnaître qu'il complétait une mesure donnée. Temps d'apprentissage et temps d'enseignement ne coïncidant pas, une tension entre ces deux pôles pourrait nuire à une « connexion » entre les actions des élèves et une institutionnalisation qui a un sens pour les élèves. Ainsi, Germain qui sait illustrer 3/4 de 1, se trouve devant une impasse devant l'ensemble de 12 framboises, une rupture de contrat didactique. À l'instar des travaux de Mary et al. (2014), nous observons l'influence des actions des élèves sur la négociation de sens. Sans la prise en compte des actions des élèves, ces derniers ne sont pas en mesure de donner sens aux savoirs et de reconnaître leur valeur pour jouer leur rôle d'apprenti (DeBlois, 2014a).

CONCLUSION

Nous avons voulu expliquer autrement que par des phénomènes sociaux ou affectifs les difficultés comportementales observées dans les classes en considérant les situations au moment d'une réaction d'évitement, d'anxiété ou d'agitation. Cette avenue semble prometteuse. Nous avons pu constater que le mouvement de l'élève vers le savoir fait surgir des ruptures au moment où une connaissance ne fonctionne plus. Toutefois, en adoptant le rôle de l'élève, il semble que le mouvement se sclérose, les empêchant de transformer leurs connaissances. Le mouvement de l'enseignant consistant à s'éloigner d'un savoir institutionnel pour s'approcher des connaissances de l'élève pourrait contribuer à organiser un environnement conceptuel. Ce dernier enrichit les interventions réalisées auprès des élèves qui manifestent des réactions d'évitement, d'anxiété ou d'agitation pour accéder à un rôle d'apprenti, plutôt que de demeurer dans celui de l'enfant ou de l'élève.

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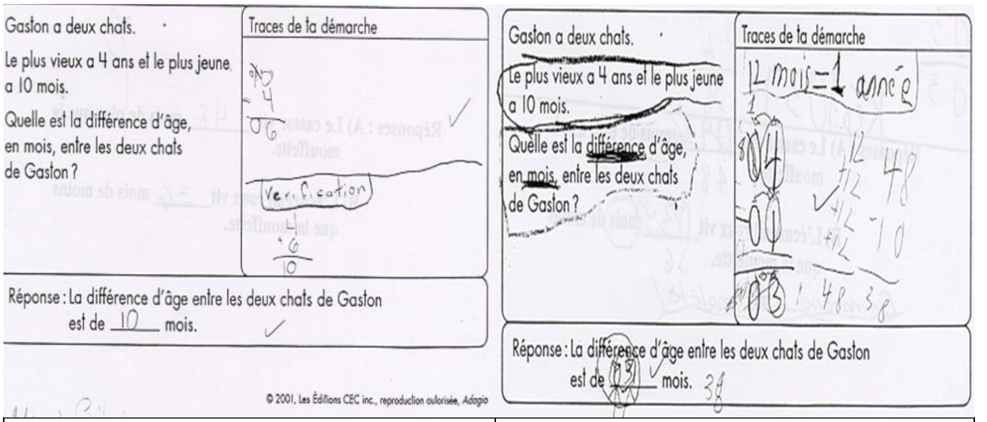
*My research on the interpretation of students' cognitive activities in mathematics (DeBlois, 2003, 2014a) and on teachers' sensitivity towards students' errors (DeBlois, 2006, 2009) has provided evidence of the emergence of tensions in classroom interactions (DeBlois, 2014b). It has been my hypothesis that the range of students' errors stem from classroom interactions (DeBlois, 2008, 2012) and the didactical contract that the latter give rise to. I then opted to study these difficulties in regard to when and how students exhibit anxiety, agitation and task avoidance. My results appear to show that the development of ingrained cognitive rules and habits may contribute to the emergence of tensions. For example, associations between words and operations, as opposed to relationships between the data given in problems, could originate from tensions between the teaching phase and the learning phase. At that point, it becomes important to identify roles played by the different classroom actors in regard to the meaning and value ascribed to mathematical knowledge (*savoirs*).*

CONTEXT

Across society, there is growing talk about behavioural problems in regular classes. Massé, Desbiens, and Lanaris (2006) have noted that the prevalence of behavioural problems appears to be on the rise. Furthermore, according to Royer (2009), such problems often stem from students' physical and social environment. More often than not, this sort of analysis results in institutional, physical, social or emotional adaptations being made to the students' environment (DeBlois & Lamothe, 2005). For example, adapting the physical environment may entail reducing the quantity of materials made available to students, or setting desks at greater distance from one another. Adapting the social environment could take the form of rewarding students for certain expected behaviours. Likewise, various proactive behavioural strategies are available for implementation, including: reducing the duration of the task at hand, segmenting the learning contents, or alternating between quiet and active activities, individual and group work, or listening and participatory activities (Masse & Couture, 2012). For my part, I have strived to address the question of whether it was possible to adopt a different approach altogether.

My research has led me to observe the relationships between the interpretations of teachers and their preferred strategies and interventions (DeBlois, 2006, 2009). It is my observation that when, in order to understand the origin of an error, teachers examine the gap between the results arrived at by a student and the expected outcome, they ascribe the error to a problem of attention on the part of the student. When, on the other hand, they view the error as the extension of previous learning, they instead try to create a cognitive conflict so as to enable students to distinguish between different mathematical situations. Furthermore, I have noticed that when teachers look into the way students interact with the task at hand, they also demonstrate a desire to understand students' representations of the task and to identify the origins of these representations and how they were constructed. All in all, different approaches to interpreting the erroneous solutions of the following two 8-year-old students mean that different types of intervention are likely to be adopted.



Translation: Gaston has two cats. The older one is 4 years old and the younger one is 10 months old. What is the difference in age, in months, between Gaston's two cats? **Problem-solving area** Verification **Answer:** The difference in age between Gaston's two cats is 10 months.	Translation: Gaston has two cats. The older one is 4 years old and the younger one is 10 months old. What is the difference in age, in months, between Gaston's two cats? **Problem-solving area** 12 months = 1 year **Answer:** The difference in age between Gaston's two cats is 3 months.

Figure 1

Thus, by opting for an interpretation of the two errors that goes beyond problems of attention, there is then room to consider what students have previously learned regarding operations of addition and subtraction with natural numbers. Such previous learnings were the subject of hypotheses by the teachers concerned. To perform a subtraction, the child whose work appears on the left focused on the larger number (10) and the smaller number (4), at the expense of the meaning of these numbers. The student whose work appears on the right interpreted the number representing the longest duration (years) and the shortest duration (months), "adapting" the number 10 (01) in order to proceed with the subtraction. Both of these errors manifest the students' expectations regarding the task to be performed. For this reason, I believe it is important to examine students' expectations; to that end, I work from the hypothesis according to which such expectations could, among some students, contribute to cognitive or behavioural disorganization or the avoidance of mathematical situations. At the same time, such expectations are a part of students' cognitive activities, with the implication being that in order to properly study them, it is critical to first situate them in context.

THEORETICAL FRAMEWORK

With the goal of discerning students' cognitive activities, I relied on Piaget's (1977) theory of *reflecting abstraction* (*abstraction réfléchissante*) to perform research on numeration and problems involving an additive structure (DeBlois, 1996, 1997a, b). Thus, during a project bearing on numeration, I noticed that the alternation occurring between students' representations of the situation, their counting procedures, and some of their conscious realizations (*prises de conscience*) concerning the characteristics of numeration all contributed to their learning process. For example, the attention of students who illustrated a quantity using manipulatives, such as opaque envelopes and tokens, alternated between the tokens (the contents of the envelope) and the groups of objects (the envelope as container) before coming to the conscious realization of equivalence relations, in particular between 1 ten and 10 units, and then again between 10 tens and 1 hundred. At that point, equivalence relations contributed to the use of counting various groups illustrating a number based on 100, 10 or 1. However, whenever students placed little trust in counting, they attempted to arrive at the 'expected' answer and skipped counting as a way of obtaining a result. On the basis of these observations, I was able to see how students' learning processes are framed by certain expectations.

My research concerning problems involving an additive structure (Vergnaud, 1981) again serves to highlight the significance of students' representations of the situation, as well as their procedures, conscious realizations and expectations. For example, a problem involving the complement of a set[1] requires students to consider the logico-mathematical relationship involving the inclusion of subsets within a set. However, in order for such understanding to develop, it would appear that students must first grasp numbers as being the representatives of a quantity so as to then be able to compare the times in the narrative (before – after) and thereafter each of the subsets or numbers; at that point, they can finally come to the conscious realization of the missing quantity allowing them to go back to the total (DeBlois, 1997a). While the manipulation of learning materials representing this situation appears to contribute to these reorganizational operations, it does not suffice by itself. Indeed, the manipulation provides a basis for illustrating numbers, thus drawing students' attention to organization in hundreds, tens and units, but at the expense of the logico-mathematical relationship between the data (i.e., inclusion of subsets in a set). A conscious realization of this kind would appear to be necessary in order for an evolution in students' procedures to occur.

I have observed a similar sequence of development at work when students had to tackle problems involving the comparison of sets[2] (DeBlois, 1997b). A qualitative evaluation (i.e., a little vs. a lot) initially prompted students to establish a one-to-one correspondence between the terms of two sets before coming to the conscious realization of the 'difference' between the sets. This awareness then enabled them to count what remained, or was lacking, and then move on to constructing the relation of implication (i.e., if.., then…). At that point, the student was able to recognize that if he had 5 coloured pencils less than his teacher, then she had five coloured pencils more than he. On the basis of these investigations, I have outlined a model for interpreting students' cognitive activities that highlights student's mental representations, procedures, conscious realizations and expectations (Figure 2).

This model makes it possible to begin inquiry on the basis of the student's work rather than on the proposed mathematical situation. In order to identify the expectations of students regarding a situation, the concept of *didactical contract*[3] (Brousseau, 1988) is required. This type of contract is useful for identifying the way students think that knowledge

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[1] You have a basket with 118 pieces of fruit in it. You know you have 37 apples. The remaining pieces of fruit are kiwis. How many kiwis do you have?

[2] You have 8 coloured pencils. You have 5 coloured pencils less than your teacher. How many coloured pencils does she have?

[3] The glossary of terms used in *didactique* translated by Warfield (Brousseau, 2003) defines didactical contract: "This is the set of reciprocal obligations and "sanctions" which each partner in the *didactical situation* imposes or believes himself to have imposed [...] on the others, or are imposed [...] on him with respect to the knowledge in question."

(*connaissances*)[4] is brought into play in a situation, the role students believe they are required to play[5] (Dencuff, 2010), and the role they expect the teacher to assume. Thus, the mathematical content of situations is constitutive of the didactical contract, thus distinguishing it from the pedagogical contract. The didactical contract is framed in terms of the situation at hand, whereas the pedagogical contract applies generally to the student's relationship to school, teachers and learning. Distinguishing between the didactical contract and the pedagogical contract makes it possible to discern not only the confrontation occurring between the teacher and a student who becomes cognitively or behaviourally disorganized when grappling with a situation, but also the cognitive conflict stemming from a break in the didactical contract[6]. Whereas the confrontation situates the student in the role of *child*—i.e., a social actor who challenges social rules—the break in the didactical contract situates this same student in the role of a *learner* who must grapple with *knowledge* that no longer functions—i.e., knowledge cast in terms of rules[7] or habits[8] (DeBlois & Larivière, 2012). By engaging in the role of *learner*, students would be able to free themselves of previously developed cognitive rules and habits and move on to mathematical knowledge (*savoirs*).

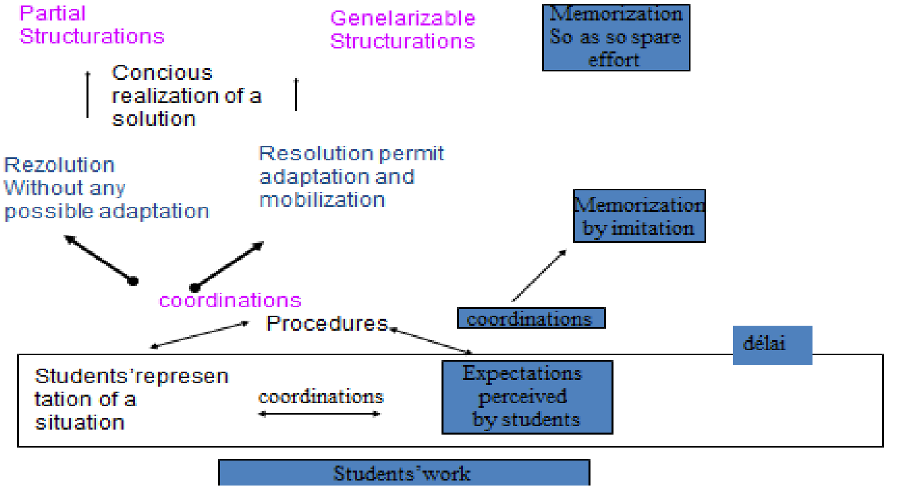


Figure 2. Interpretive model of students' cognitive activities (DeBlois, 2003).

[4] Translator's note: In acknowledgement, generally, of a terminological distinction in French (*connaissances* vs. *savoirs*) with no exact equivalent in English (knowledge), and, more specifically of the work of G. Brousseau and his theory of *didactique* 'connaissances' will refer here to personally devised, discrete elements of knowledge, and 'savoirs' will refer to knowledge in its shared or institutional form. For a considerably more nuanced explanation of these semantic issues, English-speaking readers would be well-advised to consult V. Warfield's enlightening *Introduction to Didactique* (2006), and specifically section 13 entitled "S-knowledge and c-knowledge" (pp. 109-111).

[5] Drawing on the work of Dencuff (2010), I define the *child* as a social actor who is developing social and family skills, *student* as a classroom actor who is seeking to complying with school rules and classroom routines, and *learner* as an actor who is engaged in a situation that risks transforming his or her knowledge (*connaissances*).

[6] A break in the didactical contract offers an opportunity for learning. The break occurs whenever a *connaissance* no longer works. For example, in order to understand that 0 – 3 = -3, a student must abandon the conception according to which 0 means 'nothing' and expand it, such that 0 now corresponds to a point of origin on the number line.

[7] Defined here as a piece of knowledge (*connaissance*) developed by a student on the basis of his or her personal observations.

[8] Defined here as a piece of knowledge (*connaissance*) developed by a student on the basis of class routines and rules.

However, Brousseau (1988) has pointed out that a number of "effects of the didactical contract" can limit or constrict students' learning processes. To that end, he has documented a number of such effects, including the "Topaze" effect, which consists in giving the student so many hints so as to enable him to come up with the right answer, narrowing his learning process in the process. However, Mary (2003) has noted that a teacher's introductory statements regarding a situation can be construed by students to mean the instructions pertaining to that situation, thereby generating an effect of the didactical contract. Students, out of a desire to meet what they construe their teacher's expectations to be, rack their brains and organize their work on the basis of their existing knowledge. Students' expectations take shape on the basis of cognitive rules and/or habits that have developed often unbeknownst to teachers. Following testing of my hypothesis, I believe it is possible to interpret cognitive or behavioural disorganization, anxiety or task avoidance as a manifestation of students' expectations.

The point here is not to determine whether students with behavioural problems create rules different from those of other students; it is a question, instead, of how certain rules stemming from the didactical contract appear to provide a different explanation of the inappropriate behaviours of some students, particularly those who are often identified as having behavioural problems. Thus, all students who have created rules of this kind do not develop behaviour problems, but all the students whom I met and who manifested avoidance behaviours did indeed create such rules. Closely examining the learning contents on which students were working when their behaviour became unacceptable provides a basis possible to train particular attention on the 'inner discourse' of students who undergo cognitive or behavioural disorganization. This discourse brings into play the habits and rules that students have developed throughout their entire schooling. What expectations do students have when they do math? What rules and habits do these expectations stem from? Can these rules or habits be used to explain the avoidance mechanisms and behavioural problems observed in the classroom?

TESTING MY HYPOTHESIS

During three years of experiments, I was able to perform a total of 46 *interchanges*[9] with students aged 6-7 years (2011), 8-9 years (2012) and 11-12 years (2013). The content of the interchanges, which lasted from 10 to 15 minutes, dealt with learning contents that had been examined by the entire class in the company of the teacher and were recorded using a flip video camera. Student errors were formally anticipated via a prior analysis of learning contents. Open-ended questions were developed. Students were selected to take part in an interchange on the basis of reactions of anxiety[10], avoidance[11], or agitation[12], as exhibited by them in their classroom. In addition, these interchanges took place in the presence of the teacher.

An examination of interchanges made it possible to identify students' expectations regarding a broad range of mathematical situations—e.g., problems involving natural numbers,

[9] I have preferred the term of *interchange* (*médiation*) to that of *interview,* since I was unable to know ahead of time with whom I would work or what content would be the focus of attention. General questions were planned, but major adaptations had to be made. Likewise, potential errors were anticipated to the degree permitted by the status of the research conducted to that time.

[10] The student talks to himself out loud, saying that he does not understand or does not know what to do.

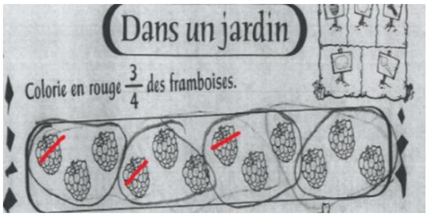
[11] The student looks around him but does not start in on the task, lays his head down on the desk, or has his "head in the clouds."

[12] The student takes his pencil and attempts to draw the attention of his neighbour sitting opposite. He shoves a plastic basket separating them.

fractions, statistics, geometry and probability. On this basis, I was able to note, first of all, a break in the didactical contract, thus confirming my research hypothesis.

GERMAIN, AGE 9 YEARS

Germain heaved sighs, declaring that his solution did not work. This agitation, coupled with anxiety, prompted the choice of this student for the purposes of an interchange. When confronted with the task of colouring ¾ of the illustrated raspberries, he associated the denominator with an equal sharing among friends and proceeded to make this distribution on the basis of his knowledge of the factors of 12 (i.e., 6 + 6…, groups of 3). However, in order to illustrate the numerator, he referred to his experiences with regard to the "*part of a whole*" fraction, explaining that: "*They said to colour 1 in each…*" He used a familiar piece of knowledge (*connaissance*) to identify the numerator but "*had the feeling*" that it did not work, whence his feelings of agitation. He expresses a break in the didactical contract. Having this break, Germain thus appeared to have adopted the posture of *child*.



Translation:

In a Garden

Place a red mark on ¾ of the raspberries.

Figure 3

ALEX, AGE 7 YEARS

I was also able to identify other phenomena underlying students' reactions. Thus, taking a cue from the work of Mary (2013), the effects of the didactical contract triggered an avoidance reaction, although anxiety was not necessarily involved. For example, when I started off the interchange, Alex, a 7-year-old, was playing with his pencils and talking with the other students at his table. He drew circles to illustrate the numbers figuring in the task. However, he came up with the solution of 8 to the subtraction of 8 minus 2 (DeBlois & Larivière, 2012). This error stemmed from his focus on the method of work (i.e., underlining important words) and numbers (which he highlighted in yellow), effectively ignoring the relationship of taking away implied by the problem. According to his explanation, he had identified the operation to be performed based on the order of the presentation of the numbers appearing in the statement of the problem. Thus, he chose to subtract because 8 is bigger than 2 and he was confronted with 2 in the problem. The focus on the method of work and the order in which the numbers were presented—as opposed to the logico-mathematical problem of taking away— manifests his posture as *student* as opposed to *learner*. He expresses an effect of the didactical contract. As such, this posture does not prompt a validation of his solution.

ALBERT, AGE 11 YEARS

Lastly, I was able to note that some student reactions stemmed from an extension of their knowledge (*connaissances*) but without entailing a break in the didactical contract. For example, knowing the volume of the tip of an iceberg (the tip being 587 m^3), Albert was asked to find 100% of the volume of this iceberg. He wrote down 90/100 on his sheet, thus acknowledging that 90% corresponds to 90/100, a familiar notation carried over from a previous activity. He then stared at the wall, toyed with his pencil, sighed and talked out loud. This avoidance reaction, from which anxiety was absent, prompted an interchange. At that time, this student was unable to interpret the measurement of 587 m^3, which corresponded to 10% of the volume of the iceberg. He expresses a break in the didactical contract because 10% and the large number 587 can't be in relationship. He pointed to the entire iceberg with his pencil when asked to show what 90% of the iceberg represented. Once having managed to illustrate 90%, thanks to a square figure which he turned into a grid, and then the ten 10% portions making up the total, he resorted to multiplication. That being said, while this familiar procedure enabled Albert to come up with the total volume of the iceberg, no interpretation of the meaning of the number 587 m^3 (10%)— i.e., what is lacking from 90% of the below-water volume to obtain 100%—can be observed. The focus was on the multiplication algorithm rather than on the logico-mathematical relationship of the complement of a total expressed in terms of a percentage. We have here a manifestation of Albert's role as *student*. As such, this posture does not prompt a validation of his solution.

Table 1 serves to show that breaks in the didactical contract play a role in the reactions of avoidance, anxiety and agitation among the students encountered. However, other phenomena, such as the effects of the didactical contract or the extension of a piece of knowledge (*connaissance*) (DeBlois & De Cotret, 2005), are also to be considered.

Phenomena underlying the interchange	Effect of didactical contract	Extension of a piece of knowledge	Break in the didactic contract
Primary – Cycle 1 (/15) Ordinary class	2	9	5
Primary – Cycle 2 (/16) Ordinary class	9	1	6
Primary – Cycle 3 (/15) Specialized class	4	6	5
Total (47)	15	16	16

Table 1. Phenomena underlying the interchange.

Thus, the development of work methods that emphasize a strategy or conformity toward teachers' prescriptions may have contributed to the emergence of habits that limit the learning process. A tension between learning support, which had figured in class planning from the start and was targeted at the group—as is illustrated the case of Alex—and help for the student as an individual did not allow Alex to enter into the role of *learner* to validate his solution. An additional contributing factor appears to have been an emphasis on associations rather than on relationships, as is shown in the case of Albert, who identified 100% of the volume of an iceberg, without having been able to recognize that he was totalling a given measurement. As learning time and teaching time do not coincide, the resulting tension between these two poles could hamper the development of a connection between the actions of students and an *institutionalization*[13] that makes sense for students. Thus, for example, Germain was able to illustrate ¾ of 1 but was at a loss when confronted with the set of 12 raspberries. Much in keeping with the research of Mary et al. (2014), I notice the influence of students' actions on the negotiation of meaning. Where no allowance is made for the actions of students, the latter are unable to make sense of mathematical knowledge (*savoirs*) and recognize its importance for playing their role of *learner* (DeBlois, 2014a).

[13] To borrow from Brousseau (as translated by Warfield (2003)), *institutionalization* may be defined as: "The passage of a *connaissance* from its role as a means of resolving a situation of action, formulation or proof to a new role, that of reference for future personal or collective uses."

CONCLUSION

It has been my aim to explain behavioural problems seen in the classroom not on the basis of social or emotional phenomena but instead through an examination of situations in which reactions of avoidance, anxiety or agitation occur. This avenue would appear to be a promising one. For one, I was able to observe that the student's movement toward knowledge (*savoir*) triggers breaks in the didactical contract once a piece of knowledge (*connaissance*) no longer works. However, this movement ossifies with the adoption of the role of *student*, thus hampering the ability of subjects to transform their knowledge (*connaissances*). Where the teacher's movement consists of distancing him-/herself from an institutional knowledge (*savoir*) and coming into closer contact with the student's knowledge (*connaissances*), the result could be to help structure a conceptual environment—i.e., an environment serving to enrich the interventions conducted among students who manifest reactions of avoidance, anxiety or agitation. At that point, these subjects could move into the role of *learner*, instead of remaining confined to that of *child* or *student*.

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MAWKINUMASULTINEJ! LET'S LEARN TOGETHER! DEVELOPING CULTURALLY-BASED INQUIRY PROJECTS IN MI'KMAW COMMUNITIES

Lisa Lunney Borden
St. Francis Xavier University

INTRODUCTION

When I began my doctoral work in 2005, after ten years of teaching secondary mathematics in We'koqma'q, a Mi'kmaw community, I believed that I would work with elders to develop culturally-based mathematics lessons and implement these in Mi'kmaw schools to determine the impact such learning experiences would have on students. I had often told my own students that there was mathematical knowledge within the community even though Mi'kmaw mathematical knowledge had not been written down in our textbooks. I wanted my students to see themselves as capable of doing mathematics and to understand that mathematics is an important part of many community practices. Although culturally-based mathematics units were my goal at that time, it is only now, in 2014, that this is coming to fruition—and this work is still very much in the beginning stages. For my topic study group presentation, I chose to share some of my current work on culturally-based inquiry units and how my doctoral work and the *Show Me Your Math* program has enabled this new work to emerge.

CONTEXT

This work is situated within the Mi'kmaw Kina'matnewey (MK) communities in Nova Scotia. MK is a collective of 12 communities who work collaboratively for education of Mi'kmaw students in K to 12 schooling and tertiary education. MK is an organization that is commonly known as a Regional Aboriginal Education Authority. MK is somewhat unique in that MK partner communities have a jurisdictional agreement with the Canadian federal government that provides these communities with control over education. Since the jurisdictional control is granted to each community, these communities must work together to make decisions. As a result of working together in this way, the communities have been able to secure funds for second level services to provide consultants in the areas of numeracy, literacy, assessment, Mi'kmaw language and so on. They have also been able to share capital dollars to ensure schools are built in communities in a more timely manner.

Currently there are five communities that have K to 12 programming at their own community schools, and two communities that have K to 6 schooling. For those communities that do not have schools, MK works with the public schools that serve these communities to provide additional supports and resources for Mi'kmaw students. MK has also partnered with the faculty of education at St. Francis Xavier University (StFX), where I now teach, to implement teacher training programs and graduate certificates as well as numerous research projects to support Mi'kmaw education broadly. As a result, there are over 120 teachers of Mi'kmaw ancestry who are now employed in these schools—Mi'kmaw teachers teaching Mi'kmaw children in Mi'kmaw communities. All of this work has led to an incredible success rate in MK schools where the graduation rate in the past 5 years has been between 87.0% and 89.3% (Mi'kmaw Kina'matnewey, 2014) which stands in stark contrast to the national graduation rates for Aboriginal children which are often reported to be about 48% (Assembly of First Nations, 2010). These schools are highly successful and high numbers of students go on to tertiary education, yet there is still room for growth in the areas of mathematics and science and this is where my work is situated.

A FRAMEWORK FOR ABORIGINAL MATHEMATICS EDUCATION

While my goal during my doctoral work was to do culturally-based mathematics units, I realized before I could do this I needed to inquire more into the challenges and complexities that were confronting Mi'kmaw students and their teachers while learning mathematics. Thus, during my doctoral work, I worked collaboratively with teachers and elders in two MK schools to explore these tensions. Through conversations in the form of *mawikinutimatimk* (coming together to learn together) I developed a framework (see Figure 1) for transforming mathematics education (Lunney Borden, 2010). Four key areas of attention for transformation emerged as themes: 1) the need to learn from Mi'kmaw language, 2) the importance of attending to value differences between Mi'kmaw concepts of mathematics and school-based mathematics, 3) the importance of attending to ways of learning and knowing, and 4) the significance of making ethnomathematical connections for students. Each theme is briefly described below.

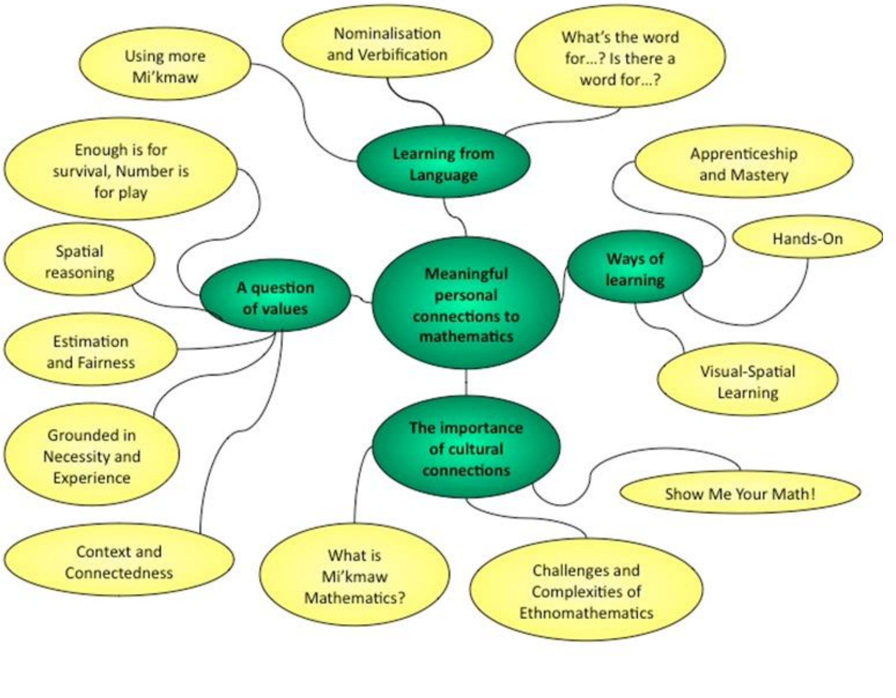

Figure 1. Framework model.

LEARNING FROM LANGUAGE

Although interconnected, each of the themes can be linked to the idea of learning from language, which emerged as an overarching theme in the model development. Examining the indigenous language of a given community context provides a starting place for transforming mathematics teaching and learning. Given that the ways of thinking are embedded in indigenous language, it can be helpful for teachers to understand how the language is structured and used within the community.

I have written about the importance of asking, "*What is the word for*…?" or "*Is there a word for*…?" to better understand how mathematical concepts are described in the language (Lunney Borden, 2013). I have argued that gathering words that can be used to describe mathematical concepts provides insight into concepts that may prove to be potential strengths for building a mathematics program. Similarly, awareness of mathematical concepts that have no translation in the indigenous language exposes the taken-for-granted assumptions that are often present in existing curricula.

I have also written about how understanding the underlying grammar structures of an indigenous language can also support teaching and learning. The prevalence of nominalisation in mathematics stands in direct contrast to the verb-based ways of thinking inherent in the Mi'kmaw language (see Lunney Borden, 2011). This is an important issue for teachers to consider. Looking to 'verbification' as an alternative may help to create a more engaging and rich curriculum for indigenous learners.

Embedding language-learning opportunities into the community-based inquiry units has been an important consideration. The role of elders in this work is essential to provide greater language learning opportunities.

A QUESTION OF VALUES

I have learned that it is also important for educators to think about how mathematical ideas are used and valued in the community context. It is important to understand how numerical and spatial reasoning emerge in the context of the community culture. My doctoral study demonstrated that spatial reasoning was highly valued as it pertained to matters of survival. Numerical reasoning was seen as useful in play. If we consider mathematics to be about examining quantity, space, and relationships (Barton, 2008) then it becomes important to build learning experiences that value these concepts in a way that is consistent with, rather than in opposition to, the way these concepts are valued within the culture. There is a need to build mathematics learning experiences for Mi'kmaw students from a basis of spatial reasoning. In most inquiry units the mathematics emerges from creating in a spatial way—building paddles, birch bark biting, doing bead work or quill work, and so on.

WAYS OF KNOWING

Language and values influence the preferred ways of learning in any community context. It was evident in this community context that a mathematics program should provide children with opportunities to be involved in learning focused on apprenticeship with time for mastery, and hands-on engagement with concrete representations of mathematical ideas. Furthermore, building from a valuing of spatial reasoning, a mathematics program should place visual spatial learning approaches on equal footing with the already dominant linear-sequential approaches, providing more ways to learn so that more students can learn. Again, each of the inquiry projects is rooted in community practices that involve a considerable amount of hands-on learning and learning from elders through traditional apprenticeship models.

CULTURAL CONNECTIONS

While the culturally-based inquiry units draw from the other three areas of the model, the focus of the work is rooted in thinking about the importance of cultural connections. It is essential to make meaningful and non-trivializing connections between the community cultural practices and school-based mathematics. This involves examining how the school-based mathematics can emerge naturally through investigating community practices. It also means creating learning experiences that help students see that mathematical reasoning is a part of their everyday lives, and has been for generations. The success of *Show Me Your Math* suggests that inviting students to be mathematicians who investigate mathematics in their own community contexts could also be an important component of a culturally-based mathematics program.

SHOW ME YOUR MATH

In 2006-2007, Dave Wagner, from the University of New Brunswick, and I began working with MK teachers to implement the *Show Me Your Math* (SMYM) program in MK schools. SMYM invites students in MK communities to find the mathematics in community practices by working with elders, craftspeople, trades people, and so on. This program is an attempt to address the marginalization of Mi'kmaw youth from mathematics by helping them to see that there is a considerable amount of mathematical (and scientific) knowledge in their own community heritage. This work is inspired by Doolittle (2006) who, at CMESG in 2006, suggested it would be helpful to "consider the question of how we might be able to pull mathematics into indigenous culture rather than how mathematics might be pushed onto indigenous culture or how indigenous culture might be pulled onto mathematics" (p. 22). Thus, with SMYM and work emerging from SMYM, I have aimed to begin with community practice as a starting point.

In recent years, SMYM has become increasingly popular with MK students and their teachers, however curriculum pressures have resulted in a call for closer alignment with provincial curriculum expectations so that teachers can integrate SMYM projects into their classroom practices while still addressing specific curriculum outcomes. Over the years, I have heard from teachers that these projects can be beneficial when the whole class works together on a project with an elder coming into the classroom. One example of such an approach involved a grade 10 class learning how to make traditional woven wooden baskets while also learning about the geometry of packaging and, in particular, looking at the economy rate of a package which is a ratio of the volume to the surface area. During this unit, the teacher called upon her aunt, a basket maker, to come and teach the children how to make baskets. Her aunt asked if she had a large class or a small class, and upon learning that there were 23 students, the basket maker decided that since she did not have many basket strips she would make a certain basket. When she came in to do the basket making with the students, they were surprised to learn that the basket she decided to make with them was the answer to the unit problem—the most economical container. This basket is cylindrical where the height is equivalent to the diameter of the base. One student, during his SMYM presentation remarked how surprised he was that the elder knew this without having to do "all the math" that he and his classmates had been learning. The elder had knowledge that was rooted in a philosophy of *tepiaq* (enough) and knew how to maximize capacity with limited basket strips.

The vignette of the basket-making project is one of many classroom based SMYM ideas that has inspired my more recent work in developing culturally-based inquiry units. Dave Wagner and I have seen that one of the benefits of the SMYM program is the development of a sense of "Wholeness [that] resists fragmentation" and creates "quality mathematics experiences [that] require cultural synthesis, bringing together cultures and values from mathematics and the community, personal holism including the child's experiential, conceptual and spiritual development, and intergenerational interaction." (Lunney Borden & Wagner, 2011, p. 379) It is this sense of wholeness that I seek to achieve with the inquiry unit work we are doing now within MK schools.

MAWKINUMASULTINEJ! LET'S LEARN TOGETHER

As a result of recent funding obtained from the Tripartite Forum Fund for Economic and Social Change for the expansion of SMYM, I have been able to develop community-based inquiry units that begin within community practices. These projects draw from our conception of wholeness and build upon Doolittle's (2006) idea of pulling in mathematics by beginning in aspects of community culture where the already present, inherent ways of reasoning within the culture can help students to make sense of the 'school-based' concepts of mathematics in the curriculum. One goal of this work is to have teachers and students learning alongside one another as they explore practices that are relevant to the community, typically with at least one elder or community member with expertise in a certain skill or topic, guiding them along this journey. As such, these projects have been called *Mawkinumasultinej! Let's Learn Together!*

BIRCH BARK BITING

One of the first projects I began working on was birch bark biting. The idea emerged somewhat unexpectedly when my summer research assistant, Sarah, and I were meeting with two elders to discuss some of the projects students had done at the annual math fair to think about ways these might be expanded upon to create classroom teaching ideas. As I spoke with one elder, Josephine, we discussed games that might have been traditionally played by children. The intent of this discussion was to build from games to possible counting and quantity concepts, however this suddenly changed when Josephine declared "*You know, when I was young, my mother would peel thin strips of bark off the logs and ask us if we could fold it and bite shapes into it.*" Aware of the art of birch bark biting in other cultural communities in Canada and intrigued by the mathematical reasoning that would be needed to bite shapes into folded bark, I excitedly asked Josephine more about this. Josephine shared that she recalled some people doing birch bark biting as a past-time when she was young but was unsure if anyone still was able to do it.

She showed me how to fold paper to model how one might fold bark for biting, always folding through the centre. We folded the bark in half and then folded it in half again by folding the original seam onto itself. I asked her if there was a word to describe that action of folding and she replied, "*Yes, Tetpaqikatu!*" I asked her what that meant and she laughed and said, "*Fold it the right way.*" After sharing ideas and looking up some birch bark biting pictures online together, she told me I should try to learn more about it. So, with Sarah's help, that is what I set out to do. We spent most of the rest of June and July learning about birch bark biting.

In doing some library research I came across an article that demonstrated that birch bark biting was indeed a historical part of the Mi'kmaw community:

> *That she was "the last one that can do it" was the same phrase echoed in 1993 by Margaret Johnson, an Eskasoni Micmac elder from Cape Breton. Continuing research has revealed that two other Micmac women – including Johnson's sister on another reserve – can also do it.* (Oberholtzer & Smith, 1995, p. 307)

I had known both Margaret, who was affectionately known as Dr. Granny, and her sister, Caroline Gould, who had resided in the community where I had taught and was a well-respected elder who often visited the school. Unfortunately both women had already passed away but both had been highly respected elders in Mi'kmaw communities known for their commitment to language and culture. This inspired me to share what I was learning with teachers and students and by learning together, birch bark biting was revived in Mi'kmaw communities.

I connected with a teacher in one MK school who was keen to try the birch bark biting with her students and invited me into her classroom to begin this inquiry. I introduced the unit by sharing the story of the conversation I had with Josephine. I showed students pictures and videos of birch bark bitings I had found online and shared the story of discovering the article describing the two Mi'kmaw women who had been birch bark biters—women the students themselves knew. I also demonstrated to students what I had learned from Josephine about folding and even brought a few of my own first attempts at creating birch bark bitings. In sharing my story of learning with the students, I invited them and their teacher to learn along with me. We would learn together.

Birch bark biting involves folding thin pieces of bark and biting shapes into the bark to create designs. The act of folding the bark presents an opportunity for students to think about fractions, angles, and symmetry. Creating the designs draws in geometric reasoning and visualization of geometric shapes. One really needs to understand that a circle is the locus of points equidistant from a given centre to be able to create a circle when biting bark—you cannot see what you are doing, you must be able to visualize. In this work, the teacher and I have seen a deepened understanding of fractions as part of a whole, of geometric properties of 2D shapes, of symmetry, and of transformational geometry. In fact almost all of the grade 5, 6 and 7 curriculum outcomes for geometry can be discussed when exploring birch bark biting, however we did not begin with the outcomes; we began with the birch bark bitings. In the topic group discussion, we also tried our hand (or rather teeth) at doing some birch bark biting to better understand the reasoning involved.

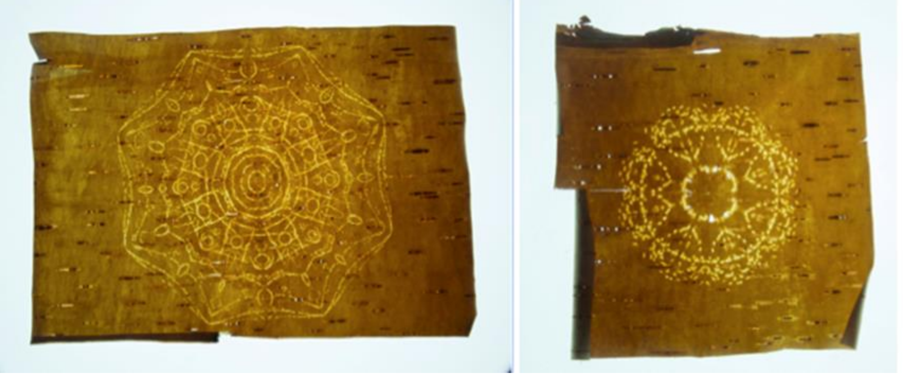

Figure 2. Birch bark bitings by Kim (left) and Phoenix (right), grade 7 students.

Many of the students took to this quickly and showed real talent for doing the bitings. I include here (see Figure 2) two pictures of bitings done by two Grade 7 students. In these designs you can see examples of both rotational and reflective symmetries, as well as geometric shapes, lines and angles. One interesting story shared with me by the teacher is how Phoenix came to her one day to explain that he had figured out how to make an 8-point star (evident in his design). The teacher explained that he had been working at it for some time and once he had done several attempts, trying different angles of biting, he had determined a bite that would result in an 8-point star. The teacher noted that she observed him develop a sense of ownership over it once he figured it out and was then able to create these patterns with intention, knowing what the design would look like.

A second project that students have worked on is paddle making. Students in Grade 8 made a canoe paddle beginning with a plank of wood and carving it into a paddle. The process of carving was done in a few hours, one day a week over a period of 3 or 4 weeks, taking about 6 to 8 hours in all. Students were required to measure themselves "*toes to nose plus four inches*" so that the paddles were proportional to their own bodies. They chose the design for their paddles and had to transfer the template onto their plank by first finding the centre line and then tracing one half of the template, flipping the template, and tracing on the other half to ensure the paddle would be symmetrical. The rough cut was done with a band saw. The students then needed to identify centre lines, quarter lines, and so on, to determine where to carve the paddle. There was a significant amount of measurement and fraction concepts involved in preparing the paddle for carving.

The students who engaged in the paddle-making activity truly enjoyed the experience and the teachers helping with this project noted how engaged and enthusiastic many students were as they did this work. They developed a real sense of pride in their craftsmanship and one student remarked how she "*felt real Mi'kmaq making [her] own paddle*." What was interesting about this project was how the students in this class did not believe they were doing any math. Their teacher commented that they were measuring, calculating, using fractions, and so on, every day yet they did not think it was math. These students had a concept of math that involved paper and pencil, calculations, and being "*not fun*". The paddle-making unit was fun and therefore could not be math. In reflecting on this phenomenon, the teacher and I agreed that it would be important to reference the paddle-making tasks when teaching later concepts to help these students make more explicit connections to the math they were using.

In the past year, MK schools have proposed new project ideas that will be expanded upon this year. One school tapped maple trees and made their own maple syrup, learning the math and science related to this project. This year we are using some grant funds to purchase additional materials for tapping trees so this school can expand this project this coming winter. Another school examined the mathematics in beadwork projects and hopes to do more with this in the coming year. One other school is about to begin a snowshoe-making project with kits purchased with grant funds. Other project ideas being explored include basket making, quill work, exploring eels and eel fishing, examining the data associated with language loss, creating drums, and making wampum belts. It is our hope that eventually every student in an MK school will have the opportunity to explore some aspect of indigenous knowledge, while working alongside elders, to develop a deep appreciation for the mathematical knowledge that exists within community practices.

FUTURE DIRECTIONS

In the winter of 2015, it is expected that at least five MK schools will conduct an inquiry project with a few grades of students at the school. My goal will be to work alongside the teachers and their students, support the projects with funding for materials and funding to bring elders into the classroom, and to help them create videos and other content to be added to the www.showmeyourmath.ca website. This will create a resource that other schools will be able to look to for ideas. Additionally, I will conduct interviews with teachers and conduct focus groups with students to determine how this experience is impacting their learning. I am interested in exploring how this work impacts teachers' understanding of the role of indigenous knowledge in mathematics education and how they might then be inspired to transform other aspects of their practice. I am also interested in how students are impacted by this learning both with respect to their academic achievement and with respect to their sense of cultural identity development. I am particularly interested to learn if students see this way of learning mathematics as being culturally consistent, enabling them to see strong connections between mathematics and Mi'kmaw community practices. The long term goal is to create a series of case studies that will show examples of culturally-based mathematics education that pulls math into community cultural practices but begins by honouring community knowledge.

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New PhD Reports

Présentations de thèses de doctorat

PROBLEM POSING AS STORYLINE: COLLECTIVE AUTHORING OF MATHEMATICS BY SMALL GROUPS OF MIDDLE SCHOOL STUDENTS

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The metaphor at the heart of my PhD dissertation is that of a tapestry—an overall design created by the weaving together of colourful, and sometimes disparate, threads. Accordingly, in this paper I have chosen to follow only a few threads of what is discussed in the dissertation in hopes of provoking thought about other ways in which we might view the process of mathematical learning that occurs in middle school classrooms. I briefly discuss the nature of groups, offer improvisation theory as a framework for viewing the behaviour of groups as that of a single learning agent, and suggest how the problems developed in a literary storyline parallel the kind of problem posing that I believe drives the process found in collective work on mathematical problem-solving tasks. I then describe set-up of the study, the analysis method (a 'blurring of data') and then provide a quick list of the results. To view the dissertation, go to: http://circle.ubc.ca/bitstream/handle/2429/43920/ubc_2013_spring_armstrong_alayne.pdf?sequence=1. To view what's been emerging from the study since, go to https://ubc.academia.edu/AlayneArmstrong.

INTRODUCTION

Mathematics weighs heavily on students in our schools. It is a subject with a reputation as being difficult and abstract, a solitary task meant only for those who have a natural capacity for it (Lafortune, Daniel, Pallascio, & Sykes, 1996; Sinclair, 2008). It is perceived by many students as a series of rules imposed by an outside authority, be it textbook or teacher, with little recognition that student thinking itself generates mathematics. Yet mathematics itself is a living and creative act (Boaler, 2008) and mathematicians themselves often collaborate in their work (Burton, 2004). So what is holding school mathematics back? Are we so conditioned to expect the act of mathematizing in school to proceed in a certain formalized way that we are neglecting other ways in which mathematical learning may emerge? How else might we frame what it is that students are doing in mathematics?

Povey, Burton, Angier, and Boylan (1999) offer an alternate viewpoint when they decenter the term *authority*—the traditional view of mathematical knowledge as external, fixed and absolute—to play with the concept of *author/ity*, splitting up the word to foreground the idea of there being an author (or authors) behind the scenes who negotiates this knowledge, and thus positioning teachers and students as potential members of this negotiating mathematics community. As a mathematics educator, I have noticed that those students who appear to be most fully engaged in collectively solving mathematics tasks are the ones who are reformulating an assigned task into their own problems. This serves to not only break the task down, so they are working with smaller, more manageable chunks, but to develop a kind of a 'storyline', one that differs from the storylines that are developed by other groups who are working on the same task. I consider these groups to be 'authoring' their mathematics, with their emergent problem posing both engaging their interest and motivating them to work towards solving the original task.

CONSIDERING COLLECTIVE BEHAVIOUR

As educators, we teach people, often a lot of people, and these people are often grouped, whether it be as a school district, a school, a class, or a small group within a class. These groups are nested within one another with boundaries that are fluid and changing, depending on the time of the observation and the perspective of the observer. The cohesiveness of a group is also fluid. At one end of the spectrum is the *collection*—people who happen to be in the same location but who are acting independently of one another, such as individuals waiting at a bus stop. Yet events may take place (for instance, the bus gets stuck in the snow) so that the members of a collection begin to cohere in order to work towards a common goal (freeing the bus). This may occur to different degrees. Roschelle and Teasley (1995) discuss how in cooperative groups the members divide up the task into individual responsibilities in order to meet the goal, while in collaborative groups there is "the mutual engagement of participants in a coordinated effort to solve the problem together" (p. 70). The group with the highest degree of cohesion is the *collective*. When we look closely at the behaviour of this kind of group, we may notice how attuned the members are to one another—through watching, through listening, and perhaps through touch (depending on the task at hand), group members constantly monitor and react to each other. At peak performance levels, the collective appears to act as a single agent.

Group work is often the means to an end in a classroom, not the end in itself. Part of the reason for this is that the idea of a group as a single agent is difficult to conceptualize (Stahl, 2006) and, as a result, it is an "often overlooked learner" (Davis & Sumara, 2005). For instance, if one follows an acquisitionist view (Sfard, 1991) where the mind is seen to function as a container and learning is a matter of pieces of knowledge being transmitted from the teacher's mind to the student, and then stored in the student's mind, then the idea of group learning makes no sense. Once the group breaks up, as it inevitably must, and the members go their different ways, where does the group's learning go? There is no permanent structure—for instance, a group brain—to contain it. Thus, studies of small groups have often tended to focus on how working within the group affects the learning of the individuals within the group rather than on how it affects the group itself (Stahl, 2006).

An effective framework for considering collective behaviour comes from improvisation theory. One might argue that improvisation is the way that we live. A conversation, for instance, is an example of unplanned give-and-take between two or more people—there's no script to follow and the direction of discussion can change at any time and in many ways. The concept of improvisation has been taken up very enthusiastically by researchers who study organizations, be they businesses, communities, or schools (see Armstrong (2013), Chapter 3, for further discussion) and the developing theory has been particularly helpful in describing the collective behaviour of students in mathematics classes (Martin & Towers, 2009; Martin, Towers, & Pirie, 2006). Take, for instance, the popular but misguided belief that improvisation springs from nothing. In fact, to be an effective improviser requires training in certain routines (or 'riffs'), and an expertise in one's area. It may be an oxymoron, but one needs to prepare in order to be spontaneous. Also important is the concept of the student as a *bricoleur*. This term comes from the anthropological work of Claude Lévi-Strauss (1966), and speaks to the idea of people working with what they have on hand rather than following a pre-planned, scripted routine. Turkle and Papert (1990) define *bricolage* as "a style of organizing work that invites descriptions such as *negotiational* rather than *planned in advance*" (p. 144), and Watson and Mason (2005) describe the mathematical 'surprises' that can result by giving students some room to work when doing a task, rather than seeking a specific prescribed outcome. Improvisation theory helps to set out the potential for a collective's work to emerge to be more than the sum of its parts.

PROBLEM POSING AS STORYLINE

For an artist, the act of pursuing and then working with problems is improvisational; it involves "constantly searching for her or his visual problem while [for instance] painting" (Sawyer, 2000, p. 153). The same is true of improvisation in the mathematics classroom, where students may be spurred on by problems that they solve, but in the process may pose further problems, which then trigger the posing of more problems, and so on. In a similar way, what may drive a storyline in literary works is a sense that something needs to be resolved, and this sets in motion further 'somethings' that also require resolution. It may be a disagreement, an uncomfortable gap in understanding, or a conflict, but it is this 'something' that provides an impetus to further action. William Shakespeare's play, *Romeo and Juliet*, provides a good example of how the central conflict of a storyline can generate a number of other conflicts, eventually driving the story to its conclusion. A boy (Romeo Montague) and a girl (Juliet Capulet) meet at a feast hosted by the Capulets and fall in love. Each belongs to opposite sides of a long-time feud between the Montagues and the Capulets and thus their friends and families will not approve of the match. How can they be together? They secretly marry and decide to wait for an opportune time to reveal this news to the world. However, their meeting soon precipitates other conflicts, including the following: Mercutio versus Tybalt regarding Romeo's disguised and unauthorized presence at the Capulet feast; Romeo versus Tybalt regarding Tybalt's slaying of Mercutio; Juliet versus Lord Capulet regarding his wish to marry Juliet to Count Paris; and so on until the final conflicts of how either member of the pair could live without the other.

There has been much research done about problem posing in mathematics education, with studies largely focusing on the nature of the problems produced (their number, their quality) by individuals (see Armstrong (2013), Chapter 4, for a review of the literature). I take a different tack. Drawing on the idea of how a storyline works, I am interested in the process of problem posing, and in how this process works on a collective level, a level that would necessarily make the process a public one. My research questions are: What problem posing patterns emerge as small groups of students work collectively on a mathematics task? What are the characteristics of problem posing as a collective process?

METHODOLOGY

The research took place at a Grade 6-8 middle school in a large suburban school district in British Columbia. The middle school age group is known for its high energy and for its enthusiasm for socializing, making its members well-suited for working in groups while tackling mathematics tasks. Sixteen students from each of two Grade 8 classes of 30 students (i.e., just over half) participated in the study for a total of 32 subjects. The study occurred in the spring of the school year, with session tapings taking place roughly every two weeks depending on the school schedule, for a total of five sessions for each class, with each session lasting approximately 40 minutes. The four groups that my dissertation focuses on are: JJKK, DATM, NIJM and REGL. I refer to these groups by their acronymic names as a way of characterizing each group as an entity in itself

Two stationary video cameras were each focused on a group. Also visible in the background were other groups participating in the study, meaning that each video-taped group was in fact being recorded by two cameras, each with a different angle. In addition, I audio-recorded two additional student groups—as the workings of any group cannot be predicted, these groups served as a back-up in case they had active on-task discussions but the two videotaped groups did not.

Groups were assigned structured "Problem of the Day" tasks that were appropriate for the middle school level. The task that is the focus of this case study reads as follows:

The Bill Nye Fan Club Party

> The Bill Nye Fan Club is having a year-end party, which features wearing lab coats and safety glasses, watching videos and singing loudly, and making things explode. As well, members of the club bring presents to give to the other members of the club. Every club member brings the same number of gifts to the party. If the presents are opened in 5-minute intervals, starting at 1:00 pm, the last gift will be opened starting at 5:35 pm. How many club members are there?

To capture some of the characteristics of collective problem posing, I 'blurred' the transcript data in order to better see the group's conversation as a whole, and how the problems shift and interweave over time. While some have employed the metaphor of a crystal to describe the potential for multiple interpretations that qualitative research admits (Janesick, 2003), the metaphor that I used to document the patterns of collective problem posing, and reduce the transcript to its 'visual essence', is that of the 'tapestry'. This metaphor is helpful because it offers the idea of different vantage points. Look at a tapestry up close and you can see individual threads, just like when you look at a transcript up close (as we normally do when we read it), you can see what the individual group members say. However, to get a sense of the whole pattern you need to step back from the tapestry, and one way to do that with a transcript is to physically shrink it. Consider the tapestries of the four small groups who worked on the Bill Nye problem (see Figure 1). Even though each group is able to correctly solve this fairly structured task, it is visually evident through the different colour patterns in each tapestry that each group develops its own pattern of posed problems (storyline) in doing so. These patterns of interaction, rather than the number of problems posed or the quality of problems posed, are what my study focuses on in considering students' mathematical behaviour.

The production of each tapestry started with the transcript itself. After multiple iterations of reading and comparing transcripts from the four groups' sessions, I identified the posed problem categories. For this study, there were 31 categories in total (see Figure 2) I colour-coded the utterances in the transcripts according to the problem posing category they best fit (the colours were assigned randomly). The colour-coded transcripts were then shrunk in size, using computer screenshots, to the point where the words of the transcript were no longer visible and the lines of colour coding appeared as a visual pattern. The resulting tapestry provided an overall image of the problems posed during the course of the group's session. The process of analysis was a dynamic one, in which I moved back and forth between the two viewpoints: the panorama provided by the tapestry, and the close-up provided by the transcript.

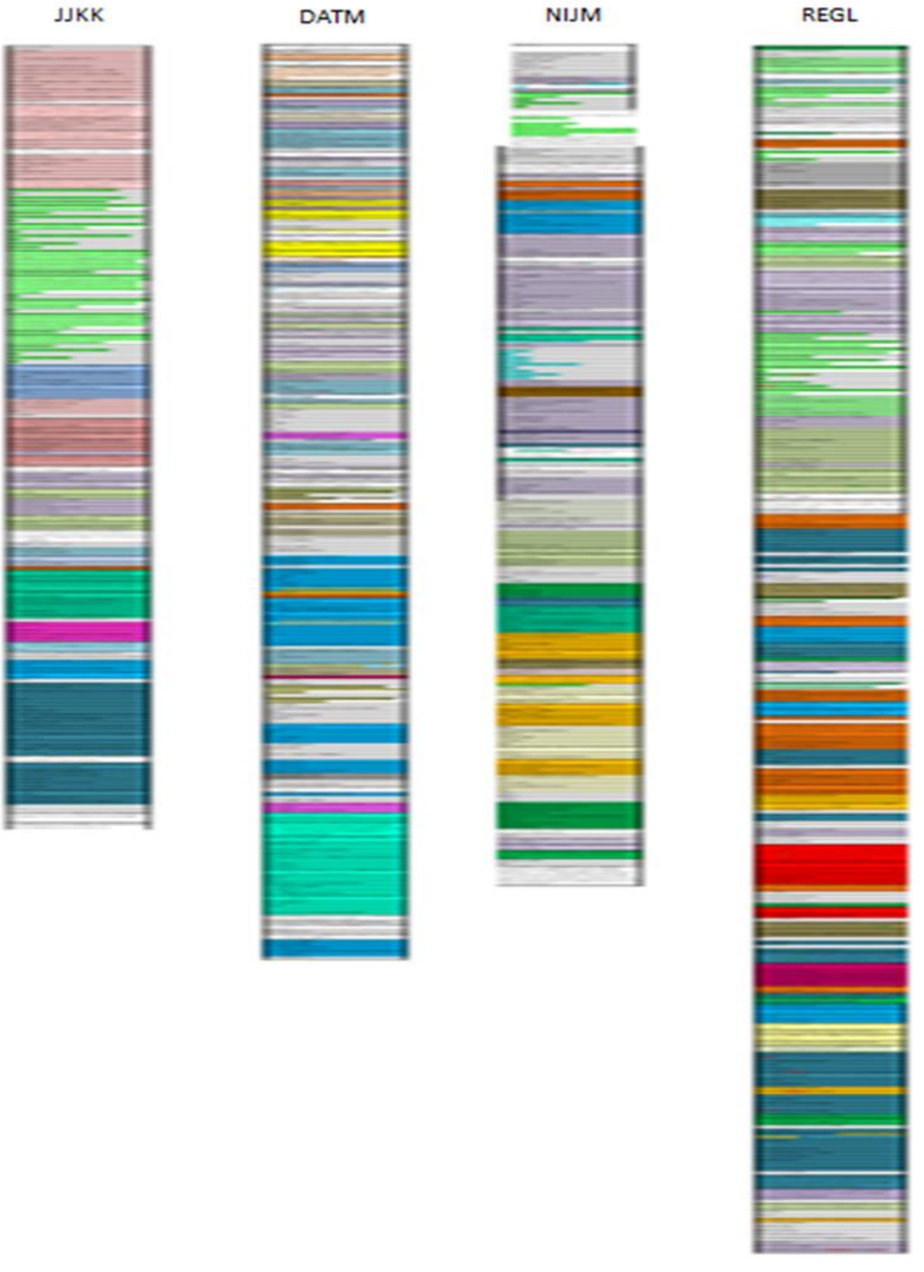

Figure 1. Tapestries.

Colour name	Problem posed (generalized)	JJKK	DATM	NIJM	REGL	#
Lavender	Do we use time and divide by 5 [number of intervals]?	X	X	X	X	4
Bright blue	What about if everyone brings x gifts each?	X	X	X	X	4
Wheat	Is there an extra 5 minutes? (because last gift is opened starting at 5:35)	X	X	X	X	4
Light blue	How many people are there?	X	X	X	X	4
Medium blue	What are the factors of *x*?	X		X	X	3
Lime green	What is meant by an interval?	X		X	X	3
Taupe	Do all members give to everyone?		X	X	X	3
Goldenrod	Do they also bring gifts for themselves?		X	X	X	3
Orange	Does everyone bring the same amount of gifts?	[X][1]	X	X	X	3
Sky blue	How many gifts are there?			X	X	2
Pale yellow	What if there are *x* people?			X	X	2
Pine green	How do we think outside the box?			X	X	2
Teal	Is it a square root?	X		X		2
Fuchsia	Why did we get *x*?	X	X			2
Coral	How long does it take to open all the gifts?	X	X			2
Periwinkle	Can they take breaks in between opening gifts?	X	X			2
Ivory	Does it start at one o'clock?		X	X		2
Gray	What is a tournament?				X	1
Red	What if it's an exchange?				X	1
Light green	How long does it take to open one gift?				X	1
Forest green	Can't we just count how many people?				X	1
Light purple	How many gifts does each person bring?	X				1
Pink	How many gifts are opened in an hour?	X				1
Brown	Is another group's answer right?			X		1
Purple	Can they bring partial gifts?			X		1
Light pink	What if someone doesn't get a gift?			X		1
Dark gray	How do we know if we're right?		X			1
Khaki	What if there are x people and gifts?		X			1
Tan	Does it take 5 minutes to open one gift or 5 minutes to open all the gifts that one person brings?		X			1
Aquamarine	How can we use the 24 hour clock?		X			1
Yellow	Can they open gifts at the same time?		X			1

Figure 2. Colour coding chart of posed problems by frequency.

[1] This problem is posed by the classroom teacher during JJKK's session.

RESULTS

In this section, I will discuss the results pertaining to each of the research questions. (For descriptions of the particular storylines developed by each group, see Armstrong (2013), Chapter 6.)

WHAT PROBLEM POSING PATTERNS EMERGE AS SMALL GROUPS OF STUDENTS WORK COLLECTIVELY ON A MATHEMATICS TASK?

While a few of the problems posed are common to all four groups, each group also poses problems that are unique to it. For instance, DATM discusses whether members can open gifts at the same time, and how to use a 24-hour clock to figure out the time intervals, while REGL discusses what an “exchange” is. It is interesting to consider how four groups who have been in the same math environment all year can each draw on a variety of ideas and experiences, and tackle the same task using different routes.

Each of the groups’ tapestries is unique in its colour patterns (Figure 1). This provides visual evidence that posed problems emerge for each of the groups in different sequences, they are considered for different lengths of time, they may or may not re-emerge in the course of further discussion, and if they do re-emerge, the frequency and length of time that this occurs varies.

In terms of the tapestry patterns, two distinct types may be identified: ‘chunky’ patterns (as seen in JJKK’s tapestry) and ‘thready’ patterns (as seen in REGL’s tapestry).

Chunky patterns indicate that a lengthy period of time is being spent discussing one specific posed problem. This could indicate that one member is dominating the discussion, a debate is occurring that is not moving towards resolution, or that the group is negotiating agreement. In the case of JJKK, reading the transcript reveals that the group is negotiating agreement, and that the group tends to discuss one problem at a time before posing and considering a subsequent one. It is noteworthy, as well, that this group rarely reposes problems—once a problem has been discussed, it rarely re-emerges in the conversation.

Thready patterns in a tapestry indicate that problems are discussed very briefly before the group moves on to other ones. In general, this might indicate that group members are not picking up on their peers’ contributions, the group is generating many problems to put ‘on the table’ for consideration, the posed problems are quickly triggering the posing of other problems, and/or that problems are being consciously juxtaposed with other problems during the discussion. In the case of REGL, the group has a tendency to discuss more than one problem at the same time, posing and reposing problems to make connections and comparisons between them as the group digs deeper into the task. Tapestries (with the exception of JJKK’s) tend to be thready at the beginning (when groups are considering how to approach the task), at the end (when groups are checking over their solutions), and at points when the group has run into an issue (for instance, REGL realizes midway through that it has not arrived at a correct solution and poses and reposes problems until it is able to make a correction in its thinking).

WHAT ARE THE CHARACTERISTICS OF PROBLEM POSING AS A COLLECTIVE PROCESS?

As mentioned earlier, groups do not necessarily draw on the same shared experiences (*bricolage*). For instance, only two of the groups considered the idea of square roots, and only one brought in the use of 24-hour clock (which had been employed in a previous task during the study).

Problem posing is a generative process, with the discussion of problems seeming to trigger the posing and reposing of other ones. For NIJM, one particular problem (such as "*Do we use time and divide time by 5?*") appeared to provide structure (or a 'central thread') for the emergence of other posed problems—each problem was discussed, and then the group would return to "*Do we use time and divide time by 5?*" which would then trigger other problems to be considered. In a case like this, problem posing appears to be self-structuring.

The purpose of a problem evolves each time it is reposed during the session according to its new context. For example, "*Do we use time and divide time by 5?*" first may have been posed as a suggestion about what to try, then reposed as a suggestion to begin calculations, then reposed again as a reminder of what had been already discussed, and finally reposed as a rechecking of calculations when reviewing the group's completed solution.

DISCUSSION AND CONCLUSIONS

This study offers a description of collective behaviour that focuses on the group as a single agent. It also describes an analysis technique for considering collective behaviour that introduces time as an element, blurring the data in order to provide visual evidence of emergent problem-posing patterns. These patterns, unique for each group, point to the development of individual storylines.

One might wonder: it is one thing to author a literary story, generating a storyline based on conflicts, but is it another to author the solution to a mathematics task? Just how original can you be in solving, for example, the *Locker Problem*, a task that thousands and thousands of students have been assigned over the years and one that has a single, correct answer? Again, there is a parallel to this situation in one of the 'classic' storylines that recur time and time again in literature. Shakespeare's central problem of "star-cross'd lovers" in *Romeo and Juliet* is echoed in our contemporary *West Side Story* and even in the more recent *High School Musical*. Shakespeare's play itself is a descendant of Arthur Brookes' 1562 poem *The Tragicall Historye of Romeus and Juliet*, which is itself a translated interpretation of one of Bandello's Italian short stories *Novelle*. Yet each author has made the story his/her own by varying the storyline. While the overarching conflict is the same (young couple from opposite sides of warring worlds comes to a tragic end), it is how the smaller conflicts, or problems, are settled that makes each text unique. This parallels what happens on a smaller scale in this study as each of the groups tackles the Bill Nye problem.

This study addresses a gap in the problem-posing literature by providing a description of the collective problem-posing process, noting the patterns that may occur, how problems are reposed and how the role of these reposed problems evolves as the session continues. It suggests that perhaps the strength of problem posing is not the generation of a list of problems at the end of the task, but the emerging patterns of problems as the group's discussion continues and how the problems help to structure pathways to a solution.

Although its limited scope necessarily precludes generalizations about the behaviour of mathematics students, my dissertation points out the creative processes that may be occurring in our mathematics classes right under our very noses. For instance, focusing on a group's process, rather than its final product, in the form of collective problem-posing patterns helps to highlight the presence of spontaneity even in what appears to be a structured task. When a song is performed, we do not listen only to the final note; when a play is developed we do not attend solely to the final word. So when students engage in school mathematics, why would we focus only on their final answer?

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DES MANIÈRES DE FAIRE DES MATHÉMATIQUES COMME ENSEIGNANTS ABORDÉES DANS UNE PERSPECTIVE ETHNOMÉTHODOLOGIQUE POUR EXPLORER LA TRANSITION SECONDAIRE COLLÉGIAL

AN ETHNOMETHODOLOGICAL PERSPECTIVE ON TEACHERS' WAYS OF DOING MATHEMATICS TO EXPLORE THE TRANSITION FROM SECONDARY TO POSTSECONDARY LEVEL

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Le sujet abordé dans la thèse est celui des transitions interordres en enseignement des mathématiques et cible plus spécifiquement la transition secondaire postsecondaire (le collégial au Québec). Cette transition est abordée par le biais des manières de faire des mathématiques des enseignants à chacun des ordres, et d'une perspective à travers lesquelles les regarder, celle d'une harmonisation. La clarification progressive de cet objet de recherche s'appuie sur une réflexion se situant au carrefour de trois mondes : le monde institutionnel, le monde de l'enseignement des mathématiques (de la pratique) et le monde de la recherche.

PROBLÉMATIQUE : LE CHOIX D'UN OBJET DE RECHERCHE ET D'UNE PERSPECTIVE POUR L'ABORDER

Cette étude s'insère dans un champ de recherches en didactique des mathématiques portant sur les questions de transitions interordres et cible plus spécifiquement la transition secondaire collégiale en mathématiques. L'étude de la transition secondaire postsecondaire a jusqu'alors été abordée d'un point de vue institutionnel par l'analyse de programmes, d'évaluations nationales ou de tâches de manuels aux deux ordres (par exemple, Bosch, Fonseca, et Gascón, 2004 ; Corriveau, 2007 ; Gueudet, 2004 ; et Najar, 2011) ou encore par les difficultés rencontrées par les étudiants du postsecondaire (par exemple, Praslon, 2000 ; Vandebrouck 2011a, 2011b ; de Vleeschouwer et Gueudet, 2011). La perspective globale de comparaison entre les deux ordres qui sous-tend ces travaux a conduit à mettre en évidence les ruptures qui caractérisent cette transition et un certain « vide didactique » laissé à la charge des élèves (Praslon, 2000). Ces recherches font entrer sur la partie explicite d'une culture mathématique caractérisant chacun des ordres (Artigue, 2004). Or les plus grandes différences interculturelles, dit Hall (1959/1984), relèvent du plan *informel* de cette culture et sont à chercher du côté des manières de faire, souvent implicites, qui la caractérisent. Cette entrée sur ce qui se fait à chacun des ordres constitue le point d'ancrage de cette recherche. L'intérêt est d'explorer la transition avec des enseignants des deux ordres, du point de vue de leurs manières de faire des mathématiques, et ce dans une perspective d'harmonisation. Il s'agit de mieux comprendre ainsi la partie implicite de cette culture mathématique qui se constitue à chacun des ordres.

DE L'OBJET « MANIÈRES DE FAIRE DES MATHÉMATIQUES » AUX FONDEMENTS ET CONCEPTS THÉORIQUES PERMETTANT DE L'ÉCLAIRER

Trois entrées théoriques permettent d'explorer l'objet *manières de faire des mathématiques comme enseignants* (MFM) en prenant en compte l'enjeu de transition interordres dans lequel il est étudié (théorie de la culture), la manière dont ces manières de faire se constituent (ethnométhodologie), et le contexte dans lequel elles trouvent leur ancrage, soit l'enseignement à un ordre donné (cognition située comme entrée complémentaire).

En abordant la transition comme un changement de cultures, notre étude puise à un premier fondement, celui de la théorie de la culture de Hall (1959/1984). Dans la perspective de Hall, la culture apparaît comme l'imbrication de trois plans, dits « formel, informel et technique », renvoyant à un ensemble organisé, assurant sa cohérence. Le plan formel de cette culture (dans notre cas la culture mathématique) renvoie à des convictions, des allants de soi, des évidences à propos des mathématiques. Le plan informel correspond à ce qui est intégré dans l'action et qui relève de l'implicite, des MFM balisées par des règles implicites d'action. Le plan technique fait appel à un système de justifications explicite, organisé et institutionnalisé dans les programmes et les manuels. Les trois plans de la culture de Hall permettent d'aborder les MFM comme un ensemble lié : des manières de faire qui relèvent de l'action, implicites, mais aussi sans doute imbriquées à des convictions, des allants de soi, renvoyant dans certains cas à des justifications explicites, qui constituent la culture mathématique de chacun des ordres d'enseignement. Ce cadre permet notamment, en entrant sur le plan informel de cette culture, d'aller au-delà de ce qui est visible dans cette culture (lorsqu'on en observe à première vue les contenus et les tâches), de manière à mieux comprendre un ordre par rapport à un autre. Si la théorie de la culture permet de situer les MFM à l'intérieur d'une certaine culture, elle ne permet pas toutefois de comprendre leur constitution fine dans l'action par les enseignants. Les concepts proposés par Garfinkel (1967/2007), fondateur de l'ethnométhodologie, permettent d'entrer plus finement sur celles-ci.

Les fondements ethnométhodologiques occupent une place centrale au plan théorique. Contrairement à ce que le nom laisse présager, l'ethnométhodologie n'est pas une approche méthodologique, mais bien une théorie des phénomènes sociaux. Elle sert de fondement conceptuel à notre étude en se proposant d'étudier les *ethnométhodes* que les acteurs (enseignants) mobilisent afin de réaliser leurs actions de tous les jours (Coulon, 1993). Le concept d'ethnométhode permet d'entrer sur les MFM qui se constituent dans l'activité professionnelle des enseignants. Il se précise à travers une constellation d'autres concepts imbriqués : *accountability*, indexicalité, circonstances, procédures interprétatives, rationnels et membres. Cette entrée théorique permet de prendre en compte de manière fine ces MFM telles qu'elles se constituent. Les « membres », soit les enseignants d'un ordre donné, sont appelés, dans le cadre ordinaire de leurs interactions professionnelles, à attester de leurs manières de faire des mathématiques comme enseignant à leur ordre d'enseignement, à préciser les circonstances de ces manières de faire, le rationnel imbriqué (les ethnométhodes mathématiques). En continuité avec l'ethnométhodologie, et de manière complémentaire, les travaux de Lave (1988, 1996) poussent l'idée du contexte en venant préciser ce qu'ont de particulier ces mathématiques de l'enseignant du secondaire et du collégial.

Ces différents éclairages nous ont amenée, à partir de la problématique de départ, à formuler trois questions. Les deux premières questions sont centrales et sont éclairées par l'ethnométhodologie et la théorie de la culture de Hall. La troisième question est davantage exploratoire.

- Comment se particularisent les ethnométhodes mathématiques (manières de faire, circonstances de l'action, rationnels, procédures interprétatives) dont attestent les enseignants d'un ordre donné (membres) lorsqu'ils explorent la transition avec des enseignants d'un autre ordre ?
- Comment se distinguent les cultures mathématiques dont témoignent ces ethnométhodes mathématiques à chacun des ordres ?
- De quelles façons l'harmonisation se constitue-t-elle dans cette exploration ? Comment se développe-t-elle ?

MÉTHODOLOGIE : LA DÉMARCHE POUR ABORDER EMPIRIQUEMENT LES ETHNOMÉTHODES MATHÉMATIQUES EN CONTEXTE DE TRANSITION

C'est par le biais d'une recherche collaborative (Bednarz, 2013 ; Desgagné, Bednarz, Couture, Poirier, & Lebuis, 2001) qu'a été menée l'exploration de ces manières de faire des mathématiques comme enseignants. Une activité réflexive de six rencontres d'une journée a servi de matériau d'analyse. Ces rencontres ont rassemblé six enseignants (trois de chacun des ordres), lesquels ont été conduits, dans l'interaction entre eux et avec la chercheuse, à attester de leurs manières de faire des mathématiques comme enseignants par le biais de situations puisées à même leurs actions professionnelles quotidiennes. Les enseignants ont aussi été amenés à aller plus loin dans une perspective d'harmonisation : en collaboration avec la chercheuse, se constitue et se développe au sein du groupe une harmonisation entre ces manières de faire les mathématiques aux deux ordres. Les *verbatims* des rencontres forment le corpus de données (plus d'une trentaine d'heures de *verbatims*).

ANALYSE AUTOUR DE TROIS THÈMES ÉMERGENTS : SYMBOLISME, UTILISATION DE CONTEXTES ET FONCTIONS

Une analyse émergente a été menée à partir d'un découpage de l'ensemble des données en trois thèmes : l'utilisation du symbolisme en mathématiques, l'utilisation de contextes et le travail sur les fonctions. Différents concepts de l'ethnométhodologie se sont avérés porteurs pour l'analyse (circonstances des MFM, rationnel de cette action, procédures interprétatives, *account*, indexicalité). Ainsi, à un premier niveau d'analyse, nous proposons une conceptualisation, en termes de territoire d'ethnométhodes mathématiques, des particularités de chacun des ordres. À un deuxième niveau d'analyse, ce qui est dégagé au premier niveau est regardé du point de vue de l'organisation d'une certaine culture. Ce point de vue permet de mettre en évidence les distinctions entre les ordres et des enjeux de transition. Enfin, à travers les échanges entre les enseignants et la chercheuse, des trajectoires d'harmonisation ont été dégagées et analysées de manière à comprendre comment cette harmonisation se constitue dans le groupe.

UN TERRITOIRE D'ETHNOMÉTHODES MATHÉMATIQUES EN CE QUI A TRAIT À L'UTILISATION DE CONTEXTES

Le territoire d'ethnométhodes mathématiques relatif à l'utilisation du symbolisme chez les enseignants du secondaire impliqués dans cette recherche se constitue à travers des MFM prenant place dans certaines circonstances et à travers un rationnel qui les compose. La constitution de ce territoire dont parlent les enseignants amène à voir l'utilisation du symbolisme selon trois aspects caractéristiques de ce que veut dire pour eux « faire des mathématiques avec un symbolisme » : un *symbolisme processus*, qui apparaît graduellement, en passant parfois par des notations intermédiaires non conventionnelles; un *symbolisme transparent*, les enseignants du secondaire choisissent de symboliser de façon à ce que le sens soit facilement accessible et l'information visible à travers ce symbolisme; et un *symbolisme choisi*, les enseignants se donnent la liberté de ne pas toujours introduire le symbolisme associé à un objet mathématique ou à une définition, préférant alors des registres intermédiaires comme des tableaux ou encore des mots. Chez les enseignants du collégial ayant participé à la recherche, le symbolisme a un rôle de premier plan en mathématique et il faut le rendre accessible aux étudiants. Le territoire d'ethnométhodes mathématiques se caractérise aussi autour de trois aspects : un *symbolisme explicité*, les enseignants présentent les objets mathématiques et leurs définitions par l'entremise du symbolisme dont ils vont préciser le sens en traduisant en langage courant; un *symbolisme conventionnel*, les enseignants utilisent un symbolisme reconnu par une communauté scientifique qu'ils ne choisissent pas; un symbolisme général et compact, une des qualités du symbolisme pour les enseignants du collégial est qu'il n'apporte pas de lourdeur et qu'il soit le plus général possible. On voit dès lors apparaître des enjeux potentiel de la transition : un symbolisme amené comme un processus au secondaire et considéré comme achevé au collégial; une façon de symboliser « parlante » au secondaire, alors que le symbolisme utilisé au collégial est compact et général (il faut donc le traduire et le « faire parler »); choisir de ne pas introduire les définitions de manière formelle, éviter ou retarder le symbolisme au secondaire alors qu'au collégial, on présente concepts et définitions de manière formelle, cherchant systématiquement à tout symboliser.

En ce qui concerne l'harmonisation, nous avons pu reconstituer une trajectoire informelle d'harmonisation de laquelle il se dégage un certain sens à ce que cela peut signifier. Dans la discussion entre les enseignants du secondaire, les enseignants du collégial problématisent leur utilisation du symbolisme en mathématiques. En se prêtent au jeu d'enquêter sur le symbolisme à la manière des enseignants du secondaire, ils passent d'un *symbolisme allant de soi* à un *symbolisme problématisé*. Il se constitue alors chez ces enseignants une nouvelle manière d'enquêter sur le symbolisme (par la lunette de la cohérence à long terme) qui fait sortir du territoire d'ethnométhodes usuelles (au collégial) pour se rapprocher de celui du secondaire.

UN TERRITOIRE D'ETHNOMÉTHODES MATHÉMATIQUES EN CE QUI A TRAIT À L'UTILISATION DE CONTEXTES

À travers différents types d'*accounts* (accomplir une tâche en contexte, relater sa pratique en lien avec l'utilisation de contextes, commenter une tâche et converser à propos de l'utilisation de contextes en général dans l'enseignement), des territoires d'ethnométhodes mathématiques liées à l'utilisation de contextes par les enseignants ont pu être dégagées à chacun des ordres. Le territoire des ethnométhodes mathématiques des enseignants du secondaire est composé de MFM liées au contexte (imager les mathématiques, travailler en contexte, des manières de parler des concepts mathématiques connotées par une situation, transformer le contexte en lien avec les mathématiques à travailler, considérer les mathématiques exploitées et le contexte dans un rapport dialectique, etc.) et de raisons de son utilisation (le contexte comme support au raisonnement, comme moyen d'assurer un engagement dans l'activité mathématique, comme permettant une démarche de recherche). Chez les enseignants du collégial, on retrouve aussi des MFM (faire la correspondance entre les éléments du contexte et les outils mathématiques, projeter les mathématiques dans une tâche, décontextualiser pour travailler sur des objets mathématiques, prévoir les contextes d'utilisation en présentant les concepts mathématiques, rechercher les éléments mathématiques dans un modèle, etc.) et de raisons d'utilisation (le contexte comme permettant de voir si les étudiants peuvent appliquer ce qui leur a été montré et pour mettre en application des outils). En jetant un regard transversal sur le territoire constitué par les enseignants du secondaire, des traits d'une culture mathématique contextuelle émergent : une culture dans laquelle les mathématiques sont « agglutinées » à un contexte, dans laquelle des processus et des contenus sont travaillés dans le contexte, dans laquelle le contexte agit comme ressource pour les mathématiques, et des mathématiques qui sont parlées et imagées à travers le contexte. Dans le territoire constitué par les enseignants du collégial, un rôle particulier est assigné aux mathématiques. Les concepts mathématiques ont un certain rôle de préalables par rapport aux sciences. Les concepts sont amenés en amont (par rapport aux sciences) par les définitions et les propriétés/théorèmes qui s'y rapportent, pour être ensuite utilisés dans des problèmes. Autrement dit, il y a les mathématiques, et il y a l'application. Nous avons choisi de caractériser cette culture par des mathématiques *illustrées*. *Illustrées*, d'abord au sens d'exemplifiées, parce que les notions mathématiques sont mises en application dans des problèmes à titre illustratif. Ensuite parce que *illustrées* (de *lustre*) renvoie à *éclairées*, et au collégial, l'éclairage est justement mis sur les mathématiques (et non sur le contexte comme au secondaire).

Des tentatives d'harmonisation ont été menées au sein du groupe, mais restent inachevées. Un certain sens est tout de même mis de l'avant dans la manière dont l'harmonisation est pensée *a priori* : la chercheuse propose aux enseignants de partir de leur territoire (des mathématiques contextualisées au secondaire; des mathématiques utilisées comme outils au collégial) pour aller vers le territoire de l'autre. Or, on n'entre pas vraiment dans le territoire de l'autre, on reste dans son territoire.

UN TERRITOIRE D'ETHNOMÉTHODES MATHÉMATIQUES EN CE QUI A TRAIT AU TRAVAIL AVEC LES FONCTIONS

Le troisième thème, celui des fonctions, se distingue des deux précédents puisqu'il correspond à un objet d'enseignement, apparaissant dans les programmes de formation aux deux ordres. Ainsi, les enseignants ont certaines directives en ce qui concerne ce qui doit être fait avec les fonctions. Par l'analyse, nous avons pu mettre en évidence certaines MFM en lien avec le travail sur les fonctions. Chez les enseignants du secondaire, dans cette recherche, les MFM sont liées à l'idée de permettre aux élèves d'étudier des situations dans lesquelles des variables sont en relation en s'appuyant sur des familles de fonctions de référence (affine, quadratique, rationnelle, exponentielle, trigonométrique) et leurs caractéristiques particulières. Chez ceux du collégial, il ressort qu'ils doivent permettre aux étudiants d'analyser n'importe quelle fonction plus complexe, d'un point de vue global comme d'un point de vue local. Ainsi, l'analyse se fait par l'entremise de caractéristiques plus générales, pouvant être attribuées à n'importe quelle fonction complexe (continuité, variations, extrémums, tangentes, asymptotes, concavité, etc.). Or, si cette analyse permet de montrer que les contraintes institutionnelles marquent les MFM des enseignants à propos des fonctions, l'entrée par l'indexicalité et les procédures interprétatives permet de soulever les implicites entourant ces MFM avec les fonctions. L'analyse met donc aussi en évidence que les enseignants ont certaines marges de manœuvre, des manières fines d'enquêter, de donner sens aux objets connotés par le contexte précis du secondaire ou du postsecondaire. Ainsi, leurs manières d'appréhender les fonctions, leur ensemble de définition, leurs tableaux de valeurs ou de variation, leur représentation graphique sont imbriquées aux MFM et apportent d'autres particularités à chacun des ordres.

En ce qui concerne l'harmonisation, la reconstruction d'une trajectoire permet un rapprochement entre les territoires constitués par les enseignants des deux ordres. Ces derniers, avec la chercheuse, élaborent des tâches à travers lesquelles apparaissent de nouvelles MFM avec les fonctions qui ne se situent ni complètement au secondaire, ni complètement au collégial, mais qui allient les deux façons d'appréhender les fonctions. Ce faisant, une harmonisation se développe à travers un repérage de lien et l'établissement de contraste; à travers l'établissement de ponts possibles (on revisite son propre territoire avec comme horizon celui de l'autre); à travers un élargissement de son propre territoire (des MFM empruntées au territoire de l'autre); par des harmonisations ponctuelles (des manières d'enquêter nouvelles qui prennent en compte ce qui est fait à l'autre ordre); par un questionnement conjoint qui invite les enseignants du collégial dans le territoire du secondaire et ouvre alors sur un élargissement du territoire du secondaire (et vice versa); avec une chercheuse qui contribue en suggérant parfois un changement de regard ou en proposant une manière de concevoir l'harmonisation; par l'élaboration de tâches à chacun des ordres en expérimentant, en prenant ce lieu de dialogue pour expérimenter et discuter de ce qui peut se faire; par l'exploration des limites de chacun des territoires.

INTERPRÉTATION ET DISCUSSION

Dans cette section de la thèse, les résultats provenant de l'analyse autour des trois thèmes (symbolisme, contexte et fonctions) sont repris au regard des trois questions de recherche dans l'idée d'aller plus loin pour faire ressortir les avancées de cette recherche au plan empirique et théorique. Ainsi, lorsque mis en parallèles, les trois territoires d'ethnométhodes mathématiques du secondaire se particularise par des MFM qui s'articulent autour de l'idée de « mettre en forme » un symbolisme, des mathématiques à travers des contextes, des fonctions; des circonstances de ces MFM associées à un projet d'enseignement, et un rationnel de l'ordre de « donner du sens ». Chez les enseignants du collégial ayant participé à la recherche, c'est un territoire d'ethnométhodes mathématiques qui se particularise avec des MFM qui s'articulent autour de l'idée de « donner accès » à un symbolisme, aux domaines d'application des mathématiques, aux outils pour étudier n'importe quelle fonction, avec des circonstances de ces MFM associées à un plan mathématico-institutionnel, et un rationnel l'ordre de « préparer les étudiants à des études en sciences ».

L'analyse met aussi en évidence que les MFM des enseignants sont enracinées dans une culture mathématique, et ce, dans une dialectique entre les plans *formel*, *informel* et *technique*. L'entrée privilégiée était celle du plan *informel*, mais des éléments qui relèvent des plans *formel* et *technique* ont pu être dégagés dans les analyses. Bien que celles-ci aient permis de montrer l'imbrication des trois plans, elles montrent aussi—à travers la symbolisation et l'utilisation du symbolisme, la contextualisation et l'utilisation de contextes, le travail avec les fonctions—la richesse de se situer au plan *informel* de la culture pour aborder les questions de transition. Cela confirme en quelque sorte l'hypothèse d'Artigue (2004) et de Hall (1959/1984) voulant que les éléments du plan *informel* de la culture soient centraux.

De plus, l'entrée ethnométhodologique sur l'harmonisation, permet de voir celle-ci comme un processus qui se constitue dans le groupe. Aborder les questions de transition requiert en ce sens d'abandonner l'idée selon laquelle les différentes MFM pourraient être harmonisées. Elle n'est pas un lien extérieur qui est établi entre deux identités déjà constituées, mais bien une perspective, un processus qui constitue les vides à combler en même temps qu'il constitue les ponts. Les résultats à propos de l'analyse des trajectoires d'harmonisation selon les thèmes soulèvent plusieurs questions sur ce qui fait que l'harmonisation ait pu aboutir sur de nouvelles façons de faire ou de nouvelles façons de voir dans deux cas (symbolisme et fonctions) et n'a pas mené à une harmonisation dans l'autre cas (contexte). Il y a là une avenue prometteuse pour d'éventuelles recherches.

CONTRIBUTIONS ET RETOMBÉES DE CETTE RECHERCHE

Cette recherche contribue au champ de la didactique des mathématiques en proposant une nouvelle manière d'aborder les questions de transition interordres, non restreinte à la dimension explicite et institutionnelle. Cette dernière permet d'enrichir la compréhension du phénomène de transition, en le regardant de l'intérieur de la pratique de l'enseignement (et non uniquement de manière externe, par les programmes, les tâches, etc.), en entrant sur la dimension implicite de ce qui se fait en mathématiques (ce que veut dire faire des mathématiques comme enseignants) et en considérant le point de vue des acteurs eux-mêmes pour aborder cette transition. En faisant cela, la recherche contribue à préciser un territoire d'ethnométhodes mathématiques constitué par les enseignants des ordres secondaire et collégial, dont ils attestent lorsqu'ils abordent la transition. Elle a permis aussi de comprendre comment ces ethnométhodes s'organisent au plan informel d'une culture mathématique à chacun des ordres. De plus, l'entrée privilégiée, celle d'une harmonisation, a permis de mieux comprendre quel sens peut prendre l'harmonisation lorsque des enseignants des deux ordres sont amenés à travailler ensemble autour de la transition. Les situations élaborées par les enseignants des deux ordres avec la chercheuse autour du concept de fonction offrent à cet égard, sur le plan pratique, des pistes prometteuses. D'un point de vue théorique, le développement du concept d'ethnométhodes mathématiques a par ailleurs été précisé et enrichi. Ce concept est né de l'objet « manière de faire des mathématiques comme enseignant ». Le travail réalisé tout au long de cette thèse vient préciser comment ces ethnométhodes mathématiques se particularisent elles-mêmes sur un plan théorique. Il s'agit là d'une contribution importante de la thèse.

Du côté de la pratique, comme le mentionne Morrissette (2009) en considérant le modèle de recherche collaborative, un dispositif pouvant « survivre » en dehors de la recherche est mis sur pied. Ce dispositif peut tenir lieu de développement professionnel (Bednarz & Barry, 2010) pour des enseignants qui souhaitent aborder les questions de transition. De plus en plus de commissions scolaires mettent sur pied des programmes de liaison entre les deux ordres et la création d'espaces réflexifs dans lequel les enseignants peuvent échanger et être accompagnés. Le dispositif mis en place ici paraît ainsi prometteur pour le type de liaison souhaitée. D'autres formes de diffusions sont à envisager pour d'une part, mettre en avant l'intérêt d'une démarche comme celle qui a été faite (la mise en dialogue entre des enseignants des deux ordres autour des manières de faire des mathématiques, d'une réflexion en termes d'harmonisation entre ces manières de faire), mais aussi, comme base de discussion avec d'autres enseignants pour poursuivre le dialogue sur ces questions, le but étant davantage de décloisonner les ordres d'enseignement, de favoriser le dialogue, de voir la richesse possible d'un tel dialogue. Une des recommandations qui peut être faite à la suite de ce travail est certainement de faire la promotion de lieux de discussions dans lesquels les enseignants de plusieurs ordres se côtoient et collaborent, d'installer une culture de développement professionnel interordres, voire même de faire de ce décloisonnement, de cette mise en dialogue, une partie intégrante de la formation initiale des enseignants de mathématiques.

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MAKING SENSE OF SENSE MAKING: LEARNING, ANXIETY, AND EDUCATION

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> *It may be noticed that I am concerned with unconscious motivation, something that is not altogether a popular concept. The data I need are not to be culled from a form-filling questionnaire…. This where those who have spent their lives doing psychoanalysis must scream out for sanity against the insane belief in surface phenomena that characterizes computerized investigations of human beings.* (Winnicott, 1971/2005, pp. 192-193)

I have long been afflicted with anxiety over the question: "*What is your dissertation about?*" Having just now mined that 480-page volume—with six clusters of some 200 notes attesting to the effort—I think I have an answer: *At bottom, it most concerns unconscious motivation.*

The work began as a speculative essay and emerged as a comprehensive theoretical treatise on learning, spiraling anxiety in learning systems, and the implications for education in general and teaching in particular. Its passages echo the non-linear, reiterative form of learning—the very emergence it studied. In the present apportioning, I attempt a briefer amenable re-rendering.

THEORETICAL UNDERPINNINGS: AUTOPOIESIS AND SINGULAR-PLURAL-BEING

One must begin somewhere. I begin and began from the principle of *autopoiesis* (Maturana & Varela, 1972): the understanding of life systems as functionally structured and bound, relational coordinations. Autopoietic (literally self-composing) systems stay alive, and are constituted in their aliveness, by self-changing in fitting-enough responses to provocations 'felt' from within and without—that is, as arising out of the mediums within which and of which they exist (Maturana & Verden-Zöller, 2008; Thompson, 2007). Put differently, autopoietic systems are coherences that learn (Davis & Sumara, 2006).

A second principle of autopoietic systems is that they are nested (Maturana & Varela, 1972). That is, learning systems have a characteristic self-similar, fractal-like organisation, with smaller totalities nested in and collectively composing larger ones; these larger ones themselves being nested in and forming larger, more comprehensive ones yet, and so on. Nesting arises out of the evolutionary, selected-for, survival strategy of the functional coupling of same-level totalities, inclusive of their milieu. Thus, all autopoietic systems operate in two domains of existence at once: the domain of operation of the elements that compose and constitute each totality and the domain within which each totality exists and operates as a totality (Maturana & Verden-Zöller, 2008, p. 23). This means that each living system simultaneously operates as a totality that learns and as a partiality of a larger totality that likewise learns.

In approaching this dissertation, I considered that if learning, as self-change, distinguishes all life systems and if life systems are nested, then, to sufficiently study anxiety in mathematics learning, and to do so non-reductively, one would want to understand the common scale-*in*dependent features of learning systems *and* to consider the mutually influencing, transphenomenal effects of specific scale-dependent learning. This led to research at four levels of emergence (see Jablonka & Lamb, 2005): (1) at the level of cell bodies, the work was informed by the neurophysiology of sense making and of human affect in sense making. (2) At the level of multicellular organisms, I drew from research in human developmental science, relational psychoanalysis, and attachment theory. (3) At the level of larger collective bodies, my concern was: locally, for the nature and evolution of social systems such as classrooms and schooling systems; and, more broadly, for the historical and cultural conditioning of the possible and the permissible in what anyone can and cannot be. And finally, (4) at the level of a body of knowledge bootstrapped to a human collective body, I would find in mathematics the markings and behaviour of an autopoietic system (Davis & Sumara, 2006).

Jean-Luc Nancy's (1996/2000) human paradox of singular-plural-being seemed to differently capture, but also reinforce, the above conundrums of being and becoming. One could say that my project was that of understanding the articulation and mediation of human selves' co-becoming into life's unavoidable *with* of being, all the way up and down. *Our mind-bodies hold interwoven biological and cultural histories that act us—past into and with the present—doing so most often beyond our conscious knowing. Deep etiologies matter and affect is never far.*

PROVOCATIONS, ANXIETY, AND RISK IN LEARNING

Provocations, conscious to awareness or not, are those impetuses to potentially trigger disruption and its possible resolution as learning. Yet, unresolved disruption becomes trauma and the condition of future anxiety. Given what a life form can and cannot do, and the degrees of admitting any said provocation, some responses are more adaptive than others.

In particular, one could imagine a viable-enough response to an admitted provocation as some combination of:

1. the totality's *performative insistence* of its 'known' through self-*repetition* and a refusal of the provocation's existence in the totality's governing awareness. Either the provocation's situated effects are, over time, dissipated across and out of the system or they remain localised to potentially compromise grander coherence and/or coordination.
2. *mimetic entrainment*. The totality *learns by amplifying* what it already knows to do and be. It becomes 'stronger' through effortful practice in those aspects—even sometimes to the point of imposing its change into the broader medium within which it exists and onto other totalities with whom (*or which?*) it shares that medium.
3. *poietic emergence* as the kind of *learning*, also effortful, *that is sense making anew*. The totality, or a part thereof, undergoes a 'phase change' into a qualitative difference. It admits the perturbation and resolves its effects by incorporating that difference into an accommodating adaptation as coherent-enough revision of self.

These responses are as much given by a cell, a human, a social system, or a discipline. Each level operates at a time scale commensurate with size. The grander the system, the greater its relative inertia, and the more difficult is adaptive self-revision. For an example at a macro-level, consider science: Repetition and amplification characterise long periods of *normal science*. Rarer paradigmatic change describes the new sense making of *scientific revolutions* (Kuhn, 1996).

In education too, we find tremendous inertia. At a macro-level and despite the rhetoric of change, schooling systems seem to respond to socio-cultural provocations by repeating themselves with notable virtuosity. Indeed, the unconscious biosocial motivations underpinning human behaviour operate at an evolutionary time scale (Jablonka & Lamb, 2005) and may well explain the performative insistence governing repetitions of curricular practice. At the same time, mechanical systems are not subject to the same rules. It strikes me that the evolutionary inertia of human systems greatly exceeds the lithe acceleration of technologically enhanced, cyborg-like, mechanisms propelling social change. Fractures in grander systems can and do compromise personal experiences of security. Anxiety signals these effects.

In the 'math wars', a return to 'the basics' of rote learning and procedural training presses a kind of anachronistic performative insistence. Here, repetition at the cultural level asks a return to practices of strengthening (memetic entrainment) at the individual one. On the other hand, poietic emergence seems to be the learning hoped to arise out of participatory and inquiry-based curricular practices. Yet curricular practices resist change. Critical, it seems, is the recognition that current 'end goals' are neither participatory nor inquiry-based practices per se, but the sense making of poietic emergence that these approaches are thought to engender.

In shaping conditions for poietic emergence, there is no technique that can substitute for a pedagogue knowing her or his students well. To have influence, any difference installed by a provocation must be discernable to the admitting learning system—in this case the learner and/or the collective of learners—at some critical level. That is, the experience to be recognised must be sufficiently similar to and sufficiently different from other experiences. If too similar or too strange, it goes unnoticed; or, if noticed, yet too impossible to tackle, it risks being forcibly rejected. In other words, any disruption a teaching milieu affords must be amenable as a difference that can be incorporated into the learning system without the system being rendered irreparably damaged as a result.

These principles apply to autopoietic systems writ large, and as much to the learner whether that learner is a teacher or not, and whether the learning object is pedagogical content knowledge or some other knowledge. Just as with students, it does little good to tell teachers what to do, if what to do is already alien to their experience. Effective teaching acts are those to position a difference at the 'admissible' borders of any learner's realm of experience. And admissibility turns on the learner's prior history as a learner in the situational and conceptual domain in question, including the learner's anticipation of potential growth and/or damage, and the mitigating support that might be rallied, in case of failure. It will be a history infused with and inseparable from affect. In sum, *affective valence governs and limits the admissible in learning*.

In regards affect, of the three responses to perturbations, poietic emergence feels the riskiest. To effect the paradigmatic change of new sense making, an already coherent-enough self must first undergo some measure of self-dismantling. One cannot know the nature or the viability of the future self one-seeks-as-oneself until after its newness has emerged. Anxiety is a protective physiology to resist risky change. Under conditions of perceived threat, such as those times when the world shifts out of synchrony and pace with selves, it makes sense to curtail poiesis (the very thing we think we want more of these days) in favour of repeating and amplifying the surer self that one is, knows, and does. It follows that, in present times, gestures-to-entrain increasingly seem to extend outward in manner to coerce, control, and manage others.

INDIVIDUAL INCLINATIONS: TOWARD FIXITY OR GROWTH

We cannot usefully separate affect from cognition and learning. Anxiety in learning and teaching, and individual tendencies to greater fixity or growth, mathematics or otherwise, have roots in affect that reach back to the interplay of infant attachment relationships and human biological proclivities toward assemblage and away from disassemblage.

Prompting a revisionist approach to Piaget's developmental theory, Klin and Jones (2007) highlight the primacy of "affect and predispositional responses" as cognition's "mental fuel" (p. 42). They reiterate the radical proposal of embodied cognition "that all mental representations […] are proxies for the actions that generated them and for which they stand (Lakoff & Johnson, 1999; Thelen & Smith, 1994; Varela, Thompson, & Rosch, 1991)" (pp. 36-37). This includes generative actions that occurred in the first year of life, prior to language, when all was steeped in diffusely understood affect. Problematically, first affective experiences, being unformulated and unorganised in language, will remain outside the reaches of subsequent conscious thought and memory (Stern, 2003). These early histories are critically present and tacitly accessible in the shared *relational unconscious* (Gerson, 2004) that structures teaching and learning interactions.

The findings of research in primary attachments are clear: Attachments formed in the first year of life are seminal to later securities and to a transgenerational phenomenon of security passed from parent to child, and child, become new parent (Wallin, 2007). That first safe other seeds self-trust (Lehrer, 1999; Wallin, 2007), doing so through counterbalancing acts of support and gentle disillusionment, departure and return in good-enough measure, and always holding cataclysmic consequence at bay (Winnicott, 1971/2005).

In dynamic systems theory we say, "the consequences of choices sediment" (Juarrero, 2002, p. 253), though not in stone. Paths bifurcate—more momentously earlier-than-later in life. Teaching matters in ways we hardly realise. Secure adults in children's lives can and do help children veer in the direction of security and agentive vitality when facing and negotiating novel sense-making challenges (Steele et al., 2007). The converse is also true.

Without the resilience born of secure-enough attachment, the threat of failure to fit and/or to potentially disassemble in the face of demands to be, know, and do, cannot but functionally jam precious agency, curiosity, and creative apperception (Winnicott, 1971/2005). The condition of anxiety in learning should be understood as a reasonable, protective, foreclosing response to an anticipated, expectation-as-assumption of a self as having already produced itself 'in the know'; that is, to have pre-emptively assembled a sense *made* of a seemingly nonsensical 'sense' *given* by the world (Winnicott, 1971/2005).

CULTURAL INCLINATIONS: TOWARD FIXITY OR GROWTH

Degrees of security in individuals, adults and children, come to be recursively reinforced bottom-up and top-down in the systems that each individual inhabits and that inhabit each individual (Imamoglu & Imamoglu, 2010). A self's becoming cannot be teased apart from that of the intimate, familial, educational, and cultural collective any more than the activity of bodily cells can be cleaved from experiences of consciousness and mind.

In the absence of safe-enough places to risk agency, a façade of certainty can make better sense. A learner or a teacher, wanting fail-safe assurance of *being* right, and thus *being found right*, proclaims, "*Tell me what to do.*" Herein echoes a desire repeated in accessions to an omnipotent and omniscient father/Father, as definitive Other, upon whom one can rely for response-ability, and whose powerful cloak one might assume as protective *pretence* in the face of threatening disintegration. These 'fixing' responses to anxiety are at play in matters educational, religious, patriarchal—and ultimately in the, still present, Cartesian belief that there could be absolute foundational truth, if not accessible through Reason, then some other way.

Butler (2005) develops that performative insistence at the level of the grander collective—born of fear of change and a desire to cling to past certainties—could be thought a kind of anachronistic violence. When a once-collective ethos is no longer commonly shared, it can impose its claim to commonality by performing the ethical violence of forcing its anachronistic self upon the present, and in so doing refusing to become past (Adorno in Butler, 2005, p. 4). The violence is effected in escalating attempts to manage away or otherwise suppress threats of newly experienced unknowns as too unpredictable, unruly, and untrustworthy. In education, this kind of violence seems today enacted in a neo-Tylerian implementation of technologically re-designed renditions, of old 'scientific management' themes, brought to bear on new problems. Escalating anxiety, resonating everywhere in schooling, seems strangely related to 'no tolerance' attempts—of a still-Modern self as over-caring anxious parent to next generations—to dictate what to do and how to do it, even if ironically, what to do is to be creative.

Descartes' illusion, that all knowing can and must be built from immutable foundational truth, sets the conditions for existential crises of disillusionment (Bernstein, 1983) and their reactive expression—these reflective of, and reflected in narcissistic (and mathematics) anxieties about not being enough. Curiously these same notions of illusion and disillusionment figure central in psychoanalytic understandings about the child's coming into well-being with the good enough mother, the transitional object, and an expanding circle of pedagogues, from parent to teacher and into broader socio-cultural contexts (see Winnicott, 1971/2005). The difference as I see it, and as developed below, is critically one of scale and contingency. At the moment-to-moment micro-level discernments of individual lives, appropriate disillusioning provocations give of micro-level, accessible, and resolvable incoherences. These prompt and encourage a learner in the direction of making cascading sense, anew. That is, newer sense becomes the favoured real, but absent any absolute claim to foundation truth. Rather, this newer illusion is recognised as a contingent foundation, good enough for now, but at the ready for revision in adaptive response to an ever-shifting world.

A MODEL OF LEARNING

Consider Figure 1 below. It describes a dialectic between illusion and disillusionment. Illusion, literally *in ludere* as 'not play,' names a ludicrous mockery that plays with playing. Psychoanalytic theory deems critical the value of illusion and disillusionment, finding the sense of nonsense, and the serious work made possible in, especially, the latitude of childhood play as pretend (see Stern, 2003, pp. 65-79 and Winnicott, 1971/2005, pp. 13-19). On the side of disillusionment, the world, aided by reason, comes to reject prior illusion for its revealed unrealness. The autopoietic system formulates instead, through some combination of creative ingenuity, a revised illusion as the newer real. In a world stripped of absolute foundational truths, a revaluing of both illusion and disillusionment seems timely.

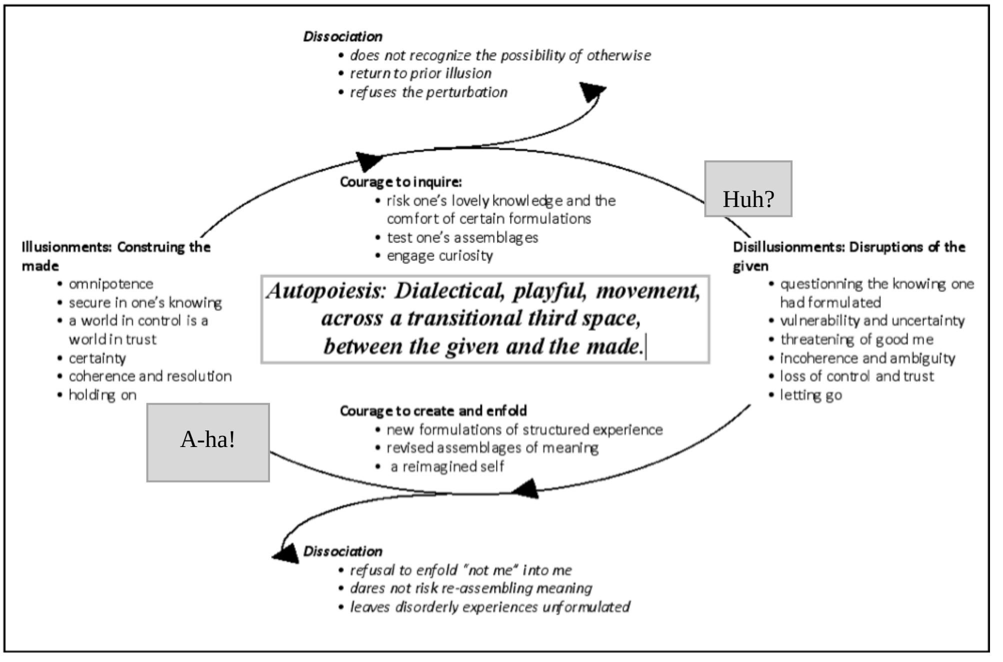

Figure 1. Recursion in learning: An expanding cycle of sense making occurs in movement in the spaces between structures, from given to made and back round again.

Figure 1 narrates a recursion where each cycle returns, neither, to the same *made* illusion, nor, to the same disillusioning *given*. On the side of illusion is consolidation, as present meaning and contingent truth, assembled through the sense made of an ambiguity resolved—the "a-ha" moment that immediately reduces cresting tension and with it anxiety. On the side of disillusionment is a perturbation, a difference, a "huh?" possibility of noticing incoherence to disrupt precious illusion's story. With each dialectic return to reimagined illusion (as both real and not real), the learning model traces a recursive enfolding of "what we *can* make *now* [the possibility of the present] out of what we *have* made *then* [the given, now embodied in that present]" (Stern, 2003, p. 4).

Consider the child's 'security blanket'. As transitional object—not unlike any cultural object, including a discipline—the blanket's meaning is both real and illusion, both of the world and of the child's making (Winnicott, 1971/2005). In the transitional space "it is unclear whether truth is invented or discovered" (Benjamin, 2005, p. 197). The impossible question, "*Was this given to you or did you make it?*" is not to be asked. Doing so would contract the sense-, world-, and self-making cycle to a single point, obliterating the play of movement, and with it the freedom to experiment safely in "*what if*" conjectures. *Where a demand to produce forecloses pretend play, there pretence, absence, and anxiety find entry*.

Across and between moments of illusion and disillusionment, are numerous experiences left unformulated. This is because "understanding is an act, and it is easier, less effortful, not to carry it out than to carry it through" (Stern, 2003, p. 76). If, as a regular matter of course, one excuses oneself from invitations to consider otherwise, then there exists an opting for stasis as defence against unknowns that might erupt. Yet "disorder is the condition of the mind's fertility" (Valéry in Stern, p. 75). Sealing it underground buries motivating affect and also denies unbidden novel experience from emerging into its creative potential.

Under the covering cloak of impermeable stories about oneself and the world, dissociation is that gesture that enacts a refusal to engage autopoiesis. Stern (2003) describes two forms of dissociation. In Figure 1, I show these as retreats: (1) to favoured illusion and (2) to abject disillusionment. To dissociate is to prevent oneself the touch of a disillusioning external world that could tear asunder one's lovely knowledge. It is also the inability or the refusal to formulate new and difficult knowledge into any workable revised sense.

In the first instance (see top of the model), Stern describes *dissociation in the weak sense as narrative rigidity* (2003, pp. 129-146). Here, in apparent lack of originality, it seems not to occur to an individual to consider competing narratives to a conventional story—the story one knows, wants to know, and accepts as known. Any perturbing influence remains outside awareness. In narrative rigidity one accedes to tacit power relations and the "invisible interpretive predispositions that represent the shaping effect of culture" (Stern, 2003, p. 77). It marks a preference for trusting structures, experienced as 'in control'. In so doing, it forecloses the ability to make meanings differently. Dissociation in the weak sense may well express blind allegiance to aforementioned anachronistic violence.

Lampert (1990) describes conventional patterns of children's "nonmathematical ways of knowing mathematics in school" (p. 55). Narrative rigidity underpins these students' responses to the question "*Why?*": "*Because that's what Tommy said, and he's usually right*" (p. 56); because that is what the rule says (p. 56); "*I just know…. It's none of your business or anyone else's how I got my answer*" (p. 57); and it's right because "*it's my way of doing it*" (p. 57).

In the second movement of the learning cycle is the work of sense-making anew, the return to, and creation of, a revised illusion. The learner, who has indeed admitted a perturbation and experienced disillusionment, has lost what was once meaningfully understood. Wanting the fortuitous making of 'instant new sense', yet failing to produce himself as thus knowing, he feels himself beside himself. Stern describes this as *dissociation in the strong sense*—a condition of *not-spelling-out* (Stern, 2003, pp. 113-128). Preferring the familiar chaos of unformulated experience the learner absents himself, insisting against the process of thinking itself. In so doing, attention eschews "'feelings of tendency'… the only direct manifestations of … [nonverbal unconscious] phenomena that we can perceive in our verbal, reflective mode" (Stern, 2003, p. 16). The learner keeps a wide berth from both the potential and the disruption of "creative disorder" (p. 76).

Not-spelling-out means saying "whatever" in a way that both names and denies care. Trauma's very presence refused leaves it beyond formulation's reach—this in favour of wandering aimlessly in a haze of unconscious anxiety. Repeated experiences of finding oneself bereft of the certainty one thought one knew—in a world that anticipates, expects, and will measure that knowing—installs an intolerable experience of profound failure and presses a fragmented exorcising of the unworthy, failed self. Under such irresolvable incoherence, the survival response is surely to excuse oneself out of presence and announce if not me, then you.

It takes courage for a learner—conditioned that to count means knowing right answers—to expose herself to being wrong, to admit any perturbation and then to attempt its resolution. Why step outside the safety of given structures only to risk losing one's footing in the messy unformulated between spaces?—No reason at all if one understands the situation as, not really learning, but producing oneself in the face of a demand to perform pre-determined measured and measurable outcomes. Under circumstances, where any space between the given and the made is foreclosed from the start, it would take quite the trusting environment to prompt a learner to risk being 'called out'.

TOWARD AN ETHICS OF RISK AND A PEDAGOGY OF ATTUNEMENT

How then might the attentive teacher wisely mete worldly disillusionment and still hold the learner safe from a terror that freezes any capacity to sense make? In answer, my two-year doctoral work with one highly anxious student returned the question to the situation of mathematics, that canary in the coalmine of something deeply amiss in education. Through these considerations, in reflective dialectic with a 30-year history in teaching, I came to understand present cultural need, not for greater control and *techne*, but for, respectful and patient, wisdom and phronesis. In the very doing of this work and in conversation with developmental research in parental attunement, affective mirroring that is containing, and a "natural human pedagogy" (Csibra & Gergely, 2011; Fonagy, Gergely, & Target, 2007) an *ethics of risk* in education and a *pedagogy of attunement* took shape.

There is not space to take up these newest formulations. Suffice it to say that an ethics of risk asks that disillusioning provocations be drawn from the cultural and disciplinary stock of knowledge, with mindful teacher attention to, individual and collective, learner variations in security, discomfort with ambiguity, and the capacity to shape resolutions. Critical is the teacher's ability and freedom to be co-present, secure, and attuned in the doing—giving audience and bearing witness not only to the learner(s)' grappling but also to the structure and laxity of the curricular scripts, scores, and choreographies on offer as objects of play.

I imagine classrooms as buttressed theatres—holding environments of just-so, reality suspension where trying on worldly acts, in pretend and not pretence, can safely occur, and where performances emerge, not according to demand, but in timely celebration of the taking up of unbidden experience into creative sense-making that is critically both self- and world-making.

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LISTENING TO STUDENTS: A STUDY OF ELEMENTARY STUDENTS' ENGAGEMENT IN MATHEMATICS THROUGH THE LENS OF IMAGINATIVE EDUCATION

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INTRODUCTION

Most educators and researchers accept student engagement to be an important part of learning (Furlong & Christenson, 2008; Taylor & Parsons, 2011). Given this view, it seems reasonable to infer learners engaged in learning are more likely to absorb and understand lesson content. This applies to any subject, including mathematics. Moreover, the significance of considering affect as an important factor in students' engagement is recognised within mathematics education (e.g., DeBellis & Goldin, 2006; Hannula, 2002, 2006). However, implementation of this recognition is not widespread. These aspects of educational inquiry rebounded against experiences in my teaching practice when I used the theoretical framework of Imaginative Education (IE) (Egan, 1997, 2005). It appeared that when I taught mathematics from an IE perspective, the subject became more appealing and engaging for students. Thus I chose to examine the issue of student engagement in elementary mathematics through the framework of IE with the question at the heart of the study being what meaning the use of IE and imaginative lesson planning frameworks had for children and their engagement in elementary mathematics. It is this PhD study that is at the centre of this article[1].

THEORETICAL FRAMEWORK

The theory of IE (Egan, 1997, 2005) involves a fundamental reconceptualization of education and its purpose, where the focus is on the development of understanding rather than on the acquisition of pieces of knowledge. In Egan's view, this occurs during five phases of gradually increasing internalisation, through the use of cultural tools such as language and communication systems. Importantly, over time, cultural tools become an individual's cognitive tools and mediate within and between cultural and educational development. While language allows expression and receipt of understanding, it is also, in Egan's view, a means of enlarging the mind achieved through progressively sophisticated use of language forms, such as oral, written and theoretic use of language, leading to acquisition of greater understanding. Drawing on a Vygotskyian view of language development, Egan sees cognitive tools as aids to thinking, gained through living as part of a society and cultural group. Developed over long periods in cultural history, cognitive tools aid development of an individual's understanding.

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[1] The full dissertation on which this article is based can be found at https://circle.ubc.ca/handle/2429/45522

FIVE PHASES OF UNDERSTANDING

An important component of IE theory (Egan, 1997, 2005) are the five phases of understanding; somatic, mythic, romantic, philosophic and ironic understanding, which are lenses or ways of seeing and understanding the world. Indeed, children may well show some characteristics of one phase before fully entering another. Although ages for the appearance of characteristics are suggested, these are only guidelines. Because of space limitations only the two phases of understanding primarily at use in the elementary school years are detailed here.

Mythic understanding—As children develop the skills of spoken language, they move into mythic understanding. Generally beginning around age 2, characteristics of the previous phase do not completely disappear but become less influential as other cognitive tools develop. Examples of commonly used cognitive tools here include language tools such as binary opposites, e.g., good/bad and captivity/freedom, which help children to organize understanding and knowledge, and stories. Language becomes a dynamic tool where children become aware that the written language with which they are becoming familiar can help them to further understand, represent, and converse with people and the world around them.

Romantic understanding—The third phase occurs between approximately 8 and 15 years of age. Children become increasingly aware of and competent in the use of written language. Literature and stories progress from the largely fantasy world and fairy stories to have more sense of reality. Children become more aware of their independence in an increasingly diverse and complex world, moving from a focus on themselves to awareness of self in relation to the world. A cognitive tool of use here includes association with heroes. Discoverers of knowledge, such as Pythagoras, can be introduced through inference of emotions they possibly experienced when making their discoveries, which can in turn be incorporated into lessons. Children begin to understand and identify with the emotions and qualities that a role model/hero embodies without going deeply into abstract characteristics of topics being studied. The sense of mystery experienced in mythic understanding now develops into a sense of wonder, encouraging children to ask a range of questions about things they notice and experience.

Two key concepts of the IE theory are that of *imagination* and *emotions* which are believed to orient and establish the development of understanding. Egan (1992) defines imagination as

> *the capacity to think of things as possibly so: it is the intentional act of mind; it is the source of invention, in the construction of all meaning; it is not distinct from rationality but is rather a capacity that greatly enriches rational thinking.* (p. 43)

This definition creates opportunity for exploration of ideas with room for creativity, growth, and the reaching of individual and collective potential. For example, the renowned geometer Donald Coxeter saw imagination as a necessary and fundamental part of inquiry and mathematical discovery: "As for the role of imagination, I should say that all discovery requires imagination" (personal communication, September 7, 2002).

Emotions are as central to the IE theory as imagination. Consideration of knowledge as a product of human minds generated from hopes and fears, joys and sorrow, leads to acceptance of a human purpose to knowledge creation. There have been, and will continue to be, discoveries and creations that began with an emotional reaction to a situation. Our emotional response is a way of gauging a stimulus effect. Therefore, emotions are involved in both the creation of and reaction to knowledge and must be considered as having an important effect on learning. The blending of imagination and emotions, two fundamental human faculties, creates a dynamic form of learning that clearly involves the affective domain, which includes our emotional responses. Hannula (2002) believes that when active attention is given to the affective domain and when affect is combined with cognitive learning, there is tremendous potential for even greater development of a learners' understanding than there is with cognitive learning alone. There is therefore, opportunity to place mathematical learning in a humanized, sociocultural, and real-world context from which it has become detached for many individuals (Boaler, 2000; Nardi & Steward, 2002a, 2002b, 2003). Within an IE theoretical context, learning can become an educational experience in which there is opportunity for "the having of wonderful ideas […] the essence of intellectual development" (Duckworth, 2006, p. 1).

METHODOLOGY

An instrumental qualitative case was designed to characterize the phenomenon of student engagement for six Grade 4 and Grade 5 students during a regular mathematics curriculum unit on shape and space. The goal was to gain more understanding about what engages students in elementary mathematics when a particular theoretical framework was used. I considered it important to do this from the students' perspective as expressed in their semi-structured individual and group interviews, mathematics journal entries and activity pages.

I took the role of Teacher/Researcher, acting as a participant observer with a Research Diary serving as a medium to record, and later reflect upon, observations noticed during lessons. In addition, a very detailed unit overview of lessons and individual lesson plans was prepared to align mathematical concepts with the theoretical framework. A *critical friend* (Costa & Kallick, 1993) was utilised during the data collection who noted her observations of a selection of lessons, later providing an oral commentary of her notes and observations.

The unit of 15 lessons lasted approximately 6 weeks. The introductory section of four lessons began with a vision walk activity where, in partners, students took turns wearing an eye mask to navigate their way around the school, and upon return to the class students wrote about their experiences in their math journals. After all lessons were completed and regular assessment and report cards had been submitted, individual semi-structured interviews and a group interview took place.

Data analysis took place in four stages, including transcription of interviews, examination of characteristics of student engagement including affective and participatory domains of learning with the use of *Participatory Affective Engagemen*t (PAE) (Hagen, 2007; Hagen & Percival, 2009), coding for use of the cognitive tools of the IE theory, and an emergent theme analysis.

FINDINGS

Bringing the cross case analysis together, as shown in Table 1 below, three themes emerged from the corpus of data across all students to form a focus of discussion. These themes were: making connections, developing self-confidence, and cultivating mathematical awareness.

The evidence from cross-case analysis demonstrated all students were able to form connections between themselves and the mathematical content in personally meaningful and relevant ways. Courtney, Jason and Freddie all drew connections between themselves and their families. For example, Jason replicated activities at home related to the discovery of the properties of angles, and took great pride in teaching his parents something they did not already know.

Student	PA Engagement Descriptors	Thematic Characteristics of Engagement
Grade Four		
Courtney	Passive positive engagement	1. Drawing on emotions 2. Making connections 3. Developing self-confidence 4. Cultivating mathematical awareness
Jordan	Active positive engagement	1. Demonstrating a sense of wonder 2. Developing self-confidence 3. Drawing on emotions 4. Making connections
Kee	Passive positive engagement	1. Demonstrating a sense of wonder 2. Making connections 3. Developing self-confidence 4. Cultivating mathematical awareness
Grade Five		
Freddie	Active positive engagement	1. Demonstrating a sense of wonder 2. Developing self-confidence 3. Making connections 4. Cultivating mathematical awareness
Grace	Active positive engagement	1. Drawing on imagination 2. Making connections 3. Cultivating mathematical awareness 4. Developing self-confidence
Jason	Active positive engagement	1. Cultivating mathematical awareness 2. Making connections 3. Developing self-confidence 4. Demonstrating a sense of wonder

Table 1. Cross-case comparison of emergent themes.

All the students were able to develop self-confidence in personally relevant ways becoming more aware of their learning styles and abilities, aided by their meta-awareness of emotive reactions to different learning situations. Levels of self-confidence were, appropriately, not commensurate across each of the students, but individual in nature, reflecting the development of the students' mathematical understanding during the unit.

The students' work samples indicated their depth of use and their adoption of cognitive tools, as summarised in Table 2 below. A key component of the IE theory (Egan, 1997, 2005) emerging as important in this study was the students' use of the cognitive tool of *wonder*. For the students, wonder had acquired significance.

A comment from Jordan during the group semi-structured interview exemplifies the sense of wonder that the students collectively indicated had helped to lay a foundation on which they could build their cognitive understanding of the shape and space concepts. Jordan stated

> Cause we all, kids, like to day dream, and then teachers don't really want them to day dream during class, but while we were doing the imagination thing, it kind of lets you do that during class so that, so that it helps you think, because you can just sit there and think, the teacher would let you, because that is the thing to do.

Boaler (2000) attributes students' disengagement from mathematics to teaching practices that do not foster connections to the world outside the mathematics classroom. Here two emerging themes—making connections and cultivating mathematical awareness—were prevalent across all six students. Throughout the unit, participants began forming connections between themselves and the mathematical concepts in personally meaningful and relevant ways. The children formed connections with historical contexts and figures, their families and everyday interests such as the environment. In addition, as these students began to connect with the concepts of shape and space in these multiple ways, not only did they appear to cultivate a growing mathematical awareness, but they explicitly spoke about their expanding views (e.g., "*math includes shapes and words*"; "*math is fun*"; "*you can put emotions in math*"; "*math is all around us*"). While some of this breadth could be attributed to the 'new' topic of shape and space (i.e., "*not just numbers, old boring math*"), these children consistently pointed to aspects of the IE theory framed unit, as seminal to their growing awareness of mathematics.

Phase of Understanding	Cognitive Tool	Frequency
Mythic	Rhyme/Metre/Pattern	9
"	Formation of Images	9
"	Mystery	5
"	Play (role)	2
Romantic	Wonder	17
"	Sense of Reality	11
"	Context	8
"	Humanization	7
"	Hero Association	7
"	Narrative	6

Table 2. Summary of student use of cognitive tools.

Likewise, these children's strong connections to mathematicians and artists (e.g., Pythagoras, Coxeter, Escher) introduced in this unit, concurs with Ward-Penny's (2011) view that humanised mathematics "Can help pupils explore mathematical ideas in a more well-rounded way and reconnect many mathematical concepts to the exploration and enquiry from which they originally emerged" (p. 147). These innovators of mathematical ideas and concepts in the past became people with whom these students felt a current human connection. Mathematics became a subject that had a rich historical past that was no longer inaccessible and clouded in obscurity, or stuck in a textbook. Mathematics also became a relevant and meaningful subject; the students were now seeing mathematics in places that they did not see math before. Consequently, using IE theory seemed to mean that for these children their engagement was such that mathematics gained a relevancy and connection to their everyday lives and their awareness of what mathematics is and where it might be found.

Nardi and Steward (2003) give an emphatic warning that, "[i]n the absence of mathematical experiences suited to individual needs and consequent feelings of success and self-esteem, students become alienated from the subject and eventually choose not to study it" (p. 5). Here, children reported feelings of pride in their work and in themselves and espoused their beliefs of doing better in mathematics. They reported that IE helps children "learn more mathematics" because "we usually do lots of cool stuff "and "since we did stuff in like a fun way, … I [we] can remember all of it." During the focus group interview the confidence these children exhibited through their frequent raising of hands, a plethora of voluntary contributions and a barrage of interconnected comments was impressive. It was as if the self-confidence they were developing, albeit to varying degrees, had reached a crescendo in this context.

What appears unique to this study then is the way in which IE features of the "mathematical experiences" served to address children's "individual needs and consequent feelings of success" (Nardi & Steward, 2003, p. 5). That is, the invitation to draw on their emotions and imagination or to demonstrate their sense of wonder (i.e. all emergent themes for individual children in this study), served to open up the ways in which each of the six children engaged with and recorded their mathematics. Thus, it appears children's self-confidence was related to the more individual ways they could address the content and express their solutions. There was, therefore, a dynamic interplay between the use of imagination and emotions, which is congruent with both Egan's (2005) view that successful learning requires emotional involvement, and that of Hannula (2002) that the combination of affect and cognitive learning provides tremendous potential for even greater development of a learner's understanding than cognitive learning alone.

An important tenet of the IE theory is the need for educators to draw upon ways of thinking which are already familiar to learners, and which the students already use (often intuitively) to make sense of the world around them. While not surprising, considering the explicit focus on IE, all participants in this study used various cognitive tools such as pattern, wonder, hero association, both in their work samples and in their interview comments. Egan's (1997, 2005) premise that cognitive tools resonate with ways in which children come to know the world was reaffirmed. It appears using the IE theory means children can and do draw on these cognitive tools to engage with mathematics, at least with shape and space. What this study adds to Egan's theory, however, is that in this context, the cognitive tool of wonder was predominant and although planned within mythic understanding, these children used a greater variety of cognitive tools from the romantic phase of understanding.

IMPLICATIONS AND RECOMMENDATIONS

FURTHER RESEARCH IN DIFFERENT CONTEXTS

While this study provides beginning insights into IE theory and children's engagement with mathematics, the number of students, while appropriate for a case study, was small. There is a need to continue research with different sized groups. Furthermore, participants were students in the intermediate years of education (i.e., Grades 4 and 5). However, additional research with students in other grades, e.g., the early years of learning, such as kindergarten or older grades, where there is even more tendency for students to disengage from learning mathematics (Boaler & Greeno, 2000; Ward-Penny, 2011), could shed light on which aspects of the IE experiences resonate in particular grade bands. It is also acknowledged that the shape and space unit may have been particularly conducive to the six students producing creative, artistic representations of their mathematics understanding. Therefore, further research about IE theory use in other mathematics strands, such as number concepts and operations, is recommended to capture how extensively it might be used to support children's engagement with mathematics. Finally, since the study focussed on student perspectives, research into teacher perspectives is also warranted.

FURTHER RESEARCH INTO IMAGINATIVE EDUCATION AND AFFECT IN MATHEMATICS

Within mathematics education much is already known about the importance of considering affective responses (DeBellis & Goldin, 2006; Malmivuori, 2006). What remains unclear is how to situate imagination in this literature so that we might better understand how children's use of imagination, likely in combination with other affective responses, can contribute towards fostering mathematical engagement. This study suggests that further research into the complementary characteristics of imagination and emotions is needed. Since within the IE theory (Egan, 1997, 2005) itself the relationship between emotions and imagination is yet to be fully explicated, both mathematics education researchers and IE researchers would likely benefit from further understanding of how these two concepts interrelate and under what circumstances they continue to foster positive engagement, as was found in the current study.

FURTHER RESEARCH INTO MATHEMATICS ENGAGEMENT

Evidence from the study pointed to the students' self-confidence, connections and mathematical awareness as important characteristics of their engagement with mathematics. In addition to identifying these characteristics, further research into how, and in what way, such characteristics influence students' ability to engage with their mathematics learning or vice-versa is warranted. For example, is self-confidence an outcome of mathematical engagement, or a necessary pre-requisite for engagement?

FUTURE PRACTICE WITH IMAGINATIVE EDUCATION

Overall, this study suggests using the IE theory meant these children engaged positively with mathematics. Also, these students' use of both their imagination and emotions seemed related to their developing self-confidence. And, making connections (often through stories) contributed to these children's increasing awareness of mathematics around them. This study implies that, while more understanding about the IE theory in mathematics education contexts would be helpful, teachers are encouraged to consider ways in which they might implement IE, in part or whole, in mathematics lessons.

Results also suggest that cognitive tools were facilitative to the students' engagement with mathematics; therefore it is recommended that teachers incorporate cognitive tools in regular mathematics lessons. The cognitive tools of wonder and rhyme, metre, and pattern were shown to be most significant to students' engagement during this study.

Another contribution of this study to future practice is drawn from the methodology. The research focus on the student perspective allowed richness and depth to be added to what this IE Shape & Space Unit meant to both the children and their engagement in elementary mathematics. At times during the study it was striking how active consideration of the students' perspective served to extend and clarify my understanding and knowledge emerging from the data. Therefore, in addition to joining Taylor and Parsons (2011) in their call for inclusion and expansion of students' perspectives in research endeavours, it is recommended to include and expand the use of students' perspectives in classroom practice. As such our practice is likely to be enriched by such consideration, as will our reflections on the intended and experienced curriculum, which students will inevitably lead.

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SOCIETAL VIEWS OF MATHEMATICS AND MATHEMATICIANS AND THEIR INFLUENCE ON ELEMENTARY STUDENTS

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INTRODUCTION

For years, mathematicians and mathematics education researchers (Furinghetti, 1993; Lim & Ernest, 1999) have expressed concerns about the ‘image problem’ of mathematics, a field perceived to be defined by rigid rules, irrelevant to everyday life, and only for a select few. In North America, many people dislike or fear mathematics, yet feel unashamed of their lack of mathematical skills (Boaler, 2008; Paulos, 1988). Mathematicians tend to be seen as socially inept individuals who narrowly focus on mathematics to the detriment of the rest of their lives (Mendick, Moreau, & Hollingworth, 2008; Picker & Berry, 2000, 2001). Popular media typically perpetuate these stereotypes about both mathematics and mathematicians (Applebaum, 1995; J. Hall, 2008; Mendick, Epstein, & Moreau, 2007). Since most people have neither met a mathematician in ‘real life’ nor studied higher mathematics, they lack the means to challenge media representations. Such representations may unduly influence children’s views of mathematics and mathematicians before they have a chance to form their own opinions, based on their own experiences. With these concerns in mind, I sought to understand how children may be impacted by growing up in a culture that is rife with negative views surrounding mathematics and mathematicians.

THEORETICAL FRAMEWORK

I assume a social constructivist and feminist epistemological stance; accordingly, I view the discipline of mathematics and conceptions about mathematics and mathematicians as socially constructed, gendered, and historically and culturally situated. I align with Cobb’s (1994) complementarist position on social constructivism, in which neither individual nor social aspects are given primacy. I apply Cobb’s stance on mathematics learning to learning in general, which is viewed as “both a process of active individual construction and a process of enculturation into the [mathematical] practices of the wider society” (Cobb, 1994, p. 13). Feminism aligns with a broader social constructivist stance due to the understanding of gender as a key factor in the construction of knowledge. Certainly, feminism is a complex term with many interpretations and schools of thought. In eschewing association with a particular label of feminism, I align with Lather (1988) who posited that, “Feminism argues the centrality of gender in the shaping of our consciousness, skills, and institutions as well as in the distribution of power and privilege” (p. 571).

My conceptual framework (Figure 1) focuses on the various views and representations of mathematics and mathematicians that my study examines, and represents the dynamic and reciprocal relationships among these multiple perspectives, provided by the study’s ‘actors’.

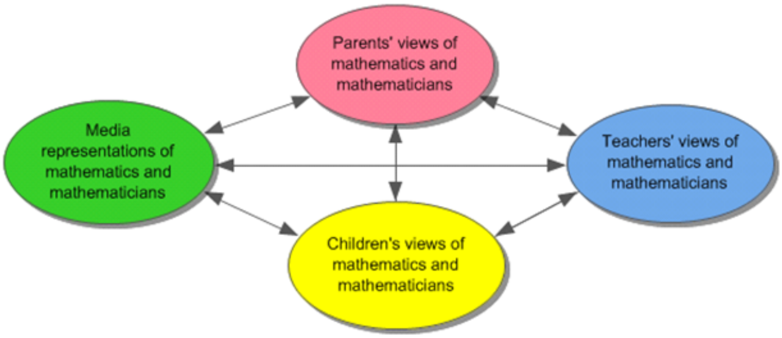

Figure 1. Conceptual framework, showing relationships among media representations, parents' views, teachers' views, and children's views of mathematics and mathematicians.

The influence of one actor on the other may not be equal in both directions, due to unequal power relationships. In each pairing, an actor plays the role of both the producer and consumer of understandings about mathematics and mathematicians. Producers create and disseminate representations, which involve "the production of meaning through language" (S. Hall, 1997, p. 28). This theory of representation, the constructionist approach, is highly linked to social constructivism: "Things don't *mean*: we *construct* meaning" (S. Hall, 1997, p. 25, emphasis in original). Consumers are actively involved in sense-making of the ideas presented to them (Bickham, Wright, & Huston, 2001; Huntemann & Morgan, 2001). Sense-making is highly individualized, based on previous experiences and current beliefs and understandings. When a perturbation occurs during an interaction, the active consumer interprets the information and then assimilates it into or makes accommodations to existing cognitive schemas.

REVIEW OF THE LITERATURE

In this section, due to the space constraints of this paper, I focus solely on studies that examined young people's views of mathematics, mathematicians, and gender, in order to provide an overarching understanding of the mathematics-related views that young people hold, and hint at the messages to which they have been exposed. Research has found that young people tend to hold stereotypical views of mathematics and mathematicians, such as mathematics as being equated to arithmetic (e.g., Perkkilä & Aarnos, 2009) and mathematicians perceived as being nerdy White men (e.g., Hekimoglu & Kittrell, 2010). Mathematicians are typically viewed as doing the same types of mathematics as students, and working in a classroom setting and/or as a teacher (e.g., Gadanidis, 2011; Rock & Shaw, 2000). Even when young people view mathematics and/or mathematicians favourably, their views of mathematicians become negative when they are personally implicated, which suggests a disconnect between young people's views of school mathematics and of mathematicians (e.g., Picker & Berry, 2001). Since mathematicians are typically viewed as men, mathematics becomes seen as a field that is misaligned with femininity. The findings from the studies examined suggest that more exploration is needed into these topics to better understand how young people form views of mathematics and mathematicians and to examine the sources that may influence their views.

METHODOLOGY

The purpose of my study was to investigate elementary students' views of mathematics and mathematicians, and the ways these views may be influenced by children's media, parents, and teachers, in order to better understand the complex interplay between outside sources and children's views. I also sought to understand gender and age-related trends in children's views. Specifically, my study was guided by the following research questions:

1. What images of mathematics and mathematicians exist in a child's world as produced by: a) media portrayals? b) parents' views? c) teachers' views?
2. How do elementary students view mathematics and mathematicians?
    a) Are there differences between boys' and girls' views?
    b) Are there differences between middle (Grade 4) and senior (Grade 8) elementary students' views?
3. What is the relationship between elementary students' views of mathematics and mathematicians and: a) media portrayals? b) parents' views? c) teachers' views?

To address my research questions, I collected several types of data, involving multiple types of participants. Specifically, my participants were Grade 4 (ages 9-10) and Grade 8 (ages 13-14) students, teachers, and parents from Ontario. Online questionnaires about media habits and views of and experiences with mathematics and mathematicians were completed by 156 students. Additionally, 94 students drew pictures of mathematicians, with written explanations of their drawings. Thirteen parents and ten teachers participated in interviews about their views of and experiences with mathematics and mathematicians, as well as their interactions with their children/students and mathematics. Children's media (selected based on the students' questionnaire responses) were analyzed for representations of mathematics and mathematicians, and some of these media examples, as well as some of the mathematician drawings, were used as prompts in focus group interviews with students (five focus groups, involving 22 students).

Due to the paucity of space in this paper, methods of analysis will not be discussed. Interested readers can consult my thesis for these details.

FINDINGS

Below, I discuss my findings in terms of overarching themes that encompass multiple data types. These themes address views of (1) mathematics, (2) mathematicians and mathematically proficient people, (3) school mathematics, and (4) gender and mathematics.

MATHEMATICS

The student, parent, and teacher participants consistently expressed a strong view that mathematics is an important subject to learn and understand—for use in everyday life, for several fields of study and careers, and for the life skills it teaches. The student questionnaire participants reported viewing mathematics as important, which reflected the strong messages the students reported receiving from their parents. These views were corroborated by the parent participants' statements and their reported actions (e.g., homework assistance). Similarly, the teacher participants reported placing high importance on mathematics in their teaching. All teacher participants stated that they regularly discuss the importance of mathematics with their students (e.g., real-world applications).

However, the importance placed on mathematics was tempered by the ways in which mathematics was defined by the participants—often, simply defined or depicted as being numbers and/or arithmetic. This narrow viewpoint was surprising, given that the students and teachers (and parents, through homework assistance) would have been exposed to five different strands of mathematics (Number Sense and Numeration; Measurement; Geometry and Spatial Sense; Patterning and Algebra; and Data Management and Probability) in the Ontario Mathematics Curriculum (Ontario Ministry of Education, 2005). This representation was clearly apparent in the students' drawings and in the children's media examined, where numerals and simple one-digit arithmetic questions were commonly shown. Such narrow views of mathematics were also present in the parent interviews, with approximately half the parent participants defining mathematics as (involving) numbers. Similarly, the examples provided in the teacher interviews often focused on number sense.

Mathematics was also frequently associated with money and finances. The third most common reason reported on the student questionnaire for hiring a mathematician was to help with personal finance. Similarly, several teacher and parent participants claimed that those involved in financial professions (e.g., accountants, bankers) were mathematicians. When discussing why it is important to learn and understand mathematics, many of the examples provided by the parents and teachers related to money. Many media examples of mathematics provided by the parent and teacher participants involved finances, such as stock prices, and money-related topics were common in the children's media as well (e.g., wizard currency in *Harry Potter*). This focus on money and finances reflects a broader societal focus on financial literacy, particularly in the wake of the recent global financial crisis. Overall, mathematics was not featured frequently in most of the children's media examined, which may suggest that mathematics is irrelevant to the characters' daily lives.

MATHEMATICIANS AND MATHEMATICALLY PROFICIENT PEOPLE

By far, the most common description of mathematicians was that they are highly intelligent. This notion was discussed by all participant groups, as well as portrayed in the media representations. Mathematical ability was associated with the ability to rapidly do calculations in one's head. Such ideas were common in the children's media examined (e.g., card-counting in *The Hangover*). Another stereotypical notion that featured prominently in the participants' discussions was the idea that mathematicians work alone. For example, none of the student participants drew a mathematician working with colleagues, and many were shown working alone on chalkboards full of equations. Several parent and teacher participants pictured mathematicians in a similar fashion.

A high level of mathematical ability and overall intelligence were seen as barriers to social interactions, as these characters were often portrayed as thinking about such esoteric topics that others could not relate to them. Such notions are linked to stereotypes of mathematicians being socially inept 'geeks' who lack street smarts. However, in all methods of data collection with the student participants, the trend was to not consider mathematicians to be geeks. This was particularly the case when the students' views were queried directly, although some contradictory stances were apparent in less explicit ways. Similarly, although the parent and teacher participants indicated an awareness of the 'math geek' stereotype, many lacked any alternative representations and admitted that they pictured mathematicians in stereotypical ways. Several of the media examples were reflective of the 'math geek' trope (e.g., *The Big Bang Theory* main characters).

Mathematicians were commonly associated by all participant groups as being teachers, although the association depended on the level of the teacher. Interestingly, such representations were not common in the media examples. Teachers were the second most common 'character category' in the student drawings, representing more than one-quarter of the drawings. When the student questionnaire participants were asked why one would hire a mathematician, three of the four most common responses (representing more than 70% of the responses) related to teaching. While the student focus group participants tended to think that elementary teachers were not mathematicians, they generally agreed that high school mathematics teachers and university mathematics professors were mathematicians, views aligned with those espoused by the teacher and parent participants.

While several of the student, teacher, and parent participants self-identified as being good at and enjoying mathematics, they seemed disconnected from personally being implicated as a mathematician. For instance, the student questionnaire participants generally reported that they enjoy doing mathematics and feel confident in their abilities, but they tended not to be interested in becoming or marrying a mathematician. However, participants who had met a mathematician were more likely than participants who had not met a mathematician to want to become one, indicating the importance of personal connections. Notably, a parallel outcome did not occur for those students who had seen a mathematician in the media. No teacher participants viewed themselves as mathematicians overall, and only two felt that they were mathematicians in any way (e.g., only when teaching mathematics). Although most parent participants had high levels of mathematics education and a few used university-level mathematics in their careers, only one parent participant identified as a mathematician. For the adult participants, narrow views of mathematicians tended to underpin their dissociation from the career.

SCHOOL MATHEMATICS

Both the mathematics content and the manner in which today's school mathematics is taught differed vastly from the mathematics the parent and teacher participants experienced as students, with the latter focusing on number sense, memorization, and repetition, and the former focusing on a variety of mathematics topics, understanding, and multiple representations. While the participants generally agreed that the way mathematics was currently being taught was an improvement due to the focus on understanding and multiple approaches to answering a question, there were concerns raised by parents who struggled to help their children with their mathematics homework. However, the teacher participants reported that parents were usually receptive to 'new' ways of doing mathematics once explanations of the underlying theories were provided, whereas the parents initially tended to resist the changes. In contrast, the teacher participants were initially receptive to these changes, but they often struggled to alter their teaching style, due to their familiarity with teaching the way they were taught. Once the teacher participants relearned mathematics in these new ways and garnered a much greater understanding, they were far more receptive to teaching in such ways. This suggests the importance of personally experiencing learning mathematics in these ways before a true appreciation of its value can be gained.

GENDER AND MATHEMATICS

The links made between mathematics and gender in this study were complex and varied: In some cases, mathematics was purported to be a gender-neutral field, while in others, stereotypes about gender and mathematics emerged. Encouragingly, in most instances, the student participants tended to link mathematics with gender equality. For instance, nearly all of the focus group participants, upon seeing a clip from *The Simpsons* about gender stereotypes, argued that boys and girls are equally capable in mathematics. Similarly, with regard to the student drawings, the mathematicians drawn were nearly evenly divided by gender.

Even with these encouraging findings, some gender stereotypes were evident. Within the nearly gender-balanced drawings of mathematicians, the professions to which the mathematicians were assigned were highly gendered: Most teacher mathematicians were drawn as women and most professional mathematicians were drawn as men. The student questionnaire participants indicated that they viewed fathers more favourably than mothers with regard to mathematics. Fathers tended to assist with mathematics homework, as they were perceived as more skilled at the subject area.

While most of the media examples provided by the parent and teacher participants featured men, several of the children's media that were analyzed featured girls and women who are mathematically proficient (e.g., Hermione in the *Harry Potter* series, Thea in the *Geronimo Stilton* series). In what could be seen as both a positive and negative sign, these characters were depicted as being conventionally attractive, feminine, and socially adept. There were also a few stereotypical representations in the children's media of girls and women who struggle with mathematics and/or have a great deal of mathematics anxiety (e.g., Penny on *The Big Bang Theory*). Overall, while some indications of progress with regard to gendered representations in the media are present, several stereotypes continue to be perpetuated, suggesting narrowly defined ways to be a girl/woman or a boy/man, particularly in a highly gendered field like mathematics.

CONCLUDING REMARKS

The images of mathematics and mathematicians to which elementary students are exposed tend to be few in number and stereotypical in nature. This paucity of examples leads to a lack of mathematical focus in our cultural consciousness, as well as more weight being placed on each example viewed. While a variety of messages are produced by different media types (as well as by parents and teachers), the focus remains on stereotypes of mathematics as numbers/arithmetic or money/finances and mathematicians as nerdy White men.

Students' views of mathematics, including confidence and gendered views, were overwhelmingly very positive. However, the students tended to view mathematics in the narrow ways discussed above, which is indicative of the influence of the cultural milieu in which they live. Students' views of mathematicians were mixed, but they seemed to distance themselves from becoming personally implicated with the career; again, such views were evidenced in the parents' and teachers' discussions. Encouragingly, and in contrast to prior research, few gender differences were found in this study, and when they did occur, they tended to relate to perceptions of mathematicians and of their parents and mathematics. Similar to prior research, grade-level differences were common in this study, with older students holding more negative and stereotypical views than younger students.

This study contributes to the extant body of mathematics education research in several ways. First, it provides a detailed description of representations of mathematics and mathematicians in children's media sources, providing a better understanding of how societal views are co-constructed by the media, and how these views may influence young people's views. Second, the study increases our understanding of elementary students', parents', and teachers' views of mathematics and mathematicians. These understandings may suggest ways to increase young people's interest in mathematics, which could lead to increased achievement and participation in mathematics. Furthermore, this study contributes to what is currently a very small body of literature that has investigated young people's views of mathematicians. Finally, as far as I am aware, this is the first study to investigate representations of mathematics and mathematicians in media sources that are popular with young people, as opposed to analyzing media that are already known to include representations of mathematics and mathematicians.

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PERCEPTIONS, PEDAGOGIES, AND PRACTICES: TEACHER PERSPECTIVES OF STUDENT ENGAGEMENT IN GRADE 9 APPLIED MATHEMATICS CLASSROOMS

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PURPOSE

Fewer than 4% of students who fail Ontario's Grade 9 Applied level English or Mathematics complete secondary school within four years (King, Warren, Boyer, & Chin, 2005). The literature emphasizes the importance of student engagement and its direct implication on student retention and academic success (e.g., Lowe et al., 2010). Indeed, much research posits that students are at risk of dropping out if they do not form positive relationships with their peers (Janosz, Le Blanc, Boulerice, & Tremblay, 1997) or if they lose interest in schooling and do not feel as though there is a personal benefit from staying in school (Ensminger, Lamkin, & Jacobson, 1996). Framed by Willms, Friesen, and Milton (2009) and Janosz, Archambault, Morizot, and Pagani's (2008) notions of engagement, this study explored the teaching practices and perceptions presented by three Grade 9 Applied Mathematics teachers to increase student engagement and enhance student learning. Specifically, the social and academic domains of student engagement are used to examine the factors that teachers considered as they strived to increase student engagement.

PERSPECTIVES

Quinn (2005) defined students to be engaged when they are "captured, heart and mind in learning [and] are cognitively and affectively connected with the learning experience" (p. 12). Student engagement is important in the school setting because disengagement can lead to rebellion, disruptive behaviour, and academic disinterest and failure (Hand, 2010). This is of particular importance during adolescence, a developmental period in which the individual undergoes many changes (e.g., physical and emotional) and, as such, is at greater risk for potentially negative repercussions (Archambault, Janosz, Morizot, & Pagani, 2009).

Students in early secondary school need a range of supports to be academically successful. Adolescents often require strong emotional and social support as well as social engagement (including social isolation/rejection, quality of student-teacher relationships, and participation in extracurricular activities) to be successful (Janosz et al., 2008). It is important that adolescents feel like they belong, as a sense of comfort with their peers can increase the chance that an adolescent will be successful (Osterman, 2000). Schools should be an environment in which students feel invited and supported; specifically, teachers can and should create a classroom-learning environment that fosters a sense of community (Hargreaves, Earl, & Ryan, 1996).

Adolescent learners appreciate the chance to take ownership of their learning (Chapman, Skinner, & Baltes, 1990). Hand (2010) encourages teachers to allow students ample opportunity to construct knowledge through exploratory, student-centered activities. Teachers should also use tasks and activities that are matched to their students' interests. By making mathematics learning relevant to the students' lives, students can become more invested in their learning and view learning as something worthwhile to engage in rather than solely a task to be completed (Davis, 2006). Teachers can also increase the relevance of mathematics to the adolescent learner through the use of technology (Gee, 2003).

METHODS AND DATA SOURCES

This study investigated the ways in which Grade 9 Applied level mathematics teachers attempt to engage their students. To this end, I endeavoured to capture how teachers engaged their students and their rationale for their pedagogical choices. I followed an exploratory case study method (Yin, 2009) and used qualitative methods of data collection (teacher interview, classroom observations, and teacher journals) and analysis (constant comparison). The participants were three Grade 9 Applied Mathematics teachers who teach at public secondary schools within a large urban centre in central Canada. All participating teachers were part of a professional development initiative that focused on improvement of instructional strategies in Grade 9 Applied Mathematics (Jao & McDougall, 2011, in press).

Data were collected during the first semester of the 2011-2012 academic year and generated through three sources: semi-structured teacher interviews, field notes from classroom observations, and teacher journals. An initial interview determined the background of the teacher and their impressions of student engagement. Participating teachers were asked to keep a journal for the duration of the study. The structure of the journal was open-ended to allow teachers to articulate their reflective and emotional impressions about their class in a way that best suited them. Each teacher was observed teaching bi-weekly and field notes were made about the teaching strategies that were used, how they were used, and how the students responded. Following each classroom observation semi-structured interviews were conducted with the teachers. Each interview started by asking the teacher to reflect on his or her lesson, followed by questions specific to the teacher based on emerging themes from journal entries, classroom observations, and previous interviews.

The data analysis included an initial exploratory review of the data and a constant comparative analysis (Miles & Huberman, 1994) of interview transcripts, field notes, and teacher journals. Computer qualitative research software, *nVivo9*, was used to assist in the analysis of the data. The initial inductive coding scheme was based on themes from literature about student engagement and was deductively elaborated on based on the emerging themes from the data.

FINDINGS

The cases of Benjamin, Mathieu, and Nadia are used to describe ways in which Grade 9 Applied Mathematics teachers increase student engagement. Consistent across all three cases is the use of similar instructional strategies, including the Ontario Ministry of Education's (2005) *TIPS4RM* resource and the use of technology. Findings suggest that these teachers consider aspects of both social and academic domains of student engagement in their teaching, but to varying degrees and with different emphases.

THE CASE OF BENJAMIN

To increase student engagement in his Grade 9 Applied Mathematics classroom, Benjamin integrates many factors into his teaching. Specifically, these factors fall under both the social and academic domains for student engagement. In the following quote, Benjamin describes one student whom he described as going from not-engaged at the beginning of the semester to being fully engaged by the end:

> I have one student who… I thought would be a non-attender. But he is engaged by a number of the strategies that we have used and he does not miss class ever. [T]here is certainly a proudness [sic] when I see him…I think that in providing engagement and just giving him some sort of validity in the classroom has been huge.

Benjamin's description of this student demonstrates that he considers factors in both the social and academic domains for student engagement. Within the social domain, Benjamin mentions that there has been an increase in this student's self-confidence. In the academic domain, Benjamin notes that the inclusion of a variety of teaching strategies had a positive influence on the student. Examples of strategies used by Benjamin include cooperative group work, use of technology and manipulatives, and rich learning tasks. As evidenced by an improved attendance record, Benjamin believes that the strategies he uses in class are compelling enough for this student to make the decision to attend class more often.

THE CASE OF MATHIEU

Formerly a self-confessed 'traditional teacher', Mathieu changed his teaching approach as a result of the *Collaborative Teacher Inquiry Project*. Since then, he has noticed a change in the engagement of his students, something that he directly attributes to his reformed teaching methods. Thus, Mathieu continues to focus on the academic domain for student engagement.

> The kids are always engaged and do the richer-type problems where they have to collaborate, work together, and try to come up with a solution. So it is mostly student driven. The teacher is just a facilitator who gives directions. [The students] are doing stuff. Hands-on. So they are not sitting there being bored.

Through Mathieu's description, his focus on factors within the academic domain is evident. Primarily using group work to increase student engagement, Mathieu also includes different technologies in his teaching to appeal to his students' personal interests. Through the use of graphing calculators, interactive whiteboards, *i>clickers*, and online simulations, Mathieu notices that his students enjoy experimenting and discovering the course material. These teaching strategies allow students to become motivated to learn the material and support their developing mathematics understanding. Additionally, Mathieu finds that these teaching strategies allow the students to be more socially at ease in the mathematics classroom as they are given the flexibility to chat and strengthen relationships with their peers and feel more secure within their peer group.

THE CASE OF NADIA

For Nadia, student engagement extends beyond student interest in mathematics content. Nadia explains that, especially for Applied level students, students show their engagement on a personal level. If students are engaged, they will connect with their peers and their teacher on a personal level. Even after students have completed the mathematics work for the class, Nadia says that students who are engaged will linger in the classroom after the period is over. Nadia says that it is important to get to know each of her students on an individual basis and finds that her 'role' changes based on the individual student's needs. For some students, Nadia finds that they are looking to be challenged academically while others look to her for personal advice. This is not to say that each student has only one need but that the priority of needs is different from student to student.

Nadia says that she is still relatively new to the Grade 9 Applied Mathematics course and the instructional strategies shown to increase student engagement:

> Everything is new. I cannot do everything. I need my own time to do well. I need to be comfortable and to be comfortable, I need to be able to put in my own time. As a teacher, you need to be comfortable first. And it has to be your style.

In addition to Nadia's belief that social engagement is the key priority for student engagement for the Applied level student, Nadia's tentativeness with certain instructional strategies further emphasizes Nadia's focus in the social domain.

CONCLUSION AND SCHOLARLY SIGNIFICANCE

This study's findings suggest that all three Grade 9 Applied Mathematics teachers were cognizant of attributes of their early adolescent learners as they sought to increase student engagement. There were five major findings from the cases of Benjamin, Mathieu and Nadia. First, developing student self-confidence is critical to increasing student engagement for early adolescent learners. In addition to the insecurity in academic ability felt by at-risk students, the doubt that surfaces during this phase in their personal development can have negative effects (Archambault et al., 2009). Second, teachers use the *TIPS4RM* resource (OME, 2005) to increase student engagement. This research-based resource is comprised of lesson plans and summative tasks that integrate multiple components of mathematics curricula and a variety of learning experiences for students, both characteristics of teaching that support academic success (Sullivan et al., 2009). Third, and reinforcing existing literature (e.g., Gee, 2003; Weaver, 2000), this study found that technology is also an effective and relevant way to increase student engagement. Fourth, domains for student engagement and the factors found within these domains are not independent. Thus, the efforts of a teacher choosing to focus on one domain will have a broader impact than the teacher may realize (McDougall, 2004). Finally, teachers may prioritize one domain over another as a result of their personal comfort with that domain. This finding reminds us of the impact of teacher self-efficacy on practice (Bruce & Ross, 2008) and the natural tendency for teachers to individualize their practices (Siegel, 2005). Having implications for our teacher education practices, explicit opportunities for teachers to internalize new strategies should be provided. Traditional, one-day workshops are limited in the resources and experiences they provide for teachers to implement the presented ideas into their practice (Stein, Smith, & Silver, 1999). Professional development initiatives where teachers can engage in collaborative models such as co-teaching and peer coaching to follow up on new strategies learned may counter these challenges (Jao, 2013).

This study provides insight into how some teachers endeavor to increase student engagement for their at-risk students. Teachers can reflect on the commonalities and diversities among the classroom practices of the cases of Benjamin, Mathieu and Nadia. Researchers can use the findings from this study to further investigate factors that increase student engagement. As this study looked at the teacher perspective, follow-up work taking into account the student perspective and student performance will be important. While this study examines student engagement and learning within the context of Grade 9 Applied Mathematics, the insights which it illuminates can support student engagement and enhance student learning in other courses and inform educational research related to student engagement and learning across the curriculum.

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FROM FRUSTRATION TO UNDERSTANDING: AN INQUIRY INTO SECONDARY MATHEMATICS TEACHERS' EXPERIENCES WITH GOVERNMENT MANDATED EXAMINATIONS

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INTRODUCTION

This research focused on the lives of secondary mathematics teachers within a context of government-mandated student examinations (GMEs). This study was conducted in Alberta where students write provincially mandated exit examinations called diploma examinations, worth 50% of their final grade in several grade 12 courses, including mathematics. These examinations are considered high-stakes for students because they significantly impact final grades, graduation, and post-secondary admission (Agrey, 2004; Barksdale-Ladd & Thomas, 2000; Dager Wilson, 2007; Webber, Aitken, Lupart, & Scott, 2009). These examinations are considered moderate-stakes for teachers, as there is no formal consequence for teachers based on student results, yet results from the examinations are made publicly available and are often reported in the media (Abrams, Pedulla, & Madaus, 2003). Three secondary mathematics teachers in different contexts participated in this research. My dissertation explored several different aspects of the teachers' experiences while teaching a grade 12 mathematics course that included a GME.

PURPOSE OF THE STUDY

The purpose of the study was to broaden understanding of secondary mathematics teachers' teaching lives within a context of GMEs; specifically, how secondary mathematics teachers make sense of high-stakes student examinations. One way that teachers make sense of GMEs is illustrated by relationships that are formed with the examination and how they incorporate the examination into their teaching practices. Previous research has reported on teacher identity (see Rex & Nelson, 2004; Upadhyay, 2009; Walls, 2008), teacher emotion (see Brady, 2008; Steinberg, 2008), and teacher practice (see Abrams et al., 2003; Agrey, 2004; Yeager & von Hover, 2006) within a context of GMEs, however none of this work fully attends to the whole experience of the teacher within that context. This particular paper focuses on the relationship that teachers have developed with respect to the examination.

THEORETICAL FRAMEWORK

The methodology for this study is based in Gadamer's (1975/2004, 1976/2008) *philosophical hermeneutics*. Philosophical hermeneutics is concerned with developing understandings of texts and takes a broad definition of what texts might be. Essential to philosophical hermeneutics, the researcher exposes existing prejudices or pre-understandings regarding the topic of the study (Fleming, Gaidys, & Robb, 2003; Paterson & Higgs, 2005). As such, within this study, I have a particular orientation to and view of GMEs based on my own experiences with them as a secondary mathematics teacher and as a developer of such examinations. I have a trusting relationship with and deep knowledge of the secondary mathematics student examinations in Alberta that stems from my continued engagement with the examination.

From my interactions and conversations with other secondary mathematics teachers in Alberta, I became aware that not every teacher had the same, or even similar, understanding of the GMEs that I had developed. This research was based on my desire to understand how other secondary mathematics teachers make sense of these examinations and to broaden my perspective of thinking around GMEs. What I noticed through this work is that teachers' relationships with the examination influenced how those teachers talked about and incorporated aspects of the examination into their practice. This paper presents the meaningful understandings (Prasad, 2005) that I have reached regarding secondary mathematics teacher relationships with GMEs.

PARTICIPANTS

This paper presents data and understandings of three secondary mathematics teachers', Wanda, Naomi, and Taryn's, experiences with GMEs. Each of them had taught GME courses previously and would be teaching one during the time period of our conversations. At the time of this study, Wanda had been teaching for approximately eight years and was teaching mathematics at a mid-sized rural school. Students in the school are from an affluent community and the school has a tradition of academic excellence on the high-stakes mathematics examination. Naomi had approximately eight years of teaching experience all at the same school whose student population was drawn from many low socio-economic families and families that had recently immigrated. The student results on the mathematics GMEs in Naomi's school have been consistently below the provincial average. Taryn had 12 years of teaching experience in three different schools. She was currently teaching in an online learning environment whose students demonstrated an inconsistent performance on GMEs.

An important factor in selecting participants for this study was that the teachers had not participated in the marking of or the developing of any of the GMEs. I had the opportunity to engage with the examination in those ways and I wondered if, by becoming familiar with the development and marking process, my relationship with the examinations was particularly influenced.

METHOD

Conversation was used as the basis for interaction with the participants (Carson, 1986). Carson describes conversational research as a way to open up spaces for thoughtful reflection by all parties engaged in the research. The conversations were unstructured and free-flowing, allowing for the participant to explore thoughts and to make connections as they arose. Questions that were "open and reflexive" (Vandermause, 2008, p. 72) were asked during the conversations for clarification purposes and to further explore ideas that participants mentioned. Multiple conversations were had over time (Glanfield, 2003) to allow for the participants to reflect on earlier conversations and to develop an understanding of their experiences at different moments in the school year. I had three, one to two-hour conversations with Wanda between November 2011 and April 2012, four conversations with Naomi between January and June 2012, and one conversation with Taryn in January 2012.

Both the audio and transcription of the conversations were used as texts for data sources. I found that the feel of the conversation could not be adequately captured by the transcript, thus I would listen to our conversations to reconnect to the time and place of the conversation and the participant herself (Fleming et al., 2003). Thoughts, comments, and questions were written in a research journal after the conversation and were also used as texts for analysis. Once the conversation was transcribed and checked, transcripts were sent to the participants to check and edit as she wished. When the transcript was returned, passages were highlighted, comments were made on why that passage was highlighted, and early understandings that were being developed were recorded. Throughout the analysis phase, I engaged in a dialogue with the texts (Fleming et al., 2003) that served as a way to clarify my developing understandings and as my way of engaging in the hermeneutic circle (Ellis, 1998; Gadamer, 1975/2004, 1976/2008) in my search for "an interpretation as coherent, comprehensive, and comprehensible as possible" (Ellis, 1998, p. 27). It was through engaging in the hermeneutic circle that I began to understand that how each teacher talked about GMEs was reflective of the relationship she developed with it and the perception she had of it. Figure 1 represents the process that I engaged in to develop my understanding.

Figure 1

I did not necessarily begin or end my engagement with the hermeneutic circle in any one way. I entered and exited the circle at different points depending on my needs at that time. I continued with the process of listening, reading, writing, reflecting, until I reached an understanding of each teacher's relationship with mathematics GMEs.

Whitehead (2004) notes, "theoretical saturation is not sought in hermeneutic studies" (p. 514). What is sought are "field-based, thick descriptions" (Rex & Nelson, 2004, p. 1293) that inform our understanding of an experience or set of experiences. The data that illustrates the relationships with the high-stakes student examination are represented by a pastiche of each teacher's words. Rex and Nelson (2004) use *pastiche* as a way "to represent the teacher's professional position in his or her own voice" where segments of different transcripts are "linked together to produce coherent texts" (p. 1297) called *pastiches*. These rich descriptions are not meant to be comprehensive of the teacher's perspective but are purposefully created to maintain the teacher's voice and to illustrate the understanding that has been developed of the relationship that the teacher has with the examination.

RESULTS

Through engaging in the hermeneutic circle, I noticed that the language each teacher used to talk about GMEs was representative of how they perceived and how they made sense of teaching within their context alongside the examination. Below I present pastiches of each of Wanda, Naomi, and Taryn's words with a discussion of my understanding of what those words represent.

Wanda: I'm not afraid of the diploma in any way because I know our program is solid and I know my kids and I know where they are. I don't have a problem with testing the curriculum. I don't have a problem with how the questions are asked. The diploma exams don't bug me. I have no problem. I cover the curriculum; my kids come out above the provincial average. I have no problem as long as we're bigger than the provincial average. Big worry if I was less or something, like if the average comes out at 76 or something and I'm at 74, my admin will haul me in and I'll have to justify it somehow, but I've done it like ten times. I just felt validated that I wasn't way off right 'cause you get that examination booklet and you're thinking please don't let there be something on here that I forgot. Ever thought that maybe you forgot a unit (*chuckle*) and you're sitting there and you're like I had an awful lot of time to review this year. Did we not do all the perms and combs, what did I miss? But I was really happy that I was able to tell my students the types of questions that were going to be on the diploma right. I can kind of predict and I know what's going to be on there.

Wanda's comments illustrate a complex and contradictory relationship with GMEs. Wanda claims that she has 'no problem' with the examination, yet she states that she worries about being 'below the provincial average'. Because students in her school have traditionally performed well on the examination, Wanda worries that she will have to defend herself if her students score less than the provincial average. She feels "vulnerable against the judgment" (Kelchtermans, 1993, p. 453) of the school administration. She is confident in her knowledge of the examination as she feels she can 'predict' the kinds of questions that are going to be on the examination, yet she worries that 'maybe she forgot a unit'. Wanda, like the teachers in Jeffrey and Woods (1996), doubts her "competence and adequacy" (p. 329) when being faced with what is seen as an external measure of her teaching.

Naomi: I kind of feel like I don't know what's going to be on the diploma. This semester I've been trying to emphasize I don't know what kind of questions you're going to get. You don't know what kind of questions you're going to get. I don't like the anxiety that I feel about my results. Especially this year being the only teacher, our results are going to be me, that's it, and in that case we're really just analyzing how I did. I had a sense of hope this semester that maybe everyone will pass but it's a scary thought though. I don't want to be that hopeful because whenever I get that idea in my head, I'm usually disappointed, so I'm trying not to be but, but I felt like I worked really hard with them this semester. Because it's so frustrating to work so hard trying to get these students through and then they bomb the diploma. So it seems frustrating because I feel like I'm working really hard and I don't feel like the results reflect that effort and it seems that the only measure of success is how we do compared to the rest of the province. Eight years I've been teaching and eight years we have the same conversation.

Naomi expresses a troubled relationship with GMEs. Although Naomi pays close attention to the examination—she reviews it the day students write it and analyzes results when available—she does not believe that she can predict exactly how questions will be worded. For her students, she emphasizes strategies that they can use to approach any question because the questions on the GME are an unknown to her. Also, in Naomi's view, her administration has only one measure of success: improvement in examination results relative to the rest of the province. Any effort Naomi might make that does not lead to improved scores is not acknowledged. Abrams et al. (2003) state "high-stakes assessments increase stress and decrease morale among teachers" (p. 20). Naomi's morale is clearly decreased. "I don't have memories of them celebrating what we've achieved," she said; she has heard the same focus from her administration in improving results. Her work will not result in success so she wonders why she should even try.

> **Taryn:** I haven't loved the diploma. I chose not to teach 30 Pure. I wasn't interested in that pressure that seemed to trickle down from the diploma. The diploma dictates how Grade 12 will go and then the Grade 11 teachers look at that and say well we'd better make Grade 11 look like Grade 12 or they won't be ready for that etcetera, etcetera and I was working in opposition to that. I wanted learning to look differently and the test take care of itself and I felt like if I got into the twelve business that I would not be enjoying myself anymore. But when I started teaching at the online school, we had a disparity between our teacher mark and our diploma exam mark of forty percent. Up until then, I had been saying these diplomas, what use are they? I can't stand these diplomas. They're constraining my teaching and blah-blah-blah. They're making other teachers crazy and I was pissed but now I'm kind of a big proponent of the diploma (*chuckle*) because I understand that it was created for this purpose exactly: to get people in line.

Taryn has expressed a reformed relationship with the GME. She initially avoided teaching courses where students must write the GME, yet she now sees the value in having an externally mandated examination to ensure curriculum standards are being met. Nevertheless, Taryn actively resists letting her teaching be influenced by the examination. Unlike previous colleagues that Taryn felt allowed the examination to infiltrate their practice, Taryn wants 'learning to look differently'. In her current context, Taryn witnessed not attending to curriculum standards could impact student learning as evidenced by performance on the GME. She learned that accountability to the curriculum could be surfaced by attending to examination results. Taryn had a major revelation that contributed to her reformed relationship with the examination.

IMPLICATIONS AND FUTURE DIRECTIONS

From delving into teachers' experiences with teaching a course where the government mandates an external examination, I developed an appreciation of how complex and varied teachers' understandings of GMEs are. I recognize that context and experience are major factors in how teachers make sense of their daily work. There are implications that have arisen out of this research for examination developers, school administrators, teachers, and teacher educators. Examination developers and government officials need to be sure that they present a clear and consistent message about the purposes behind examinations and conclusions that can be supported by results so that teachers are not being evaluated based on student results on the GME. School administrators need to be aware of the potential messages that their teachers are interpreting from constant calls to improve or maintain results. These constant pressures have the potential to decrease motivation and morale of teachers when they feel like they cannot meet the demands that are being asked of them. Teachers can recognize that there are alternate ways of living and teaching within context of GMEs and to have open conversations with their colleagues about how they are making sense of the messages and demands that they are being given. Teacher educators have an opportunity to acknowledge the context that pre-service teachers will be working within and encourage conversation about how to mitigate effects of GMEs on their teaching and on their perception of themselves.

This study illustrates three teachers' relationships with GMEs. There are many teachers across the globe, living and working within a context of external examinations. Teachers have developed relationships with the examinations based on their experiences and are influenced by personal and contextual factors. These relationships are complex and require attention to how the relationships impact teacher practice and morale. Further study with respect to teacher relationships with GMEs and how those relationships are developed would play an important role in informing us as a community of educators working within a context of GMEs. Further research would serve to broaden our perspectives on how teachers relate to the examinations and can help us better understand how to best support teachers in their work and how to best prepare pre-service teachers to work within that context.

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TOWARDS THE PEDAGOGY OF RISK: TEACHING AND LEARNING RISK IN THE CONTEXT OF SECONDARY MATHEMATICS

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A qualitative case study was presented in order to explore an inquiry-based learning approach to teaching risk in two different grade 11 mathematics classes in an urban centre in Canada. The first class was in an all-boys independent school (23 boys) and the second class was in a publicly funded religious school (19 girls and 4 boys). The students were given an initial assessment in which they were asked about the safety of nuclear power plants and their knowledge of the Fukushima nuclear power plant accident. Following the initial assessment, the students participated in an activity with the purpose of determining the empirical probability of a nuclear power plant accident based on the authentic data found online. The second activity was then presented in order to determine the impact of a nuclear power plant accident and compare it to a coal power plant accident.

The findings provide evidence that the students possess intuitive knowledge that risk of an event should be assessed by both its likelihood and its impact. The study confirms the Levinson, Kent, Pratt, Kapadia, and Yogui (2012) pedagogic model of risk in which individuals' values and prior experiences together with representations and judgments of probability play a role in the estimation of risk. The study also expands on this model by suggesting that pedagogy of risk should include five components, namely: 1) knowledge, beliefs, and values, 2) judgment of impact, 3) judgment of probability, 4) representations, and 5) estimation of risk. These components do not necessarily appear in the instruction or students' decision making in a chronological order; furthermore, they influence each other. For example, judgments about impact (deciding not to consider accidents with low impact into calculations) may influence the judgments about probability.

INTRODUCTION

The thesis explores ways in which mathematics educators can foster secondary school students' understanding of risk. Specifically, I investigate students' interpretation, communication, and decision making based on data involving risk in the classroom setting. Informed by the investigation, I consider ways in which curriculum and pedagogy can assist in the development of the teaching and learning of risk. For the purpose of exploring risk in the classroom, I draw from definitions of risk from various disciplines as well as the frameworks of statistical and probability literacy, particularly those of Gal (2005). Further, students' learning is studied within the pedagogic model of risk (Levinson et al., 2012; Pratt et al., 2011). The thesis is framed as a qualitative case study of teaching risk in two grade 11 classrooms using an inquiry-based learning approach to pedagogy (Bybee et al., 2006).

Despite calls for teaching risk in the classroom, and despite the explorations by the TURS research group, there remains a lack of research in the mathematics classroom setting and involving students. The purpose of this study is to address the lack of research by exploring the ways in which risk could be taught within the mathematics classroom. Specifically, this study explores the ways that secondary school mathematics instruction can support students' developing understanding of risk, and I focus on the following guiding question: How do secondary school students reason and make decisions about risk?

LITERATURE REVIEW

Risk is a concept that is prevalent in many disciplines and the term *risk* has been used in many distinct yet connected ways. Hansson (2009) distinguishes between five different definitions of risk: 1) risk as an unwanted event which may or may not occur; 2) the cause of an unwanted event which may or may not occur; 3) the probability of an unwanted event which may or may not occur; 4) the fact that a decision is made under conditions of known probabilities; and 5) the statistical expectation value of unwanted events which may or may not occur.

The third, fourth, and fifth definitions are the most common in mathematics. The third definition aligns with the view that a risk associated with an event is a quantifiable uncertainty (Gigerenzer, 2002), which is equivalent to the likelihood or probability of the event. This definition of risk is suitable when the events have similar consequences, but it becomes problematic if the impact of each event is different. For example, the likelihood of a person catching a cold is relatively large but its impact on the person's life is most likely to be minimal, whereas the likelihood of getting killed in a terrorist attack is relatively small but the impact is immense. In order to account for both likelihood and impact, proper understanding of risk requires the coordination between judgments of probability and impact (Pratt et al., 2011), which corresponds to the fifth definition, the statistical expectation. This coordination can be done informally, but also formally using mathematical representations. Symbolically, the fifth definition of risk can be written as $R = \sum_{i=1}^{n} p_i d_i$ where the overall risk, *R*, of a hazard, is the sum of the products of the probability (*p*) and disutility or impact (*d*) of each event associated with the hazard (Pratt et al.,2011). For example, to assess the overall financial risk of owning a car, one would find the probability of each outcome (e.g., flat tire), multiply those by the financial impact, and then obtain the total sum of all the products. The approach based on the above formula is known as the utility theory of risk (Levinson et al., 2012) and is the standard approach in technical risk analysis (Moller, 2013).

CULTURAL PERSPECTIVES

Utility theory and technical risk analysis are not the only approaches to risk. Technical risk analysis, which is a domain of philosophy, statistics, and economics, has been extended to risk governance which involves actors' understanding and handling of risk (Lidskog & Sundqvist, 2013). However, risk governance is a complex task, particularly in the case of global risks such as terrorism, catastrophic weather due to climate change, financial meltdown, and nuclear accidents such as the radiation leakage due to the Fukushima nuclear disaster. The anticipation of global risks can seldom be determined using methods of science. The less we are able to calculate risk, the more the balance shifts toward the cultural perspectives on risk (Beck, 2009). It follows that assessing risk goes beyond the utility theory.

Assessing risk in the vast majority of social situations involves more than individual considerations of maximizing utility; rather, it is a dynamic consensus-making political process involving diverse actors and contexts (Douglas, 1992). Consistent with the cultural perspective on risk, "sociology opposes any kind of reification of risks, in which risks are lifted out of their social context and dealt with as something uninfluenced by the activities, technologies, and instruments that serve to map them" (Lidskog & Sundqvist, 2013, p. 77).

For the purpose of exploring the pedagogy of risk, researchers involved in the Institute of Education's TURS Project (promoting *T*eachers' *U*nderstanding of *R*isk in *S*ocio-scientific issues) developed a computer microworld called *Deborah's Dilemma* (Levinson et al., 2011; Levinson et al., 2012; Pratt et al., 2011). In *Deborah's Dilemma*, students were engaged in a narrative involving a fictitious person, Deborah, who suffers from a spinal cord condition. Based on the data about the side effects of a surgery and the consequences of not having the surgery, pairs of math and science teachers had to choose the best possible course of action for Deborah. One of the outcomes of the research program was the development of the pedagogic model of risk (Levinson et al., 2012).

According to this model, probabilistic judgments lead to the estimation of risk but the judgments are informed by values, experiences, personal and social commitments, as well as representations. This is in contrast with the utility model of risk, where values are separate from the probabilistic judgments and may only play a role in risk management (following an analysis of risk). Relevant findings from the study have been used throughout this literature review to outline the elements of the pedagogy of risk.

METHODS

I applied a qualitative case study approach as I explored the teaching of risk in two grade 11 classrooms that were using an inquiry-based learning approach to pedagogy. The methods chapter of my dissertation begins with a justification of my selection of a qualitative case study methodology for conducting research in the classroom, followed by my reasoning for the use of inquiry-based learning. I then outline the selection of the school, teachers, and participants, as well as the chronology of research, including the initial interviews with teachers, initial assessment of students, inquiry-based activities, final assessment, and final interviews with teachers. The chapter concludes with a detailed description of methods used for data collection and analysis, and also the ethical considerations relevant to my research. Due to space limitations, full details are not provided here.

The first research setting was Dale Academy, an all-boys private secondary school following the International Baccalaureate curriculum. Every student at the Dale Academy had access to many educational resources, including laptop computers and wireless internet. The reason why this school was chosen was to be able to see how risk pedagogy can be approached in settings in which there is no lack of resources. The grade 11 class was chosen because the International Baccalaureate probability and statistics unit was a good place for teaching and learning about risk. In order to have a greater diversity of participants, the second school was St. Hubertus Secondary School, a co-educational school with no direct access to laptops and no wireless internet access. I did most of the teaching in the study—the two teachers, Breanna and Clarissa, were there to help me plan the lessons, observe them, and assist me with the logistics and classroom management. Thus, the case study centres on the students and me, whereas the role of the classroom teacher was not explored in the study.

FINDINGS

REASONING ABOUT RISK

There is evidence that students possess pre-existing informal (intuitive) knowledge of impact. From the pedagogic view, this is very encouraging because, in many other domains (such as assessment of probability), there is strong evidence that individuals' intuitions are often erroneous (see, for example, Kahneman, Slovic, and Tversky, 1982). As was seen from the study, this informal knowledge has potential to be used in instruction. The initial assessment in both classrooms shows that students use different language to talk about impact (e.g., "*massive*" and "*dangerous*"). Other words used to express impact include: "*big*", "*astronomical*", and the students even used the phrase "*a barren landscape*" to visualize the impact of the Chernobyl accident. This corresponds to what Pratt et al. (2011) label as "fuzzy qualitative descriptors" (p. 339), which students in his study used for the purpose of a rough quantification of impact ("*serious*", "*massive*", "*bad*", "*fine*", "*big*").

This is also consistent with the Levinson et al. (2012) study in which the teachers used phrases such as "*impact on her life*", "*pain threshold*", and "*prohibitively dangerous option*" to describe impact. The authors state that there was no opportunity for teachers to quantify impact. They also suggest that a meaningful quantification of impact and probability can only be done in an inquiry-based approach where the students can apply their values, representations, and experiences. This is the reason why the students in my study were given the impact statistic, and why I operationalized impact in terms of accidents and fatalities.

Reasoning based on the magnitude of impact leads to reasoning about rational numbers—proportions, rates, and reciprocal values. The findings showed that the students were making use of equivalent fractions, rates, and percentages. Sometimes they were correct but sometimes the use of rational numbers was incorrect. For example, some students were incorrectly using percentages.

Some students did not recognize that statements were equivalent: 1.72 is the same as 31/18, which is the same as saying that the ratio is 1/18 to 1/31. However, it could be that the students thought that those statements, although mathematically equivalent, conveyed different contextual information. This stresses the differences between mathematical reasoning and quantitative reasoning in context (Mayes, Peterson, & Bonilla, 2012). The study as presented presents the case for quantitative reasoning in context.

Another quantitative concept that the students were having difficulties with was the concept of reciprocal values. For example, St. Hubertus students understood that 31 fatalities/accident was a greater risk than 18 fatalities/accident. However, they did not understand that 1/31 accidents/fatality was a greater risk than 1/18 accidents/fatality.

Mathematical instruction underplays the importance of units. In quantitative reasoning in context, however, the units are very important. My study shows that units were seldom used by Dale Academy students, while they were more often used by St. Hubertus students, which may be a consequence of them having been explicitly instructed to use units.

THE ROLE OF BELIEFS, FEELINGS, AND VALUES

The affective factor is very important in individuals' risk-based reasoning. Slovic (2010) talks about the *dread factor* that creates mental images about the hazards of interest (e.g., nuclear power plant accidents). We can infer the feeling of dread in some imagery expressed by the St. Hubertus students, for example, a student talking about the impact of nuclear power plants as "*a barren landscape*". Similarly, Gregory et al. (2012) have shown that beliefs and values have to be an integral part of risk assessment and that the choice of data and the presentation of data depend on values. This can be seen in my study when Christine's group encountered the table of fatalities and argued about whether it was valid to only consider the accidents that resulted in five immediate fatalities. One of the students had a stern belief that "*every death should count*", and the other one was more pragmatic. Finally, the student drew on her personal experience, saying that it would matter to her if she was the person or if she knew the person. The students did not draw as much on personal experience as did the students in the Pratt et al. (2011) study. The reason is that the question was framed in terms of logical statements: *Are nuclear power plants safe?* This can be compared to the decision statement: *Should we have nuclear power plants, or more specifically, should we build more power plants in a certain area?* Students did draw on their personal experiences, however. Particularly, one of the Dale Academy students was in the region (Hong Kong) when the Fukushima accident happened and he drew on this experience when making a decision about the safety of nuclear energy. In addition, another student at St. Hubertus stated that she would not like to live next to the nuclear power plant. However, because of how the question was construed, the students did not draw too much on personal experiences. Some students did show empathy (suggesting to "*pray for Japan*").

The pattern in both case studies was that the students did not seem to shift their beliefs about nuclear power plants. There were instances in which the exposition of quantitative data did cause students to question their beliefs. For example, some students were very surprised to find out that the fatalities for coal power plants were higher than those for nuclear plants. This is consistent with Kolsto's (2006) claim that students should be confronted with diverse information and viewpoints. However, students tended to include auxiliary information in order to 'salvage' their beliefs.

CONCLUSION

This research documents the complexity of the concept of risk and decision making based on risk. It also suggests how risk can be taught in the mathematics classroom. The study contributes to educational research by shedding light on the teaching and learning of risk in the mathematics classroom, whereas there is a lack of research in this area (Pratt et al., 2011). Another major contribution of the research is the identification of the understanding of rational numbers as being crucial to understanding risk.

An important lesson to take from my research is that decision making about risk is an interplay between quantitative reasoning, experiences, values, beliefs, and content knowledge. Restricting the instruction to any of these single components without meaningful consideration of other components will trivialize and reduce the effectiveness of the teaching. Risk is all around us and the pedagogy of risk should play an important role in mathematics education.

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TRANSITIONS BETWEEN REPRESENTATIONAL MODES IN CALCULUS

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This article argues for a shift in how researchers discuss and examine students' uses of representations during their calculus problem solving. An extension of Zazkis, Dubinsky, and Dautermann's (1996) Visualization/Analysis-framework to include contextual modes of reasoning is proposed. An example that details how transitions between visual, analytic and contextual reasoning inform students' problem solving in a calculus context is discussed.

BACKGROUND

The dichotomy between analytic and visual reasoning has been a theme in mathematics education literature since at least the 1970s (Krutetskii, 1976). Compatible distinctions have a long history in this literature, for example visualizer/non-visualizer (Presmeg, 1986), visual/analytic (Vinner, 1989), depictive/descriptive (Schnotz, 2002) and semantic/syntactic (Weber & Alcock, 2004). These distinctions continue to catalyze advances in education research.

The role of non-mathematical contexts, such as those borrowed from physics, are commonly either ignored or subsumed under the visual category within the above dichotomies. In this paper I argue that reasoning based in these non-mathematical contexts, which is referred to here as contextual reasoning, deserves to be treated as separate from visual and analytic reasoning. I propose a model that treats contextual reasoning in this way and illustrate its utility for analyzing student data in a calculus setting.

Motivation for making such a distinction can be found in the literature. For example, Roth and Bowen (2001) found that expert scientists, when working on graph interpretation tasks, interpret the graphs in these tasks as describing a familiar context from their research. This contextualization in terms of a familiar scenario occurs regardless of whether the graphs given are de-contextualized or situated in a less-familiar context. Subsuming contextualization behaviour under the visual category obfuscates differences between those working in purely mathematical domains and those who contextualize mathematical situations during their problem-solving processes. Additionally, subsuming contextual reasoning under the visual category relies on the assumption that contextualization co-occurs with visualization, which I show later is not always the case.

Researchers have explored understandings of the connection between visual and analytic modes through tasks that prompt students to transition between these modes (e.g., Knuth

2000; Haciomeroglu, Aspinwall, & Presmeg, 2010). Additionally, there is a growing literature on mathematization of real and imagined scenarios and interpretation of what graphs imply about the situations they describe (e.g., Gravemeijer & Doorman, 1999; Nemirovsky, Tierney, & Wright, 1998; Wawro, Rasmussen, Zandieh, Larson, & Sweeney, 2012). However, a model that treats contextualized reasoning as a separate entity from visual and analytic reasoning does not currently exist in the literature. The creation of such a model, which is the goal of this article, bridges research on contextualization/mathematization and research on visual/analytic reasoning. As can be seen in reviews of these respective areas of research (e.g., Presmeg, 2006; Roth & McGinn, 1998) these research areas currently remain quite disjoint. However, given that both of these areas focus on how one representation of a problem scenario informs another there is good reason to try to unify these two bodies of work.

THEORETICAL PERSPECTIVE

In this section I introduce the Visualization/Analysis-model (VA-model) and suggest an extension of it that includes contextual modes of reasoning. I argue that a more detailed look at transitions among different modes of reasoning is needed and I exemplify different types of transitions in calculus tasks.

THE VA-MODEL

My perspective extends the VA-model (Zazkis, Dubinsky, & Dautermann, 1996), which views the development of visual and analytic modes of reasoning as complementary rather than disjoint processes (see Figure 1). The modes, which may start as wholly separate entities, build on one another as reasoning develops. As students progress, their ability to translate between these modes becomes more common and the connections between the modes become stronger. In other words, the model contends that a back-and-forth relationship between the modes of reasoning does not develop overnight; it develops over time, and as it does, the transition between modes becomes progressively more natural for students to make. Figure 1 illustrates this process via a path through successive levels of visualization and analysis in which the 'distance' between visualization and analysis decreases as the levels advance.

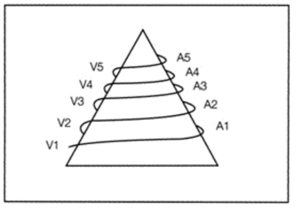

Figure 1. The VA-model diagram (from Zazkis et al. (1996)).

Although the VA-model was used in prior research, these studies tend to focus on classifying individual students as visual thinkers, analytic thinkers or harmonic thinkers (e.g., Haciomeroglu et al., 2010). I see this classification as inconsistent with the VA-model since the model contends that all students make transitions between modes during their mathematical development. In other words, the model contends, implicitly, that all students are harmonic thinkers.

The tendency to shy away from examining transitions between representation modes is also present in work that does not subscribe to the VA-model. For example, Zandieh's (2000) derivative framework classifies individual students in terms of whether or not they have expressed various modes of thinking. That is, rather than focusing on transitions between modes of thinking and how one mode informs another, students were classified in terms of the modes they expressed.

Focusing specifically on transitions between representations stays true to the VA-model. Additionally, it helps illuminate how harmonic thinking develops, even in students who rarely use certain modes of thinking.

THE EXPANDED VA-MODEL

In line with Zandieh's (2000) work and the general trend in calculus education research to include real or imagined contextualized scenarios, my model adds representations that are based on real or imagined contextualized scenarios to the VA-model. This is consistent with a growing body of work that emphasizes the importance of context in students' understanding of calculus, such as the relationship between acceleration, velocity and position (Nemirovsky et al., 1998). In order to reflect this change, the model will henceforth be referred to as the VAC-model. Mathematics is often motivated by connections to contextualized scenarios. So I see this addition to the model, which was originally not developed for calculus, as applicable to other areas of mathematics.

The VA-Model diagram shows levels that spiral up a triangle as reasoning advances. Visualization and analysis become closer to each other as one moves to more advanced levels. The VAC-Model diagram is a tetrahedron, to accommodate the addition of a contextual mode. The path between modes also spirals up with levels getting closer to each other, however, in the VA-diagram there is an orderly path that moves from visualization to analysis and back. In the VAC-diagram the path moves upward between three modes, but does so through a disorderly unpredictable path. This signifies that the transitions between visual, analytic and contextual modes do not follow a specified sequence.

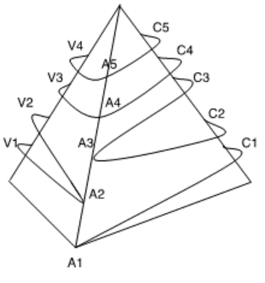

Figure 2. The VAC-model diagram.

Students' transitions between the three modes of thinking, in the context of problem solving, are of particular interest because they inform how students use multiple modes of representation in conjunction with each other. The VAC-model contends that these modes inform each other, but what this looks like, when it happens, or how such transitions can be fostered by instruction are not predicted by the model.

Instead of classifying individual students as predominantly preferring one mode of reasoning over another, I contend that the classification should be of students' claims and justifications and whether those are visual, analytic, or contextual in nature. In other words I regard all students as harmonic reasoners to some extent, regardless of which representational mode is predominant in their thinking. Within such a classification I place a special emphasis on back-and-forth transitions between modes of reasoning.

CATEGORIZING TRANSITIONS

In the secondary grades translation tasks are often solely about the translation itself. In a calculus context, however, translation between modes is often part of the task and not the task itself. Some calculus tasks are stated in one representational mode and require an answer in another. For example, consider the following task: "*If* $f(0)=1, f'(0)=1, f(3)=7, f'(3)=-1$ *and* $f''(3)=-1$, *sketch a possible graph of* $f(x)$." The task is stated in terms of the analytic mode, since the information about the function is given symbolically, and the answer is supposed to be provided in a graphical mode. Note that in order to complete the task a student is required to transition between representational modes. Solving the task requires moving from one edge of the VAC-diagram to another.

It can also be the case that translation between modes is not necessarily required in order to complete the task. However, spontaneous transitions between representational modes may occur anyway during students' problem solving. For example if a student is given the following integral to solve, $\int_{-3}^{3} x\sqrt{9-x^2}\,dx$, she may solve it using standard methods, such as *u*-substitution. The problem can, however, be solved by reasoning about the shape of the graph of $x\sqrt{9-x^2}$. The graph has a 180º rotational symmetry about the origin (odd function). Therefore every region above the *x*-axis has a corresponding region below the axis on the other side of the *y*-axis. Since the bounds of integration are symmetric with respect to the origin, the integral evaluates to zero. Even though the problem is stated in symbolic/analytic terms and requires a symbolic/numerical answer, the second solution makes extensive use of the graphical mode. If a student solves the task in this way, her thought process moves from one edge of the VAC-diagram to another, but this transition is not specifically required by the problem itself.

I refer to transitions between modes that are not required by the task, as *unprompted transitions*. Further, I refer to transitions that are part of the problem itself, that is, when a problem is stated in one mode and requires an answer stated in another, as *prompted transitions*. Note that prompted transitions are an attribute of a task and unprompted transitions are an attribute of a solution. So it is possible to have an unprompted transition occur within the context of a prompted transition problem.

METHOD

DATA AND PARTICIPANTS

This study followed a group of three average students as measured by their scores on the standardized *Calculus Concept Readiness* (CCR) test (Carlson, Madison, &West, 2010). The group consisted of two males and one female, who were given the pseudonyms Carson, Brad and Ann. These students were observed over the course of a semester-long technologically enriched calculus class taught at a large university in the southwestern United States. The class had approximately 70 students. The three students in this study worked together during in-class group work, which was recorded daily. Each of the three students also participated in three individual problem-solving interviews throughout the semester. The data in this article come from these interviews.

ANALYSIS

All interview tasks were coded for which representation modes were prompted. Student work on these tasks was also coded for representational mode with special attention paid to when transitions occurred and how these transitions informed students' problem solving. The metonymy of many mathematical terms necessitated the use of a neutral code. The code was applied when it was unclear which mode of reasoning was being used.

RESULTS

I begin by discussing the students' use of representations as a whole and how they relate to the representations in the task statements. Then I shift to discussing a particular student in detail.

Table 1 documents which transitions (if any) were explicitly required by each of the interview tasks and the transitions between representational modes students used in their solutions. Visual, analytic, and contextual are indicated with V, A, and C, respectively.

	Interview 1					Interview 2							Interview 3						
Task #	1	2	3[1]	4	5	1	2	3	4	5	6	7	1	2	3	4	5	6	7
Task Type	AC	AC	V	V	VC	AC	V	V	V	V	AC	VAC	AC	VC	AC	V	VAC	VC	VC
Ann	VAC	VAC	V	VA	VC	VAC	VC	VC	VAC	VAC	VAC	VAC	VAC	VC	VAC	VC	VAC	VC	VAC
Brad	VAC	VAC	V	V	VC	VAC	V	V	VA	VA	VAC	VAC	VAC	VC	VAC	V	VAC	VC	VC
Carson	VAC	VAC	VAC	VC	VC	VAC	VC	VC	VAC	VAC	VAC	VAC	VAC	VC	VAC	VC	VAC	VC	VC

Table 1. Prompted and unprompted transitions.

Note that in solutions to tasks that incorporate all three modes, unprompted transitions are not possible. In Table 1 two thirds of student solutions that could have shown an unprompted transition did. One important thing to notice from Table 1 is that most of the instances where students made an unprompted transition involved the addition of a visual or contextual mode. The instances that incorporated unprompted transitions to the visual mode typically involved drawing a graph to aid with reasoning. Unprompted transitions to the contextual mode typically involved reasoning about a graph as if it were describing the motion of some particle or vehicle, which was not part of the specified problem. A specific transition to the contextual mode is explored in more detail below. The addition of an unprompted analytic component was rare but did occur in the data. This took the form of reasoning that a graph (or part of a graph) appeared similar to a known analytic function and then finding a related function analytically before translating back to the graphical mode. This behaviour, which typifies analytic thinkers in Presmeg and her colleague's work, was fairly uncommon in this data set.

Table 1 documents what modes were used for particular questions but does not detail the specifics of how representational modes inform one another during the problem-solving process. The VAC-model contends that these transitions between modes are central to students' development. The next section details a particular set of transitions used to solve a graphing task.

[1] This is the task shown in Figure 3.

CARSON AND THE DERIVATIVE SKETCHING TASK

Although each of the three interviews involved several tasks, only one task will be discussed due to space limitations. The derivative sketching task (Figure 3) is a graphing task that asks students to sketch the graph of a derivative given a particular original function. The task is similar to a task discussed by Aspinwall and Shaw (2002) that asked students to sketch the derivative of a continuous symmetric 'saw-tooth' graph that alternated between a slope of negative one and one. The graph in Figure 3, unlike Apsinwall and Shaw's graph, alternates between several different slopes. These graphs have no simple translation into analytic notation. They therefore discourage an analytic approach. Aspinwall and Shaw observed that students they classified as analytic thinkers had difficulty with the saw-tooth task. As presented, the task below does not force any transitions between modes of representation since it can be solved using only visual reasoning.

Sketch the graph of the derivative of the following function. Think aloud as you sketch your graph.

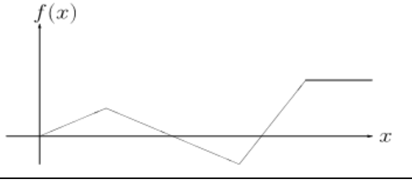

Figure 3. Derivative sketching task.

Below is the transcript of Carson working on the derivative sketching task (Figure 3). This task is stated in graphic terms and requires a graphic solution. The transcript details a solution that does not stay solely within the confines of graphical thinking.

Carson: Alright, so I know that the derivative is the slope and I took physics so I know that this is distance [writes a *d* under *x*-axis] over, no that's wrong this is time [crosses out *d* and writes *t* under the *x*-axis]. This is time over distance, which is your speed. Speed is distance over time. So this is time and this is your speed [labels axis on derivative function] and so as your distance…I'm sorry… So this is constant so you know that velocity is constant. So your velocity is something like this [draws short horizontal line segment above *x*-axis] and then later when it hits this tip it's at zero [marks a dot on the *x*-axis after previously drawn segment]. And then later when it's decelerating. Ya this is a negative speed so the graph. And it's a straight line so you know it would be something like this [draws a horizontal line under the *x*-axis]. And then again at this point it's zero [draws another dot on the *x*-axis after the second segment]. And then um accelerating. But this time it's more this is steeper. So it would be higher because your velocity would be faster. Your speed would be faster. So it would be like this [draws a third horizontal line segment above the *x*-axis higher than the first]. And then again right here it's zero [draws another dot on the *x*-axis]. And then now this one your distance isn't changing. Since your distance isn't changing. This equation [$s=d/t$] looks like zero over time. So the rest of the graph would look like this [draws a fourth line along the *x*-axis.]

In the above transcript Carson is presented with a question that makes no mention of a physical context, however, Carson attributes the function in the question to a function that describes a moving object. This transition to contextual thinking is unprompted by the question. Carson does not simply shift into a contextual mode and remain there. He continually moves back and forth between visual and contextual modes. More specifically, he interprets a section of the given graph as corresponding to a physical motion, reasons about the velocity/speed of that motion and then translates that into a velocity graph. This cycle occurs several times throughout the transcript.

When dealing with the last segment of the function Carson switches to an analytic mode. He reasons that a non-changing position corresponds to $0/t = 0$, no change in distance over a non-zero change in time, before sketching the last segment of the derivative. So the above transcript shows unprompted transitions to both contextual and analytic modes within a graphical problem. Relating this back to the VAC-diagram, Carson's reasoning continually alternates between the contextual and analytic edges of the tetrahedron before moving to its analytic edge. Carson's solution is shown in Figure 4.

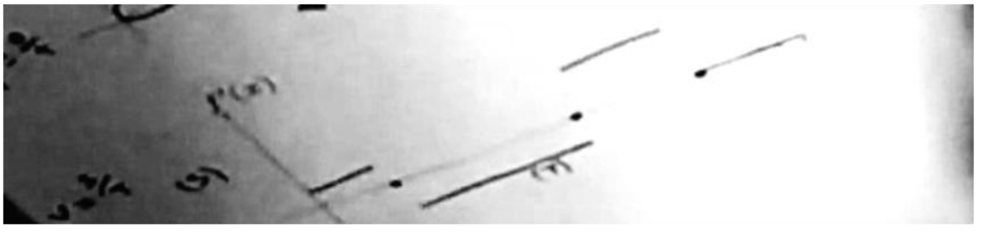

Figure 4. Carson's solution to the derivative sketching task.

The connections that Carson makes between physical and graphical contexts led to some interesting artifacts. The given graph cannot represent the position graph of a physical object. Physical objects cannot instantaneously change directions and so it does not make sense to discuss the contextual interpretation of what happens at those points. Carson, however, does not abandon the contextual-graphical link. This leads him to conclude erroneously that there are zeros at the points where the graph instantaneously switches direction. This phenomenon is similar to one noted in Aspinwall, Shaw, and Presmeg (1997), which they termed *uncontrollable mental imagery*. This is where visual images associated with students' graphical interpretations interfere with their analytic interpretations.

Further questioning revealed that Carson's translation into the contextual mode only appears to affect his ability to deal with sudden transitions from increasing to decreasing or vice-versa. Consequently, his errors are limited to several discrete points. So, his use of the connection between graphical and contextual modes appears to help him more than it hinders.

DISCUSSION

In my view, calculus curricula that make non-trivial attempts to incorporate graphical and contextual modes carry with them the implicit goal of fostering representational fluency. In other words, the goal of incorporating visual/graphical and contextual elements into a calculus course is not to expose students to separate modes of thinking, each of which targets a specific class of problems. Rather, the goal is to expose students to ways of approaching problems that can complement and elaborate each other. Simply exposing students to multiple representations does not ensure that they can translate between them. In order to better understand how to foster a rich back and forth relationship between modes of reasoning, researchers need to understand what such transitions look like, how they evolve over time and what kinds of tasks and teaching actions help foster them. This paper is a contribution to the first of these goals, illuminating what these transitions look like, both when they are prompted and unprompted. These transitions, at least in this particular data set are not uncommon. They occurred in two thirds of student solutions that could have shown an unprompted transition.

If I were to stick solely to classifying general tendencies, Carson would be labeled a contextual thinker because he used contextual reasoning as part of his solution to every task. However, this labeling would have completely overlooked the rich unprompted transitions between modes of thinking that were integral to his problem-solving processes.

These transitions between modes, in which one mode of reasoning informs another, are central to how the VAC-model views the development of analytic, visual and contextual modes. More importantly, they shed light on what these transitions look like and may be used as a launching point for developing curricula and instruction that strengthens students' ability to make such transitions.

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Ad Hoc Sessions

Séances ad hoc

A WAR ZONE: THE FRAMING OF MATHEMATICS EDUCATION IN PUBLIC NEWS REPORTING

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INTRODUCTION AND CONTEXT

Mathematics education is often in the news. Recent months have seen news reports relating to the publication of the 2013 PISA results. In this paper, we present an initial investigation designed to better understand the reporting of mathematics education in Canadian newspapers. Issues related to mathematics education often make news headlines, mostly highlighting negative aspects of the issue (Camara & Shaw, 2012). This kind of negativity has been noted specifically in relation to the publication of PISA results. To analyse how Canadian mathematics education is portrayed in the news media, and in relations to the PISA results, we draw on two concepts of *framing* and *interpretive repertoires*.

THEORETICAL FRAMEWORK

Our research is broadly framed by a critical discourse perspective (e.g., Edwards & Potter, 1992). From this perspective, the language of news reporting constructs particular versions of the world (e.g., of mathematics education). These versions of the world are seen as reflecting particular interests, as designed for particular audiences and as constructing particular realities. To better analyse the news construction of mathematics education, we used the media concept of *framing*. The term *framing* refers to "modes of presentation that journalists and other communicators use to present information in a way that resonates with existing underlying schemas among their audience" (Scheufele & Tewksbury, 2007, p. 12). Framing is based on the assumption that how an issue is characterized in news reports can have an influence on how it is understood by audiences. By examining framing of news reports, we can develop an understanding of how mathematics education is portrayed in news media.

METHODS

We selected three Canadian national news media (i.e., *The Globe and Mail*, *The National Post* and *Macleans*) to collect articles in a six-month period (September 2013 – March 2014) on mathematics education. In our *ad hoc* session, we shared our analysis of one report entitled "*Math wars: The division over how to improve test scores*" (Blaze Carlson, 2014), selected because it discusses mathematics education across the country.

FINDINGS AND DISCUSSIONS

In our selected report, mathematics education in Canada is framed as a war zone with two sides confronting each other; 'back to basics' and 'discovery learning'. The war framing is combined with a narrative of long-term decline in PISA ranking to portray a 'national emergency', encompassing 'a vast swath of the country'. We identified two groups of participants, '*actors*' (e.g., governments, parents) and '*acted-ons*' (e.g., curriculum, students). Actors take action and sides, and fight the battles. For example, parents launch petitions to win back territory lost to the forces of discovery learning. We argue that the framing of mathematics education as a 'fractured battle ground' forms a simplistic view of mathematics teaching and learning. We plan to systematically examine the whole data set, to give a better picture of how mathematics education is framed in Canadian public newspapers.

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RETHINKING LESSON PLANNING IN MATHEMATICS CLASSROOMS

Martha J. Koch
University of Manitoba

Studies indicate that mathematics achievement can be enhanced when teachers use information about student thinking to guide their instructional moves during a lesson. Formative assessment, including careful observation of student work, asking questions to reveal student thinking and listening to their responses is a key part of effective math teaching (Wiliam, 2007). In this *ad hoc* session, participants discussed how these assessment principles relate to lesson planning in K to 12 classrooms. Many lesson-planning strategies such as those associated with backward design and the widely adopted 'three-part lesson plan' produce a detailed sequence of instructional activities. With such a sequence in hand, teachers may be more inclined to adhere to their plan than to respond to what takes place during the lesson. In this way, lesson planning may become a barrier to the use of formative assessment. At the same time, carefully planning the sequence of activities in a lesson can significantly enhance math learning, especially for pre-service teachers who have a more limited repertoire of problems and examples to draw on as a lesson unfolds. I wonder if more flexible forms of lesson planning that enable math teachers to change course during a lesson in response to student thinking may be a response to this conundrum.

EMERGING INSIGHTS

The individuals who attended the session included K-12 teachers, graduate students, teacher educators, mathematicians and researchers. The tone of the conversation was set from the outset as one participant stated, "*lesson planning is definitely important because it structures the whole way you think about learning*". The conversation soon turned to how research might be done on this topic. I suggested gathering data on: how lesson planning is taught in mathematics teacher education courses; approaches to lesson planning recommended by ministries of education and in math methods textbooks; and interviewing experienced teachers to understand how they approach this conundrum. With respect to pre-service teachers, one participant noted that what is emphasized by faculty advisors who evaluate candidates during practica could be a valuable data source. Some faculty advisors or pre-service teachers may place too much value on not deviating from the plan. For experienced teachers, a participant noted that gathering artifacts such as daybooks might reveal limited information because these teachers "*keep their lesson plans in their minds*". Gathering lesson plans prepared for substitute teachers could help to address this. Another participant suggested insights might come from the book, *Finnish Lessons* (Sahlberg, 2011), particularly with respect to how lessons emerge from the joint actions of teachers and students. The complexity of the issue and need for further research were reiterated.

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UNPACKING STUDENTS' MEANING DURING LEARNING OF NEW CONCEPTS AND APPLICATION TO TASK

Lydia Oladosu
University of Calgary

During this session I addressed the issue of unpacking students' pre-formal instruction meanings based on findings from a case study that indicated many students' pre-formal meanings did not reflect the mathematical meanings of the concepts or support problem solving that involves contextual situations. Prior knowledge can actually interfere with learning new concepts (NRC, 2000; Resnick, 1983) because they continue to attach their naïve meanings to technical terms in the new learning.

From a constructivist perspective (von Glasersfeld, 1996), meaning of concepts involves students' sense making of the concepts based on their own way of relating to them. From this perspective, meaning is unique to each individual, believed to be contextually constructed, built on prior understanding, and influenced by learning experiences. Using circle geometry taught in grade 9 as an example, students' pre-formal instructional meanings were influenced mainly by their real-world experiences and prior learning of school geometry. These meanings thus reflect visual representation, taking measurements, using definitions, and associating a concept to real-world objects. Examples of such meanings are: Circle:"*is like a basketball*", "*is like a pie chart*", "*You use circles in gym for games*"; Chord: "*that would remind me of band class*", "*I don't play a string instrument, it's like a guitar*"; Arc: "*arc is like a dome shape*".

These meanings were understood to impact students' learning of new concepts both positively and negatively. Those that have positive impact are those meanings that draw connections and establish relationships between the properties of the concepts. While those that have negative impact focus on the physical real-world object, which sometimes includes non-representatives of the concept. Since these meanings provide a basis for learning of new concepts (Shuell, 1990), it is important to consider and unpack those initial meanings when introducing new concepts.

In learning new concepts, formal instruction that includes unpacking non-productive meanings can reshape and deepen understanding of many other fundamental concepts upon which formal meanings build. Unpacking students' meaning is needed to establish the relationship between the properties of the concepts, draw connections between prior understandings and new learning, and help build justifications for a solution to a task when solving problems related to the new concept.

One approach to unpacking meaning held during formal instruction is to analyze students' supplied pre-formal meanings. This approach involves: class discussion, guided questioning, making necessary connections to and re-teaching previously learned concepts, supporting reasoning that uses relationships between properties and suggests productive habits of mind, analyzing assumptions and generalizations made about the concepts, watching communication and the use of mathematical terms, interpreting visual and algebraic representations, justifying the problem-solving process (approach) used during class examples and addressing incomplete meanings.

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MATHEMATICAL ARTS: CHANGE THE NAME… CHANGE THE LENS (…?) … CHANGE THE EXPERIENCE… (?)

Jamie Pyper
Queen's University

MOTIVATIONS

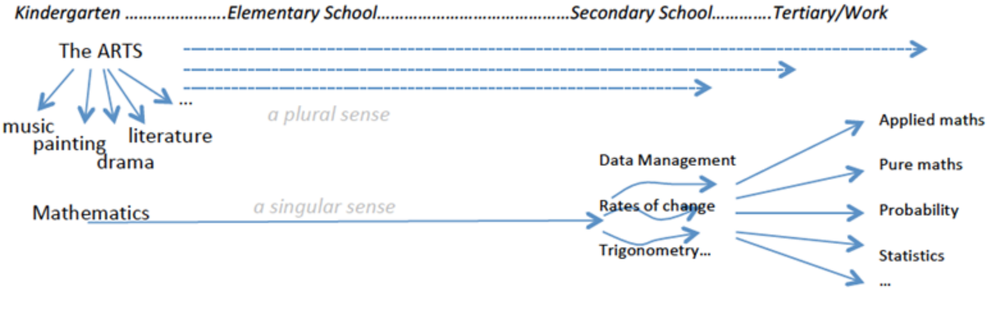
- UNESCO International Network for Research in Arts Education, roundtable talk, March 31, 2014, Queen's University (Larry O'Farrell, Professor Emeritus and holder of the UNESCO Chair in Arts and Learning, Faculty of Education, Queen's University) for the 'aha' moment
- Mathematics Knowledge for Teaching (e.g., Ball, Hoover Thames, & Phelps, 2008) for content perspective
- Mathematics for Teaching (e.g., Davis & Simmt, 2006) for content in context perspective
- Teacher efficacy, teacher concerns, teacher orientation (Pyper, 2012) for teachers' sense in classroom practice
- The Math Wars (in Canada) for the social/societal 'crisis'
- Curriculum conceptions – e.g., aesthetic, (Eisner, 2004; Pinar & Bowers, 1992)
- Pedagogy of care (Nodding), pedagogical thoughtfulness (van Manen)

SELECTED COMMENTS REMEMBERED FROM ROUNDTABLE CONVERSATION AND THOUGHTS…

…the arts… it is how I make sense of the world, and the world around me…
…mathematics is how I make sense of the world and the world around me…
…Arts without walls …rebuked for noise in my classroom
… current curricula impart a sense of control – of the Arts, of the artistic,…
…being comfortable within chaos (with noise), knowing that harmony (not order) will emerge

…when social/political/economic/technological/etc. crises appear, we look towards creativity to find a resolution/solution/approach… why is 'the arts' not considered an integral component of such creativity and/or the development of the abilities and expressions of such creativity?

A SENSE OF THE ARTS, AND MATHEMATICS



WHAT IF…?

We re-label (re-conceptualize) 'mathematics' as 'mathematical arts'—considering the thoughtfulness and appreciation and common understanding of arts-creativity-enjoyment-risk taking-etc…. What might some outcomes be?

i. for teachers who come into elementary classrooms with non-math/science degrees?
ii. for parents who are trying to understand?
iii. for curriculum designers?
iv. for implementation of 'reform-based' mathematics—manipulatives, technology, critical thinking, diverse thinking,…?

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DAY-TO-DAY, MOMENT-TO-MOMENT TEACHING: WHAT DOES IT MEAN TO TEACH MATHEMATICS?

Amanjot Toor
Brock University

> *In all of our interaction with children, we are constantly involved, whether we like it or not, in distinguishing between what is good and what is not good for them (in contrast, educational research is usually more interested in distinguishing between what is effective and what is in-effective). Yet even (or especially) the best educators temper their practice with the knowledge that we all often fall short and do not know what is best.* (van Manen, 1993, p. xii)

As a teacher, I try to make mathematics more humanistic by telling stories to my students that relate to the mathematical topics at hand. For example, once for Halloween, I decided to talk about the invention of the circle and the impact of such an invention on the world. Therefore, I dressed up like Cleopatra to bring the historical perspective, and began to talk about how the circle might have developed. First, I talked about how one day someone might have realized that there is a thing called *line*, which may have led to someone exploring the idea of two lines intersecting and developing a corner. Next, I talked about the idea of four lines where two lines at a time intersect, which might have led to the development of four-sided figures, like rectangles and squares. Further, I said that there might have been one day when someone might have asked, "*Can a line intersect itself?*", and in the process of examining that they might have connected the two ends of the line, which might have made a circle. Further, while expressing my personal curiosity, I said that this invention of a circle might have contributed to the invention of wheels. The moment I made the connection between circles and wheels, a lively class discussion ensued where one student said, "*It may also be the reason why Christopher Columbus discovered the world because he might have realized that there are other shapes than four sided figures*." Then another student made a reference to spirituality and said, "*It could also be when people realized the karma …you know what goes around is what comes around*." As a grade 7 mathematics teacher, what would you do with these kinds of conversations, other than saying that they are good? What does this have to do with teaching mathematics? Even though my intention was to bring a humanistic aspect to mathematics, I knew then that what had just happened was bigger than my intention.

Reflecting on this and other similar experiences teaching mathematics, embedded in them is the question, "*What does it mean to know how to teach mathematics on day-to-day and moment-to-moment bases?"* Even though researchers seem to agree on the importance of teachers' knowledge of mathematics for students' learning, there is little agreement over what teachers actually need to know to teach mathematics. Further, there is little research that focuses on the particularity and uniqueness of the day-to-day, moment-to-moment nature of the work of teaching mathematics. The purpose of this *ad hoc* was to discuss and explore what day-to-day and moment-to-moment teaching in mathematics classes entails and what informs teachers' decisions in class.

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Mathematics Gallery

Gallérie Mathématique

STUDENTS' IMAGES OF MATHEMATICS

Jennifer Hall, *University of Calgary*
Jo Towers, *University of Calgary*
Lyndon C. Martin, *York University*

This poster shared initial findings from a large-scale research project that is investigating students' lived experiences learning mathematics in Canadian schools and the roles that schools and teachers play in shaping students' mathematical experiences. The project involves the collection of mathematical autobiographies in a variety of formats (e.g., interviews, drawings) from Kindergarten to Grade 12 students, post-secondary students, and members of the general public. Data collection is taking place in Alberta and Ontario, and is ongoing.

For this poster presentation, we reported on findings from interviews with 94 Kindergarten to Grade 9 students that took place in the Spring of 2014 in two Alberta public schools. Our analysis focused on one interview question: "*When you hear the word mathematics, what images come to your mind?*" The participants' responses were analyzed through emergent coding, and counts were calculated for each code. Categories with more than one response were inserted into a *Wordle*, a visual representation of response frequencies (see Figure 1).

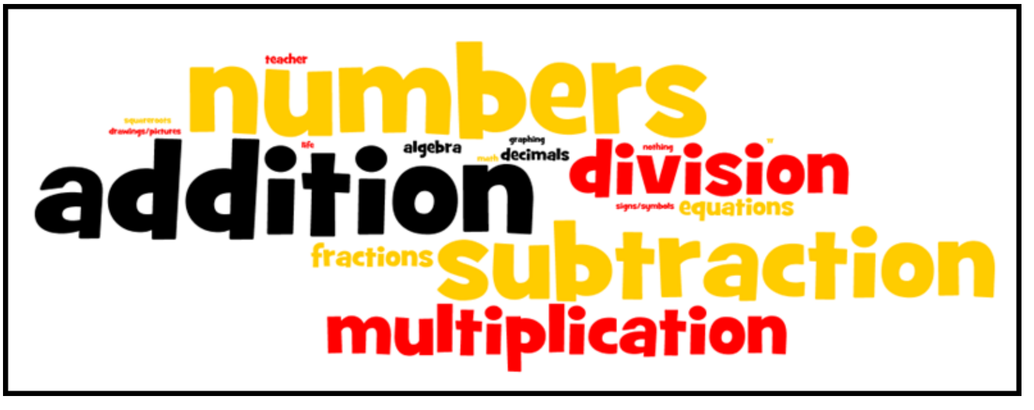

Figure 1. *Wordle* representation of students' images of mathematics.

As demonstrated by the *Wordle* above, the students' images of mathematics were narrowly focused on number sense and numeration—specifically, numbers and the four basic operations. These five responses were far more commonly cited (often tenfold) than the other responses. The narrow focus was surprising in a number of ways. In Alberta, where these data were collected, the mathematics curriculum covers four strands: (1) Number, (2) Patterns and Relations, (3) Shape and Space, and (4) Statistics and Probability. Yet, nearly all of the students' responses aligned with the first strand, Number. This trend could be indicative of the emphasis placed on this topic by teachers. We were also surprised to find that students' images of mathematics did not broaden with age. Presumably, students are exposed to more mathematics topics over time so it is troubling to continue to see narrow views through to Grade 9. Our continued research is exploring other aspects of the students' autobiographies in order to shed light on these findings.

UNFOLDING OF DIAGRAMMING AND GESTURING BETWEEN MATHEMATICS GRADUATE STUDENT AND SUPERVISOR DURING RESEARCH MEETINGS

Petra Menz
Simon Fraser University

In the fall of 2011 I began my PhD journey jointly between the Department of Mathematics and the Faculty of Math Education at SFU. Through various readings (e.g., McNeill, 1992, 2008; Radford, 2008, 2009) and reflections on my 20-odd years of teaching, I became intrigued with the study of gestures and diagrams, and how they link to mathematical thinking and creation.

The more recent work in this area (e.g., Bailly & Longo, 2011; de Freitas & Sinclair, 2012; Sinclair, de Freitas, & Ferrara, 2013) resonates with my own teaching and learning experiences as a mathematics teacher. I have come to believe that mathematical invention and intuition emerge long before a symbolic representation of the mathematics is encountered. I find it intriguing that a diagram should hold more meaning than the few lines that are needed to draw it, and that the very act of drawing the diagram and engaging with it can possibly provide keys about one's understanding of the mathematics that led to the diagram.

My research is therefore based on the ideas of the philosopher Gilles Châtelet, which are explained in his work *Figuring Space – Philosophy, Mathematics, and Physics* (1993/2000). Châtelet lays the foundation of the concept of *virtuality* as something that pushes the material aspects of mathematics and where the diagram is the connection between the virtual and actual. Châtelet interprets gesture as even more than a visible, non-verbal, bodily action that carries meaning; indeed, a gesture is the articulation between the virtual and the actual and as such is immediate and embodied. While Châtelet's understanding of gestures and diagrams is at odds with some of his contemporaries (Lakoff & Núñez, 2000; Nemirovsky & Ferrara, 2009), their gesture studies, like Châtelet's, support the view that an embodied approach is needed to understand abstract thinking because gestures play a significant role in mathematical thinking and learning.

Since Châtelet based his analysis on manuscripts left behind by famous mathematicians without access to live observations or interviews of these mathematicians about their work, I became fascinated by whether one can see mathematical creation through diagramming in live mathematicians. Therefore, as my title indicates, I decided to study the unfolding of diagramming and gesturing between mathematics graduate students and their supervisors during research meetings. I have collected data from the research meetings of two supervisor-graduate student pairs and am in the process of analyzing the data. My poster (Figure 1) depicts images where the participants of my study engage with a diagram. The titles for the images suggest ways in which these engagements can be organized to shed light on who attends to what.

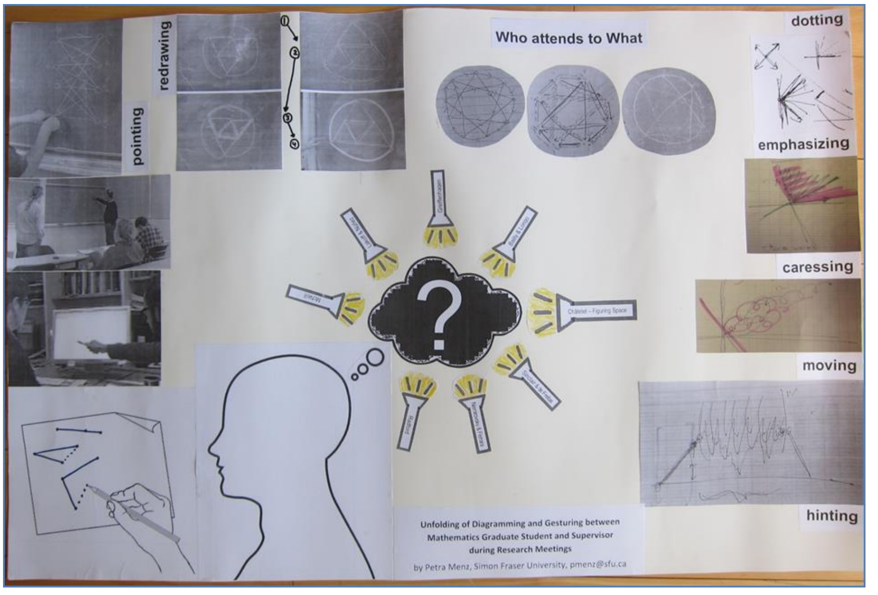

Figure 1. Unfolding of diagramming and gesturing between mathematics graduate student and supervisor during research meetings.

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EXPLORING MATHEMATICS THROUGH NARRATIVE/STORIES: A HUMANISTIC APPROACH FOR TEACHING MATHEMATICS

Amanjot Toor
Brock University

Humanistic mathematics involves interdisciplinary connections between mathematics and other worlds of thought and methods of learning (Tennant, 2014). Humanizing educational content involves using the principles of humanistic recognitions while taking into consideration the interests and abilities of its learners (Cernajeva, 2012). Literature supporting humanistic perspective reveals that, "narrative is a way of specifying experience, a mode of thought, a way of making sense of human actions or a way of knowing" (Chapman, 2008, p. 16). The type of stories a society narrates is a mirror of what information is considered important, which contributes to the beliefs and the values to which its members adhere (Schiro, 2004). Evidentially, narratives in mathematics classrooms, unfolding the importance of failure as an essential step to be successful in mathematics, may influence one's perception of mathematics as a humanistic subject.

Both fictional and non-fictional stories can emerge in various places in the mathematics classroom. Stories may provide the background for a mathematical activity, they may provide explanation, and they may be presented in a way that poses a question. Narrative creates an environment of imagination, emotion, and thinking, which makes mathematics more enjoyable and more memorable. Additionally, it creates a comfortable and supportive atmosphere in the classroom, and builds a bond between an educator and learners. Furthermore, narrative sparks one's interest in mathematics, assists in memory, and reduces one's anxiety related to mathematics. Stories may convey passion and enthusiasm, which may result in engaging students by creating excitement, mystery or suspense, and may motivate students to think about a particular problem. Stories in which students are able to identify themselves may also make the lesson more relevant and more vivid for students. Stories that involve specific examples may give students a sense of belonging where one may relax as it provides them with something to hold to when moving to general theory.

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Appendices

Annexes

Appendix A / Annexe A

WORKING GROUPS AT EACH ANNUAL MEETING / WORKING GROUPS DES RENCONTRES ANNUELLES

1977 *Queen's University, Kingston, Ontario*

- Teacher education programmes
- Undergraduate mathematics programmes and prospective teachers
- Research and mathematics education
- Learning and teaching mathematics

1978 *Queen's University, Kingston, Ontario*

- Mathematics courses for prospective elementary teachers
- Mathematization
- Research in mathematics education

1979 *Queen's University, Kingston, Ontario*

- Ratio and proportion: a study of a mathematical concept
- Minicalculators in the mathematics classroom
- Is there a mathematical method?
- Topics suitable for mathematics courses for elementary teachers

1980 *Université Laval, Québec, Québec*

- The teaching of calculus and analysis
- Applications of mathematics for high school students
- Geometry in the elementary and junior high school curriculum
- The diagnosis and remediation of common mathematical errors

1981 *University of Alberta, Edmonton, Alberta*

- Research and the classroom
- Computer education for teachers
- Issues in the teaching of calculus
- Revitalising mathematics in teacher education courses

1982 *Queen's University, Kingston, Ontario*

- The influence of computer science on undergraduate mathematics education
- Applications of research in mathematics education to teacher training programmes
- Problem solving in the curriculum

1983 *University of British Columbia, Vancouver, British Columbia*

- Developing statistical thinking
- Training in diagnosis and remediation of teachers
- Mathematics and language
- The influence of computer science on the mathematics curriculum

1984 *University of Waterloo, Waterloo, Ontario*

- Logo and the mathematics curriculum
- The impact of research and technology on school algebra
- Epistemology and mathematics
- Visual thinking in mathematics

1985 *Université Laval, Québec, Québec*

- Lessons from research about students' errors
- Logo activities for the high school
- Impact of symbolic manipulation software on the teaching of calculus

1986 *Memorial University of Newfoundland, St. John's, Newfoundland*

- The role of feelings in mathematics
- The problem of rigour in mathematics teaching
- Microcomputers in teacher education
- The role of microcomputers in developing statistical thinking

1987 *Queen's University, Kingston, Ontario*

- Methods courses for secondary teacher education
- The problem of formal reasoning in undergraduate programmes
- Small group work in the mathematics classroom

1988 *University of Manitoba, Winnipeg, Manitoba*

- Teacher education: what could it be?
- Natural learning and mathematics
- Using software for geometrical investigations
- A study of the remedial teaching of mathematics

1989 *Brock University, St. Catharines, Ontario*

- Using computers to investigate work with teachers
- Computers in the undergraduate mathematics curriculum
- Natural language and mathematical language
- Research strategies for pupils' conceptions in mathematics

1990 *Simon Fraser University, Vancouver, British Columbia*

- Reading and writing in the mathematics classroom
- The NCTM "Standards" and Canadian reality
- Explanatory models of children's mathematics
- Chaos and fractal geometry for high school students

1991 *University of New Brunswick, Fredericton, New Brunswick*

- Fractal geometry in the curriculum
- Socio-cultural aspects of mathematics
- Technology and understanding mathematics
- Constructivism: implications for teacher education in mathematics

1992 *ICME–7, Université Laval, Québec, Québec*

1993 *York University, Toronto, Ontario*

- Research in undergraduate teaching and learning of mathematics
- New ideas in assessment
- Computers in the classroom: mathematical and social implications
- Gender and mathematics
- Training pre-service teachers for creating mathematical communities in the classroom

1994 *University of Regina, Regina, Saskatchewan*

- Theories of mathematics education
- Pre-service mathematics teachers as purposeful learners: issues of enculturation
- Popularizing mathematics

1995 *University of Western Ontario, London, Ontario*

- Autonomy and authority in the design and conduct of learning activity
- Expanding the conversation: trying to talk about what our theories don't talk about
- Factors affecting the transition from high school to university mathematics
- Geometric proofs and knowledge without axioms

1996 *Mount Saint Vincent University, Halifax, Nova Scotia*

- Teacher education: challenges, opportunities and innovations
- Formation à l'enseignement des mathématiques au secondaire: nouvelles perspectives et défis
- What is dynamic algebra?
- The role of proof in post-secondary education

1997 *Lakehead University, Thunder Bay, Ontario*

- Awareness and expression of generality in teaching mathematics
- Communicating mathematics
- The crisis in school mathematics content

1998 *University of British Columbia, Vancouver, British Columbia*

- Assessing mathematical thinking
- From theory to observational data (and back again)
- Bringing Ethnomathematics into the classroom in a meaningful way
- Mathematical software for the undergraduate curriculum

1999 *Brock University, St. Catharines, Ontario*

- Information technology and mathematics education: What's out there and how can we use it?
- Applied mathematics in the secondary school curriculum
- Elementary mathematics
- Teaching practices and teacher education

2000 *Université du Québec à Montréal, Montréal, Québec*

- Des cours de mathématiques pour les futurs enseignants et enseignantes du primaire/Mathematics courses for prospective elementary teachers
- Crafting an algebraic mind: Intersections from history and the contemporary mathematics classroom
- Mathematics education et didactique des mathématiques : y a-t-il une raison pour vivre des vies séparées?/Mathematics education et didactique des mathématiques: Is there a reason for living separate lives?
- Teachers, technologies, and productive pedagogy

2001 *University of Alberta, Edmonton, Alberta*

- Considering how linear algebra is taught and learned
- Children's proving
- Inservice mathematics teacher education
- Where is the mathematics?

2002 *Queen's University, Kingston, Ontario*

- Mathematics and the arts
- Philosophy for children on mathematics
- The arithmetic/algebra interface: Implications for primary and secondary mathematics / Articulation arithmétique/algèbre: Implications pour l'enseignement des mathématiques au primaire et au secondaire
- Mathematics, the written and the drawn
- Des cours de mathématiques pour les futurs (et actuels) maîtres au secondaire / Types and characteristics desired of courses in mathematics programs for future (and in-service) teachers

2003 *Acadia University, Wolfville, Nova Scotia*

- L'histoire des mathématiques en tant que levier pédagogique au primaire et au secondaire / The history of mathematics as a pedagogic tool in Grades K–12
- Teacher research: An empowering practice?
- Images of undergraduate mathematics
- A mathematics curriculum manifesto

2004 *Université Laval, Québec, Québec*

- Learner generated examples as space for mathematical learning
- Transition to university mathematics
- Integrating applications and modeling in secondary and post secondary mathematics
- Elementary teacher education – Defining the crucial experiences
- A critical look at the language and practice of mathematics education technology

2005 *University of Ottawa, Ottawa, Ontario*

- Mathematics, education, society, and peace
- Learning mathematics in the early years (pre-K – 3)
- Discrete mathematics in secondary school curriculum
- Socio-cultural dimensions of mathematics learning

2006 *University of Calgary, Calgary, Alberta*

- Secondary mathematics teacher development
- Developing links between statistical and probabilistic thinking in school mathematics education
- Developing trust and respect when working with teachers of mathematics
- The body, the sense, and mathematics learning

2007 *University of New Brunswick, New Brunswick*

- Outreach in mathematics – Activities, engagement, & reflection
- Geometry, space, and technology: challenges for teachers and students
- The design and implementation of learning situations
- The multifaceted role of feedback in the teaching and learning of mathematics

2008 *Université de Sherbrooke, Sherbrooke, Québec*

- Mathematical reasoning of young children
- Mathematics-in-and-for-teaching (MifT): the case of algebra
- Mathematics and human alienation
- Communication and mathematical technology use throughout the post-secondary curriculum / Utilisation de technologies dans l'enseignement mathématique postsecondaire
- Cultures of generality and their associated pedagogies

2009 *York University, Toronto, Ontario*

- Mathematically gifted students / Les élèves doués et talentueux en mathématiques
- Mathematics and the life sciences
- Les méthodologies de recherches actuelles et émergentes en didactique des mathématiques / Contemporary and emergent research methodologies in mathematics education
- Reframing learning (mathematics) as collective action
- Étude des pratiques d'enseignement
- Mathematics as social (in)justice / Mathématiques citoyennes face à l'(in)justice sociale

2010 *Simon Fraser University, Burnaby, British Columbia*

- Teaching mathematics to special needs students: Who is at-risk?
- Attending to data analysis and visualizing data
- Recruitment, attrition, and retention in post-secondary mathematics
  Can we be thankful for mathematics? Mathematical thinking and aboriginal peoples
- Beauty in applied mathematics
- Noticing and engaging the mathematicians in our classrooms

2011 *Memorial University of Newfoundland, St. John's, Newfoundland*

- Mathematics teaching and climate change
- Meaningful procedural knowledge in mathematics learning
- Emergent methods for mathematics education research: Using data to develop theory / Méthodes émergentes pour les recherches en didactique des mathématiques: partir des données pour développer des théories
- Using simulation to develop students' mathematical competencies – Post secondary and teacher education
- Making art, doing mathematics / Créer de l'art; faire des maths
- Selecting tasks for future teachers in mathematics education

2012 *Université Laval, Québec City, Québec*

- Numeracy: Goals, affordances, and challenges
- Diversities in mathematics and their relation to equity
- Technology and mathematics teachers (K-16) / La technologie et l'enseignant mathématique (K-16)
- La preuve en mathématiques et en classe / Proof in mathematics and in schools
- The role of text/books in the mathematics classroom / Le rôle des manuels scolaires dans la classe de mathématiques
- Preparing teachers for the development of algebraic thinking at elementary and secondary levels / Préparer les enseignants au développement de la pensée algébrique au primaire et au secondaire

2013 *Brock University, St. Catharines, Ontario*

- MOOCs and online mathematics teaching and learning
- Exploring creativity: From the mathematics classroom to the mathematicians' mind / Explorer la créativité : de la classe de mathématiques à l'esprit des mathématiciens
- Mathematics of Planet Earth 2013: Education and communication / Mathématiques de la planète Terre 2013 : formation et communication (K-16)
- What does it mean to understand multiplicative ideas and processes? Designing strategies for teaching and learning
- Mathematics curriculum re-conceptualisation

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- Mathematical habits of mind / Modes de pensée mathématiques
- Formative assessment in mathematics: developing understandings, sharing practice, and confronting dilemmas
- Texter mathématique / Texting mathematics
- Complex dynamical systems
- Role-playing and script-writing in mathematics education: practice and research

Appendix B / Annexe B

PLENARY LECTURES AT EACH ANNUAL MEETING / CONFÉRENCES PLÉNIÈRES DES RENCONTRES ANNUELLES

1977	A.J. COLEMAN	The objectives of mathematics education
	C. GAULIN	Innovations in teacher education programmes
	T.E. KIEREN	The state of research in mathematics education
1978	G.R. RISING	The mathematician's contribution to curriculum development
	A.I. WEINZWEIG	The mathematician's contribution to pedagogy
1979	J. AGASSI	The Lakatosian revolution
	J.A. EASLEY	Formal and informal research methods and the cultural status of school mathematics
1980	C. GATTEGNO	Reflections on forty years of thinking about the teaching of mathematics
	D. HAWKINS	Understanding understanding mathematics
1981	K. IVERSON	Mathematics and computers
	J. KILPATRICK	The reasonable effectiveness of research in mathematics education
1982	P.J. DAVIS	Towards a philosophy of computation
	G. VERGNAUD	Cognitive and developmental psychology and research in mathematics education
1983	S.I. BROWN	The nature of problem generation and the mathematics curriculum
	P.J. HILTON	The nature of mathematics today and implications for mathematics teaching

1984	A.J. BISHOP	The social construction of meaning: A significant development for mathematics education?
	L. HENKIN	Linguistic aspects of mathematics and mathematics instruction
1985	H. BAUERSFELD	Contributions to a fundamental theory of mathematics learning and teaching
	H.O. POLLAK	On the relation between the applications of mathematics and the teaching of mathematics
1986	R. FINNEY	Professional applications of undergraduate mathematics
	A.H. SCHOENFELD	Confessions of an accidental theorist
1987	P. NESHER	Formulating instructional theory: the role of students' misconceptions
	H.S. WILF	The calculator with a college education
1988	C. KEITEL	Mathematics education and technology
	L.A. STEEN	All one system
1989	N. BALACHEFF	Teaching mathematical proof: The relevance and complexity of a social approach
	D. SCHATTSNEIDER	Geometry is alive and well
1990	U. D'AMBROSIO	Values in mathematics education
	A. SIERPINSKA	On understanding mathematics
1991	J .J. KAPUT	Mathematics and technology: Multiple visions of multiple futures
	C. LABORDE	Approches théoriques et méthodologiques des recherches françaises en didactique des mathématiques
1992	ICME-7	
1993	G.G. JOSEPH	What is a square root? A study of geometrical representation in different mathematical traditions
	J CONFREY	Forging a revised theory of intellectual development: Piaget, Vygotsky and beyond
1994	A. SFARD	Understanding = Doing + Seeing ?
	K. DEVLIN	Mathematics for the twenty-first century
1995	M. ARTIGUE	The role of epistemological analysis in a didactic approach to the phenomenon of mathematics learning and teaching
	K. MILLETT	Teaching and making certain it counts
1996	C. HOYLES	Beyond the classroom: The curriculum as a key factor in students' approaches to proof
	D. HENDERSON	Alive mathematical reasoning

1997	R. BORASSI	What does it really mean to teach mathematics through inquiry?
	P. TAYLOR	The high school math curriculum
	T. KIEREN	Triple embodiment: Studies of mathematical understanding-in-interaction in my work and in the work of CMESG/GCEDM
1998	J. MASON	Structure of attention in teaching mathematics
	K. HEINRICH	Communicating mathematics or mathematics storytelling
1999	J. BORWEIN	The impact of technology on the doing of mathematics
	W. WHITELEY	The decline and rise of geometry in 20th century North America
	W. LANGFORD	Industrial mathematics for the 21st century
	J. ADLER	Learning to understand mathematics teacher development and change: Researching resource availability and use in the context of formalised INSET in South Africa
	B. BARTON	An archaeology of mathematical concepts: Sifting languages for mathematical meanings
2000	G. LABELLE	Manipulating combinatorial structures
	M. B. BUSSI	The theoretical dimension of mathematics: A challenge for didacticians
2001	O. SKOVSMOSE	Mathematics in action: A challenge for social theorising
	C. ROUSSEAU	Mathematics, a living discipline within science and technology
2002	D. BALL & H. BASS	Toward a practice-based theory of mathematical knowledge for teaching
	J. BORWEIN	The experimental mathematician: The pleasure of discovery and the role of proof
2003	T. ARCHIBALD	Using history of mathematics in the classroom: Prospects and problems
	A. SIERPINSKA	Research in mathematics education through a keyhole
2004	C. MARGOLINAS	La situation du professeur et les connaissances en jeu au cours de l’activité mathématique en classe
	N. BOULEAU	La personnalité d’Evariste Galois: le contexte psychologique d’un goût prononcé pour les mathématique abstraites
2005	S. LERMAN	Learning as developing identity in the mathematics classroom
	J. TAYLOR	Soap bubbles and crystals
2006	B. JAWORSKI	Developmental research in mathematics teaching and learning: Developing learning communities based on inquiry and design
	E. DOOLITTLE	Mathematics as medicine

2007	R. NÚÑEZ	Understanding abstraction in mathematics education: Meaning, language, gesture, and the human brain
	T. C. STEVENS	Mathematics departments, new faculty, and the future of collegiate mathematics
2008	A. DJEBBAR	Art, culture et mathématiques en pays d'Islam (IX^{e}-XV^{e} s.)
	A. WATSON	Adolescent learning and secondary mathematics
2009	M. BORBA	Humans-with-media and the production of mathematical knowledge in online environments
	G. de VRIES	Mathematical biology: A case study in interdisciplinarity
2010	W. BYERS	Ambiguity and mathematical thinking
	M. CIVIL	Learning from and with parents: Resources for equity in mathematics education
	B. HODGSON	Collaboration et échanges internationaux en éduction mathématique dans le cadre de la CIEM : regards selon une perspective canadienne / ICMI as a space for international collaboration and exchange in mathematics education: Some views from a Canadian perspective
	S. DAWSON	My journey across, through, over, and around academia: "...a path laid while walking..."
2011	C. K. PALMER	Pattern composition: Beyond the basics
	P. TSAMIR & D. TIROSH	The Pair-Dialogue approach in mathematics teacher education
2012	P. GERDES	Old and new mathematical ideas from Africa: Challenges for reflection
	M. WALSHAW	Towards an understanding of ethical practical action in mathematics education: Insights from contemporary inquiries
	W. HIGGINSON	Cooda, wooda, didda, shooda: Time series reflections on CMESG/GCEDM
2013	R. LEIKIN	On the relationships between mathematical creativity, excellence and giftedness
	B. RALPH	Are we teaching Roman numerals in a digital age?
	E. MULLER	Through a CMESG looking glass
2014	D. HEWITT	The economic use of time and effort in the teaching and learning of mathematics
	N. NIGAM	Mathematics in industry, mathematics in the classroom: Analogy and metaphor
	T. KIEREN	Inter-action and mathematics knowing – An elder's memoire

Appendix C / Annexe C

PROCEEDINGS OF ANNUAL MEETINGS / ACTES DES RENCONTRES ANNUELLES

Past proceedings of CMESG/GCEDM annual meetings have been deposited in the ERIC documentation system with call numbers as follows:

Proceedings of the 1980 Annual Meeting	ED 204120
Proceedings of the 1981 Annual Meeting	ED 234988
Proceedings of the 1982 Annual Meeting	ED 234989
Proceedings of the 1983 Annual Meeting	ED 243653
Proceedings of the 1984 Annual Meeting	ED 257640
Proceedings of the 1985 Annual Meeting	ED 277573
Proceedings of the 1986 Annual Meeting	ED 297966
Proceedings of the 1987 Annual Meeting	ED 295842
Proceedings of the 1988 Annual Meeting	ED 306259
Proceedings of the 1989 Annual Meeting	ED 319606
Proceedings of the 1990 Annual Meeting	ED 344746
Proceedings of the 1991 Annual Meeting	ED 350161
Proceedings of the 1993 Annual Meeting	ED 407243
Proceedings of the 1994 Annual Meeting	ED 407242

Proceedings of the 1995 Annual Meeting	ED 407241
Proceedings of the 1996 Annual Meeting	ED 425054
Proceedings of the 1997 Annual Meeting	ED 423116
Proceedings of the 1998 Annual Meeting	ED 431624
Proceedings of the 1999 Annual Meeting	ED 445894
Proceedings of the 2000 Annual Meeting	ED 472094
Proceedings of the 2001 Annual Meeting	ED 472091
Proceedings of the 2002 Annual Meeting	ED 529557
Proceedings of the 2003 Annual Meeting	ED 529558
Proceedings of the 2004 Annual Meeting	ED 529563
Proceedings of the 2005 Annual Meeting	ED 529560
Proceedings of the 2006 Annual Meeting	ED 529562
Proceedings of the 2007 Annual Meeting	ED 529556
Proceedings of the 2008 Annual Meeting	ED 529561
Proceedings of the 2009 Annual Meeting	ED 529559
Proceedings of the 2010 Annual Meeting	ED 529564
Proceedings of the 2011 Annual Meeting	*submitted*
Proceedings of the 2012 Annual Meeting	*submitted*
Proceedings of the 2013 Annual Meeting	*submitted*
Proceedings of the 2014 Annual Meeting	*submitted*

NOTE

There was no Annual Meeting in 1992 because Canada hosted the Seventh International Conference on Mathematical Education that year.