Lost in 4-space.

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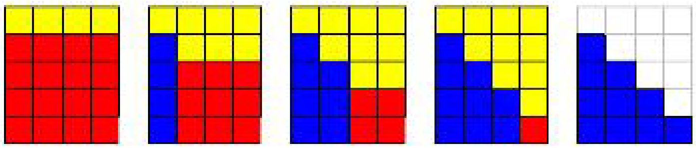
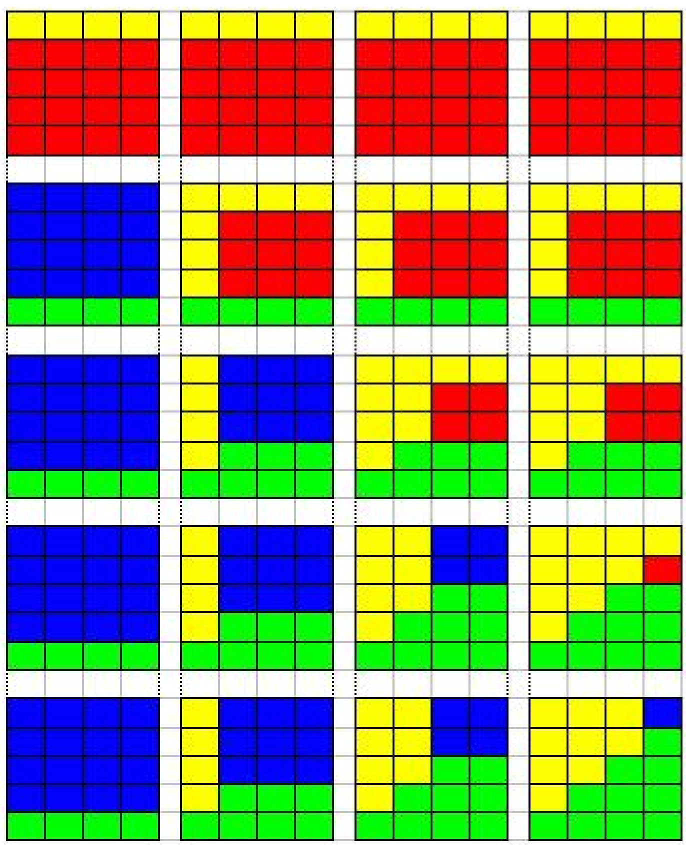
Amongst the many thoughts, insights and ideas, I took away from GCEDM/CMESG 2000, there was some mathematics, stimulated by the topic session offered by Vicki Zack and David Reid. In their session they talked about some work they have been doing with Vicki's class on sums of n and sums of n squared. Actually, I do not remember the discussion too clearly, since I became absorbed by an image which David showed us, which he had used with Vicki's class.

The image, which was three dimensional, was related to sums of ‘pyramid' consisting of a 4x4 square layer, then a 3x3 layer, a 2x2 layer and a 1x1 layer. Using three such pyramids, David fitted them all together to make something that was almost - but not quite - a cuboid. We thought about how this image can be used to develop the formula for Σn^2. I became fascinated and began to wonder if the same approach could be extended to finding the formula for Σn^3, a move which requires four dimensions. First, I needed to understand how the three pyramids so magically fitted together. To work on this, I needed a way of examining the whole structure of the nearly-cuboid, rather than just looking at the exterior. I therefore sketched each layer of the solid, represented as five rectangular 4x5 grids, with each square shaded according to the colour of the cubes. It is this activity of visualising how the three pyramids fit together in order to produce the layer representation that absorbed me for much of the rest of the session. I now had a new image. [See diagram 1]



I also had a new challenge - to think about what happens in four dimensions. What does a pyramid look like in 4D? How can it be represented? How many are needed to fit together? How do 4D pyramids fit together? These and related questions not only absorbed me for the rest of Vicki and David's session, but for the rest of the conference and the several months which have followed. At first I found it difficult to work with the extra dimension. During the breaks I thought and I talked with anyone who was interested and I sat around with glazed eyes, lost in 4-space, my brain aching.

Later in the conference, instead of attending a plenary discussion, I chose to go to the coffee room where there was a crate of multilink cubes. I prepared sets of cubes, as representations of $1^3 + 2^3 + 3^3 + 4^3$. Another way of thinking about this is as four slices through one dimension of a 4-pyramid, in the same way that slices through a 3-pyramid produces increasing squares. A small group of us sat around seeking a way to fit them together, as well as thinking what fitting-together is like in 4-space. I got left behind, particularly at a point where the group had a moment of insight which I could not see, whereupon things moved very fast. The result was four cuboids pieced together from parts of four 4-pyramids. Another image. An image of images, in fact. We looked at the cuboids, remarking some of the patterns which move through and across from one cuboid to another, starting to make sense of them. It feels important that I was still able to do this, despite having been left behind during the construction of the images. Later I sat and produced layer diagrams for each cuboid, side-by-side in my notebook. (See Diagram 2) I started to see more patterns and make more sense of where the pyramids were. I realised that cuboids can be sliced in more than one way and produced a second set of slices which I could compare with the first, allowing me to see more general patterns.



What can I say about this process of seeing more? Although the original motivation of putting pyramids together was to think about formulas for sums of n or n-squared or n-cubed, for me there was also the mathematics of increasing dimensions, a process of increasing generalisation and abstraction. In this, the patterns seem to be important: my access to the mathematics is made possible by the rich patterns found in these images. I am also aware that the process of seeing more is a joint activity. What I have described here came about through talking, thinking and seeing with others. At different moments, different participants had different insights which moved all of us forward. We shared various images, yet transformed them, and so ourselves, through our talk, so that colours and shapes became patterns and then patterns of patterns.

And so the process continues. I carried the notebook and my images home to the UK. I am still absorbed by them - there is so much that can be seen in them. I have reconstructed the cuboids using multilink cubes, gradually making more sense of what is happening, at some stage moving beyond the point where I was originally left behind in the coffee room. I have started to extend the ideas to 5 and 6 dimensions, creating more complex patterns, through which yet more general patterns emerge. Josee, my wife has, grown used to heaps of multilink cubes all over the place, and to me staring at them, lost in space.