CANADIAN MATHEMATICS EDUCATION
STUDY GROUP

GROUPE CANADIEN D'ÉTUDE EN DIDACTIQUE
DES MATHÉMATIQUES

PROCEEDINGS / ACTES
2016 ANNUAL MEETING /
RENCONTRE ANNUELLE 2016

CMESG
GCEDM

Queen's University
Kingston, Ontario
June 3 – June 7, 2016

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Proceedings of the 2016 Annual Meeting of the
Canadian Mathematics Education Study Group /
Groupe Canadien d'Étude en Didactique des Mathématiques
are published by CMESG/GCEDM.
They were published online in May 2017.

ISBN: 978-1-77287-020-6

PROCEEDINGS OF THE 2016 ANNUAL MEETING OF THE CANADIAN MATHEMATICS EDUCATION STUDY GROUP / ACTES DE LA RENCONTRE ANNUELLE 2016 DU GROUPE CANADIEN D'ÉTUDE EN DIDACTIQUE DES MATHÉMATIQUES

40th Annual Meeting
Queen's University
Kingston, Ontario
June 3 – June 7, 2016

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INTRODUCTION

Olive Chapman – President, CMESG/GCEDM
University of Calgary

The 40th Annual Meeting of the Canadian Mathematics Education Study Group/Groupe Canadien d'étude en didactique des mathématiques [CMESG/GCEDM] was an inspiring, memorable academic and social event that lived up to the theme of *Celebrating the Past, Inspiring the Future*.

On behalf of CMESG/GCEDM executive and participants of the meeting, I acknowledge the outstanding work of our hosts at Queen's University. The meeting was excellently organized and managed from beginning to end. The Town Crier to open the meeting, the Kingston 1000 Islands dinner cruise on the Island Queen triple-deck paddle-wheeler that allowed us to enjoy the remarkable scenery and learn more about the geography and history of the region, and the conference dinner at the beautiful Isabel Bader Centre for the Performing Arts with floor to ceiling windows to enjoy the scenic shores of Lake Ontario on which it is located, are some of the highlights of the social program that made the event special, memorable, and celebratory.

Thanks to our colleagues, Jamie Pyper, who in his fearless manner took the lead on everything for the local organizing of the meeting, and Peter Taylor, who noted that he could hardly stop marveling at what 40 years hath wrought. Thanks to the graduate students and post-docs (members of the organizing team and volunteers during the conference) who did a tremendous amount of work both before and during the meeting: Heather Braund, Meghan Dale, Steve MacGregor, Asia Matthews, Andrew McEachern, Matt Norris Deena Salem, Adelina Valliquette and Judy Wearing. Thanks for the financial support and considerable administrative support before, during and after the conference that was provided by the Mathematics, Science and Technology Education Group (MSTE) and by the Mathematics and Statistics Department. Thanks for the timely support that was offered by Queen's University and the City of Kingston, and general support, encouragement and advice that were freely bestowed by Rebecca Luce-Kapler, the Dean of the Queen's University Faculty of Education.

I also acknowledge the CMESG/GCEDM executive for organizing a special scientific program to honour our 40th anniversary meeting. As a significant landmark of the annual gathering of our members, we returned to Queen's University, Kingston, the birthplace of our organization, where the first three meetings were held. This landmark meeting was also an excellent opportunity to celebrate our history and provide inspiration for the future of our Group. Thus, the choice of *Celebrating the Past, Inspiring the Future* as the theme for the scientific program. Consistent with this theme, the executive developed a program that highlighted past and current senior members of our Group as key presenters and leaders. This resulted in a significant deviation from our regular program of having two plenary sessions (one with a mathematician and one with a mathematics education researcher as speakers), separate sessions for small-group discussion of plenaries and preparation of questions that are presented to the plenary speakers in separate plenary question sessions, and topic sessions. These sessions were replaced with special anniversary plenary sessions that involved past and current senior members of our Group as plenary and plenary panel presenters.

On behalf of the executive and members, thanks to these colleagues who were instrumental in realizing the vision of the meeting by serving as plenary and plenary panel speakers. Our four plenary speakers inspired us in a variety of ways. For example, Bernard R. Hodgson emphasized the importance of the contribution of mathematicians in the education of mathematics teachers at all levels and the need for a context that furthers and strengthens the links between mathematicians and mathematics educators involved in the preparation of teachers for this contribution to fully flourish. Carolyn Kieran highlighted the importance of task design in mathematics education with a focus on its history, its frameworks, and its heuristics. Eric Muller explored implications of *a third pillar of scientific inquiry of complex systems* and the connection to two mathematics courses at Brock University. Peter Taylor treated participants to a unique artistic, touching rendition of his plenary, which consisted of an allegory of aspects of his journey as a mathematician and was held at the impressive Isabel Bader Centre Performance Hall. The presenters also contributed to the theme of the meeting by making connections to their experiences with CMESG/GCEDM that provided lenses for us to look back and to consider future directions for CMESG/GCEDM and mathematics education in Canada.

The goal of the two plenary panels was to focus specifically on the theme of the meeting. The opening plenary panel, coordinated and chaired by Olive Chapman, focused on the portion of the theme: “Celebrating the Past.” The five panelists, Ed Barbeau, Bill Higginson, Bernard Hodgson, Tom Kieren and Peter Taylor who were present at the first meeting of the Group (i.e., founding members) inspired and entertained us with memories of the years they were (Peter continues to be) actively engaged in the Group as participants and leaders. Their ways of looking back reminded us of our rich past of excellence, collaboration, and collegiality and illustrated the kind of organization we want CMESG/GCEDM to continue to be. The closing plenary panel, coordinated and chaired by Peter Liljedahl, focused on the other portion of the theme: “Inspiring the Future.” The four panelists, Nadine Bednarz, John Mason, Anna Sierpinska, and Walter Whiteley, in looking forward, opened some rich avenues for our Group to think about and consider acting on regarding possible futures of the work of the Group and mathematics education in Canada.

The presence and contributions of these colleagues at the meeting were highly appreciated. Their participation and interactions with participants throughout the meeting helped us to achieve our goal of making this meeting one of the most memorable. With their presence, the meeting was an exciting reunion for some of us and an opportunity for others to engage with some of our prominent and inspiring Canadian researchers in mathematics and mathematics education.

In addition to these plenary sessions, other aspects of the scientific program (working groups, new PhDs, gallery walk, and ad hoc discussion sessions) remained the same. Over the last 40 years, the Working Groups, which form the core of the scientific program, have engaged participants in learning activities and deep discussions on a variety of timely and significant topics in mathematics education from Kindergarten to post-secondary levels. Thus, for this 40th anniversary meeting, working group topics were selected to reflect trends and key aspects among these topics of our past meetings that continue to be relevant and important to our community and the field of mathematics education. Thanks to our 15 experienced colleagues who took on the demanding roles of leaders of the six Working Groups. We are honoured to have such colleagues who are willing to commit their time and expertise to guarantee the success of this central aspect of our meeting.

Thanks to the seven new-PhDs who presented their research, the Ad Hoc discussion leaders, and Math Gallery Walk presenters whose contributions helped to make the meeting a meaningful and worthwhile experience. The Gallery Walk had an impressive number of high quality poster presentations that made the session a rich scientific and social event.

As an anniversary gift to participants, a special 40th anniversary proceedings (now available at www.cmesg.org) was published to provide a snapshot of events/activities from our history that exemplify the nature of CMESG/GCEDM as an organization and of the annual meetings. As noted in the introduction:

> *Through a selection of excerpts from past proceedings we have stitched together a partial history of our time as an organization. This serves not only as a summary of our collective history, but also serves as an introduction to the activities of CMESG/GCEDM through our first 40 years.* (p. v)

Thanks to Peter Liljedahl for taking the lead on this project and to the other members of the editorial team (Darien Allan, Olive Chapman, Frédéric Gourdeau, Caroline Lajoie, Susan Oesterle, Elaine Simmt, and Peter Taylor) for their valuable contributions that made it possible.

This was one of our largest meetings and largest number of invited presenters that reflected our membership of mathematicians, mathematics teacher educators and mathematics education researchers, and the bilingualism (French and English) of the Group. The combination of participants resulted in an atmosphere that encouraged a significantly high level of social interactions and networking among them that made the meeting truly an anniversary celebration and reunion. So, finally, on behalf of the CMESG/GCEDM executive and the local organizers, thanks to all the participants who were instrumental in making this special anniversary meeting live up to the landmark event we wanted it to be.

This publication of the proceedings of this special anniversary meeting of our Group offers readers the opportunity to learn about the scientific activities of the meeting and some of the mathematics education research and past and future interests of our community. It is hoped that it also provides a means for participants of the meeting to further reflect on and build on their contributions and experiences at the meeting and for others to share in and be inspired by the work of the mathematics education community in Canada.

Horaire

Vendredi 3 juin	Samedi 4 juin	Dimanche 5 juin	Lundi 6 juin	Mardi 7 juin
FLM Pré-rencontre jeudi, 2 juin 18h30 – 21h00 vendredi, 3 juin 9h00 – 15h00	8h45 – 10h15 **Groupes de travail**	8h45 – 10h15 **Groupes de travail**	8h45 – 10h15 **Groupes de travail**	8h45 – 10h15 **Panel de clôture**
14h30 – 18h45 **Inscription**	10h15 – 10h45 Pause café	10h15 – 10h45 Pause café	10h15 – 10h45 Pause café	10h15 – 10h30 Pause café
17h00 – 18h45 **Souper**	10h45 – 12h15 **Groupes de travail**	10h45 – 12h15 **Groupes de travail**	10h45 – 12h15 **Groupes de travail**	10h30 – 11h00 **Séances *ad hoc***
18h45 – 19h30 **Séance d’ouverture**	12h30 – 13h45 Dîner	12h30 – 13h30 Dîner	12h30 – 13h45 Dîner	11h45 – 12h30 **Séance de clôture**
19h30 – 21h00 **Panel d’ouverture**	13h45 – 15h00 **Plénière 1**	13h30 – 14h30 **Café, dessert et galerie mathématique**	13h45 – 14h15 **N^{lles} thèses (2)**	
21h00 – ? **Réception**	15h15 – 15h45 **N^{lles} thèses (1)**	14h30 – 15h45 **Plénière 2**	14h30 – 15h45 **Plénière 3**	
	15h45 – 16h15 **Séances *ad hoc* d'anniversaire spéciaux**	17h00 – 21h00 **Excursion** *Croisière 1000 Îles* **Souper á bord** 21h00 – ? **Espaces sociaux downtown**	15h45 – 16h30 **Groupes de travail - préparation pour la présentation de clôture**	
	16h15 – 16h45 Pause café		16h30 – 17h30 **Social au Centre Isabel Bader**	
	16h45 – 18h00 **Assemblée générale annuelle**		17h30 – 19h00 **Souper**	
	18h00 – ? **Souper libre**		19h00 – 20h15 **Plénière 4**	
			20h15 – ? **Social/Danse**	

Schedule

Friday June 3	**Saturday June 4**	**Sunday June 5**	**Monday June 6**	**Tuesday June 7**
FLM Pre-Conference Thursday, June 2 18:30 – 21:00 Friday, June 3 9:00 – 15:00	8:45 – 10:15 **Working Groups**	8:45 – 10:15 **Working Groups**	8:45 – 10:15 **Working Groups**	8:45 – 10:15 **Closing Panel**
14:30 – 18:45 **Registration**	10:15 – 10:45 Break	10:15 – 10:45 Break	10:15 – 10:45 Break	10:15 – 10:30 Break
17:00 – 18:45 **Dinner**	10:45 – 12:15 **Working Groups**	10:45 – 12:15 **Working Groups**	10:45 – 12:15 **Working Groups**	10:30 – 11:00 ***Ad hoc Session***
18:45 – 19:30 **Opening Session**	12:30 – 13:45 Lunch	12:30 – 13:30 Lunch	12:30 – 13:45 Lunch	11:45 – 12:30 **Closing Session**
19:30 – 21:00 **Opening Panel**	13:45 – 15:00 **Plenary 1**	13:30 – 14:30 **Coffee, Dessert and Mathematics Gallery**	13:45 – 14:15 **New PhDs (2)**	
21:00 – ? **Reception**	15:15 – 15:45 **New PhDs (1)**	14:30 – 15:45 **Plenary 2**	14:30 – 15:45 **Plenary 3**	
	15:45 – 16:15 **Special Anniversary *Ad Hoc***	17:00 – 21:00 **Excursion** *1000 Islands Cruise* **Dinner on board** 21:00 – ? **Downtown social spaces**	15:45 – 16:30 **Working Groups – Preparation for the Closing Session**	
	16:15 – 16:45 Break		16:30 – 17:30 **Social at the Isabel Bader Centre**	
	16:45 – 18:00 **Annual General Meeting**		17:30 – 19:00 **Dinner**	
	18:00 – ? **Dinner on your own**		19:00 – 20:15 **Plenary 4**	
			20:15 – ? **Social/Dance**	

Plenary Lectures

Conférences plénières

APPORT DES MATHÉMATICIENS À LA FORMATION DES ENSEIGNANTS DU PRIMAIRE : REGARDS SUR LE « MODÈLE LAVAL »

Bernard R. Hodgson
Université Laval

En hommage à ma collègue et amie Linda Lessard,
à l'occasion de son départ à la retraite

My situation as a mathematician hired in a math department, but in a position devoted to the mathematical preparation of primary school teachers, is of a peculiar type—probably even today. Being invited to give a plenary lecture at the 40th anniversary meeting of CMESG/GCEDM was for me an opportunity for reflecting on my involvement for more than four decades in mathematics education in general, and teacher education in particular. The links that I have developed over the years with colleagues from various mathematics and mathematics education communities—in Québec, in Canada (notably within CMESG), as well as at the international level—were an exceptional source of support and inspiration which greatly influenced my perception and understanding of fundamental aspects of the teaching and learning of mathematics, especially in relation to the education of teachers.

I concentrate in this paper on the pre-service mathematical preparation of primary school teachers offered by the Department of Mathematics and Statistics at Université Laval. After a reminder of the context, both provincial and local, in which this endeavour was launched, I offer a survey of the two mathematics courses established to that end. In so doing, I aim at presenting the 'mathematical message' intended for prospective primary school teachers that we are proposing in the teacher education model set up at my university. This paper does not pretend to offer a detailed or comparative study of various approaches to the mathematical preparation for primary school teaching. The reader is invited to see it as a personal testimony, based on more than forty years of experience, about the contribution that mathematicians may bring to the education of teachers, including those of the primary level.

INTRODUCTION

L'invitation de présenter une conférence plénière dans le cadre de la rencontre du 40e anniversaire du GCEDM/CMESG s'est offerte à moi comme une occasion propice pour réfléchir à certains jalons qui ont marqué ma vie de mathématicien œuvrant depuis plus de quatre décennies en éducation mathématique. Non seulement mes débuts comme professeur à l'université coïncident-ils presque avec la naissance du GCEDM, mais surtout ma participation au fil des ans aux rencontres du groupe a fortement marqué ma perception et ma compréhension d'aspects fondamentaux de l'enseignement et de l'apprentissage des mathématiques, notamment en lien avec la formation des maîtres.

Mon statut de mathématicien engagé en 1975 dans un département de mathématiques, mais dans le cadre d'un poste destiné à la préparation mathématique des enseignants du primaire, a évidemment une connotation particulière—et ce, encore même dans le contexte d'aujourd'hui. Ce serait certes un euphémisme de dire qu'à mes débuts je me sentais un peu dépourvu… Mais l'appui et la stimulation que j'ai pu trouver par la fréquentation de diverses communautés mathématiques et didactiques—québécoise, canadienne (notamment au GCEDM) et éventuellement internationale—m'ont permis de cheminer et d'en venir à me sentir pleinement dans mon élément. J'en ai développé une double conviction : autant les mathématiciens ont une contribution importante et spécifique à apporter à la formation mathématique des enseignants, autant cette contribution ne saurait s'épanouir pleinement que dans un contexte encourageant et renforçant les liens entre mathématiciens et didacticiens impliqués dans la préparation à l'enseignement—j'ai déjà soutenu un tel point de vue dans (Hodgson, 2001).

En toile de fond de ma présentation lors du congrès se retrouvait une « équation humaine »—pour reprendre un slogan emprunté à la Faculté (des sciences et de génie) à laquelle j'appartiens—*mon* équation humaine, dans laquelle interviennent de nombreux paramètres : des domaines mathématiques qui m'ont allumé (logique mathématique, histoire des mathématiques) ; un cadre scientifique globalement axé sur l'éducation mathématique ; une implication assidue et soutenue en formation des enseignants du primaire et du secondaire ; et des contacts exceptionnellement riches avec une foultitude de collègues, ici et ailleurs, qui m'ont apporté énormément et incité à aller plus loin dans mon domaine.

J'ai choisi dans le présent texte de me concentrer sur un volet de cette équation : la formation initiale des enseignants du primaire en mathématiques offerte par le Département de mathématiques et de statistique de l'Université Laval. Après avoir rappelé le contexte, tant provincial que local, dans lequel cet enseignement a pris naissance, j'offrirai un survol des deux cours de mathématiques créés à cette fin. Je vise ce faisant à présenter le « message mathématique » destiné aux futurs enseignants du primaire que nous proposons dans le cadre du modèle de formation des maîtres mis en place à l'Université Laval. Ce texte ne se veut en aucune façon une étude détaillée ou comparative de diverses approches à la préparation à l'enseignement au primaire sur le plan mathématique. Le lecteur est invité à plutôt y voir un témoignage personnel, basé sur plus de quarante ans d'expérience, à propos de la contribution que peuvent apporter les mathématiciens à la formation des enseignants, y compris de l'ordre primaire.

LA FORMATION DES ENSEIGNANTS AU QUÉBEC

Quelques mots tout d'abord sur la formation des enseignants au Québec—voir aussi, sur le même sujet, (Hodgson & Lajoie, 2015). Compte tenu de la responsabilité provinciale en matière d'éducation, la toile de fond qui suit est substantiellement différente de ce qui se passe dans les autres provinces canadiennes. On trouvera dans (Bednarz, 2012), à cet égard, une étude comparative de différents modèles mis en place au Canada pour la formation mathématique des enseignants.

DES ÉCOLES NORMALES À LA FORMATION UNIVERSITAIRE

Le rapport Parent

Jusqu'au début des années 1970, la formation des enseignants au Québec s'effectuait dans le cadre d'un réseau d'*écoles normales* graduellement mis en place à compter du milieu du 19^{e} siècle. Ces écoles normales relevaient pour la plupart de communautés religieuses œuvrant en éducation. Au cours des années 1960, le Québec a connu une réforme profonde et globale de son système d'éducation à la suite des travaux de la *Commission royale d'enquête sur l'enseignement dans la province de Québec*—mieux connue sous le vocable de « Commission Parent », du nom de son président, Alphonse-Marie Parent (1906-1970). Cette commission, mise en place par le gouvernement québécois élu en 1960—en rupture, il convient de le souligner, avec l'ère politique qui précédait—est née dans un contexte de renouveau qui a mené à ce que l'usage dénomme la « révolution tranquille », un moment-charnière de l'histoire du Québec au cours de la dernière moitié du 20^{e} siècle. Durant la période 1963-1965, la Commission Parent a publié un imposant rapport en cinq volumes visant à promouvoir tant la démocratisation que la qualité de l'éducation au Québec. Notons entre autres que c'est à l'occasion de cette réforme que fut lancé dès 1967 le réseau des *collèges d'enseignement général et professionnel* (cégeps), un ordre d'éducation spécifique au Québec et reconnu pour sa pertinence quant au développement des jeunes, tant sur le plan personnel qu'intellectuel.

L'une des visées du rapport Parent, directement relié au thème du présent texte, était l'enrichissement de la qualification des enseignants du primaire et du secondaire. Cet objectif se manifeste dans l'une des recommandations fortes du rapport, proposant le transfert à l'ordre universitaire de la responsabilité de la formation des maîtres.

> *Nous recommandons, par conséquent, que la formation des maîtres soit intégrée à l'enseignement supérieur et ne soit confiée qu'aux établissements universitaires, c'est-à-dire aux universités actuelles, aux nouvelles universités à charte limitée et aux centres d'études universitaires. Ces institutions assureront la formation des maîtres, rôle que remplissaient jusqu'ici en partie les écoles normales.* (Parent, 1964, article 397)

On assurait ainsi que les futurs enseignants auraient un minimum de 16 années de scolarité, une nette augmentation par rapport aux différents systèmes jusque-là en vigueur. On voyait de plus la recherche se déroulant en milieu universitaire comme favorisant l'émergence d'un climat et d'un contexte bénéfiques au futur enseignant.

Dans la foulée du rapport Parent et du transfert à l'université de la formation des maîtres, le Département de mathématiques et de statistique de l'Université Laval fut invité par la direction de l'université, au début des années 1970, à participer activement à la formation mathématique des futurs enseignants du primaire—une invitation peut-être un peu insolite, à tout le moins dans le contexte de l'époque. Il s'agissait là bien sûr d'un mandat tout à fait nouveau, pour lequel le département ne possédait pas *a priori* d'expertise pertinente. Mais la réponse fut tout à fait positive et il en résulta la création de deux cours de mathématiques conçus spécifiquement en lien avec l'enseignement au primaire. Ces deux cours ont bien sûr évolué depuis, mais ils forment toujours le cœur de la contribution de mon département à la préparation à l'enseignement primaire.

La réforme de la formation à l'enseignement des années 1990

Une vingtaine d'années après la création des premiers programmes universitaires de formation à l'enseignement, il fut jugé que ceux-ci étaient mûrs pour une réforme en profondeur. L'un des changements les plus visibles de cette réforme fut l'ajout d'une année d'étude. Il faut souligner ici qu'en raison de la création du réseau collégial, les premiers programmes de formation des maîtres résultant de la mise en application du rapport Parent étaient d'une durée de trois ans, comme l'étaient (et le sont encore) la plupart des programmes universitaires québécois—sauf certains programmes professionnels tels médecine ou génie.

Les programmes actuels de formation à l'enseignement au Québec, lancés dans la première moitié des années 1990 sous les nouvelles directives du ministère de l'Éducation, se déroulent donc sur quatre années et ont entre autres comme particularité d'être de type concomitant, la formation disciplinaire allant de pair avec la formation pédagogique. En conséquence, la décision de devenir un enseignant se prend dès l'admission à l'université. (Voir Tatto, Lerman et Novotná (2009, p. 18) pour de plus amples commentaires sur le modèle concomitant, par opposition au modèle dit consécutif. Le choix du modèle concomitant a eu au Québec un impact marqué sur la formation à l'enseignement secondaire, davantage que pour le primaire où il était déjà dans les faits la norme.)

Au cœur de la réforme des années 1990 se retrouve l'objectif de renforcer la « professionnalisation » de l'enseignant :

> *Cette réforme, dont l'urgence avait été reconnue par tous les partenaires, avait pour objet de faire de l'acte d'enseigner un acte professionnel. Il fallait donc poser autrement le problème de la formation du personnel enseignant et procéder différemment tant dans les programmes et les approches de formation que dans les objets et méthodes de recherche.* (MEQ, 2001, p. 22)

La formation initiale des enseignants doit donc viser à permettre aux enseignants d'acquérir une autonomie et une expertise telles qu'ils soient en mesure de poser un regard critique sur les situations d'apprentissage, de prendre des décisions pédagogiques judicieuses et de poser des gestes pertinents en lien avec les objets d'enseignement—notamment en mathématiques—relevant de leur compétence.

L'ajout d'une quatrième année d'étude universitaire a entre autres permis d'augmenter de manière substantielle le temps consacré aux stages en milieu scolaire par les futurs enseignants. Nous touchons ici à l'un des deux thèmes (« *Learning in and from practice* ») sur lesquels repose l'*Étude 15 de l'ICMI* sur la formation et le développement des enseignants (Even & Loewenberg Ball, 2009). Répartis tout le long du programme avec une intensité croissante, les stages culminent dans la prise en charge par le futur enseignant durant tout un trimestre, à l'automne de la quatrième année, d'une véritable classe en milieu scolaire. Les situations d'apprentissage vécues durant les divers stages par l'étudiant-enseignant sont perçues de manière très positive. Combinées au caractère concomitant des programmes de formation à l'enseignement, les activités de stage permettent au futur maître de faire appel, dans les cours des dernières années du programme de formation, à ses expériences propres d'enseignement à l'école, enrichissant ainsi considérablement les discussions dans de tels cours.

LA FORMATION MATHÉMATIQUE DES ENSEIGNANTS DU PRIMAIRE À L'UNIVERSITÉ LAVAL

La plupart des composantes de tout programme de formation initiale à l'enseignement offert dans une université québécoise reposent sur les paramètres déterminés par le ministère de l'Éducation (MEQ, 2001). Ceux-ci se cristallisent autour d'un « référentiel de [douze] compétences professionnelles de la profession enseignante » auquel doivent contribuer les cours du programme—voir MEQ (2001, pp. 57 *sqq*.). Mais les universités disposent néanmoins d'une certaine marge de manœuvre quant à l'articulation et à l'organisation du programme, permettant ainsi d'y introduire un ingrédient de couleur locale.

L'une des particularités du programme de *Baccalauréat en éducation au préscolaire et en enseignement au primaire* (BÉPEP) de l'Université Laval, touchant au cœur même de mon propos, est bien sûr la présence de deux cours obligatoires de mathématiques créés et offerts à l'intention des étudiants du programme par le Département de mathématiques et de statistique. Il convient d'insister sur le fait que les deux cours en question ne sont pas des cours généraux de mathématiques destinés à un vaste public et auxquels s'inscriraient les étudiants du programme, mais bien des cours de mathématiques donnés par des mathématiciens et s'adressant de façon spécifique à de futurs enseignants du primaire. Ce modèle n'est pas si fréquent sur les scènes canadienne ou québécoise, comme le montrent les tableaux 1 et 2 de (Bednarz, 2012, pp. 20, 23-24). Ces mêmes tableaux font de plus ressortir une grande variabilité quant aux cours de mathématiques que doivent suivre les futurs enseignants du primaire : par exemple, parfois aucun cours, parfois un cours mais non spécifique, parfois un cours spécifique mais offert par la Faculté des sciences de l'éducation, de sorte qu'il n'y a alors pas de contact direct des futurs enseignants avec des mathématiciens.

Bien sûr, un certain nombre de cours de didactique des mathématiques feront toujours partie d'un programme de formation à l'enseignement primaire. Dans les cas du BÉPEP de l'Université Laval, il s'agit de trois cours offerts par la Faculté des sciences de l'éducation et portant respectivement sur la didactique des nombres naturels et des entiers relatifs, la didactique des nombres rationnels et de la mesure, et la didactique de la géométrie. Ce programme contient donc au total cinq cours obligatoires visant directement la formation initiale en vue de l'enseignement des mathématiques au primaire. Hodgson et Lajoie (2015) proposent des réflexions sur les liens possibles dans un tel cadre entre les cours de mathématiques proprement dits et ceux de didactique, en utilisant comme cas d'espèce le cours portant sur l'arithmétique.

COURS DE MATHÉMATIQUES EN VUE DE L'ENSEIGNEMENT AU PRIMAIRE : LE CAS DE L'UNIVERSITÉ LAVAL

Depuis plus de quatre décennies, le Département de mathématiques et de statistique de l'Université Laval contribue donc à la formation initiale des enseignants du primaire par l'offre de deux cours spécifiquement conçus en vue de l'enseignement primaire, l'un portant sur l'arithmétique et l'autre sur la géométrie. Un aspect crucial de ces cours est qu'ils sont vraiment conçus et donnés en ayant constamment à l'esprit le fait qu'ils sont destinés à de futurs enseignants de l'ordre primaire. Il ne s'agit donc pas du tout de simplement leur permettre de rafraîchir ou d'enrichir leur bagage mathématique en leur faisant faire « plus de maths », mais bien de les mettre en situation d'une véritable réflexion, du point de vue d'un adulte se destinant à l'enseignement, sur les thèmes mathématiques reliés à l'enseignement primaire. Si la fin ultime de cette réflexion, à savoir l'intervention auprès des élèves du primaire, se trouve toujours en toile de fond de nos cours, la réflexion elle-même se situe résolument dans une authentique perspective mathématique.

Je signale d'entrée de jeu, afin de mieux faire sentir au lecteur le cadre dans lequel se déroulent les deux cours dont il va être ici question, que la taille typique d'un groupe d'étudiants donné se situe entre 90 et 100—avec des variantes allant de 50 à 140, au gré des trimestres. Il s'agit donc d'un contexte de grand groupe, ce qui du coup impose des contraintes importantes quant aux modes d'enseignement praticables. En prenant connaissance des commentaires qui suivent sur les visées mathématiques au cœur de nos cours, il peut être utile de garder ces données à l'esprit, afin de bien percevoir le milieu pédagogique de notre démarche.

À PROPOS DE MATHÉMATIQUES POUR L'ENSEIGNANT DU PRIMAIRE

Conceptions sous-jacentes à notre démarche mathématique et pédagogique

L'esprit visé dans nos cours créés pour le programme de BÉPEP repose sur la conviction que pour remplir adéquatement son rôle de guide et être un communicateur efficace, l'enseignant du primaire doit avoir atteint un niveau de compétence tel en mathématiques qu'il se sente en pleine possession de l'outil mathématique avec lequel il va travailler, en un mot qu'il se sente *autonome* sur le plan mathématique. Nous lui proposons donc dans ces cours une occasion de reconstruction personnelle de l'arithmétique et de la géométrie élémentaires, par le truchement d'un cheminement mathématique lui permettant de clarifier et de vraiment faire siennes les notions qui sous-tendent l'enseignement des mathématiques à l'ordre primaire. Nous sommes conscients que plusieurs de ces notions sont sans doute relativement familières chez plusieurs, voire une majorité, de nos étudiants. Mais il nous paraît opportun de leur en proposer un « regard d'adulte », de manière à les amener bien au-delà de simples automatismes qu'ils ont pu retenir de leur propre parcours scolaire et à leur permettre de mieux accompagner et guider leurs élèves dans leurs apprentissages.

Un objectif primordial dans ces cours est de favoriser chez le futur enseignant le développement d'une attitude positive à l'égard des mathématiques—qui n'est malheureusement pas forcément présente chez certains des étudiants que nous accueillons. Nous souhaitons lutter contre l'horrible vision réductrice qui ferait des maths un ramassis de règles plus ou moins arbitraires à apprendre par cœur et à appliquer aveuglément. En l'aidant à prendre conscience d'une logique interne et d'une cohérence propres aux mathématiques qu'il aura à travailler avec ses élèves, nous cherchons à ce que l'étudiant-maître se sente en pleine mesure d'amener ses élèves à percevoir eux-mêmes les mathématiques de manière positive. Celui-ci doit être conscient de l'influence tant de son attitude que de son comportement sur ses élèves, face à ce que sont les mathématiques et à leur importance.

Le choix des thèmes mathématiques abordés dans nos cours, ainsi que les méthodes pédagogiques mises en place, ont pour objectif d'aider le futur enseignant du primaire à démystifier—voire démythifier—les mathématiques de manière à en faire un objet qui lui soit totalement familier, en ce qui regarde l'enseignement au primaire. Nous visons donc une recherche de clarté conceptuelle, sans doute exigeante pour l'étudiant, mais nettement récompensée par un sens accru de maîtrise et d'aisance. Nous souhaitons qu'en tant que professionnel de l'enseignement, le futur maître ait acquis un sentiment de confiance non seulement dans ses compétences mathématiques proprement dites, mais aussi dans son jugement critique, dans une perspective tant mathématique que pédagogique. Bref, nous visons à ce qu'il se voie vraiment comme l'« expert » des mathématiques destinées aux élèves du primaire, de sorte à pouvoir entre autres développer une appréciation personnelle face aux diverses tendances dans l'enseignement des mathématiques qui se retrouveront inévitablement sur sa route, tout au long de sa carrière d'enseignant. À cette fin, nous proposons aux étudiants des situations d'apprentissage ayant pour but de les faire plonger dans des expériences mathématiques significatives non seulement sur le plan théorique, mais aussi d'un point de vue pratique, étant amenés à véritablement « mettre la main à la pâte ». Comme souligné dans Hodgson et Lajoie (2015, pp. 308-309), une telle approche s'insère dans la lignée de divers travaux de recherche des dernières décennies, notamment ceux de (Ball & Bass, 2003) portant sur le savoir mathématique en vue de l'enseignement.

Les deux cours dont il sera ici question s'articulent autour d'activités d'enseignement et d'apprentissage comportant plusieurs éléments : leçons magistrales ; résolution d'exercices et de problèmes par les étudiants en dehors des heures de cours ; séances d'exercices et de problèmes en classe exigeant une participation active des étudiants ; et en géométrie, travaux pratiques visant à l'auto-apprentissage de certains thèmes géométriques choisis et retour en classe sur ceux-ci. À cela s'ajoutent des séances de récupération au cours de laquelle un auxiliaire d'enseignement se tient à la disposition des étudiants pour une aide personnalisée. Plusieurs étudiants nous parlent de cette « clinique sans rendez-vous », exclusive au BÉPEP, comme ayant joué un rôle majeur dans leur apprivoisement de la matière présentée en classe.

Apprendre à enseigner les mathématiques : du côté de chez Pólya

Notre démarche pour la préparation à l'enseignement peut aussi être reliée à des réflexions plus anciennes, notamment aux commentaires de l'éminent mathématicien George Pólya (1887-1985) à propos de l'enseignement ainsi que des qualités requises chez l'enseignant. S'il est vrai qu'au fil des âges nombre de mathématiciens ont su manifester un grand souci pédagogique par la haute qualité du matériel qu'ils produisaient en lien avec l'enseignement—le cas du grand Leonhard Euler (1707-1783) vient ici spontanément à l'esprit—peu d'entre eux, avant Pólya, se sont livrés à des réflexions explicites sur les contextes et attitudes propres à l'enseignement des mathématiques. (On notera au passage que si Pólya écrivait en ayant à l'esprit l'enseignement secondaire, plusieurs de ses commentaires se transposent de façon manifeste à l'ordre primaire.)

Déjà dans *Comment poser et résoudre un problème*, Pólya (1957/1962) énonçait, un peu sur un ton de boutade peut-être, ce qu'il percevait comme constituant des « règles de l'enseignement » :

> *La première de ces règles, c'est bien connaître ce qu'on est censé enseigner ; le seconde, c'est en connaître un peu davantage.* […] *N'oublions pas qu'un professeur de mathématiques doit connaître ce qu'il enseigne et que, s'il désire inculquer à ses élèves la tournure d'esprit qui convient pour aborder les problèmes, il doit d'abord avoir lui-même acquis cette attitude.* (p. 199)

Les mêmes idées se retrouvent dans ses célèbres « Dix commandements du professeur » (Pólya, 1967) tirés d'un chapitre intitulé *Apprendre, enseigner et apprendre à enseigner*. Parmi ces commandements figurent entre autres les règles suivantes : « (1) Soyez intéressé par votre sujet. (2) Possédez votre sujet.[…] (5) Ne leur donnez pas uniquement du savoir, mais du "savoir-faire", des attitudes intellectuelles, l'habitude d'un travail méthodique » (p. 299). Et Pólya commente ses dix commandements en ces termes :

> *Sur quelle autorité ces commandements sont-ils fondés ? Cher collègue enseignant, n'acceptez aucune autorité sauf celle de votre expérience bien assimilée et de votre propre jugement. Essayez de voir clairement ce que signifie chaque conseil, dans votre situation particulière, éprouvez sa justesse dans vos classes, et formulez votre jugement après un essai loyal.* (p. 298)

Pólya insiste sur la distinction entre *savoir* et *savoir-faire* : « Tout le monde admet qu'en mathématiques, le savoir-faire est plus important et même beaucoup plus important que la simple possession du savoir » (p. 295). D'où sa conclusion un brin provoquante, en lien avec le cinquième commandement : « Par là, en mathématiques, la façon dont vous enseignez est plus importante que ce que vous enseignez » (p. 301). (On notera que le texte original de Pólya est un peu moins catégorique : « Since know-how is more important in mathematics than information, it may be more important in the mathematics class how you teach than what you teach » (Pólya, 1965, p. 118).

Remarque : Dans le même esprit, on peut rappeler ici un passage bien connu où Pólya, souhaitant mettre en évidence « quelques ficelles du métier de professeur », souligne certains points communs que l'enseignement partage avec l'art théâtral. Il est essentiel, selon lui, de maintenir un climat d'intérêt dans la salle de classe, même dans le cas où l'enseignant aurait pu devenir lassé, au fil des ans, d'aborder pour une énième fois un thème donné. À cette fin, Pólya lui prodigue l'exhortation suivante : « Vous devriez un peu jouer la comédie par égard pour vos étudiants ; ils peuvent apprendre plus, à l'occasion, de vos attitudes que de la matière enseignée » (Pólya, 1967, pp. 281-282).

<u>Plongeon d'un mathématicien dans le merveilleux monde de la formation des enseignants</u>

Bien sûr, ces réflexions de Pólya peuvent être vues comme un peu naïves sur un plan didactique. Ce n'est pas là-dessus, pourrait-on penser, qu'on peut faire reposer une vision robuste de la formation mathématique en vue de l'enseignement au primaire….

Et pourtant… Si j'ai tenu à en parler, c'est que dans mon cas personnel elles ont néanmoins été à mes débuts une précieuse source d'inspiration et de motivation : elles me suggéraient sinon des directions à suivre, du moins des pistes à explorer en vue de créer, tant par mon enseignement que par mon attitude dans ma salle de classe, un environnement propice à un véritable cheminement de la part de mes étudiants-maîtres. Ce n'était certes pas chez un Bourbaki, le mathématicien mythique de mes années de doctorat, que je pouvais espérer trouver ce que je recherchais comme « climat mathématique » pour mon travail auprès des enseignants du primaire. Non plus que chez un Dieudonné et son tonitruant « À bas Euclide! », ou encore chez les autres mathématiciens prosélytes de la réforme des « maths modernes » des années 1960.

J'aimerais souligner ici la réflexion proposée au début des années 1960, à contre-courant du bourbakisme et des maths modernes de l'époque, par le mathématicien Alexander Wittenberg (1926-1965), professeur à l'Université Laval de 1957 à 1963 puis à l'Université York de Toronto, alors récemment créée. Wittenberg (1963) présente un cours d'« Introduction à la philosophie des mathématiques » donné à l'Université Laval à l'intention des futurs enseignants et visant à leur permettre d'amener leurs élèves « *to learn mathematics through genuine insight—not through rote learning and mechanical drill* » (p. 1091). Basée sur l'enseignement génétique, la vision proposée par Wittenberg se veut en résistance à un enseignement « "bourbakisé" » (Wittenberg, Sœur Sainte-Jeanne-de-France, & Lemay, 1963, p. 91) des mathématiques, résultant d'entreprises de réforme enfermées « dans une surprenante et naïve conviction que c'est assez, dans ce domaine, de méditer la pensée d'un seul homme (multiplement réincarné, il est vrai), et que : *qui a lu Bourbaki, a tout lu* » (p. 11). Aux dires de Wittenberg,

> *l'enseignement génétique, s'il sacrifie forcément nombre de notions et de chapitres afin d'avoir les coudées franches pour approfondir patiemment ceux auxquels il se consacre, représente en revanche une pratique incessante du muscle [intellectuel] mathématique.* (p. 91)

L'enseignant qui aura vécu une approche génétique aura eu l'occasion de véritablement apprendre à penser.

> *Il faut apprendre au maître ou au futur maître à voir les mathématiques élémentaires avec des yeux neufs—c'est-à-dire à les voir comme les verra son élève. Il faut l'amener à réfléchir par lui-même sur leur structure interne, sur l'enchaînement des idées, sur les raisons qu'il y a de penser d'une façon plutôt que d'une autre. De ces mathématiques, il doit faire une expérience renouvelée—une expérience informée, non pas formée, par ce que les mathématiques supérieures lui auront appris.* (p. 13)

Je ne prétendrai certes pas avoir moi-même pratiqué l'enseignement génétique tel que l'entend Wittenberg. Mais je souscris volontiers aux objectifs qu'il se donne en vue de la formation des futurs enseignants.

Je souligne au passage que lors de mon tout premier plongeon dans l'enseignement de l'arithmétique et de la géométrie aux futurs enseignants du primaire, je me suis dans une large mesure reposé sur du matériel développé par des collègues avant mon arrivée au département. Petit à petit, j'en suis venu à apporter une contribution plus personnelle. Je me suis alors abreuvé abondamment à diverses sources—au premier chef figurent ici les contacts et échanges si riches que j'ai pu avoir lors des rencontres du GCEDM. J'ai été aussi un lecteur assidu de nombreuses revues (d'ici et d'ailleurs) destinées aux enseignants tant du secondaire que du primaire, afin de mieux « sentir » les démarches mathématiques susceptibles d'être proposées à des élèves du primaire et d'aider mes étudiants à s'y préparer : ces lectures m'ont servi d'inspiration pour plusieurs des activités et problèmes que j'ai introduits au fil des ans dans mes cours. (On pourra consulter Hodgson (1985) pour des exemples de résolution de problèmes destinés à des enseignants du primaire, et Hodgson (2004) pour une discussion à propos d'une brochette de problèmes que j'ai utilisés en formation des enseignants, tantôt du primaire tantôt du secondaire.) Certains livres m'ont aussi été utiles afin de mieux structurer ma vision d'ensemble de la matière à présenter aux futurs maîtres. J'en mentionne deux seulement, parmi une flopée que j'ai alors utilisés : Bell, Fuson, et Lesh (1976) du côté arithmétique, et O'Daffer et Clemens (1992) pour le volet géométrie. Toutes ces sources m'ont grandement inspiré et influencé quant à la façon dont j'en suis venu à aborder, en tant que mathématicien, la formation des enseignants du primaire.

Remarque : En lien avec le rôle important joué par le mouvement des « maths modernes » dans la réforme des programmes d'enseignement des années 1960 mise en place dans divers pays, on trouvera dans Mura (2003) à la fois un témoignage personnel quant à la satisfaction éprouvée au contact de certains ingrédients de cette vision des mathématiques, mais aussi un rappel des principaux éléments expliquant qu'au plan didactique, ce mouvement fut un échec.

ARITHMÉTIQUE POUR L'ENSEIGNEMENT AU PRIMAIRE

Le cours MAT-1905 *Arithmétique pour l'enseignement au préscolaire/primaire* est normalement suivi au tout premier trimestre du programme de BÉPEP. Certains étudiants nous avouent avoir ressenti un certain choc en réalisant qu'un tel cours de *mathématiques (!)* était obligatoire dans leur parcours de futurs enseignants du primaire….

J'ai déjà eu l'occasion de présenter et commenter quelques ingrédients de ce cours—voir Hodgson et Lajoie (2015) et Hodgson (2017). J'en reprends ici la trame principale tout en faisant ressortir certains aspects-clés. La démarche du cours s'appuie sur des notes de cours que j'ai rédigées en collaboration avec Linda Lessard (Hodgson & Lessard, 2002).

Une vue d'ensemble du cours d'arithmétique

J'ai mentionné plus haut l'optique de reconstruction personnelle de l'arithmétique élémentaire que nous proposons à l'étudiant dans ce cours. Par une exploration systématique de la numération, des différents ensembles de nombres, d'opérations qu'on y effectue ainsi que de leurs principales propriétés, le cours vise à la fois la consolidation de notions arithmétiques déjà connues et l'acquisition de concepts nouveaux afin d'aider le futur enseignant à se préparer à intervenir auprès des élèves du primaire.

Partant d'une définition explicite et constructive du nombre naturel et des opérations d'addition et de multiplication—on s'appuie alors sur la notion de suite de « bâtons » dont il sera question plus bas—on dégage progressivement les lois fondamentales régissant ces dernières. L'accent mis sur l'apprentissage de ces lois prend sa source dans le fait qu'elles sont presque toutes conservées à travers les extensions successives de notre système de nombres : passage de $\mathbb{N}$ à $\mathbb{Z}$, puis à $\mathbb{Q}$, et enfin à $\mathbb{R}$. (Une telle démarche peut être vue comme répondant à l'esprit du « principe de la permanence des formes équivalentes » énoncé par George Peacock (1791-1858) au milieu du 19^{e} siècle.) En même temps, ces lois nous permettent de comprendre le fonctionnement des divers algorithmes arithmétiques basés sur notre système de numération à valeur positionnelle, algorithmes qui constituent, il va sans dire, une partie substantielle de l'apprentissage de l'arithmétique au primaire. Nous présentons donc les divers ensembles numériques en cause (des naturels aux réels) comme répondant à des besoins à la fois théoriques, sur le plan mathématique, et pratiques quant à leurs applications.

Le cours porte sur les concepts arithmétiques de base, en insistant sur les liens qui les unissent et les propriétés dont ils jouissent. Il ne s'agit pas de faire un simple catalogue de telles propriétés, mais bien d'expliciter et de justifier leur validité. Autrement dit, nous souhaitons qu'en plus de savoir exécuter correctement divers « *comment?* » reliés à l'arithmétique élémentaire, l'étudiant soit en mesure d'expliquer les « *pourquoi?* » qui se cachent derrière les tâches à accomplir.

Le cours d'arithmétique propose également de nombreux *exercices* et *problèmes*, les premiers visant à s'assurer de la simple maîtrise technique des notions étudiées tandis que les seconds invitent à l'exploration de questions de nature plus ouverte—l'étudiant se retrouvant alors en situation de résolution de problèmes, approche que nous cherchons à étayer avec lui. À noter que les problèmes que nous avons choisis sont tous tirés ou inspirés de la littérature destinée aux enseignants du primaire et éventuellement utilisables avec les jeunes du primaire, bien sûr dans un contexte bien adapté.

Je veux maintenant donner une idée de l'articulation générale du cours d'arithmétique du BÉPEP en présentant certains de ses points forts.

Premiers regards d'adulte sur l'arithmétique élémentaire

Il nous tient à cœur que les étudiants plongent « pour vrai » dans le cours d'entrée de jeu. À cette fin une double démarche est lancée dès la première semaine du cours.

D'une part nous invitons les étudiants à une réflexion sur le « phénomène mathématique » vu dans sa globalité. Pour ce faire, un court texte leur est remis—dont ils doivent faire une lecture personnelle—où l'essence du *processus mathématisant* est décrite en utilisant comme cas de figure le rôle joué par les nombres naturels en tant que *modèles* de diverses situations concrètes. On en vient à dégager un schéma (Figure 1) illustrant les liens entre la réalité (physique) et le monde (abstrait) des mathématiques. Le processus de modélisation mathématique—et son pendant, l'interprétation—sont au cœur du va-et-vient entre ces deux « univers ». (La forme du diagramme de la Figure 1 reprend en partie des éléments se retrouvant dans Davis et Hersh (1981/1985, p. 124).)

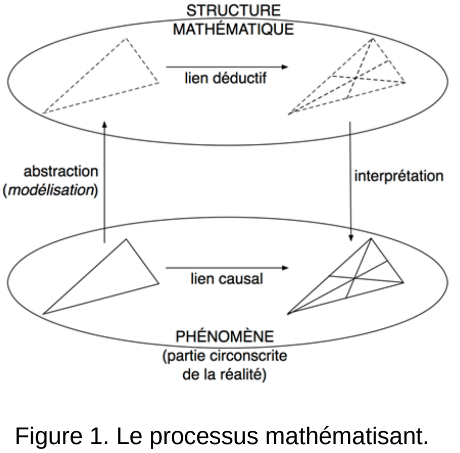


Figure 1. Le processus mathématisant.

Simultanément, de manière à provoquer une immersion immédiate des étudiants dans une démarche arithmétique signifiante, nous leur donnons dès le tout premier jour du cours deux tâches spécifiques à réaliser avant le cours suivant.

La première de ces tâches consiste à effectuer des calculs arithmétiques élémentaires (+, −, ×, ÷) dans une base autre que dix — disons, en base huit. Bien sûr nos étudiants savent exécuter les « quatre opérations » en base dix, mais souvent sous forme de simples automatismes : pouvoir expliquer les gestes qu'ils posent, voire les justifier clairement, est une autre histoire…. Après une présentation sommaire en classe de l'idée de système de numération « comme le nôtre » mais reposant sur la base huit, nous présentons la tâche, tout en encourageant les étudiants, au cours de leur démarche, à rester constamment dans la base indiquée, sans « tricher » par exemple en faisant intervenir des informations en base dix—il y va ici d'une question d'honnêteté envers les jeunes du primaire qui, dans leur apprentissage et exécution des algorithmes arithmétiques usuels, ne peuvent avoir recours à une telle « béquille ». Rapidement surgit l'importance d'avoir accès efficacement aux faits arithmétiques élémentaires : il devient donc utile d'avoir à sa disposition les tables d'addition et de multiplication en base huit, les fameuses « tables de Pythagore ».

Lors du cours suivant, on effectue un retour sur l'activité, tout en envisageant le cas d'autres bases, notamment des bases supérieures à dix. (On rend alors disponibles aux étudiants les tables pour les bases deux à neuf, onze et douze.) Nous ne leur cachons pas que l'objectif de tels calculs, effectués dans une base non décimale, est de provoquer un effet déstabilisateur afin de les ralentir un peu dans l'exécution routinière d'un algorithme, de les forcer à réfléchir au sens des gestes qu'ils posent et ainsi les aider à saisir de manière plus profonde le fonctionnement des algorithmes. Plusieurs étudiants, revenant plus tard sur cette activité au caractère quasi initiatique, en parle comme d'un moment fort dans leur cheminement en tant que futurs enseignants.

L'autre tâche lancée dès le premier cours vise à envisager les nombres naturels en tant qu'outils pour compter. À cette fin, après avoir abordé l'idée générale de *liste de comptage*—notamment à la lumière d'une chanson enfantine telle *Au clair de la lune*—nous leur demandons de construire une liste de comptage de leur choix à partir de trois symboles donnés, par exemple A, B et C. Le retour sur cette activité, lors du cours suivant, permet de mettre en relief les caractéristiques d'une « bonne » liste de comptage.

On compare alors leurs productions avec la liste de comptage résultant de la représentation des naturels en base trois, et du rôle particulier qu'y joue le chiffre 0. On fait aussi ressortir un symbole numérique comme émergeant d'un processus de regroupement—former des *rondes* de trois enfants, puis des *grandes rondes* faites de trois rondes, puis des *très grandes rondes*, etc.—chaque regroupement correspondant à une position dans le symbole. On examine également la possibilité de représenter un regroupement donné à l'aide d'un échange, par exemple par un code couleur : une ronde correspond à un jeton blanc, une grande ronde à un jeton rouge, une très grande ronde à un bleu, etc. On peut même systématiser de tels échanges basés sur la couleur, ce qu'illustre entre autres la « méthode des cargaisons » de Fernand Lemay, reposant sur une utilisation assez particulière des réglettes *Cuisenaire* (voir à ce sujet Hodgson, 2017).

Ces réflexions autour de notre système de représentation des nombres comportent aussi une brève composante historique, où il est question des principales caractéristiques de la numération romaine, égyptienne, mésopotamienne ou chinoise. Un aspect crucial pour nous est que nos étudiants aient vraiment à l'esprit la signification générale d'un symbole numérique donné, quel que soit le système de numération en cause. De manière particulière, bien sûr, on s'intéresse ici au lien entre un symbole numérique usuel en base dix, par exemple 12 345, et son développement décimal :

$$1\times10^4+2\times10^3+3\times10^2+4\times10^1+5\times10^0$$

Certains des exercices reliés à cette partie de matière portent sur l'exécution de changements de base, par exemple passer à la représentation en base onze d'un nombre donné en base six. Dans certains cas, le passage par l'intermédiaire de la base dix est permis, parfois on demande d'aller directement d'une base à l'autre. Il est intéressant de noter que laissés à eux-mêmes pour effectuer de tels changements de base, les étudiants mettent en œuvre spontanément l'une ou l'autre des trois méthodes classiques : la *division* par la base cible (effectuée dans la base donnée), l'*évaluation* (effectuée dans la base cible), ou l'*épuisement* par les puissances de la base cible (effectué dans la base donnée). Sous le volet problèmes, deux exemples reliés au thème de la numération (décimale) sont les suivants : trouver le nombre de pages d'un livre, sachant que sa pagination a nécessité précisément 6065 caractères (ou chiffres); trouver le nombre de fois où l'on écrit le chiffre 7, si on écrit tous les nombres de 1 à 9999.

Vers une définition du nombre naturel

Une source potentielle de confusion, s'agissant d'arithmétique élémentaire destinée à des adultes, serait de se restreindre à voir les nombre naturels comme étant tout simplement la « bonne vieille » liste (en base dix) que tout le monde connaît :

0, 1, 2, …, 9, 10, 11, …, 19, 20, 21, …, 30, …, 98, 99, 100, 101, …, 999, 1000, …

Ou encore de ramener la commutativité de l'addition au fait qu'on obtient la même somme lorsqu'on effectue les deux additions suivantes (à l'aide d'un algorithme approprié, par calcul mental ou écrit, voire en pitonnant sur la calculatrice) : $345+67$ et $67+345$.

Bien sûr, il y a une part de vrai dans ces propos. Mais ce n'est pas sur de telles considérations, nous semble-t-il, que doit s'appuyer une véritable réflexion sur les enjeux fondamentaux de l'arithmétique élémentaire : une telle vision présuppose de fait beaucoup quant à la nature des nombres naturels, comme s'ils surgissaient spontanément avec un schème de représentation donné et des opérations à l'avenant.

Bref, il nous semble souhaitable, dans notre approche au cours d'arithmétique, de ne pas tenir les nombres naturels *a priori* pour acquis : nous cherchons donc à les introduire correctement, dans le sens où nous voulons en donner une *définition* en bonne et due forme.

Fondements ensemblistes de l'arithmétique

La première version du cours, créée au début des années 1970—heures de gloire des fameuses maths modernes—était tout à fait dans l'esprit du temps : les ensembles étant vus comme le concept de base par excellence, c'est donc sur cette assise qu'il fallait faire reposer les nombres naturels et leur arithmétique. Cette vision est bien cernée dans la citation suivante, tirée du rapport d'une rencontre soutenue par l'ICMI (*Commission internationale de l'enseignement mathématique*) et organisée par l'UNESCO en 1971, dans un chapitre portant sur « La mathématique dans l'enseignement élémentaire » :

> *L'introduction des ensembles dans l'enseignement de la mathématique à l'école élémentaire est sans conteste l'un des traits les plus visibles des changements actuels. (…) Les ensembles sont aujourd'hui utilisés par tous les promoteurs de la réforme. (…) La tendance universelle est à l'utilisation des ensembles pour le développement du concept de cardinaux ou ordinaux et des quatre opérations élémentaires sur les naturels.* (UNESCO, 1973, p. 6)

Lors des premières années où le cours était donné, le nombre naturel était donc vu comme la cardinalité d'un ensemble fini, et les opérations arithmétiques conséquemment définies en prenant appui sur des opérations ensemblistes. Mais les ensembles et leurs opérations sont-ils vraiment plus « primitifs » que les nombres et les opérations arithmétiques élémentaires ? Même si une approche ensembliste pouvait avoir certains attraits psychopédagogiques, et était de surcroît au goût du jour, nous en sommes vite venus à la conclusion qu'elle était accompagnée d'une certaine lourdeur et était loin d'être optimale pour nos étudiants. (Soulignons au passage que la notion d'ensemble est restée présente dans la version actuelle du cours, mais essentiellement comme simple outil « linguistique » visant à faciliter la communication.)

Une vision « primitive » du nombre naturel

Revenant au nombre naturel comme servant au comptage, et prenant inspiration notamment dans des méthodes de comptabilité remontant à la préhistoire, on en vient à associer un nombre naturel donné à des marques que l'on ferait en dénombrant une certaine population : il s'agit de la technique de l'*entaille*, vieille « d'au moins quarante millénaires » (Ifrah, 1994, p. 161).

Nous concevons donc l'existence d'une « unité » abstraite, appelée (dans le cadre du cours) le *bâton*, à partir de laquelle peuvent être engendrés par répétition les autres nombres. Un nombre naturel est donc une suite de bâtons—suite *finie*, comme il va de soi. On accepte aussi qu'une telle suite de bâtons soit éventuellement vide, ce qui ne pose pas vraiment de problème conceptuel pour les étudiants et permet directement l'introduction de 0, d'importance primordiale quand l'addition entrera en jeu. L'ensemble $\mathbb{N}$ des nombres naturels est donc vu comme comprenant toutes les suites finies de bâtons, ce qui nous donne une définition en bonne et due forme à partir de laquelle travailler avec nos étudiants.

On peut noter au passage que si cette vision « unaire » du nombre naturel a un côté ancien et primitif, elle a aussi un volet moderne car elle se retrouve dans la modélisation du concept de calculabilité via les machines de Turing (Kleene, 1952, p. 359). Elle a également une saveur didactique, étant par exemple au cœur de la réflexion proposée par (Wittmann, 1975, p. 60) sur l'enseignement et l'apprentissage des nombres naturels.

Tel que défini, le nombre est un concept abstrait. Mais nous avons évidemment besoin de le représenter symboliquement, de l'écrire. À cette fin, le bâton-unité est représenté par un trait vertical « | » —que l'on pourrait nommer lui aussi (un peu abusivement) le *bâton*. Mettant de côté la suite vide de bâtons (un symbole spécial tel un triangle inversé pourrait être introduit à cette fin), voici donc la représentation des premiers membres de la suite (infinie) des nombres naturels :

$$| \qquad || \qquad ||| \qquad |||| \qquad ||||| \qquad |||||| \qquad \cdots$$

Afin de pouvoir s'exprimer avec un certain degré de généralité, un symbolisme tel

$$\overset{a}{\overline{||\cdots||}}$$

est introduit dans le cours afin de représenter une suite finie de bâtons de longueur arbitraire, a.

Un support géométrique utile

La liste de suites de bâtons qui précède représente de manière non équivoque l'essence de l'ensemble (infini) des nombres naturels (non nuls) —ou à tout le moins un fragment de cet ensemble. Mais il devient rapidement utile d'introduire une autre méthode de visualisation fort évocatrice du nombre, de nature géométrique et davantage maniable : la *droite numérique* (*naturelle* en l'occurrence). Il s'agit alors de structurer une droite donnée en désignant certains de ses points comment ayant des « étiquettes » numériques associées à la suite des nombres naturels. Un tel support géométrique est aisément conçu en se dotant d'un « segment-unité » dont les deux extrémités sont vues comme correspondant respectivement à 0 et à 1, et en reproduisant successivement cette unité à partir du point nommé 1.

Ce modèle des nombres naturels sera particulièrement fertile en vue d'illustrer des relations et opérations dans $\mathbb{N}$, de même que lorsqu'il s'agira d'enrichir notre cadre numérique à des ensembles plus vastes de nombres.

Les bases de l'arithmétique élémentaire

Nous avons maintenant à notre disposition tout l'appareillage nécessaire pour lancer sur des bases solides l'arithmétique des nombres naturels. On notera en particulier, dans les commentaires qui suivent, que tous les concepts introduits sont de fait *définis* et leurs propriétés, bel et bien *démontrées* (et non seulement observées comme « tombant du ciel »).

Ainsi en est-il de l'*égalité* de deux nombres naturels, qui revient au fait que les suites de bâtons déterminant ces deux nombres peuvent être mises en correspondance terme à terme. Cette dernière notion est facilement acceptée comme étant tout à fait naturelle et ne requérant aucun fondement ensembliste. On peut aussi voir l'*ordre* (strict) entre deux naturels comme revenant au fait qu'une suite de bâtons s'épuise avant l'autre, dans une recherche de correspondance terme à terme.

L'opération d'*addition* de deux naturels est aisément introduite, correspondant tout simplement à la juxtaposition des deux suites données :

$$\overset{a+b}{\overline{||\cdots||}} \;=\; \overset{a}{\overline{||\cdots||}}\;\overset{b}{\overline{||\cdots||}}$$

On voit immédiatement que la somme $a + b$ est bien un nombre naturel. De même façon, la *multiplication* de deux naturels a et b peut être vue comme le résultat du remplacement de chaque bâton de la suite pour a par une réplique de la suite pour b. Il peut être commode, au plan conceptuel, de visualiser la suite de bâtons ainsi obtenue

$$\overset{a\times b}{\overline{||\cdots||}}$$

sous forme d'un arrangement rectangulaire (ou matrice) de bâtons :

$$\begin{array}{c|c} & \overset{b}{\overline{||\cdots||}} \\ a & ||\cdots|| \\ & \vdots \\ & ||\cdots|| \end{array}$$

La définition du nombre naturel en termes de suite de bâtons—et le symbolisme écrit qui l'accompagne—donnent immédiatement accès aux propriétés de base de l'arithmétique élémentaire. Ainsi la *commutativité de l'addition*, correspondant à l'égalité $a + b = b + a$, se démontre en vérifiant que l'ordre de juxtaposition des suites de bâtons n'importe pas—comme le montre par exemple la correspondance terme à terme associant le bâton le plus à la droite de b, dans la suite $a + b$, au bâton le plus à la gauche de b dans $b + a$, et ainsi de suite. Les deux suites de bâtons pour $a + b$ et $b + a$ s'épuiseront ainsi simultanément.

De la même manière, toutes les « lois » fondamentales concernant l'addition et la multiplication de naturels peuvent être démontrées à l'aide de raisonnements s'appuyant sur les suites de bâtons. On en vient ainsi à un tableau tel celui de la Figure 2, résumant les « règles de jeu » à propos des opérations élémentaires dans l'ensemble $\mathbb{N}$ des nombres naturels.

A0 : 0 EST LE SEUL NON-SUCCESSEUR	**M0** : 0 EST ABSORBANT $a \times 0 = 0$ et $0 \times a = 0$
A1 : 0 EST NEUTRE + $a + 0 = a$ et $0 + a = a$	**M1** : 1 EST NEUTRE × $a \times 1 = a$ et $1 \times a = a$
A2 : COMMUTATIVITÉ + $a + b = b + a$	**M2** : COMMUTATIVITÉ × $a \times b = b \times a$
A3 : ASSOCIATIVITÉ + $a + (b + c) = (a + b) + c$	**M3** : ASSOCIATIVITÉ × $a \times (b \times c) = (a \times b) \times c$
A4 : COMPATIBILITÉ DE = AVEC + $b = c \;\Rightarrow\; a + b = a + c$	**M4** : COMPATIBILITÉ DE = AVEC × $b = c \;\Rightarrow\; a \times b = a \times c$
A5 : SIMPLIFICATION POUR + $a + b = a + c \;\Rightarrow\; b = c$	**M5** : SIMPLIFICATION POUR × $[a \neq 0 \text{ et } a \times b = a \times c] \;\Rightarrow\; b = c$
	M6 : DISTRIBUTIVITÉ ×/+ $a \times (b + c) = a \times b + a \times c$

Figure 2. Les propriétés de base de l'arithmétique.

Je souhaite insister à nouveau sur le fait que ces règles sont dûment justifiées à l'aide d'arguments qui nous paraissent tout à fait appropriés pour des enseignants du primaire, et non pas observées sur des cas concrets, voire tout simplement imposées.

Parmi les exercices portant sur cette partie de la matière figure le fait de pouvoir analyser un calcul ultra-détaillé en cherchant à identifier une à une les nombreuses applications implicites de ces propriétés que recèle tout calcul, si simple soit-il (voir la Figure 3, en lien avec l'évaluation du produit 23 × 15). Quant aux problèmes, ils comprennent notamment l'examen d'algorithmes pour la multiplication ayant une importance historique, tels l'algorithme de la jalousie, l'algorithme égyptien ou celui du paysan russe.

$$
\begin{aligned}
23 \times 15 &= (2 \times 10 + 3) \times (1 \times 10 + 5) && (1)\\
&= ((2 \times 10 + 3) \times (1 \times 10)) + ((2 \times 10 + 3) \times 5) && (2)\\
&= ((2 \times 10) \times (1 \times 10) + 3 \times (1 \times 10)) + && (3)\\
&\qquad\quad ((2 \times 10) \times 5 + 3 \times 5) &&\\
&= ((2 \times 10) \times 10 + 3 \times 10) + ((2 \times 10) \times 5 + 3 \times 5) && (4)\\
&= ((2 \times 10) \times 10 + 3 \times 10) + (5 \times (2 \times 10) + 3 \times 5) && (5)\\
&= (2 \times 10 \times 10 + 3 \times 10) + (5 \times (2 \times 10) + 3 \times 5) && (6)\\
&= (2 \times 10 \times 10 + 3 \times 10) + ((5 \times 2) \times 10 + 3 \times 5) && (7)\\
&= (2 \times 10 \times 10 + 3 \times 10) + (10 \times 10 + 15) && (8)\\
&= (2 \times 10^2 + 3 \times 10) + (10^2 + 15) && (9)\\
&= (2 \times 10^2 + 3 \times 10) + (1 \times 10^2 + 15) && (10)
\end{aligned}
$$

Figure 3. Fragment initial du calcul de 23 × 15 (version détaillée).

Nul besoin d'insister sur le fait qu'il n'est pas question de demander à nos étudiants de construire par eux-mêmes des manipulations comme celles de la Figure 3, et encore moins de suggérer qu'ils adoptent une telle approche avec leurs élèves ! Mais nous pensons que de voir (au moins une fois dans sa vie !) ces propriétés en action de manière explicite peut être un élément d'apprentissage pertinent afin de bien saisir l'importance et le rôle des propriétés en cause.

On observera que les relations d'égalité et d'ordre, de même que les opérations d'addition et de multiplication, s'interprètent immédiatement sur la droite numérique. À ce stade-ci du cours, nous tenons donc pour acquis que nos étudiants sont à l'aise avec une triple vision de l'ensemble $\mathbb{N}$ (et des opérations qu'on y exécute) : les naturels vus comme des suites de bâtons ; les naturels vus comme des points sur une droite numérique ; et les naturels tels que représentés dans un certain système de numération (principalement, notre système à valeur positionnelle de base dix).

Un commentaire d'ordre épistémologique

L'approche aux fondements de l'arithmétique élémentaire que nous proposons à nos étudiants se veut donc dans une perspective structuraliste : nous souhaitons mettre en évidence de manière explicite l'idée de *structure* sous-jacente à cette arithmétique. Qui plus est, nous souhaitons que cela se fasse dans un contexte où les systèmes numériques en jeu sont bel et bien définis sur des bases intuitivement adéquates, et leurs propriétés, démontrées dans un cadre argumentatif approprié pour nos étudiants. Mais il ne s'agit pas pour autant de faire de la notion de structure un objet d'étude en soi, ni de lui accorder toute la place. Bref, le concept de structure ne joue pas le rôle qu'on a pu lui prêter dans le cadre de certaines démarches typiques des maths modernes—voir par exemple Ziglon (1971). Les structures sont présentes dans le cours, mais en tant que paradigmes en vue de bien cerner certains cadres mathématiques donnés et les principales propriétés qui y interviennent.

Bien sûr de nombreuses questions d'ordre didactique pourraient être soulevées en lien avec la vision que nous proposons—voir à ce sujet Hodgson et Lajoie (2015). Mais la démarche au cœur de notre cours d'arithmétique se situe vraiment dans une perspective mathématique, afin de favoriser chez l'étudiant le développement d'une vision cohérente de l'arithmétique élémentaire. D'autres choix mathématiques seraient bien sûr ici possibles, tel que le commente notamment Klein (1924/1932, pp. 10-13). Mais l'expérience suggère que l'approche que nous avons développée offre une solution plausible et pertinente en vue de la préparation à l'enseignement primaire.

On aura noté que la notion de principe de récurrence—voire l'axiomatique de Peano en tant que telle—est présente implicitement dans les propos qui précèdent. Nous nous bornons dans le cours à une très brève allusion à une telle perspective, mais sans en faire un objectif d'apprentissage en tant que tel.

Autres aspects de l'arithmétique des nombres naturels

L'introduction des nombres naturels et des opérations arithmétiques de base, tels que cristallisés dans les propriétés de la Figure 2, couvre environ les trois premières semaines du cours. Si j'ai mis un tel accent sur ces éléments dans le présent texte, c'était afin de bien faire ressortir la « philosophie » qui sous-tend notre démarche, la manière d'appréhender le savoir mathématique avec lequel nous souhaitons mettre nos étudiants en contact. Je veux maintenant indiquer l'approche que nous proposons vers d'autres aspects de l'arithmétique dans $\mathbb{N}$, tout d'abord en ce qui concerne la soustraction et la division.

Étant donné deux naturels a et b, les opérations d'addition et de multiplication peuvent être vues comme apportant une réponse aux questions : $a + b = ?$ et $a \times b = ?$.

Dans le même esprit, on peut voir le problème de la soustraction (dans $\mathbb{N}$) comme la recherche d'un nombre naturel satisfaisant à l'égalité : $a + ? = b$; et le problème de la division comme correspondant à : $a \times ? = b$. À noter que dans ce dernier cas, on impose, comme il va de soi, que a doit être non nul.

Une telle vision, reliant explicitement la soustraction à l'addition et la division à la multiplication, présente l'avantage de mettre à notre disposition, dans les diverses manœuvres à exécuter, les propriétés des deux opérations de base. Contrairement à ce qui se passe avec l'addition et de la multiplication, ces problèmes ne trouvent pas toujours de réponse dans $\mathbb{N}$. Mais lorsque la réponse existe, elle est unique !

La soustraction dans $\mathbb{N}$

Dans le cas de la soustraction, *la différence de b et a* (lorsqu'elle existe) est donc le naturel, dénoté $b - a$, tel que $a + (b - a) = b$. Autrement dit, par définition même, l'égalité $c = b - a$ revient au fait que $a + c = b$. La relation d'ordre entre deux naturels a et b est directement reliée à l'existence ou non de la différence $b - a$. La droite numérique naturelle fournit un cadre propice pour illustrer le problème de la soustraction et interpréter le sens de $b - a$ (lorsque cette différence existe).

S'appuyant sur les définitions mêmes ainsi que sur les propriétés de la Figure 2, on peut alors démontrer les principales propriétés de l'ordre : transitivité, compatibilité avec l'addition, etc. On peut aussi établir que la multiplication se distribue sur la soustraction, et qu'on a une espèce d'« associativité +, – » : $(a + b) - c = a + (b - c)$ (la soustraction n'est bien sûr ni associative, ni commutative). De même on peut directement démontrer, sur la base des propriétés de la Figure 2, la validité de la fameuse « règles des signes » :

$$a - (b + c) = (a - b) - c \quad \text{et} \quad a - (b - c) = (a - b) + c.$$

On observe au passage que lorsqu'il s'agit d'évaluer une soustraction donnée, notre système de numération positionnelle se prête à de nombreux algorithmes, certains un peu étranges, ce qui ne manque pas d'étonner plusieurs étudiants.

La division dans $\mathbb{N}$

Une démarche semblable peut être faite en lien avec la division. Le quotient de b et a (lorsqu'il existe) est le naturel $b \div a$ tel que $a \times (b \div a) = b$ (cette situation se visualise aisément sur la droite numérique).

Lorsque le quotient $b \div a$ existe dans $\mathbb{N}$, nous sommes donc dans le cas où on peut exhiber un naturel c tel que $a \times c = b$. Autrement dit, a est un *diviseur* (ou *facteur*) de b, et b est un *multiple* de a : il s'agit donc ici de la relation de *divisibilité* dans $\mathbb{N}$, notée $a \mid b$. Cette relation, qui mérite qu'on s'arrête à ses principales propriétés (transitivité, comportement par rapport aux autres opérations), mène directement aux notions de PGCD et de PPCM, bien présentes dans les mathématiques du primaire.

Même lorsque le quotient de b et a n'existe pas comme tel dans $\mathbb{N}$, il est possible de mettre en place une autre vision de la division au sein des nombres naturels, où l'on accepte qu'un certain partage puisse donner lieu à un « reste » : il s'agit alors de la *division euclidienne*, si fondamentale dans l'enseignement primaire. Il n'est pas question dans notre cours de démontrer l'existence et l'unicité du quotient et du reste dans une division euclidienne : nous acceptons cela comme un fait allant de soi et s'illustrant fort bien sur la droite numérique.

La division euclidienne occupe une place importante dans notre cours, notamment en lien avec son utilisation dans l'*algorithme d'Euclide* pour le calcul d'un PGCD. Il nous paraît important de présenter cet algorithme aux étudiants : il s'agit d'un algorithme inconnu de la majorité d'entre eux et nous voyons un tel apprentissage d'un nouvel algorithme portant sur les nombres naturels comme une expérience personnelle hautement pertinente, pour un futur enseignant qui aura à aider ses élèves dans la maîtrise des algorithmes arithmétiques élémentaires. De plus, la recherche d'un PGCD à l'aide de l'algorithme d'Euclide, en « remontant la chaîne » de calculs, peut être reliée l'*identité de Bézout* permettant d'exprimer un PGCD comme une différence de multiples de deux naturels donnés. On est ainsi amené directement vers des problèmes de type « transvidage », bien présents au primaire, où il s'agit par exemple d'obtenir une certaine quantité d'eau à l'aide de contenants de volume déterminé. Bézout ouvre la porte à des stratégies permettant d'aller au-delà du simple tâtonnement.

Plusieurs étudiants manifestent une réelle satisfaction de connaître l'algorithme d'Euclide et de savoir expliquer pourquoi il fonctionne, tout en appréciant son efficacité en pratique. Signalons à cet égard que nous imposons dans le cours un modèle de calculatrice conçu en vue de l'enseignement primaire et muni, en plus de la touche usuelle de division (avec la partie fractionnaire sous forme de développement décimal), d'une touche de division euclidienne fournissant directement le quotient (entier) et le reste d'une division donnée. L'emploi de cette calculatrice a rendu cette partie du cours davantage porteuse, car les étudiants n'ont plus à consacrer d'énergie à trouver le quotient et le reste pour deux naturels donnés, même lorsqu'ils sont assez grands, et peuvent ainsi se concentrer sur l'exécution de l'algorithme en tant que tel.

Parmi les problèmes que nous proposons aux étudiants en lien avec le thème de la divisibilité, l'un des plus féconds est celui qui, dans notre folklore local, porte le nom de *L'Hôtel Longpré*—mais il se retrouve dans la littérature sous diverses appellations. La scène se passe dans un interminable corridor d'hôtel, le long duquel sont situées n portes à la queue leu leu. Les n hôtes de ces chambres décident de se livrer à un jeu du type « ouvrons/fermons les portes », l'hôte #k changeant l'état de chaque k-ième porte (en commençant à la porte #k)—les portes sont toutes fermées au début de l'histoire. Le fait qu'à la fin du processus, une porte donnée se retrouvera ouverte ou non selon la parité du nombre de ses diviseurs donne lieu à une belle exploration portant sur une situation mathématique non banale, mais tout à fait appropriée pour le primaire. (Une variante de ce problème, mais se déroulant cette fois à *L'Hôtel Rondeau*, de forme circulaire, est discutée dans Cassidy et Hodgson (1982). Cette version s'adresse cependant davantage à l'enseignement secondaire.)

Nombres premiers et factorisation première

Le thème suivant abordé dans le cours d'arithmétique concerne les *nombres premiers*. Après un bref regard sur la notion générale de nombre figuré, l'attention se porte sur les nombres qui, tel 7, ne peuvent donner lieu, géométriquement parlant, qu'à un simple arrangement linéaire de points—tandis qu'avec 36 points, par exemple, on peut obtenir des formes géométriques plus variées : ligne, triangle, rectangle(s), carré. La notion intuitive de premier ne pose évidemment aucun problème pour l'étudiant, mais il faut quand même la définir correctement (de manière à exclure 1, tout en indiquant pourquoi il « doit » en être ainsi). L'identification des premiers jusqu'à une limite donnée peut se faire par un criblage à la Ératosthène.

Or un élément de folklore mathématique qui semble relativement peu connu parmi les enseignants du primaire—voir Hodgson (1986) —est qu'en disposant les naturels à cribler sur six colonnes, on se retrouve dans un cadre de travail fort fructueux. En effet, non seulement le criblage est plus simple à effectuer, mais de plus une telle répartition mène à une observation jolie et non banale : tout premier supérieur à 3 est un « voisin » d'un multiple de 6. La figure résultant du processus de criblage constitue en elle-même une preuve de ce résultat, ce qui fournit l'occasion d'une discussion féconde avec nos étudiants autour de la notion de preuve—notamment en lien avec son rôle au primaire dans une perspective argumentative—ainsi que sur la place des preuves visuelles dans la classe de maths. Bien sûr, s'agissant dudit crible, on « savait » qu'en disposant les naturels sur six colonnes, on obtiendrait un cadre intéressant, puisqu'on peut aisément démontrer par des arguments arithmétiques élémentaires que tout premier supérieur à 3 est de la forme $6k \pm 1$ (arguments que nous présentons à nos étudiants).

La notion de factorisation première d'un naturel est directement mise en lien avec les questions de divisibilité, ce qui fournit un contexte utile pour revisiter les notions de PGCD et de PPCM.

L'étude des nombres premiers se termine par la démonstration de deux résultats théoriques de base : l'unicité de la factorisation première d'un naturel, et l'infinitude des nombres premiers (par un argument à la Euclide). J'aimerais souligner à ce propos, sans aucunement prétendre que nos étudiants sont tous emballés par des telles preuves, que nous pouvons ressentir chez plusieurs d'entre eux une véritable satisfaction à suivre la finesse du développement de ces arguments, même si on est alors sans doute un peu loin de la classe du primaire. Je dirais que plusieurs y voient de notre part comme une marque de respect intellectuel envers eux : il n'est pas du tout question alors d'assimiler des « trucs » pour se débrouiller face à diverses manipulations, mais bien de viser à une compréhension profonde de certains objets mathématiques, de leur essence même.

Un des problèmes que nous proposons, en lien avec la factorisation première est un « tour de magie » numérique : pourquoi, en répétant un nombre de trois chiffres (par exemple 493 devient 493 493), obtient-on un nombre de six chiffres divisibles à la fois par 7, par 11 et par 13? Un autre demande de trouver le nombre de zéros qui se retrouvent à la queue d'un nombre obtenu à l'aide de la factorielle, disons 75!.

L'« arithmétique de l'horloge »

Il est présentement 15 h 30 ; quelle heure sera-t-il dans 1000 heures ? Ou encore : *C'est aujourd'hui mardi ; quel jour de la semaine serons-nous dans 18 jours ?* De telles questions, reliées à une espèce d'arithmétique du quotidien, mettent en relief le fait que dans certains contextes de division euclidienne, le reste a parfois une importance plus grande que le quotient. La familiarité avec le comportement arithmétique de l'horloge—de 12 ou de 24 heures—est un apprentissage de base pour l'enfant, ce qui met en lumière l'intérêt, de manière plus générale, de l'arithmétique modulaire en lien avec la préparation à l'enseignement primaire.

L'horloge usuelle de 12 heures se transforme aisément, en remplaçant le « 12 » par « 0 », en une horloge *modulo* 12. Et on peut alors généraliser à un module m quelconque—par exemple $m = 7$ (Figure 4) lorsqu'il s'agit des jours de la semaine.

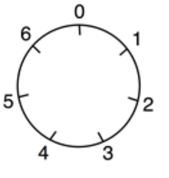


Figure 4. Une « horloge » modulo 7.

On peut aussi, dans le même registre, s'intéresser à l'utilisation d'une comptine en vue de désigner l'enfant qui jouera un rôle particulier dans le cadre d'un jeu :

Am, stram, gram,
Pic et pic et colégram,
Bour et bour et ratatam,
Am, stram, gram.

Après cette valorisation du rôle du reste dans certains contextes, la relation de *congruence* modulo m entre deux naturels a et b est définie en bonne et due forme par le truchement de l'égalité des restes obtenus dans la division de a et de b par m. Nous ne mettons pas alors l'accent sur le rôle des classes d'équivalence (un aspect fondamental lorsque c'est la notion même d'*équivalence modulaire* que l'on souhaite faire ressortir), mais plutôt sur l'*arithmétique* *des restes* qui peut être développée dans un tel contexte. On en dégage ainsi les règles d'un calcul nouveau, où on vérifie que le reste de la somme de deux naturels, dans la division par m, est égal à la somme des restes de chacun de ces naturels (au besoin réduisant à nouveau modulo m) —et de même pour le produit.

L'application principale de ces notions concerne un thème précis de l'enseignement primaire, les critères de divisibilité—bien sûr par 2, 5 ou 10, mais aussi par 3 et 9, voire 11, ainsi que par 4, 8, 25 ou 125. Nous visons à ce que les étudiants saisissent bien les principes mathématiques sur lesquels s'appuient ces critères, en particulier en lien avec le fonctionnement du système de numération décimale. Nous souhaitons aussi qu'ils y voient des activités amusantes et enrichissantes à faire avec leurs élèves. Il en est de même pour la célèbre *preuve par neuf*, qui fournit l'occasion d'une belle parenthèse historique.

Prolongements de l'arithmétique

Rendu à ce stade-ci du cours, il reste environ 4 à 5 semaines à notre disposition. L'objectif est alors de revenir sur certaines « carences » de l'ensemble des naturels sur le plan arithmétique et de voir comment il est possible d'y remédier. J'ai bien sûr ici à l'esprit le fait que $\mathbb{N}$ n'est pas fermé pour la soustraction et la division, d'où le projet de prolongement de cet ensemble à $\mathbb{Z}$ et à $\mathbb{Q}$ afin de se doter d'un cadre arithmétique plus complet. Nous ne doutons pas que pour plusieurs de nos étudiants, le calcul sur les fractions ou les nombres négatifs ne pose pas vraiment de difficulté (à tout le moins au plan algorithmique). Le point central devient alors de les amener à bien comprendre pourquoi et comment ces nouveaux nombres surgissent, et aussi de faire ressortir les principes qui sous-tendent le fonctionnement de cette arithmétique prolongée, principalement dans le cas des fractions. Le lien avec l'arithmétique des naturels, et le souci de préserver la permanence de ses propriétés de base, sont au cœur de la démarche que nous proposons. Nous voulons aussi faire clairement ressortir l'effet de ces prolongements sur la droite numérique, qui demeure encore ici un support géométrique fort utile, voire essentiel.

On se rappellera ici que nous avons cherché à cerner les nombres naturels en les voyant comme des suites finies de bâtons, ce qui nous a permis d'établir les principales propriétés qui surgissent en lien avec leur rôle dans le cadre d'un système numérique donné. Notre approche à $\mathbb{Z}$ et à $\mathbb{Q}$ est un peu différente, car mis à part leur interprétation sur la droite numérique, nous ne nous intéressons pas alors à ce que *sont* ces nouveaux nombres dans leur essence, nous concentrant plutôt sur ce qu'ils *font* (un peu dans l'esprit de la Figure 2). Un tel point de vue est explicitement soutenu par exemple par le médaillé Fields Timothy Gowers dans son ouvrage de vulgarisation (Gowers, 2002, pp. 17-28). Il n'est pas question ici de présentation de ces ensembles en tant que structures quotients, fût-ce, dans le cas de $\mathbb{Z}$, sous le joli déguisement pédagogique des « batailles de nombres rouges et bleus » de Frédérique Papy (Papy, 1970, pp. 203 *sqq*.), approche basée sur la relation de dominance—voir aussi Bell et al. (1976, pp. 108 *ssq*.).

Certains des thèmes qui suivent ont une place relativement légère dans les classes du primaire. J'ai entre autres à l'esprit, à cet égard, le fait que l'idée de nombre entier peut y être présente, mais pas vraiment les calculs sur les négatifs. Il en est de même du concept général de nombre réel. Mais il nous apparaît important de permettre au futur enseignant de se bâtir une intuition solide et une vision adéquate de tels concepts, même s'ils ne figurent pas comme objets d'apprentissage en tant que tels par leurs élèves.

L'arithmétique des entiers

Après avoir rappelé la présence des entiers négatifs dans notre quotidien (températures hivernales, bilans financiers, etc.), nous voulons examiner comment traiter de telles situations sur le plan mathématique.

Revenant au fait que la différence de deux naturels, $b - a$, n'est pas toujours définie dans $\mathbb{N}$, nous présentons comme « projet » l'identification d'un ensemble de nombres le plus simple possible englobant $\mathbb{N}$ et dans lequel tous les problèmes de soustraction auraient une solution. On demande de plus que l'arithmétique de $\mathbb{N}$ puisse être étendue à ce nouvel ensemble, dans le sens où y demeurent valides les propriétés arithmétiques déjà observées dans $\mathbb{N}$ (entendre, les « lois » telles celles de la Figure 2).

Or pensant à la différence $b - a$ comme étant reliée à l'écart entre les points a et b sur la droite numérique (naturelle), on voit que cet écart peut se réinterpréter comme concernant 0 et un certain naturel n. (On aura compris qu'on se repose ici sur l'idée de *distance*, c'est-à-dire de l'écart entre deux points de la droite numérique, pris en « valeur absolue »). Par exemple, déplaçant point par point l'écart entre les nombres 24 et 41, on aurait de la sorte

$$24 - 41 = 23 - 40 = 22 - 39 = \ldots = 1 - 18 = 0 - 17$$

Bref, l'accent est mis de la sorte sur les différences du type $0 - a$, où a est un naturel, c'est-à-dire sur les problèmes de soustraction de la forme $a + ? = 0$. On est ainsi amené à introduire l'opération d'opposition qui, à tout naturel a, associe son *opposé* ${}^{-}a$, un « nouveau nombre » satisfaisant à l'égalité $a + {}^{-}a = 0$.

Une droite numérique « étendue » (au-delà du 0) fournit une intuition robuste pour interpréter cette notion d'opposé d'un naturel donné. Et c'est l'interprétation géométrique de l'addition dans $\mathbb{N}$ qui guide ici notre intuition. Nous passons ainsi à la *droite numérique entière*.

Utilisant le symbole $\mathbb{Z}$ pour désigner l'ensemble de tous les *nombres entiers* qui en résultent, on voit que ceux ci se répartissent en trois catégories : les entiers positifs (c'est-à-dire les naturels non nuls), le nombre 0, et les entiers négatifs (qui sont les opposés des naturels).

Étendant l'opposition à tous les entiers, on en tire, en s'appuyant sur les propriétés arithmétiques de base, les deux règles fondamentales de la nouvelle arithmétique dans $\mathbb{Z}$:

$${}^{-}({}^{-}x) = x \quad \text{et} \quad x - y = x + {}^{-}y.$$

Notons en particulier que la soustraction devient ainsi tout bonnement l'addition d'un opposé.

Notre étude de l'ensemble des entiers se termine par de brefs commentaires à propos de l'ordre dans $\mathbb{Z}$, ainsi que sur la relation de divisibilité entre entiers.

Remarque : Pour le bénéfice du lecteur (peut-être) agacé par le fait que nous ayons eu recours au stratagème géométrique de la droite numérique pour motiver l'égalité $24 - 41 = 0 - 17$ (et plus généralement le passage à une différence de la forme $0 - a$, c'est-à-dire à l'opposé ${}^{-}a$), signalons que le tout pourrait se faire sur une base plus précise, formellement parlant : il s'agirait de démontrer, à l'aide des propriétés de base de l'arithmétique des nombres naturels, que si des naturels u, v et k sont tels que les trois différences $u - v$, $u - k$ et $v - k$ existent toutes dans $\mathbb{N}$, on a alors $u - v = (u - k) - (v - k)$. (Le cas particulier $k = 1$ est ici fort utile en ce qui concerne la série d'égalités qui précède : $24 - 41 = 23 - 40 = \ldots$.) Dans notre cours, une telle vision est maintenant reléguée à une activité d'enrichissement facultative et nous préférons faire reposer notre démarche sur l'intuition—somme toute raisonnable—fournie par la droite numérique.

L'arithmétique des rationnels

La démarche menant à l'ensemble des *nombres rationnels* repose sur un « projet » qui à maints égards ressemble à celui pour $\mathbb{Z}$. On revient au problème de division $a \times ? = b$ entre nombres naturels (avec $a \neq 0$). Même si le quotient $b \div a$ n'existe pas toujours dans $\mathbb{N}$, rappelons que la division euclidienne peut être vue comme apportant une « réponse » dans $\mathbb{N}$, mais exprimée en termes d'un quotient et d'un reste :

$$b = a \times q + r, \text{ où } 0 \leq r < a.$$

Ce que nous visons maintenant est de prolonger $\mathbb{N}$ (et $\mathbb{Z}$ de même) à un nouvel ensemble de nombres qui constituent les réponses à tous les problèmes de division d'entiers. Dans un tel contexte, le quotient résultant du problème $a \times ? = b$ (où a et b sont des entiers) est noté, comme le veut l'usage, sous la forme d'une *fraction* représentée à l'aide d'un *numérateur* et d'un *dénominateur*.

Nous dénotons donc par $\mathbb{Q}$ l'ensemble formé de tous les quotients d'entiers—c'est-à-dire comprenant toutes les fractions $\frac{b}{a}$ avec a et b entiers et a non nul. On a donc, par définition, que la fraction $\frac{b}{a}$ est telle que $a \times \frac{b}{a} = b$.

On établit aisément, sur une telle base, que l'égalité de deux fractions revient à l'égalité (dans $\mathbb{Z}$) des « produits croisés » des numérateurs et dénominateurs.

De manière analogue au passage de $\mathbb{N}$ à $\mathbb{Z}$, un type particulier de problèmes de division joue un rôle fondamental, à savoir les problèmes de la forme $a \times ? = 1$. On est ainsi amené à considérer le quotient $\frac{1}{a}$, appelé l'*inverse* de l'entier a. De là, toujours en analogie avec le cas de $\mathbb{Z}$, on peut, en s'appuyant sur les propriétés arithmétiques de base, établir que toute fraction est le produit d'un entier par un inverse d'entier :

$$\frac{b}{a} = b \times \frac{1}{a}$$

Dans le même esprit, appliquant l'inversion à une fraction quelconque, on montre que l'inverse de $\frac{b}{a}$ est la fraction $\frac{a}{b}$. Ces deux constats servent de guides pour la mise en place d'une « arithmétique des fractions » : le résultat de l'une quelconque des quatre opérations appliquée à deux fractions est ainsi lui-même une fraction qui se trouve complètement déterminée par les règles arithmétiques. De tels rappels, selon une perspective mathématique, à propos du comportement arithmétique des fractions nous semblent utiles en vue de préparer le terrain pour une réflexion didactique sur le sujet.

Signalons au passage que sur la calculatrice utilisée dans le cours, spécialement conçue en vue de l'enseignement primaire, se retrouvent quelques touches permettant la manipulation des fractions. Si une telle calculatrice est utile quand vient le temps de faire un calcul concret (sur des fractions pas trop compliquées), elle ne rend pas pour autant caduque—et les étudiants en sont fort conscients—la nécessité de bien posséder les principes sous-tendant l'arithmétique des fractions ainsi que le fonctionnement même des algorithmes de calcul.

L'ensemble $\mathbb{Q}$ peut s'interpréter directement sur une droite numérique encore une fois « enrichie » : *la droite numérique rationnelle*. On peut dans un premier temps s'intéresser aux rationnels entre 0 et 1—y situant tout d'abord, en vertu de l'égalité

$$a \times \frac{1}{a} = 1\,,$$

les fractions *unitaires* (c'est-à-dire les inverses d'entiers positifs), puis les fractions *propres* $\frac{b}{a}$ (avec un numérateur inférieur au dénominateur) ; le cas d'une fraction quelconque peut alors suivre par translation en faisant intervenir le quotient euclidien de *b* par *a* ainsi qu'une fraction propre—par exemple, la fraction $\frac{26}{3}$ revient à $8+\frac{2}{3}$; on passe enfin aux opposés de ces fractions. On voit ainsi tous les rationnels se répandre sur la droite numérique. Et une fois introduite la relation d'ordre entre fractions, on observe que les rationnels sont densément distribués sur la droite numérique, ce qui amène une distinction spectaculaire avec le cas de la droite numérique entière.

La droite numérique rationnelle peut servir d'amorce à une autre vision de $\mathbb{Q}$, s'appuyant cette fois sur un prolongement de la représentation des nombres dans le système de numération décimal : on en vient ainsi à la notion de développement décimal avec partie dite *fractionnaire*—c'est-à-dire à l'utilisation de puissances négatives de dix pour une fraction propre (positive). Les situations reliées à la monnaie, de même que les manipulations faites sur une calculatrice usuelle, rendent de telles notations familières.

Ce changement dans l'écriture des nombres rationnels a bien sûr un impact majeur sur l'exécution des calculs faisant intervenir les quatre opérations, et il convient de s'y arrêter. Mais par-delà ces habiletés algorithmiques, des questions à la fois jolies et importantes demeurent, sur lesquelles nous tenons à nous pencher. Par exemple, peut-on prédire à l'avance, sans faire de calcul, si le développement décimal d'une fraction $\frac{b}{a}$ donné est fini ou infini ? Et s'il est infini, que peut-on dire à propos de sa période ? (Et au fait, pourquoi doit-il forcément y avoir une telle période ?) Ou encore, comment retrouver, à partir d'un développement décimal infini et périodique, la fraction correspondante ? Autant de questions qui aident à comprendre le comportement de notre système de numération décimal ainsi que la nature même des nombres rationnels.

Coda arithmétique : les réels… et un soupçon de prob & stat

Nous en sommes à la fin du cours d'arithmétique : il ne nous reste quelques heures, dans le trimestre, pour clore notre propos. Il nous faut bien sûr regarder un instant la notion de *nombre réel*. La motivation principale pour l'ensemble $\mathbb{R}$ repose sur la droite numérique, qui a été successivement enrichie en passant de $\mathbb{N}$ à $\mathbb{Z}$, puis à $\mathbb{Q}$. Une question naturelle s'impose : tous les points de la droite de départ ont-ils maintenant reçu une « étiquette » ? La réponse, on le sait—et on le démontre dans le cours—est négative. La discussion porte alors sur la célèbre irrationalité de $\sqrt{2}$, que l'on montre à l'aide de l'une des nombreuses preuves élémentaires (par exemple la preuve « classique », ou encore la variante s'appuyant sur la factorisation première). Un autre regard sur l'irrationalité repose sur l'examen d'un développement décimal non périodique. J'aime bien alors, au passage, présenter mon irrationnel préféré, si je puis dire : 0,707007000700007…. Nous sommes alors un peu loin de l'école primaire en tant que telle, mais de telles observations sont pertinentes pour le futur enseignant, et en général fort bien reçues.

S'agissant des probabilités et de la statistique, je dois faire un aveu : en aucune façon nous ne pouvons prétendre rendre justice à ces sujets dans la version actuelle du cours. Même si les exigences officielles des programmes québécois pour l'enseignement primaire demeurent somme toute assez modestes quant à ce domaine, il s'agit néanmoins d'un mode de pensée fort important dans la vie citoyenne, et il faut y initier les élèves dès le primaire—et ce, correctement ! Nous souhaiterions fournir au futur enseignant l'occasion d'acquérir une vision renouvelée des probabilités, et surtout de la statistique, tout en rafraîchissant ses connaissances. Le défi est d'aller au-delà des propos usuels concernant par exemple les fameuses mesures statistiques (moyenne *vs* médiane, etc.), que nos étudiants ont rabâchées tout le long de leurs études secondaires. C'est donc vers une vision globale de modélisation de situations aléatoires que nous souhaiterions nous diriger, portant attention à des aspects tels l'analyse des données ou les bonnes pratiques dans la représentation graphique des données. Mais pour l'heure le temps limité que nous accordons à ce sujet nous amène à le survoler, de même que les notions de base des probabilités, et partant de combinatoire, sur lesquelles s'appuie une partie de ces réflexions. Bref, un volet à approfondir… et peut-être à transformer en une sorte d'auto-apprentissage sous forme de « travaux pratiques », comme nous le faisons depuis longtemps dans le cours de géométrie.

GÉOMÉTRIE POUR L'ENSEIGNEMENT AU PRIMAIRE

En présentant le cours d'arithmétique avec autant de détails, je souhaitais bien faire ressortir les choix que nous avons faits, au fil des ans, à propos des fondements mathématiques sur lesquels repose la démarche proposée dans le cours. À cet égard, le cours de géométrie du programme de BÉPEP est plus simple quant aux cadres conceptuels dans lesquels se déroule le cours. Mais les tâches rencontrées en géométrie demeurent néanmoins substantielles en termes de réappropriation par le futur enseignant.

Le cours MAT-1906 *Géométrie pour l'enseignement au préscolaire/primaire* est un cours obligatoire habituellement pris à l'avant-dernière année du BÉPEP. Dans la séquence des cours préparant à l'enseignement des mathématiques, les étudiants ont alors suivi le cours d'arithmétique ainsi que les deux cours de didactique portant sur ce même sujet. Il leur restera donc, après MAT-1906, à prendre un cinquième et dernier cours, portant sur la didactique de la géométrie.

La démarche du cours de géométrie d'appuie sur deux documents de notes de cours que j'ai rédigés également en collaboration avec Linda Lessard (Hodgson & Lessard, 2006a, 2006b).

Une vue d'ensemble du cours de géométrie

Tout comme le cours d'arithmétique, le cours de géométrie du BÉPEP vise à offrir à des étudiants adultes l'occasion d'une reconstruction personnelle des notions mathématiques de base en vue de l'enseignement au primaire. Par une exploration systématique de l'univers géométrique élémentaire, il vise à la fois la consolidation de concepts géométriques déjà connus et l'acquisition de notions nouvelles afin d'aider les futurs enseignants à se préparer pour leurs interventions auprès des élèves.

Le choix des thèmes abordés est fait en fonction de l'intérêt qu'ils revêtent pour l'enseignant, tant pour sa formation personnelle qu'en lien avec le programme de mathématiques pour le primaire en vigueur au Québec. La démarche suivie est de manière générale de nature inductive et intuitive—mais pas floue pour autant ! —plutôt que déductive et formelle : il s'agit avant tout de favoriser une sensibilisation à certaines facettes géométriques de la « réalité » (concrète ou abstraite) dans laquelle nous baignons, et non pas d'effectuer la construction achevée d'une somme de connaissances. Une telle approche n'entre bien sûr pas en contradiction avec le souci de se doter d'outils de communication clairs, ou encore de justifier adéquatement, de façon locale, certaines propriétés observées. Un volet du cours porte d'ailleurs explicitement sur l'idée de preuve en géométrie.

Le cours part du constat que toute la démarche géométrique au primaire prend sa source dans l'espace à trois dimensions où évoluent les enfants, par le biais de l'exploration physique (visuelle ou tactile) d'objets réels à l'aide d'actions concrètes. Cependant, il devient rapidement commode d'y substituer une exploration mentale et surtout de travailler avec des êtres géométriques habitant l'univers à deux dimensions (par exemple celui de la feuille de papier) : une telle abstraction trouve sa motivation dans un souci de clarté et de simplicité, les phénomènes géométriques devenant ainsi plus abordables. Une partie importante du cours se déroule donc dans le cadre de l'univers géométrique 2D.

Il serait par ailleurs absurde de penser acquérir le sens géométrique visé dans le cours en se confinant dans un rôle de simple « spectateur » : il faut vraiment mettre la main à la pâte afin que le processus d'apprentissage débouche sur le développement d'un modèle robuste des concepts en jeu. Le cahier *Travaux pratiques de géométrie* vise justement à faire jouer par l'étudiant un rôle actif dans une exploration personnelle de l'univers géométrique. Plusieurs outils et contextes d'exploration sont alors mis à contribution : on y revient plus bas.

L'enseignement de la géométrie étant parfois un laissé-pour-compte dans les écoles (primaires ou secondaires) du Québec, nous sommes conscients qu'il pourra arriver que le futur maître n'ait retenu de sa propre scolarisation pré-universitaire qu'un bagage relativement faible de connaissances géométriques. Le cours vise donc à contribuer à combler de telles lacunes, tout en favorisant le développement d'une attitude positive vis-à-vis de la géométrie.

<u>Étude et pratique de la géométrie : l'organisation du cours</u>

La démarche proposée dans le cours de géométrie peut être vue comme s'articulant autour de trois volets principaux.

D'une part nous voulons faire un retour sur les êtres géométriques de base, afin d'aider l'étudiant-maître à y « mettre de l'ordre » tant au plan conceptuel qu'en ce qui concerne le vocabulaire même utilisé pour les nommer, où il présente parfois de sérieuses lacunes : il nous paraît en effet important à cet égard de favoriser chez le futur enseignant l'emploi des termes corrects pour « parler géométrie ». Ce volet du cours se situe dans une large mesure dans une perspective à la fois exploratrice et descriptive à propos des concepts géométriques en cause.

Une autre partie du cours vise à proposer une approche déductive en lien avec certains des thèmes vus dans le cours. La notion de cadre argumentatif était déjà présente dans le cours d'arithmétique, et nous voulons en géométrie pousser un cran plus loin tant la pratique que la réflexion autour de la notion de preuve. Nous visons ainsi à permettre au futur enseignant de se préparer à l'utilisation de situations argumentatives dans les classes du primaire dans le but de valider certaines observations. Mais nos objectifs demeurent à ce chapitre relativement limités, comme il va sans doute de soi.

Le troisième volet vise à faire voir la géométrie « en action », pour ainsi dire. Il occupe une place importante dans le travail à faire en dehors des heures de cours et repose sur une série de travaux pratiques (TP) offrant une occasion d'exploration personnelle de divers cadres géométriques. Ces TP, inspirés dans une large mesure de la littérature pour les enseignants, se déroulent en mode individuel ou en équipe et se veulent une occasion d'action et de réflexion sur des objets géométriques parfois familiers, parfois un peu inusités, mais toujours inspirants, nous l'espérons. Les thèmes géométriques alors abordés sont utilisables avec les élèves du primaire, bien sûr en adaptant le rythme et la démarche.

Ces trois volets du cours ne sont pas traités de manière consécutive, mais s'entrecroisent constamment tout au long du trimestre : l'intensité d'un volet donné peut certes varier d'une semaine à l'autre, mais à tout moment l'étudiant doit accorder de l'attention à chacun d'eux.

<u>Le démarrage du cours de géométrie</u>

Souhaitant (tout comme en arithmétique) que les étudiants plongent rapidement dans le cours, nous lançons celui-ci en initiant trois activités différentes dès les premières heures de contact.

D'une part les notes de cours proposent un texte où un premier « regard géométrique » est porté sur des contextes divers, parfois reliés à la vie courante. Sous forme de lecture personnelle, ce texte permet à l'étudiant de voir par exemple la géométrie présente dans le monde physique qui nous entoure—que ce soit par le truchement d'objets de la nature (toiles d'araignée, rayons de ruche, cristaux de neige, …), ou par des artéfacts (dallage d'un plancher, architecture d'édifices ou de ponts, forme et ornementation de vases, etc.). Il y est aussi question de géométrie et art—depuis les travaux d'un Mondrian ou d'un Escher jusqu'à « la géométrie secrète des peintres » (Bouleau, 1963). La géométrie est également mise en lien avec la perception visuelle, notamment en ce qui concerne les illusions d'optique, ou encore dans les casse-têtes géométriques. Bref, il s'agit ici de rendre l'étudiant sensible à diverses manifestations géométriques qui nous entourent, innombrables et variées à l'infini.

Afin d'assurer que les étudiants ne demeurent pas dans un mode contemplatif, deux TP sont lancés dès le premier cours, avec commentaires partiels lors du cours suivant et correction à la deuxième semaine. Je reviens plus bas sur ces TP.

Et enfin un aspect plus théorique du cours est lui aussi entamé à la première semaine, par un premier contact (à la fois concret et formel) avec les « premiers habitants » de l'univers géométrique.

Une fois le cours lancé, la démarche se poursuit selon les trois volets présentés plus haut, que je commente maintenant à tour de rôle.

<u>Les figures géométriques</u>

L'un des objectifs de base du cours est d'assurer que le futur enseignant soit à l'aise avec les thèmes géométriques figurant dans le programme du primaire, au plan tant conceptuel que terminologique. À cet effet, un cheminement est proposé, qui dure tout le trimestre, au cours duquel les divers êtres géométriques sont introduits intuitivement et définis adéquatement, puis examinés sous divers aspects. Au cours de cette étude, la notion de transformation géométrique intervient afin de mieux cristalliser certaines des propriétés géométriques en cause.

À noter que la familiarisation avec ces objets géométriques est reprise abondamment dans les deux autres volets de la démarche du cours, tantôt dans un cadre déductif et tantôt sous forme d'activités de TP.

Les premiers habitants de l'univers géométrique

Afin de bien ancrer la démarche dans un environnement concret, la réflexion démarre à partir d'un objet usuel, par exemple une boîte à souliers. Contemplant les diverses parties de ce prisme rectangulaire droit—sans pour autant le nommer ainsi pour le moment—on en extrait l'idée de *point*, de *droite* et de *plan*, respectivement incarnés par les sommets, arêtes et faces de la boîte. Et cette boîte existe dans notre environnement 3D, ce qui nous amène à la notion d'*espace*.

Chemin faisant, on observe (et accepte) quelques propriétés dont jouissent ces premiers êtres géométriques : sur une droite donnée, on peut choisir autant de points que l'on veut, et entre deux points donnés, on peut aussi en choisir autant que l'on veut ; deux points distincts déterminent une et une seule droite ; par une droite donnée passent autant de plans que l'on veut ; etc. Nous avons ainsi décrit les quatre « ancêtres », à partir desquels seront dûment définis tous les autres êtres—ou *figures géométriques*—peuplant notre univers géométrique (abstrait).

S'ensuit un premier examen de descendants qu'engendrent ces ancêtres. Tout d'abord des descendants en ligne directe, tels demi-droite, angle et segment de droite ; puis de « véritables » figures géométriques, tels polygone et cercle. Relaxant la stricte exigence de linéarité et de planarité, la droite et le plan mènent aux notions de courbe et de surface.

Très tôt surgit le besoin de comparer entre elles les figures géométriques, ce qui entraîne notamment la relation de congruence entre figures géométriques, définie tout bonnement—dans le cas de figures planes—en termes de superposabilité. Nous n'hésitons pas ici à recourir à une superposition concrète, pouvant s'effectuer fort bien en 2D en reproduisant une première figure sur du papier pelure puis, par des mouvements appropriés du papier, en venir à superposer son calque à l'autre figure ; mais nous évoquons aussi un schème de « calquage abstrait » (et « parfait », à l'abri de l'imprécision de nos manipulations concrètes avec du papier pelure), qui serait exécuté sur des figures existant en tant qu'êtres qui habitent dans un « monde géométrique ».

À cet égard, nous souhaitons que les étudiants gardent bien présent à l'esprit que l'étude d'un triangle équilatéral, par exemple, ne porte pas vraiment sur le dessin qu'ils en ont fait sur une feuille de papier, mais bien sur une figure abstraite. Sans que le phénomène de l'abstraction devienne une sorte d'obsession, il nous apparaît utile de le mettre en toile de fond dans nos cours—et ce, tant en arithmétique qu'en géométrie. Il nous semble en effet important que les futurs enseignants se sentent bien équipés pour accompagner leurs élèves dans leurs premiers pas sur le chemin de l'abstraction en mathématiques.

Le transfert de la notion de congruence en 3D pose une difficulté conceptuelle, car il n'y a pas d'équivalent immédiat du calquage 2D. On peut bien sûr, dans un premier temps, voir des figures congruentes comme étant issues d'un même « moule », mais cela ne règle pas le cas élémentaire de la congruence des mains gauche et droite. Mais un gant (souple) recouvrant la main gauche peut être inversé (intérieur/extérieur), par diverses contorsions, de manière à pouvoir l'enfiler sur la main droite. On touche ici à la notion (relativement complexe) de transformation géométrique 3D, qui sera reprise partiellement dans un TP.

De la notion de congruence découlent divers attributs, dont le concept fondamental d'angle droit (et donc de perpendicularité), défini non pas en termes de degrés, mais de manière plus primitive—comme le fait d'ailleurs Euclide dans ses *Éléments*—par la congruence des angles déterminés par l'intersection de deux droites. Deux droites remarquables sont aussi définies, qui interviennent à moult reprises et très tôt dans le cours : la *médiatrice* d'un segment et la *bissectrice* d'un angle.

Ce premier contact avec l'univers géométrique se termine par des rappels sommaires sur l'idée de mesure (mesure de longueur d'un segment et mesure d'angle), afin d'avoir accès à ces notions dès le début du cours. Notons qu'une réflexion plus en profondeur sur le concept mathématique de mesure est proposée vers la fin du cours, où il est question en plus de mesure d'aire et de mesure de volume (voir plus bas). Soulignons de plus que le contexte de l'enseignement primaire fait que nous exprimons volontiers la mesure d'un angle en fraction de tour, surtout lorsqu'il s'agit d'un angle en lien avec un mouvement de rotation ; mais bien sûr les degrés sont aussi de la partie, comme il se doit. Parmi les exercices touchant cette partie du cours, l'un demande de déterminer l'angle formé, à une heure donnée, par les deux aiguilles d'une montre ; et un autre, d'obtenir, avec justification à l'appui, un angle droit par pliage d'une feuille de papier (deux cas de figure : le sommet de l'angle est un point quelconque de la feuille, ou un point sur le rebord de la feuille).

Courbes et polygones

L'idée de figure géométrique prend une saveur particulière un peu plus tard dans le cours, alors que la notion générale de courbe est examinée de manière systématique. S'ensuivent notamment diverses notions fondamentales—courbe simple et fermée, intérieur et extérieur, convexité et concavité—en lien avec les premiers rapports topologiques que peut avoir l'enfant avec son environnement (Sauvy & Sauvy, 1972). Les jeux de labyrinthe ou d'accès à un trésor sont ici de la partie.

On propose aussi une exploration systématique de la famille des polygones, principalement des triangles et des quadrilatères, afin de bien cristalliser les concepts et le vocabulaire liés à ces figures occupant une place primordiale dans l'enseignement primaire. Une attention particulière est portée aux polygones réguliers, et à leurs divers angles (angle intérieur, extérieur ou au centre). Afin d'apporter une vision rafraîchissante de ces figures si familières, on propose par exemple de classifier les quadrilatères convexes en termes de leurs diagonales.

Il convient de souligner ici, s'agissant du cercle, que nous avons fait le choix de creuser ce thème principalement dans le cadre d'un TP, plutôt que comme matière vue en classe, à l'instar des polygones. Il est alors entre autres question, dans ce TP, de diamètre (l'étudiant montre que c'est la plus longue des cordes) et de tangente (perpendiculaire au rayon au point de contact). On y propose aussi une démonstration guidée du théorème sur l'angle inscrit dans un cercle, et on y parle des polygones réguliers vus comme inscriptibles dans un cercle.

Les solides

La géométrie 3D est en soi fort complexe et il ne saurait être question d'en faire une étude systématique dans le cadre du cours. Mais en lien avec l'ancrage de la géométrie du primaire dans l'environnement de l'enfant, il apparaît important de démystifier les objets ne pouvant vivre que dans un monde 3D et de fournir les outils pour une amorce de classification.

On propose donc dans le cours un inventaire des principaux solides intervenant dans un premier contact avec l'espace vu comme cadre géométrique : polyèdres, prismes, cube, pyramides, cylindre, cône… et sphère. On fait ressortir l'aspect développement en 2D de certains d'entre eux, alors que d'autres sont vus en tant que solides de révolution.

La présentation en classe des solides demeure relativement sommaire—et s'accompagne d'objets concrétisant ces diverses formes : modèles des solides platoniciens, boîtes de divers formats, cône sectionné, etc.

Deux TP poursuivent la réflexion sur les solides. Dans un premier cas, c'est par le truchement du problème du « cube mouillé », vu à l'occasion d'un travail sur la notion de polyminos (Golomb, 1965). Après avoir considéré divers polyminos (principalement sur du papier quadrillé, mais aussi leurs « cousins » sur du papier isométrique triangulé), on porte une attention particulière aux hexaminos, formés de six carrés assemblés de sorte que chacun en touche un autre par au moins un côté complet. On cherche dans un premier temps à identifier les hexaminos qui sont des développements du cube—voir par exemple la Figure 5(a)—les carrés étant alors numérotés comme sur un dé à jouer, de sorte que la somme de deux faces opposées du cube donne 7.

(a) (b)

Figure 5. Un développement du « cube mouillé ».

On demande ensuite d'indiquer sur ce développement la partie qui serait mouillée si le cube était plongé à moitié dans l'eau, disons avec la face 4 au-dessous—Figure 5(b). Ce passage du 3D au développement 2D présente un beau défi de visualisation spatiale.

Un deuxième TP a pour thème la notion de symétrie dans l'espace. Dans le déroulement du cours d'autres TPs ont alors déjà porté sur la symétrie dans le plan (rosaces, frises et pavages—j'y reviens plus bas), et on souhaite maintenant faire un transfert au 3D. Le défi est substantiel, tant en ce qui concerne l'intuition des phénomènes en cause que la manière de les représenter adéquatement sur papier. Bien sûr, l'exploration de ces symétries se fait à partir d'objets concrets ayant la forme d'un cube, d'un prisme rectangulaire, d'un cylindre ou d'un cône (tous droits), ou encore d'une sphère. En disséquant étape par étape les cas les plus simples, on peut amener les étudiants à développer une bonne appréciation de phénomènes élémentaires de symétrie en 3D. Mais bien sûr le sujet demeure plutôt difficile, ce qui vient illustrer pourquoi la géométrie 2D n'est pas si artificielle que ça et présente de clairs avantages, tant au plan conceptuel que dans une perspective purement pédagogique.

Les transformations géométriques

Les transformations géométriques sont présentes à plusieurs titres dans le cours, principalement en relation avec le rôle des isométries dans l'étude des symétries d'une figure géométrique. Ainsi la symétrie 2D fait l'objet de quelques TP, où les situations de symétrie sont mises en évidence à l'aide d'outils divers : papier calque, compas, miroir, *Mira*, …. Le lien avec les isométries correspondantes se fait alors naturellement. Des commentaires semblables, mais dans un registre plus sommaire, sont aussi faits pour la symétrie 3D.

L'un des TP vise explicitement à illustrer le phénomène de la congruence par le truchement du papier calque, qui permet de modéliser le déplacement « rigide » d'une figure. Des mouvements systématiques du calque sont alors introduits—glissement, pivotement et retournement—qui mènent directement aux notions de translation, de rotation et de réflexion en tant qu'isométries. Une synthèse de ces notions est reprise dans le cours, avec une visée de classification des « mouvements rigides » dans le plan où est mis en relief, au passage, le rôle de la réflexion glissée.

Les homothéties sont aussi sommairement abordées dans le cours, à la fois, dans un cadre plus intuitif, en lien avec la relation de *similitude* entre figures géométriques, et aussi par le truchement de papier quadrillé d'échelles variées. (On en profite au passage pour suggérer l'idée de transformations géométriques plus générales par le jeu de grilles à structure géométrique diversifiée.)

Remarque : Dans une version antérieure du cours, l'un des TP proposait une étude guidée et systématique des isométries, selon qu'elles inversent ou non les figures, afin de démontrer que toute isométrie dans le plan peut être vue comme la composée d'au plus trois réflexions—voir Hodgson (1981). Les contraintes de temps nous ont amenés à mettre ce travail de côté, de sorte que nous nous bornons maintenant à mentionner aux étudiants que l'importance accordée aux notions de translation, rotation et réflexion résulte du fait que toute isométrie est de l'une de ces formes, ou encore une réflexion glissée.

Une étude quantitative des figures géométriques

Outre les notions de mesure de longueur et d'angle, introduites tôt dans le cours pour des raisons de commodité, l'essentiel de la démarche géométrique proposée au fil des semaines est davantage de nature qualitative que quantitative. Le dernier thème abordé sous le volet de l'étude générale des figures géométriques est la notion générale de *mesure de grandeur* appliquée à une figure géométrique donnée.

La démarche repose sur deux principes de base : *le principe de congruence* (une mesure de deux figures congruentes est la même) et le *principe d'additivité* (une mesure est préservée par un « collage approprié » de deux figures, de sorte que les mesures de chacune d'elles peuvent être additionnées). On revient, sur la base de ces deux principes, sur les mesures de longueur et d'angle, puis on passe aux mesures d'aire et de volume—l'idée de « collage approprié » étant à clarifier dans chaque cas. Étant donné une unité de longueur, par exemple le segment de 1 cm, on met en relief les deux étapes en vue de passer à une mesure d'aire ou de volume. Tout d'abord, on introduit une convention menant aux nouvelles unités de mesure : le carré de côté 1 cm a, par définition, une aire de 1 cm^2 ; et le cube d'arête 1 cm, un volume de 1 cm^3. Puis on fait appel à un principe permettant d'appliquer les nouvelles unités de mesure à des figures plus générales. Dans le cas de la mesure d'aire, il s'agit d'un principe « *aire vs longueur* » stipulant qu'un rectangle dont les côtés ont pour longueur a et b a pour aire $a \times b$. Le cas du volume se traite de même en lien avec le volume d'un prisme rectangulaire droit d'arêtes a, b et c. Pour plusieurs étudiants, cette prise de conscience des principes régissant le phénomène de la mesure (et le choix des unités conventionnelles d'aire et de volume) est reçue comme un apprentissage important.

Cette partie du cours se termine par un regard sur le périmètre et l'aire des principales figures au programme du primaire—en insistant sur l'importance d'aller nettement plus en profondeur que la simple mémorisation de « formules » —et aussi par un clin d'œil… à Pythagore.

<u>Une vision déductive de la géométrie</u>

Le second volet du cours porte sur le rôle de la déduction en géométrie. Il va de soi qu'il est hors de question d'appuyer notre cours sur une approche à forte saveur axiomatique. Nous croyons cependant qu'il est approprié de profiter du cadre géométrique qui s'offre à nous afin de souligner l'importance de savoir justifier de façon adéquate un raisonnement mathématique. Nous visons ainsi à permettre à nos étudiants de bien saisir l'apport et l'à-propos de cet aspect dans une démarche mathématique—y compris éventuellement dans l'enseignement primaire.

Nous approchons la déduction en géométrie en deux temps, afin d'aborder le thème par petites bouchées, pour ainsi dire. Tout d'abord, après une brève allusion à l'idée générale de système déductif et l'introduction de la notion de *somme d'angles*, nous présentons le premier axiome (ou principe) de notre démarche : *la somme des angles du triangle*. Comme tout axiome, il pourrait être introduit tout simplement de manière arbitraire. Mais nous essayons bien sûr de le « soutenir » par diverses observations : promenade d'un objet « orienté » (par exemple un crayon couché sur la table) autour des trois sommets d'un triangle ; ou encore le fait de déchirer les trois coins d'un triangle de papier et de les replacer judicieusement ; ou encore d'observer le treillis obtenu en traçant un réseau de droites parallèles aux côtés d'un triangle donné.

Nous en profitons au passage pour inviter nos étudiants à une réflexion sur l'observation d'un élève (histoire vécue!) : « Le fait que l'angle plat vaut 180° est une décision de l'Homme; mais le fait que l'angle plat correspond à la somme des angles du triangle est inscrit dans la Nature ».

Cette première partie déductive du cours mène à diverses applications, tel le théorème sur l'angle extérieur d'un triangle ou sur l'angle inscrit dans un demi-cercle. Il est aussi question de perpendicularité et de parallélisme, par exemple le fait que deux droites perpendiculaires à une troisième sont parallèles (de sorte que les côtés d'un rectangle sont parallèles deux à deux). Et on termine par le lien entre parallélisme et congruence d'angles (disons, alternes-internes).

Le second contact avec la déduction en géométrie, qui survient quelques semaines plus tard, est axé cette fois sur les cas usuels de congruence de triangles. Ceux-ci sont motivés comme axiomes en s'inspirant d'un « jeu du téléphone » (à l'ancienne…) : quelle information faut-il fournir (oralement) à un interlocuteur pour qu'il puisse reproduire le triangle que nous avons sous les yeux : que lui faut-il savoir des côtés ? des angles ?

S'ensuivent une foultitude de résultats possibles à explorer, parmi lesquels il faut faire un choix, l'objectif du cours n'étant pas de « tout » couvrir à ce chapitre. On y traite par exemple la question des angles du triangle isocèle, ou encore les caractéristiques des côtés et diagonales des parallélogrammes. On y présente aussi la caractérisation de la médiatrice d'un segment en termes d'équidistance (d'où il suit que les trois médiatrices d'un triangle sont concourantes), et de même pour la bissectrice d'un angle. On observe au passage que s'agissant de la médiatrice d'un segment, celle-ci peut donc être *définie* comme étant la droite perpendiculaire passant par le milieu du segment, ou encore comme l'ensemble des points équidistants des extrémités du segment. C'est là l'occasion d'une réflexion stimulante à propos du rôle et du choix des définitions en mathématiques, notamment dans une perspective pédagogique.

Autant nous croyons que la déduction a sa place dans notre cours, autant nous ne voulons pas provoquer un contexte où les étudiants se mettraient à apprendre par cœur démonstration après démonstration…. Par ailleurs la question de déterminer quels résultats peuvent être acceptés « gratuitement » dans un argument donné et dans un cadre donné demeure toujours fort délicate à trancher, pédagogiquement parlant (et ce, quel que soit le niveau).

Afin de créer pour nos étudiants un cadre de raisonnement plus en harmonie avec leur spécificité d'enseignants du primaire, nous leur fournissons lors de l'examen un document (qu'ils connaissent à l'avance) contenant des « résultats choisis » qu'ils peuvent invoquer à tout moment dans une démonstration, sans avoir à les justifier en soi : de tels résultat prennent alors pleinement le statut d'« outils » pour établir un autre fait. Pour chaque examen, la liste des résultats choisis en comprend environ une vingtaine. (Parmi ceux-ci, quelques-uns sont pointés comme pouvant faire l'objet d'une démonstration en soi lors de l'examen.)

La mise en place de ce cadre d'évaluation dans le cours de géométrie a fortement contribué à dédramatiser la présence de démonstrations en examen, permettant ainsi aux étudiants de se concentrer davantage, dans leurs apprentissages, sur le rôle même de la démonstration en mathématiques—une prise de conscience essentielle même pour un enseignant du primaire.

Les connaissances développées en lien avec les aspects déductifs en géométrie sont notamment mises en pratique lors de l'étude de quelques constructions de base à la règle et au compas, ou encore lors de constructions avec d'autres instruments : équerre, pliage, miroir, etc.

Travaux pratiques en géométrie

Tel que mentionné précédemment, une composante importante du cours vise à inciter les étudiants à vraiment « mettre la main à la pâte » dans une démarche exploratoire et personnelle à propos de contextes géométriques divers. C'est là le but des quatorze travaux pratiques, effectués (seul ou en équipe, au choix des étudiants) en dehors du cadre de la classe. Chaque TP se veut donc une activité d'auto-apprentissage guidée, qui se conclut habituellement par un bref commentaire en classe où sont soulignés les aspects les plus marquants du travail proposé. Plusieurs « outils d'exploration » sont mis en œuvre dans les TP : instruments standard de géométrie (règle, compas, équerre, rapporteur), pochoir, miroir de poche, *Mira*, petits cubes emboîtables, objets ayant la forme de solides de base (cube, cylindre, etc.), patron des sept pièces du tangram, papier isométrique (quadrillé, triangulé ou hexagoné, en version pointillée ou à ligne pleine), papier pelure (pour le calquage), carton, ciseaux, quelques crayons de couleur… voire l'ordinateur.

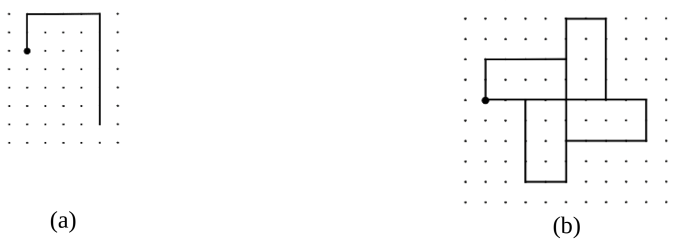
Chaque TP vise le double objectif de permettre à l'étudiant de se familiariser avec un cadre géométrique donné (souvent nouveau, voire un peu déstabilisant), et aussi d'y trouver inspiration pour d'éventuelles activités avec ses élèves. De telles activités pourraient même déborder le « classe de mathématiques » proprement dite, car certains des TP—notamment ceux portant sur la symétrie de rosaces, frises ou pavages—se prêtent admirablement bien à servir de support pour des activités d'arts plastiques, une partie intégrante de la formation au primaire.

Plusieurs des TP ont été évoqués dans les pages qui précèdent. J'en présente maintenant quelques autres.

Spirolatérales et taxi-distance

Les *spirolatérales* (ou « spirales à côtés ») sont des courbes obtenues en exécutant, sur une grille donnée, des séries de déplacements entrecoupés de rotations d'angle déterminé (Odds, 1973). Nous proposons dès le tout premier cours à nos étudiants d'explorer par eux-mêmes l'univers des spirolatérales, d'une part afin de les faire s'engager rapidement dans le cours, mais aussi en souhaitant leur faire prendre conscience que pour « faire des maths », il suffit de bien peu : crayon, papier, un brin d'imagination, le goût de la découverte… et évidemment un soupçon de matière grise !

Travaillons d'abord sur une grille quadrillée. Fixant un point de départ sur la feuille et une orientation donnée (disons le Nord, c'est-à-dire le haut de la feuille), répétons (autant que nécessaire…) une séquence de déplacements donnés, tournant après chaque déplacement d'un angle de 90° (disons, dans le sens horaire). On voit par exemple à la Figure 6(a) l'effet de l'exécution de la séquence [2, 4, 6], et en 6(b), la spirolatérale obtenue en répétant cette séquence quatre fois (ou indéfiniment…).



(a) (b)

Figure 6. La spirolatérale [2, 4, 6] sur du quadrillé.

On observe immédiatement que certaines spirolatérales, telle [2, 4, 6, 8], se propagent à l'infini ! Voilà donc un autre contact avec ce concept fondamental et quasi omniprésent, accessible aux jeunes du primaire de manière spontanée et variée.

On peut aussi changer de grille. La Figure 7 montre le résultat de la séquence [1, 2, 3, 4, 5] sur du papier triangulé, en faisant une rotation de 60° après chaque déplacement (on suppose ici que l'orientation initiale est vers l'Est, c'est-à-dire la droite de la page).

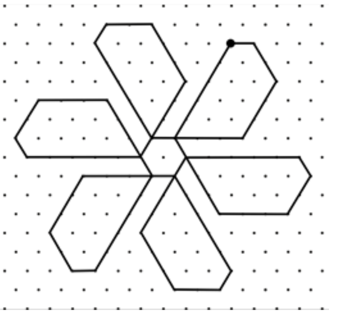


Figure 7. Une spirolatérale à 60°.

Outre le fait d'obtenir de fort jolies figures, une question se pose naturellement à propos des spirolatérales : comment caractériser leur comportement? Nos étudiants sont ainsi amenés à distinguer, dans le cas du quadrillé, qu'il y a trois cas possibles : la spirolatérale va à l'infini (lorsque le nombre d'étapes est un multiple de 4), ou bien elle est complétée après deux répétitions (c'est le cas où le nombre d'étapes est pair, mais non divisible par 4), ou encore après quatre répétitions (nombre impair d'étapes). Cette caractérisation n'est pas seulement observée, elle est susceptible d'une démonstration par une analyse des déplacements selon les quatre points cardinaux. Plusieurs étudiants nous rapportent que pouvoir ainsi donner une justification de ce résultat, qui n'est pas évident *a priori*, constitue pour eux une expérience mathématique stimulante et positive. (Pour ce qui est du papier triangulé, on se limite, dans le cadre du TP, à simplement explorer le comportement des spirolatérales avec angles de 60° ou de 120°.)

Un second TP proposé aux étudiants dès le début du cours se veut un clin d'œil à une géométrie non euclidienne—sans pour autant insister sur ce phénomène en tant que tel. Il s'agit du contexte où la distance entre deux points dans le plan cartésien s'obtient par des déplacements parallèles aux axes de coordonnées. Cette approche, popularisée sous le nom de *taxi-distance*—voir Byrkit (1971) ou Krause (1975) —prend inspiration du fait qu'une telle façon de voir l'idée de distance correspond à un déplacement dans une ville dont les rues et les avenues s'intersectent perpendiculairement : la distance entre deux points s'obtient non pas selon le théorème de Pythagore (la distance « à vol d'oiseau »), mais plutôt en vertu d'une « métrique de Minkowski » (la distance pédestre… ou en taxi). Le but de ce TP se limite à regarder sommairement la *taxi-géométrie,* une géométrie différente de celle à laquelle nous sommes habitués, et à réaliser ce qu'y deviennent, en vertu de leur définition même, certains objets familiers, ce qui mène par exemple au « taxi-cercle ».

Symétries dans le plan : rosaces, frises et pavages

Le thème de la symétrie (dans le plan) est abordé dans le cadre de trois TP. Dans un premier temps les étudiants sont invités à explorer le monde des *rosaces*, c'est-à-dire des figures géométriques bornées possédant un centre et symétriques par rotation autour de ce centre. Certaines rosaces ne possèdent que des symétries de rotation (Figure 8), tandis que d'autres sont de surcroît laissées fixes par réflexion.

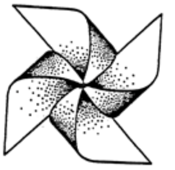


Figure 8. Une rosace possédant quatre angles de symétrie (90°, 180°, 270°, 360°).

Une approche classique à la symétrie de telles figures met l'accent sur les isométries elles-mêmes laissant fixe une rosace, qui forment une structure de groupe—voir par exemple (Hodgson & Graf, 2000). S'ensuit une terminologie axée sur ces groupes. Dans le cas d'une rosace symétrique à la fois par rotation et par réflexion, il s'agit du groupe dièdre D_n comprenant $2n$ éléments (où n varie selon la complexité de la rosace) : n réflexions dans des droites concourantes et n rotations de même centre (le points de concours des droites) et ayant pour angles les multiples de $\frac{360°}{n}$. Lorsqu'une rosace n'est symétrique que par rotation, on est restreint au groupe cyclique C_n formé des n rotations, un sous-groupe de D_n. Le choix fait dans le TP est d'identifier une rosace non pas en termes de son groupe de symétrie, mais plutôt en se concentrant sur la figure en tant que telle. À cette fin, une terminologie adéquate devait être développée.

J'en profite ici pour souligner combien le choix d'un bon vocabulaire mathématique peut aider à s'exprimer à propos d'un sujet donné, et ultimement à bien le concevoir (il en est de même bien sûr du choix d'une bonne notation). Et à l'occasion il ne faut pas hésiter à introduire, à des fins pédagogiques, un vocabulaire original—mais avec prudence et modération, il va de soi. C'est ce qu'a fait mon collègue Ghislain Roy dans le cas des rosaces, en prenant inspiration du mot « trilobé » (un terme de botanique) afin de désigner explicitement le type d'une rosace. Il a donc proposé d'appeler *k-lobe* une rosace fixe par rotation et par réflexion—le paramètre k correspond à la fois au nombre d'axes de symétrie et d'angles de symétries de la figure. Et poursuivant sur la même lancée, il a créé le terme *k-pied* pour nommer une rosace fixe seulement par rotation. La Figure 9 illustre clairement la motivation derrière cette terminologie.

(a) (b)

Figure 9. Rosaces : un 3-pied et un 4-lobe typiques.

J'aimerais insister sur le fait que l'emploi d'un tel vocabulaire en « pied » et en « lobe » a un petit côté amusant pour les étudiants, mais surtout rend la discussion sur ce sujet beaucoup plus limpide. Un bon choix pédagogique, me semble-t-il, malgré son côté local (dont les étudiants sont avertis).

En plus d'analyser diverses figures quant à leur nature en tant que rosaces (polygones divers, polygones réguliers, voire logos commerciaux), nous demandons à nos étudiants de construire des rosaces d'un type donné. Un exemple est de compléter la Figure 10 en y ombrageant le plus petit nombre possible de carrés, de sorte que la rosace qui en résulte soit un 4-pied, ou encore un 2-lobe.

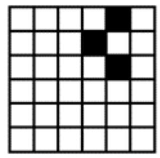


Figure 10. Une rosace à compléter.

Le contact avec les *frises* se veut une autre occasion de rencontre avec l'infini. Une telle figure géométrique étant, par définition, laissée fixe par des mouvements de translation dont les vecteurs sont tous de même direction, elle correspond à une sorte de ruban doublement infini qui se prolonge dans les deux sens de parcours déterminés par cette direction. Le TP porte à la fois sur la construction de frises—tantôt de façon naïve, tantôt dans une démarche guidée—et sur l'identification de frises données, certaines issues de contextes culturels et historiques particuliers. Il est intéressant de souligner le contraste entre les rosaces, qui se retrouvent en une infinité de variantes en fonction de la valeur du paramètre k, et les frises, pour lesquelles il y a précisément sept types possibles (selon la famille des isométries laissant fixe une frise donnée). Nous ne démontrons pas ce résultat, mais nous en proposons une sorte de vérification expérimentale. Une telle étude fournit une occasion de revenir sur la notion de réflexion glissée, qui est la caractéristique fondamentale de l'un des types de frise. Le TP vise à ce qu'en bout de parcours, les étudiants soient familiers avec la schématisation des sept types de frises proposée dans Coxeter (1969, p. 48) —voir Figure 11.

Figure 11. Les sept types de frises.

Quant aux pavages, ils sont explorés en deux temps. Dans un premier TP, fait assez tôt dans le cours, les étudiants sont invités à identifier les formes que peuvent prendre des tuiles permettant de paver le plan (à l'infini). Après s'être entendu sur les règles d'un « bon pavage », on se concentre d'abord sur les polygones réguliers, parmi lesquels on peut identifier aisément trois cas qui fonctionnent—le pochoir est alors un instrument utile, ou encore le papier isométrique. Plus intéressant est le fait de démontrer qu'il n'y a que ces trois cas qui sont possibles, ce que font les étudiants par le biais d'une preuve guidée. On examine ensuite d'autres polygones pouvant servir de tuiles : s'il est assez simple de se convaincre que tout triangle peut paver le plan, le résultat est nettement plus étonnant dans le cas d'un quadrilatère quelconque. Ce TP se termine par une activité (facultative) proposant aux étudiants des pistes pour produire des pavages à la Escher. Dans tous les cas, on invite les étudiants à s'inspirer de ces formes pour expérimenter d'éventuelles activités en arts plastiques : les crayons de couleur sont ici de mise pour mettre en relief les régularités désirées.

Le thème des pavages est repris plus tard dans le cours du point de vue de leur symétrie. Contrairement aux frises, il n'est pas question de faire l'inventaire de tous les (dix-sept!) types de pavages. On se contente ici de regarder sommairement les isométries laissant fixes certains pavages donnés—notamment tirés des chefs-d'œuvre de la tradition artistique islamique, tels les pavages ornant l'Alhambra de Grenade.

La géométrie du kaléidoscope

On propose dans ce TP une étude du kaléidoscope en tant qu'outil géométrique. Il est intéressant de constater combien cet instrument créé il y a précisément 200 ans—son inventeur, David Brewster, a tenté en 1817 d'obtenir un brevet, mais sans succès—conserve encore aujourd'hui tout son pouvoir de fascination. Lors du travail sur ce TP, j'apporte en classe quelques spécimens de ma collection personnelle de kaléidoscopes, ce qui ne manque pas de ravir les étudiants. J'espère ce faisant les rendre sensibles au fait qu'une étape cruciale en vue d'inciter leurs élèves à plonger dans une démarche de découverte en mathématiques est d'abord de capter leur attention, ce que le kaléidoscope accomplit à coup sûr.

Le travail en TP repose pour l'essentiel sur la manipulation de deux miroirs de poche. Si ceux-ci sont placés en parallèle et qu'un objet (ou motif) est placé entre eux, on obtient une frise. Le cas qui nous intéresse ici est celui où les miroirs sont concourants et la question qui se pose est la suivante : s'il est vrai que n'importe quel angle entre les miroirs produit un certain effet visuel, quels sont les angles qui donnent de bons résultats, notamment quant à la symétrie de la figure engendrée par le jeu des miroirs ?

Le TP est structuré autour d'une exploration guidée à la recherche des « bons angles » (Hodgson, 1987), et on en vient ainsi en mettre en lumière les « règles de Brewster » pour un kaléidoscope idoine : afin d'obtenir une belle symétrie, l'angle entre les miroirs doit être de la forme $\frac{180^\circ}{n}$, avec $n \geq 2$. (Notons à cet égard que l'angle de la plupart des kaléidoscopes que l'on trouve sur le marché est de 60°, 45°, 36° ou 30°.)

S'agissant d'un kaléidoscope de 60°, on observera que si un motif symétrique est placé symétriquement entre les deux miroirs, la rosace qui est résulté est de type 6-lobe (Figure 12(a)). Mais sinon—ce qui est le cas usuel d'une image dans un véritable kaléidoscope—on obtient un 3-lobe, comme le montre la Figure 12(b).

(a) (b)

Figure 12. Deux figures kaléidoscopiques (60°).

Le kaléidoscope se prête à diverses généralisations, dont l'un est due à Brewster lui-même et que l'on traite dans le TP : que dire de prismes kaléidoscopiques, faits de plus de deux miroirs ? Le cas de trois miroirs est particulièrement intéressant, car on peut assez facilement montrer que seules trois configurations respectent les règles de Brewster à propos des bons angles : 90° – 60° – 30°, 90° – 45° – 45° et 60° – 60° – 60° (Hodgson, 1987). Une autre généralisation possible, mais que l'on ne fait que mentionner, est le passage à l'ordinateur, ce qui permet l'utilisation de « miroirs » étonnants (Graf & Hodgson, 1990 et Hodgson & Graf, 2000).

Bien comprendre les principes mathématiques qui sous-tendent le kaléidoscope est, je crois, un défi tout à fait approprié dans le cadre de notre cours. Il n'est pas si fréquent que des étudiants destinés à l'enseignement primaire puissent développer une connaissance profonde, et à maints égards complète, d'un objet mathématique d'une certaine complexité (et aussi d'une beauté frappante sur le plan de l'esthétique). Pour plusieurs d'entre eux, maîtriser la « théorie du kaléidoscope », en tant que contenu mathématique, représente une expérience d'apprentissage vraiment marquante, qui a un impact positif tant sur leur perception des mathématiques que sur leur relation personnelle avec elles.

Autres thèmes de TP

Les documents de support pour la plupart des TP évoqués jusqu'ici sont relativement substantiels, faisant chacun en moyenne plus d'une vingtaine de pages (y compris l'espace laissé sur le papier pour l'exécution des diverses tâches faisant partie de la démarche proposée). On a regroupé dans un TP « pot-pourri » une série de thèmes d'exploration davantage concentrés. Outre le cube mouillé dont il a été question plus haut, j'en mentionne deux : le tangram, tant du point de vue de casse-tête divers à résoudre avec les sept tans, que comme support pour de jolis problèmes d'aire ou de périmètre ; et le golf miniature (Figure 13), où on voit le phénomène de la réflexion sur les bandes nous guider pour réussir un trou d'un coup.

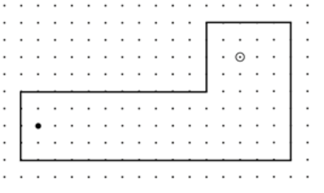


Figure 13. Le golf miniature.

Et l'ordinateur dans tout cela?

Dans sa version actuelle, le cours de géométrie ne se veut pas vraiment une occasion de travail substantiel avec l'ordinateur, même si on peut facilement imaginer à cet égard de nombreux contextes pertinents en lien avec l'enseignement primaire.

Il convient sans doute de souligner que dans le TP sur les spirolatérales, on utilise des commandes de type *Logo* (Papert, 1980/1981), faisant même appel à l'avatar de la tortue… mais sur papier. On y parle aussi des « spirales de la tortue » et de « géométrie de tortue ». Ainsi la spirolatérale de la Figure 7 correspond à la commande

REP 6 [AV 1 DR 60 AV 2 DR 60 AV 3 DR 60 AV 4 DR 60 AV 5 DR 60].

Pendant quelques années, des séances de cours se sont déroulées au laboratoire d'informatique, les étudiants étant appelés à y faire de la programmation en Logo. Mais la taille des groupes et l'état rudimentaire des environnements informatiques de l'époque nous ont vite fait prendre conscience que le jeu n'en valait pas vraiment la chandelle.

L'étape suivante, sur le plan informatique, fut l'utilisation d'un logiciel de géométrie dynamique—*Cabri-Géomètre* dans notre cas. À quelques reprises durant le cours, des manipulations étaient présentées en classe afin d'illustrer certaines propriétés de base des figures géométriques, par exemple à propos des droites remarquables d'un triangle ou encore, concernant les homothéties, l'effet de la position du centre ou du rapport de l'homothétie. De plus un TP (facultatif) a été rédigé, dans lequel les étudiants étaient invités à se rendre au labo d'informatique afin d'exécuter quelques manipulations *Cabri*, notamment en revisitant certains des thèmes vus dans les autres TP (par exemple, trouver le centre d'un cercle donné ou encore construire une rosace d'un type donné).

Aujourd'hui la disponibilité du gratuiciel *GeoGebra* fait en sorte que les étudiants ont tous facilement accès à un environnement informatique pour la géométrie. Pour l'heure, l'utilisation de *GeoGebra* se limite à des illustrations par l'enseignant en classe et à des activités facultatives par l'étudiant. Outre le fait de faire prendre conscience aux étudiants de l'existence même de tels gratuiciels, l'idée est aussi de les sensibiliser à la présence d'une communauté d'utilisateurs internautes qui partagent des constructions diverses en lien avec l'enseignement primaire. On envisage d'éventuellement intégrer plus profondément ce logiciel dans la démarche du cours.

EN GUISE DE CONCLUSION

Mon but, en rédigeant ce texte, était de permettre au lecteur d'apprécier le cadre dans lequel se déroule à l'Université Laval l'intervention des mathématiciens auprès des futurs enseignants du primaire, et aussi d'avoir une idée des contenus que nous avons développés au fil des ans. Je ne voudrais pas ici sembler prétendre à une originalité éclatante dans notre enseignement dans le cadre du programme de BÉPEP : j'ai déjà mentionné plus haut combien, dans mon cas personnel, je me suis abreuvé à de nombreuses sources d'inspiration. Et bien sûr des approches semblables à la nôtre se retrouvent sous divers déguisements ailleurs, par exemple sous la forme de « cahiers de laboratoire » pour des explorations en géométrie.

Il demeure quand même frappant que depuis plus de quatre décennies, le Département auquel j'appartiens offre une formation (obligatoire) en mathématiques conçue spécifiquement en vue de l'enseignement primaire, et que cela est reconnu comme une partie intégrante de sa mission. Je souhaiterais vivement, pour ma part, que davantage de mathématiciens, ailleurs au Québec et au Canada, soient ainsi présents auprès des futurs enseignants du primaire : une telle contribution est importante pour ces enseignants, en vue de les aider à consolider leurs connaissances mathématiques en rapport avec les divers thèmes du programme du primaire, mais elle est aussi importante pour les mathématiciens eux-mêmes, de sorte qu'ils soient davantage sensibilisés aux défis mathématiques de l'enseignement primaire et au caractère essentiel de la collaboration entre mathématiciens et didacticiens à cet égard.

À l'intention de mes collègues mathématiciens, je tiens à les assurer que l'enseignement des mathématiques aux futurs enseignants du primaire, outre les satisfactions pédagogiques qu'on peut en retirer, apporte à coup sûr des moments riches sur le plan de la démarche mathématique même. En un mot, on y trouve abondamment de « belles » maths ! Ce n'est certes pas parce qu'un thème mathématiques donné est élémentaire qu'il est banal pour autant et qu'on n'y trouve pas des éléments de grande subtilité. Guider les futurs enseignants du primaire en vue de bien appréhender ces subtilités procure des moments fort gratifiants.

Cela dit, il est indéniable que le défi pédagogique, pour un mathématicien se présentant devant des étudiants qui se destinent à l'enseignement primaire, demeure substantiel. Savoir rejoindre les étudiants peut s'avérer parfois un peu malaisé. Mais l'attitude du prof, et le respect qu'il manifeste à leur égard, forment somme toute les assises d'une relation pédagogique et mathématique fructueuse.

EN TÉMOIGNAGE DE RECONNAISSANCE

S'il est vrai que j'ai moi-même été fortement impliqué dans le développement des deux cours dont il est question dans ce texte, je tiens à souligner l'apport des nombreux collègues de l'Université Laval qui ont contribué de manière essentielle à cette belle aventure. D'entrée de jeu, j'aimerais saluer Norbert Lacroix qui, à titre de directeur, a su manifester une grande

ouverture devant l'invitation lancée au Département de mathématiques, dans la foulée du rapport Parent, de participer à la formation mathématique des enseignants du primaire. La toute première version de ces cours a été créée en 1974 en collaboration par Cameron McQueen, Claude Lemaire et Ghislain Roy. D'autres collègues y ont contribué de manière importante au fil des ans, notamment William S. Hatcher au début des années 1980, en lien avec les fondements mêmes du cours d'arithmétique, Charles Cassidy, et plus récemment Frédéric Gourdeau. Nos cours ont aussi été enrichis par les nombreux commentaires offerts, selon une perspective didactique, par Roberta Mura et Caroline Lajoie (aujourd'hui à l'UQAM).

Mais j'aimerais surtout rendre hommage ici à une collègue de grande valeur, Linda Lessard, qui a été pendant plus de 35 ans chargée de cours au Département de mathématiques et de statistique, avec comme tâche spécifique la formation à l'enseignement primaire. Notre collaboration durant toutes ces années a, j'ose le croire, contribué à l'évolution harmonieuse et à la qualité des cours d'arithmétique et de géométrie offerts par mon Département aux futurs enseignants du primaire.

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TASK DESIGN IN MATHEMATICS EDUCATION: FRAMEWORKS AND EXEMPLARS[1]

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INTRODUCTION

The field of mathematics education could be said to have been involved in design ever since its beginnings—going back to the time of Euclid, and perhaps even Pythagoras, for whom the term *mathema* meant *subject of instruction*. However, as Wittmann (1995) has remarked, in a paper titled "Mathematics Education as a Design Science", the design of teaching units was never a focus of the educational research community until the mid-1970s. Artigue (2009) too has argued that "didactical design has always played an important role in the field of mathematics education, but it has not always been a major theme of theoretical interest in the community" (p. 7). While it could be maintained that pre-1970 didactical design has been rooted in thoughtful tinkering, growing out of intuition and classroom development, the movement toward theoretically-based research from the 1970s onward has benefitted largely from the emergence of an international research community in mathematics education, as well as from ongoing contributions from the disciplines of mathematics and psychology (Kilpatrick, 1992).

To illustrate these influences and their evolution within the community, this chapter is divided into three sections. The first section provides a historical overview of the theorizing initiatives with respect to task design in our field over the past half-century. The second section, which is the heart of the chapter, presents a conceptualization of current frameworks for task design in mathematics education and describes the characteristics of the design principles/tools/heuristics offered by these frames—a conceptualization that distinguishes among the three levels of grand, intermediate, and domain-specific frames, and that involves consideration of the constructs of *design as intention* and *design as implementation*. These lenses, accompanied by examples drawn from the international body of research literature, help to clarify the various ways in which theory and task design are related, as well as point to the crucial role of instructional support. The third section, in acknowledging that frameworks and principles cannot account for all aspects of task design, concludes with the recommendation that knowledge about design grows in the community as design principles are explicitly described, discussed, and refined, and that more work in this direction is vital.

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[1] The content of this paper is based partly on my contribution to a chapter (Kieran, Doorman, & Ohtani, 2015) in the volume, *Task Design in Mathematics Education* (Watson & Ohtani, 2015).

BRIEF HISTORY OF THE EMERGENCE OF DESIGN-RELATED WORK FROM THE 1960S ONWARD

In 1969, the first International Congress on Mathematical Education (ICME) took place in Lyon. A round table at that congress set the stage for the formation in 1976 of what was to quickly become the largest association of mathematics education researchers in the world, the International Group for the Psychology of Mathematics Education (PME). The emergence of this community was accompanied by the creation of several research journals, as well as research institutes in many countries. The late 1960s and 1970s thus signalled a huge surge and interest in research in mathematics education.

INFLUENCES FROM PSYCHOLOGY

This surge in research in mathematics education had to rely almost exclusively in its early days on psychology as a source of theory (Johnson, 1980). Piaget's (1971) cognitively-oriented, genetic epistemology is but one example of the psychological frames adopted by the emerging mathematics education research community in its studies on the learning of mathematics. Other psychologists with an interest in education also had an influence on early design efforts in mathematics education. For example, in 1965 Robert Gagné published *The Conditions of Learning*. Based on models from behaviourist psychology, Gagné's (1965) nine conditions of learning were viewed as principles for instructional design. In parallel with the instructional design approach being developed by Gagné and others, advances in design considerations were stimulated by the theorizing of the cognitive scientist and Nobel laureate, Herbert Simon (1969), in *The Sciences of the Artificial*. Robert Glaser (1976), in his "Components of a Psychology of Instruction: Toward a Science of Design", distinguished between the descriptive nature of theories of learning and the prescriptive nature of theories of instruction. In integrating design considerations into instructional research, he argued that the structure of the subject-matter discipline may not be the most useful for facilitating the learning of less expert individuals—a point of view that was questioned somewhat by researchers in mathematics education. Thus, mathematics education researchers would need to develop, during the years to come, their own scientific approaches to designing environments for the learning of mathematics and to generating frameworks for task design in particular.

EARLY DESIGN INITIATIVES OF THE MATHEMATICS EDUCATION RESEARCH COMMUNITY

During the 1970s, the focus within the mathematics education research community was on the learning of mathematics and the development of models of that learning. For example, the paper that the mathematician Hans Freudenthal presented at PME3 in 1979 (one of the 24 research reports presented in 1979 at the recently formed PME), dealt with the growth of reflective thinking in learners (Freudenthal, 1979). Nevertheless, that paper sowed the seeds for a mathematical-psychological approach to task design—an approach that was to develop during the late 1980s and 1990s into the instructional theory specific to mathematics education known as Realistic Mathematics Education.

Two of the 1979 PME3 papers did touch specifically upon issues related to design. One of these, by Alan Bell (1979), focused on the learning that develops from different teaching approaches with various curriculum units that had been designed for the South Nottinghamshire project. However, in Bell's paper, which reflected the early days of thinking about design within the mathematics education research community, design considerations were seen more through the lens of particular teaching methods than as approaches to the design of tasks per se.

The 1980s brought some evolution in this regard with, for example, the work of Erich Wittmann. In his 1984 *Educational Studies in Mathematics* paper (a modified version of his opening address at the 14th annual meeting of German mathematics educators in 1981), titled "Teaching Units as the Integrating Core of Mathematics Education", Wittmann (1984) argued for tasks displaying the following characteristics: the objectives, the materials, the mathematical problems arising from the context of the unit, and the mostly mathematical, sometimes psychological, background of the unit. He suggested that a teaching unit is not an elaborated plan for a series of lessons; rather it is an idea for a teaching approach that leaves open various ways of realizing the unit.

During the years 1985 to 1988, one of the PME working groups focused on the extent to which its activities had established principles for the design of teaching. In 1988, a collection of papers from this working group was put together by the Shell Centre under the title *The Design of Teaching –Papers from a PME Working Group,* and subsequently published in a special issue of *Educational Studies in Mathematics* in 1993. In his editorial for the special issue, Alan Bell wrote:

> *Experimental work on the development of understanding in particular mathematical topics is relatively easy to conduct* [...] *but studies of the general properties of different* teaching methods *and* materials *are more difficult to set up.* [...] *Types of research on teaching which have been found productive, albeit in different ways, are the following: (1) basic psychological studies of aspects of learning* [...]*; (2)* developmental activities in which teaching materials are designed on the basis of theory and practical experience and are then taken through several cycles of trial and improvement, [...]*; and (3) comparative studies in which the same topic is taught to parallel classes by different methods*. (Bell, 1993, pp. 1-2, emphasis added)

Notice in the above quote the integration of *teaching methods* and *teaching materials* (the latter implicitly including *tasks*)—in other words, principles of teaching practice that are in harmony with principles that have been incorporated into the design of the teaching materials. This integration of two types of principles will continue to be important in task design within the community over the decades to come.

Another important development during the decade of the 1980s with respect to design was the emergence within France of *didactical engineering*, an exacting theory-based approach to conducting research that had didactical design at its heart (Artigue, 1992). Despite its success as a design-based research practice, certain problems were encountered, according to Artigue (2009), when the rigorous designs were implemented in everyday classroom practice throughout its first decade. It was observed that the original designs went through a certain mutation in practice, leading her to note that "the relationships between theory and practice as regards didactical design are not under theoretical control" (Artigue, 2009, p. 12). This awareness pointed to one of the inherent limitations in theorizing about task design in isolation from considerations regarding instructional practice.

THE 1990s AND EARLY 2000s: DEVELOPMENT OF *DESIGN EXPERIMENTS*

The term *design experiment* came into prominence in 1992 with the psychologist Ann Brown's (1992) paper on design experiments. Brown's paper signalled a kind of tipping point with respect to interest in design in the mathematics education research community (Lesh, 2002). Several factors had fallen into place, including the maturing of the community over a 20-year period and an evolving desire to be able to study within one's research not just learning or not just teaching. Design experiments aimed at taking into account the entire learning picture.

As Cobb, Confrey, diSessa, Lehrer, and Schauble (2003) pointed out:

> *Design experiments ideally result in greater understanding of a learning ecology.* [...] *Elements of a learning ecology typically include the tasks or problems that students are asked to solve, the kinds of discourse that are encouraged, the norms of*

> *participation that are established, the tools and related material means provided, and the practical means by which classroom teachers can orchestrate relations among these elements.* (p. 9)

Within this conception of design experiments, the task or task-sequence is considered but one of a larger set of design considerations involving the entire learning ecology—*task* or *task-sequence* (which could take an entire lesson, or more) being characterized in the ICMI Study-22 Discussion Document as "anything that a teacher uses to demonstrate mathematics, to pursue interactively with students, or to ask students to do something […] also anything that students decide to do for themselves in a particular situation" (Watson et al., 2013, p. 10).

FROM EARLY 2000 ONWARD

Theorizing related to design in mathematics education research developed considerably during the 2000s (Kelly, Lesh, & Baek, 2008). Contributing to this development was the recommendation put forward by Cobb et al. (2003):

> *General philosophical orientations to educational matters—such as constructivism—are important to educational practice, but they often fail to provide detailed guidance in organizing instruction. The critical question that must be asked is whether the theory informs prospective design and, if so, in precisely what way? Rather than grand theories of learning that may be difficult to project into particular circumstances, design experiments tend to emphasize an intermediate theoretical scope.* (pp. 10-11)

Cobb et al. also argued that design experiments are conducted to develop theories, not merely to tune empirically 'what works': "a design theory explains why designs work and suggests how they may be adapted to new circumstances" (p. 9).

In addition to the evolution in theoretical perspectives on design during these years, the term *task design* came to be more clearly present in discussions of research related to design. For example, at the 2005 PME conference, a research forum was dedicated to task design, having as its stated theme, "The significance of task design in mathematics education" (Ainley & Pratt, 2005). At ICME-11 in 2008 the scientific program included for the first time a Topic Study Group (TSG) on task design: "Research and Development in Task Design and Analysis". The excitement generated regarding this research area was such that a similar TSG was put on the program for ICME-12 in 2012, as well as for ICME-13 in 2016. This interest was further illustrated by the holding of the 2013 ICMI Study-22 Conference on the same theme, as well as within our own CMESG when its 2015 annual meeting featured a working group with the title, "Task Design and Problem Posing" (Mamolo & McLoughlin, 2015).

TWO KEY ISSUES

In closing this section on the historical overview of the emergence of research related to design activity, I would like to elaborate on two central issues regarding theory and task design within design research: i) *design as intention* versus *design as implementation*, which is related to the complex role of theory as both a resource for and a product of design research, and ii) the status given to the initial design of the set of tasks. These issues underpin and run through much of the discussion that follows in the next section on frameworks and principles.

In a recent article on design tools in didactical research, Ruthven, Laborde, Leach, and Tiberghien (2009) expanded upon the distinction between *design as intention* and *design as implementation* (Collins, Joseph, & Bielaczyc, 2004). *Design as implementation* focuses attention on the process by which a designed sequence is integrated into the classroom environment and subsequently is progressively refined, whereas *design by intention* addresses specifically the initial formulation of the design.

While many studies address both, the distinction can be useful for understanding certain nuanced differences between one study and another. Ruthven et al. (2009) state that design as intention emphasizes the "original design and the clarity and coherence of the intentions it expresses" (p. 329). Design as intention makes use, in general, of theoretical frames that are well developed so as to provide this clarity and coherence. Although Ruthven et al. (2009) add that "the availability of design tools capable of identifying and addressing specific aspects of the situation under design can support both the initial formulation of a design and its subsequent refinement in the light of implementation" (p. 329), their examples cast light in particular on the *design as intention* orientation. In so doing, they illustrate clearly the role that theoretical tools play in the initial design.

In contrast to the front-end importance given to theory-based design tools by Ruthven et al., Gravemeijer and Cobb (2006) put the focus more toward the *development* of theory and its role as a product of the design research. In their design experiment studies, the initial theoretical base for the study, and its accompanying instructional plan, undergo successive refinements by means of the implementation process. The description of the entire process constitutes the development of the theory. Because of the centrality of the implementation process in the development of the resulting theory, such studies are characterized as *design as implementation* studies, even though they also have a strong initial theoretical base. The complexity of the dialectical role played by theory in such research warns, however, against equating, on the one hand, design as intention and theory as a resource or, on the other hand, design as implementation and theory as a product.

Cobb and Gravemeijer (2008) emphasize that the

> *products of design experiments typically include sequences of activities and associated resources for supporting a particular form of learning, together with a domain-specific, instructional theory that underpins the instructional sequences and constitutes its rationale; a domain-specific, instructional theory consists of a substantiated learning process that culminates with the achievement of significant learning goals as well as the demonstrated means of supporting that learning process.* (p. 77)

In brief, and as mentioned earlier, Cobb et al. (2003) insist that "design experiments are conducted to develop theories" (p. 9).

Put another way, theories are both a resource and a product. As a resource, they provide theoretical tools and principles to support the design of a teaching sequence (e.g., Ruthven et al., 2009) and, as a product of design research, theories inform us about both the processes of learning and the means that have been shown to support that learning (Cobb et al., 2003).

A second issue related to the role and nature of theory in design is the significance given to task design itself within the design process. When theory and its design tools are viewed as a front-end resource in the design process, the way in which task design is informed by these theory-based tools moves to center stage (Ruthven et al., 2009). By way of contrast, when theory development is viewed as the aim of design experiments, task design tends to be less central: "One of the primary aims of this type of research is *not* to develop the instructional sequence as such, but to support the constitution of an empirically grounded local instruction theory that underpins that instructional sequence" (Gravemeijer & Cobb, 2006, p. 77, emphasis added). This is not to say, in the latter case, that task design is unimportant (it clearly is); rather the design of the instructional sequence is only one of several all-encompassing considerations within the whole interactive learning ecology.

In practice, most design experiments combine both orientations: the design is based on a conceptual framework and upon theoretical propositions, while the successive iterations of implementation and retrospective analysis contribute to further theory building that is central to the research.

These two issues, that is: i) *design as intention* and *design as implementation* and ii) the status given to the initial design of the set of tasks, allow for nuancing the relations that can exist between theory and task design within the design process. In the next section, I interweave aspects of these notions into my discussion of the nature of the frames adopted, adapted, and developed by the mathematics education research community—frames that are illustrated by means of selected examples.

A CONCEPTUALIZATION OF CURRENT THEORETICAL FRAMEWORKS AND PRINCIPLES FOR TASK DESIGN IN MATHEMATICS EDUCATION RESEARCH

INTRODUCTION

The historical look at the early research efforts related to task design hinted at a mix of task and instructional considerations. However, the extent to which instructional aspects are factored into task design is but one of the ways in which design frameworks can vary. Frameworks can also differ according to the manner in which they draw upon cognitive, sociological, sociocultural, discursive, or other theories. In addition, frameworks are distinguishable according to the extent to which they can be related to various task genres, that is, whether the tasks are geared toward i) the development of mathematical knowledge (such as concepts, procedures, representations); ii) the development of the processes of mathematical reasoning (such as conjecturing, generalizing, proving, as well as fostering creativity, argumentation, and critical thinking); iii) the development of modelling and problem-solving activity; iv) the assessment of mathematical knowledge, processes, and problem solving, and so on.

As well, some frameworks may be more suited to the design of specific tasks; others to the design of lesson flow; still others to the design of sequences involving the integration of particular artefacts. Because several considerations enter into an overall design—considerations that include the specific genre of the task, its instructional support, the classroom milieu, the tools being used, and so on—each part of the design might call for different theoretical underpinnings. Thus, the resulting design can involve a *networking* of various theoretical frames and principles (Prediger, Bikner-Ahsbahs, & Arzarello, 2008).

A more holistic way of thinking about frames is to conceptualize them in terms of different levels or types, for example, grand frames, intermediate level frames, domain-specific frames (i.e., frames related to the learning of specific mathematical concepts and reasoning processes), and frames related to particular features of the learning environment (e.g., frames for tool use)—all of them together constituting any one theoretical base for the design of a given study (Gravemeijer & Cobb, 2006). Conceptualizing frameworks in terms of these levels is, of course, not the only way to parse the diversity of design frames that exist (see, e.g., Watson, 2016).

GRAND THEORETICAL FRAMES

Mathematics education research has tended in large measure to adopt such grand theoretical perspectives as the cognitive-psychological, the constructivist, and the socioconstructivist. However, as pointed out by Lerman, Xu, and Tsatsaroni (2002), these are but three from the vast array of theoretical fields, in addition to those from educational psychology and/or mathematics that have backgrounded mathematics education research.

In line with Cobb (2007), who has argued that such grand theories need to be adapted and interpreted in order to serve the needs of design research, and because so much has already been written about these grand theories, I will not go into any further discussion of them, except to say that no discussion of design theories would be complete without acknowledging the grand theoretical frames within which they are situated.

INTERMEDIATE LEVEL FRAMES

Intermediate level frames have a more specialized focus than the grand theories and, as such, have the property that they can contribute in a more refined way to the design of particular curricular areas as, for example, that of school mathematics. Many of these intermediate level frameworks have, in fact, been developed by mathematics education researchers for explicit application to the field of mathematics education. In brief, intermediate level frames are located between the grand theories and the more local, domain-specific frames, the latter of which will be seen to address distinct mathematical concepts, procedures, or processes of mathematical reasoning.

The multitude of intermediate level frames that are being applied, some in adaptive ways, to design research in mathematics education include, for example, Realistic Mathematics Education theory (Treffers, 1987), the Theory of Didactical Situations (Brousseau, 1997), the Anthropological Theory of Didactics (Chevallard, 1999), Lesson Study (Lewis, 2002), Variation Theory (Runesson, 2005), Conceptual Change Theory (Van Dooren, Vamvakoussi, & Verschaffel, 2013), and so on.

In general, intermediate level frames can be characterized by explicit principles/heuristics/tools that can be applied to the design of tasks and task-sequences. Because these frames tend to be highly developed, they are often used in *design as intention* approaches. In addition, intermediate level frames can also be characterized according to whether their roots are primarily theoretical or whether they are based to a large extent on deep craft knowledge. The two examples of intermediate level frames and their accompanying principles for task design that I will now describe reflect these two roots. The first is the Theory of Didactical Situations and the second is that of Lesson Study. For both types, I examine the framework and associated principles that support the process of task design.

An example of a theory-based intermediate level frame: Theory of Didactical Situations

The Theory of Didactical Situations (TDS) is generally associated with Guy Brousseau (1997, 1998); however, its development over the years has been contributed to by the French mathematical *didactique* community at large. A central characteristic of TDS research is its framing within a deep *a priori* analysis of the underlying mathematics of the topic to be learned, integrating the epistemology of the discipline, and supported by cognitive hypotheses related to the learning of the given topic. TDS is considered to be an intermediate level theory that draws upon the grand theory of Piaget's work in cognitive development.

According to Ruthven et al. (2009), one of the central design tools provided by TDS is the *adidactical situation*, which mediates the development of students' mathematical knowledge through independent problem solving. The term *adidactical* within TDS refers specifically to that part of the activity "between the moment the student accepts the problem as if it were her own and the moment when she produces her answer, [a time when] the teacher refrains from interfering and suggesting the knowledge that she wants to see appear" (Brousseau, 1997, p. 30).

A situation includes both the task and the environment that is designed to provide for the adidactical activity of the student. According to the TDS frame, the adidactical situation tool furnishes guidelines as to:

> *the problem to be posed, the conditions under which it is to be solved, and the expected progression toward a strategy that is both valid and efficient; this includes the process of 'devolution' intended to lead students to directly experience the mathematical problem as such and the creation of a (material and social) 'milieu' that provides students with feedback conducive to the evolution of their strategies.* (Ruthven et al., 2009, p. 331)

During the early years of the development of the TDS, it was found that the frame needed some modification so as to take into account the necessary role played by the teacher in fostering the later institutionalization of the student's mathematical knowledge acquired during the adidactical phase—the term *institutionalization* referring specifically to the process whereby the teacher gives a certain status to the ideas developed by students by framing and situating them within the concepts and terminology of the broader cultural body of scientific knowledge.

Identifying a suitable set of problem situations that can support the development of new mathematical knowledge is absolutely central to the design of a TDS teaching sequence. The adidactical situation must be one for which students have a starting approach, but one that turns out to be unsatisfactory. Students must be able to obtain feedback from the milieu that both lets them know that their approach is inappropriate and also provides the means to move forward. The "enlargement of a shape puzzle" (see Ruthven et al., 2009, pp. 332-334) is a paradigmatic example of the design of an adidactical situation.

In this task, students are asked to make a larger puzzle of the same shape (a square cut into triangles and trapezoids of different dimensions and which are pieced together in the form of a square), but with the edge of the component trapezoid whose length is 4 cm to be enlarged to 7 cm. It is expected that the students would use additive reasoning to construct the enlargement of each of the individual pieces of the puzzle. But the feedback provided by the attempt to put the enlarged puzzle pieces back together lets the students know that their way of solving the enlargement problem is incorrect. Eventually, with the help of the teacher, students come to arrive collectively at a multiplicative model for the enlargement puzzle.

In addition to the adidactical situation tool, TDS-based design is also informed by a second design heuristic, that of the *didactical variables* tool. This supplementary design tool allows for choices regarding particular aspects of the main task and how it is to be carried out (e.g., the shape and dimensions of the pieces, the ratio of the enlargement, the various pieces of the puzzle having to be constructed by different students), aspects that are subject to modification as a result of successive cycles of the teaching sequence. Although certain modifications are made to those aspects of the task that are found to improve the learning potential of the situation (i.e., that students are more likely to learn what is intended), the initial design of the task is absolutely central to the TDS-framed *design as intention* process.

<u>An example of a craft-based intermediate level frame: Lesson Study</u>

Lesson Study is typically associated with Japanese education where its roots can be traced back to the early 1900s (Fernandez & Yoshida, 2004). It is a culturally-situated, collaborative, approach to design—one where teachers with their deep, craft-based knowledge are pivotal to the process and which at the same time constitutes a form of professional development (Krainer, 2011; Ohtani, 2011). Fundamental to Japanese teachers' ability to design and implement high-quality mathematics lessons that are centered on high-quality mathematical tasks is a detailed, widely-shared conception of what constitutes effective mathematics pedagogy (Jacobs &

Morita, 2002). With its cultural and collaborative foundations, Lesson Study is considered to be situated within the grand theory of socioculturalism.

Lesson Study consists of the following phases: (1) collaboratively planning a research lesson; (2) seeing the research lesson in action; (3) discussing the research lesson; (4) revising the lesson (optional); (5) teaching the new version of the lesson; and (6) sharing reflections on the new version of the lesson. Phases 4 to 6 are sometimes replaced with a single phase of *consolidating and reporting*. While the unfolding of the research lesson and its evaluation takes only one day, its planning can occupy anywhere from one to two months. In that the ultimate description of a research lesson characterizes not only the detailed design of the task itself, but also its careful progression in class by means of the teacher's well-thought-out questioning, Lesson Study is a frame devoted as much to *design as intention* as it is to *design as implementation.*

Three design principles constitute the Lesson Study frame: *kyozaikenkyu*, structured problem solving, and task evaluation. *Kyozaikenkyu* means literally "instructional materials research." As pointed out by Fujii (2013, 2015), *kyozaikenkyu* involves examining teaching materials and tasks from a mathematical point of view (mathematical content analysis), an educational point of view (considering broader values such as 'skills for living'), as well as from the students' point of view (readiness, what students know, anticipated students' thinking and misconceptions, etc.). It includes studying other textbook series treating the same topic, thinking about the manipulatives being used, and analyzing what the curriculum standards and research have to say about the topic and its teaching and learning. *Kyozaikenkyu* is, in fact, not only a key component of the craft-based frame for task design but also central to Japanese teachers' everyday practice.

A second design principle concerns the actual form that the research lesson takes. Referred to as *structured problem solving* by Stigler and Hiebert (1999), the research lesson involves a single task and the following four specific phases: i) teacher presenting the problem (*donyu*, 5 - 10 min), ii) students working at solving the problem without the teacher's help (*jiriki-kaiketsu*, 10 - 20 min), iii) comparing and discussing solution approaches (*neriage*, 10 - 20 min), and iv) summing up by the teacher (*matome*, 5 min). During students' independent working, the teacher walks between the desks (*kikan-junshi*) and silently assesses students' work; she is in the process of making a provisional plan as to which student contribution should be presented first in order to make clear the progress and elaboration from simple idea to sophisticated one: this is the core of *neriage*, a phase during which students' shared ideas are analyzed, compared, and contrasted. During the fourth phase of the research lesson (*matome*), the teacher will usually comment as to the more efficient of the discussed strategies, as well as the task's and the lesson's mathematical and educational values. As an aside, it is noted that Japanese teachers use these specific didactical terms to discuss their teaching and that such didactical terms not only mediate the activity of the various participants involved in Lesson Study but also lead to the co-construction of deep craft knowledge.

After the research lesson has been observed by other teachers, school administrators, and sometimes an outside expert, it is then discussed and evaluated in relation to its overall goals. This process of lesson evaluation, and in particular *task evaluation,* is considered a third design principle. The post-lesson discussion focuses to a large extent on the effects of the initial task design with respect to student thinking and learning. The teacher's thought-out key questioning receives much attention. Another of the main aspects discussed is whether the anticipated student solutions were in fact evoked by the task and its accompanying manipulative materials, or whether improvements in specific parts of the task design are warranted.

DOMAIN-SPECIFIC FRAMES

In contrast to intermediate level frames whose characterizations do not specify any particular mathematical reasoning process or any particular mathematical content area, domain-specific frames for the design of tasks or task-sequences do specify particular reasoning processes (e.g., conjecturing, arguing, proving) or particular content (e.g., geometry, integer numbers, numerical concepts, algebraic techniques). Task design research involving domain-specific frames typically draws upon past research findings in a given area, in addition to being situated within certain intermediate-level, and more general grand-level, frameworks. As such, domain-specific frames for task design research tend to be more eclectic than their intermediate level counterparts.

Note that some researchers use the term *local theories* or *local frames* for what I am referring to as domain-specific frames. In general, domain-specific frames are associated with *design as implementation* in that the main aim of the research is the further development of the theoretical domain-specific frame by means of the implementation process. However, this is not a hard distinction. For those studies with a dual primary aim that includes both the design of the instructional task or sequence itself as well as the detailed description of the learning ecology in which the designed instructional task or sequence unfolds, the approach is as much *design as intention* as it is *design as implementation*.

A domain-specific frame for fostering mathematical argumentation within geometric problem solving

Prusak, Hershkowitz, and Schwarz (2013) recently reported on a year-long, design-research-based course for 3rd graders in mathematical problem solving that aimed at instilling inquiry learning and argumentative norms. The researchers investigated if, and in which ways, principled design is effective in promoting a problem-solving culture, mathematical reasoning, and geometric conceptual learning.

Their design was situated in a multi-faceted framework that drew upon principles from the intermediate level, educational theory of Cognitive Apprenticeship, as described in Schoenfeld (1994), and from domain-specific research in geometric reasoning (Hershkowitz, 1990) and argumentation (Arzarello & Sabena, 2011; Duval, 2006), as well as from multiple studies with a sociocultural orientation. The Prusak et al. study was, in fact, one that articulated explicitly two design components: one for the task and one for the learning environment.

The task that Prusak et al. (2013) discuss in their paper is the *Sharing a Cake* task. The task worksheet included 9 square grids where the students, who worked first individually and then in groups, could draw their own particular cuttings of the square cake into four equal pieces in a variety of ways, followed by an explanation of their thinking on the several blank lines below each grid. Prusak et al. stated that the design of this task, as well as that of the others used within their year-long study, relied on the following five principles:

- encourage the production of multiple solutions;
- create collaborative situations;
- engage in socio-cognitive conflicts;
- provide tools for checking hypotheses; and
- invite students to reflect on solutions.

Setting up a problem-solving culture in the classroom was an integral part of the Prusak et al. design study. By bringing into play Schoenfeld's (1994) use of the Cognitive Apprenticeship model, they scaffolded students' problem solving in a classroom culture that emphasized communication, reflective mathematical practice, and reasoning rather than results. The following instructional-practice principles thereby constituted a second overall design frame for the Prusak et al. study:

- emphasize processes rather than solely results;
- use a variety of social settings (individual, small group, and whole class);
- develop a critical attitude towards mathematical arguments using prompts like, “Does it convince me?”;
- encourage students to listen and try to persuade each other and, thus, to develop ideas together; and
- have students learn to report on what they do, first verbally, then in written form, explaining their solutions to their teammates or to the entire class.

The Prusak et al. study presents an example of the use of well-defined, even if quite general, principles as a front-end resource for the design of the tasks. A second set of principles provided the frame for the design of the learning culture in which the tasks would unfold. Both sets of principles make their study one that could be described as *design as intention*. The authors conclude their paper with a theoretical model for learning early geometry through multimodal argumentation in a problem-solving context—a model that includes the description of the learning process and the demonstrated means of supporting that learning process. In this sense, the study is also an example of *design as implementation*. They emphasize that the designed task served as a principle-based research tool, one that was central to the elaboration of their domain-specific model.

<u>A domain-specific frame for the learning of integer concepts and operations</u>

The design research of Stephan and Akyuz (2013) involved creating and implementing a hypothetical learning trajectory and associated sequence of instructional tasks for teaching integers in a middle-grades classroom over a five-week period. Grounded in the researchers’ deep knowledge of past research on the learning of integers and integer operations, the design of their instructional sequence was underpinned by the following three heuristics from the intermediate level frame of Realistic Mathematics Education:

- Guided reinvention – “To start developing an instructional sequence, the designer first engages in a thought experiment to envision a learning route the class might invent with guidance of a teacher” (p. 510);
- Sequences experientially real for students – “Instructional tasks draw on realistic situations as a semantic grounding for students’ mathematizations” (p. 510); and
- Emergent models – “Instructional activities should encourage students to transition from reasoning with models of their informal mathematical activity to modeling their formal mathematical activity, also called emergent modeling (Gravemeijer & Stephan, 2002)” (p. 510).

The plan of the sequence of instructional tasks involved six phases of integer activity, with each phase including the following categories of items: tool, imagery, taken-as-shared interest and activity, possible topics of mathematical discourse, and possible gesturing and metaphors. To situate their interpretation of observed classroom events, the researchers used a version of social constructivism, called the *emergent perspective* (Cobb & Yackel, 1996), where learning is considered both an individual, psychological process and a social process.

Stephan and Akyuz’s description of the entire process, which constitutes an empirically-sustained, domain-specific theoretical model for the teaching of integers and integer operations, is a classic example of *design as implementation*. In the spirit of Cobb and Gravemeijer (2008), the researchers generated an initial domain-specific, instructional theory that embodied the classroom-based, activity-oriented, process of learning a specific mathematical content and which included a very detailed description of the representational tools, classroom interactions, and teacher interventions that were to sustain this learning. In describing the unfolding of their research study, Stephan and Akyuz stress the manner in which students engaged with the tasks. This emphasis, and the way in which teachers actually facilitated that activity, is central to *design as implementation*. It also helps to shed an explanatory light on Gravemeijer and Cobb's (2006) earlier statement that one of the primary aims of design research is *not* to develop an instructional sequence as such. More precisely, the description of the entire design process (including initial design, implementation, and revision) is intended to foster an understanding of why and how the final sequence is supposed to promote learning. The whole description supports others in implementing the sequence in other contexts and as such constitutes its theoretical role: that of a local instructional theory for a specific mathematical domain.

<u>A domain-specific frame for the CAS-supported co-emergence of technique and theory within the activity of algebraic factorization</u>

In our past research on the use of CAS technology in algebra learning (team of C. Kieran, A. Boileau, D. Tanguay, & J. Guzmán, and which has also included at various times: F. Hitt, P. Drijvers, L. Saldanha, M. Artigue, & A. Solares; see website: http://www.math.uqam.ca/~APTE/TachesA.html), we carried out several studies with classes of Grade 10 students, with each study involving multiple sets of CAS-supported task-sequences. In Kieran and Drijvers (2006), we reported on the design and classroom implementation of two of these task-sequences (see also Hitt & Kieran, 2009). (Note: For further discussion of task-design frames related specifically to the integration of tools, see Leung & Bolite-Frant, 2015.)

The design of our task-sequences drew upon:

- The intermediate level frameworks of:
    - The Anthropological Theory of Didactics (ATD) (Chevallard, 1999) with its task-technique-theory (TTT) tool;
    - The Instrumental Approach to Tool Use with its dual Vygotskian and Piagetian roots (Vérillon & Rabardel, 1995; Artigue, 2002);
    - Pólya's (1945/1957) mathematical problem-solving frame (especially the phase of "looking back", i.e., reflecting); and
    - Didactical Engineering (Artigue, 1992), the theory-based approach to conducting research with its emphasis on a priori mathematical and epistemological analyses for shaping not only the design of individual tasks but also their ordering;
- As well as previous domain-specific research in:
    - algebraic reasoning (Kieran, 1992, 2007) – in particular, Kieran's (2004) domain-specific model for conceptualizing algebraic activity in terms of its generational, transformational, and global/meta-level activity;
    - classroom interaction from a sociocultural perspective, in particular, the focus on the reasoning processes provoked by collective discussion in the mathematics classroom (e.g., Herbel-Eisenmann & Cirillo, 2009); and
    - tool-based activity involving CAS technology for symbol manipulation in algebra (e.g., Artigue, 1997; Lagrange, 2002).

In line with the ATD frame, which is an integral part of the instrumental approach to tool use, our focus was on the interplay between the technical and the conceptual, that is, on the techniques and theories that students develop while using technological tools and in social interaction. Crucial to the notion that conceptual understanding can co-emerge with technique, and in line with the Kieran (2004) model of algebraic activity, the transformational aspects of algebra (involving factoring, expanding, etc.) need to be linked to—especially during their early phases of learning—the global/meta-level activity of algebra (involving, e.g., noticing structure, generalizing, analyzing relationships, predicting, justifying, proving). As Lagrange (2003), has argued:

> *Technique plays an epistemic role by contributing to an understanding of the objects that it handles, particularly during its elaboration. It also serves as an object for conceptual reflection when compared with other techniques and when discussed with regard to consistency.* (p. 271)

Reflecting the above frameworks that underpinned our research, the design of the multiple task-sequences always involved the following five principles:

- Integrate a dialectic between technical and theoretical activity within a predominantly exploratory, inquiry-based approach;
- Integrate the CAS as an epistemic motor for developing students' theoretical thinking and as a tool for generating and testing conjectures;
- Interweave paper-and-pencil work with CAS activity with the aim of coordinating the technical and theoretical aspects of the mathematics;
- Include questions of a reflective nature where students write about how they are interpreting the content they are working on and eventually talk about and explain their way of thinking;
- Integrate such processes as pattern seeking, looking for different ways of structuring a given expression, conjecturing, predicting, testing, generalizing, and justifying.

One of the two specific task-sequences described in the Kieran and Drijvers (2006) article involved an elaboration of the factoring task of $x^n - 1$, a task inspired by the earlier work of Mounier and Aldon (1996). The design of this particular task-sequence, because of its strong focus on generalization, also drew upon additional domain-specific frames related to the processes of generalizing (e.g., Cañadas, Deulofeu, Figueiras, Reid, & Yevdokimov, 2007; Mason, 1996)—frames that shaped the following three-phase sequence for the individual tasks we designed:

1. Seeing patterns in factors and moving toward a generalization;
2. Refining a generalization – with conjecturing and reconciling; and
3. Proving a generalization.

The first phase, which involved CAS as well as paper and pencil, linked students' past experience with factoring to the generalization that they would be working towards regarding the factoring of $x^n - 1$. The beginning set of tasks was oriented towards noticing a particular regularity in the factored examples of the $x^n - 1$ family of polynomials for positive integral values of n and then justifying the form of these products. As is illustrated by the sample questions provided in Figure 1, the tasks aimed at promoting an awareness of the presence of the factor $(x - 1)$ in the given factored forms of the expressions $x^2 - 1$, $x^3 - 1$, and $x^4 - 1$. To promote *generalization* of the form $x^n - 1 = (x - 1)(x^{n-1} + x^{n-2} + \ ... + x + 1)$, students were then asked to judge the validity of the equality presented in Question 6. After students began to conjecture a general rule for the factorization of the $x^n - 1$ family, they were requested to reflect on how they might express this conjecture by means of symbolic notation, using the symbol n for the exponent, rather than specific integers.

1. Perform the indicated operations: $(x - 1)(x + 1)$; $(x - 1)(x^2 + x + 1)$.
2. Without doing any algebraic manipulation, anticipate the result of the following product: $(x - 1)(x^3 + x^2 + x + 1) =$
3. Verify the above result using paper and pencil, and then using the calculator.
4. What do the following three expressions have in common? And, also, how do they differ?
5. $(x - 1)(x + 1)$, $(x - 1)(x^2 + x + 1)$, and $(x - 1)(x^3 + x^2 + x + 1)$
6. How do you explain the fact that when you multiply: i) the two binomials above, ii) the binomial with the trinomial above, and iii) the binomial with the quadrinomial above, you always obtain a binomial as the product?
7. Is your explanation valid for the following equality:
8. $(x - 1)(x^{134} + x^{133} + x^{132} + \ldots + x^2 + x + 1) = x^{135} - 1$? Explain.

Figure 1. Some of the initial tasks from the first phase of the $x^n - 1$ task-sequence.

The next phase of the task-sequence involved students' *confronting* the paper-and-pencil factorizations that they produced for $x^n - 1$, for integer values of n from 2 to 6 (and then from 7 to 13), with the completely factored forms produced by the CAS, and in *reconciling* these two factorizations (see Figure 2).

In this activity, each line of the table below must be filled in completely (all three cells), one row at a time. Start from the top row (the cells of the three columns) and work your way down. If, for a given row, the results in the left and middle columns differ, reconcile the two by using algebraic manipulations in the right-hand column.

Factorization using <u>paper and pencil</u>	Result produced by the <u>FACTOR command</u>	Calculation to reconcile the two, if necessary
$x^2 - 1 =$		
$x^3 - 1 =$		
$x^4 - 1 =$		
$x^5 - 1 =$		
$x^6 - 1 =$		

Figure 2. One of the factorization tasks from the second phase of the $x^n - 1$ task-sequence.

An important aspect of this phase of the task-sequence involved reflecting on and *forming conjectures* (see Figure 3) on the relations between particular expressions of the $x^n - 1$ family and their completely factored forms.

Conjecture, in general, for what numbers n will the factorization of $x^n - 1$:

i) contain exactly two factors?

ii) contain more than two factors?

iii) include $(x + 1)$ as a factor?

Please explain.

Figure 3. A conjecturing task from the second phase of the $x^n - 1$ task-sequence where students examine more closely the nature of the factors produced by the CAS.

The third phase of the task-sequence (see Figure 4) focused on students' proving one of the conjectures that they had generated during the previous phase of the task-sequence.

Prove that $(x + 1)$ is always a factor of $x^n - 1$ for even values of n.

Figure 4. The proving task from the third phase of the $x^n - 1$ task-sequence.

The nature of the students' reflections related to the proving task was revealed by having some of them present and explain their proofs at the board, and by encouraging classroom discussion, query, and reaction to the presented proofs. Other instructional practice principles, which were more fully described in the teacher guide that we designed than in the Kieran and Drijvers (2006) article, included: (a) allowing enough time for students to grapple with and think through the given tasks (both individually and group-wise) before initiating collective discussion of this work; (b) eliciting students' thinking during collective discussions and encouraging them to share their ideas, questions, and conjectures, rather than accepting quick and easy answers or rapidly giving them the answers; and (c) supporting students in presenting their work and in having them justify their thinking.

The TTT framework guided the data analyses towards the identification of students' going back-and-forth between theoretical thinking and technical growth, and involving the interplay between CAS and paper-and-pencil. While this study could be characterized as primarily one of *design as intention*, its description of the entire process from the design of the task-sequences and learning environment, through to the classroom implementation with its details on the tools, classroom interactions, and teacher interventions that sustained the learning, make it also an example of *design as implementation*. The entire description constitutes its theoretical role—a domain-specific theoretical model for the co-emergence of technique and theory in a combined CAS and paper-and-pencil environment for algebraic factorization—one that can serve as a basis for further design research in the recursive process of domain-specific frame-development in this particular content area.

The five examples that I have just presented with respect to intermediate and domain-specific levels of frames all included attention to instructional support, some in the form of quite explicit principles, as in the Prusak et al. (2013) example—principles that delineated a clear set of indices related to the way in which the instructional environment and the designed task were to mutually support each other. However, Ruthven (2015) goes farther than this in his commentary chapter in the ICMI Study-22 volume. He argues that any framework for task design ought to address explicitly the following four aspects related to the *staging of the task in the classroom*: i) a template for the unfolding of the various phases of a task activity (e.g., as in Japanese Lesson Study); ii) criteria for devising a productive task (e.g., as in the Stephan & Akyuz (2013) example with its experientially-real task sequence, framed within Realistic Mathematics Education); iii) organization of the task environment (e.g., as noted in the Prusak et al. (2013) study that drew upon the Cognitive Apprenticeship frame); and iv) management of crucial task variables (e.g., as in the Theory of Didactical Situations with its *didactical variables* tool). Ruthven maintains that few, if any, frameworks for task design explicitly address all four of these aspects regarding the staging of the task in the classroom.

CONCLUDING REMARKS

Up to now I have examined the design process and task design from the perspective of the frameworks and principles that have underpinned a significant amount of the design-oriented research in mathematics education. The particular perspective that was used was that of grand, intermediate, and domain-specific levels of frames—a perspective that aimed at elaborating the ways in which frames and task design are related. However, such frameworks and principles do not tell the whole story of task design. Some of the scholars and reflective practitioners who engage in task design see it as being a much more eclectic activity than has been suggested thus far—activity that includes factors that have not yet been accounted for in the design process, such as its artistic and value-laden aspects.

According to de Lange (2012), educational design research does not provide usable knowledge for designers nor practical suggestions for design; neither does it offer a theoretical underpinning for educational design at the micro level. De Lange (2013, 2015) argues that this limitation of theory is due to the artistic aspects of task creation. In addition, values and aims enter into the design process. Frameworks and principles for task design will vary relative to philosophies of mathematics education. In this regard, Burkhardt (2014) points out that different groups of people, for example, *basic skills people*, *mathematical literacy people*, *technology people*, and *investigation people*, have different priorities with respect to curricular aims or goals in mathematics—aims that are reflected in the objectives of their design initiatives. Furthermore, Artigue (2015) has commented that, in addition to design where theoretical outcomes are often oriented towards the description of learning trajectories, task design can also be engaged in for the "production and reproduction of didactical phenomena attached to the functioning of didactic systems, such as phenomena resulting from the paradoxes of the didactical contract" (p. 327). For further discussion of these various aspects, see Kieran et al. (2015).

This chapter began with a description of the history of task design in mathematics education and went on to elaborate how more recent work with its levels of frameworks for task design has yielded theories that are both a resource for and a product of design research. It would seem appropriate at this moment to draw the chapter to a close by summing up the progress that we —as a community—have made over the past five decades with respect to task design.

We have come to see that our frames tend to be either holistic or multi-dimensional in nature. That is, the inspiration for our designs can come primarily from one quite global, intermediate-level framework (e.g., TDS, ATD, VT) or from a constellation of theories of different levels and different types (e.g., the various examples of domain-specific frames for the learning of particular concepts, procedures, or reasoning processes). But not all design frames are theoretical in nature. Lesson Study is a classic example of a craft-based design frame based on teaching practice, one where teachers with their deep, experiential knowledge are central to the design process.

We have also come to see that the task or task-sequence, which unfolds within orchestrated classroom activity, involves—at least implicitly—design principles related to instructional support that can help to realize the potential of the task. However, as Ruthven (2015) has commented, aspects related to the staging of a task in the classroom—aspects that figure in the design process and which underpin assumptions about the manner in which the task is to be shaped—need to be more fully developed in most design frameworks.

We have come to see that theories are both a resource for and a product of the design process. As a resource, they provide theoretical tools and principles to support the design of a teaching sequence. As a product of design research, theories inform us about both the processes of learning and the means that have proven to be effective for supporting that learning. Related to this dual and often dialectical role of theory in design research is the distinction between *design as intention* and *design as implementation*—*design as intention* addressing more specifically the initial formulation of the design, and *design as implementation* focusing attention on the process by which a designed sequence is integrated into the classroom environment, subsequently refined, and then theorized about. This distinction also has a bearing on the relative nature of the significance given to the design of the task-sequence or task itself within the design process.

We have become aware too that the grain size for describing principles for task design is an area for further reflection and development. Many of the design principles that were delineated in the examples given in the second section of this chapter, while being situated within specific frameworks, tended to be phrased in rather general terms that are subject to broad interpretation. The work of the educational designer Kali (2008) suggests the feasibility of considering, and possibly integrating, a much finer grain size of principles into our design work. We are reminded of the critical question that, according to Cobb et al. (2003), must be asked of our frames, that is, whether their principles inform prospective design and, if so, in precisely what way. Clearly, further theoretical work on grain size of principles for task design is needed.

Finally, we have come to see—based, in particular, on the activity of the ICMI-22 study on task design—that knowledge about design grows in the community as design principles are explicitly described, discussed, and refined. Although the examples cited in this chapter all specified the frames and principles underlying their designs—and also provided, for one case, some sample tasks to illustrate these principles—such is not common in the majority of papers presented at mathematics education research conferences or reported in research journals (Sierpinska, 2003). Despite the recent growth spurt of design studies within mathematics education, specificity of the principles that inform task design in a precise way remains both underdeveloped and, even when somewhat developed, under-reported. A possible obstacle that stands in the way of specificity can be traced to length constraints on published papers and the extended amount of space that the provision of specific details requires. Were it not for websites such as *Educational Designer* (http://www.educationaldesigner.org), there are few avenues for presenting the explicit and detailed thinking that lies behind the final versions of designed tasks or task-sequences. Nevertheless, it seems reasonable to expect that mathematics education researchers could be more explicit in their published research papers about the principles that underlie the tasks they design for their research studies. More work remains to be done in documenting such practice.

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A THIRD PILLAR OF SCIENTIFIC INQUIRY OF COMPLEX SYSTEMS—SOME IMPLICATIONS FOR MATHEMATICS EDUCATION IN CANADA

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ABSTRACT

The European Mathematical Society (EMS) was founded in 1990 and consists of about 60 national mathematical societies. Very recently EMS, in a Position Paper on the European Commission's Contributions to European Research *(EMS, 2011), stated: "Together with theory and experimentation, a Third Pillar of scientific inquiry of complex systems has emerged in the form of a combination of modelling, simulation, optimization and visualisation." (p. 2). As I explore some implications of this Third Pillar to all levels of mathematics education in Canada, I will also draw on the recent work by Weintrop et al. (2016) that proposes a set of classroom practices, which map onto the components of the Third Pillar in a number of ways. CMESG has a rich 40 year recorded history of Plenary and Working Group reports, and I will refer to a small number of these. For further insight, I will point to some of our research with Chantal Buteau which has focused on a set of* Mathematics Integrating Computers and Applications (MICA) *courses implemented by the Department of Mathematics and Statistics at Brock University in 2001. We have argued that these courses may provide an effective way for undergraduates to develop proficiency, through programming, in the Third pillar of scientific inquiry of complex systems.*

INTRODUCTION

For many of us 40 years spans a career. In the case of CMESG, the recorded proceedings provide an insight into a rich and vibrant organisation that was envisioned and established by David Wheeler and John Coleman. CMESG has and continues to influence mathematics education in Canada at all levels and for me it has, with the ICMI Studies and the MAA, provided an evolving set of platforms onto which to build my understanding in mathematics education. My focus was not so much on research in mathematics education but rather on implementation of mathematical education strategies. In publications, ideas are credited to their original authors, however in implementation, whether it be in classrooms, in programs, or for the general public, the originator is rarely acknowledged. In my case I have a multitude of CMESG colleagues who shared their ideas and experiences with me and on which I built my contributions to the execution of mathematics education throughout my career.

In this presentation I will explore how important mathematics, which has emerged through the use of technology, challenges the evolution of mathematics education in Canada. Unfortunately, I will only have time to touch on a few of the 40-year record of CMESG plenaries. However, there are other plenaries and recent working group reports that have addressed important education issues arising from mathematics in the Third Pillar.

WHAT IS THE THIRD PILLAR?

The Third Pillar of scientific inquiry of complex systems was proposed by a large body of mathematicians, namely the European Mathematical Society (EMS). The society groups together sixty national mathematical societies, forty research centres and has over three thousand members. A recent Position Paper on the European Commission's Contributions to European Research (EMS, 2011) states, "Together with theory and experimentation, a third pillar of scientific inquiry of complex systems has emerged in the form of a combination of modelling, simulation, optimization and visualisation" (p. 2).

These concepts are not new in the sense that mathematicians have undertaken these activities before and students are exposed to these ideas in 'traditional' mathematics courses. What is new is that the focus is on complex systems, systems which we were unable to analyze before the advent of the computer. Examples of complex systems involve very large sets of data, or they demand many variables and parameters which are deemed of equal importance in the model, or are discrete in nature, or their visualization requires dynamic representations, and so on. It is my view that because the Third Pillar points to mathematics that cannot be done without the computer, programming is intertwined with these concepts. Therefore, I propose that when paying attention to the teaching and learning of mathematics in the Third Pillar, students should be able to program and not have just a 'black box' computer experience.

In parallel with the developments around the Third Pillar, the mathematics education community has focused on the concept of *computational thinking*. In a recent publication, Weintrop et al. (2016) suggest a set of classroom practices, which from Figure 1 can be seen to map onto the components of modelling, simulation, optimization and visualisation in the Third Pillar.

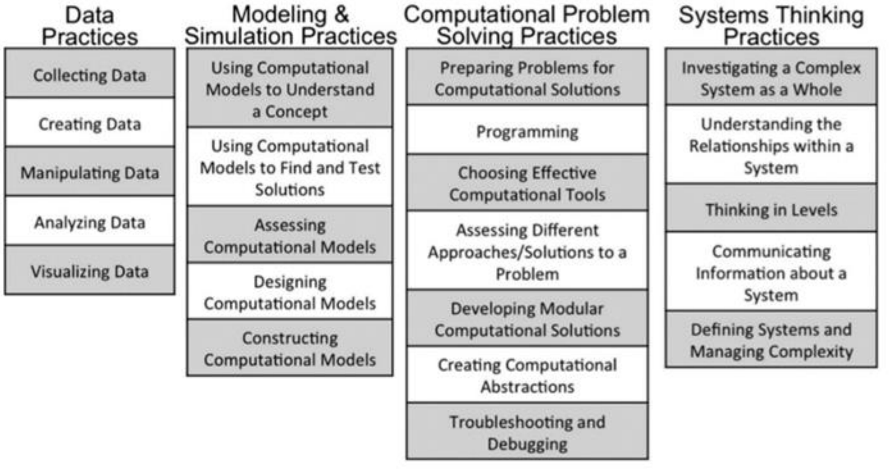

Figure 1. Computational Thinking Practices mapped out by Weintrop et al. (2016), p. 135.

At first sight mathematics in the Third Pillar appears to be distant from mathematics presently listed in Canadian mathematics curricula found in schools and universities. However, because mathematics in the Third Pillar builds on our knowledge of mathematics presently taught, it is my view that the gap is bridgeable as I propose in the following example.

A THIRD PILLAR CLASSROOM ACTIVITY

It may help us to identify with mathematics in the Third Pillar by taking an example of a problem that was reported in the press during the 1990s and that could be tackled in a secondary school mathematics classroom.

THE PROBLEM

In the 1990s, I was invited to make a presentation to over 100 mathematics teachers at a leadership conference of the Ontario Association for Mathematics Education. It was a time when probability was being introduced in the school curriculum and it was also soon after Marilyn von Savant had discussed, in her column of *Parade Magazine*, the previously known Monte Hall Problem. Her proposed solution generated a flurry of contradictory responses from lay and professional mathematicians (see http://marilynvossavant.com/game-show-problem/). In the OAME conference, we explored the Monte Hall Problem within the context of probability. However, in this presentation we look at it from mathematics of the Third Pillar.

The Monte Hall Problem was familiar to viewers of the popular TV game show *Let's Make a Deal* which was hosted by the Canadian actor, Monte Halperin, who used the name Monte Hall in the show. The winner in an episode of the TV show was awarded one of three prizes (either a Cadillac or one of two goats). The prize was selected as follows:

- the winner had to choose one of three doors behind which were hidden the prizes;
- Monte Hall knew behind which door the Cadillac was and would open a door not selected by the winner but which revealed a goat;
- Monte Hall would then ask the winner whether he/she wished to change to select the other unopened door.

The question for us is, "*Should the winner change his/her selection?*"

What advice would we provide and how would we support our recommendation?

THE ENVIRONMENT

For my OAME presentation I brought a set of manipulatives to enable the participating teachers to model the game and for them to experience, through simulation, the role of probability in the Monte Hall Problem.

1. Teachers were divided into fours which were further divided into pairs.
2. No verbal communication was allowed between pairs.
3. One pair was given 1) a six-sided die; 2) three identical cards, each blank on one side and marked with \$50, \$50, and \$500 000, on the other side; and 3) a fourth card that described the Monte Hall Problem, together with a probability p taken from the set {0, 1/6, 1/3, 1/2, 2/3, 5/6, 1}, which the winner would use to decide whether or not to change the selected door. This pair of teachers developed instructions to simulate repetitions of the prize selection by winners of the TV game show. The written instructions were then given to the second pair of teachers which they used to implement 25 simulations. An example of a possible outcome of the game is shown in Figure 2.
4. After some time, simulation results were recorded and displayed on a large board in columns based on the probability *p*.

1	2	3	Step 1: Winner chooses card 1
1	2	3 $50	Step 2: Host turns card
1	2	3 $50	Step 3: Should winner change?

Figure 2. Possible game.

Once all the results had been displayed we had an 'in-class' discussion and some of the main points that arose were:

- Not all the groups developed and performed simulations which were consistent with the problem and its additional probability constraint. In the incorrect simulations, it was noted that either the model and/or instructions were faulty or that the directions were incorrectly implemented.
- The game instructions developed by one pair for the second pair of teachers had to be very clearly written, i.e. explicit language used since there was to be no verbal communications.
- The data generated by the number of consistent simulations threw some light on a solution which was hotly debated. Many individuals had previously made up their minds on what the outcome should be. Some were of the opinion that it should make no difference, others had made up their minds that the winner should never change, while still others suggested that the winner should always change.
- Admittedly, the resulting data was rather sparse to be really convincing and more simulation results were needed. Unfortunately, we were running out of time and manpower. Fortunately, I had set some time aside to perform a previously programmed simulation.

I performed an Excel simulation of 400 repetitions at each of the seven probabilities from 0 to 1. Running the program many times and graphing the results provided more persuasive results. Even though for successive simulations the number of wins could change at each *p* value, the number of wins consistently increased as *p* ranged from 0 to 1 as is illustrated in Figure 3.

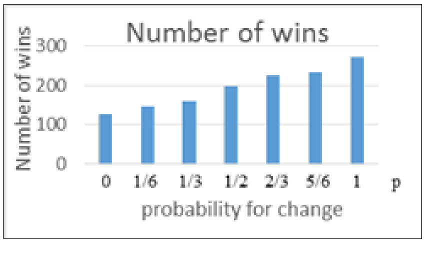

Figure 3

One of the things that made this problem engaging for the teachers was that individuals had differing opinions about its solution. This was also clearly demonstrated by the range of responses that were sent to Marilyn von Savant after she published the Monte Hall Problem in her column of *Parade Magazine*. Professional mathematicians also provided solutions which were not in agreement. The following account provides an interesting example.

Andrew Vazsonyi, a pioneer in Operations Research, was a friend of Paul Erdös, one of the most prolific mathematics problem solvers. They had grown up together in Hungary and were both successful mathematicians. As a feature editor of *Decision Line*, he recalled mentioning the Monte Hall Problem with its solution to Erdös. To his surprise Erdös said, "*No, that is impossible, it should make no difference*" (Vazsonyi, 1998, p. 18). Vazsonyi then showed him the results of a simulation of the problem to which "Erdös objected that he still did not understand the reason why, but was reluctantly convinced that I was right" (p. 18). What is important for us to note is that it was the result of a computer simulation that raised doubt in Erdös' mind about his assumed solution. Simulations can clearly play an important role in the learning of mathematics. It is interesting that the cover of Vazsonyi's (2002) autobiographical book entitled *Which Door has the Cadillac: Adventures of a Real-Life Mathematician* had an illustration of the Monte Hall Problem on its cover.

A SUMMARY OF THE THIRD PILLAR IN TEACHERS' ACTIVITIES

Today our reflections about the teacher activities focus on the mathematics of the Third Pillar. We note that one pair of teachers developed a *model* of the problem by employing the manipulatives that were provided—three cards and a die. The other pair of teachers performed a *simulation* of the problem following the instructions provided by the first pair. Data from the simulations were accumulated and displayed, both in tabular and in graphical form—*visualization*. In the resulting discussion, the teachers searched for the best solution, that is, the case that provided highest probability of winning the big prize—*optimization*.

Within the teachers' activities we can also identify some of the Computational Thinking Practices mapped out by Weintrop et al., (2016). Under *Data Practices* the teachers created, collected, manipulated, analyzed, and visualized data. Within the *Modelling and Simulation Practices* they were involved in designing a computational model that was first driven by individuals generating the data and were then exposed to a model where the data was generated by a computer. In the discussion that followed the activities, the teachers debugged models by identifying errors in instructions that had been written or that had been performed. I found that the model that I developed and then programmed for the Excel simulation had to be far more mathematically and logically detailed than the communication between teachers. In fact, in order for me to turn mathematical ideas into algorithms for programming, I needed to have a nearly perfect understanding of the underlying mathematics. For me, this makes a case for introducing some form of programming into mathematics curricula. Students simply cannot write functioning programs with a superficial understanding of the mathematics and its applications. This brings us to *Computational and Problem Solving Practices* which were not experienced by the teachers but which were central to my Excel simulation. The Monte Hall Problem that teachers explored was, for them, not a complex system, as I had broken it down to manageable bits. Therefore, the teachers were not involved to any extent with *Systems Thinking Practices* as identified by Weintrop et al. (2016).

I found relations to the Third Pillar in two plenary presentations in the adolescent years of CMESG. These significantly influenced my development in mathematics education.

TWO EARLY CMESG PLENARIES – MATHEMATICS AND TECHNOLOGY

In some of the initial meetings of CMESG, the role of computers in mathematics education was examined. In 1981, Kenneth Iverson, Canadian mathematician and computer scientist, was invited as a plenary speaker. While working at IBM he had developed APL (Address Programming Language). The title of his presentation was "Mathematics and Computers – Reflecting on Notation as Food for Thought". In the early 1960s and in one of my own PhD projects, I had programmed computers to explore numerical solutions of integral equations in the theory of liquids. Later at Brock, I was involved with the initial development of Computer Science courses. Even these experiences did not prepare me for Iverson's remark, that

> *Mathematical notation provides perhaps the best-known and best-developed example of language used consciously as a tool of thought* […] *Nevertheless mathematical notation has serious deficiencies. In particular, it lacks universality, and must be interpreted differently according to the topic, according to the author, and even according to the immediate context. Programming languages, because they were designed for the purpose of directing computers, offer important advantages as tools for thought. Not only are they universal (general-purpose), but they are also executable and unambiguous.* (p. 5)

For me, his remarks sparked a new vision for the role of programming in mathematics education. It also gave me an incentive to focus much more conscientiously on notation. At that time Tom Jenkyns, a Brock colleague, and I were developing a (two-term) course in Discrete Mathematics. By integrating the use of pseudo-code for developing algorithms we felt that the communication of the students' knowledge of mathematics would become more explicit or unambiguous, but of course not executable. We addressed this topic in one of the papers that we presented at the 1985 first ICMI Study entitled "The influence of computers on mathematics and its teaching" (Jenkyns & Muller, 1985). The study was held in Strasbourg and I attended it with Bernard Hodgson.

Recently, when I visited the cathedral in Strasbourg and viewed its monumental clock I was reminded of the 1988 CMESG plenary by Christine Keitel. In her presentation entitled "Mathematics and Technology", she focused on the "relationship between mathematics education and the social use of mathematics" (p. 1). What brought her plenary to mind was her example of a technology, the mechanical clock. Keitel described,

Photo by Alice Muller

> *Its construction is based on a particular perception of one aspect of nature, namely, time in relation to the movement of the planet system. This approach is generalized and condensed to a mathematical model, transformed into a technological structure, and as such installed outside its original limited realm of significance.* (p. 5)

The Strasbourg Cathedral 12-metre astronomical clock (shown at left) was completed in 1354 and occupies a wall of one of the chapels. Its mechanisms run moving displays that display time, a calendar, and astronomical indicators. It is ornately decorated with both static and moving sculptures, paintings depicting the influence of time, space, life, etc. It is my view that some of the ideas that Keitel developed in her plenary can be extended to the mathematics in the Third Pillar.

First Keitel proposed that "…the mechanical clock changes the relation between mankind and reality far beyond its original domain of application […] devaluating the authority of individual and collective (subjective) experience or insight" (p. 6). Analogously, integrating mathematics of the Third Pillar within computer technology has changed "the relation between mankind and reality far beyond its original domain". Sixty years ago, when computers were under early development, only scientists programmed computers for mathematics and scientific applications. Since then mathematicians have used mathematics in the Third Pillar to study complex systems, some of which had been formulated previously, but which could not be analysed through human power alone. Few foresaw the contributions to an expanding digital reality which has devalued 'the authority of individual', i.e. doctors, teachers, parents, and "collective (subjective) experience or insight", i.e. politics, news, weather.

Second, with the development of the mechanical clock, Keitel suggested that: “Earlier human perceptions of time, which had grown out of both individual and collective experiences and remained bound and restricted to these, were now rivalled and ultimately substituted by this novel kind of perceiving time” (pp. 5-6). Now with computer technology and the applications of mathematics in the Third Pillar, we find that previous human experiences of communications, warfare, privacy, etc., which had evolved from both individual and collective practices before these applications, are now “rivalled and ultimately substituted” as are they revolutionized by Wi-Fi communications, positioning systems, pilotless aircraft, encryption methods, etc. It is the mathematics of modelling, simulation, visualization, and optimization which powers most of these developments.

THE THIRD PILLAR AND MATHEMATICS EDUCATION

In this presentation, I used an example of the Monte Hall Problem to indicate where mathematics in the Third Pillar could be introduced at the school level. For an illustration at the university level, I propose a set of three core courses developed and implemented in 2001 by the Department of Mathematics and Statistics at Brock University. These original and innovative courses, called *Mathematics Integrating Computers and Applications*, or MICA (Ralph, 2001), were the result of the department’s response to technology and the new areas of mathematics which were later identified in the Third Pillar. MICA I is a first-year, one-semester mathematics course where students learn in parallel to program, design and use dynamic, interactive computer environments, called *Exploratory Objects* (EOs). The EOs capitalize on visualization and are developed to explore a mathematical concept, conjecture, or real-world situation. In the upper-year MICA courses students apply this knowledge to mathematics topics of increasing complexity (Muller, Buteau, Ralph, & Mgombelo, 2009).

Printed with permission of Bill Ralph.

The MICA experience includes students developing 14 EOs, in 11 of which the mathematics topic is assigned by the instructor, while for the remaining three, at the end of each term, the topic is selected by the student. In the past ten years Chantal Buteau, some of her students, colleagues and I have undertaken reflective and research work focused on MICA courses that were implemented and have been evolving since 2001. These include the following:

- One of the aspects of constructionism pioneered by Seymour Papert is that learning can occur most effectively when students are involved in making concrete objects. In the MICA courses students design, program and use their EOs to explore a mathematical concept, conjecture, or real-world situation. EOs are concrete objects of the digital world and we have explored the MICA students’ constructionist experiences of learning mathematics (for example, Buteau, Muller, & Marshall, 2015);

- The MICA courses differ in many ways from the ‘traditional’ university mathematics lecture-centered courses. First, the content is not traditional, it involves student mastery of programming, it focuses on mathematics that can only be reasonably undertaken with a computer; and second, the role of the faculty and TA is both one of mentor and of teacher. For university departments aiming to provide their students with experiences of mathematics in the Third Pillar, we have addressed the question: “*How do the students, teaching assistants, individual faculty, and the department perceive the nature of the MICA courses?*” (for example, Buteau, Marshall, & Muller, 2014);

- Faculty in the first-year MICA course face a major challenge of providing experiences for their students that develop their programming skills in parallel with their exploration of new and increasingly complex mathematical concepts. This first-year course has been running for over 15 years and faculty consensus is that students are most successful in their learning when programming requirements are introduced only as needed for the mathematics under consideration. In a number of publications, we have explored aspects of the instrumental integration of programming in the first-year MICA course (for example, Buteau & Muller (2014) and Buteau, Muller, & Ralph (2015));
- What competencies do students develop as they progress through the MICA courses? Marshall and Buteau (2014) suggested a list of potentially developed competencies which were then used in a student survey aimed at generating estimations of their acquisition and/or improvement of competencies potentially developed in their EO activities. Results of this survey suggest that the student's purchase of these selected competencies is enhanced by the repetitive development and use of EOs (Buteau, Muller, & Marshall, 2014);
- To gain some insights into the procedures that students follow to create and use EOs, we have developed a graphic that illustrates the task analysis of student activities (Buteau & Muller, 2010). The model was refined in a subsequent publication (Marshall & Buteau, 2014);
- In the technology intensive MICA courses, the instructor takes on the role of a mentor, as the students seek knowledge through communications technology and develop their understanding of the mathematics through their designing, programming and using of their EOs. For example, in MICA I, "The role of the tutors in this course is to carefully guide the students' instrumental genesis of programming technology for the investigation of both mathematics concepts and conjectures, and real-world applications" (Buteau & Muller, 2014, p. 163);
- There are few publications that explore how projects are assessed in university mathematics courses, and we did not find any that reported on assessment of projects which involved programming. Buteau and Muller (in press) have a paper accepted for publication in the DEME journal which details assessment in MICA courses at the time of writing. In university mathematics courses, assessment is the responsibility of the instructor and can therefore be subject to change with different instructors ; and
- We recently reported (Buteau, Muller, Marshall, Sacristán, & Mgombelo, 2016) the results of a case study that followed a student's learning experiences in the three terms of MICA courses, using a post-analysis of her 14 EO projects together with a post-MICA questionnaire. As a result, we suggested that

  > *The design of the MICA courses can be seen to provide a sequence of 11 stepping stones to guide the student's learning of programming as a tool within a context of increasingly complex mathematical ideas identified in the Third Pillar of scientific inquiry.* (p. 162)

ALICE RETURNS

In my 2013 CMESG plenary (Muller, 2013), I called on *Alice in Wonderland* and *Through the Looking Glass* (Carroll, 1960) to provide illustrations for the points I was making. As we have recently celebrated the 150th anniversary of the publication of these stories, I would be remiss if I did not bring her back. Let me evoke one of the incidents when Alice met, not for the first time, the grinning Cheshire cat and this time it

> *... vanished quite slowly, beginning with the end of the tail, and ending with the grin, which remained for some time after it had gone. "Well! I've often seen a cat without a grin," thought Alice, "but a grin without a cat! It's the most curious thing I ever saw in all my life!"*(p. 91)

Martin Gardner, in the annotated volume of Alice, suggests

> *The phrase "grin without a cat" is not a bad description of pure mathematics. Although mathematical theorems often can be usefully applied to the structure of the external world, the theorems themselves are abstractions that belong in another realm "remote from human passions," as Bertrand Russell once put it!* (p. 91)

In a recent *Mathematics Today* article, "Through the Looking Glass – Programming Environments for Advanced Mathematics", we (Muller, Buteau, & Sacristán, 2015) discuss the MICA courses for a broad audience of mathematicians and suggest that

> *Now with the Third Pillar of scientific enquiry, we can amaze Alice even more: by starting with the grin alone, we are able to reconstruct what the cat would most likely look like. For example, using* simulation *and* modelling*, and starting with weather data from satellites and other sources, we can predict what the weather will most likely be in the coming days*. (p. 280)

The weather predictions are *optimized* to reflect local conditions and generally displayed in an attractive *visual*.

CONCLUDING REMARKS

Both mathematics in the Third Pillar and 'computational thinking' pose significant challenges for the mathematics education community. In his CMESG plenary, Bill Ralph (2013) suggested that we should be considering two types of mathematical knowledge.

> *Autonomous knowledge would concentrate on traditional elements of mathematics like theorems and proofs, as well as solving problems by hand. Linked knowledge would revolve around Technology Assisted Problem Solving and Information Retrieval* [...] *society's need for this approach is so great that eventually mathematics teaching will have to be extended to include activities like the following:*
>
> 1. *Using expert systems to explore concepts and perform calculations*
> 2. *Visualizing and analyzing large data sets*
> 3. *Writing and using computer programs to build models, create simulations and investigate mathematical problems. ...* (p. 22)

We note that the attribute of 'complex systems' is included in both the Third Pillar, for mathematical sciences research, and in the work of Ralph (2013) and Weintrop et al. (2016), for mathematics education. These are complex systems within mathematics and its applications and they include mathematics of modelling, simulation, and visualization. Writing and using computer programs is specifically mentioned by Ralph (2013) and Weintrop et al. (2016)

As mathematics educators why should we introduce mathematics identified in the Third Pillar in school and post-secondary mathematics? Part of the answer may be found in Keitel's (1988) observation that we are living with a contradiction that has been with us for a long time, namely "*no modern society can exist without mathematics, but the overwhelming majority in a modern society can and do live quite well without doing nearly any mathematics*" (p. 7). This is addressed in more detail and from differing points of view by Jablonka and Gellert (2007) in the volume entitled *Mathematisation and Demathematisation*. It is obvious that education systems need to respond to meet the needs of both groups of individuals, namely, educate those individuals whose capabilities and interests will engage them in the mathematics necessary for their society to progress and also inform the overwhelming majority of individuals who "live and do quite well without doing nearly any mathematics". The gap between the mathematics requirements for these two groups has been widening as recent technologies are rapidly replacing the small amount of mathematics that was necessary to function on a daily basis. These observations have been in the forefront of curriculum reform, producing heated debates about how much of the 'traditional' mathematics should be retained and how it should be taught now that computer technology is readily available, a technology which produces consistently correct results for mathematics presently taught in schools and in the first two years of tertiary mathematics.

It is my view that by introducing mathematics of the Third Pillar in the curricula at all levels, and also by attending to 'computational thinking', mathematics educators will better address the needs of both groups. More specifically, individuals who have the capabilities and interest to contribute to mathematical knowledge and its applications using technology, in the broadest sense, will be better equipped to make their contribution in the digital world. Furthermore, for the overwhelming majority of individuals who are not aware of mathematics in their daily lives, an education that provides experiences of mathematics from the Third Pillar and of 'computational thinking' could provide them with some appreciation of the mathematics in the technology they use so frequently. Similar arguments were put forward for introducing probability and statistics in school curricula, namely that society (the large second group) would become better informed as individuals develop some experiences in the way their lives are affected by variation, odds, and by decisions made on the basis of this mathematics.

Although programming is not mentioned explicitly in the Third Pillar, it is included in 'computational thinking', and it is my view that by introducing some form of coding into mathematics curricula it can provide benefits, such as insight on how to communicate with certain technologies, different approaches to learning mathematics, and immediate feedback on logical and mathematical accuracy.

ACKNOWLEDGEMENTS

Many of these reflections have been developed after my retirement from Brock University in 2004. The research on the Brock courses would not have materialized without the collaboration with Chantal Buteau. I am sincerely grateful for her leadership in this work. I also wish to thank the CMESG Executive for inviting me to once again share some visions with the Group.

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STRUCTURE—AN ALLEGORY

Peter Taylor
Queen's University

When I was invited to give a plenary talk at this wonderful anniversary meeting, I was pleased and honoured. In addition, I felt challenged—challenged because I knew right away that I wanted to do something different—because that's what I also wanted for my current work with the high school math curriculum, something quite different.

See, there's this idea, this generally accepted idea of what a high school math class looks like that seems wrong to me, out of tune, in a sense, from what mathematics really is. I wondered if the plenary talk would open up some new avenues there.

I had some beginning ideas of what this talk might involve. It should live in the world of the arts as I often feel that the arts and humanities can provide a powerful but often unrecognized "ground" for the growth of mathematical ideas.

It should be deeply personal. Fundamentally that's what we all are, but in a large formal gathering, considering the audience as well as myself, I tend to be wary of working with deeply personal experiences. And thus I began to think about an allegory. I have always realized myself as a rabbit.

I wanted it to have a strong narrative structure because that's what I aspire to in the classroom as well. And as part of that I wanted another character to interact with, to highlight similarities and differences. Someone in the same universe as a rabbit, but in some sense on the other side: a squirrel perhaps.

By chance I encountered Judy Wearing a few weeks before the event and she turned out to be both a squirrel, and a writer. By chance? All my life I've viewed apparently chance events as flowing from (or through) some fundamental structure. I have always defined mathematics as the abstract study of structure, too. The notion of structure turned out to be the driving idea behind this allegory that Judy and I then wrote together. Along the way we learned about how rabbits and squirrels relate to structure in real life as we collaborated to create a piece that captured our philosophies, ways of life, and interactions with others in and out of education.

In the presentation we used slides, music, and lighting. We also drew in a major way from two sources that resonated deeply with the personal and educational messages we wished to work with—a poem, *The woman in this poem*, by Bronwen Wallace, a Kingston poet, and a book, *True and False,* by the playwright David Mamet.

The script in its entirety is available here.

Structure—an allegory

Written by Peter Taylor and Judy Wearing

The characters in an allegory are not real, but they are all the more real for that. They are the selves that stay hidden inside, deep at the core, the selves we fear, the selves we love, the selves we hide from others. Their behaviour is whimsical, juvenile, even ridiculous, but their interactions structure our lives and give it meaning, the meaning we are always striving for, but can never quite figure out where it could possibly come from. That structure rules but does not dictate, is powerful enough to make purpose out of randomness, but remains nevertheless so hard to capture that the allegory itself is the only truth that remains.

Acted by:

Peter Taylor (Rabbit)

Judy Wearing (Squirrel)

ACKNOWLEDGEMENTS

To those who contributed to this project:

- The Queen's Mathematics & Statistics Department
- The staff of the Isabel, particularly Kathryn Palaic and Kevin Tanner
- Kelsey McNulty and Sarah Dufour for the translations
- Jo Ambrose and Jacob Wilde for ideas and inspiration
- Helen Bretzke for her discovery of *True and False* and her acute understanding of the relationship between the character, the actor and the person.

♦♦♦♦♦♦♦♦♦♦

MUSIC: *Free to be*

Rabbit dances to music.

Music ends and Rabbit talks as he walks to podium.

RABBIT: Quand j'étais un petit lapin, je pouvais danser toute la nuit. Mais je ne dansais presque jamais. Tabernacle!

But when I did dance, that was my song! It talked of a land where I was free to be myself, and it painted that land as a light and joyful place. Now as an elder rabbit I know a lot more about that land. I know it is not what it seems to be. I know it lies well beyond the garden wall, and yes, it does indeed “lie” in both senses of the word. That’s a powerful pun that we will meet again later. Certainly the glimpses I have of the world beyond the wall are chaotic and scary—not at all the land of my childhood.

[Enter Squirrel. She walks to her podium.]

The world outside is where Squirrel lives. The garden wall is quite high but I can see her when she is in the branches. She scampers out to the end and leaps onto the power line over to the next tree. Sometimes though she just sits on the branch and looks down into the garden. Even after the snow comes and there is ice on the branches, she is up there heading for the power line. I can hardly watch.

Sometimes the garden gate gets left open and I can see the rough grass and dandelions in the park across the street. Trees but no bushes—nowhere to hide! A busy road with trucks and squealing tires. Dogs sometimes off the leash. I hardly dare to venture out. What if the gate closed before I got back in?

But when I’m safe in the garden, I dance to that song.

SQUIRREL: I’ve been in this tree for a while. It’s alright. There’s a good view of the stars. Good access to power lines, though I don’t trust them—I have learned not to trust much in this world. Thirteen times last night a bat touched down at the entrance to my nest, and each time I woke up and sank deeper into the tree. Once I heard the gentle click of claws on the bark—not a bat but an owl.

It’s the dogs I hate most. They want a piece of me, purely for amusement. I’m not fast, like Rabbit, and I have few places to hide, but I can change direction on a dime. That’s how I’ve survived this long.

From the high branches I can see Rabbit in his garden. He munches on carrot tops warily. Once in a while Mr. McGregor appears with his wheelbarrow and hoe, and then Rabbit bolts to the bushes. But it never looks as though there is any real threat. It’s more like a chess game on an orderly board with brown and green squares. I envy Rabbit, with his protective wall and his secure source of food. I wonder what I might accomplish if I did not have to worry so much.

RABBIT: I remember clearly that first day of term. I got there early, before 8—I wanted to make sure my room was set up—but already there were students at their lockers, milling the halls.

I have to say I was nervous—university I understand, and they’re just one year younger than my first-year students, but it seemed so different here, and I had to wonder why on earth I hadn’t just stayed in my comfy well-worn office chair. Well a bunch of us are thinking about the school curriculum, trying to give it some sort of new life, and if you’re going to play that game, I guess you gotta spend some time in the trenches.

And suddenly—on my way to my room—there was Squirrel. Could it be?—surely not, but she looked at me and said "Rabbit?" just as surprised as I was. She must be a high-school teacher—I'd always wondered what she did. But no, turns out she's here just for the term, like me.

Crazy 'cause we live right across the street from one another, though we've never actually talked, perhaps nodded to one another from time to time. But after the term got going we'd sometimes walk home together and share the happenings of the day. If it wasn't for the school we'd probably still only be nodding over the garden wall. Funny how things happen. I think there's a structure to the events of our lives—have you ever noticed?

That was early February, four months ago. Four months. I'm not at the school anymore. Didn't quite make it to the end of term. [shakes his head]

They're good kids. I miss them.

SQUIRREL: Odd running into Rabbit like that but nice to have someone to talk to on the walk home. No surprise to find out he's a math prof—he always seems to be counting carrots, measuring lettuces, estimating his distance from the bushes. I respect him for stepping out of his university life, trying to have an impact out of the ordinary.

I am aware, as usual, that he has a safe home to go to. I, however, am out on a limb. If I'm going to teach for a career, this maternity gig is my chance to show what I can do. I am pretty pumped about teaching adolescents. They're so adoringly unpredictable. All that zest for life, wearing their emotions on their sleeves, the self-consciousness. But at the same time, scared. I relate. And this is English 10, college stream. Lots of kids who are not at all sure about English, but have to take a provincial test this year anyway. If they don't pass, they don't get a high school diploma.

Maybe, just maybe, I'd find it this time. A tree I can call home. A place to fit in, where I can make a difference. Like my high school teachers did for me. They didn't treat me like dirt because I smelled like cigarette smoke and lived on the wrong side of the tracks. They gave me good grades. Because I deserved them. Now I'm grown up, maybe I can help other young squirrels find their way in the world. And get paid for it! All the nuts I can eat. My own house! Perhaps even—*a garden*.

RABBIT: I wanted to do something really different from what seems to happen in math class—not just different examples and problems—a whole new structure. So that it wasn't about what you learned but what you could make—with your hands, so to speak. 'Cause it seemed to me that math really didn't have any concrete meaning to most of them, not all of them but most of them. See the ones that get it, the few that actually get it, well I know those kids and it doesn't matter what you do with them as long as the quality is high, because they just grab hold and run with it, and come back later with questions. It's the others I worry about. Long ago they missed some critical connection and they can't make sense anymore. As a result, they lower their standards of what it means to make sense, and it's really hard to recover from that.

I wanted them to really care about what we were doing! All of them. I wanted them to locate mathematics at the centre of their lives. I wanted to capture them—no that's confusing 'cause I wanted to set them free.

Funny, that. I wonder if that's where things went wrong.

SQUIRREL: I love words. Reading, and writing. And I intended all along to help this English 10 class pass their provincial literacy test, but I've also wanted to show them how they might use words to find value in themselves and their ideas. I wanted to give them stories that resonate with soul and sinew. I wanted to wrap them in the emotional threads that bind us all. In joy, fear, and sorrow, we are the same. Each and every one of us.

This English 10 is my favourite class of all time. One third of the students have individualized learning plans, learning disabilities, behavioural issues. Several are repeating. Josh, in the back refused to participate at first. He handed in quizzes blank without his name. But I love this class because they're hungry. They gobble up attention, affection, like it is ice cream. I never tire of the sparkle in their eyes when I've found them a poem that speaks to them—you know what I mean—*right – to – them*. Like it was written—*just – for – them*. In less than a month, Josh was eager. And the Goth girl had put away her phone and her makeup kit. Kevin told me shyly he'd quit smoking. I can do it. I CAN make a difference.

…and the story proceeds from here.

Opening Panel

Panel d’ouverture

A VERY CANADIAN ORGANIZATION: SOME INDIVIDUAL OBSERVATIONS ON THE FIRST FORTY YEARS OF CMESG/GCEDM

William Higginson

My remarks as a member of the Special Anniversary Panel at the June 2016 Meeting were entitled: *A Very Canadian Organization: Some Individual Observations on the First Forty Years of CMESG/GCEDM*. My speaking notes on that occasion were about 8 pages long, roughly divided into four sections, being, respectively: *Context*, *Why CMESG is the Way it Is*, *What We've Done Well*, and, *Where We Might Do Better*. The first two sections had considerable overlap with my Elder Lecture from the 2012 Conference at Laval and are easily available in the excellent proceedings to that meeting. In this compacted form of my remarks I will, therefore, pass over the points of those sections relatively quickly and expand rather more on some parts of sections three and four.

The social context of four decades ago in this country was significantly different in very many ways from the age we live in currently. The rapidly growing tertiary education level of the mid-seventies was open to possibilities which it never had been before, and never has been since. The mathematics/mathematics education spheres were extremely fortunate to have the energies, imaginations, and influences of two outstanding educators, David Wheeler and John Coleman, who worked together very effectively to create an unusual organization. The framework and the Group's prevailing values were largely due, in my view, to David Wheeler, and these in turn had been very largely formed through his extensive work with the Association of Teachers of Mathematics in the United Kingdom before he immigrated to North America.

From my perspective, the Group has done a wide range of things exceptionally well for quite a long time. There would be widespread agreement among members, I would think, that for many the annual meeting has had very satisfying aspects on both personal and professional levels. [Think *grad-school refresher* crossed with *family reunion*.] I certainly enjoyed and benefited from this characteristic a great deal. What surprised me more, however, in retrospect, was the influence of the Group on my understanding of my native land. Partly, this was just geography. After a few decades I had a much better feel for many of the country's major metropoli—Montreal, Vancouver, Toronto, Edmonton, Ottawa, Calgary. But these I might well have come to know anyway. It was the insights into the medium and small cities that made the most impact on me. So memories of St. John's, Sherbrooke, Fredericton, Thunder Bay, St. Catherine's, Quebec City and Wolfville are particularly strong. And then there was the personal richness of colleagues—especially in the early years—who I might never have encountered without the Group. This was particularly true of individuals who brought a strong European (or, perhaps more accurately, non-Anglophone) and most enriching perspective to meetings. I am profoundly grateful to have been able to share the ideas of scholars like Dieter Lunkenbien

(Sherbrooke/Germany), Gerry Vervoort (Lakehead/Holland), and the remarkable team from Laval over the years, Fernand LeMay, Claude Gaulin and Roberta Mura among many others.

Let me turn then in my final section to a few reflections on some areas where I think that we might ‘do better’. It is here that the ‘very Canadian’ aspect of my title emerges, for CMESG has, for all of the reasons commented on above, been a *Canucky* type of enterprise from its earliest days: open, idealistic, diverse, enjoyable. But then why should it not be? Its members have the best job in the world (cf. the *Wall Street Journal*, no less, over many years) in the second best ‘country brand’ on the planet. To them much has been given. Behind the brand though there are general concerns about productivity. And here, I think, the Group does not bear close scrutiny either. Despite all of these advantages and resources, just what materials or initiatives can we point to that we have been instrumental in bringing forth? Have we succumbed to a version of what might be called the “Bankster Temptation”—namely—“*Aren’t these wonderful goodies??? Perhaps I’ll have them for myself???*”

The STEM fields in recent years have been raised almost above criticism, but for some there is a view that the version of mathematics which we have been (to varying extents, admittedly) complicit in promoting has been utterly corrosive and central to the neo-liberal nightmare that is unfolding. It probably was no accident that that ultimate shopkeeper’s daughter, Margaret Thatcher, succeeded in forcing her ‘shopkeeper’s arithmetic’ (in its various forms, such as contemporary economic modelling—what is coming to be known as latter-day astrology, or autistic mathematics) to the centre of her savage, nature-ignoring, world.

If there is any truth to such views, is there anything that a Group like ours might do to counterbalance it, with particular reference to the question of ‘productivity’ suggested above? My view is that much could be done and should be done. My reason for feeling that way is that I know of no other place where the human virtues and strengths of mathematics have consistently been displayed in stronger and clearer forms. Here there has been a robust balance between context/empathy and abstraction, unlike the bloodless version that has been the experience of almost all formally educated individuals in the twentieth century. So this is my challenge to the upcoming members of the group—don’t hide your way of doing mathematics from the world. Share and build on the many wonderful and modestly hidden examples that have circulated privately at our meetings for many years. In finishing let me commend to you the advice of one of the 20th century’s greatest mathematics educators, Israel Gelfand, late in a long career: *It is important not to separate mathematics from life*. Happy joining!

MES PREMIÈRES ANNÉES AU GCEDM

Bernard R. Hodgson
Université Laval

Je dois avouer tirer une certaine fierté, que j'espère légitime, de me retrouver au sein du groupe plutôt restreint des membres fondateurs du Groupe canadien d'étude en didactique des mathématiques. Malgré mon fort jeune âge (à l'époque…), je suis en effet du nombre des 33 participants à la toute première rencontre de ce qui allait devenir le GCEDM, rencontre tenue à l'Université Queen's du 31 août au 3 septembre 1977.

Je dois cependant à la vérité historique de signaler que j'y étais à titre de substitut, puisque c'est de fait Norbert Lacroix, directeur de mon département et mathématicien proche de John Coleman, le principal instigateur de cette rencontre, qui avait reçu une invitation. Étant dans l'impossibilité d'être présent, Norbert m'avait, avec l'accord de Coleman, en quelque sorte « désigné volontaire » pour y participer. Il faut dire que cet « envoi en mission » était naturel dans l'esprit de mon directeur—et peut-être aussi dans le mien, malgré la hauteur de la marche à gravir en vue d'une telle participation : je commençais alors tout juste une carrière de prof mathématicien, engagé dans un département de mathématiques et avec comme responsabilité principale… la formation des futurs enseignants du primaire ! J'étais vraiment conscient d'avoir des croûtes à manger à cet égard et j'étais à l'affût de toute occasion pouvant m'aider à mieux naviguer dans ces eaux toutes nouvelles pour moi. L'offre de mon directeur en était manifestement une que je ne pouvais refuser !

Et quelle expérience unique ce fut me rendre à Kingston, en cette fin d'été 1977 !

Je me rappelle très bien avoir été au préalable passablement intimidé à l'idée de me retrouver au milieu d'un tel aréopage d'experts en éducation. Ainsi je voyais, parmi les trois conférenciers pléniers au programme, mon collègue de la Faculté des sciences de l'éducation Claude Gaulin, que j'avais commencé à côtoyer à l'Université Laval, notamment dans le cadre du programme de perfectionnement en mathématiques pour le primaire *PPMM-Laval* qui venait tout juste d'être lancé—une belle aventure qui durera finalement près de vingt ans. J'étais fort conscient que Claude était une authentique « grosse pointure », particulièrement sur la scène internationale.

Et puis il y avait à titre d'hôte de cette rencontre le directeur du Département de mathématiques de Queen's, John Coleman… comme dans « Rapport Coleman »!

Durant mes études de doctorat (en logique mathématique), j'avais participé à un sondage dans le cadre de cette vaste enquête (Beltzner, Coleman, & Edwards, 1976) faisant un tour d'horizon des mathématiques au Canada. La version française de ce que tous, dans la communauté mathématique canadienne, nommaient tout simplement *le rapport Coleman* venait tout juste de paraître, et ce que j'y lisais ne contribuait certes pas à atténuer la sorte de désarroi que je ressentais par rapport à la rencontre devant se tenir à Kingston. Comment en effet ne pas être impressionné à la lecture d'un passage tel :

> *Le raisonnement mathématique est un plaisir. C'est l'emploi radieux d'un des pouvoirs les plus remarquables de l'intellect, l'utilisation explicite des structures implicites du cerveau noétique.* (Beltzner et al., 1977, p. 92)

Ouf! Heureusement le texte original anglais, lancé sur une sorte de cri du cœur typiquement à la Coleman, était de lecture plus accessible :

> *To mathematize is to joy! It is to exult in one of the powers distinctive of our humanity. It is to employ explicitly structures which are implicit in the higher levels of our brain.* (Beltzner et al., 1976, p. 126)

En outre, le rapport Coleman, dans son ensemble, présentait des perspectives fort inspirantes sur les mathématiques et leur enseignement. On y lit par exemple :

> *La plupart des professeurs de mathématiques du 1er cycle admettent qu'ils visent, non à inculquer des faits, mais plutôt à enseigner un mode de penser. C'est ce que nous entendons par la 'cogitation mathématique'.* (Beltzner et al., 1977, p. 107)

Et en anglais, cette cogitation mathématique devenait tout bonnement le « *mathematizing* » (Beltzner et al., 1976, p. 147).

Enfin, comme co-organisateur de la rencontre, outre un dénommé William Higginson dont j'ignorais alors l'existence (*so sorry Bill*—mais cette lacune fut vite corrigée!), il y avait surtout David Wheeler. Que dis-je : il y avait « Le Wheeler » !

C'est en effet sous ce vocable qu'était connu au Québec un de mes principaux livres de chevet de l'époque, le célèbre *Mathématique dans l'enseignement élémentaire* (Wheeler, 1970) — version française de (ATM, 1967). La version originale anglaise est présentée comme l'œuvre de « *members of the Association of Teachers of Mathematics* » et dans une « *Prefatory note* », on y donne la liste de plus de deux douzaines de collaborateurs à cet ouvrage, avec comme mention « *executive editor* » à côté de l'un des noms, D. H. Wheeler. Mais l'édition française a fait disparaître cette note liminaire et seul le nom « Wheeler » —sans prénom ni même initiale ! —figure sur la page couverture. Bref, ce livre fort populaire dans les années 1970 était connu comme « Le Wheeler », et j'allais donc me retrouver à Kingston en compagnie du « Wheeler » lui-même !

Mais finalement le tout s'est déroulé de façon on ne peut plus extraordinaire. Cette rencontre de 1977 fut pour moi sans l'ombre d'un doute un point marquant de ma carrière : une rencontre enrichissante, inspirante, stimulante… et tellement conviviale ! D'entrée de jeu, je suis tombé sous le charme du groupe et en ai été l'un de ses fidèles participants pendant de nombreuses années—jusqu'à ce que mon engagement dans l'ICMI/CIEM, entre autres, me force à une présence plus sporadique. Je me suis même retrouvé dès 1979 pour une première ronde sur le comité exécutif du GCEDM : j'ai beaucoup appris au fil des ans, tant aux plans humain que scientifique, d'une telle implication dans les « cuisines » du Groupe.

Sous le volet purement scientifique, mes premières années au GCEDM ont exercé une influence profonde sur ma façon d'appréhender les maths en tant qu'objet d'enseignement et d'apprentissage, tout en m'ouvrant des horizons insoupçonnés et fructueux vers des contextes pédagogiques possibles, voire inattendus, et remarquablement propices à la relation maître-élève. Mon travail à titre de mathématicien auprès des enseignants, notamment du primaire, s'en est trouvé profondément et indélébilement marqué.

J'ai déjà eu l'occasion de mentionner devant ce groupe combien les contacts s'établissant lors de rencontres du GCEDM peuvent mener à des moments charnières sur le plan personnel. Un tel épisode demeure vivement gravé dans ma mémoire (voir Hodgson, 2011, pp. 41-42), alors que David Wheeler, dans le cadre d'une conversation tout à fait informelle, m'avait quasi intimé de soumettre une contribution à la toute première *Étude de l'ICMI*. On était au milieu des années 1980, et jamais je n'aurais pu imaginer qu'un tel cadre était accessible à moi, le tout jeune prof. Les encouragements alors prodigués par David ont à l'évidence exercé une influence déterminante sur la suite de mon cheminement professionnel.

Les participants aux rencontres du GCEDM ont su créer une tradition exceptionnelle quant à la qualité des échanges et de la collaboration qu'on peut y développer. Mais comment en faire profiter davantage de nos collègues au Canada, notamment parmi la communauté des mathématiciens ? Je viens d'évoquer comment je me suis retrouvé, presque par accident, à la toute première rencontre du GCEDM. Et vous, qui—ou qu'est-ce qui—a provoqué votre première participation ? Et quels gestes concrets pouvons-nous poser, collectivement et individuellement, afin d'inciter, au cours des années prochaines, de nouveaux collègues, étudiants, amis… à vivre l'expérience de *leur première rencontre GCEDM/CMESG* ?

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TRANSFORMATION GEOMETRY IN GRADE 10

Peter Taylor
Queen's University

This panel session is supposed to be about looking back, but I will instead be looking forward and I must begin by apologizing for that. The reason for this is that I have happily been invited to make a plenary presentation at this meeting and it's better for me to use that occasion to look back.

I want here to talk briefly about *Transformations10* which is the first phase (hopefully of many more) of ProjectMath9-12 a research-based study designed to bring new life and sophistication to the high-school mathematics curriculum. We are part of a multi-university network and are supported by the Fields Institute, through the Fields Centre for Mathematics Education; by SSHRC, through a Partnership Development Grant involving seven Canadian universities; and by Queen's University.

Each project has three phases:

- curriculum development: work with students in a series of intensive sessions
- teacher training and interaction: prepare the teacher and facilitate development of teacher resources
- classroom research: document the teaching activity and the student learning.

Our general approach is inquiry-based, with an emphasis on student involvement in design and construction. The process of designing and building sophisticated structures out of simple components is our version of what is known as *computational thinking*. The transformation unit also draws from *spatial reasoning*, another fundamental human capacity. Both *computational thinking* and *spatial reasoning* are currently active areas in mathematics education, particularly at the elementary school level. In a real sense, the objective of Project Math9-12 is to construct a high school math curriculum that will be in place and waiting for the students who have graduated from the richer elementary school experience that we are now working to create.

SOPHISTICATION

One thing that might strike you is that the Transformation10 unit is at a higher level of what I call sophistication than is currently found in the high-school math curriculum, and on reviewing it you might suspect that the material will surely be too hard for most grade 10 students. I think 'hard' is the wrong word here, but it's certainly more challenging. That extra challenge comes from the sophistication of the subject—the power of the ideas and the complexity of the arguments the students will encounter, and in mathematics that is long overdue.

THE ARTS AND THE HUMANITIES

To get a sense of what I'm talking about here, consider that in art, music and literature the student has serious encounters with sophisticated works of art—works that the teacher herself finds interesting and engaging, works that inspire the students to construct comparable works with their own creative stamp. This is not generally the case in mathematics and I believe that is the reason so many students feel alienated from the discipline. In 1997, I wrote a *Facts and Arguments* about that in *The Globe* when my grade 12 daughter came home discouraged with a poor math test result. "*Don't forget*," I reminded her, "*that none of this is mathematics. This has nothing to do with your success in math. I know this because I am a mathematician, and these things are definitely not what I do.*"

…*none of this is mathematics*? I guess I'd better explain what I believe mathematics is, and since we are talking here about Grade 10, we'll take a look at that.

GRADE 10 MATH

The Ontario Grade 10 Academic Math Curriculum (Ontario Ministry of Education, 2005) is a pretty good document. It starts off with a Math Processes section that lays out the activities for a young math student to be engaged in—a familiar but important list: problem-solving, reasoning, selecting tools and strategies, connecting ideas, communicating. It goes on to give the details of its three strands: Quadratic Functions, Analytic Geometry and Triangle Trig—all important components of any mathematician's toolkit. The problem is that it doesn't tells us very much about how the tools can be used to do mathematics—it doesn't bridge the gap between the tools and the processes detailed on the first page.

MATHEMATICIANS

Well maybe that's okay. Perhaps the Ontario Ministry has wisely left this critical task to the mathematicians, the practitioners of the discipline who are presumably involved in the creation of tasks and activities and the writing of text-books. Well there are wonderful problems and activities out there on the web, but they are too often piecemeal and not aligned with the curriculum, so not easy for the teacher (who needs to 'cover' the curriculum) to effectively use. And the text-books seem to me to be collections of exercises in using the tools rather than sophisticated projects which involve us in, for example, *selecting strategies* and *connecting ideas*.

The tools are important for mathematicians because they allow us to design, construct and analyze structures of great beauty and sophistication, *and it is those structures that we call mathematics*. If the tools did not provide this, they'd be of little interest to us, just as a chisel and a lathe would be of little interest to my carpenter friend David Ambrose if they didn't allow him to build a beautiful table.

As I have observed, the humanities and the creative arts have important tool kits as well, but somehow they manage to have their students learn how to handle them and at the same time study, and even design and create real works of art. Why does that seem so much harder to do in mathematics? —that's a good question to meditate upon and your conclusions might surprise you.

I of course believe that in mathematics we *can* build beautiful tables and I put forward the Transformations10 curriculum as an example—with a hope of many more to follow. In a sense I might say that my overall objective is to craft a high school math curriculum modeled on the curricula in the arts and the humanities. For me it would be equivalent to say that I want the problems I bring in to the high school math classroom to be ones that catch the attention of and engage mathematicians.

And finally, in a sense these ideas *are* in fact 'looking back' as they accord well with many of the ideas we at CMESG have talked about over the past 40 years.

Transformations10 has been designed to effectively replace the last two of the three strands in the Ontario Grade 10 Academic Math Curriculum, Analytic Geometry and Triangle Trig. We are actively looking for adventurous teachers who might like to try this out.

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Closing Panel

Panel de clôture

INTRODUCTION

Peter Liljedahl, *Simon Fraser University*

CMESG has given much to our community and the field of mathematics education over the last 40 years. Looking forward, what can we offer over the next 40 years? This panel, rooted in our individual and collective experiences and history with CMESG aims to offer the members of CMESG/GCEDM, CMESG/GCEDM as an organization, and the field of mathematics education research a set of possible futures for mathematics education in Canada.

Au cours des 40 dernières années, Le GCEDM a beaucoup apporté à notre communauté dans le domaine de l'enseignement des mathématiques. En nous tournant maintenant vers l'avenir, que pouvons-nous envisager pour les 40 prochaines années ? Cette table ronde, ancrée dans l'histoire du GCEDM et dans nos expériences individuelles et collectives, vise à offrir aux membres du GCEDM, et au champ de la recherche en enseignement des mathématiques, un ensemble de réflexions et de pistes possibles pour le développement de l'éducation mathématique au Canada.

CMESG: FUTURE CONTRIBUTIONS TO MATHEMATICS EDUCATION

John Mason, *The Open University, UK*

CMESG is my conference of choice, and *flm* is my journal of choice. Why is that the case? I like CMESG because of the fundamentally phenomenological basis of the 9 hours of work on a single topic, and I prefer *flm* because it publishes thoughtfully insightful articles which are based on experience and which try to communicate something of that experience. In what follows I elaborate on these themes and why CMESG is so well placed to continue its significant contribution to mathematics education.

COMING TO KNOW

How do we come to know 'things' in mathematics education? Certainly there are no theorems, much to the annoyance of mathematicians. What sorts of 'things' is it possible to 'know'? I suggest that what is most needed is rigorous study of personal experience, tested against the experience of others, yet it is the most over-looked source of knowing in our field.

Large-scale statistical studies are all very well and often informative about general trends, but they are at best only indicators of what might be possible. They are highly unreliable as the basis for policy. The reason is that not only do they say very little about the individual (think mortality tables and life insurance) but such studies necessarily make dubious assumptions about the nature of human beings.

Many social science studies, hoping to inform policy and practice, assume that humans are rational; that they 'make choices' when planning and in-the-moment. In actual fact, as economists have realised only relatively recently, but as ancient psychologists knew only too well, human beings very often behave like machines, acting out of habit rather than out of informed choice. This is why behaviourists like Skinner were successful to the extent that they were, and why fitness and athletic coaches are able to be successful. People's behaviour can be trained. But when all you do is train behaviour, you end up with partial machines who, especially in mathematics lessons, do not remember all of the steps in procedures, and who cannot work out what procedure to use when the context changes. The issue is often referred to as the *transfer problem*. How near-transfer becomes far-transfer has perplexed researchers since the distinction was first made. Latterly it was transmuted into *situatedness* by socio-culturists, but unfortunately the transfer problem does not go away. It is simply transformed into "*how does situatedness become expanded or extended?*", because clearly it does, at least some times.

As Gattegno so appropriately noted, only awareness is educable (Gattegno, 1970; see also Young & Messum, 2011), which is part of three protases based on the western psychological distinctions of *enaction*, *affect* and *cognition*, namely,

> Only awareness is educable;
> Only behaviour is trainable;
> Only emotion is harnessable.

By awareness I believe Gattegno meant *that which enables action*. So an awareness of some phenomenon, of some emerging situation, can bring various associated actions to the fore, ready to be enacted. This is where true freedom lies: participation in a moment of genuine choice of action, rather than being caught in a flow of automated and emotionally triggered (re)actions. The importance of informed awareness, including awareness of awareness (Mason, 1998) cannot be overstated.

Mathematics education is plagued with a plethora of people reporting that subjects 'cannot do' this or that mathematical procedure, and an ever greater refinement of distinctions observed in subjects' behaviour. To do research you need a framework, a collection of distinctions to make. I often get confused when reading papers as to whether the researchers are trying to validate some distinctions they have made (by showing that indeed the distinctions distinguish between behaviours), or whether they are uncovering some distinctions that they think have not been previously made.

An observation is made, a distinction is formed, a phenomenon is identified. But does the distinction inform future actions? There is no 'best' action in any situation, only possible choices. The greatest effectiveness as well as the greatest experience of freedom appear when the choices are richest. When a researcher delineates several different categories (e.g. young children's drawings of something like a house or a grid of squares), can a teacher really be expected to internalise these and add them to their repertoire of distinctions to notice, in the hope that they might then inform their actions accordingly?

For example, mastery teaching has recently emerged as a policy-based slogan in England, and its associated rhetoric seems to require teachers to do two things simultaneously:

- To recognise what each learner is currently doing and saying, so as to know what task to present next;
- To keep all learners working on the same topic rather than accelerating some and being resigned to others not keeping up.

To do both of these requires great sensitivity to particular learners, not simply 'evidence-based actions' developed by studies of large populations, because it is about the teacher acting in the moment, according to particularities of the specific situation. This in turn requires a repertoire of possible pedagogic actions which will stimulate useful actions by learners, leading to activity in which they experience significant mathematical themes, the use of their own natural powers, specific mathematical concepts and through which they practise the use of important procedures.

All of this leads me to the question:

> How, as contributors to mathematics education, can we establish agreed ways both to negotiate meaning for technical terms and to critique their misuse, especially when they enter policy statements?

This is where I think CMESG has made, and could continue to make, a substantial contribution to the field of mathematics education.

PROMOTING A PHENOMENOLOGICAL STANCE

Surely no-one would give a lecture on 'teaching investigatively', nor on 'Problem Solving in the Classroom'. Surely the only sensible approach is to engage people in mathematical investigation and in problem solving. At various times participants can be invited to withdraw from the action and to reflect upon that action, to become aware of that action as a useful action.

Becoming aware of what was effective and what not so effective, of how it might be useful in the future, and considering what stimulated it or brought it to the fore contributes to educating awareness. Effective actions can be noted and labelled, and those labels can be used in the future to invoke those same actions. My conjecture is that it is only through becoming sensitised to notice opportunities to respond rather than react that effective teaching will develop. This is the basis for the *discipline of noticing* which is an elaboration of lessons learned experientially from J. G. Bennett, and expounded as experientially as I could manage in Mason (2002).

CMESG, through its intensive working groups, will, I trust, continue to promote an experiential approach to working on issues and concerns in mathematic education. Unfortunately, despite most courses which prepare people to teach being largely experiential (if not, what on earth are they doing, and how is their philosophy consistent with what is known about effective teaching?), educational research is stymied by the dominance of large-scale statistical studies as the source of knowledge, as ‘what is worth knowing’.

One thing that emerges from a phenomenological stance is scepticism about what credence can be assigned when people are asked to fill in a questionnaire, do some mathematical tasks, or respond to interview questions. Does the subject experience the probes in the way that the researcher experiences them? Does the subject react, enacting the first action that becomes available, or do they carefully consider their response to the question-task, selecting from among possible responses in the way that the researcher imagines they will? Perhaps their (re)action is emotion-led, or perhaps it arises from a different source beyond the interplay of cognition, affect and reaction.

SEEKING TO UNIFY DISPARATE FRAMEWORKS FOR TEACHING & LEARNING MATHEMATICS

Mathematics Education abounds with collections of distinctions (frameworks). Yet we use different words for pretty much the same thing and we use the same words for rather different things. These both inform and limit what can be discerned and related. Partly because of this, much of mathematics education seems to me to be in the doldrums: there is a plethora of theoretical distinctions, an emphasis on evidence-based actions, and an implicit requirement that to progress you need to refine previous distinctions or create new ones. It is hard to see therefore how the domain is to advance. Most papers published now refer to previous work no earlier than 2000, and usually much more recent. Yet the same issues and concerns have been worked on as long as I have been involved, and indeed much longer! I agree that each generation has to re-articulate the issues and concerns, and possible approaches, in their own vernacular, which includes their own situations, but to me the field, as presented by research papers, is currently much more like a merry-go-round than a cross country ski adventure or a harrier run.

As an example of unhelpful distinctions, consider Kirschner, Sweller, and Clark (2006) which claims that “unguided instruction [is] normally less effective” than “strong instructional guidance” (pp. 83-84). Surely no-one actually proposes ‘unguided instruction’? Even the much maligned *discovery learning* espoused by Bruner (1966) never meant learners being left on their own to ‘discover’ without any intervention or guidance. What is meant by terms, often used as slogans, such as *direct instruction*, *formal teaching*, *teaching through discussion*, *teaching by listening* and so on need to be elaborated so as to make them actionable.

Instead of accepting gross labels such as *discovery, constructivist, mastery, direct instruction,* …, it would be of real help to the community of teachers, educators, researchers, and policy makers to provide informative structure to these ambiguous and misinterpreted notions. Notable steps in this direction have been taken by Towers and Proulx (2013) in proposing three distinguished types of pedagogic practices: *Informing*, *Shepherding* and *Orienting*. The idea is, I think, to replace the unhelpful distinction between 'telling' and 'not-telling' with their inescapable negative and positive values, by recognition of the positive and effective roles of directing learner attention in different ways, from directly pointing (direct telling) through indirect indicating (shepherding) to supportive contexts for independent initiative. The role of tasks could be seen as providing experience as preparation in order to be able to hear, to make sense of, what the teacher can tell you. One thing that constructivism provides is a question as to whether, having told someone something, you think and act as if they now 'know it' in the same way that you do. Of course this is an example of *scaffolding* and *fading* (Brown, Collins, & Duguid, 1989), of direct, then increasingly in-direct prompts, eventually ending with learners initiating action for themselves (Love & Mason, 1992), which is what van der Veer and Valsiner (1991) suggest is what Vygotsky meant by the *Zone of Proximal Development*.

As an example of probing, connecting and integrating, consider the slogan *teaching by listening*. Apparently contradictory, it is, like so many slogans in mathematics education, vulnerable to multiple contradictory interpretations. However, the notion of hermeneutic listening used by Davis (1996) refers to the action of a teacher *listening-to* what learners are saying, and *watching-what* learners are doing, rather than *listening-for* what they want to hear, or *watching-for* what they want to see. One way to sensitise yourself to *listening-to* is through recognising the phenomenon that Malara and Navarra (2003) called *babbling*, by analogy with a young child in a cot making the sounds of sentences without yet having the words. The label *babbling* can alert you to trying to hear what may be behind the words, what learners may be trying to express, even though they may not be using terms correctly. So *babbling* can serve as a trigger for hermeneutic listening. The didactic tension (Mason & Davis, 1989) which arises from the work of Brousseau (1997) suggests that the more clearly and precisely a teacher specifies the behaviour they want learners to display (what they are looking and listening for), the easier it is for learners to display that behaviour without actually generating it for themselves. This explains why hermeneutic listening, 'teaching by listening', can be so effective. But it is all too easy to fall into 'training learner behaviour' rather than providing conditions in which learners 'educate their awareness' (Gattegno, 1970; Mason, 1998). As Towers and Davis (2002) say,

> *These attentive and tentative modes of engagement are offered in contrast to those that frame classroom interaction in terms of causal actions and control—which, once again, we might characterise in terms of a shift from architectural to biological senses of structure. An important element in this manner of pedagogy is its embrace of ambiguity and contingency.* (p. 338)

Retaining the complexity of teaching is vital, responding to and making use of the rich complexity of the human psyche, rather than trying to simplify acts of teaching as if on an assembly line. A contribution to structuring teacher-learner interactions can be found in the six modes of interaction proposed by Mason (1979) based on the systematics of Bennett (1956-1966) and the notion of mathematical themes and powers. To these can be added the five strands of mathematical proficiency proposed by Kilpatrick, Swafford, and Findell (2001), the five dimensions of mathematically powerful classrooms proposed by Schoenfeld (2014) and the habits of mind of Cuoco, Goldenberg, and Mark (1996). The overlap is significant and the expositions distinctive. I conjecture that for mathematics education to progress as a field, these need to be communally integrated. Put another way, more work is needed on simplifying and coordinating the many different ways of preparing oneself to make effective pedagogic choices when planning and in the moment, preserving the complexity of the human psyche but not over complicating it.

One domain of pedagogic choices that seems not to be mentioned very often has to do with learner engagement. By getting learners to make significant mathematical choices, by getting them to construct mathematical objects, exercises and examples for themselves, they can push themselves just as much as they feel capable of, rather than depending on the teacher to provide a range of examples suitable to different learners (Watson & Mason, 2005). These and other pedagogic strategies could be brought to teachers' attention more widely, through engaging them in effective personal experiences.

CMESG could use its working groups to work on integrating, refining and unifying frameworks of distinctions and associated particular pedagogical actions as a service to and model for the wider community. CMESG members could continue to promote 'staying with the complexity' of teaching and learning mathematics as an antidote to the policy-makers' drive to simplify and mechanise what is an intensely human, and hence complex activity. This could be done by developing an agreed approach to validating and using distinctions, and negotiating paradigmatic examples.

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INSPIRONS LE FUTUR

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Lorsque j'ai reçu l'invitation de participer à ce panel de clôture, la question qui s'est posée à moi portait sur la manière dont je pouvais aborder ce futur possible du groupe canadien et de la recherche en didactique des mathématiques. Il y avait là tout un défi. Ma manière d'y répondre, et ce en cohérence avec la perspective adoptée dans mes travaux récents, fut d'approcher une telle question en l'ancrant dans le passé ; un tel développement ne pouvant selon moi se comprendre que s'il prend racine dans une connaissance profonde de ce passé. Il s'agit là en effet, dans les études que j'ai menées au cours des dernières années, d'un enjeu central, comme en témoignent le travail mené, en collaboration avec Caroline Lajoie, sur une mise en perspective historique de la résolution de problèmes en enseignement des mathématiques (Lajoie & Bednarz, 2012 ; 2014) ou encore celui portant sur la genèse de la didactique des mathématiques au Québec (Bednarz, 2007). J'ai donc souhaité tout d'abord me replonger dans ce passé, pour en faire ressortir quelques éléments clés.

QUELQUES CARACTÉRISTIQUES SIGNIFICATIVES DU GCEDM

En parcourant, d'une part, différents écrits portant sur le GCEDM (voir notamment Coleman, Higginson, &Wheeler, 1977 ; Wheeler, 1992 ; Simmt, 2015 ; les différents compte-rendus des rencontres annuelles depuis 1977), et en partant, d'autre part, de ma propre expérience in situ dans différents groupes de travail, plusieurs caractéristiques émergent. (1) Un premier fait significatif saute rapidement aux yeux, celui de la présence au sein de ces rencontres, et ce depuis le début, à la fois de mathématiciens, intéressés par les questions d'enseignement des mathématiques, de chercheurs-formateurs en didactique des mathématiques, d'enseignants de mathématiques, tous impliqués dans ces rencontres, notamment dans les groupes de travail. Il est en effet plutôt rare de retrouver une telle participation dans les autres congrès en enseignement des mathématiques. Le groupe de travail auquel j'ai participé en 1996, sur le rôle de la preuve au postsecondaire, en est un bel exemple. Il illustre la fécondité de discussions entre différents acteurs sur un thème d'intérêt commun et qui dépassait nettement, dans nos discussions, l'ordre postsecondaire. (2) Un certain climat caractérise par ailleurs ces groupes de travail, qui constituent le pivot des rencontres du GCEDM : les interactions qui y prennent place tablent en effet sur la diversité d'expériences des participants du groupe, permettant en cela de générer quelque chose de très 'rafraîchissant' au regard du thème abordé, et ce en maillant ces connaissances et expériences que chaque participant amène au regard du thème abordé. (3) Ces groupes de travail sont aussi l'occasion d'intégrer de façon 'naturelle' de nouveaux chercheurs, enseignants, mathématiciens, etc. Les étudiants s'y sentent en effet interpellés au même titre que les autres. Se développe ainsi, au sein de ces groupes de travail, une certaine culture implicite, prenant forme dans des manières de faire partagées, qui se constituent dans l'action. (4) Enfin certains thèmes récurrents reviennent d'année en année, comme en témoignent les différents actes de ces rencontres. C'est le cas notamment de la formation des enseignants et des enjeux associés.

QUEL FUTUR POSSIBLE EN S'ARTICULANT SUR (EN PRENANT EN COMPTE) CES COMPOSANTES CLÉS ?

Comment tirer parti de ces faits significatifs et de cette culture mis en évidence précédemment ? Il nous semble ici porteur de pousser plus loin cette dynamique particulière au groupe canadien, en croisant les voix de mathématiciens, de chercheurs en didactique des mathématiques et d'enseignants de mathématiques pour investiguer certaines questions de recherche. Cette idée d'un croisement de regards s'inspire des travaux que nous avons menés en recherche collaborative depuis les années 1990, elle table sur l'installation de ce que Wasser et Bresler (1996) nomment une *zone interprétative partagée*. Dans un souci de double vraisemblance rejoignant les uns et les autres tout au long de la démarche (Dubet, 1994), cette approche de recherche maille les compréhensions, pour avancer, sur le long terme, sur des enjeux qui se posent en enseignement des mathématiques, les différents acteurs impliqués permettant d'en développer une compréhension profonde, de l'intérieur de leur pratique (telle qu'elle se constitue dans les cours de mathématiques, les cours de didactique des mathématiques, les études menées avec les enseignants, etc.), et ce en ayant eux-mêmes des ressources à offrir pour nourrir cette investigation. Il s'agit là d'une idée clé de la recherche collaborative, une approche que nous avons essayé de conceptualiser, de préciser sur un plan théorique et méthodologique (Bednarz, 2013).

UN PREMIER EXEMPLE : LA TRANSITION SECONDAIRE-POSTSECONDAIRE EN ENSEIGNEMENT DES MATHS

La plus grande partie des recherches menées sur ce thème ont adopté une perspective institutionnelle, se centrant sur l'analyse des curriculums, de tâches précises proposées à chacun des ordres dans un domaine spécifique, ou encore d'évaluations. Si ces travaux contribuent à éclairer les ruptures dans le passage d'un ordre à l'autre, d'autres aspects sont ici peu pris en compte, ceux se situant davantage au plan informel de la culture mathématique. Ils nous renvoient, comme le montre bien Corriveau (2013), aux *manières de faire les maths* (MFM) qui se constituent dans l'action, souvent dans l'implicite, à chacun des ordres. Pour comprendre ces MFM qui se constituent au quotidien, de l'intérieur de leur pratique, mathématiciens, didacticiens, enseignants de mathématiques au secondaire, au collégial, à l'université, sont ici des acteurs clés, pour avancer sur les difficultés de passage d'un ordre à l'autre et les pistes d'articulation. Un tel croisement de regards permettrait de nourrir les réflexions de la communauté (sur un exemple de tel travail pour le secondaire/collégial, voir Corriveau, 2013).

UN DEUXIÈME EXEMPLE : MATHÉMATIQUES 'ACADÉMIQUES' ET MATHÉMATIQUES AU TRAVAIL

Un important corpus de recherches a contribué à la clarification de ce que l'on appelle les *maths au travail*, via différentes études ethnographiques menées auprès de différents groupes de professionnels : infirmières, ingénieurs, banquiers, techniciens en électronique, etc. (Hall & Stevens, 1995 ; Janvier, Baril, & Mary, 1993 ; Kent, Bakker, Hoyles, & Noss, 2011 ; Masingila, 1994 ; Millroy, 1992 ; Noss, 2002 ; Noss, Bakker, Hoyles, & Kent, 2007 ; Noss & Hoyles, 1996 ; Zeverenberg, 2011). Ces études mettent en évidence la nature de ces maths situées, ancrées en contexte, 'différentes' sur plusieurs plans des mathématiques académiques (un 'bruit' important de la situation, du contexte de travail, donnant lieu à une restructuration conceptuelle, une abstraction située…). Pour avancer sur les possibles ponts entre ces différentes mathématiques, là encore le croisement de regards de différents acteurs clés que sont les mathématiciens, mathématiciens appliqués, professionnels, didacticiens des mathématiques, peut jouer un rôle important : il s'agit ici d'éclairer des MFM dans chacun des contextes, de l'intérieur de chacune des pratiques, pour explorer les articulations possibles, venant nourrir, en retour, les réflexions au sujet de la formation professionnelle.

UN TROISIÈME EXEMPLE : LA RÉSOLUTION DE PROBLÈMES DANS UN CONTEXTE D'ENSEIGNEMENT

Qu'est-ce qu'un 'bon' problème en mathématiques pour un enseignant au secondaire, au primaire ? Pour un didacticien des mathématiques ? Pour un mathématicien ? En quoi est-ce un 'bon' problème ? Quelles intentions guident le choix de 'bons' problèmes pour chacun d'entre eux ? Autant de questions pour lesquelles le croisement de regards de différents acteurs, mathématiciens, didacticiens, enseignants peut s'avérer fort riche, comme le montre le groupe de travail sur la résolution de problèmes tenu lors de cette rencontre annuelle auquel je participais.

Quels défis pose, par ailleurs, la résolution de problèmes en classe ? Dans un projet de recherche en cours mené par une équipe de chercheurs (Lajoie, Bednarz, Saboya, et Bacon), en collaboration avec des conseillers pédagogiques en mathématiques au primaire, l'un des enjeux majeur qui ressort des réflexions conduites conjointement est celui du pilotage du problème en classe, une activité qui mérite qu'on s'y attarde, notamment lors de moments clés : l'orchestration des discussions autour du problème et des stratégies de résolution, le retour sur ce problème, le séquençage des stratégies sur lesquelles revenir, la synthèse et l'institutionnalisation. Un enjeu complexe pour l'enseignant et l'accompagnement des enseignants, pour lequel le maillage de différentes compréhensions, peut s'avérer là encore porteur.

SUR UN THÈME RÉCURRENT, LA FORMATION DES ENSEIGNANTS : CROISER LES REGARDS POUR INVESTIGUER DE NOUVELLES QUESTIONS

Avant de revenir sur certaines de ces questions, je prendrai l'exemple d'une conversation entre un mathématicien et un didacticien, ayant donné lieu à une présentation lors d'un colloque portant sur la formation mathématique des enseignants (Gourdeau, Proulx, 2012). Une immersion de Jérôme Proulx dans certains cours de mathématiques donnés par Frédéric Gourdeau, auprès des futurs enseignants du secondaire, suivie de discussions entre Frédéric et Jérôme, a mis en évidence le potentiel d'un tel croisement de regards. Je reprends ici quelques-uns des éléments qui ont émergé de leur discussion : (1) au-delà des contenus abordés, ce qui se dégage est avant tout l'idée de faire des maths, de faire vivre une expérience mathématique ; (2) à propos de l'utilisation d'exemples génériques (très présents dans le cours de géométrie mais aussi dans les cours de didactique) : quel type de réflexion est amorcée dans un contexte mathématique et poursuivie dans les cours de didactique ? (3) d'autres types de raisonnements clés avec lesquels les étudiants s'engagent dans ces cours, et sur lesquels il serait intéressant d'aller plus loin, par exemple l'exemplification, la généralisation, la formulation de conjectures. Comment là encore s'articulent les expériences menées de part et d'autre ? ; (4) l'écriture mathématique et le processus de symbolisation : comment cet enjeu est-il abordé dans les cours de maths ? Comment prend-on en compte ce qui a été fait dans les cours de didactique à ce sujet quand on traite ce processus dans les cours de maths ? Cette brève incursion permet d'entrevoir le potentiel de ce croisement de regards entre mathématiciens et didacticiens.

QUELQUES ENJEUX À INVESTIGUER EN CROISANT CES DIFFÉRENTS REGARDS

Les questions de formation des enseignants sont souvent abordées par la formation mathématique, didactique, ou pratique. Mais des problèmes se posent, sur le plan de la recherche, dans l'articulation de ces différents aspects.

Une préparation mathématique, dans les cours de maths, qui prend la forme de cours de maths avancés ou de cours dédiés (destinés à des futurs enseignants)	Plusieurs travaux de recherche mettent en évidence la spécificité des mathématiques que les enseignants mobilisent en contexte professionnel (Ball & Bass, 2003; Ball, Charalambous, Thames, & Lewis, 2009)
Pourquoi considérer cette articulation ? Une formation à caractère professionnel qui interroge toutes les composantes de la formation (pas seulement la formation didactique et les stages, mais aussi la formation mathématique)	Une fréquentation estudiantine versus professionnelle : une rupture documentée (double discontinuité)

Articulation préparation math/ pratique professionnelle.

Quelles articulations possibles ? Il y a là un enjeu à documenter (sur le plan de la recherche) en faisant appel à un regard croisé de différents acteurs (mathématiciens intervenant dans ces cours de maths, enseignants, didacticiens).

ARTICULATION MATHS/DIDACTIQUE

Quelle articulation entre la formation math travaillée dans les deux cas ? Il n'y a pas en effet que dans les cours de maths que la formation mathématique est travaillée, les cours de didactique sont aussi une occasion de revisiter les concepts mathématiques et de travailler certains raisonnements clés. Quelle articulation, par ailleurs, entre les composantes mathématiques et didactiques ? Quelles imbrications ?

Il y a lieu d'interroger la manière dont les formation math et didactique sont souvent pensées (il n'est possible de s'engager dans un certain travail didactique que si une base math solide est présente). Plusieurs travaux de recherche mettent en effet en évidence l'imbrication profonde de ces différents aspects dans la pratique professionnelle des enseignants (aspects mathématiques, didactiques, voir institutionnels) (Margolinas et al., 2005 ; Bednarz, Proulx, 2009). Dès lors, celles-ci ne pourraient-elles pas être davantage pensées en interrelation (comme tend à le suggérer par exemple la conversation reprise au tout début de cette partie (Gourdeau & Proulx, 2012).

Dans cette articulation, des zones sensibles sont à prendre en compte, à la source de discontinuités possibles (à mieux comprendre) : par exemple les enjeux relatifs au symbolisme (défaire le symbolisme, les notations, conventions…de manière à mieux comprendre leur pertinence, leurs différents significations, ce qu'ils exigent, est un aspect central de la formation didactique ; explorer, par contraste, la nécessité et l'importance de la symbolisation et de la formalisation pour définir, opérer, démontrer, est sans doute central dans les cours de maths). Comment concilier ces deux aspects ?

POURQUOI LE GCEDM COMME COMMUNAUTÉ EST-ELLE BIEN PLACÉE POUR S'ENGAGER DANS DE TELLES INVESTIGATIONS ?

Qu'est-ce qui fait que le groupe canadien est bien placé, comme communauté, pour entreprendre de telles investigations ? Il existe, dans ces rencontres, un habitus de discussions, d'interactions, qui a pris place au fil du temps, des manières de faire qui se sont constituées dans l'implicite, permettant que s'installe un véritable rapport de symétrie entre les partenaires, dans la construction de nouvelles compréhensions, de nouveaux savoirs basés sur une négociation de sens.

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MATHEMATICS EDUCATION

Anna Sierpinska, *Concordia University*

At last year's CMESG meeting in Moncton, Working Group C (WG C) on "Theoretical frameworks in mathematics education research" attracted as many as 23 participants. A domain's growing concern with its methodological tools and theoretical foundations does not augur well for its future. It is never healthy to withdraw into oneself. Little worries become huge problems, while real problems remain unattended.

What are the big problems in mathematics education? We (Jérôme Proulx and I—leaders of WG C in Moncton) asked this question of the participants. One of them, Sophie René de Cotret, mentioned these very concrete problems of MATHEMATICS teaching: "*How to design instructional sequences that would entice students to develop mathematical knowledge that would be both applicable and meaningful for the students? What to take into account? Instructional variables? Instructional contract? Instructional environment? (The tasks, their context, the mathematical content, the available tools). Institutional constraints?*", and: "*Why are people not using the knowledge learned at school outside of school even in situations where it would be useful to do so?*" And Geneviève Barabé raised the issue of absence of mathematical content in the discussions about the teaching of mathematics: "*the debate is often about ways of teaching, without thinking about the contents.*" This remark applied also to the debates we had in the plenary sessions in WG C. Issues related to teaching particular mathematical content were sometimes discussed in small groups but they were never brought up when representatives of the small groups summarized these discussions in whole group sessions.

So this is my point: bring mathematics back into mathematics education; give it the attention it deserves. Social, cultural or affective issues are important but they are often not specific to mathematics education and if we do not address them, researchers specializing in sociology, anthropology or psychology or general education will and they have been doing it much longer than us. But nobody can replace us in critically analyzing the mathematical content of teaching and learning in lessons or courses or textbooks and in designing sensible mathematical learning environments. In doing so we may need knowledge of the social, cultural and affective aspects of the learning environments we study or design but this knowledge should remain at the service of understanding and resolving issues specific to mathematics teaching and learning rather than the latter being evoked only in passing to illustrate the more general problems of education.

I think it is urgent to problematize, to question the mathematical content of teaching at all levels; it is important to look at the textbooks, the exercise books, study them critically with future teachers, and, if necessary, publicly denounce the nonsense that is sometimes found in them. Examples are not difficult to find: the diagonal of a square assumed to have the same length as its side; confusion of quantities with abstract numbers; drawings defying the laws of gravity such as this one:

In the opening session we were reminded of David Wheeler's repeated calls for more visibility of the results of mathematics education research, of his calls on members of CMESG to publish, to write.

But today's world is flooded with publications, while people read less and less. They are too busy texting, tweeting, or posting their photos on *Facebook*. Decision makers are also too busy to read messages of more than 140 characters. Publishers allow the publication of textbooks and exercise books without checking them for inconsistencies. Today, it is not enough to publish to be heard. We would have to scream and shout. But how? I leave it to the younger generation to figure out the means of making our voices heard and our results to matter for day-to-day teaching and learning of mathematics at all levels.

MISSING PEOPLE—IMPORTANT OPPORTUNITIES

Walter Whiteley, *York University*

There are people who are missing from the CMESG gatherings and working groups. I notice three missing groups: (i) Mathematics Graduate Students, (ii) Math Teaching-Stream Faculty, and (iii) Teaching Mentors in Math Departments. Sustaining the vital collaborations of mathematics educators and mathematicians has been a key part of CMESG over the last 40 years, and is essential to our future.

I want to describe some future opportunities that might lie ahead: (i) support for developing teaching communities within mathematics and statistics departments, (ii) continuing a focus on research in tertiary math education, (iii) offering better mathematics experiences within mathematics programs for future teachers.

I give some links to examples of relevant activities which surround us and may be linked into CMESG planning and programs:

- 2015 MAA/NSF National Study of College Calculus generated an important report, with recommendations, which is available on-line at: http://www.maa.org/programs/faculty-and-departments/curriculum-development-resources/national-studies-college-calculus. The report recommends developing a culture in mathematics departments which use local data (research) to improve teaching in first-year calculus in colleges, small universities, and research universities. The report highlights the value of teaching as a collaborative community effort.
- The Carl Wieman Science Education Initiative (CWSEI) offers Science Education Research Post-Docs for recent math and science PhDs (see http://www.cwsei.ubc.ca/). We notice that Math Departments in universities often live within Science Faculties. The initiative offers post-doc training for future teaching-stream faculty, and these post-docs strengthen the CVs of strong mathematics PhDs preparing for teaching-stream faculty positions. The CWSEI mission is "Achieving the most effective, evidence-based science education" (CWSEI Home, 2017). The CWSEI helps departments take a scientific approach to teaching. The model is to help departments follow these steps: "(i) establish what students should learn, (ii) scientifically measure what students are actually learning, (iii) adapt instructional methods and curriculum [to incorporate pedagogical research], (iv) disseminate and adopt what works" (CWSEI Home, 2017, para. 2).
- MAA Project NExT supports newly appointed mathematics faculty members through mentoring and gatherings. Here are some themes they promote:
    - *Innovative approaches to a variety of introductory and advanced courses;*
    - *Using writing to help students learn mathematics;*
    - *Attracting and retaining students from under-represented groups;*

  - *Involving undergraduates in mathematical research* [Math Ed Research?]*;*
  - *Preparing future K-12 teachers of mathematics;*
  - *Writing grant proposals; and*
  - *Balancing teaching and research.* (Project NExT, 2017, para. 3)

- The MAA Special Interest Group in Research in Undergraduate Mathematics Education holds annual conferences, with proceedings on line; offers on-line resources to support publishing in this area; links to pages on TA training; and provides community resources to improve undergraduate mathematics education. (See http://sigmaa.maa.org/rume/Site/About.html)

- With respect to efforts to develop a teaching/learning culture in mathematics departments, here are some samples of initiatives: (i) openness to non-lecture classes (e.g. Moore Method: https://en.wikipedia.org/wiki/Moore_method); (ii) classes with group work, using vertical, non-permanent surfaces for active classrooms; (iii) an Ontario requirement for a culture of Learning Objectives and Assessments in all programs; (iv) the potential for hiring for leadership in teaching; and (v) being attentive to selecting/mentoring teaching-stream faculty. Departments are struggling with whether this culture is a priority.

What role is there for CMESG in supporting these developments in mathematics departments, and in supporting the growing community of teaching stream faculty? Here are a few that come to mind:

a) Pay regular attention in planning working groups, plenaries, … and inclusion of these themes in FLM.
b) Act to give career support for math grad students, post-docs who want a teaching career: e.g. Post-Doc Math Ed Talks.
c) Consider some collaboration with Science Education within the program.
d) Encourage collaboration of mathematicians and mathematics educators on teacher prep.
e) Consider cross-faculty support for Math Ed in Faculties of Education and Education people in Mathematics Departments.
f) Pay attention to developing positive collaborations between mathematicians and mathematics educators: our best route to avoid the Math Wars is by collaborations.

Let me close by highlighting a different theme that I think is important to the future of CMESG—selective advocacy. This issue was raised by Bill Higginson in his opening comments. This theme is highlighted by the *Manifesto on Unstuffing the Curriculum*, selected for the 40th anniversary book. This Manifesto was rejected as a CMESG position paper, by the CMESG executive, citing the principle, "We do not take stands!" In the future, I hope that CMESG will consider taking stands.

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Working Groups

Groupes de travail

COMPUTATIONAL THINKING AND MATHEMATICS CURRICULUM

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To reading, writing, and arithmetic, we should add computational thinking to every child's analytical ability. (Wing, 2006, p. 33)

INTRODUCTION

In our working group we studied computational thinking (CT) and its integration with mathematics learning and teaching at all levels, from preschool to undergraduate. To frame our discussion, we focused on the following themes:

- Conceptualizing CT in the context of, and in reaction to, the needs of citizens of a 21st century society;
- Meaningful mathematics tasks in the context of CT;
- Looking ahead concerning CT in math education: research and teaching.

Throughout the three days, various mathematics activities of different CT types (screen-based, off-screen pseudo-code, and tangibles) concerning different education levels grounded our discussion, geared toward exploring the three themes as summarized next.

CONCEPTUALIZING COMPUTATIONAL THINKING

'Talking mathematics' to a computer… is to learning mathematics what living in France is to learning French. (Papert, 1993, p. 6)

We started by attempting to conceptualize CT. What is CT? Is it a new way of thinking, and if so, what are its defining features? What CT looks like in education is not well-defined, as it has not really been integrated widely in mathematics curricula (Grover & Pea, 2013; Lye & Koh, 2014). Nevertheless, Wing (2006) argues that "computational thinking… represents a universally applicable attitude and skill set everyone, not just computer scientists, would be eager to learn and use" (p. 33). Furthermore, Hinsliff (2015) asks if a child with no programming skills will indeed be left behind. One of our goals was to discuss what computational thinking is, and more specifically what computational thinking in mathematics education is.

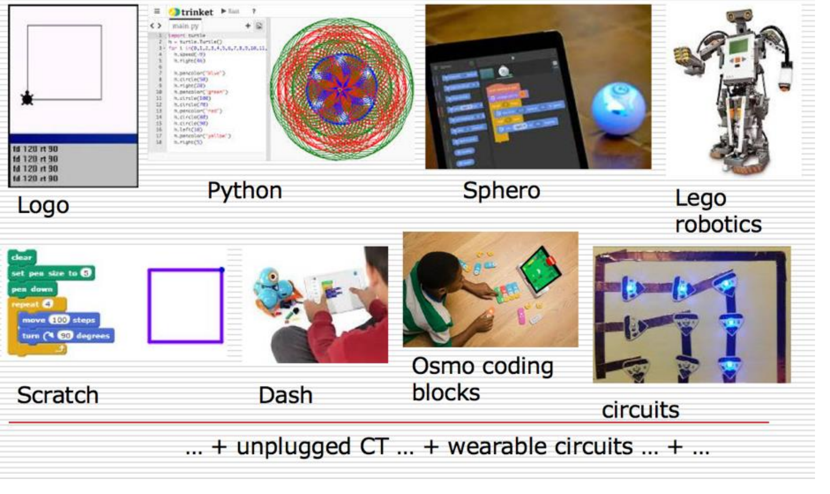

Figure 1. Different forms of CT.

CT can come in a variety of forms; see Figure 1. To start off our reflection, we engaged in the first day in using *Scratch* (https://scratch.mit.edu/) and *Python* (http://cscircles.cemc.uwaterloo.ca/console/) to investigate patterns with squares, and using unplugged probability activities that lead to the binomial distribution. Our guiding questions were:

- What is common/different in these activities?
- How differently from using paper & pencil do I engage in 'a square'?
- What is this? What else like this is possible?
- Does it complement mathematics? Or, is it mathematics? How?
- What role does programming play in CT?
- What role do algorithms play in CT?
- What kind of technologies can be connected to CT?

To further our reflection on the nature of CT and CT in mathematics (learning), participants were also invited to read one or two publications of their choice from a list, many of which appear in the reference list. A selection of key excerpts was also shared during our session. For example, concerning CT in general, Brennan and Resnick's (2012) three-dimensional framework of CT comprising

> computational concepts *(the concepts designers engage with as they program, such as iteration […]),* computational practices *(the practices designers develop as they engage with the concepts, such as debugging projects […]), and* computational perspectives *(the perspectives designers form about the world around them and about themselves).* (p. 3)

As for CT in mathematics as a discipline, the European Mathematical Society (2011) has identified as the third pillar of scientific inquiry: “Together with theory and experimentation, a third pillar of scientific inquiry of complex systems has emerged in the form of a combination of modelling, simulation, optimization and visualization” (p. 2). Furthermore, the synopsis of a 2016 six-month-long thematic semester on Computational Mathematics in Emerging Applications at Centre de Recherches Mathématiques mentions:

> *Un changement fondamental est en train de se produire* […]. *Le rapport entre la modélisation, l’analyse et les solutions des problèmes mathématiques dans les applications a changé* […]. *Dans les applications émergentes, le choix des modèles va de pair avec les outils computationnels et l’analyse mathématique* […]. *Des domaines tels que* […] *l’imagerie médicale font un usage de plus en plus fréquent d’algorithmes et requièrent de nouvelles méthodes de traitement, utilisables en pratique et efficaces. Les techniques des mathématiques computationnelles sont particulièrement adaptées à ces problèmes.* (CRM, 2016)

Finally, about CT in mathematics education, Weintrop et al. (2016) provide a taxonomy of CT practices in the school mathematics and science classroom; see Figure 2.

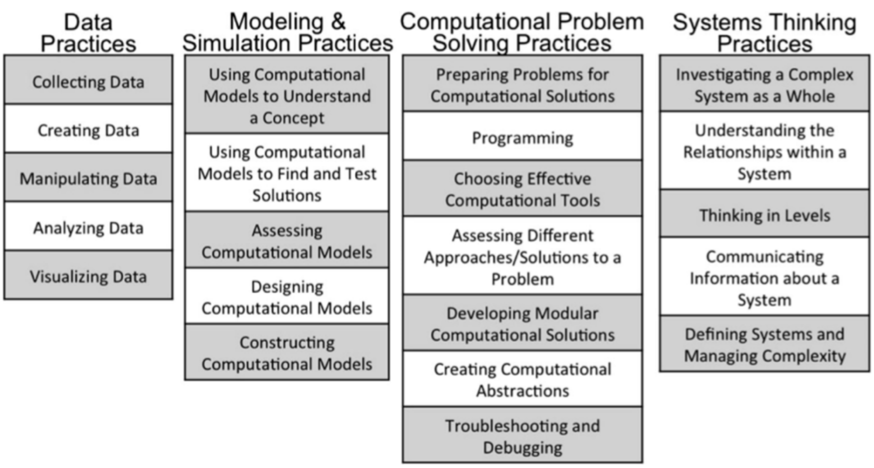

Figure 2. Computational thinking practices, taken from Weintrop et al. (2016), p. 135.

On the last day, we engaged in a Venn diagram exercise involving digital circuits that would illustrate the overlap of mathematics, CT, programming and digital tangibles. The discussion led to an agreement that programming was included in CT, and that there was some intersection with mathematics (mathematical thinking), although details of the nature of this intersection remained open to debate.

Below are some themes that arose from our discussion over the three days of our working group.

1. What is CT (in/for math learning)?

    a) *Kinds of ‘Thinking’ in CT*. Algorithmic thinking, logical thinking, innovative thinking, and also scientific thinking and efficient thinking.

b) *Rephrasing the question to 'What CT is not?'* Is CT always related to a task and is mathematical thinking more than that?
c) *Programming as tangible mathematics.* Coding brings an intermediary representation of mathematics between the teacher and the student.
d) *Similarities and differences between mathematics and CT*. Syntax, if-then, iff: there are anecdotal reports that some students with programming experience seem to be having more difficulties with induction proofs.
e) *Problem solving.* What distinguishes CT from other tools for problem solving? For example, does CT, and more specifically programming, assist students to break down a problem into smaller components and does it stimulate the development of intuition? Can one provide a math problem that is not CT? What distinguishes problem solving in CT? 1) Is it the power to do iterations? 2) Is it that abstract math concepts may become 'tangible' coded artefacts?

2. What may CT contribute to mathematics education?

a) *Agency*. For the learner, where does the knowledge (or information?) reside? There seems to be a change in CT. It is no longer in the 'teacher' but results from the 'interaction computer–student'. CT may contribute to deeper engagement in mathematics, and/or application of logic and attention to detail.
b) *Investigation affordance.* One of the affordances of coding is the visualization of the procedure. It allows an investigation approach: What if I change this? What if...? It's a conjecturing affordance. It also encourages persistence.
c) *Instant feedback.* Computing may provide an environment that is more accessible (different cognitive models) for students to engage in the mathematics. Could this be related to the instant feedback affordance from the computer? When coding a problem, one can code little bits then test. This small steps-testing isn't possible in a paper-pencil environment.
d) *Making mistakes is 'OK'*. Perspectives: e.g. making mistakes is OK, it is part of learning. Debugging might be part of any discipline, but according to Papert, programming is a natural environment. How we experience making mistakes in mathematics is different from when coding.
e) *Abstraction*. How is abstraction important in CT? What is 'abstraction' when programming or more generally when engaging in CT activities? How is CT abstraction similar to or different from mathematical abstraction?
f) *Logic learning.* Programming can teach logic (much needed, and often not covered in curricula).
g) *Equity issue.* Do highly user-friendly programming environments that look like game environments (e.g. *Scratch* versus *Python*) increase the social differences in the access to developing CT?

MEANINGFUL MATHEMATICS TASKS IN THE CONTEXT OF CT

The role of the teacher is to create the conditions for invention rather than provide ready-made knowledge. – Seymour Papert

We were also interested in the pragmatic matter of integrating CT into mathematics curricula at all levels of education, and in particular, in meaningful CT-based mathematics task design. By creating engaging problems and tasks, we could envision what the teaching of mathematics through CT would look like. For example, one could seek to find ways to transfer the 'fun' of constructing and 'playing' with computer code into mathematics; i.e., to investigate CT as a tool to stimulate students' interest and increase their motivation to do mathematics. We could envision that CT problems and tasks could: (a) be integrated into existing curricular objectives; (b) suggest novel approaches to teaching certain topics; or (c) introduce new topics into mathematics curriculum. As an example of (a): students could use programming to analyze one-variable or two-variable data in their data management course; by having to write a program to calculate various statistical quantities they will be able to understand the formulas that are used (and which are hidden when they use *Excel* or other ready-made software). As an example of (b): some truly exciting and ‘low floor, high ceiling’ problems come from number theory. By using a given simple code, and modifying it as they see fit, students could investigate patterns hiding in the sequences produced by examining the Collatz conjecture. Such activities fall, for instance, within *Number Sense and Numeration* and *Patterning and Algebra* strands in Ontario Grades 1-8 Curriculum (Ontario Ministry of Education, 2005). As an example of (c): by using computer code (again, given in a rudimentary form, and then student-modified), to investigate how a certain quantity reacts to changes in another quantity, students could be introduced very early to the concepts of a variable and a function. Introduced in high school mathematics courses in somewhat abstract ways, students find these two concepts quite challenging, and some struggle with them even in university math classes. ‘Tangible’, ‘real’ experiences with variables and functions could help them overcome these challenges.

In order to stimulate reflection on this topic on integration of CT, working group participants were offered a menu of CT-based mathematics problems on the second day. As illustrations, we mention three problems:

1. Estimate the surface area of a lake, drawn on a grid paper. No specific method has been suggested.
2. Write a code for *n* steps of a random walk in two dimensions. Write a code for a self-avoiding random walk (meaning that no location can be visited more than once), which stops when it is impossible to move any further.
3. Using the interactive visual *Geometer Sketchpad* (GSP) simulator, find the sequence of transformations that transform a given shape into its transformed version.

We briefly discuss problems 2 and 3 since they attracted the most attention and engaged most participants. Problem 2 is an example of a ‘low floor, high ceiling’ problem, i.e., it requires very little mathematics to start but can lead to deeper mathematical concepts (Gadanidis, Hughes, Minniti, & White, 2016). The one-dimensional random walk is a classical problem that could be given to early elementary school students. Here, we suggested that the participants study a two-dimensional walk. This example could be viewed as representative of what a true integration of CT and mathematics entails: it is not only that solving mathematics problems requires programming, but also the code testing phase requires that we go back to mathematics and use additional results. In Appendix 1 we provide a detailed description and analysis of the problem.

Problem 3 uses transformational geometry (usually discussed in university classes), adapted for grade 9 and 10 students by one of our participants (Peter Taylor). As introduction, students are given the set of basic ‘building blocks’, i.e., elementary planar transformations: translations (dilations), reflections, rotations and shears. Once they internalize the meanings and properties of each building block, students work on constructing more complicated transformations. They use a combination of pencil-and-paper calculations and the software that enables them to track the outcome of their calculations. For example, a student might be given the transformation of a square (Figure 3) and is asked to construct it as a sequence of elementary transformations.

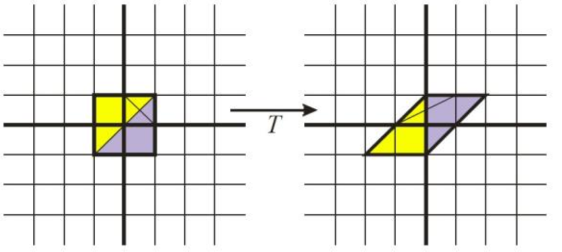

Figure 3. Investigating transformations in a plane.

The guiding questions for these activities concerned looking at three different perspectives:

- How do these tasks change the role of the student (how differently do they engage in the math)?
- How do they change the role of the teacher?
- And what about the curriculum? How does mathematics change? What technology is best suited and for which tasks? What are the characteristics of a 'meaningful math task' for the development of CT (design principles)?

Our engagement in these tasks provided rich grounds for discussion. Here are some themes that emerged from the discussion.

READY-MADE SIMULATIONS

Problem 3 uses a simulation that is already implemented, and thus does not require programming. Students use it to investigate transformations, and check the answers they obtained by using pencil and paper. There are many ready-made pieces available on the Internet (for instance, applets that help investigate graphs of families of functions). What are advantages and disadvantages of working with ready-made simulations?

PEDAGOGICAL CONCERNS

Engaging with CT demands that students learn a programming language. What are good teaching practices: do we teach mathematics first and then programming, the other way around, or simultaneously? There are computer programming packages that are very intuitive and easy to learn (such as *Scratch*), but which might not be suitable for math problems at high school or university levels. As well, we need to make students aware that the aim is to know necessary programming to engage with math and not to become a professional programmer.

ROLE OF ALGORITHMS AND CODING

Translating a math problem into a form suitable for programming (i.e., creating algorithms) is definitely a worthwhile activity, as it supplements our understanding of mathematics. However, the syntax aspect of coding (computers still cannot read our minds, so we need to use correct, non-ambiguous syntax) is potentially a large issue, as debugging for syntax is time consuming, and could take our attention away from important aspects of the program, such as its logical structure.

GENERAL CONCERNS ON VALUES

How important is CT for living in our technologically-driven world? By focusing on CT, are we neglecting other aspects of education that might be equally, or more, relevant? What social justice or equity issues are affected by presence, or absence, of CT?

REAL APPLICATIONS

A significant, but not often mentioned, benefit to using programming in mathematics is related to working with true applications. Feurzeig, Papert, Bloom, Grant, & Solomon (1969) observe: "The richness of non-numerical examples open to programming can be exploited to enlarge the cultural base of the mathematics course by bringing it into contact with physical and biological science, language study, geography, economics, and other subjects" (p. 11).

IMPLEMENTING CT INTO A UNIVERSITY MATH CURRICULUM

Successful implementations—such as the *MICA* program at Brock (see Appendix 3), or the problem-solving course *Math 3G3* at McMaster (Lovric, 2016) —transcend the boundaries of traditional courses, not just by introducing different types of activities and novel approaches to looking at math, but also because they include topics from a wide variety of areas of mathematics. As non-traditional math courses (for instance, not having a precise, rigid, course outline, nor the exact place within the course structure) they are close to what we might think as ideal (in Wing's words, to have a course which seamlessly incorporates geometric, numeric, algebraic and CT). Perhaps that is the farthest we can go, and not insist on injecting CT into, say, calculus or linear algebra courses.

RESEARCH, FUTURE OF MATH AND CT, CURRICULUM THOUGHTS AND IDEAS

I am convinced that the best learning takes place when the learner takes charge.
– Seymour Papert

In recent years we have witnessed the introduction of programming and/or CT in school curricula in various countries around the world—e.g., Australia (Cameron, 2015), England, Estonia, France (Chiprianov, 2016), New Zealand (Parsons, 2016), Sweden—and in a number of provinces in Canada—e.g., British Columbia (Silcoff, 2016), Nova Scotia (Willick, 2015).

As mentioned previously, what CT looks like in education is not yet well-defined, as it has not really been integrated widely in mathematics curricula (Grover & Pea, 2013; Lye & Koh, 2014). France might provide an example with their recent integration of what they call 'algorithmic thinking' in their school mathematics curriculum (Ministère de l'éducation nationale, 2009). Overall, the current insertion of CT in education is more of its own curriculum area, as an end in itself (e.g. in England), rather than integrated with existing subject areas. So, how do we effectively integrate CT into mathematics teaching and learning? What questions should be at the core of our work? What are the tools that we want, or need?

In the last day, we mainly focused on reflecting on the research that we believe would be helpful to develop a better understanding of CT and its possible integration into mathematics education. Participants formed groups to suggest pertinent research areas and implementation strategies. The following ideas were generated:

IMPLEMENTATION STRATEGIES

1. *Professional development (and pre-service education).* Invest in professional development for implementation of CT in schools:
   a) E.g., having math teachers work with CT activities (programming-based, unplugged, etc.) and the same day, trying it in a classroom, coming back to their group to discuss: *What kind of thinking is this supporting?*
   b) Have teachers involved in the design of CT tasks;

c) Work with teachers by first asking them what they need help with, and then explore if/how CT could be integrated.

2. *Curriculum.* In designing national/provincial curriculum, should CT be a separate area (e.g., in England) or should it be cross curricular (e.g. in the US?), or primarily as math education (e.g. in France)?

   a) *Cross-disciplinary approach.* Using CT development as a unifying theme across disciplines e.g. around problem solving. Can the problem-solving focus be reframed both in mathematics and other disciplines thereby providing a unifying theme of developing CT (transversal competence)? Is it possible to specify disciplinary concepts and processes in mathematics and CT that would be desirable at each level concurrently and conceive/develop/adapt/validate a progression of appropriate activities that would help to develop understanding within the unifying theme?
   b) *Mathematics curriculum re-structuring.* What kinds of CT could/should be incorporated into (existing?) mathematics curricula and how can this be done? Investigate the possibility of making CT a backbone around which we could build a cohesive mathematics curriculum, unconstrained by the traditional 'boxing' of math sub-disciplines and topics. What kind of assessment should be used to determine whether or not the introduction of CT in mathematics classes has resulted in improvement (improvement of what?)?
   c) *Learning by making.* Using coding as a means to making. This may provide a different perspective to look at existing topics in the mathematics curriculum.

RESEARCH AREAS/QUESTIONS/DIRECTIONS

1. Does students' engagement in CT activities impact their ability to think mathematically?
2. Does the incorporation of CT by teachers in their classroom change their mathematics instruction?
3. What are the affordances of specific environments (e.g., *Scratch*, *Python*, off-screen pseudo-algorithms, etc.) for learning mathematics?
4. How are task design principles for mathematics, for CT, or for CT-math similar? In particular, are the design principles for CT-math tasks unique?
5. Can CT be used to provide different pathways to mathematics concepts under consideration? Does coding require a different, perhaps more detailed and in depth, approach by the student to the mathematics under consideration?
6. How do the affordances of CT (abstraction, dynamic modeling, automation, …) interplay with the mathematics experienced, student thinking, teachers' pedagogy, ...?
7. (How can we identify) which areas (topics) of mathematics that are most meaningful for CT integration? (research/implementation)
8. How does CT as 'a making experience' interplay with the mathematics experienced, student thinking, teachers' pedagogy, ...?
9. Historically, where has CT been in (math) curriculum? Where is it today? Where is it heading to? Who are the stakeholders?
10. It has been suggested that spatial reasoning develops better mathematical thinking generally. Does CT also develop better mathematical thinking?
11. What are the competencies developed from math-CT tasks?
12. How does the 'low floor-high ceiling' nature of CT disrupt the grade specific math curriculum and teaching that is common in schools?
13. How does the discourse about integrating CAS, dynamical geometry software, statistical software, etc. in our math instruction relates/fits in/enrich our discussion about CT? What are elements of these environments that are related to CT? How could

we build on those tools to promote CT? E.g. a teacher might prepare worksheets for students, do all the coding, and then students use them. Could/should it be modified for CT-rich math activities?

14. Looking at the different levels of coding (coding, modifying a code, working on the interface, …), how do students interact with computer tools? What are the affordances proper to each level?
15. What are the social and affective aspects of integrating CT in the mathematics classroom?
16. Can the rich literature on technology integration in mathematics education inform the integration of CT?

CONCLUSION

Throughout our working group sessions, we explored the potential of CT as a vehicle to bring excitement, exploration and experimentation as routine activities (and not 'add-ons') into mathematics curriculum.

In short, this is what CT in mathematics (learning) is for some of us:

- Innovation and creation and construction
- Problem solving
- Student agency affordance
- Deductive and inductive reasoning
- Making abstraction tangible and/or more concrete
- *Penser avec les contraintes et les possibilités d'un ordinateur*
- *Concevoir l'ordinateur comme une extension de soi*
- Modeling and simulating
- Externalizing our thinking
- Debugging one's thought process
- Testing versus proving
- Creating a low floor and high ceiling for mathematics
- Dealing with large data
- Structure operationalizing and enriching
- Programming / coding
- Visualizing

APPENDIX 1

ANALYSIS OF A MEANINGFUL MATH TASK AT A TERTIARY LEVEL

Miroslav Lovric, *McMaster University*

The random walk problem turned out to be the most attractive, and several groups of participants worked on it. To discuss various aspects of this problem, we provide sample code in *Python* (participants were invited to use whichever coding language they were most comfortable with). A typical code to produce a random path could look like the code on the left in Figure 4. The array called *path* initially consists of a starting point, and then, after each iteration, another point is appended to the array. Thus, after all required steps of the random walk are completed, the entire walk is saved as a sequence of points, and can be graphed, as in Figure 4 (right).

```
N=25 #number of steps
x=0 #starting point, x-coordinate
y=0 #starting point, y-coordinate
path=[[0,0]] #list of all visited points

for i in range(1,N+1):
    a=ra.randrange(1,5)
    if a==1:
        x=x+1
    elif a==2:
        x=x-1
    elif a==3:
        y=y+1
    else:
        y=y-1
    path.append([x,y])
```

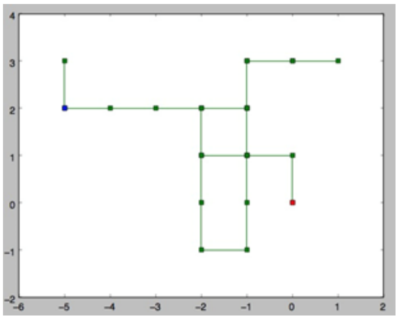

Figure 4. Python code for a random walk and a sample output.

Alternatively, we could use *Scratch*, Figure 5, with self-explanatory code.

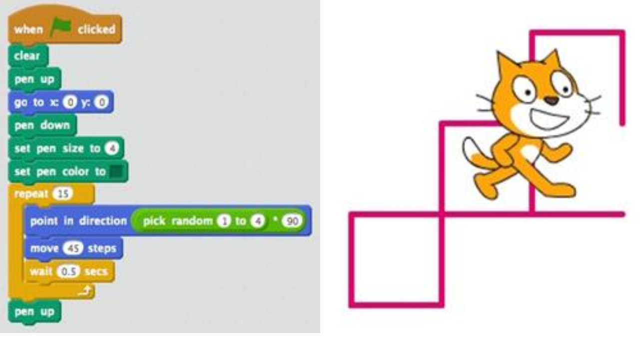

Figure 5. Scratch code for a random walk and a sample output.

Naturally, a question of testing the code emerges (testing and debugging, see Brennan & Resnick, 2012): how can we ensure that the code is correct, i.e., that it gives a correct output? Of course, we could go over the lines and verify the logical structure, but how can we be sure that the `randrange` command in *Python* or the `pick random` command in *Scratch* indeed return each of the four numbers 1, 2, 3 and 4 with equal probability?

For a deterministic code, we could check the output against the cases that we can compute by hand, or by using other means. But what are the appropriate cases to use, and how many of these do we need to be confident that our code is correct? Testing a given code (*main practice*, see Brennan & Resnick, 2012) is generally not an easy task, as we need to test it on number of *representative* cases, which are selected so that they ensure that all possible situations are verified. For instance, we need to check all different paths through conditional statements, or make sure that the loops terminate after a finite number of steps.

Testing a code suggests an interesting and worthwhile mathematical activity. In a standard first-year university Calculus course we often do not discuss proofs (mostly because they are too complex; for instance, the Extreme Value Theorem is routinely covered in Calculus, but its proof is deferred to upper-level courses). What we could do in our first-year courses is to ask students to generate representative test cases, and verify theorems in those cases. Such activity would motivate students to generate different functions which display different types of behaviour.

Going back to the random walk—how do we test the code? In the one-dimensional case, we rely on the theoretical math result (thus this coding question brings in mathematics!), which states that the distribution of ending locations of random walks (all with the same number of steps) is normally distributed (i.e., bell-shaped), with the mean located at the initial point. This experience suggests that we seek an equivalent mathematical result for a two-dimensional random walk. And there is one—the average distance, after n steps (assumed of unit length), is equal to the square root of n. Thus, we repeat the random walk many times, say 10 000 times, record the end points, and compute the average distance (standard deviation).

Several working group participants engaged in coding the self-avoiding walk (in which one cannot return to a point that has been previously visited). Confident that the standard random walk code was correct, they now used it as a starting point to create the new code (*modular approach*, another main coding practice according to Brennan and Resnick, 2012). The self-avoiding restriction means that the random walk could get stuck, unable to advance (i.e., come to a location all of whose neighbouring locations have been previously visited). One easily discovers that the more steps requested, the easier it is for a random walk to get stuck at some location. An interesting extension of the problem consists in finding a relation between the number of steps and the probability of the walk to get stuck.

We can extend this concept further, for instance by investigating random walks in three dimensions, or random walks where the probabilities of moving in different directions are different (i.e., we sample random numbers not from a uniform distribution, but from some other distribution).

Clearly, starting with a very simple concept, we can create 'high ceiling' activities, which motivate students to ask their own questions, experiment, hypothesize, visualize; in other words, they have a chance to practice habits of mind (Cuoco, Goldenberg, & Mark, 1996). Another reason which makes this problem a *good programming problem* is that it brings mathematics back into it (through the process of testing the code, as discussed earlier).

APPENDIX 2

TAKEN FROM THE WORKING GROUP WIKI

A preliminary list of software and robotics was produced to indicate the wealth of materials that may be of use in the (mathematics) classroom.

Robotics: *Arduino*, *Bee-Bots*, *Cubelets*, *Dash and Dot*, *Sphero*, *Ollie*, *Ozbit*, and *Webb*;
Software: *Code.org*, *Hopscotch*, *Hours of Code*, *Minecraft*, *Python*, *Scratch*, *Scratch Junior*, *Snap*, and *Tickle*.

APPENDIX 3

CT IN BROCK UNIVERSITY UNDERGRADUATE MATHEMATICS COURSES

Chantal Buteau and Eric Muller, *Brock University*

In 2001 the Mathematics and Statistics Department at Brock University implemented a set of core undergraduate courses called *Mathematics Integrating Computers and Applications* (MICA) (Ralph, 2001; Buteau, Muller, & Ralph, 2015) that, on further study, incorporate many of the attributes defining CT.

Students enrolled in MICA I-III courses learn mathematics as they engage in designing, programming and using interactive, computer-based environments, called *Exploratory Objects* that are "interactive and dynamic computer-based model[s] or tool[s] that capitalise on visualisation and [are] developed to explore a mathematical concept or conjecture, or a real-world situation" (Muller, Buteau, Ralph, & Mgombelo, 2009, p. 65). A student completing the three MICA courses normally undertakes 14 EO projects over the three terms. In 11 of these, the mathematics topic is specified by the instructor who provides some background both for the mathematics and the programming. For each of the other three EOs, scheduled at the end of each term, the student is responsible to select a topic or area of interest. We have recently completed a case study that reports on a student's 14 EO projects that she finished in her MICA courses (Buteau , Muller, Marshall, Sacristán, & Mgombelo,. 2016). As a result, we suggested that

> *The design of the MICA courses can be seen to provide a sequence of 11 stepping stones to guide the student's learning of programming as a tool within a context of increasingly complex mathematical ideas identified in the Third pillar of scientific inquiry [as described by the EMS (2011)].* (p. 162)

Since this publication, Weintrop et al. (2016) have suggested a set of CT classroom practices divided into four categories as illustrated in Figure 2 of this report. In the following we briefly illustrate the MICA students' CT experiences by picking a few examples, from our case study mentioned above. These, we recently argued, can closely be mapped to authentic professional CT practices (Broley, Buteau, & Muller, 2017).

DATA PRACTICES

From the start of the MICA courses students learn to generate data through simulations to explore mathematical concepts/applications. In addition, MICA II students from the case study were involved in seeking and analyzing actual stock market data, for example by providing a histogram summary of S&P 500 from 1950 to 2012, and further statistical analysis.

MODELLING AND SIMULATION PRACTICES.

Students in the MICA courses are involved in modelling (understood not only in an applied but also in a pure mathematical context) and simulation in every EO they undertake. For example, MICA III students apply cellular automata to "create simulations of epidemics to examine the effects of inoculation on the spread of epidemics, and their cost" (Buteau et al., 2016, p. 156).

COMPUTATIONAL PROBLEM SOLVING PRACTICES

From their very first lab students develop concurrently the necessary programming knowledge and skills to tackle the mathematics problems at hand. For example, MICA I students are introduced to discrete dynamical systems and using their "new programming skills learned (arrays, graphics), students create an EO about the logistic function $f(x) = kx(1 - x)$ (one parameter k involved), and are guided by the instructor to systematically explore, graphically and numerically, its behaviour" (Buteau et al., 2016, p. 151).

SYSTEMS THINKING PRACTICES

In most of the EOs defined by the instructor, students focus on the mathematics concepts/applications as the instructor has broken down the programming/mathematics/simulation into manageable parts, and the students do little *System Thinking Practices*. However, some of the more talented students in their original EOs, for which they self-select the topic, undertake situations in which they develop these practices. For example, a group of three MICA III students "decided to create an EO to explore the changes in the water supply of Lake Erie (Ontario) over time and explain why and how it changes" (Buteau et al., 2016, p. 156).

Readers who are interested in further details about reflective and research work on the MICA courses are directed to Muller's plenary published in these proceedings.

APPENDIX 4

REFLECTION ON 'COMPUTATIONAL THINKING'

Paul Muir, Mathematics and Computing Science, *Saint Mary's University*

The title of the Working Group leads us to ask the central question, "*What value can there be in introducing computational thinking into the mathematics curriculum?*" And we must then ask, "*What is meant by the term* computational thinking*?*" It turns out that there has been and continues to be much debate regarding the meaning of this term. In this appendix, I would like to provide some discussion around the term *computational thinking*. I am particularly interested in attempting to discern the types of thinking that might be characterized as computational thinking.

Unfortunately, it appears that the term *computational thinking* has become overloaded with many different meanings. In Wing (2006), the author says "Computational thinking involves solving problems, designing systems, and understanding human behavior, by drawing on the concepts fundamental to computer science" (p. 33) and then gives a number of examples of what she considers to be computational thinking. Her list includes recursion, parallel processing, type checking, error correction, pre-fetching, caching, etc., as examples of computational thinking. This is, I believe, too wide a definition; it would seem to say that computational thinking involves any type of thinking that a computer scientist might do.

As well, I think it would be useful to not include within *computational thinking*, necessarily anything that involves the use of computers, technology, software, etc. That is, I suggest that we not include within *computational thinking* the introduction of technology into the mathematics curriculum or into the 'doing' of mathematics. For example, there is a long history of using computers and software to solve mathematical models but I would not want to consider that type of activity to be particularly associated with the type of thinking we might identify as *computational thinking*.

In order to attempt to focus specifically on the ‘thinking’ part of computational thinking, we seek a narrower interpretation of the term. In the National Research Council (2010) report, we are told that many computer scientists believe that computer literacy, computer programming, and programming applications such as games, robots, and simulations “should not be confused with learning to think computationally” (p. vii). Furthermore, the report refers to the idea that computational thinking is “not equivalent to computer science” (p. 28), rather “computational thinking is a certain part of computer science” (p. 28). The report provides what I find to be a useful definition of computational thinking as “a collection of mental tools and concepts from computer science that help people to solve problems” (p. 10). The report also refers to computational thinking as being “cognitive and intellectual skills that human beings can use to understand and solve problems” (p. 10).

A type of thinking that I think should be included in what we consider to be computational thinking—and that seemed to be the interpretation of our Working Group—is what might be referred to as *algorithmic thinking*. In the National Research Council (2010) report, it is stated that “computational thinking was closely related to, if not the same as, the original notions of procedural thinking” (p. 11) and that procedural thinking involves “a detailed step-by-step set of instructions” (p. 11). This could include instructions that involve conditional tests and branching (if-then-else statements) or deterministic or conditional iterations (loops). The Working Group considered many examples involving the development of algorithms (and in some cases implementations of the algorithms in a specific computer programming language, e.g., *Python* or *Scratch*). A key point here is that procedural thinking is only part of computational thinking.

An idea less well explored by our Working Group is the more general view of the thinking associated with the development of computer software known as *object-oriented programming*. This type of computational thinking involves Wing (2006) “using abstraction and decomposition when attacking a large complex task” (p. 33). The National Research Council (2010) report states that “the notion of handling and manipulating intangible abstractions for problem-solving purposes [is] at the core of computational thinking” (p. 12). However, neither document explicitly refers to *object-oriented programming*.

A standard characterization of object-oriented programming identifies it as a programming framework or perspective based on the concept of ‘objects’, which contain the data to be processed or manipulated, and ‘methods’, which are implementations of algorithms that perform operations on the objects. (Note the procedural view of programming is included within the methods but there is a wider view of programming in this context which focuses firstly on articulating (in software) an abstract representation of the data to be processed by the methods.)

This brings us to the recognition of a second type of thinking within the umbrella of computational thinking, namely the notion of a precisely specified abstract representation of an object and the corresponding precise definition of what actions might be performed on that object. Of course abstraction is central to mathematics but one might argue that the contribution from computational thinking is the recognition that the abstract definition of the mathematical object, e.g. a matrix or a graph, and its ‘data’ or properties, needs to be coupled with a precise definition of the allowable actions on said object. In the National Research Council (2010) report, it is stated that “computational thinking focuses on the process of creating and managing abstractions, and defining relationships between layers of abstraction” (p. 16).

A third element of computational thinking is associated with the observation that since, in computer science, one typically develops software that implements an abstract representation of an object and a set of actions that can be performed on that object, there is the opportunity for an experimental or discovery-based investigation of the mathematical object through the software that implements it. (The Working Group activities included a number of examples of the application of this type of activity.)

This computational thinking connects with the type of thinking we might associate with a traditional scientific discipline, where experimentation is usually featured as a key element of what it means to 'do' the science. When this element of computational thinking is considered in the context of learning mathematics, it opens the door to using experimentation—on a larger scale than usual—as a way of learning mathematics. The implementation of the mathematical object in software can allow for a more efficient exploration of the properties of the object. Evidence of the importance of this experimentally focused way of thinking about mathematics comes from the Experimental Mathematics movement, see, for instance, *The Journal of Experimental Mathematics*, http://www.tandfonline.com/loi/uexm20.

In summary, in this appendix we have made an effort to limit the definition of computational thinking to focus specifically on certain types of thinking people commonly undertake in the context of computer science, and we have undertaken the exercise of attempting to identify some fundamental elements of computational thinking. *Procedural thinking*, *object-oriented thinking*, and the *perspective of experimentation* have been identified as key elements of computational thinking that might be relevant to the learning of mathematics. Furthermore, it is likely that further investigation of types of thinking used in computer science will lead to the identification of additional applications of computational thinking that could be relevant to the learning of mathematics.

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MATHEMATICS IN TEACHER EDUCATION: WHAT, HOW… AND WHY

LES MATHÉMATIQUES DANS LA FORMATION DES ENSEIGNANTS : QUOI, COMMENT… ET POURQUOI

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PREAMBLE

The main focus of this working group was on the mathematical education of elementary and secondary prospective mathematics teachers. It is a complex topic for discussion, and one which involves several dimensions of knowing and doing. We were fortunate to have a large and varied group of participants who brought their expertise and energy to our discussions and, as facilitators, we wish to thank them. It was a truly representative CMESG working group, with seasoned members and newcomers, and it was a positive and stimulating working group to lead.

INTRODUCTION

Many researchers have paid attention to, and developed models for, describing the complexity of the mathematical education of teachers in ways which help us understand these dimensions better. Among these, Ball's *Mathematical Knowledge for Teaching* (MKT) is probably the most well-known in Canada (see, for example, Ball, Thames, and Phelps (2008)). MKT is often contrasted to *Advanced Mathematical Knowledge* (AMK), with many studies concluding that AMK courses are not useful and, in some cases, are even detrimental to teachers (see, for example, Hart & Swars (2009). Even if the precise definition of AMK courses varies with authors and educational systems, it is often used to describe traditional university service courses which focus on algorithms, techniques and methods, often at the expense of an emphasis on understanding. Research also points to the reality that teacher education courses can talk *about* how best to teach and learn mathematics without providing ample opportunities for prospective teachers to learn mathematics and to learn it *through* these approaches.

In this working group, key questions relating to the education and preparation of K-12 mathematics teachers were asked—these questions include: What mathematics do (we believe) teachers need to know?, How do they need to know it?, and last but not least, Why do we think this way?

To frame our discussions, we considered four frameworks or approaches

- Big Ideas in/of mathematics
- Deep understanding of mathematics
- Quality engagement with mathematics
- Using a noticing framework in the mathematical education of K-12 teachers

These approaches were chosen as they can be claimed to promote a fruitful engagement with mathematics in teacher education, and because we felt that they could appeal to the whole range of educators from K to 12. We endeavoured to work on these approaches by using activities which enabled us to draw on participants' experiences as well as their active engagement during the working group.

DAY 1

After a round of introductions by participants and working group leaders, we started our work with *Big Ideas in Mathematics*. To set the stage, Kathy first explained what is meant by Big Ideas before leading participants into an activity to explore this further. She presented three characteristics of what Big Ideas are/are not, according to several sources:

- A Big Idea is not a topic (such as, equations); it is not a strand (such as, geometry); it is not a curriculum outcome/objective (Small, 2009, 2011). It is a statement of an idea that is central to the learning of mathematics, one that links numerous mathematical understandings into a coherent whole (Charles, 2005).
- Big Ideas are the enduring understandings that underpin the K-12 mathematics curricula. Each one is represented in many, if not all, grades. They are similar to the *Mathematical Processes* in that they are fundamental content concepts that repeatedly emerge and grow in the study of mathematics (GAINS, 2010).
- Big Ideas should be the foundation for one's mathematics content knowledge, for one's teaching practices, and for the mathematics curriculum. Grounding one's mathematics content knowledge on relatively few Big Ideas establishes a robust understanding of mathematics (Charles, 2005).

Kathy then led the group into an activity based on collections of different mathematical questions, all pertaining to some Big Idea which participants were asked to try to identify through solving the questions and discussing them. The work was done in small groups which were divided according to the interest of the participants. This activity was meant to serve both as an example of teaching/learning mathematics using a Big Ideas framework, as well as an opportunity to engage in the discussion of what Big Ideas were.

The precise instructions were that each working group participant select a level that they would prefer to work at with respect to mathematics problems: primary/junior, junior/intermediate, or intermediate/senior. The participants were asked to self-select into groups such that the result would be even distribution across 9 groups to work on three Big Ideas across three levels.

BIG IDEA 'A'	BIG IDEA 'B'	BIG IDEA 'C'
primary/junior	primary/junior	primary/junior
junior/intermediate	junior/intermediate	junior/intermediate
intermediate/senior	intermediate/senior	intermediate/senior

A sample of questions corresponding to one of these nine situations is as follows.

Big Idea A – junior/intermediate

- Choose a percent greater than 100. Represent it in two different ways. Tell how each representation might be better.
- Represent 4/5 in many different ways. Which of those ways help you see that 4/5 is the same as 80%? Which do not?
- Jane's father drove 417 km in 4.9 hours. Leah's father drove 318 km in 3.8 h. Who was driving faster? By how much?
- Describe three situations when it might be useful to know that ½ can be written as an equivalent fraction.
- The answer is 10%. What is the question?
- What numbers might be missing? $x / 7 = a / 30$
- Two equivalent fractions have denominators that are 10 apart. What can they be? What can they not be?
- You want to make a scale drawing of a regular hexagonal patio which is 5 m on a side. What is the largest drawing you can make on a 22 cm x 29 cm piece of paper?
- A recipe for 8 muffins uses 2 cups of flour. Choose a different number of muffins—either 10, 12, or 14 muffins. Tell how you will change the amount of flour. How many extra muffins did you make? Why do you not add that many cups of flour to the recipe?

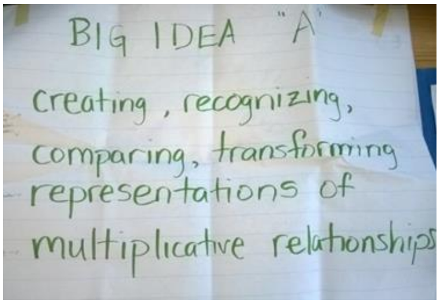
Kathy gave the following instructions to the participants.

- In your small groups, discuss and solve the mathematics problems provided at your table. As you solve, *note the characteristics or threads that are common across the problems you have been given*. That is, create a list of mathematical concepts/ideas that you think are being learned/emphasized through these problems.
- Create at least one question/problem of your own that 'fits' with the rest of your problems; that is, create a problem that emphasizes similar mathematical concepts/ideas as the ones you noted in the problems provided to you.
- Then, we will 'jigsaw' the groups such that new 'mixed-level' groups are formed having one member from each of the 3 levels (staying in the same Big Idea group A, B, or C). In these new mixed-level groups, compare the mathematical concepts/ideas lists generated from your first small groups. Using the points of intersection and overlaps in your lists (we hope there are some!), formulate (decide upon) the Big Idea that your mixed-level group thinks aptly describes all of the activities across these 3 levels.
- A spokesperson for each small 'mixed-level' Big Idea group (for each of A, B, C) will be asked to report back to the larger group on the Big Idea they decided upon.

In the discussions which followed in small groups, many questions were asked and ideas discussed, which led to very interesting ways of looking at Big Ideas. Here is what came out on the flipchart prepared by the small groups.

<u>Big Idea A</u>

- Creating, recognizing, comparing, transforming (all about the "thinking and the doing" of mathematics).
- Representations of multiplicative relationships, touching on the communication and content of mathematics.



<u>Big Idea B</u>

- Making comparisons and Revealing **Relationships** among **Quantities** through the use of **Multiple Representations**.

The group also talked about strategies, what they are and how they can help (e.g., 'bench-marking').

<u>Big Idea C</u>

There were 3 smaller groups working on Big Idea C, so more variety in responses was received.

C1. Multiplicative structures—in what ways do multiplication and division represent building and taking apart quantities?

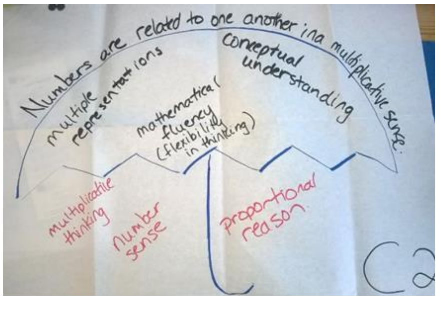
Commenting on this with the lens of processes and content, the Big Idea was seen as content under multiplicative structures.

C2. An umbrella image was produced.

Top idea: Numbers are related to one another in a multiplicative sense.

Inside umbrella: multiple representations – conceptual understanding – mathematical fluency (flexibility in thinking).

Under the umbrella: multiplicative thinking – number sense – proportional reasoning



C3. The overall title was "*Champs/structures multiplicative*", which consisted of

Différentes écritures – *Expressions équivalentes* – and, centrally, *Propriétés des opérations*

This was all inside a process which went from $\mathbb{N}$ to $\mathbb{Q}$ to $\mathbb{R}$.

After this small group sharing, Kathy shared the resource's version of the Big Ideas (GAINS, 2010).

> **Big Idea A**
>
> There are many equivalent representations for a number or numerical relationship. Each representation may emphasize something different about that number or relationship.
>
> **Big Idea B**
>
> Numbers are compared in many ways. Sometimes they are compared to each other; other times, they are compared to benchmark numbers.
>
> **Big Idea C**
>
> There are many algorithms for performing a given operation.

This led to a lively and active discussion, as participants questioned if these Big Ideas were satisfactory to them.

For Big Idea A, there was a sense that it is hard to say the same thing for a number and a numerical relationship: a relationship is the cognitive work going in your mind as a result of the representation; it is not a representation in the same way as that of a number. What you see or get when you represent a number has a more direct relationship or connection to the number. One area of reflection was that the Big Idea of the resource *Math Gains* was more about the facts of mathematics, and that our own attention was more with the pedagogy and idea of relationship as opposed to a static, outside view.

For Big Idea B, there was unease with the lack of distinction between numbers and quantities, and the absence of any reference to the use of multiple or different representations. Some participants noted that there may be confusion between number and quantity—they noted that if you compare two quantities additively, the answer is quantity, but if you compare multiplicatively, it is a scale factor.

In the discussion of Big Idea C, there was a strong sense that the issue here was more political: the Big Idea of different algorithms did not emerge from the small groups, and to see this as a Big Idea in itself was disconcerting to many of the participants. It did not correspond to Big Ideas in mathematics—perhaps in the curriculum.

The use of the word *algorithm* was questioned by some who claimed that there is a major difference between situations which lead to a standard algorithm and situations where one is finding ways to do something. One reformulation could be that "*there are many ways to perform a given operation*."

There was also some explicit criticism of Big Ideas as proposed by *Math Gains*. Seeing mathematics as a web of interconnected ideas, teachers need to understand the interconnections between ideas, the concepts as a whole. If we describe it as distinct ideas, a teacher can take it as a union of different parts: if the teacher doesn't have a more holistic view, they might take the material and go with it as it is. These Big Ideas may give good topics to discuss and limit ways to teach it in a mathematical thinking way.

Kathy then led the group into a discussion on using a Big Ideas approach in teacher education without focusing on the *Math Gains* particular take on it. She started by sharing her recent experience in using this approach in her classes with students at both the graduate (in-service teachers) and undergraduate (pre-service teachers) level. In the graduate course offering, Kathy drew on the teaching experiences of her students to form groups across grades K-12 (K-2, 3-5, 6-8, 9-12), forming strand-focused groups. The strand groups selected one of the four Saskatchewan curriculum strands (Number, Patterns & Relations, Shape & Space, and Statistics & Probability) and one big idea within that strand to focus on, with the ultimate goal of constructing a teaching guide (K-12) as well as an interactive workshop for their colleagues in the class for the big idea. The progression of outcomes and activities was highlighted in both the guide and the workshop, with students having an opportunity to experience mathematics connections and understandings that they had not previously been aware of due to their focused experience of teaching mathematics primarily at one or two grade levels. For example, kindergarten teachers saw threads of their own mathematics concepts appearing in high school mathematics, giving them a strong sense of validation for the important work done in the early years. Similarly, high school mathematics teachers confessed to being previously unaware of how early and how often mathematics concepts had been seen by their students prior to reaching high school, giving them a sense of freedom to emphasize building on knowledge rather than starting with a 'blank slate' when teaching their students.

Kathy shared that facilitating a similar assignment with her undergraduate middle years and high school mathematics pre-service (student) teachers was also an eye-opening experience for these students, as they grappled with broadening their perspective to include progressive K-12 understandings of curriculum rather than a limited focus on the grade they 'plan' to teach when they begin their teaching careers.

Participants in the working group responded to Kathy's discussion of this Big Ideas approach in her classes by sharing some of their own experiences, in both their classroom practice and in their research. One participant made comparisons to the Common Core Curriculum (in the United States) which is Big Idea-based and has become quite politically charged due to the existence of both proponents and opponents to the approach.

Other participants compared Big Ideas to *enduring understandings* and to *understanding by design*, illustrating how various forms of language have permeated discussions over the years. Reference was also made to Steen's (1990) *On the Shoulders of Giants* as an approach to numeracy that emphasizes deep ideas of mathematics—all to say that Big Ideas have been a big idea for quite some time now, even though there is no clear agreement on how (or if) this is actually influencing what we are doing in teacher education with pre-service teachers. In other words, the idea of Big Ideas presents a big challenge.

In the end, the plan to spend time discussing the strengths and shortcomings of a Big Ideas approach and whether participants were using (or would consider using) a similar approach in their own mathematics courses or teacher education programs was cut a bit short due to Day 1 of the working group coming to an end. We concluded the day with a brief presentation of the approach chosen to discuss *deep understanding of mathematics,* to be the focus of Day 2.

DAY 2

Frédéric led the group, briefly outlining what was meant by *deep understanding of mathematics* for our working group.

He first referred to the important work of Liping Ma (1999), *Knowing and Teaching Elementary Mathematics: Teachers' Understanding of Fundamental Mathematics in China and the United States*. Ma's work with elementary school teachers led her to define *Profound Understanding of Fundamental Mathematics* (PUFM). She argued that PUFM supports teaching and learning which has the following four properties:

- Connectedness
- Multiple perspectives
- Basic Ideas
- Longitudinal coherence

Mathematical attitudes are part of this description, which includes aspects included in pedagogical content knowledge.

Another take on deep understanding of mathematics is to take it as more or less equivalent to Klein's (1924/1932) *Elementary mathematics from an advanced standpoint*. This is a way of describing relevant mathematics for teaching which has been used by some mathematicians and mathematics educators.

In other studies, there seems to be an assumption that what is learnt in *advanced mathematical courses* is more or less deep understanding, which would then require a useful definition.

Given this context, we decided to take a pre-service teacher student perspective from students who experienced a curriculum which stated *deep understanding of mathematics* as a specific goal (Adler et al., 2014). For these students, the main features of a *deep understanding of mathematics* can be summarized as

- Understanding basic ideas in mathematics
- Seeing the connectedness in mathematics
- Seeing multiple perspectives on a problem
- Reasoning in mathematics

If we expand, for these students *reasoning in mathematics* is

- Being able to work from first principles, providing a proof, an explanation.
- Understanding mathematical reasoning: definitions, proofs, properties, etc.
- Being able to reason school mathematics from and advanced standpoint: seeing connections to more advanced mathematics.
- Linked to unpacking, knowing where it comes from.

As for *seeing the connectedness in mathematics,* Adler et al. qualify it in different ways:

- Deep subject knowledge is connected, flexible knowledge.
- It is reflected in the ability to make links across different areas of mathematics, and to the real world.
- It enables you to explore different kinds of mathematical problems, and helps you to approach problems flexibly and solve them.

Frédéric then led the working group participants in a small group activity. Participants were offered five choices of activities or teaching sequences that they could consider in small groups—these were made available at the end of Day 1. In each small group, they were asked to consider the activity or teaching sequence with the idea of furthering *deep mathematical understanding*, and discuss what the activity could bring to the fore as well as what it did not provide. Four of these activities were adapted from Frédéric's work with preservice secondary school teachers, and one (the fifth) adapted from Adler et al. (2013).

Activity Choices

1. A learning sequence on area, starting from a definition based on expected properties of area. It aims to get students to explain why the area of a rectangle is given by the product of the length of the sides. From there, the sequence moves into volume, with a similar approach, leading to volume of a cone.
2. Two examples drawn from Movshovitz-Hadar & Webb (1998), where one has to find and correct mistakes in erroneous resolutions of equations. (Numbers 2 and 7 of this reference.)
3. Examples of student's mistakes in dealing with logarithm and exponentiation. Identifying the mistakes and how to correct them.
4. A learning sequence on exponentiation, starting from the definition of a^n for n a positive integer. The sequence focuses on properties and coherence, leading to an extension of the domain of definition of exponentiation, drawing attention to what can be proved and what is adopted as a definition.
5. An outline of a teacher's approach to teaching parabola. It starts by students positioning themselves as equidistant from a point and a line, and moves to the blackboard at a second stage.

In their discussions, participants identified some of the affordances of such activities with respect to dimensions of *deep understanding of mathematics*. Here are some of the aspects which came out, loosely regrouped.

- Difference between definition and property (affordances and constraints)
- Consider other/different definitions (contrast – compare), challenging acronyms
- Attending to own misunderstandings or surface understandings
- Breaking down understandings to recreate understandings that are more connected/complex

- Multiple solutions/modelling
- Representations helping to make connections (is that deeper?)
- Multiple representations
- Using graphs to understand equations

- Breaking down (e.g., one-dim and natural numbers only)
- Developing skills in reasoning
- Multiple pathways
- Perseverance to solution
- Never being finished; extending; open-ended
- Students 'doing' mathematics

- Relationships between numbers
- Recognizing fundamental structures
- The notion of a unit as fundamental to own thinking
- Connecting (definitions) to historical development
- Equivalence of equations

- The more we know, the more we know we don't know
- Flexible/adaptable
- Differentiable for many levels of students
- Difference between picture and diagram

The group then engaged in a discussion prompted by the question: What is missing in our discussion that is related to *deep understanding*?

One aspect mentioned which wasn't brought to the fore is that having deep understanding means asking new questions and not just working with what has been given. What can we do with this thing now that we have a deeper understanding? How can we deepen our understanding by going forward/beyond what we now know? And what does it mean to go deeper? The word *deep* may not convey as much as we want to the idea of interconnectedness, and may not reflect adequately the fact that gaining a deeper understanding may be different for each person.

One participant noted: the way I use the expression *to understand* may not work for every student... it's a lot to ask but we need to know in multiple ways. Models have constraints. There is no one model and we can let go of a model to move to another model. Eventually we see that our concrete models will not suffice, and this helps us see why we move to the abstract in mathematics. Deep understanding relies on understanding underlying structure of mathematics (e.g., with place value). Deep shouldn't imply that we've reached something, as we are constantly building on our understanding. We continuously realize the complexity and what we still don't know, and part of our work as teachers is helping students see the complexity.

Frédéric then connected back to our WG schedule as we headed for our next approach, *quality engagement with mathematics*. Frédéric gave some background explaining why this became part of our working group's proposed agenda. In a colloquium at UQAM in 2012, *Formation mathématique pour l'enseignement des mathématiques* (Proulx, Corriveau, & Squalli, 2012), one of the recurring themes was *Faire faire des mathématiques*. This literally translates into *Make do mathematics*, which doesn't sound very good in English. The idea is to get someone (students for us) fully engaged in the doing of mathematics, and the quality of this engagement becomes a focal point of discussion. Some of the aspects we can think of in relation to this are duration, breadth and depth, as well as emotional aspects. *Mathematical Habits of Mind* (MHoM) can also be useful in thinking about quality of engagement. Drawing from Susan Oesterle's (a member of this working group) presentation at *Changing the Culture* in 2016, we say that we demonstrate MHoM when we habitually choose actions and strategies, pose questions and display attitudes that are productive in a mathematical context, and MHoM can be thought of as attitudes, actions, strategies and questions. There was a working group at CMESG in 2014 on MHoM (Gourdeau, Oesterle, & Stordy, 2015).

In the 2012 colloquium, quality engagement with mathematics (as meant in *Faire faire des mathématiques*) was seen partly in contrast to *teach and drill*, to skills acquisition. It was used to describe activities where learners were active, engaged, and often reflective. It was often not limited to the classroom.

Many reasons were brought forth to support the importance of such type of engagement with (pre-service) teachers.

- Develop the capacity of teachers to be fluent, comfortable in mathematics.
- Help them become able to solve the problems they will face in teaching.
- Develop their mathematical curiosity.

- Develop their capacity to give meaning to mathematics.
- Help them become life-long learners, to continue their development.

In looking at mathematical learning though this lens instead of *deep understanding*, there was less emphasis on the content and more emphasis on the engagement of students with mathematics, the process of developing mathematics, and with the activities themselves.

As we were concluding our Day 2 session, Frédéric invited participants to reflect individually, and think of one lesson, activity or project that they had used or experienced in teacher education, and that they would describe as "providing quality engagement with mathematics" in the sense described here, as a preparation for Day 3.

DAY 3

Participants were invited to share in small groups an activity they had thought of following yesterday's suggestion, and to consider what these experiences helped achieve in the mathematical education of those taking part. This would then be reported to the whole group.

The discussions in small groups were lively, and could have continued longer! A variety of activities were shared (playing a game on a graph, problem solving over an extended period, asking students to re-explain, mathematical magic (algebra), using material to look at 3D, etc.) while focusing mostly on their characteristics. Some of the key ideas which came out from these discussions were:

- Open-ended, flexible, adaptable
- Low access points, low floor/high ceiling
- Differentiable for many levels of students
- Multiple representations and pathways, spatial and diagrammatic reasoning
- New questions, new ways of looking at something
- Attending to misunderstandings or surface understandings
- Recreate understandings that are more connected/complex
- Reconstruct meaning
- Aha moments
- Devolution of the activity
- No fear of being judged, accepting mistakes as part of the process
- Enables discussion, debate
- Invites experimentation, hands-on, and/or inquiry

One small group described important features as a direct personal experience of

1. Multiple solutions/modelling
2. Challenging acronyms
3. Recognizing fundamental structures
4. Difference between picture and diagram

As an example of 3 and 4, they presented the following problem:

> **Lamp, child, shadow**. Street lamp is 5 m high. Child is 1 m high. Child is walking around. It is night. What is the trajectory of the head of the shadow?

NOTICING FRAMEWORK IN ACTION

Kathy introduced the fourth, and final, approach to be discussed in this working group by sharing her use of a *noticing framework for professional development* in teacher education field experience (internship). After serving as a faculty advisor for interns for many years, Kathy felt that she wanted to reconceptualise the traditional model of how faculty advisors visit interns and observe their lesson from the back of the classroom. She believed that teacher educators should be more actively involved in mentoring new (becoming) mathematics teachers so she designed a research project to study her role as a faculty advisor within a learning community of interns and cooperating teachers. In the community, interns share video recordings of their own practicum classroom teaching and the group engages in a noticing process to discuss and reflect on the intern's teaching. The noticing framework used in the learning community is adapted from John Mason's (2002) *discipline of noticing*. Kathy shared a couple of key points about noticing:

- Noticing... the process of attending to, and reasoning about, important elements of classroom teaching and learning (Santagata, 2008, p. 156)
- The *discipline of noticing* (Mason, 2011) is a collection of techniques (1) for preparing to notice in the moment; (2) for reflecting on past events to understand what one wants to, or is sensitized to, notice; and (3) for learning to notice in the moment so as to act freshly rather than habitually (p. 48).

With these points and a handout providing an overview of the process, Kathy then played a brief 8-minute video segment (a Grade 8 lesson on exponents http://www.timssvideo.com/69) and invited working group participants to engage in the same 4-step process that interns and cooperating teachers do in her learning community:

Step 1: View video, taking your own 'noticing notes'.

Step 2: Remain in silence for 1 or 2 minutes, replaying the video in your mind and highlighting notes you plan to share.

Step 3: Give an **account of** what was observed in the video (only share what was observed directly, in detail; avoid all interpretation at this stage).

Step 4: Now **account for** what was observed (this is the interpretive stage where you may share possible meanings for, or questions about, what was observed in the video).

Kathy emphasized the very important distinction between steps 3 and 4 in the process—a distinction that meant a safe and productive environment, free of judgement and criticism, could be created and sustained in the learning community. While all participants in the working group were invited to take noticing notes while the video was being played, there was insufficient time for everyone to share their notes. However, as an example, we present Frédéric's noticing notes here:

- Instructions written on board.
- "it is starting to double in size"
- A kid says 2 to the 5 is 25. Not followed up on.
- Teacher says "get significant growth"
- Teacher on the board.
- Examples with 2 as base.
- "Write it as a solution of a power."
- "kid asks a question" – not followed up on.
- 2 times x and 2 times x
- Kids say no – not followed up on.
- She instructs.
- Pattern spotting.

When asked to provide reflections and feedback on the 4-step process, a lively discussion ensued among working group participants. Participants noted and discussed the differences in what was being noticed by everyone within our own small working group: some focused on mathematical language being used overall by the teacher and students, while others took a more focused approach of noticing the idea of discrete to continuous mathematics (how, for example, a working definition at one level becomes incorrect/incomplete as the mathematics concepts go deeper). Other participants focused on what/who is moving (teacher, students, hands, etc.), and still others wanted to spend time reflecting on what is missing from the process in terms of what is available for students to look at, and see, as they are learning.

As a method of transitioning toward closure of the working group, Day 4 discussions then moved into a reflection on how the noticing framework might fit (or not fit) with the other three approaches that had been emphasized in this working group (big ideas; deep understanding; quality engagement with mathematics). Plenty of questions were posed: How do we use a part of this noticing in pre-service teaching? Are there too many distracters in the video for us to focus on mathematical understanding? Is there too much to consider all at once going on in the video? How could we learn to use more precise noticing in our observations? Could we attempt to improve the process by viewing multiple videos about the same content? Could videos like these be used in mathematics classes in university departments of mathematics? What are the critical aspects of exponential learning which should be taken into account?

The discussion concluded with one participant noting that, in spite of all the ‘flaws’ we noticed about that teacher in the video, she was doing exactly what the research over the past 40 years has been recommending that teachers do. By recapping the various teacher moves in the video (use of multiple representations, manipulatives, reference to previous knowledge, etc.), this participant drew attention to the fact that the teacher did everything she was ‘supposed to do’ according to the research, and yet it seemed that she fell short of satisfying us. It was definitely food for thought, and a poignant way to end our 3-day working group.

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PROBLEM SOLVING: DEFINITION, ROLE, AND PEDAGOGY

RÉSOLUTION DE PROBLÈMES : DÉFINITION, RÔLE, ET PÉDAGOGIE ASSOCIÉE

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INTRODUCTION

In this working group, we engaged in three main activities:

- We discussed and attempted to come to a definition of a 'good' problem (specifically in educational contexts), and a definition of 'problem solving'.
- We discussed the role of problem solving in K-12 classrooms, especially problem solving as a method of teaching rather than a specific topic to teach in the curriculum.
- We discussed how problem solving should be addressed with future educators.

Dans ce groupe de travail, nous avons discuté de trois thèmes principaux :

- *Les définitions d'un « bon » problème, et de la « résolution de problèmes ».*
- *Les rôles de l'activité de résolution de problèmes dans les écoles de la maternelle à la douzième année, d'autant plus que la résolution de problèmes n'est plus seulement un objet d'enseignement, mais aussi un outil didactique pour enseigner d'autres sujets particuliers du programme.*
- *Les approches pour aborder la résolution de problèmes avec les futurs enseignants.*

DAY ONE

Here was our focus for the first day: *Discuss and attempt to come to a definition of a 'good' problem for learning, and a definition of 'problem-solving'*.

We started with this warm-up question:

> Each of the sixteen squares has a 'point value', as indicated. Your goal is to collect as many points as possible, starting at the top-left corner and ending at the bottom-right corner. You may pass through each room at most once. What is the maximum number of points that you can collect?

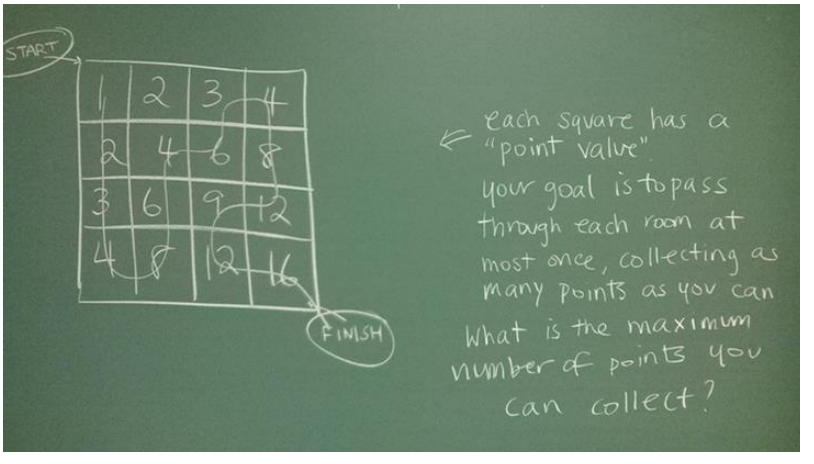

Figure 1

Participants shared various reasons why this is a good 'problem':

- it is easy to start and it is easy to make more difficult;
- it does not require algebraic tricks;
- chessboard colouring provides a nice anchor to existing knowledge;
- the grid has 'distractors' in that the sixteen numbers are irrelevant; and
- one can quickly come up with a near-optimal answer, but needs to do some work to obtain the optimal answer (of 98 points).

However, participants also mentioned a weakness of this problem: the chessboard colouring is heuristically non-transparent, i.e., how would someone come up with that approach? Of course there are ways to address this in a teaching context, but this need for a 'trick' or an 'aha! insight' makes using this problem less straightforward.

This notion of making progress is important to the teaching of mathematical problem solving. We discussed the need to present problems that get the students hooked, where they can make incremental progress and see success; this is true whether we are teaching in elementary school or high school, teaching undergraduates, or conducting pre-service or in-service work for educators.

A good problem should not rely on a single 'aha' insight that renders the problem trivial, where it is impossible to make progress without seeing the trick. In this light, the question above is not a good 'problem', in spite of its many other strengths. (For discussion of this issue in relation to proving, see Reid & Tanguay, 2012).

The problem mentioned above can be good for something, for example to introduce the applications of grid colouring, but not so good for something else, for example to convince the students that they can make progress by themselves in solving any problem. In order to use problems in class efficiently, we need to decide for each problem for what learning it is good or bad.

SHARING PROBLEMS

We split into groups of four, in which we shared our favourite problems and selected one or two to offer to the others. We posted the problems on the walls, and milled about, working on whatever problems caught our interests. The problems presented included the following:

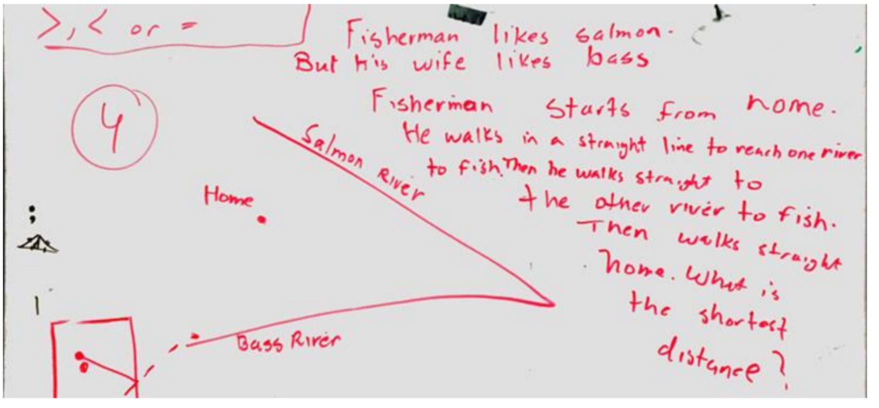

Figure 2. The Fisherman Problem.

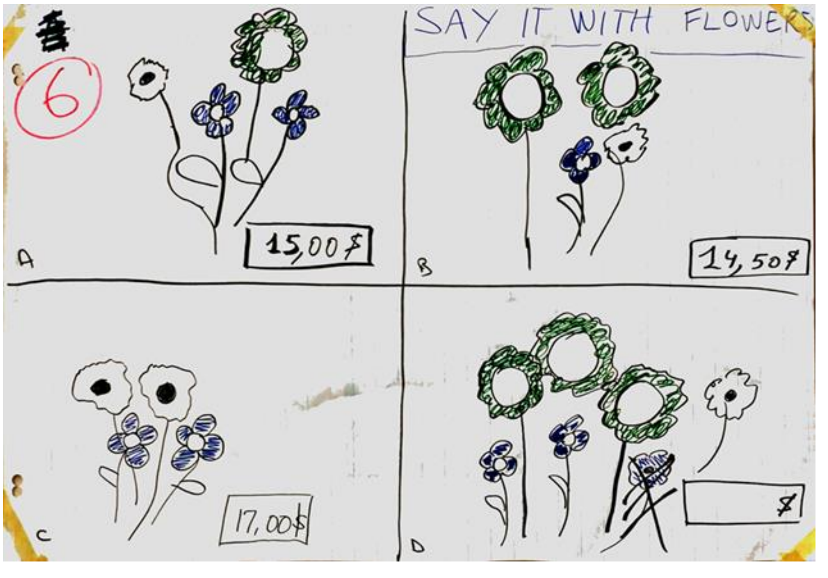

Figure 3. The Flower Problem.

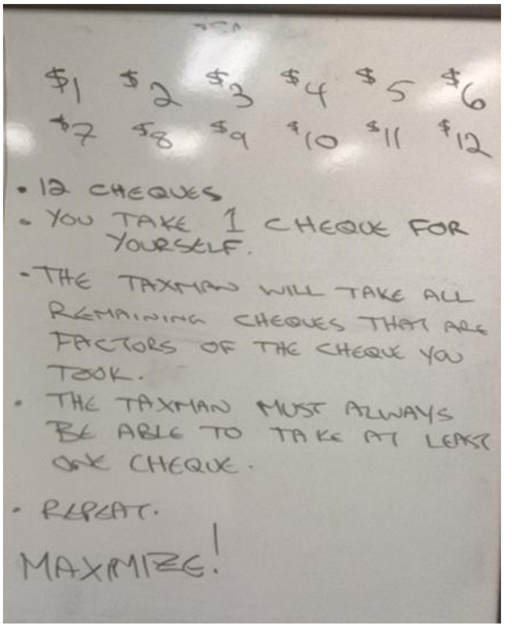

Figure 4. The Taxman Problem.

Inspired by these problems, we discussed characteristics of what makes them 'good'.

- A good problem is often one where an insight leads to further mathematics, as in the Fisherman Problem (Figure 2). The Fisherman Problem can be generalised or simplified in several ways that develop mathematical understanding.
- A good problem is often accessible and non-threatening, as in the Flower Problem (Figure 3) which was presented using a non-verbal context. This avoids the issue of having to understand the words of a problem, before being able to engage with the problem itself. As someone noted, the problem with word problems is the words. This also reminded us that what makes a problem a problem is dependent on the problem solver. For us, one possible solution path, using algebra, was obvious, and hence the problem was not a problem. We could make it one, however, by imposing constraints, such as not using algebra, or operating on the sets of flowers concretely rather than abstracting them symbolically. Following this logic, the interpretation of a task (word problem or picture problem) in a mathematical way can be problematic for someone. So the initial mathematical interpretation of a task is a cornerstone for problem solving.
- A good problem is often generalizable, as in the Taxman Problem (Figure 4) where we can replace 12 cheques with 18 cheques or with *N* cheques. And this problem is accessible to Grade 4 children while also inspiring questions that would challenge upper-year computer science majors. But we need to be aware of what each of these groups can learn through the process of solving.
- A good problem is often grounded in a real-life context (e.g. maximizing a profit), which engages students and enables them to connect the topics they learn in class to the issues they care about.

Peter introduced the term 'unicorn problems' to describe those rare problems that are not only good for student learning, but are *immediately engaging in almost any context with almost any population*. Peter offered the Taxman Problem (Figure 4) as an example of a unicorn problem.

On the other hand, the Island-Moat Problem (Burger & Starbird, 2005) below rests on a single ‘aha’ insight (or trick) of how to lay down the two planks to reach the island (Figure 5).

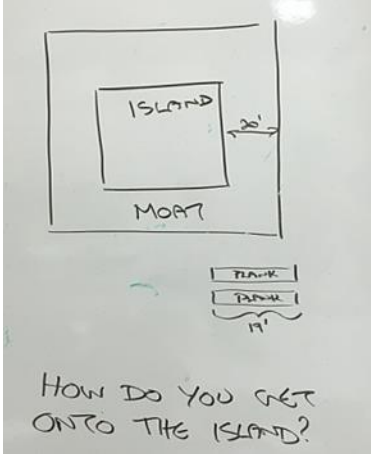

Figure 5. Island-Moat Problem (Burger & Starbird, 2005).

But we should note that in the original formulation, the Island-Moat Problem is presented orally or textually, rather than visually. Thus, there is some work for students to do in order to move from one form of mathematical representation to another; once the above visual representation is attained, a clever insight is required to solve the problem. It is also possible that the use of physical objects, planks of appropriate size, can facilitate the clever insight.

This problem can be improved, and our Working Group spent some time discussing the different ways to do that: for example, determine the shortest possible length of the two planks for which you can get onto the island.

The problems we discussed also reminded us of some definitions that have been offered by others:

> *First, a problem is only a* Problem *(as mathematicians use the term) if you don't know how to go about solving it. A problem that holds no 'surprises' in store, and that can be solved comfortably by routine or familiar procedures (no matter how difficult!) is an* exercise*. This latter description applies to most of the 'word problems' that students encounter in elementary school, to 'mixture problems', 'rate problems', or other standard parts of the secondary curriculum. Dealing with them is certainly an important part of learning mathematics, but (unless the context is unusual) working such exercises is not generally considered 'doing problem solving'.*
>
> *Second, a problem is not a* Problem *until one wants to solve it. (The presumption in most problem books is that the reader does. Why else would he or she be looking at the problem book?) Once one wants to solve a* Problem*, there is an emotional and intellectual commitment to the solution, and the risks and rewards concomitant with that commitment.* (Schoenfeld, 1983, p. 41)

A good problem should not be procedural: if it is routine for us, then it is an exercise and not a problem. We need to get ‘stuck’, to find an obstacle that we have to overcome, in order for a question to become a problem. The theory of *Zone of Proximal Development* proposed by Vygotsky (1991) explains, “The teaching oriented toward already completed cycles of development happens to be vain from the point of view of the child’s general development, it does not lead the development but follows the tail of it” (p. 386). Thus, a problem needs to present some difficulty to the learner in order to serve the purpose of knowledge development.

In our discussions, we reminded each other that a 'problem' is relative to each individual, based on their culture, experience, confidence, and background. What is a problem for me might not be a problem for you. And what was a problem for you one year ago might not be a problem for you today. However, this way to define a problem cannot be used to describe a mathematical task per se, because it requires the presence of a solver. Thus, the terminology of 'problem' is not very practical. Many researchers in the field use 'task' terminology, which is associated with the formulation of the task, and not in relation to a particular solver. A task can become a 'good problem' for some solvers in some conditions and a routine problem (or not a problem) for other solvers or in different conditions.

This connects to Brousseau's notion of *problem situations*. For Brousseau (1971), problems are tasks. The teacher needs to do something special, to create a milieu, for the task to let students develop a particular knowledge.

> *L'élève acquiert ces connaissances par diverses formes d'adaptation aux contraintes de son environnement. En situation scolaire, l'enseignant organise et constitue un milieu, par exemple un problème, qui révèle plus ou moins clairement son intention d'enseigner un certain savoir à l'élève mais qui dissimule suffisamment ce savoir et la réponse attendue pour que l'élève ne puisse les obtenir que par une adaptation personnelle au problème proposé. La valeur des connaissances acquises ainsi dépend de la qualité du milieu comme instigateur d'un fonctionnement « réel », culturel du savoir, donc du degré de refoulement a-didactique obtenu.* (Brousseau, 1971, p. 40)

Participants noted the following:

1. A mathematical problem is a situation where a solution is needed and is not obvious (requiring an exploration, a cognitive obstacle, motivational) for a person (intention, culture) according to *a priori* analysis, can/should potentially involve mathematics at some time for a specific part of it, and probably invent some.
2. Solve a problem: recognize the 'problem', use our previous knowledge and understanding of the problem, to find a solution that satisfies us.
3. A mathematical problem is what at least one person considers as such, many people choose to engage and find it meaningful.
4. A mathematical problem represents connection between mathematics and real life.
5. A mathematical problem is a question requiring a mathematical read and for which the solution does not exist.

We ended Day 1 by having each participant define 'problem' and 'problem-solving' on their own, with each participant submitting their definition on a sheet of paper. Based on the collected input, here is our first draft at our collective group's definition of these two key terms:

- A mathematical problem is a clearly-stated question whose aim is to challenge the one attempting the question by forcing them to resolve a cognitive obstacle in order to develop mathematical reasoning and maturity. A good problem requires challenge, is ideally accessible and engaging to people of most/many levels, and necessitates that the problem-solver get stuck or challenged. Often the key insight is found by making a connection between an abstract concept and a real-life context, or finding a connection between two or more fields of mathematics. A mathematical question is not a 'problem' for everyone, and is relative to an individual's prior experiences with the subject and to his/her personal characteristics as a learner.
- Mathematical problem solving is a cognitive activity that aims to resolve a challenging question by getting 'unstuck'. It is rarely a routine process (step 1 do this, step 2 do that), and instead is a non-linear process that necessitates conceptual understanding rather than routine calculations, and requires us to try different strategies and methods to discover key insights and make progress. Through our

> struggle and perseverance, sometimes we can find multiple solutions to the problem, requiring different insights and techniques that relate to our previous experiences with mathematics. An unexpected benefit is that solving these problems deepens our passion for the subject.

DAY TWO

Here was our focus for the second day: *Discuss the role of problem-solving in K-12 classrooms, especially problem solving as a method of teaching rather than a specific topic to teach in the curriculum.*

We started with a two-minute video showing a Grade 2 student solving the following task, where she got the wrong answer. The students were asked first to read the problem and then propose a strategy to solve it:

> I have 13 tokens. I hide 7 of them in my left hand and the others in my right hand. How many tokens do I have in my right hand?

In the video, the student drew 13 circles, then 7 other circles. She said that 7 are in the left hand, so the others (13 on her picture) are in the right hand.

We asked ourselves, "*Is this a real problem?*" It seems that the student had the preconceived expectation of what to do: she immediately started drawing circles. Some of our students have preconceived expectations (what Brousseau (1971) would call a « *contrat didactique* »), where they want to apply the exact recipe they learned in class to solve each question.

For the Grade 2 child, she used a procedure learned in class—drawing circles—which she could not control because of a lack of conceptual understanding of the task. What was problematic to the student was not to calculate the answer to 13 – 7, but to correctly interpret the described story in a mathematical way.

A mathematical interpretation of a situation, task, or problem, which is connecting a real-life context with mathematical concepts, is the first step to the solution and it is the first challenge elementary students have in problem solving.

Educators use problem solving to stimulate learning, to ask questions that are *complex*, rather than *complicated*. In a good problem, words are not a barrier where we need to read pages to understand what a problem is saying. Language is often a barrier, especially for students who do not speak the official school language well. However, we must ask ourselves whether the words pose a 'barrier' or pose a 'challenge'. The answer depends on the learner's language abilities and on the purpose of the solving. According to Vygotsky (1934), language is one of the main reasoning tools, thus it should be developed together with other ways of mathematical reasoning.

Once we understand what the problem is saying, we need to understand it mathematically. This is a large part of the challenge for a learner: once this understanding has taken place, much of the problem has been solved. A problem can be presented as a word story, as a picture (flower problem) or another way (equation). Sometimes the problem can be solved the way it is presented (solve the equation) or transformed into another representation (flowers into equations; wording into schema).

In these transformations, the mathematical meaning should be preserved. Different representations require different parts of the brain to be involved to understand them, where wording is decoded mainly by the left hemisphere and pictures by the right hemisphere. If we can pose problems and solve problems that require us to make connections between the left brain and the right brain, growth occurs (Wachsmuth, 1981). We saw another video, showing the "cinnamon heart" problem, where the students analyzed the following text:

> Rene has 11 cinnamon hearts. He ate 6 of them. There remain 8 cinnamon hearts.

The students realized that there was an error in the text. They made the assumption that the error is at the end, that the number 8 is wrong. However, the error could have been at the beginning or in the middle, i.e., the 11 or 6 could have been wrong. The teacher asked the students to represent the situation in order to mathematically analyze it. The task was quite challenging for students, and many of them were not successful in constructing a representation.

This task satisfies many possible definitions of a problem (1, 2, 3, and 5 in the above list). However, it does not present a connection with real-life but rather a disconnection. This disconnection creates an element of surprise and a need for further investigation (Schoenfeld, 1987). It serves to engage students in a profound mathematical analysis and allow for mathematical knowledge and development.

In this mathematical task, the class analyzed the problem, rather than just solving it. This led to an important discussion about our role as educators: Should we make a clear distinction between problem analysis/modeling and problem solving, or rather see the problem analysis/modeling as an important part of problem solving?

We said that a 'conventional' problem can become a great learning opportunity by pushing it: by putting a small twist on it. So much problem solving occurs when we as teachers have the confidence to encourage open-ended problem solving, such as giving them Pascal's Triangle and asking them to find as many patterns as they can.

Let's give our students great problem-solving experiences, which require us to be open to being surprised, having to go off script from our lesson plan. We can include non-conventional 'problem-solving' opportunities in the curriculum which can help develop the reasoning required for problem solving.

At this point, we analyzed three of the problems from Day 1, asking what they are good for, and for whom they are good. In other words, we did a didactic analysis, thinking about how we would present these problems in a lesson.

1. The Fisherman Problem (Figure 2): we presented this problem as a sequence of activities, suitable for a Grade 12 class or for an undergraduate problem-solving class (see Figure 6). In this carefully structured sequence of activities, students got stuck and had to learn how to get unstuck. The 'reflection' insight is an example of epistemological justification, which motivates to the student why we need to learn the technique (not what it is or how it works).
2. The Flower Problem (Figure 3): this problem was applicable to Grade 2 and to Grade 9 students, where they develop new knowledge and practice existing knowledge. One could solve this by bringing actual flowers, so that they can solve the problem by manipulating flowers and building the bouquet themselves. Elena had difficulty associating white flowers in two bouquets because they were drawn slightly differently. The visual representation (probably any particular representation) of a problem can be an obstacle/challenge, as well as wording in a natural language.

3. The Taxman Problem (Figure 4): this problem can be used in any grade level, from Grade 4 students to upper-year undergraduates. This is a playful game that is easy to understand, students can pay the taxman as many times as they need, and easy to make difficult by adding more bills. It is resistant to poor delivery (representation is simple), and minimal teacher input is required.

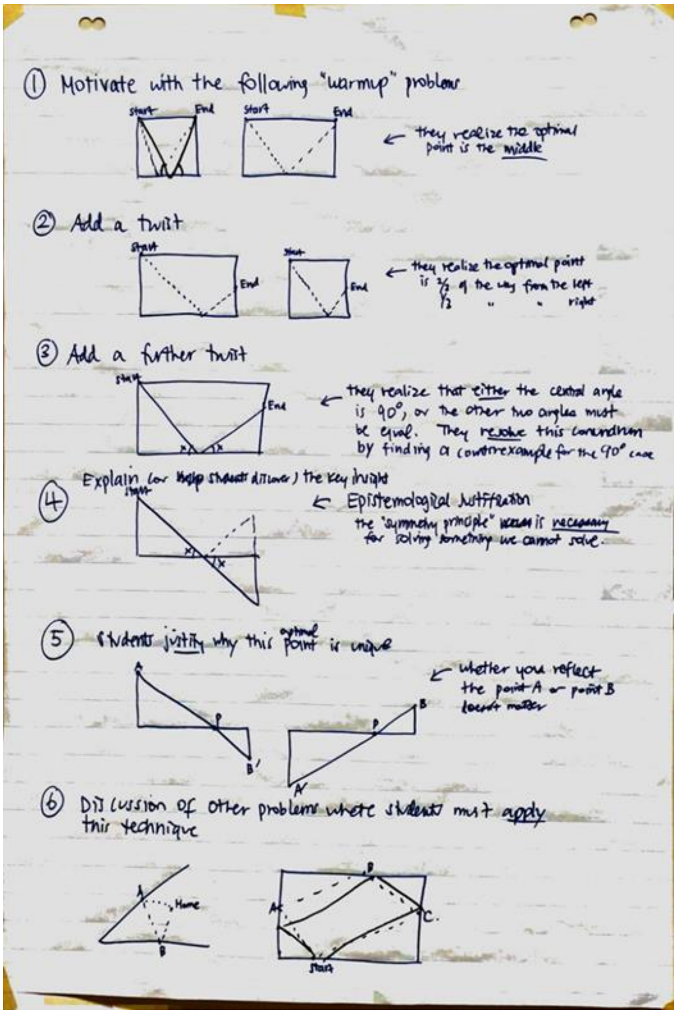

Figure 6

Here were some of our conclusions from Day 2:

- It is important for students to be stuck and realize the limitation of their knowledge.
- It is okay for teachers to propose new ideas to get students 'unstuck' in the problem-solving process.
- The way in which the problem-solving process is managed in class is sometimes more important than the problem itself.

DAY THREE

Here was our focus for the third day: *Discuss how problem solving should be addressed with future educators: in our methods courses, should we explicitly teach problem-solving heuristics and strategies, or perhaps teach content that incorporates problem solving?*

In thinking about the didactical analysis from the previous day, we realized the importance of making things complex, not complicated. Complexity is good for student learning, and there are two types: mathematical complexity (referring to the level of content) and procedural complexity (referring to the number of steps required to solve the problem).

Peter/Ann/Caroline/Nadine/Jimmy shared a poster with us that illustrated this point (See Figure 7). While this poster was created at the end of Day 2, it was fitting that Day 3 began with this discussion, because it tied so well to our focus.

We need to have a balance between task complexity and student ability: once these two items are in balance, we create *flow* (Csíkszentmihályi, 1990).

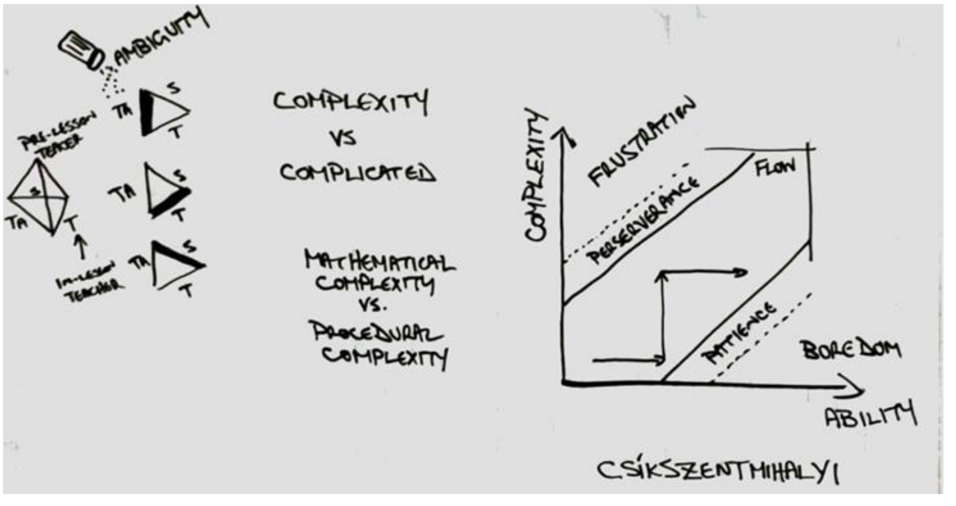

Figure 7

If the ability is too high and the complexity is too low, then the student is bored. If the ability is too low and the complexity is too high, then the student is frustrated. We want to make problems at just the right level of complexity to ensure flow. We want to introduce doubt when a student has solved the problem quickly: Are you sure this is the right answer? This way, students figure out the difference between thinking an answer is true, and knowing that an answer must be true. In other words, they have not just found an *answer*: they have obtained a *solution*.

As for who makes the complexity, sometimes it is the task itself, especially if the task (or problem) is easily understandable and broad. Sometimes it is the teacher who provides the complexity, asking provocative questions that lead to further thinking (e.g. What if you change the taxman problem from $N = 12$ to $N = 18$?) And sometimes it is the student who provides the complexity, asking themselves the questions that inspire further thinking and learning.

There are three approaches of mathematics education: teaching math techniques (e.g. basic skills) so that students can solve problems later; teaching problem-solving heuristics (e.g.

Polya's four techniques); teaching math through problem-solving. The ideal is the third one: we can and should use problem-solving to help students develop mathematical knowledge and to master it.

Many teachers have the (incorrect) belief that students cannot do problem solving unless they have been drilled on technique and content. As a result, students often do 'content' from September to May and then do 'problem solving' in June. Instead, let's integrate the two.

At this point, David asked us the following questions:

- What do we want teachers to know about what a 'problem' is?
- What do we want teachers to do with these problems?
- What do we want teachers to know about problem solving?
- How can they learn this? Through which problems?

The ensuing discussion gave rise to many observations. The following list gives a flavour of the discussion without quite capturing its diversity:

- We seek to create a need for a student to want to learn something, and a carefully-posed problem can be a way to do that. For example, asking students to figure out 28×40 in a Grade 4 class gets students to do it the hard way (for example adding 40 twenty-eight times) and then motivates the value of learning a method (or algorithm) for two-digit by two-digit multiplication.
- We need to recognize that many teachers do not want to do open-ended problem solving because it necessitates losing control, and might make the teacher look bad in front of their students—say if they are surprised with an approach they cannot 'manage' because they have not seen it before. But let's remember that vulnerability is necessary, and there is so much value to making mistakes, modelling lifelong learning. Struggle builds perseverance and character, and being comfortable with being uncomfortable.
- We want teachers to see that problem solving builds confidence, creativity, critical thinking skills, oral communication skills, and written communication skills. These five skills are so important in every child's life, no matter what endeavours he/she chooses to pursue in the future. And mathematical problem solving is arguably the best way to develop these five skills in a student.
- Let's fight the myth that teaching problem solving takes longer. Yes, some problems take longer, especially at the beginning. But if we build "thinking classrooms" (Liljedahl, 2015), then students learn content in a fraction of the time and retain the material better.
- We cannot just teach problem-solving techniques (e.g. heuristics). We need to move into much more complex problems that inspire, contextualize and motivate students to learn mathematical content. But let's remember that high school teachers teach for 200 days, six classes a day. We cannot make every class super-innovative, and it is unrealistic to do a deep inquiry-level class every day.
- We need to build problem-solving communities, which could mean at the local level (e.g. several teachers from the same school working together) or a national community like this where we have intentional discussions about what we are teaching and why, and share resources.
- Immersion, experience, and experimentation are key. We want to immerse teachers in problem-solving environments where they gain experience at solving problems themselves. But we cannot stop there. Teachers need to experiment with trying different problems, seeing which ones work in their classrooms and which ones do not, to continually refine their practice.

- We discussed the Sherlock Holmes Bicycle Tire problem, where a student can use tangent lines to figure out which direction a bicycle was going. Instead of using this as the Keystone Problem, i.e., the problem that is presented to the students after they have learned tangent lines, we can do this at the beginning—where they get to a point where they get stuck, and then they need to learn something new in order to progress further. Once they do, deep learning occurs because of this epistemological justification, much more so than if one had 400 exercises.
- In chaos theory, the most important things happen at the edge of chaos, never in the chaos, and never in a perfectly-ordered system. That applies in our teaching. Let's think about "how do we best teach problem solving?": as a problem to be solved! And let's try to solve this problem in creative and innovative ways.

We concluded the working group by discussing how we could support teachers and students in schools. We realized that one way we can do this is by having moments of 'productive disruption' together, where professors partner with local teachers to co-teach classes together, which creates community.

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MATHEMATICS EDUCATION AND SOCIAL JUSTICE: LEARNING TO MEET THE OTHERS IN THE CLASSROOM

ÉDUCATION MATHÉMATIQUE ET JUSTICE SOCIALE : APPRENDRE À RENCONTRER LES AUTRES DANS LA CLASSE

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Une version française vient à la suite.

INTRODUCTION

All human interaction involves the experience of Otherness. Mathematical activity is not an exception. Whether through history, social practices, language, aesthetic experiences or cultural practices, the experience of Otherness in mathematics comes ineluctably, consubstantial of teaching-learning. In this light, different kinds of reasoning, languages and orientations appear, as many voices claim their legitimacy and space of action in the mathematical world.

But what are these marginal voices? What actions can be taken to give them more space to be heard? What implications does this have for the mathematics classroom? What are the current prospects for research on/in this thematic? Above all, what conceptual or theoretical frameworks can help us think about mathematics education in these terms?

In this context, the mathematics classroom doesn't assign itself the role of promoting an individualistic idea of autonomy, but rather one of social engagement (cf. Arendt, 1961), where the fundamental openness to the Other and the respect of the Otherness appear central and decisive to us.

This is the deep sense in which we tried together to question the idea of Social Justice and theme of Otherness in the context of mathematics education. We tried to do so by examining various theoretical and conceptual frameworks that address mathematical education in these terms and examples of research issues and problems experienced in mathematics classrooms. The numerous and bright works from the recent *Mathematics Education and Society* (MES) conferences (Mukhopadhyay & Greer, 2015) have provided some material for our exploration.

In preparation for dialogue, a significant number of resources were collected and made available to the group through a web-based platform (https://blogs.ubc.ca/cmesg40wgd/). These resources (articles, book chapters, audio or video clips, hyperlinks, etc.), more than 30 in total, were collected by the facilitators in winter 2016 and organized under four different themes:

- Fundamental reflection on Alterity and social justice
- Cultural Identities (in)questions - on post-colonialism
- Doing research: tensions, illusions and possibilities
- Social justice in mathematics classroom – meanings and critiques

The resources represented a range of viewpoints from philosophers, researchers, educators, activists and journalists (Bakhtin, Levinas, Freire, Reynolds, Chimamanda, Dion, Donald Pinard, Memmi, Vanier, Žižek, Gutiérrez, Dixon-Roman, Radford, Atweh, Vale, Walshaw, Vithal, Skovsmose, Gutstein, Peterson, Stocker, Bigelow and Fasheh).

This platform gave tools for thinking, understood as pretexts, to stimulate dialogue for group discussions. It gave us access to a variety of authors and references so we could meet the interests and emerging questions of participants as discussion progressed.

DAY 1

It was important to begin by creating an atmosphere conducive to reflecting on social justice and the notion of Otherness in mathematics education. We hoped that drawing upon our own experiences of living-together within the Working Group would provide a provocative and respectful pathway for discussions involving the Other, social justice and mathematics education. Thus we began our Working Group with an activity for participants to introduce themselves while at the same time creating a spirit of community that would support possible difficult discussions.

Participants were organized into two circle groups each with about 9 participants. The task required the group to count to 20 with no verbal or physical cues other than to count, where one person gives one number another gives the next number. Any participant can start the count and anyone can give the next number. However, if two or more people gave the same number simultaneously the count reverted back to “one” and the count continued again.

Thus, completing a group-count to 20 required participants to be attentive to one another, to observe the other, and be aware of other ways of listening. It was also an opportunity to meet, laugh and initiate a cooperative relationship.

Figure 1. Counting to 20.

Following multiple attempts at the count, with one then the other group reaching the count to 20, participants discussed how the activity offered potential for creating a community spirit. Certainly part of the interest in the activity was the humour and laughter it instilled. But participants also remarked on how it offered opportunities to attend to each other in a different way as they considered their own contributions while anticipating contributions of others.

A second introductory activity invited group members to share their interests and motivations for studying mathematics and social justice. On a *Post-It*, participants wrote a response to the following:

- Reasons which led them into this Working Group
- Questions that animate particularly in connection with the theme of the Working Group
- The specific thematics they would like to see addressed during the Working Group.

This small activity allowed group members to start their engagement in the Working Group with introspective reflection. It seemed to enable participants to refine and highlight their personal questions, while revealing the plurality of the group's motivations. Participants shared their thoughts and questions with the group and then posted their responses on several large white sheets fixed on the walls around the room. Participants were asked to post their responses on wall sheets that contained other posted responses around a similar theme. As a result, participants asked each other to clarify their thoughts and questions. Participant's posted responses included:

- Race/math/achievement/students/racialized-visible minorities
- Research interest? / School leadership vs. equity
- I came to learn/listen different ways people conceptualize and take up social justice/invisible diversity
- What is the nature of inequities in school math?
- What do different people think *social justice* means? / How our languages and language give different senses of social justice?
- *Comment est-ce que je peux appuyer la diversité dans ma classe ?*
- Social justice and the invisible diversity of the classroom.
- How to conceptualize mathematical practice to be more inclusive, accessible.
- The world has become more tumultuous and insecure, and divided/polarised. So, how can we as math educators prepare our students for the unforeseeable challenge that awaits them after varsity school?

- Where am I going? / How do I become aware or more able to respond to social justice issues?
- Researchers' responsibilities / Teacher educators' responsibilities / *La souffrance de l'autre…*
- How social justice issues are reflected in mathematics education / Do socioeconomic factors affect mathematical communicating and thinking?
- *Comment je peux partager l'autorité dans la classe ?* / What are signs of disenfranchisement?
- Creating space for the legitimate other / We cannot make any substantial, lasting, inroads without listening.
- *Qu'est-ce que ça veut dire accepter les autres* ? / What does it mean to accept others?
- What is the Other? Otherness? Alterity? in math education?
- What role does mathematics play in this group? / Is it simply a context for questions of justice? / "it" – mathematical actions, problem solving… "interaction", power, enactivism
- Is it possible at all that mathematics education itself perpetuates inequality? If yes, then what can we do about it?
- How math got this much power to control and how that recursively and unjustly affects societies?

This second introductory activity, allowed the facilitators to learn more about the overall interests of the group. Wall sheets containing posted responses were found to be focused around general themes that included: the topic of mathematics itself, the role the participants can play and values to which they are related concerning social justice. Another important issue was the notion of otherness and the 'other' around the question of social justice.

The participants were then invited to read an article by Cynthia Reynolds published in the Canadian magazine *Maclean's* in October 2012 (http://www.macleans.ca/news/canada/why-are-schools-brainwashing-our-children/). The author brings a reactionary discourse, denouncing several cases of what she calls 'brain washing'. Reynolds depicts some primary and secondary students indoctrinated by different educational interventions where students were made aware of different social justice issues or different ideas and progressive positions. The author is outraged by the lack of neutrality within various educational environments.

We began with this reading in order to provoke discussion by exposing participants to a significantly contrary argument. Discussions around this article involved participants in articulating their own positions on including issues of social justice in school curriculum as well as possible reactions of such inclusions by parents and community members.

An important concern for the group was to clarify the very notion of social justice and what this concept entails in order to better respond to Reynolds' argument. The concepts of *tolerance* and *acceptance* were then particularly discussed and compared.

In response to the debate on the need to 'tolerate' or 'accept', we, as facilitators, presented to the group an audio recording through which Slavoj Žižek (2016) takes a position against the idea of integration, by associating some form of violence with the integration and the meeting of cultures.

Žižek (https://www.youtube.com/watch?v=b34kTAp4eY8) (from 59:00 to 1:10:00) argues for another form of multiculturalism, a multiculturalism where several cultural groups coexist without necessarily integrating themselves with one another. Again, a contradictory speech was provided in order to provoke more deep discussions, within tensioning divergent positions.

Listening to the speech of Žižek led participants to clarify their position concerning the notions *tolerance* and *acceptance*.

DAY 2

As we discussed at length social justice itself and the need to clarify its position within the curriculum, the group moved to focus more on the classroom in general and the mathematics classroom in particular. We therefore chose to show a video presenting an interview with the Brazilian educator Paulo Freire (2009) (https://www.youtube.com/watch?v=aFWjnkFypFA). Returning to his teaching career, Freire succinctly summarizes the idea of a 'collective' empowerment and stresses the need for teachers to deploy empathy and curiosity about their students and their lives.

The discussion then focused on the need for teachers to know their students and the context in which they live. Participants agreed with Freire and his emphasis that the question of social justice in the classroom must begin with the context of the students, because students may not feel concern for elements related to social justice that deviate too much from their reality. In other words, social justice issues must be 'in' the student experience.

Related to this 'student experience' concern, a participant proposed that the group read a text by Munir Fasheh (2015). Although the entire text was available to participants, an excerpt was chosen for group discussion in which Fasheh describes a school that brings 'unrooted' knowledge while ignoring the living knowledge of students along with the opportunity for students to free themselves. Instead, the author advocates a school of freedom, consciousness and social change.

But what are the contexts that could be discussed in class? The group seemed to perceive a difficulty for teachers to provide learning experiences that are rich and authentic, as proposed by Fasheh, while meeting their desires to highlight issues of social justice in their classrooms. The risk of 'false problems' in this context seemed high for participants.

Figure 2. Each day, the discussions continue despite the end of the period allocated for the Working Group. This photo is taken 30 minutes pass the end time of the Working Group.

Much of the discussion focused on the part of the text in which Fasheh talks about his mother, a textile worker, showing ‘authentic’, incarnated and rooted mathematical knowledge. He praised the authentic culture of his mother in opposition to the elitist culture promulgated by the school. The discourse of Fasheh was understood and well received by the group, but many participants felt the need to defend the school and nuanced the portrait made by the author.

DAY 3

At the beginning of the third day, the need to get closer to the mathematics classroom was felt. We had been discussing social justice and education in general and, for some in the group, there was a call to move even closer to mathematical issues. We decided to start the last day by reading a text from Gutstein and Peterson (2013) that was made available on the web platform.

This reading generated much discussion, including how to highlight social justice issues through mathematical activity. An important question raised focused on the extent to which mathematical activity was substantially present in contexts related to social justice. Indeed, the contexts presented by the authors appeared to be ‘superficial’ and ‘placed over’ the mathematics. These ways of introducing social justice in the classroom seem insufficient because students could simply ignore the context in their mathematics activities.

Therefore, a second question arose: where is mathematics in what could be called *mathematics education for social justice*? Again, the relationship between mathematics itself and social justice was questioned. The need for everyone to confront themselves with this question appeared crucial for the participants.

Following the break a few participants expressed a sense of discomfort and unease. Indeed, many felt some tension within the group. We decided, as facilitators, to face this situation, rather than avoid it by providing new suggestions to organize the Working Group. We decided to spend the last moments of the Working Group to the elucidation of these tensions. Participants were invited to express themselves on this issue of discomfort.

The exercise went well and tensions were reduced. This very emotionally and intense situation allowed us to act, miraculously, the questions that were raised in recent days. It provided opportunities to interrogate, in truly embodied and lived manners, the question of Otherness for the exploration of the concept of *social justice in mathematics education*.

What remained was for the group to decide how to present a resume of the Working Group during the plenary session. Participants chose the activity of creating *six-word memoires* where each summarized their own experience in the Working Group using six words. Their notes were presented in the form of video or audio clip at the plenary presentation:

- Mathematics fostering life awareness in community
- Tendrils of everything touch my heart
- Recognition of the other in disagreement
- Putting numbers only on local problems
- No social justice without humility
- Enacting social justice with the group
- Social justice philosophical framework – not content
- Bringing forth the world with others
- Power of maths perpetuates social injustice?
- *Mystère, violence, meute, autre, chemins, étoiles*

- Consciousness/*conscience*, feeling/*sentiment*, power/*pouvoir*, ethic/*éthique*, action, space/*espace*
- Mathematics education as reconciliation, just justice
- Doing forty times twelve in love…
- Urgent matters, profound engagement, meaningfully accomplished
- Social justice begins with authentic relationship.

CONCLUSION

We believe that the participants of the Working Group were able to grow significantly in their questions about social justice and mathematics education. We lived a rich and emotionally intense experience which has allowed us to refine our thinking and our ways-of-being in the classroom. Inevitably, several issues remained unresolved and the vast majority of documentation that we made available to participants remained unused but available to participants for future use. Nonetheless, we are particularly proud to have been able to stay closer to the emerging interests of the participants, and this, throughout the three days of activity.

Editor's Note: References follow the French version.

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INTRODUCTION

Toute interaction humaine implique l'expérience de l'Altérité. L'activité mathématique n'y échappe pas. Que ce soit à travers l'histoire, les pratiques sociales, les modalités langagières, les vécus esthétiques ou les pratiques culturelles, l'expérience de l'altérité en mathématiques se présente de manière inéluctable, consubstantielle de l'enseignement-apprentissage. Dans cette lumière, une diversité de formes, de raisonnements, de langages et d'orientations peut être mise au jour, autant de voix réclamant leur légitimité et leur espace d'action dans le monde mathématique. Nécessairement, cette perspective porte avec elle un aspect foncièrement critique en mettant à l'avant-scène des manières d'être-en-mathématiques fragiles, marginales ou minoritaires qui laissent parfois entendre des revendications sociales et politiques. En outre, elle montre qu'il n'y a pas de savoir neutre idéologiquement et que tout savoir s'insère dans une problématique éthique pour laquelle il nous faut développer notre sensibilité. Perspective qui se place en porte à faux avec l'idéologie capitaliste et son ombre, l'universalisme.

Or, quelles sont ces voix minoritaires ? Quels gestes peuvent être posés afin de leur donner la parole ? Quelles sont les implications pour la classe de mathématiques ? Quelles sont les perspectives de recherche actuelles sur/dans cette thématique ? Surtout, quels cadres conceptuels ou théoriques peuvent nous permettre de penser l'éducation mathématique en ces termes ?

Car, pour nous, la socialité du processus d'apprentissage signifie la formation et la transformation de la conscience, qui est justement (con)science, c'est-à-dire « savoir en commun » ou « savoir-avec-d'autres ». Dans ce cadre, la classe de mathématique ne peut donc se voir attribuer le rôle de promouvoir une idée individualiste d'autonomie, mais plutôt celle conçue comme engagement social (cf. Arendt, 1961/1989), où l'ouverture fondamentale à l'Autre et le respect de l'Altérité (c'est-à-dire de l'autre comme autre) nous apparait centraux et déterminants.

C'est donc dans un sens profond qu'avons tâché d'interroger l'idée de justice sociale dans le cadre de l'éducation mathématique, et ce, en examinant différents cadres théoriques et conceptuels qui permettent de penser l'éducation mathématique en ces termes, ainsi que des exemples de problématiques de recherche et de problématiques vécus en classe. Les nombreux et lumineux travaux issus du dernier colloque *Mathematics Education and Society* (MES) (Mukhopadhyay & Greer, 2015) ont fourni une partie du matériel pour notre démarche d'exploration.

Afin de préparer la discussion, une quantité importante de documents ont été rassemblés sur une plateforme web (https://blogs.ubc.ca/cmesg40wgd/). Cette documentation (articles, chapitre de livre, extraits audio ou vidéo, hyperliens, etc.) est issue d'une longue recherche bibliographique ayant eu lieu à l'hiver 2016. Cette recherche menée par les facilitateurs du groupe de travail a permis de recueillir près d'une trentaine de documents différents. Ceux-ci ont été regroupés sous quatre différentes rubriques :

- Réflexions fondamentales sur l'Altérité et la justice sociale
- Les identités culturelles en question - sur le postcolonialisme
- Faire de la recherche dans une perspective de justice sociale - tensions, illusions et possibilités
- La justice sociale en classe de mathématiques - sens et critique

De nombreux auteurs (progressistes et conservateurs) ont donc été rassemblés. Ils sont pour la plupart philosophes, chercheurs, éducateurs, militants ou journalistes (Bakhtine, Levinas, Freire, Reynolds, Chimamanda, Dion, Donald, Pinard, Memmi, Vanier, Žižek, Gutiérrez, Dixon-Roman, Radford, Atweh, Vale, Walshaw, Vithal, Skovsmose, Gutstein, Peterson, Stocker, Bigelow et Fasheh).

L'objectif de cette plateforme était de rendre accessible des outils de pensée, entendus comme des prétextes à la discussion ou des éléments émulateurs afin de susciter le débat. Avec une telle variété d'auteurs et de références, notre ambition, à titre de facilitateur, était donc de fournir une atmosphère d'ouverture en dirigeant le moins possible l'orientation des discussions.

En effet, l'idée était de fournir du matériel pour la réflexion, mais aussi de la documentation variée susceptible de répondre aux intérêts et questionnements émergents des participants au fur et à mesure de l'avancement du groupe de travail.

JOUR 1

Lors du premier jour, il était nécessaire de créer une ambiance propice à la réflexion sur la justice sociale et la notion d'Altérité en éducation mathématique. En tâchant d'intégrer l'idée même de l'Autre et la notion du vivre-ensemble qui allaient être l'arrière-plan de nos discussions, il nous fallait déployer une activité d'introduction cohérente et consistante, c'est-à-dire qu'il fallait mettre en acte ces dimensions tout en réalisant une certaine introduction de chacun au groupe et un certain esprit de communauté.

Cynthia avait alors proposé une activité à travers laquelle les participants devaient se placer debout et en cercle, et ce, afin de se faire face. La consigne était alors de compter jusqu'à 20 alors que chacun énonce tour à tour et spontanément le nombre en prenant la parole. Ainsi, le jeu commence lorsque quelqu'un lance spontanément : « un ». Une seconde personne se devait de dire spontanément « deux » au moment où elle le souhaite. Cependant, la consigne était telle que lorsque deux participants énonçaient en même temps le nombre suivant, par exemple si deux personnes (ou même trois personnes) disent ensuite le nombre « trois », le groupe devait alors recommencer à zéro.

De cette manière, afin de réaliser le jeu, les participants se devaient d'être attentifs aux réactions de chacun, d'observer les autres, de croiser leur regard. C'était aussi simplement l'occasion de se rencontrer, de rire et d'amorcer une relation de coopération.

Figure 1. Comptage à 20.

Une petite discussion a d'ailleurs émergé à la suite de ce jeu quant à l'idée d'être ensemble et au potentiel de ce dernier pour la création d'un esprit de groupe. Les participants ont conclu qu'il s'agissait d'une bonne activité d'« échauffement » pour délier chacun de sa position individuelle ou pour inviter chacun à rencontrer les membres du groupe.

Une seconde activité d'introduction a été celle du partage des intérêts et des motivations de chacun. La consigne était alors d'écrire sur un *Post-It* soit :

- les raisons qui ont amené les participants à se présenter à ce groupe de travail
- les questions qui les animent particulièrement en lien avec la thématique du groupe de travail
- les thématiques particulières qu'ils souhaiteraient que le groupe aborde lors de trois jours de discussion.

Bien évidemment, les réponses ont été variées. Elles ont permis à chacun d'amorcer sa réflexion en partant d'un exercice d'introspection leur permettant de peaufiner et de mettre en évidence leurs questionnements personnels, tout en s'ouvrant à la pluralité des motivations du groupe. De plus, chacun devait énoncer ce qu'il avait écrit sur son bout de papier avant de le coller au mur où avaient été fixées plusieurs grandes feuilles blanches dans le but d'accueillir et de regrouper les pense-bêtes de chacun. De cette façon, les participants étaient amenés à énoncer leurs réflexions ou leurs questionnements, ce qui créait déjà quelques discussions, puisque plusieurs demandaient à d'autres de clarifier certains éléments de réflexions. Les réponses des participants (facilitateurs inclus) ont été les suivantes :

- *Race/math/achievement/students/racialized-visible minorities*
- *Research interest? / School leadership vs. equity*
- *I came to learn/listen different ways people conceptualize and take up social justice/invisible diversity*
- *What is the nature of inequities in school math?*

- *What do different people think* social justice *means? / How our languages and language give different senses of social justice?*
- Comment est-ce que je peux appuyer la diversité dans ma classe ?
- *Social justice and the invisible diversity of the classroom.*
- *How to conceptualize mathematical practice to be more inclusive, accessible.*
- *The world has become more tumultuous and insecure, and divided/polarised. So, how can we as math educators prepare our students for the unforeseeable challenge that awaits them after varsity school?*
- *Where am I going? / How do I become aware or more able to respond to social justice issues?*
- *Researchers' responsibilities / Teacher educators' responsibilities* / La souffrance de l'autre…
- *How social justice issues are reflected in mathematics education / Do socioeconomic factors affect mathematical communicating and thinking?*
- Comment je peux partager l'autorité dans la classe ? / *What are signs of disenfranchisement?*
- *Creating space for the legitimate other / We cannot make any substantial, lasting, inroads without listening.*
- Qu'est-ce que ça veut dire accepter les autres ? / *What does it mean to accept others?*
- *What is the Other? Otherness? Alterity? in math education?*
- *What role does mathematics play in this group? / Is it simply a context for questions of justice? / "it" – mathematical actions, problem solving… "interaction", power, enactivism*
- *Is it possible at all that mathematics education itself perpetuates inequality? If yes, then what can we do about it?*
- *How math got this much power to control and how that recursively and unjustly affects societies?*

Cette seconde activité d'introduction, qui permettait aussi aux facilitateurs de connaître globalement les intérêts du groupe, faisait aussi en sorte que les participants puissent simplement se présenter et prendre part déjà à l'orientation des discussions. Un sujet brulant a été celui des mathématiques elles-mêmes, du rôle qu'elles peuvent jouer et des valeurs qu'elles véhiculent en relation quant à la justice sociale. Une autre question importante était celle de l'Altérité et de l'Autre dans la question de la justice sociale.

Les participants se sont ensuite adonnés à la lecture de l'article de Cynthia Reynolds paru dans la revue canadienne *Maclean's* en octobre 2012 (http://www.macleans.ca/news/canada/why-are-schools-brainwashing-our-children/). L'auteur y tient des propos réactionnaires en dénonçant plusieurs cas de ce qu'elle appelle « *brain washing* ». Elle y dépeint des élèves du primaire et du secondaire endoctrinés par différentes interventions pédagogiques où les élèves ont été sensibilisés à différents enjeux de justice sociale ou à différentes idées et positions progressistes. L'auteur s'indigne de ce manque de neutralité de la part des intervenants des divers milieux éducatifs.

Bien entendu, l'objectif de cette lecture était de provoquer les participants en les confrontant à un discours sensiblement éloigné de leur position. L'effet a été important puisque les discussions ont alors tourné autour des intérêts des participants quant aux questions de justice sociale. La discussion a alors tourné sur l'importance de la sensibilité à la justice sociale dans les milieux respectifs des participants (pour la plupart des formateurs dans les milieux universitaires).

Une question importante pour le groupe semblait être celle de la justice sociale et de ce que ce concept peut recouvrir. En d'autres mots, les participants manifestaient le besoin de bien se situer quant à la notion de *justice sociale* afin de mieux répondre à l'article de Reynolds. Les notions de *tolérance* et d'*acceptance* ont été particulièrement discutées et comparées.

En réaction à ce débat, quant à la nécessité de tolérer ou d'accepter, nous avons trouvé pertinent de présenter au groupe un enregistrement audio à travers lequel Slavoj Žižek (2016) prend position, de manière plutôt cavalière, contre l'idée d'intégration en associant une certaine forme de violence à l'intégration et à la rencontre des cultures.

Žižek (https://www.youtube.com/watch?v=b34kTAp4eY8) (de 59:00 à 1:10:00) y plaide pour une autre forme de multiculturalisme, un multiculturalisme dans lequel plusieurs groupes culturels coexistent sans nécessairement s'intégrer l'une à l'autre. Encore là, un contre-discours était recherché afin de provoquer plus de profondeur dans les discussions par la mise en tension de positions divergentes.

L'écoute du discours de Žižek a mis en évidence, pour chacun, le besoin de clarifier sa position quant à la tolérance et au vivre-ensemble en tenant compte des problématiques qui en émergent.

JOUR 2

Comme il a été question longuement de la justice sociale en elle-même et du besoin de chacun de clarifier sa position, nous sentions, à titre de facilitateurs, que le groupe souhaitait d'orienter davantage la discussion vers la classe et possiblement vers la classe de mathématiques. Nous avons donc choisi de présenter au groupe une vidéo qui présente un entretien avec le pédagogue Paulo Freire (2009) (https://www.youtube.com/watch?v=aFWjnkFypFA). Freire revient sur son parcours d'enseignant. Il résume de manière succincte l'idée d'une émancipation « collective » et souligne la nécessité pour l'enseignant de déployer une empathie et une curiosité quant aux élèves et à ce qu'ils vivent.

La discussion a alors porté sur la nécessité pour les enseignants de connaitre leur élève et le contexte dans lequel ils évoluent. Or, la question de la justice sociale en classe doit commencer, comme le souligne d'ailleurs Freire, par le contexte des élèves, car les élèves risquent de ne pas se sentir concerner par des éléments relatifs à la justice sociale qui s'éloignent trop de leur réalité. Autrement dit, les questions de justice sociale, afin qu'elles prennent sens en classe, doivent se situer dans l'expérience des élèves.

Une lecture d'un texte de Munir Fasheh (2015) a alors été proposée au groupe (suivant la proposition d'une participante). Fasheh y décrit une école abrutissante qui enseigne des savoirs déracinés et qui méprise ceux des élèves et la possibilité pour eux de s'émanciper. L'auteur plaide plutôt pour une école de la liberté, de la conscience et du changement social.

Mais quels contextes sont susceptibles d'être abordés en classe ? Le groupe semblait percevoir une difficulté pour les enseignants de proposer des situations d'apprentissage qui présentent des contextes riches et authentiques, comme le propose Fasheh, tout en répondant à leurs désirs de mettre en évidence les enjeux de justice sociale dans leurs classes. Le risque de présenter de « faux problèmes » dans ce contexte semblait grand pour les participants.

Une bonne partie des discussions ont porté sur la partie du texte dans laquelle Fasheh présente sa mère, travailleuse du textile et des connaissances mathématiques véritables, incarnées et enracinées qu'elle possédait. Il fait l'éloge de la culture authentique de sa mère en opposition à la culture factice et élitiste promulguée par l'école. D'une part, le discours de Fasheh a été compris et bien reçu par le groupe, mais plusieurs ont senti le besoin de défendre l'école et de nuancer le portrait qu'en donne l'auteur.

Figure 2. Chaque jour, les discussions continuaient malgré la fin de la période allouée pour le groupe de travail.

JOUR 3

Au début du troisième jour, le besoin de se rapprocher de la classe de mathématiques se faisait sentir. Nous avions discuté de la justice sociale et de l'éducation en général et il nous fallait nous rapprocher, pour plusieurs, de la classe de mathématiques. Nous avons donc choisi d'entamer la lecture du texte de Gutstein et Peterson (2013) que nous avions rendu disponible sur la plateforme web.

La lecture a suscité de nombreuses discussions, notamment sur la manière de mettre en évidence les problématiques de justice sociale à travers l'activité mathématique. Il était question de savoir s'il était suffisant de présenter des contextes en lien avec la justice sociale afin d'aborder ces questions avec les élèves. En effet, les contextes présentés par les auteurs semblent « plaqués » sur les mathématiques du programme. Or, il apparait pour plusieurs que ces manières d'aborder la justice sociale semblent insuffisantes, car les élèves sont susceptibles de ne simplement pas tenir compte du contexte dans leurs activités de classe.

C'est pourquoi une seconde question a été soulevée : où se trouvent les mathématiques dans ce qui pourrait être appelé l'éducation mathématique pour la justice sociale? Encore là, la relation entre les mathématiques elles-mêmes et la justice sociale a été interrogée. La nécessité pour chacun (formateurs, enseignants, directeurs…) de se confronter à cette question est apparue cruciale pour les participants.

Après la pause de la 3e journée est survenu un malaise au sein du groupe. Plusieurs participants ont manifesté un sentiment d'inconfort et de malaise. En effet, plusieurs ont senti une certaine tension et ne se sont plus sentis en sécurité au sein du groupe. Nous avons donc décidé, à titre de facilitateurs, d'affronter la situation plutôt que de l'éviter en proposant de nouvelles manières d'organiser le groupe de travail. Il en allait de la réussite de l'activité et de la bonne entente entre les collègues de travail. Nous avons donc décidé de consacrer les derniers moments du groupe de travail à l'élucidation de ces tensions. Les participants ont donc été invités à s'exprimer et à affronter courageusement la situation.

L'exercice s'est bien déroulé et les tensions ont pu être résorbées au mieux. Cette situation très intense émotionnellement a permis de mettre en acte, miraculeusement, les questions qui ont été soulevées lors des dernières journées et d'interroger « en actes », de manière véritablement incarnée et vécue, la question de l'Altérité pour l'exploration de la notion de justice sociale en éducation mathématique.

Restait pour le groupe de décider de la manière de présenter les conclusions du groupe de travail en plénière. Les participants ont choisi l'activité du *six-word memoires*, où il leur fallait résumer leur expérience dans le groupe de travail en six mots. Leurs notes ont été présentées sous forme de capsule vidéo ou audio lors de la présentation plénière :

- *Mathematics fostering life awareness in community*
- *Tendrils of everything touch my heart*
- *Recognition of the other in disagreement*
- *Putting numbers only on local problems*
- *No social justice without humility*
- *Enacting social justice with the group*
- *Social justice philosophical framework – not content*
- *Bringing forth the world with others*
- *Power of maths perpetuates social injustice?*
- Mystère, violence, meute, autre, chemins, étoiles
- *Consciousness*/conscience, *feeling*/sentiment, *power*/pouvoir, *ethic*/éthique, action, *space*/espace
- *Mathematics education as reconciliation, just justice*
- *Doing forty times twelve in love…*
- *Urgent matters, profound engagement, meaningfully accomplished*
- *Social justice begins with authentic relationship.*

CONCLUSION

Nous croyons que les participants du groupe de travail ont pu avancer de manière importante dans leurs questionnements autour de la justice sociale et de l'éducation mathématique. Nous avons vécu une expérience riche et intense émotionnellement ce qui nous a permis de grandir et de raffiner notre réflexion. Inévitablement, plusieurs questions sont restées en suspens et la grande majorité de la documentation que nous avions rendue disponible aux participants est restée inutilisée. Cependant, nous sommes particulièrement fiers d'avoir su rester au plus près des intérêts émergents des participants, et ce, tout au long des trois journées d'activité.

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ROLE OF SPATIAL REASONING IN MATHEMATICS

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INTRODUCTION

The intent of this working group was to explore the role of spatial reasoning in mathematics teaching and learning. There was a particular emphasis on elementary school mathematics, but we also examined some ways in which spatial reasoning can be extended both outside of the traditional geometry curriculum and into middle and high school mathematics.

Globally, we have drawn on recent research in mathematics education and cognitive science to inform our discussion of what spatial reasoning can look like and to understand how and why it might be significant for mathematical learning. We have drawn on philosophical and historical considerations to help us appreciate why spatial reasoning has received, in the past, much less attention both in research and practice. But, mainly we have explored some activities that have been designed to emphasise spatial reasoning and discuss how these might be further refined and extended.

On each day, we started the session by doing a different imagery exercise with the working group members and afterward the session was divided into two working blocks. This report describes globally the activities that solicited spatial reasoning that we proposed to the group and the reflection that emerged.

DAY 1

On the first day, we presented some elements about the reasons we should work on spatial reasoning, like the link that researchers make between spatial reasoning and mathematical performance, the fact that spatial reasoning seems to be a determining factor in future mathematics and scientific success, and the fact that it can be developed, so that teachers are empowered in this development (Wai, Lubinski, & Benbow, 2009). But, at this time, we have little information on ways to promote spatial reasoning in class.

The first imagery exercise proposed to the group was based on the work of Conway, Doyle, Gilman, and Thurston (2010). The group was invited to imagine the actions that we described. For example:

> Cut off the corners of an equilateral triangle as far as the midpoints of its edges. What is left over? Cut off the corners of a tetrahedron as far as the midpoints of the edges. What shape is left over?

The challenge was to be able to do the actions mentally and to remember the starting point and the sequence of actions.

DAY 1, BLOCK A: TWO EXAMPLES OF ACTIVITIES PROMOTING SPATIAL REASONING

First activity: Rectangles Everywhere!

This task proposes to represent each multiplication on the grid. Here are two samples:

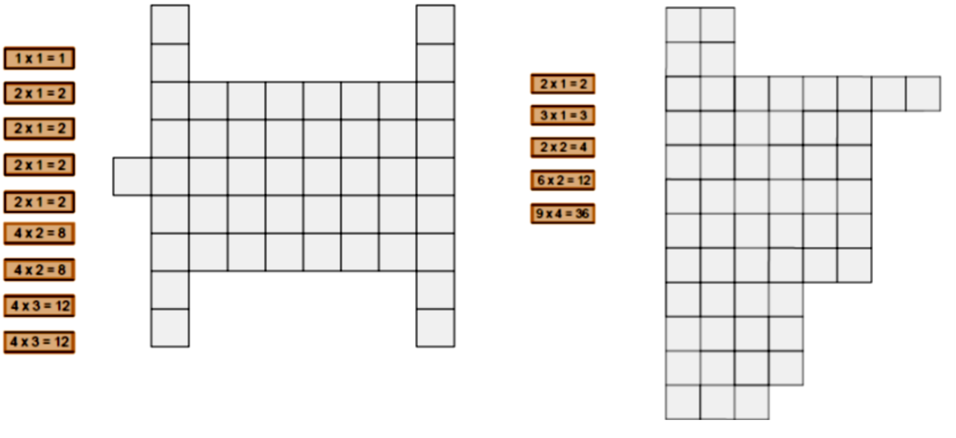


Figure 1. Two samples for representing multiplication on the grid.

Second activity: *Les tours de Valentin* (2005)

This task involves a 3x3 grid. Each team had 9 towers of different heights (3 towers constructed with one cube, 3 with two cubes and 3 with three cubes) and they had to place them on the grid to respect the four perspectives. For example, on the left of the first grid below, we see 2, 1 and 3. These numbers represent the quantity of towers that we see from that point of view (in the first column, we only see two of the three towers, in the second, only one is visible and in the last, we see the three towers). The same exercise was done with a 4x4 grid and 16 towers. Here are two samples (Braconne-Michoux, 2012)[1].

[1] For more examples: https://www.brainbashers.com/showskyscraper.asp?date=1017&size=4&diff=1 or https://www.cariboutests.com/games/skyscrapers.php

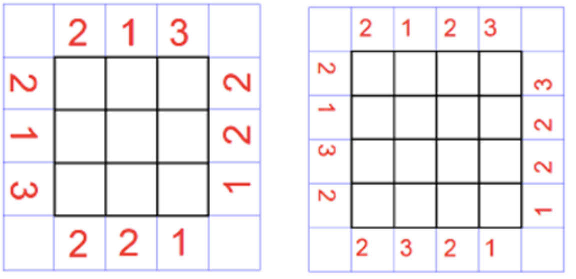


Figure 2. Two samples from the tower activity.

Here are two teams of our working group in action:

Figure 3. Two teams working on the tower activity.

These two activities were chosen to explore different aspects of spatial reasoning and the ways in which it is expressed in each case. Here are some elements of discussion that emerged from these two activities:

- *In each case, even during the imagery exercise, spatial reasoning was related to a private experience for each WG member, so the communication of what each person saw in their mind was very important. In the imagery exercise, other participants asked questions that helped them clarify and be more aware of their own mental image. So, what should the role of communication be in the development of spatial reasoning?*
- *In the three cases (including the imagery exercise), the task involved generalisation and regularity reasoning (for example,* possible *vs.* impossible *or* necessary *vs.* possible*). Is there a link between spatial reasoning and other types of reasoning (for example generalisation)?*
- *The manipulation was different in the three cases (none, writing on paper, and moving objects). Is it more appealing when we can manipulate? Do we dare more? Do objects give us a starting point to exchange and express ourselves? Touch sense has a different aspect—what could its role be in the development of spatial reasoning?*
- *Sometimes imagery requires a lot of energy and sometimes it does not. In which cases does it, and in which does it not? And how does imagery link to memory or to sound?*

- *Did we use spatial reasoning or did we reason on spatial objects?*
- *For the activity with the towers, we were tempted to take a picture of the four perspectives and also from the top view. In doing that, we have had to choose the position of the camera. Would it be meaningful to give this type of task to our students?*
- *The tower activity offered a dynamic way of approaching the concept of perspective. What impact can this variable have on the development of perspective and, more globally on the development of the spatial reasoning involved?*

DAY 1, BLOCK B: SPATIAL REASONING—FIRST MODEL AND TWO PROBLEMS

Here is the first model proposed to the working group to use to approach spatial reasoning. Uttal et al. (2013) developed a topology of spatial reasoning:

	Intrinsic	Extrinsic
Static	Relationships between parts of an object; static geometric characteristics. 	Coding the location of objects relative to other objects or a reference frame. 
Dynamic	Folding, bending, rotating, scaling, cross-sectioning, comparing 2-d and 3-d views. 	How one's perception of the relations among objects changes as one moves through the environment. 

Figure 4. A 2x2 classification of special skills and examples of each (adapted from Uttal et al., 2013, p. 354).

This 2x2 model, although not perfect, expresses spatial reasoning through *intrinsic* tasks involving only the object at hand versus *extrinsic* tasks involving relations among multiple objects to a frame of reference. And, those tasks can also be characterized by their *static* context, where the objects remain stationary, or their *dynamic* context where objects or perspectives can be transformed either physically or mentally (Davis & the Spatial Reasoning Study Group, 2015). The WG was invited to classify the two previous activities using the model.

The WG found this model to be helpful to understand psychological origins and meaning—for example, to make sense of the results of a student who has been evaluated by a psychological test involving these spatial reasoning aspects. It also gives us a framework to talk more specifically about spatial reasoning and to analyze our class tasks. It also underlines the pedagogic factor behind each task. But it doesn't allow us to have a better idea of the development of this reasoning or ways that a teacher can promote it in class. Additionally, it does not seem complete and the categories of *intrinsic* and *extrinsic* are blurred for us. Some questions remain:

- *This model is extracted from a diagnostic evaluation, but can it be used in a mathematics classroom?*
- *Is it supposed to characterize the task, or the strategies students may use to resolve the task?*
- *Can we look at this model as a grid that we can go through during the resolution of a task?*
- *How does the author of this model link imagery and remembering*?

The last activity of the WG for this first day was to analyze different productions of sixth-grade students for the following two problems:

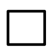
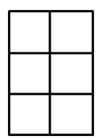
#1 *My neighbor has a rectangular plot whose length is equal to 3 times the length of my plot. I also know that its width is 2 times the width of my plot. How many times is my neighbor's plot area larger than the area of my plot? Justify your answer.* (Proulx, 2012)

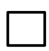
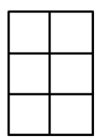
Production A: My plot 3 × 1.5 = 5.5 (calculation mistake), neighbor plot: 9 × 3 = 27
So, 5.5 + 5.5 + 5.5 + 5.5 + 5.5 = 27.5 answer: 5 times more.

Production B: The answer is 5 times more. My plot is 3 × 3 and the neighbor plot 6 × 9. So, 2 times more + 3 times more = 5 times more.

Production C: If my plot is 3 × 2, the area is 6 cm^2 and the neighbor plot is 6 × 6 = 36 cm^2. The area is 6 times more (depending on the rectangles).

Production D: My plot L = 10 m, W = 5 m so area = 50 m^2, my neighbor plot: L = 30 m, W = 10 m, area = 300 m^2. Answer = 6 times bigger. Verification: L = 15 m, W = 12 m, A = 180 m^2; L = 45 m, W = 24 m, A = 1080 m^2, so again 6 times bigger.

Production E: My plot 1 × 1:  Neighbor plot 3 × 2:  The area of my neighbor is six times bigger than mine.

Production E: My plot:  Neighbor plot:  6*x*. His plot is 6 times bigger because my plot enters 6 times in it.

#2 *Which of these mixes will taste 'orangiest':*

Mix A	Mix B	Mix C	Mix D
water: 3 cups concentrate: 2 cans	water: 4 cups concentrate: 1 can	water: 8 cups concentrate: 4 cans	water: 5 cups concentrate: 3 cans

Production G: We reduced the cans of concentrate to one and reduced the cans of water accordingly (A: 1 can of concentrate with 1.5 cans of water; B: 1 with 4; C: 1 with 2; D: 1 with 1 2/3). A is the most 'orangy' because we reduced all the cans of concentrate down to one and A had the least cans of water compared to B, C and D.

Production H: Mix A: 2 + 3 = 5 cups, × 3 = 15 cups, so 6 concentrate/15; Mix B: 1 + 4 = 5 cups, 3/12; Mix C: 4 + 8 = 12 cups, × 1.25 = 15 cups, so 5/15; Mix D: 3 + 5 = 8 cups, × 1.875 = 15, so 5.6/15. Answer is A.

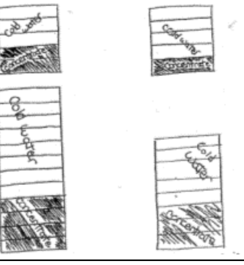
Production I:

Mixture A would have the most taste.

Production J: Mix A is the most 'orangy' because it has the least amount of water added which is 1 ½ cups of water (A: 1 can / 1 ½ water; B: ¼; C: ½; D: 1/ 1 2/3).

Production K: Mix A is most 'orangy', because… A: 100% ÷ 5 = 20%, 1 cup = 20%, 2 cups = 40%; B: 20%; C: 100% ÷ 12 = 8%, 1 cup = 8%, 4 cups = 32%; D: 100% ÷ 8 = 12.5%, 1 cup = 12.5%, 3 cups = 37.5%.

Production L: The juice that will taste 'orangy' is mix A, because it does not have as much water as mixes B, C and D.



As always, questions arose from this task:

- *How can we take into account spatial reasoning in the context of problem solving? Is it considered equivalent to a numeric solution?*
- *The generalized character of spatial reasoning emerges here also. In production F, the spatial reasoning tends to support a general solution. If we pay attention to the top of the second diagram, the area of the plot doesn't seem to be added on but rather divided from the total area. It is an evidence to show that "*I'm showing it, but I already know it's six.*"*
- *Which of those productions is closer to the algebraic solution and why?*
- *What is evidence of spatial reasoning in these types of productions?*
- *Why do students use more verbal explanation for the second problem? Is it because of the nature of the task? Is it because of the contract?*

Readings after Day 1: Okamoto, Kotsopoulos, McGarvey, & Hallowell (2015); Newcombe (2010); Marchand & Braconne-Michoux (2013).

DAY 2

On the second day, the WG explored spatial reasoning with two other examples of activities that promote this reasoning and compared the first model to a second one that was elaborated from another framework. Walter Whiteley and Ami Mamolo generously led the *Popcorn Box Activity* with the WG.

First of all, we started with an imagination exercise that was orientated around the following:

a) Take a wire frame, which forms the edges of a cube. Trace out a closed path which goes exactly once through each corner.
b) Imagine a wire is shaped to go up one inch, right one inch, back one inch, up one inch, right one inch, back one inch,.... What does it look like, viewed from different perspectives? Now, take a pencil to draw the line…

For part a), some participants had different points of view: (i) some viewed a small cube in front of them, some saw a big cube and other saw themselves in the cube. For part b), some imagined themselves walking the path and others made the path by moving their eyes.

In this exercise, the addition of a pencil in the mental representation created different reactions among the participants of the WG:

- *The pencil was seen as an intrusion in the image and its actions.*
- *I saw myself taking the pencil in my hand.*
- *I had to move backward to see my drawing and my hand.*
- *I focused more on my hand then my drawing.*
- *It didn't change anything.*
- *My hand position changed completely.*
- *The pencil made my drawing more blurry, more like a draft ('sloppy').*
- *It was difficult to preserve the information received and to mentally take the pencil.*

DAY 2, BLOCK A: A DYNAMIC EXAMPLE AND A SECOND MODEL

Third activity: Triangles

See: http://www.sfu.ca/geometry4yl/triangles.html

This activity works on the notion of triangles in a dynamic way. One person worked directly with the application and the other one drew the result:

a) "*Construct a triangle* [with three points and three vertices] *and drag the vertices*" [to see the effect of this action].
b) "*Use these triangles (two or more) to fill up the shape*" [square, pentagon or star]
c) "*Make a design of your own using these triangles. Try a house or an animal.*"

Here are some observations of the WG comparing this activity with *Rectangles Everywhere* and *Les tours de Valentin*:

- *The first two activities involved more a creation of a shape or an arrangement, and this one involved mostly a composition/decomposition of shapes and moving around the triangles. The focus is put on the fact that it takes three points to make a triangle.*
- *This activity focuses on classification, sorting, and looking for symmetry.*
- *The central question is: what does it take to make a triangle? The students are invited to move the triangles around, to stretch them and see if the result is still a triangle.*
- *We notice that the application is not precise. Sometimes, it is hard to correctly place the triangles into the shape. But, the question to address is: does this fact help students to go beyond this representation and reason more on the triangle concept, or does this constitute a limit for them?*
- *As with the two other activities, this one deals with some regularity and generalisation, reasoning in terms of what is necessary or invariant in a triangle. The students are, in this sense, invited to reason on the class of triangles and not a particular one.*

Second model in spatial reasoning

Here is the second model that we can look at to think about spatial reasoning. This model has its origins in mathematical didactics, both the French and English traditions, but it was created to treat *les connaissances spatiales*, and not spatial reasoning in particular. The model is inspired by the work of: Clements, 1999; Berthelot & Salin, 1992, 1999; Laborde, 1988; Battista, 2007; von Glasersfeld, 1991; Steffe & Cobb, 1988; and Johnson-Laird, 1998.

	Orientation		Organisation	
	Repérage 	Latéralité 	Représentation 	Articulation 
Archéologique				
Photographique				
Scénographique				

Figure 5. Second model of spatial reasoning.

Spatial sense is divided in this model into two components: *orientation* and *organisation*. The orientation component relates to the location and the lateral conventions of *right* and *left*. The second component treats the organisation of the shapes or solids on paper or mentally (as one, two, or three dimensions) and the relations between them and in space (e.g., in front, on the side, above, "the cube is on one of its vertices", "the two right angles are adjacent"). For each of these components, three levels of abstraction are possible. At the first level, the shape or the solid has to be visible in order to characterize it. Like an archaeologist, the students are extracting facts directly from the shape or the solid. At the second level, students are able to describe or relate to a shape or a solid even if it is not present. At this stage, students can see a picture of the shape or solid being considered, but they don't have the capability of moving it or transforming it in their minds. The mental representation is like a picture, but in relation to a specific shape or solid. But for the same shape, a student can imagine more than one picture in his mind. It is at the third level that students are able to mentally imagine a shape or a solid and be able to move it around (e.g., mental rotation or describing mentally the way to go home) or to transform it (e.g., to cut out one corner of a cube and look at the area).

The WG compared this model to the first one. They observed that static versus dynamic was not present in the same terms. But for the second model, that concrete action and representation (*niveau archéologique*) versus mental action and representation were more explicit. The comparison of the two models needs further exploration, but here are questions that were raised by the participants, along the same lines as for the first model:

- *This model involves* les connaissances spatiales, *but what is the difference between spatial reasoning and* connaissances spatiales*?*
- *Is it supposed to characterize the task or the strategies students may use to resolve the task?*
- *Can we look at this model as a grid that we can go through during the resolution of a task?*

DAY 2, BLOCK B: ANOTHER EXAMPLE OF ACTIVITIES PROMOTING SPATIAL REASONING

<u>Fourth activity: *Popcorn Box* presented by Walter Whiteley and Ami Mamolo</u>

Each team received two popcorn boxes and they had to predict which box would hold more popcorn. This is a well-known optimization problem: *What box will hold more popcorn?* The goal was to find a way to compare the boxes in pairs, find the one that holds the most popcorn, and then compare this box to others. The necessity for precision came when the volumes of the two popcorn boxes were close. Also, the necessity to compare extreme cases added to the reflection (too thin = 0 or too high = 0) and made the teams search for this turning point. The

WG hypotheses were 1/3 and 1/4. With the students, the case where the volumes were close was treated with sand or rice; this activity took around three to four hours in class.

Figure 6. Working on the Popcorn Box activity.

After making hypotheses and comparing some popcorn boxes built with physical materials, Walter and Ami invited the group to continue the activity with this application: http://www.sfu.ca/~nathsinc/gsp/PopcornBoxBasic/. This oriented the group to new questions:

- *What size corner should be removed from a piece of paper to maximize the volume of that paper when it is folded into a box? To treat this question, the application displayed a plan of the paper cut and the box made with this paper cut and made it possible to increase or reduce the corner and dynamically observe the effect on the volume.*
- *What is the relation between the size of the cut and the volume? At this point, a graphic was added to the application to observe the maximum point.*
- *Comparing side area with base area: here, two plans of the same sheet were presented and the participant had to observe the new one, which was focused on the 1/4 area.*
- *Comparing the graphic, the two plans, the box itself, and the volume, all together.*
- *Comparing loss and gain (sides folded in and the overlap generated).*
- *What if we tilt the sides of the popcorn box?*

This application must be presented to students only after they have had the time to explore the problem with the physical model and to elaborate a hypothesis. In fact, the WG participants observed that when the application was introduced, they lost control of the mathematical activity and it seemed like someone else was doing the reasoning for them; their role became mostly observation of the mathematical phenomena. But the purpose in using this application was to treat this question in a dynamic way and to compare two representations of the popcorn box simultaneously (i.e., the box plan versus the popcorn box itself, or the box plan with the graphic). Walter remarked that the answer was in front of us:

$$V = x(l-2x)^2 \qquad \frac{dV}{dx} = (l-2x)^2 - 4x(l-2x) = (l-2x)(l-6x)$$

If we think in terms of ratio, where should we cut the paper to maximise the volume? This reasoning is used with rectangular paper, but is it the case with a circle? With a polygon? Here again, spatial reasoning seems to be linked to regularity and generalisation reasoning.

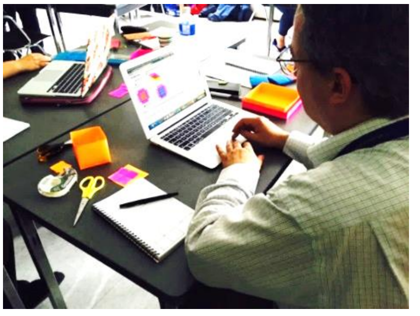


Figure 7. Working with the Popcorn Box application.

The group discussed the fact that similar activities can be elaborated in 1- or 2-dimensions, but by experience, three dimensions seem to be more relevant. This activity links spatial reasoning with differential calculus. The WG participants found that in this task, the imagery was very stimulating to promote visualizing the phenomena. And, finally, a link between this activity and Cavalieri's Principle was discussed.

Readings after Day 2: Whiteley (2002); Laborde (2005); Lunkenbein, Allard, & Goupille (1983).

DAY 3

On the last day, the imagery exercise was oriented around one of Nicolet's (1980) films: *Families of Circles in the Plane*. John Mason was kind enough to conduct this part of the WG and exchange with the group the way he approached this kind of task. At first, he asked us to view part of the film and he oriented the reflection afterwards:

1. By yourself, see how much you can re-construct in your mind. Take some notes about what you noticed when you tried to reconstruct.
2. In groups at your table try to re-construct the beginning of the film—share what you saw. Anything that is said at this point is a conjecture.
3. Collectively, we then, reconstructed the film. We did not check the film until we shared the same conjecture. Here is one hypothesis of the group: "*starting at the top of the screen, the circle wanted to catch the point. Then the circle started to change size, the circle changed color, and then the circle attached to the point and paused…*". Here the hypothesis is based on the mental image, the words that came to mind, or the feeling that was formed during the observation of the film.
4. We viewed the film again and analysed whether this at all changed the first observations, and how this second viewing is influenced by the collective reflection: the chronology was hard to remember at first; the fact that we talked about the sequence of the film made it easier to see things during the second viewing; and, what would it change if we added music to the film?

John Mason asked a question afterward about the pursuit of the activity with students: What pedagogic strategies come to your mind that we could use next? Act it out with others. Identify a particular scene of the film, sketch it out and compare the different sketches. What can I ask about those circles? (Where are the centres?) What sort of things did you notice? (Still, moving, invariance, variance, colour changes, drawing attention (thickening: look at me)). What is going on beyond and between the scenes?

Finally, the WG expressed different feelings from this exercise:

- *When viewing the film, we tended to predict the next action of the circle.*
- *When we had to say what we saw, it allowed us to put the structure before the sequence of the film. During the film, we were following the action; by describing it, we had to reconstruct the structure of the film.*
- *Some expressed feelings for the circle: the circle was attracted by the point… with the arrival of the third point, I felt bad for the circle because it was losing its freedom…*
- *We were puzzled about what was going on between the scenes, what was invisible.*
- *If we use more than one sense, it seems easier to remember the actions.*

DAY 3, BLOCK A: GROUPING SPATIAL REASONING VERBS, OTHERS EXAMPLES AND REFLECTION ON TEACHING

During the previous days, we expressed the verbs that we use to describe our spatial reasoning. On the last day, the task was to group them to characterize what we mean by spatial reasoning. Here is the list of verbs that the WG used to describe spatial reasoning:

Moving – expanding – sliding – rotating
Visualizing – imagining – creating images
Cutting
Creating
Seeing – observing
Manipulating
Counting x +
Organizing
Planning – strategizing
Evaluating – verifying
Translating

Making beautiful
Thinking about necessary + sufficient
Constraints
Trying – testing
Playing
Discussing – communicating
Pushing against norms
Composing – decomposing
Cleaning the slate
Patterning

Problem solving
Conjecturing
Explaining
Justifying – arguing
Drawing
Permuting
Questioning
Gesturing – pointing
Generalizing
Symbolizing
Deducting
Locating – orienting – positioning

Transforming
Isolating – eliminating – impossibilities
Visually coordinating
Perspective taking
Coordinating num. /visual
Scaling
Connecting
Thinking aloud – sharing
Voicing vs not voicing

These verbs were grouped differently by the participants. Here are the groupings produced by five different teams in the WG.

First grouping:

On one side, the physical, on the other, the mental, and in between, the creation. But, there was no consensus within the group regarding the placement and the relation of these three parts: some actions can only be done mentally, but not others; some actions can be done in both contexts.

Second grouping:

Here spatial reasoning is treated as a process, like problem solving in a loop pattern: verbs that are related to actions that we do at the beginning of the resolution, verbs that are involved in the core of the resolution, and verbs that translate actions that we do at the end of the resolution.

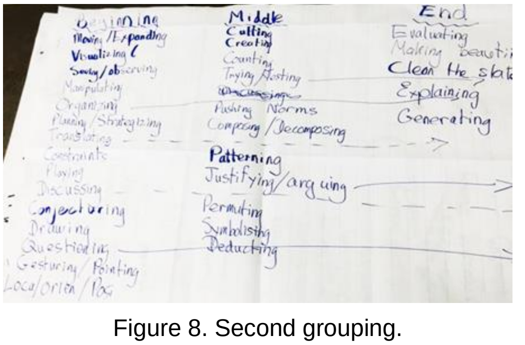


Figure 8. Second grouping.

Third grouping:

Below we have a group of verbs in relation with communication, a group in relation with manipulating, another one on exploring and a last group on a 'meta' level for verbs that are related to mental actions that we do before manipulations. See Figure 9.

Fourth grouping:

Members started by grouping the verbs and then tried to find a label to describe each group, but this task was hard because of the link and the fluidity of this set of verbs. They came up with: mental images, physical–representation, interactive (communicating), understanding, discovering, and precision. See Figure 10.

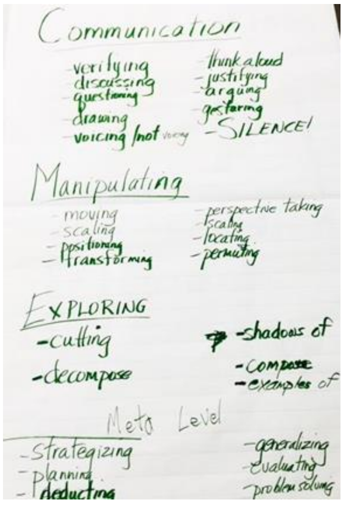


Figure 9. Third grouping.

Figure 10. Fourth grouping.

Fifth grouping:

This group worked in an opposite way by extracting the verbs, then trying mostly to express or to distinguish what we mean by spatial reasoning and presenting their work in a spatial way to bring out different perspectives and the mathematical complexity. See Figure 11.

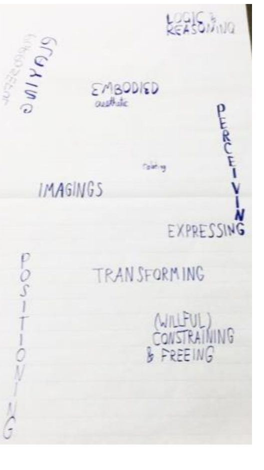


Figure 11. Fifth grouping.

DAY 3, BLOCK B: SPATIAL REASONING BEYOND ELEMENTARY SCHOOL AND GEOMETRY

In this last block, the WG explored one activity that went beyond elementary school by treating the development of a cube in a more creative and complex way: https://twitter.com/geometry4yl

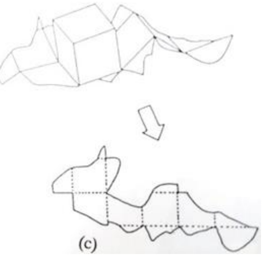


Figure 12. Day 3 activity: Cube development.

Then, we mentioned other activities where spatial reasoning has beeb used to make sense of the mathematical concepts in play, like the case presented by Pinto and Tall (2002) on the limit concept, where the student uses spatial reasoning to create his limit definition: "He compresses information in a picture, which he evokes when writing down the definition" (p. 5). For more examples, see Sinclair and Bruce (2014), and Davis et al. (2015).

CONCLUSION

In conjunction with the discussion on spatial reasoning, some 'big questions' emerged from the WG: *How can we integrate spatial reasoning in our teaching? Should it be considered as a new learning object in the curriculum or as a way of learning and teaching mathematics? What are the impacts for teachers in practice and for preservice teachers?*

Here are some elements that feed these questions that the participants shared:

- *The principle is not to add a concept or a competence in the curriculum. This addition represents a danger in our eyes. We could reflect on how it can enrich the actual curriculum. If we add some examples in the curriculum, it can create an unwanted shift from the target concept (e.g., rectangle arranging for multiplication).*
- *We have to reduce the number of words in our curriculum to make 'space' for spatial reasoning.*
- *How can we articulate the mathematics curriculum with others, like the geography program that also addresses spatial reasoning in another context?*
- *Make the mathematical concepts more dynamic and spatial in the classroom.*
- *Imagery and gesture should be encouraged in mathematics class.*
- *We have to create meaningful activities that allow students to develop their spatial reasoning.*
- *Technology tools can be explored to enrich this development.*
- *Spatial reasoning is generally always in relation to other mathematical concepts. How can we connect mathematical concepts with each other in the learning process (e.g., area and fractions)? How can we expose the link between mathematical concepts in class without treating them individually?*
- *How can we have a common vocabulary to talk about spatial reasoning?*
- *The students have to be aware that they are, themselves, a part of the space, and that they can be an object studied in this space. The classroom is a space and they are part of this space.*

- *Each day or week in class could start with an imagery exercise, "Hands off" activity.*
- *We have to find a way to reach preservice teachers and let them experience these kinds of activities. How can we reach teachers? How can we support them in this process?*
- *It seems more relevant to intervene with in-service teacher professional development then to change the curriculum.*
- *Teachers need to be aware of the importance of spatial reasoning on mathematical development, of the relation between spatial reasoning and geometric, numeric and algebraic concepts, the central role that they can play in its development and they have to be enriched with concrete ways of doing this in their classrooms.*

The working group leaders wish to acknowledge the contributions of all participants who enriched the exchange we had during the three days of this working group. The interest for the development of spatial reasoning in mathematics and in science classrooms is presently growing and, as we can notice from this report, several questions need to be addressed.

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THE PUBLIC DISCOURSE ABOUT MATHEMATICS AND MATHEMATICS EDUCATION

LE DISCOURS PUBLIC SUR LES MATHÉMATIQUES ET L'ENSEIGNEMENT DES MATHÉMATIQUES

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INTRODUCTION

Mathematics education is a cornerstone of public schooling. A look across time suggests arithmetic and geometry have been part of mathematics education since antiquity, but it is only in the last half century where mathematics education, in broader and more rigorous forms has been intended for all children and youth, in Canada, and around the world as nations accept primary education as a basic human right.

L'enseignement des mathématiques est une pierre angulaire de l'enseignement public. Un regard dans le temps suggère que l'arithmétique et la géométrie ont fait partie de l'enseignement des mathématiques depuis l'antiquité, mais c'est seulement depuis le dernier demi-siècle, où l'enseignement des mathématiques, dans des formes plus larges et plus rigoureuses, a été destiné à tous les enfants et les jeunes (au Canada, et maintenant aussi dans des nations moins nanties qui acceptent l'enseignement primaire comme un droit humain fondamental).

For a century mathematics educators and researchers have been studying, and proposing suitable curriculum and teaching methods that are intended to provide learners with meaningful experiences which will lead to highly numerate and mathematical persons. Curriculum developers and policy makers use that research to create mathematics curriculum for all learners. With the broader goals of mathematics for all (Goos, Geiger, & Dole, 2014), the teaching strategies that learners are encountering are unfamiliar to parents, and with the pervasiveness of social and traditional media, mathematics education has become heightened in the public discourse and highly politicized.

Pendant des décennies, des chercheurs en didactique des mathématiques ont étudié et ont proposé des méthodes pédagogiques et didactiques appropriées destinées à fournir aux apprenants des expériences significatives qui les conduiront à devenir des personnes très habiles en calcul et mathématisées. Les concepteurs de programmes et les décideurs politiques utilisent la recherche pour créer des programmes de mathématiques pour tous les apprenants. Avec de vastes objectifs de mathématiques pour tous (Goos, Geiger, & Dole, 2014), les stratégies d'enseignement que les apprenants rencontrent ne sont pas connues des parents. Avec l'omniprésence des médias sociaux et traditionnels, la discussion sur l'enseignement des mathématiques s'est accrue dans le discours public et hautement politisé.

At the 2014 CMESG annual meeting in Edmonton, a panel responded to the media attention to the 2012 PISA results (McGarvey, Reid, Savard, & Wagner, 2015). The print and television media were in attendance and extended the public discussion with an article about the *Math Wars*. As an organization, we have not responded. Nevertheless, the conversation continues without us. With our working group we intended to provide a space for CMESG members to:

- explore the messages in the media;
- analyse the messages;
- develop strategies for communicating in the media;
- analyse the storyline in the media
- develop a strategy to communicate a storyline by mathematics educators in the media

Lors de la réunion annuelle du GCEDM à Edmonton en 2014, un panel a répondu à l'attention des médias sur les résultats du PISA 2012 (McGarvey, Reid, Savard, & Wagner, 2015). Des journalistes provenant des médias imprimés et télédiffusés étaient présents et ont continué la discussion avec un article sur « la guerre des mathématiques ». En tant qu'organisation, nous n'avons pas répondu; toutefois, la conversation s'est poursuivie sans nous. Dans ce groupe de travail, nous allons :

- *explorer les messages dans les médias;*
- *analyser les messages;*
- *développer une stratégie de communication dans les médias;*
- *analyser le récit véhiculé dans les médias;*
- *développer une stratégie de communication dans les médias.*

In this report we outline the activities of the working group each day, including what was done, aspects of the conversations and strategies members developed to contribute to the public discourse about mathematics education.

DAY 1

We began by sharing the headlines *we would like to see* in the media about mathematics education before turning to the print media to examine the messaging and storylines that have been reported across the country since CMESG 2015.

La première journée a débuté par un partage des titres que nous aimerions voir *dans les médias à propos de la didactique des mathématiques, suivi par une étude des messages et des trames narratives qui ont été publiés dans des médias écrits au pays depuis GCEDM 2015.*

The headlines participants wrote reflected diverse interests with an optimistic tone. Imagine the headline "*School Board Meeting Addresses Increased Demand for Mathematics*." Maybe it followed an earlier one, "*It Turns Out We Can All Do It!*" or "*Canadian Students are Able to Think Mathematically and Problem Solve: Math Teachers are Doing a Great Job*." "*Ça y est ! Les élèves comprennent finalement ce qu'ils font en maths ?*" "*Nouvel expo sur les mathématiques ouverte au MBAM – On adore ça.*" Other headlines speak to possibilities for addressing issues addressed in the current media: "*Teachers and Math Researchers Work to 'Close the Gap' in Math Education*"; "*There Are More Than Two Ways to Teach Math and Learn Math*"; "*Let Good Teachers Teach: Legislating Toward the Mean.*" And yet others speak to a vision of mathematics: "*Mathematics No Longer the Gatekeeper: How Teachers and Students Changed the Game*"; "*Ministry Reassesses Math Literacy: New Committee Studies What Mathematical Understanding Is Needed for Citizenship*"; "*Parents and Students Reject Math Testing Nation Wide!*". There was reference to the political aspect of the debate: "*Rational Thought Finally Emerges in Politics*". And, there was the reflective and self-critical too: "*Math Educators Weigh in on the Math Wars-Albeit Five Years Too Late but Better Late Than Never… I Guess…?*"; "*Universities and High Schools Present United Front for Incoming Students*".

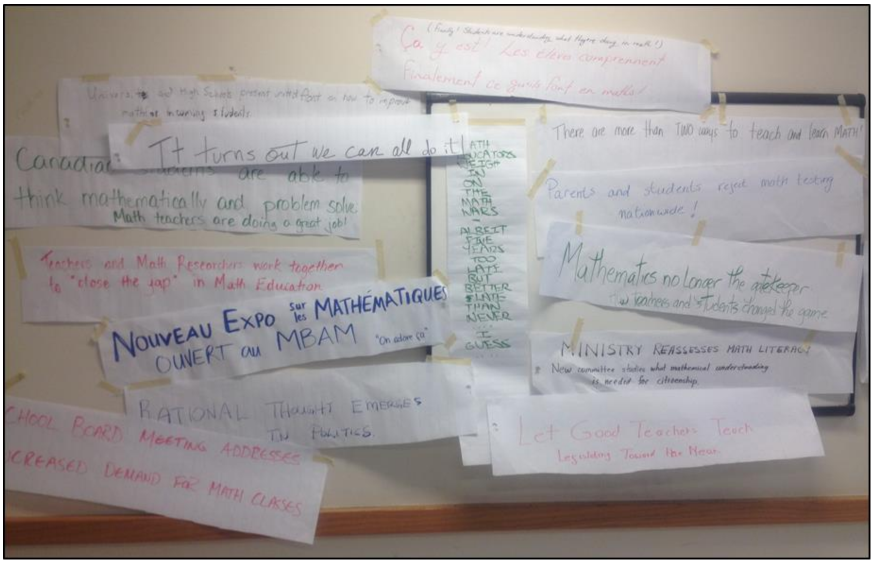

Figure 1. Headlines we would like to see in the media.

Both Lynn McGarvey (University of Alberta) and Egan Chernoff (University of Saskatchewan) have been following the media for mathematics education stories and have been archiving news articles about mathematics education in the Canadian press[1]. Members of the working group were asked to select from one of three themes in articles taken from the archives (math debate, teaching mathematics, PISA) and examine articles to learn: who are the speakers, what is the message, what are the positions they are taking, when do the articles emerge, and why are they written (reporting news, special interest piece, opinion piece proposing a position). The headlines we wanted to see are different than those we identified by scanning through dozens upon dozens of media articles. Some of the themes that working group members found in the news media articles are reflected in the headlines shown in Figure 2.

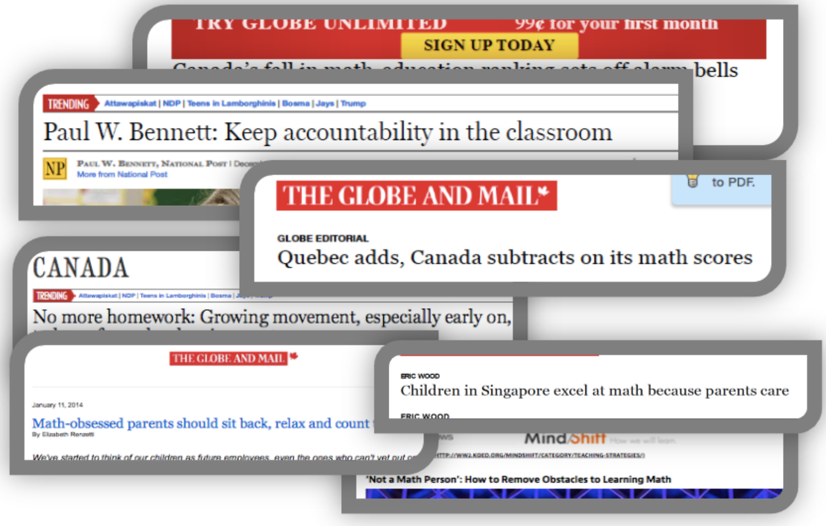

Figure 2. Headlines we found in the national media.

DAY 2

There were two primary goals for Day 2: to analyse messages in the media using Herbel-Eisenmann et al.'s (2016) framework; and to develop strategies for us as mathematics educators to communicate in different media.

Deux buts ont été poursuivis lors de la deuxième journée de travail, soit analyser des messages dans les médias en utilisant le cadre théorique développé par Herbel-Eisenmann et al. (2016) et développer des stratégies pour nous en tant que didacticiens des mathématiques pour communiquer dans différents médias.

In a 2016 paper in *Journal for Research in Mathematics Education* Herbel-Eisenmann and colleagues (including CMESG member Nathalie Sinclair) studied the messaging about mathematics education that has been created in the media over the past decade. They identified three broad shared narratives that serve as a backdrop for the positioning. They called these storylines: 1) Two dichotomous ways of teaching: basic and discovery; 2) Mathematics educational research is not trustworthy and not used; and 3) The main goal of mathematics education is to produce workers in science technology, engineering and mathematics (STEM).

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[1] For Lynn McGarvey's archives, see https://sites.google.com/a/ualberta.ca/mathnewsrepository/
For Egan Chernoff's archives, see http://matthewmadduxeducation.com/

We agreed with their view that mathematics education researchers can shift the storylines, but recognized that to do so we need to participate in the public discourse with deliberateness and intentionality. Following Herbel-Eisenmann et al. (2016) working group participants were asked to return to the collection of articles we archived and ask the following questions:

- What communication acts are invoked? What are the signs and symbols that are being projected through the media articles and that shape meaning making?
- How are the various actors positioned? Who is speaking? What right and/or duty do they have to speak?
- What are the emergent storylines about mathematics education in the articles? What broad shared narratives serve as a backdrop for the positioning the actors take?

Working group members noted the polarized storylines and antagonistic tone in the conversations about mathematics curriculum and teaching approaches in the media. Mathematicians were critiquing curriculum and teaching methods. Parents were expressing anxiety about student proficiency and the few mathematics education researchers that were cited were found to be trying to temper the conversations. The most prominent voices in the articles we sampled were mathematicians, government officials, and reporters. There were articles that included mathematics teacher educators and educational researchers, but their voices were not as common. Some parents were adding to the discourse, mostly in support of a return to 'traditional' teaching of 'basic facts'. The voices of teachers and students were strikingly absent.

There were biases noted in the reporting, in the sense that particular writers seemed to quote experts (often mathematicians) whose opinions were strong and based on their experience of working with students in college and university. Where mathematics education experts were cited, the articles tended to position the opinions based on anecdotes with the modern approaches to teaching and curriculum espoused by the educationalists. There were also the more rounded articles where educational experts were asked to respond to PISA results and those comments were more likely to appear with ones from government officials, and mathematicians. It seemed to members of the group that there were few commentators in the many articles. The same people seemed to be cited in multiple sources. This lead to some concern that the voice of mathematics educators and educational researchers was neither pervasive nor diverse in the media storylines, leading many of us in the working group to assert that we must find ways to insert ourselves into the narrative.

The second part of Day 2 involved exploring some of the ways we (mathematics educators and CMESG members) could contribute to the media discourse and alter the narratives. A number of recommendations were made that CMESG could do and what each member can do in their own context. Suggestions for CMESG:

- Add a new link on the website for public discourse
- Suggest some names to be contacted by media (experts)
- Invite people to tweet #CMESG16/GCEDM16 #ousontlesmaths

- *Ajouter un hyperlien sur le site : Discours public*
- *Nommer des personnes qui pourront être contactées par les médias*
- *Inviter les gens à twitter : #CMESG16/GCEDM16 #ousontlesmaths*

DAY 3

The primary goal of our working group was to have participants use the time and the collaborative environment to generate messages that could be contributed to various media venues from parent newsletters, to blogs, *YouTube*, tweets, print, radio, and television media. On Day 3 participants were challenged to write a piece for the media, prepare speaking points for our interactions with the television and radio media, as well as an editorial piece for the print media and video.

L'objectif premier de notre groupe de travail était de demander aux participants d'utiliser le temps et l'environnement de collaboration pour générer des messages qui pourraient être diffusés dans divers médias, des bulletins aux parents, des blogs, sur YouTube, sur Twitter et dans des médias imprimés, radiophoniques et télévisuels. Le troisième jour, les participants ont été mis au défi d'écrire une pièce pour les médias, de préparer des points de parole pour nos interactions avec les médias de télévision et de radio, ainsi qu'un éditorial pour la presse écrite et vidéo.

We began Day 3 by thinking about how journalists write. Richard Barwell (2009) has written about this. He notes that journalists use: eyewitnesses and corroboration. They provide details and have an objective style of description. With this in mind, how can we prepare for interviews that we will likely be asked for under a very short timeline? We took some advice from the University of Alberta's media relations office about speaking with the media, in particular, to prepare for an interview.

- Establish key messages
    - Two or three points you want to make
    - Point form is useful
    - Include benefits to readers/listeners
- Share a story
- Rehearse
- Be succinct and use lay language
- Anticipate questions

OUR PRODUCTS

A number of messages and products came out of our working group. To the extent that we could collect them they are presented here. The first few address myths that are propagated in the media.

<u>Making sense of media coverage of math in our schools – Alison Goss</u>

Recently there has been much coverage in the media about how math has been taught in our schools. Media suggest that there are two opposing methods being used to teach this critical subject. One is the so-called 'discovery' approach and the other is the 'back-to-basics' approach. But is this really the case? What is 'discovery math', and what is 'back to basics'? The discovery approach is reinterpreted in a number of ways, but generally it is understood to mean that students are encouraged to be their own problem solvers, with the idea and goal that meanings they discover themselves will generate a more powerful understanding of mathematical concepts. The criticism of the discovery approach is that students can flounder if they do not have their basic math skills mastered, and if they do not have the support of direct instruction from a teacher. The back-to-basics method suggests that children need a solid foundation and so-called basics of mathematics in order to be successful in their mathematical learning. For young students, the basics could be the full operations' number facts, for example. Without these basic facts, it is argued students cannot be expected to do any higher-order work such as problem solving. The criticism towards the back-to-basics approach is that it requires rote learning, is mechanistic, and does not apply higher-order thinking. So, is this the reality in our schools? Not according to the math curriculum: back-to-basics and discovery math actually are *both* part of the curriculum. If the curriculum acknowledges the importance of basic skills and recognizes their role in mathematics as being part of a concise and powerful means of communication, these are the mathematical structures, operations, processes, and languages that provide students with the framework and tools for reasoning, justifying conclusions, and expressing ideas clearly.

<u>Osnat Fellus</u>

I was very excited about the topic of our working group: *Public Discourse About Mathematics and Mathematics Education*. As we were discussing the narratives propagated by some media venues, I mentioned George Lakoff's (2004) work, *Don't Think of an Elephant*. In the book, Lakoff talks about the language of political discourse. He explains that when you ask people what not to think about, people actually think exactly what you tell them not to think about. Lakoff brings the example of him teaching a class and repeatedly instructing his students, "*Don't think of an elephant!*" Students, surprisingly, do report that they could not think of anything else but an elephant. The idea is that if we use the language, concepts, or terms that are used by the media, we are in effect perpetuating the very message advanced by the media. For example, if we use the term 'Math Wars' in interviews with the media, we in fact buttress the idea of math wars even though we are trying to convey a message that we all, in fact, not at war at all. This, as Lakoff's work shows, is very difficult to do. But with proper preparation we can direct public discourse to a message we want to push forward.

So, what can be our message then? In conversation with Florence Glanfield, during lunch, about what it is that we want our students to be able to do, she said,

> *Math education is all about pulling all stakeholders together to show that we ultimately have a common goal. We all want students who can think. We all want students who think flexibly around mathematical ideas. We all want students to develop the ability to ask questions. And we all want students to be able to see mathematics as pluralistic, embodied, and omnipresent.*

<u>Are you doing math? – Annette Rouleau</u>

Annette made a short documentary video interviewing people on the street engaged in various activities. She asked, "*Are you doing math?*" She interviewed a man working on a kayak, two pilots in a helicopter, two children playing basketball, a taxi cab driver, a server at the coffee shop and an artist in a gallery. All but the artist described some part of their activity as doing math. The artist was standing in front of her paintings that had very strong geometrical elements. For more information about the interviews please contact Annette.

<u>The Good Ol' Hockey Game: Lessons for Math Class – Jennifer Holm and Ann Kajander</u>

My friend Ann's child wants to play hockey, and Ann has the choice of three coaches. Coach John gets the children to practice drills repeatedly without them ever getting to play a game. Coach John says, "*The kids have got to have the skills before they can play!*" Coach Seymour has children skate and play in games but never spends any time on drills. Coach Seymour says, "*They just need to play the game and the skills will come from that! The kids will discover the plays on their own!*" Coach Jo blends the two together: children spend time during each practice on drills and scrimmage. Coach Jo says, "*Drills are important for fundamentals, but they need to experience playing the game for themselves! They need to find themselves as hockey players, and my job is to coach them in the areas where they need some help.*" Ann decided Coach Jo is the one for her child.

At the first practice, Coach Jo talks with the children. Then Coach Jo lets them skate in order to evaluate what each child already knows and what skills need to be addressed. Coach Jo creates drills to develop the weak areas in the children's game in order to support their growth. After spending some time learning something new, they all get together and try out their skills in a scrimmage. Again, Coach Jo is observing and guiding when the children fall in order to help them learn from their mistakes.

This makes an excellent analogy for how I 'teach' my mathematics classes. I provide a problem and let the students discuss and engage mathematically. My job is to see what they can do and where there are some struggles. Then I design experiences or lessons to assist students to further develop. I coach them when they need some support and let them experiment when they can. They need to think and reason independently because it is important to develop the skills of mathematics as needed in a context. Like on Jo's team, children do not need to master every math drill in order to be winners.

Storyboard – Manon LeBlanc and colleagues

This group produced a storyboard that represented the first step towards preparing a video about excellent classroom practice that could be shared with parents at school curriculum evenings or other public events. See Figure 3 below.

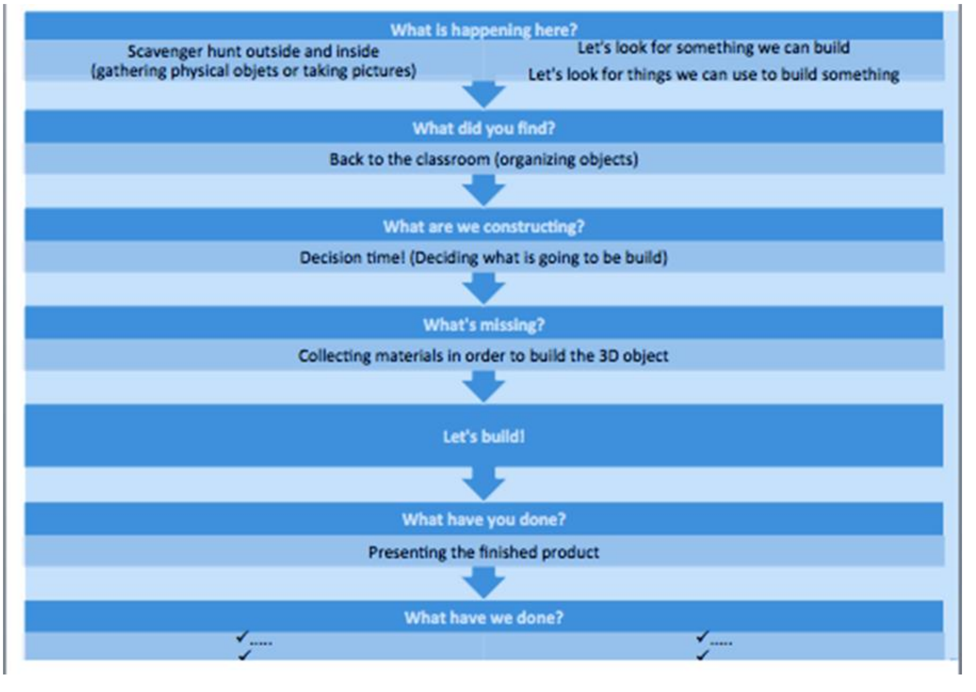

Figure 3. A storyboard.

Feeding the twitter chain – The Collective

Over the three days members contributed new content to the hashtag #CMESG16 and connected it to other networks. Some of the working group members signed up as new *Twitter* users so they could contribute. A new hashtag was created: #ousontlesmaths.

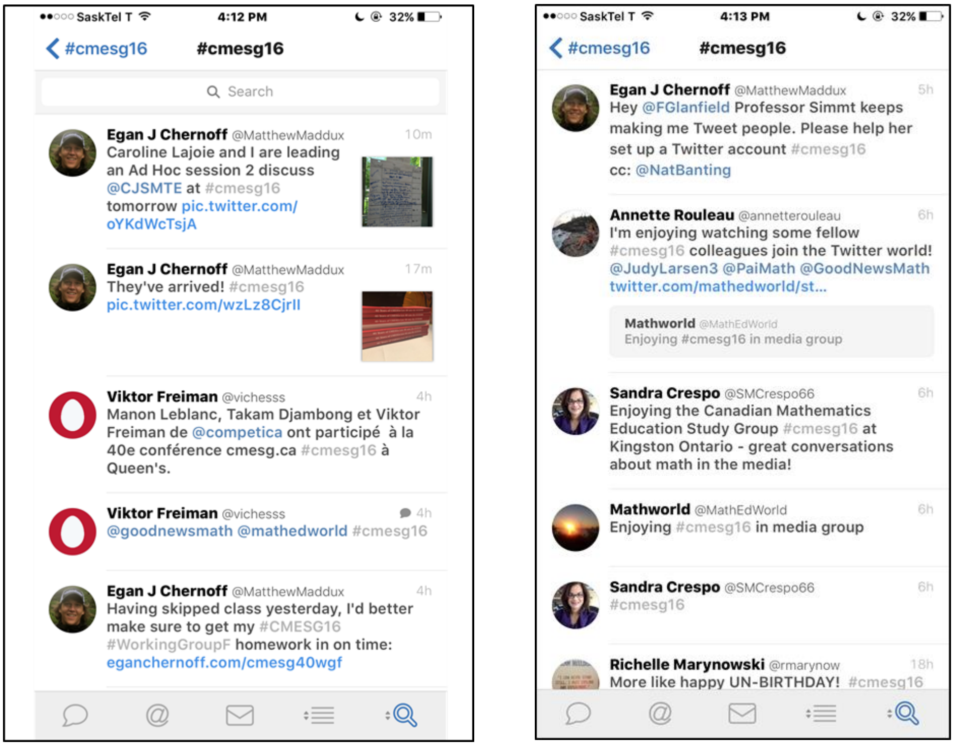

Figure 4. Tweets at #cmesg16.

<u>Connecting with and using existing social media – Egan Chernoff and others</u>

Egan has been maintaining a very active weblog *MatthewMaddux Education* that the group used and wants to encourage others in CMESG to visit.

- http://matthewmadduxeducation.com

Other blogs that were recommended by members of the group included:

- {Musing Mathematically}
- mathblogging.org
- for the love of learning

Twitter accounts to watch for include:

- Goodnewsmath
- MathEdWorld

We invite all CMESG members to take steps to influence storylines in the media.[2]

[2] We are pleased to report that at the time of preparing this document the *Canadian Journal of Science, Mathematics and Technology Education* published a pair of papers on storylines in the national media (Rodney, Rouleau, & Sinclair, 2016; Chorney, Ng, & Pimm, 2016). Annette Rouleau was a member of the working group. Nathalie Sinclair was a co-author on the Herbel-Eisenmann et al. (2016) paper used in the working group.

Lors de la présentation de notre groupe de travail à la fin de la rencontre, nous avons invité tous les membres du GCEDM à se positionner pour influencer les trames narratives véhiculées dans les médias. Cette invitation tient toujours.

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New PhD Reports

Présentations de thèses de doctorat

FROM AGENCY TO NARRATIVE: TOOLS IN MATHEMATICAL LEARNING

Sean Chorney
Simon Fraser University

My dissertation explores ideas from new materialism *as a theoretical lens for identifying and understanding the role of tools within mathematical practice. I propose that new materialism, particularly within a post-humanist perspective, offers the opportunity to articulate a non-dualist approach to mathematical thinking and learning—with a particular focus on the entanglement of tools, humans and concepts. Theoretically reshaping the traditional approach of seeing learning as occurring solely within the individual, the focus of mathematical learning in this dissertation is neither on the student nor on the tool, but on the coupled entity* student-tool. *The theoretical perspective developed in this dissertation draws mainly on scholars from outside the field of mathematics education, including the anthropologist Tim Ingold and the feminist science studies scholar Karen Barad. Both articulate forms of post-humanist materialism that attend specifically to the interplay between tools and cognition. In addition to these scholars, I draw on the inclusive materialism of de Freitas and Sinclair, who in their book* Mathematics and the Body *have extended Barad's ideas to the context of mathematics, and who argue that material engagement with a tool is an integral part of the 'assemblage' from which mathematics 'emerges'.*

INTRODUCTION

My dissertation is concerned with the 'learning' of mathematics, and, in particular, to develop a better understanding of the relation between tools and mathematics. As part of my re-thinking of tools and their relation with learners and mathematics, this thesis has aimed to draw attention to the processional activity of mathematical doing. Of course, it has long been acknowledged that it is the process and not only the product of mathematical thinking that should be valued in the classroom. Various forms of mathematizing, such as generalizing, conjecturing and looking for invariance, are widely held to be core components of mathematical activity. However, the difficulty that I encountered in describing activity in process, without imputing agency or reifying actions, suggests that *mathematics-as-process* is still a challenging idea. There are connections here to the work of researchers such as Lunney Borden (2011), who discusses both the challenges and opportunities of the highly verb-rich nature of the Mi'kmaw language; while it has been challenging to translate between Mi'kmaw and English, the Mi'kmaw ways of naming circles and squares and other mathematical objects may well have pedagogical value, while also pointing to new possible ontologies of mathematics.

Although it can be helpful in communication, to speak of mathematical objects is to ratify a specific perspective of mathematics. This can develop into a boundary-making practice in which mathematical objects have some kind of disciplinary or material agency. This boundary-making practice around mathematical objects either determines how they can be approached (disciplinary agency) or how they can be accommodated (material agency) (Pickering, 1995). To draw mathematics to the process leads to questions of how these objects are formed: from what assemblages did they emerge? How might they change? These questions do not simply relate to the historical or discursive changes in mathematics, which are many and which have been well documented, but also insist on reckoning with the material. This can be difficult in mathematics, where objects are so often, and incredibly quickly, folded into an immutable zone that refuses to be sullied by the physical world. It should be possible to examine how every shift in mathematical thinking, whether it be in the history of the discipline, or the life of a learner, occurs within particular material contexts. Rotman (2008) provides a very powerful analysis of this sort in relation to the concept of zero and the emergence of writing.

In addition, by reformulating the focus to the observable, we become less concerned with what has been learned and more concerned with what is being practiced. This identification of the observable draws attention to what is occurring in-the-now as opposed to what 'facts' the student is taking away. This reorientation towards the material assemblage unfolding in practice produces new insights.

There are existing theories that assume a dialectal relation between tools and learners and consider tools to be central in the process of learning; they maintain a dualist perspective between mind and matter. For example, in the more monist phenomenological framework of Nemirovsky, Kelton, and Rhodehamel (2013), which refuses the mind/body distinction that is so prevalent in the mathematics education literature, even amongst theorists of embodied cognition (de Freitas & Sinclair, 2014), there remains a separation between the human and the non-human. Further, often these theories seek to account for the relation between tools and mathematics, leaving the latter somehow fixed in the course of learning. Any contribution to the way in which tools might affect the emergence of concepts, while at the same time shaping the very nature of these concepts, seems a highlight relevant to mathematics education. It has implications, in particular for assessment practices, which often involve separating tools from mathematics.

THEORETICAL FRAMEWORK

The overriding theme of this dissertation is *new materialism*, especially in relation to how it can be used as a lens to inquire into mathematics education. I elaborate on new materialism as it is found in Barad's (2007) work, and also draw on de Freitas and Sinclair's (2014) *inclusive materialist* approach.

Barad's materiality develops out of her background in feminist studies where she was inspired by criticisms of science studies' representationalism that ignores "gender, race, class, sexuality, ethnicity, religion and nationality" (p. 87). Drawing extensively as well on recent research in physics, as well as the writings and findings of Neils Bohr, Barad seeks to reconfigure the nature of concepts and objects in the world, insisting on their fundamental ontological indeterminacy. This has led her to propose a critical practice of engagement that attends to the "specific material entanglements" (p. 93) involved in the production of objects and subjects. In particular, she proposes that concepts emerge out of "material-discursive boundary making practices" (p. 148).

de Freitas and Sinclair (2014) build on Barad as well as the work of the philosopher of mathematics, Gilles Châtelet (2000), to extend the concept of *new materialism* to the field of mathematics. Châtelet enables them to take some of Barad's ideas, which are articulated in the context of physics, where materiality is less problematic, and elaborate them within mathematics, where concepts are more readily seen as having little material dimension (they are abstractions of the mind). In their *inclusive materialism* approach, de Freitas and Sinclair show how mathematical concepts can also be seen as emerging from material-discursive boundary making practices. These practices are viewed as material engagement of concept, student, tool and movement. The mathematical concept then, is the intra-action, or coupling, of a human subject with the tool. The concept, therefore, is not purely abstract but partakes of the physical world. In this way, de Freitas and Sinclair adopt a “theory of matter that resists the binary divide between human agency and inert passive matter” (p. 39); they revitalize materials, which are often considered to be passive or inert, and re-animate concepts, which are often considered detemporalised (Balacheff, 1988).

These writings on new materialism invite us to reconsider ‘knowledge’. Passing on knowledge to the minds of students is the focus of education, but Barad (2007), Ingold (2011), Pickering (1995), and Sfard (2008) all challenge the common notion that knowledge is a ‘thing’ that resides in the head. Knowledge is not an abstract object that can be delivered to students by teachers or textbooks; both knowledge and thinking are practices, or processes, not entities.

In addition to an implementation of a new materialist framing of this dissertation, I also draw on a process ontology, drawing from Whitehead (1978), and post-humanism, drawing from Barad (2007), and de Freitas and Sinclair (2014).

While a process ontology may not be a focus in materialism, I argue that in mathematical practice, process—becoming of material, human, and mathematics—is an important consideration that is compatible with new materialism. I show these links by drawing on and interweaving the work of Ingold and Barad, and argue that it is not meaningful to talk of human, tool and mathematics as entities prior to a mathematical activity: activity, which involves movement and time, as well as intra-action, thus becomes the initial unit of analysis. This requires looking at what emerges from a processional activity of movement.

In addition, I suggest that a post-humanist attunement is a requirement of any new materialist account. Within a materialist framework, the human is not granted special privilege or position, given that there is a flattening of ontology in which all is matter and, as de Freitas and Sinclair write, matter itself comes to matter. The post-humanist perspective I hold in this dissertation does not ignore or deny the human or human abilities, but is more inclusive of all material entities. This is perhaps one of the most counter-intuitive aspects of my work, but it is a feature of a growing number of scholars, working in a variety of disciplines, who challenge both the taken-for-granted boundaries of the body as well as the simple models of causality in which humans effect change.

The final major theme extending from a materialist process approach that I mention here is that of *the tool*. In this study, I refer to the tool as playing a significant, dynamic role in mathematical practice. Decentralizing the human allows for the implication of the tool. The consideration of tools is particularly significant in learning situations as opposed to their significance in refined and well-worn mathematics methods. In my investigation of the role of tools, I draw on Rotman (2008), who describes mathematics and tools as co-evolving, meaning that the practice of mathematics depends on tool use, just as mathematical tool development relies on the practice of mathematics.

STRUCTURE OF THESIS

The first paper presents mathematical practice as a continued *process*, meaning that the ‘partners’ of student-tool and of mathematics are not static but processes of ‘becoming’. In this paper, mathematical practice is seen as a continued process, between the ‘partners’ of student and tool as a fused entity, *student-tool*, and mathematics. I came to this process point of view through my readings of the mathematician and philosopher Alfred North Whitehead (1978), but these readings were helpfully interpreted and extended by Ingold (2011). The paper investigates how student-tool and mathematics are not static but processes of 'becoming'. It does so by examining what it could mean for each of these 'partners' to be thought of as processes, and highlights the long-standing assumptions that make such a perspective so challenging. Working from Barad's (2007) ideas, the process lens in this paper further argues that the student, tool and mathematics are not discrete, invariable elements, but fundamental parts that comprise the process and the mathematics. From this point of view, the student is neither the manipulator of the tools, nor controlled by mathematical rules; instead, all are entangled, and this entanglement—this assemblage of student, tool and concept—defines mathematics practice. Each part of the assemblage is shaped through the process, and is thus continually becoming. Though different assemblage parts may be named or identified for convenience, such as the name of the tool, the student, or the concept, focusing on the individual aspects may result in losing sight of the meaning that emerges only from the assemblage as a whole, and the assemblage becoming through a process of movement and change.

The second paper approaches the concept of the circle through the framework of *mathematics-as-becoming*. This paper thus focuses specifically on how a concept can be thought of as a process, and on the implications that this might have for mathematics learning. Contrary to long-standing assumptions about mathematical concepts as ideal, inert, Platonic forms, the new materialist framework sees the circle originating from the animation of a life force. Temporality and mobility thus become key aspects of mathematical concepts. Indeed, using Ingold's notion of *meshwork*, I argue that circles materialize from movement; they are perhaps better thought of as verbs than as nouns.

Paper three provides an empirical study of a high school geometry classroom and uses a number of constructs from post-humanist, materialist theories to analyse the data collected from the classroom activities. It challenges the identification of student as the central, unitary agent, and instead attempts to describe and analyse the mathematical activity in terms of a student-tool coupling (where the tool in this case is *The Geometer's Sketchpad* (Jackiw, 1988)). In this paper, I utilize a very recent methodological approach called *diffractive analysis*, which is new to mathematics education, to interpret the data—it provides original insights and questions into the ramifications of studying mathematical learning activities from a non-dualist perspective. Barad originally formulated a diffractive approach using a metaphor of mirrors. Mirrors are usually used as a metaphor for representationalist approaches, but when mirrors are used to reflect back upon each other, a physical phenomenon of different patterns highlight what Barad (2007) describes as an *indefinite nature of boundaries*. She describes the approach as "thinking insights from [different] theories through one another" (p. 92). According to Barad, the approach implicates the researcher in that it is not about observing or reflecting; rather, it is about moving and knowing in a way "that is attentive to, and responsive/responsible to, the specificity of material entanglements," which "manifests the extraordinary liveliness of the world" (p. 91). In particular, instead of searching for distinctions between different materialisms in an effort to argue for the best or most consistent theory, Barad is committed to a more inclusive approach that values the differences that emerge from particular commitments, interests and expertise.

Mazzei (2014) proposes a *diffractive analysis* of interview data she collected. She writes that a diffractive analysis "leads in different directions and keeps analysis and knowledge production on the move" (p. 743). The outcome is less about resolution and more about disruptions and questions. I use a diffractive analysis that looks at data with/against the different lenses I have outlined above. The idea is that one looks at data with one lens and then looks at the results of that approach through another lens. The goal is to open up analysis from a variety of perspectives and to challenge findings that are based on the ‘objective’ reflections of a researcher.

CONCLUSIONS

The post-human starting point of this paper invites us to move away from the tendency to locate learning and thinking solely within the mind of the learner. While several theories have been proposed in which thinking and learning is distributed across other material phenomena, they have not explicitly attempted to provide methods for studying what might be entailed in a classroom situation. Because of the tendency to place the learner at the centre of thinking and learning, and to look for changes in how the learner talks, acts and moves, it can be challenging to study mathematical learning situations from a post-humanist point of view. By drawing on some of the main constructs offered in the literature, such as *resistance/accommodation* (Pickering, 1995), *assemblage, intra-action* (Barad, 2007), *process and tools as narrative* (Ingold, 2011), none of which have been explicitly designed to account for educational research, I have attempted to find out what new insights such post-humanist constructs can provide and what new questions they prompt.

The diffractive analysis shows how the construct of *resistance/accommodation* was fruitful in identifying turning points in the three episodes, but also potentially misleading in inciting a human-centric view in which the tool ends up being subordinate to the human. The Baradian constructs of *intra-action* and *assemblage* served as powerful starting points for overcoming a human-centric perspective. It also drew attention to the changing nature of relations and, in turn, to the evolving assemblages involved in each episode. However, it seems challenging to describe mathematical learning in terms of assemblages. We may be able to see difference, but know very little about whether the difference is mathematically relevant.

The shift to the physical reconfigures what it means to do mathematics. As opposed to a theorizing of how the mind is learning, responding or interiorizing, this shift draws back to the ‘in-the-moment’, embodied experience of tooling, rather than using a tool—much as we say “walking” rather than “using our legs”. Considering the material and temporal experiences is to study mathematics education in a post-human and materialist way, but one that as discussed, does have challenges. To think about mathematics in the classroom involving ‘tooling’ rather than ‘using tools’, requires avoiding the tendency and tradition of isolating nouns, like ‘student’, ‘tool’ and ‘concept’ and locating meaning in their entanglement. This is particularly difficult when wishing to discuss particular aspects of an assemblage, as the tendency is then to detach that aspect or part from the whole. This is especially true with regard to discussing the human part of an assemblage, as the western tradition grants supreme agency to the human actor. I have suggested using a diffractive approach to help overcome this tension. This approach demands that when discussing any part of an assemblage, the focus should be on the part’s role in the intra-activity, which is to say, how it acts as a verb and contributes to the assemblage’s movement.

The process ontology, the actions of student as wayfarer (Ingold, 2011), and especially the notion of tool as narrative, drew attention to movement and temporality. Movement and temporality were vital in interpreting the exercise involving the triangle and the exercise involving the almost-square; the triangle and the almost-square would not have emerged—would not have meaning—if only the student or only the tool were considered as isolated and inert nouns. It was the doing—the fused nouns participating in the temporal verb of acting—that was the mathematical practice. To look at any single, static point, or movement of transport from one static point to another, within the process of the student-tool assemblage changes the meaning entirely: one might miss the triangle entirely if, for example, it was only seen when it existed partly-off screen; one might only see a quadrilateral (noun) instead of the in-action almost-square. In analysing mathematical practice through a relational and verb-oriented process ontology, in seeing the student-tool as a line of becoming, the significance of the embodied experience and the material world are animated, consequentially changing the understanding of mathematics teaching and learning.

This research produced a shift from a twenty-year-old perspective I had reified as a mathematics teacher in which students acquire, or possess, abstract mathematical concepts through transmission, to a new materialistic perspective where concepts are physical 'arrangements'. It has been, to say the least, a significant shift. The last six years of study and putting the theories I was reading into practice has reoriented me as a mathematics teacher. I no longer see 30 individual students before me, waiting to be filled with knowledge; instead, I see a group of people who are dependent upon materials, each other and myself—myself also being dependent upon them. I had thought that if mathematics was taught 'just right' then the immaterial ideas would 'enter' the minds of my students—and my constructivist leanings did nothing to challenge this view since, in retrospect, they only offered different techniques for transmission. Now, with a new materialist sensibility, I see mathematics not as abstract rules that my students and I are entirely subject to, but as the moment-to-moment physical engagement with the concepts, textbooks and tools—material 'things'—and also each other. I have come to accept that teaching and learning are not about accommodating the transfer of knowledge, but instead, are acts of engagement with heterogeneous parts emerging, or 'becoming', through the movement of the things, concepts and people. As such, I am more inclined to see and consider Tim Ingold's distinction between the property of knowing and the practices of knowing. In other words, I am more interested in viewing mathematics through its practice and the processes through which learning occurs, including with what tools and resources, rather than as a set of objective facts to instill into the receptive brains of students.

A question for further research within this perspective will be: Can we conceive of learning as being inclusive of things outside our body, of our mind?

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UN MODÈLE CONCEPTUEL DU RAISONNEMENT MATHÉMATIQUE (À L'ÉCOLE)

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INTRODUCTION

Dans ma thèse, je me suis intéressée au concept de raisonnement mathématique (RM) selon une perspective commognitive (voir Jeannotte, 2015) dans le but de développer un modèle pour l'enseignement et l'apprentissage à l'école. Je présente ici une brève aperçue du modèle développé, modèle qui se situe dans un contexte où à l'international comme localement, le raisonnement mathématique prend de plus en plus de place dans les documents institutionnels.

En fait, que ce soit à l'international ou au Canada, une analyse de ses documents permet de constater que le raisonnement mathématique y est vu avec des sens variés et non systématisés. Juste au Québec, le programme du primaire et celui du secondaire diffèrent quant à leur vision du raisonnement mathématique. Cette différence n'est pas uniquement le fruit d'une complexification. Par exemple, au primaire, raisonnement est lié à créativité (voir MEQ, 2001). Au secondaire, on n'y fait aucunement référence (voir MELS, 2007). Au primaire, on parle de la compétence « raisonner à l'aide de concepts et de processus mathématiques » (MEQ, 2001, p. 128), au secondaire, de « déployer un raisonnement mathématique » (MELS, 2007). Comme le mentionnent Yackel et Hanna (2003) ou encore Duval (1995), le concept de RM est souvent utilisé de façon intuitive sans définition ou caractérisation. Ceci explique surement le flou conceptuel entourant ce terme qu'une rapide analyse de la littérature concernant le raisonnement mathématique permet de constater. En fait, la terminologie diffère grandement d'un auteur à l'autre. De plus, le terme *raisonnement* est souvent utilisé en assumant que tous s'entendent sur son sens, ce qui amène l'utilisation de ce terme sans clarification ou élaboration (Yackel & Hanna, 2003).

Mes premières lectures concernant le raisonnement mathématique ont mis en lumière des convergences et des divergences en ce qui a trait aux sens accordés au raisonnement mathématique dans la communauté des chercheurs en didactique des mathématiques. Toutefois, la pléthore d'écrits portant sur le raisonnement mathématique m'est apparue comme une banque de données extrêmement riches et pertinentes à explorer pour réaliser l'objectif du projet qui est d'élaborer un modèle conceptuel du raisonnement mathématique.

LA COMMOGNITION COMME GUIDE THÉORIQUE

J'ai opté pour une posture *commognitive* (Sfard, 2008, 2012) pour guider le développement autant du point de vue méthodologique que théorique. En effet, ces fondements m'ont permis de dégager différents éléments pour construire un modèle cohérent, et ainsi réduire le flou conceptuel soulevé. Premièrement, *commognition* est un néologisme formé des mots *cognition* et *communication*. Ces deux concepts sont vus comme la même activité, la cognition étant une forme individualisée de communication. Deuxièmement, dans une perspective commognitive, les mathématiques sont un discours avec ses propres règles, son propre vocabulaire, médiateurs visuels, routine et énoncés généralement acceptés par une communauté de mathématistes.

En harmonie avec le cadre commognitif, j'ai synthétisé et construit à partir des convergences discursives qui se trouvent dans le discours de notre propre communauté le modèle présenté. J'ai dégagé les principales caractéristiques du raisonnement mathématique pour que ceci puisse servir d'outil conceptuel qui relie sémantiquement différents concepts dans un réseau cohérent.

MÉTHODOLOGIE

Pour encadrer ce processus de conceptualisation, j'ai opté pour la démarche d'anasynthèse développée par Legendre (2005). Cette démarche méthodologique se veut interprétative et cyclique. Dans un premier temps, il y a sélection du corpus. Dans mon cas, les textes sélectionnés devaient avoir comme objet d'étude le RM. Les textes qui avaient *raisonnement mathématique* comme mot-clé ou associé à des mots-clés comme *raisonnement déductif*, *raisonnement inductif*, *pensée mathématique*, etc., ont été sélectionnés. Des cycles supplémentaires ont permis de dégager de nouveaux mots-clés qui ont favorisé la formation du corpus d'analyse.

Dans un second temps, il y a analyse et synthèse de l'ensemble de départ pour mettre en lumière les convergences, les divergences et les potentialités du discours qui y sont véhiculées. Les choix faits ici l'ont été fait en cohérence avec la théorie commognitive. Les lectures répétées et permettent de dégager les unités liées au raisonnement mathématique qui ont ensuite été catégoriser. En s'appuyant sur la commognition, un prototype de modèle a été développé, prototype qui est devenu un modèle cohérent du point de vue théorique.

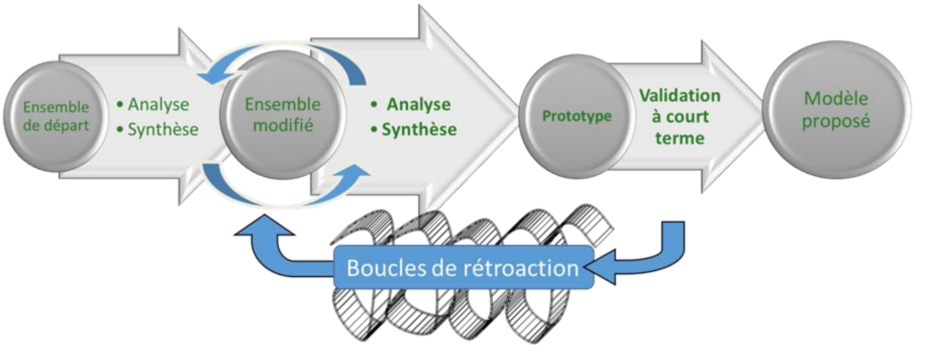

Figure 1. Anasynthèse.

LE MODÈLE

Lors de l'anasynthèse, quatre éléments majeurs ont émergé qui ont aidé à clarifier le flou conceptuel soulevé : La dichotomie activité/produit ; la nature inférentielle du RM, les buts et fonction du RM, et les aspects structurels et processuels du RM.

Le cadre commognitif m'a aidé à donner sens à ces éléments. En particulier, le raisonnement mathématique est conceptualisé comme un processus commognitif qui permet d'inférer des énoncés mathématiques à partir d'autres énoncés. Il s'agit d'une activité métadiscursive qui permet un développement de niveau objet, c'est-à-dire que le raisonnement mathématique permet un développement en extension du discours et non le développement d'un nouveau discours. La valeur épistémique de ces énoncés est un élément important qui met en jeu différentes règles discursives.

Je traiterai dans cet écrit davantage du quatrième élément mis en lumière, c'est-à-dire *l'aspect processuel et structurel du raisonnement*.

À partir de la littérature, l'aspect structurel s'est avéré plus facile à caractériser puisque plusieurs auteurs avaient théorisé à ce propos. Par ailleurs, cet aspect ne suffit pas à lui seul pour comprendre ce qu'est le raisonnement mathématique, comment il se manifeste. L'aspect structurel fait référence aux différentes structures que peuvent prendre les pas de RM (déductif, inductif, abductif), ce qui est parfois appelé la forme du raisonnement mathématique. Basé sur les modèles de Toulmin (1958) et de Peirce (n.d.) l'aspect structural est un ensemble de pas de raisonnement caractérisé par trois éléments de base : *les données*, *la règle* et *le résultat*. D'autres éléments peuvent venir s'attacher à un pas élémentaire : un fondement attaché à la règle, une réfutation et un qualificateur attaché à l'énoncé inféré. Basé sur les premiers écrits de Peirce, le pas déductif infère un résultat, le pas inductif infère une règle et un pas abductif infère des données. La Figure 2 donne un exemple de pas inductif. Une élève de 4[e] année cherche à trouver une règle pour savoir si un nombre est divisible par 4. La tâche originale est tirée de Del Notaro (2011). Pour ce faire, elle teste la divisibilité de 3 exemples et génère la règle : *Si un nombre est divisible par 4, alors il se termine par 2*. Quoiqu'il est impossible de déterminer la qualification, on peut penser que cette règle est vraie ou vraisemblable pour l'élève. Une réfutation (tel un contrexemple) pourrait ainsi changer la nature de ce qualificateur et remettre en question la règle.

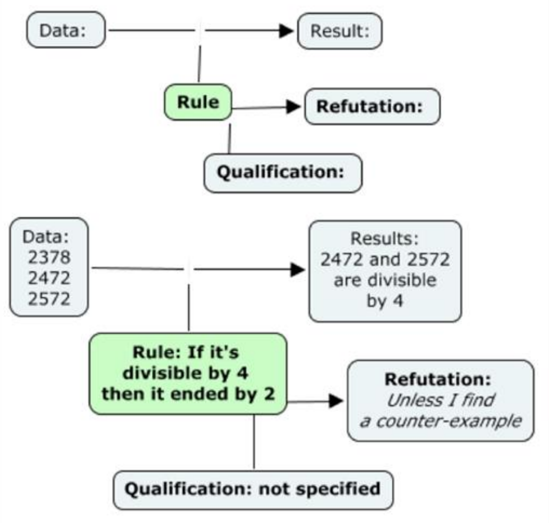

Figure 2. Un pas inductif.

À la Figure 2, le pas inductif est caractérisé par des données composées de trois exemples (2378, 2472 et 2572) dont deux sont divisibles par 4 (résultat). Une règle est inférée dont le qualificateur est inconnu.

L'aspect processuel fait référence à différentes actions discursives qui peuvent être mises en œuvre. Un processus est considéré comme une activité commognitive, régulé par un ensemble de règles et de nature récursive. L'aspect processuel est aussi lié à l'élément but et fonction mis en lumière dans la littérature. En effet, les buts et fonctions sont souvent liés à des verbes d'action. Par exemple, on dira qu'on raisonne pour valider, pour justifier. Le modèle classe les processus en deux grands groupes. Les processus de recherche de similarité et de différences sont : *identifier une régularité*, *généraliser*, *conjecturer*, *comparer* et *classifier*. Les processus de recherche de validation sont : *justifier*, *prouver* et *démontrer*. L'anasynthèse a permis d'arriver à une définition pour chacun des processus. En particulier, voici les définitions de *généraliser*, *conjecturer* et *identifier une régularité*.

> ***Généraliser** : généraliser, en tant que processus de RM, infère un énoncé à propos d'un ensemble d'objets mathématiques, ou d'une relation entre différents objets de cet ensemble, à partir d'un ensemble plus restreint d'objets contenus dans ce premier.*
>
> ***Conjecturer** : conjecturer est un processus de RM qui, par la recherche de similitudes et de différences, permet d'inférer un énoncé à propos d'une régularité, ou d'une relation, pour lequel la valeur épistémique qui lui est rattachée est vraisemblable, et qui a un potentiel de théorisation mathématique.*
>
> ***Identifier une régularité** : identifier une régularité, en tant que processus de RM, infère un énoncé à propos d'une relation récursive entre différents objets ou relations mathématiques, par la recherche de similitudes et de différences entre ces objets ou relations mathématiques.* (Jeannotte, 2015, p. 271)

La Figure 3 montre des traces liées à un processus identifier une régularité, généraliser et conjecturer. L'élève devait trouver une relation en lien avec la suite des nombres triangulaires. Pour ce faire, il a exemplifié les six premiers nombres triangulaires. À partir des exemples, il a identifié une régularité qu'il était en mesure de justifier comme vraie sur l'ensemble des positions paires (généraliser). En étendant cette régularité aux positions impaires, il conjecturait une relation (vraisemblable) pour l'ensemble des nombres triangulaires.

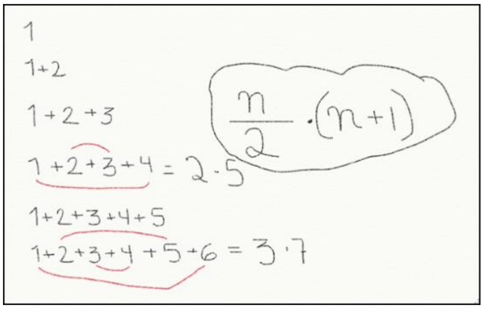

Figure 3. Identifier une régularité vs conjecturer (tiré de Jeannotte, 2015, p. 212).

Les deux autres processus de recherche de similitudes et de différences, comparer et classifier se définissent comme suit :

> ***Comparer** : comparer, en tant que processus de RM, permet d'inférer, par la recherche de similitudes et de différences, un énoncé à propos d'objets et de relations mathématiques.*
>
> ***Classifier** : processus de RM qui, par la recherche de similitude et de différences entre des objets mathématiques, permet d'inférer des énoncés à propos de classes en s'appuyant sur des propriétés ou des définitions mathématiques.* (Jeannotte, 2015, p. 271)

Les raisonnements de recherche de similitudes et de différences sont tout aussi très importants en mathématiques que les processus de validation souvent mis de l'avant comme la particularité des mathématiques (Thurston, 1996). En particulier, trois processus de validation ont émergé de l'analyse : *justifier, prouver* et *démontrer*. Comme on le retrouve à la Figure 4, ces processus visent tous un changement de valeur épistémique, démontrer étant un cas particulier de prouver et prouver un cas particulier de justifier.

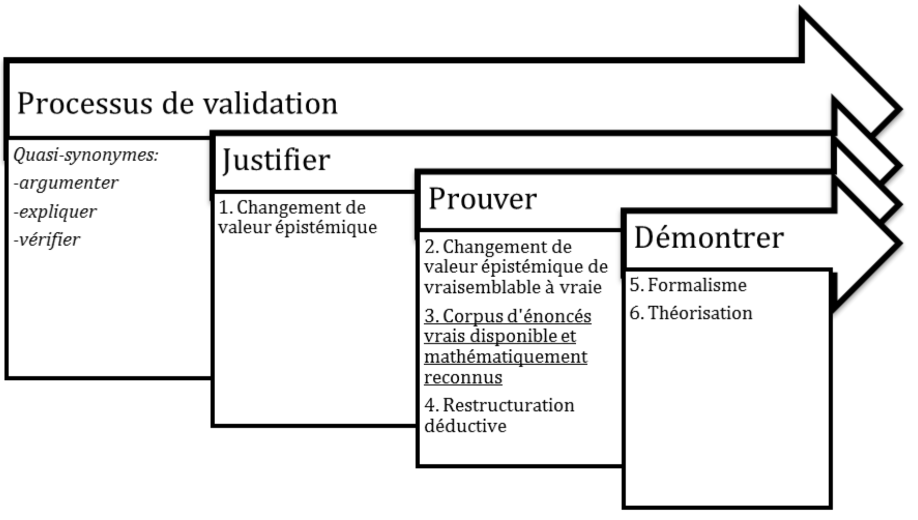

Figure 4. Processus de validation : tiré de Jeannotte, 2015, p. 255.

La Figure 5 montre une règle de divisibilité par 4 générée par la même élève dont il est question à la Figure 2. On retrouve ici des traces du processus *justifier*. L'élève s'appuie sur ses connaissances de la table de multiplication par 4 pour justifier sa règle. Toutefois, ceci ne peut être considéré comme prouver puisque l'énoncé, quoique vrai et disponible à l'élève n'est reconnu par la communauté mathématique comme étant acceptable. Ce que Balacheff (1987) appelle *exemple générique* pourrait ici servir pour prouver la conjecture (ce qui a été fait par l'élève).

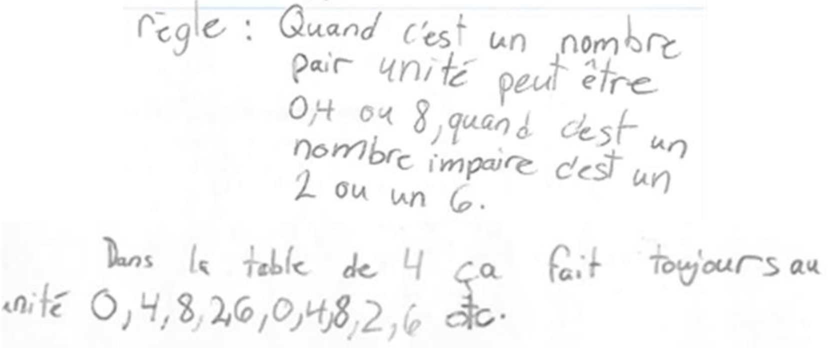

Figure 5. L'élève justifie sa règle.

Comme on peut le constater dans les exemples donnés ci-haut, exemplifier joue des rôles importants dans les raisonnements d'élèves. Parfois, le processus exemplifier supporte des processus reliés à la recherche de similarités et de différences (comme à la Figure 2) et d'autres fois (comme pour le cas de l'exemple générique), il supporte un processus de recherche de validation.

CONCLUSION

Les aspects structurel et processuel permettent de parler des raisonnements mathématiques d'élève de différentes façons. Il est un enchevêtrement de processus et de structure qu'il est parfois impossible de compartimenter. Toutefois, le développement en concomitance des processus, en recyclant des éléments générés antérieurement par d'autres, rend difficile, voire impossible, de ne s'appuyer que sur l'aspect structurel pour caractériser le RM, comme il se fait traditionnellement. Un processus s'appuyant sur un autre ou se déroulant simultanément, une structure qui bouge, change au gré des processus qui se déroulent, chacune de ces définitions permet de jeter un éclairage différent sur un même raisonnement. On peut y voir des liens entre les processus et les structures, y déceler des règles de classe (voir Jeannotte et Corriveau, 2015) des énoncés qui semblent partagées par une communauté de mathématiques.

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DIAGRAMMING AND GESTURING BETWEEN MATHEMATICS GRADUATE STUDENT AND EXPERT MATHEMATICIANS

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Rather than treating the mathematical diagram as a visual representation of already existing mathematical objects and relations, Châtelet (1993/2000) regards the diagram as a material site of engaging with and mobilizing the mathematics through his study of historical, mathematical manuscripts. His approach is employed in this study to create a window into the realm of mathematical thinking and invention by examining how a graduate student (as the less-expert mathematician) and his supervisor and two research colleagues (as the expert mathematicians) interact with diagrams. An embodied lens, based on the works of de Freitas, Roth, Rotman, Sinclair, and Streeck, exposes the similarities and differences in the way that each class of mathematician gestures and diagrams. This study achieves two purposes, namely to confirm and advance Gilles Châtelet's theory to the context of live mathematical activity and to elucidate the enculturation process of a graduate student into mathematical research.

INTRODUCTION

Little research has been done on analyzing the mathematizing that arises during research meetings between graduate student and supervisor, and what work exists focuses either on theorem and proof construction or the social and cultural aspects of such interaction. Furthermore, recent studies in semiotics (e.g., Châtelet, 1993/2000; McNeill, 1992, 2008; Radford, 2008, 2009) discuss the connection between the study of gestures and diagrams, and how they may link to mathematical thinking and creation. Châtelet's work itself is based on the surviving written manuscripts of mathematical figureheads without access to sound- or video-recordings of these mathematicians while undertaking their research, as might be produced today and that could provide further data to illuminate or clarify creative moments. The findings of the study on diagramming and gesturing between mathematics graduate student and supervisor during 'live' research meetings (Menz, 2015) were presented at CMESG 2016, and this article is a report of that presentation.

This article begins with a brief review of current research that theorises the role of mathematical diagrams in diametrically opposed ways, proposing that diagrams are either a visual representation of already existing mathematical objects and relations, or that they are the means through which mathematical objects and relations emerge. The literature review is followed by a snapshot of the methodology employed in the study. Then, two major results are presented, namely the insights gained on the enculturation process of the mathematics graduate student becoming a mathematician, and how interactions between the diagram and mathematician can be viewed as a diagram life-cycle. The article ends with concluding remarks pertaining to diagramming and gesturing, and a discussion of the implications of this study and directions for future work.

THE DIAGRAM AS THE VISUAL

Research studies of mathematizing describe visualization as "the process of producing or using geometrical or graphical representations of mathematical concepts, principles or problems, whether hand drawn or computer generated" or achieved mentally, as opposed to the pure, direct meaning of mental images in the everyday use of the term visualization (Zimmermann & Cunningham, 1991, p. 1). While visualization has often been viewed as a crutch to reach the formal, finished mathematical product in earlier research circles, some later studies upgrade visualization as being a recognized tool for mathematical reasoning and offer a more balanced view of mathematics that includes the visual and intuitive rather than just the symbolic, verbal and analytical elements. One of the visionary researchers in visualization studies, Presmeg (1986), introduces the concept of *mathematical visuality* that represents the "extent to which that person prefers to use visual methods when attempting mathematical problems which may be solved by both visual and nonvisual methods" (p. 42).

In the last decade, most visualization studies in mathematics concentrate directly on the diagram and diagrammatic reasoning, particularly those adopting a semiotic approach. For example, philosopher Radford (2008) argues that diagrammatic thinking is a deeply subjective activity embedded in the socio-cultural experiences of the individual, which constructs a reality in accordance with semiotic systems of signification out of which creation emerges. While Radford brings light to gestures by looking at semiotics through the lenses of cultural meaning and sensuous cognition, the problem remains, according to de Freitas and Sinclair (2012), that "[m]uch of this work conceives of diagrams and gestures as 'external' representations of abstract mathematical concepts or cognitive schemas" (p. 4). In this manner, the diagram becomes static and the hand that mobilized the diagram is forgotten.

To summarize, current studies of visualization in mathematics concern themselves with the production, interpretation and utilization of visual representations during mathematical problem solving or proof production from the novice to the expert-level mathematician; however, these studies take the approach that the visual representations are evoked from invisible representations that existed *a priori* for the person engaged in mathematical activities. Rather than treating the diagram as a product of mathematization, the next section reviews the literature on diagrams as the virtual that is being actualized coupled with mathematical invention as the possible that is being realized and shows how this viewpoint helps to shed light on the creative processes in mathematizing.

THE DIAGRAM AS THE VIRTUAL

In his foreword to Châtelet's *Figuring Space* (2000), Knoespel writes that the ancient Greek verb "Διαγραμμα in effect embodies a practice of figuring and defiguring" (p. xvi). Through this flexible view of figuring and defiguring, Châtelet brings new ideas and insights to the study of gestures, diagrams and mathematics. By analyzing historical manuscripts of famous mathematicians such as Oresme, Leibniz and Hamilton and without access to sound- or video-recordings of these mathematicians during their research, Châtelet delves deeply into these mathematicians' research and traces their thoughts and actions purely from their writing and their diagramming. His fundamental finding is that the virtual is evoked in historical mathematical inventions "through diagramming experiments whose sources Châtelet can trace to mobile gestural acts" (Sinclair & de Freitas, 2014, p. 563). In this way, Châtelet points to the physical nature of mathematics that is necessary for its own invention, which is provocative and challenging counterpoint to the abstract nature that has come to be customarily associated with mathematics. The key ideas of his theory are that: (1) the diagram is never really fixed, rather it is erased, drawn over, reassembled, or redrawn as it hovers between virtuality and actuality; (2) there is a mutual interaction between mathematician and diagram; and (3) it is through material interactions with the diagram that a person understands or invents mathematics, which in Châteletan terms is the possible within the person that is realized.

Recent research in mathematics philosophy suggests that the key to understanding mathematical invention is the diagram as a place of possibilities through which the mathematician's virtual gestures create and engage with the diagram (e.g., Châtelet, 1993/2000; de Freitas & Sinclair, 2012). Over the last two decades, a broad-based understanding has grown out of research in such areas as anthropology, cognitive science, mathematics education, and linguistics, that physical gestures are linked to thought (Kita, 2000; Krummheuer, 2013; Lakoff & Núñez, 2000; McNeill, 2008; Radford, 2009). More recently, through the viewpoints of embodied cognition and new materiality, both virtual and physical gestures are shown to be crucial for doing and thinking mathematics (Bailly & Longo, 2011; Barany & MacKenzie, 2014; de Freitas & Sinclair, 2013; Greiffenhagen, 2014; Menz, 2015; Roth & Maheux 2015; Rotman, 2012). This study is germane in that it connects Châtelet's theory of virtual gestures to physical gestures during live mathematizing, provides insights into the enculturation process of the graduate student on his way to becoming a mathematician, and furthermore, identifies the different functions of the interactions between mathematicians and their created diagrams in an effort to expose how a diagram is given a voice that contributes to the process of invention during mathematizing.

METHODOLOGY

The research subjects in this study comprise one male, Caucasian, mathematics graduate student (less-expert mathematician) referred to as Finn and three male, Caucasian, research mathematicians (expert mathematicians) in the field of Topological Graph Theory from two prominent North American universities. The graduate student and three expert mathematicians met nine times over a period of three months to study the class of 2-regular directed graphs and how they embed in different surfaces. The research goal was to compile a list of obstructions, for the projective plane in particular, although other surfaces such as the torus and Klein bottle were also explored at times, and to classify these obstructions for 2-regular directed graphs. All research meetings took place in one university's mathematics seminar room, well-equipped with blackboards on which all diagrams of this study were drawn. The author was present in order to video-record these nine research meetings that varied in lengths from one to two hours; to capture digital still images of diagrams by the participants using a second camera; and to make field notes during the meeting that concerned both the mathematics being developed and observations regarding diagramming and gesturing.

Based on these observations and multiple viewings of the recorded research meetings that paid attention to the emergence of diagrams and how these three mathematicians create and engage with the diagrams, the 12 hours of video-recordings were broken up into time intervals that contained at least one diagram. A total of 122 such intervals were selected for further analysis varying in length between approximately thirty seconds and six minutes, and containing a total of 128 diagrams.

ENCULTURATION PROCESS

There is no denying that the graduate student Finn was in an unusual research meeting setting. Typically, graduate-student/supervisor meetings are held in the supervisor's office, which has little board-space for writing and drawing, whereas Finn's research meetings were run like informal seminars with copious blackboard space. Barany and MacKenzie (2014) point out that the blackboard "as a semiotic technology [is] as much a stage as a writing surface" and as such public and "ostentatious so much so that colleagues in shared offices expressed shyness about doing board work when office-mates are present" (p. 12). While Finn initially exhibited the blackboard-as-a-stage syndrome because his voice was low and he hardly moved up to the blackboard, there was an assuredness in the manner in which the three expert mathematicians occupied the room, carrying themselves upright and naturally, using the blackboard space, and speaking to make themselves heard.

The discourse of the expert mathematicians often centred on diagrams and was heavily saturated with gestures of pointing, hand-pointing, touch-pointing, holding, tracing, sweeping and covering up. Finn's utterances, on the other hand, were initially tentative, short and unaccompanied by gestures other than pointing. However, as anthropologist Gerholm (1990, p. 266) points out, scientific discourse is the most significant type among six types of tacit knowledge that a graduate student must acquire to become a researcher. The expert mathematicians modelled investigative strategies, verbal and gestural communication skills, and engagement with the diagram, and the research meetings provided Finn with opportunities to mimic the expert mathematicians. Figure 1 provides a summary of Finn's successful and unsuccessful communication attempts as well as communication directed at him. Initially, Finn made no attempt to join the discussion and neither was he addressed by any of the expert mathematicians. However, over the course of the nine research meetings, Finn became more and more successful at contributing to research discussions, employing gestures other than pointing such as drawing on the blackboard, tracing diagrams, and gesturing mathematical objects during speech. Furthermore, the research mathematicians not only summarized for Finn major findings or new insights that they had gained in their diagramming explorations or discussions, but also started addressing Finn directly, either asking about some information in his notebook or engaging him in a discussion about some graphs that he had explored.

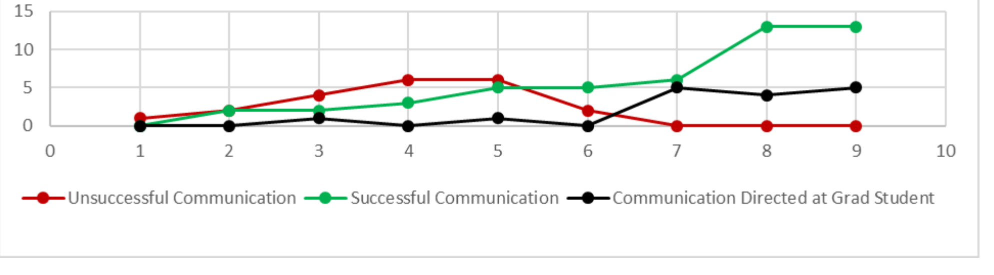

Figure 1. Summary of graduate student's communication and communication directed at him.

As supported by scholarly work on graduate studies (e.g., Golde & Walker, 2006), in order for the mathematics graduate student to grow into an expert mathematician, he should not only know the mathematics of his field, but he must also become accustomed to principles and conventions of mathematical communication so that he becomes a more effective mathematician. Through the acts of diagramming and gesturing exhibited by the three expert mathematicians during mathematical discourse, the graduate student learned how to behave and talk like a mathematician.

DIAGRAM LIFE-CYCLE

A diagram does not just simply get made, but as seen from the data, the process of producing the diagram is crucial in making the mathematics visible for the mathematicians. The mathematicians referred to the diagram by mathematical words such as "*crossing land*", "*anti-digon*", and "*surface*" or by deictic words such as "*this*", "*this guy*", "*this creature*", "*here*", and "*it*" to name but a few, which is evidence that, for the mathematician, the diagram is not just a visual product, but also a mathematical object in its own right. While Châtelet only intimates at the whole process of diagramming during mathematical invention, the analysis of the 122 time intervals allows a glimpse into this process and reveals how diagrams come into being and are engaged with. It became apparent that not only do diagrams undergo transformations through the creation and engagement by the mathematicians, but also that relationships develop between the diagram and the mathematicians. These findings enrich Châtelet's theories by being able to identify the different functions of interactions between mathematician and diagram in inventive mathematical processes, introduced as diagram life-cycle.

The diagram life-cycle is divided into three phases, which concern themselves with: (1) how the mathematician brings the diagram into being—*manufacturing phase*; (2) what the relationship is between the diagram and the mathematician—*communication phase*; and (3) how the mathematician resolves, in simplistic terms, whether or not the diagram stays—*dénouement phase*. Because the term *episode* alludes to both a period of time and a side-story being told, it was chosen to describe a particular function of interaction between diagram and mathematician during the various phases. The analysis shows that throughout the life-cycle of a diagram, from its creation to its 'establishment' or 'obliteration', there are eleven distinct diagram episodes: *emerging*, *present*, *unsupportive*, *disruptive*, *supportive*, *pulling*, *central*, *discarded*, *obliterated*, *absent* and *established* (see Figure 2). Incidentally, the two most common episodes are *diagram-is-supportive* and *diagram-is-central* at 37% and 38% respectively. This should not come as a surprise, because the mathematicians chose to diagram in order to explore the mathematics underlying their research. As Châtelet aptly puts it, "one is infused with the gesture before knowing it" (1993/2000, p. 10), which is evident when mathematicians gesture at a diagram during speech but before the chalk marks actually create it on the blackboard.

<table>
<tr><td rowspan="2">manufacturing phase:
making the mathematics real</td><td colspan="3">emerging</td></tr>
<tr><td colspan="3">present</td></tr>
<tr><td>communication phase:
moulding the diagram,
reorienting the mathematician,
mobilizing the mathematics</td><td colspan="2">unsupportive
disruptive</td><td>supportive
pulling
central</td></tr>
<tr><td>dénouement phase:
levels of mathematical acceptance</td><td>discarded
obliterated</td><td>absent</td><td>established</td></tr>
</table>

Figure 2. Episodes, phases and their relationships in the life-cycle of a diagram.

In the interest of space, only the episode *diagram-is-disruptive* is described here, while the remaining episodes are detailed in the work of Menz (2015). The premise of a disruptive diagram is that the thinking track of a mathematician is interrupted through the information that the diagram reveals, which is either unexpected or in discord with the knowledge of the mathematician. Such episodes typically begin with speech repairs or exclamations such as "*oh*" and "*ah*" in the utterances of the mathematician who is engaged with the diagram, are usually quite tumultuous, and often end with laughter. In the following example about a disruptive diagram the mathematicians are exploring the embedding of an unknown graph and are expecting the faces of the embedding to consist of four 3-cycles and two 4-cycles. One of the mathematicians is working at the blackboard and attempting to find these particular cycles. After four minutes of repeatedly walking to the blackboard, tracing edges (see Figure 3a), erasing edges, drawing over edges (see Figure 3b), stepping away from the blackboard, staring at the diagram (see Figure 3c), and otherwise remaining silent, the mathematician exclaims, "*Oh! It's not three, three, three, three, four, four. It's five threes and a five.*" The data that the diagram reveals to the mathematician after his diagramming exploration comes as a surprise, and so, this is classified as an episode of diagram-is-disruptive.

Figure 3. Mathematician tracing edges (a), drawing over edges (b), and staring at diagram (c).

The above episode is one of many in which the expert mathematicians are led to discover a parity argument of closed walks in support of uniqueness for the embedding of the reversed planar octahedron in the torus. Rather than seeing the unravelling of uniqueness as a mental process that does not partake of the physical world, the data supports that the mathematical result derives from an embodied engagement with diagrams. Considering all the episodes that are possible in the life-cycle of a diagram, the diagram transcends just being a visual. As the mathematician engages with the diagram virtually through gestures (e.g., points, traces, stares) or physically (e.g., adds, erases, draws over), the diagram constantly reassembles itself for the mathematician and thereby mobilizes mathematical invention. As Roth and Maheux (2015) point out "the virtual cannot be grasped but is that which allows grasping to occur" (p. 236).

Furthermore, the direct, intimate, and material engagement with the diagrams, and the prolific gestures that accompany the mathematician's speech are all evidence that the diagram (1) consists of material entities (e.g., edges, vertices, graphs, embeddings) and material relationships (e.g., closed walks), and (2) is engaged with, not only virtually but also physically. Through these virtual and physical gestures of the expert mathematician, the diagram is given a voice, which speaks not only to the gesturing mathematician but all of its participating onlookers including the graduate student. The anthropologist Ingold (2013) puts it quite aptly in that,

> *[T]he drawing that tells is not an image, nor is it the expression of an image; it is the trace of a gesture. […] Thus the drawing is not the visible shadow of a mental event;* it is a process of thinking, not the projection of thought. (p. 128, emphasis in original)

CONCLUSION

The idea that gestures make thought visible has strong support in the mathematical research community (Kita, 2000; Krummheuer, 2013; Lakoff & Núñez, 2000; McNeill, 2008; Radford, 2009). However, a novel concept on the horizon in recent studies on gestures and thought "is the effort to identify causal and measurable relations and interactions between bodily behavior and hypothesized internal processes and to explain these within embracing and detailed theories of kinetic, communicative, cognitive, and symbolic systems" (Streeck, 2009, p. 172). The current study reinforces that at least in the culture of research in Topological Graph Theory, gestures play a vital role in that they not only support communication among the mathematicians, but also kinesthetic and haptic experience with the diagram and the mathematical meaning that the diagram holds. The episodes that are described in the diagram life-cycle, provide evidence that the concept of *virtuality* is pushing the material aspects of mathematics, and reveal how mathematics comes into being for the mathematicians through this conceptualization. Furthermore, the acts of diagramming and gesturing during mathematizing by the three expert mathematicians offer profound insights into the culture of doing mathematics at a research level, and they contribute to how the graduate student is enculturated into the role of a mathematician.

Some possible recommendation of these findings for teaching and learning at all levels of acquiring mathematics is to create an environment that places each student in the position of a mathematician, where the student explores objects and their relationships perhaps dynamically using geometry software, or drawing by hand. During their creative phase the mathematicians needed to *see* and *feel* objects and their relationships in order to gain an understanding of them, or to explore how they could be altered or newly positioned to extract new mathematical insights. Therefore, as mathematics educators we need to allow our students to *see* and *feel* mathematics in environments, where the student is bound to point, touch, hold, trace, add and delete, which may lead to further material engagement with the mathematics being explored.

Lastly, the findings in this study regarding gesturing and diagramming pertain to the mathematization of the less-expert mathematician compared with that of experts. In an effort to gain a clearer understanding of gesturing and diagramming at all levels of mathematizing, this study can be extended to investigating the mathematizing of mathematicians ranging from novice to expert, which should include students and teachers from pre-K, grades K-12, as well as undergraduate studies. Such a study would not only raise awareness of an embodied view of mathematics, but would also shed light on a scaffolded approach to gesturing and diagramming that could provide new insights into the teaching and learning of mathematics.

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ANALYSE DU RAISONNEMENT COVARIATIONNEL FAVORISANT LE PASSAGE DE LA FONCTION À LA DÉRIVÉE ET DES SITUATIONS QUI EN SOLLICITENT LE DÉPLOIEMENT CHEZ DES ÉLÈVES DE 15 À 18 ANS

ANALYSIS OF COVARIATIONAL REASONING PROMOTING THE PASSAGE FROM THE FUNCTION TO THE DERIVATIVE AND OF SITUATIONS THAT LEAD 15- TO 18-YEAR OLD STUDENTS TO DEPLOY THAT REASONING

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INTRODUCTION

L'apprentissage de la notion de fonction occupe une place importante dans l'enseignement secondaire. Pour les élèves qui poursuivent leurs études post-secondaires dans des domaines scientifiques, l'étude approfondie des fonctions en algèbre débouche systématiquement sur l'étude des fonctions dans le contexte du calcul différentiel et intégral. Ainsi, les élèves sont amenés à approfondir leur compréhension de la notion de fonction lors d'un travail sur la fonction dérivée. Dans cette recherche, nous nous sommes penchée sur ce passage de la fonction à la dérivée dans l'optique de mieux comprendre comment favoriser l'articulation de ces deux notions dans l'enseignement.

LES ENJEUX DU PASSAGE DE LA FONCTION À LA DÉRIVÉE

Plusieurs études mettent de l'avant les difficultés rencontrées par les élèves lors de l'apprentissage du calcul différentiel (Artigue, 2000). Certains auteurs attribuent ces difficultés aux conceptions inadéquates ou incomplètes développées par les élèves à propos de certains concepts fondamentaux, tels *le taux de variation, la variable* et *la fonction*, sur lesquels s'appuie cet apprentissage. Par exemple, Weber et Dorko (2014) constatent que même les élèves qui ont été initiés au calcul différentiel conçoivent *le taux de variation* comme un objet statique associé à une formule, à la pente d'une droite ou à un quotient de différences. Ils considèrent que ces conceptions, déconnectées de la variable et de la variation, nuisent à l'apprentissage de la notion de dérivée. Pour eux, il faudrait amener les élèves à concevoir le taux de variation comme une *mesure de la covariation*. Cette recommandation rejoint celle de plusieurs autres chercheurs qui suggèrent qu'un travail sur les aspects covariants de la fonction permettrait de mieux préparer les élèves à aborder le calcul différentiel.

Par ailleurs, certains faits historiques semblent montrer que la notion de fonction a subi une évolution majeure lors de l'émergence du calcul différentiel et intégral aux XVII^{e} et XVIII^{e} siècles (Charbonneau, 1987; Youschkevitch, 1981). C'est dans ce contexte, en effet, que les mathématiciens ont défini pour la première fois l'objet *fonction*. Euler en parle alors comme d'une relation entre des quantités covariantes : « Si certaines quantités dépendent d'autres quantités de telle manière que si les autres changent, ces quantités changent aussi, alors on a l'habitude de nommer ces quantités fonctions de ces dernières (…) » (cité par Youschkevitch, 1981, p. 49). Ces considérations appuient l'hypothèse selon laquelle un travail de la fonction par la covariation puisse être propice à une co-construction des notions de fonction et de dérivée dans l'apprentissage.

UNE APPROCHE COVARIATIONNELLE DE LA FONCTION

Plusieurs études dans les dernières décennies abordent l'idée d'un travail sur la covariation (Carlson, Jacobs, Coe, Larsen, & Hsu, 2002 ; Carlson, Larsen, & Jacobs, 2001 ; Confrey & Smith, 1995 ; Hitt & Morasse, 2009 ; Saldanha & Thompson, 1998). Certains auteurs considèrent la covariation comme une notion en soit alors que d'autres la voient comme une manière d'aborder la notion de fonction. Dans cette dernière veine se situent Confrey et Smith (1995) qui parlent de la covariation comme d'un regard porté sur la fonction. Ce regard cible les changements concomitants des deux grandeurs mises en relation ; on s'intéresse généralement à décrire la variation de la grandeur dépendante lorsque la grandeur indépendante varie. Les auteurs distinguent ce regard de celui par la correspondance qui cible le lien entre les valeurs des deux grandeurs qui se correspondent ; on s'intéresse alors généralement à établir une règle de correspondance entre les deux grandeurs. Les idées de Confrey et Smith (1995) rejoignent ainsi celles de Monk (1992) qui avait associé cette dualité du travail sur la fonction à deux types de questions qui se posent lors de l'étude de phénomènes physiques, les questions « ponctuelles » (*pointwise questions*) et les questions « à travers le temps » (across-time questions). Dans cette étude, Monk suggérait de travailler le questionnement à travers le temps à l'aide d'une analyse qualitative des accroissements de la grandeur dépendante.

Nous inspirant notamment des perspectives de Confrey et Smith et de Monk, nous parlons d'une approche covariationnelle de la fonction comme d'une manière d'aborder la notion de fonction dans le contexte de la modélisation de situations réelles. Cette approche consiste alors en une étude approfondie des variations concomitantes de deux grandeurs par l'intermédiaire d'un travail sur les accroissements concomitants de ces deux grandeurs. Globalement, l'objectif de notre recherche était d'identifier les caractéristiques des situations associées à cette *approche covariationnelle de la fonction* et favorisant le passage de la fonction à la dérivée chez des élèves de 15 à 18 ans.

LE RAISONNEMENT COVARIATIONNEL

Pour Carlson et ses collègues (Carlson et al., 2002, 2001), le travail sur des situations dans lesquelles on s'intéresse à comment change une grandeur lorsqu'une autre grandeur change implique la mobilisation d'un raisonnement covariationnel. De manière générale, ce type de raisonnement est défini comme « *the cognitive activities involved in coordinating two varying quantities while attending to the ways in which they change in relation to each other* » (Carlson et al., 2002, p. 354). Ce cadre comporte cinq actions mentales (voir Tableau 1) ainsi que cinq niveaux de développement en lien avec ces actions mentales. *For the authors* « *one's covariational reasoning ability has reached a given level of development when it supports the mental actions associated with that level* and *the actions associated with all lower levels* » (p. 357) (par exemple, un élève atteint le 3^{e} niveau de développement lorsqu'il mobilise les actions mentales 1 à 3).

Mental action	Description of mental action
Mental action 1 (MA1)	Coordinating the value of one variable with the changes in the other
Mental action 2 (MA2)	Coordinating the direction of change of one variable with changes in the other variable
Mental action 3 (MA3)	Coordinating the amount of change of one variable with changes in the other variable
Mental action 4 (MA4)	Coordinating the average rate of change of the function with uniform increments of change in the input variable
Mental action 5 (MA5)	Coordinating the instantaneous rate of change of the function with continuous changes in the independent variable for the entire domain of the function

Tableau 1. *Mental actions of the Covariation Framework* (Carlson et al., 2002, p. 357)

Afin d'identifier les caractéristiques des situations associées à une approche covariationnelle de la fonction, nous nous sommes intéressée au déploiement d'un raisonnement covariationnel engendré par ces situations. Ainsi, à partir du cadre de Carlson et d'une analyse des situations proposées par plusieurs chercheurs pour travailler les aspects covariants de la fonction, nous avons identifié treize unités de raisonnement ainsi que les questions pouvant a priori solliciter la mobilisation de ces unités (voir Annexe A). Le déploiement d'un raisonnement covariationnel était alors considéré comme une articulation d'*unités de raisonnement* orientée par un questionnement spécifique.

Quatre situations de modélisation de phénomènes réels ont été construites afin de susciter la mobilisation des treize unités de raisonnement identifiées chez des élèves de 15 à 18 ans. Les questions spécifiques de notre recherche visaient alors à comprendre comment peut se déployer le raisonnement covariationnel chez ces élèves dans ces situations et à cerner l'influence des caractéristiques des situations. Plus précisément, nous avons posé les questions suivantes :

1. Quelles unités de raisonnement sont mobilisées par ces élèves, comment s'actualisent-elles et en réponse à quelles questions ?
2. Comment s'articulent ces unités dans la dynamique du raisonnement déployé par ces élèves et comment les situations influencent-elles cette articulation ?

EXPÉRIMENTATION

Afin d'apporter des réponses à notre questionnement, nous avons mené une expérimentation auprès d'élèves de la fin du secondaire (*10^th^ and 11^th^ grade*) et du début du collégial (*12^th^ grade*). Pour chaque niveau scolaire, une équipe de quatre élèves volontaires ont participé à quatre séances de 75 minutes en dehors de la classe. Les élèves étaient isolés dans un local, ils étaient filmés et ils devaient travailler ensemble sur les situations données.

Les quatre situations construites exploitaient deux contextes et suggéraient d'observer des relations entretenues par plusieurs grandeurs à travers l'étude de trois fonctions liées (voir Tableau 2) et le questionnement avait été élaboré de manière à solliciter directement, et relativement dans l'ordre, les treize unités de raisonnement identifiées préalablement. De plus, du matériel était fourni afin de permettre aux élèves de simuler les phénomènes des situations 1 et 3.

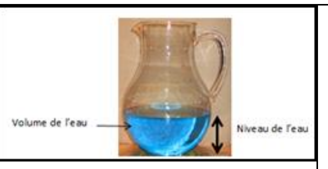
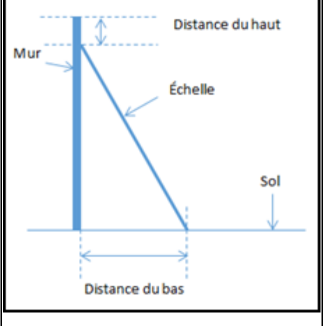
Contexte du pichet On remplit un pichet vide avec de l'eau		<u>Situation 1 (S1)</u> On s'intéresse au **niveau** de l'eau en fonction du **volume** d'eau dans le pichet.	<u>Situation 2 (S2)</u> *Le pichet est rempli avec un robinet à débit constant.* On s'intéresse d'abord au **niveau** de l'eau en fonction du **temps** puis à la **vitesse** à laquelle le niveau de l'eau monte dans le pichet en fonction du **temps**
Contexte de l'échelle Une échelle est posée contre un mur de même hauteur et on fait glisser le bas de l'échelle sur le sol.		<u>Situation 3 (S3)</u> On s'intéresse à la **distance** qui sépare le haut de l'échelle du haut du mur en fonction de la **distance** qui sépare le bas de l'échelle du bas du mur.	<u>Situation 4 (S4)</u> *Le bas de l'échelle se déplace à vitesse constante.* On s'intéresse d'abord à la **distance** qui sépare le haut de l'échelle du haut du mur en fonction du **temps** puis à la **vitesse** du haut de l'échelle en fonction du **temps**.

Tableau 2. Contextes et grandeurs observées pour chacune des situations expérimentées.

ANALYSE DES DONNÉES ET RÉSULTATS

Les données collectées lors de l'expérimentation (vidéos et productions écrites) ont permis d'effectuer une analyse détaillée du raisonnement déployé par les trois groupes d'élèves.

Dans un premier temps, nous avons procédé au repérage des unités de raisonnement mobilisées. Cette démarche a permis de préciser les descriptions de plusieurs unités de raisonnement à l'aide, notamment, de l'identification de sous-unités. Dans un deuxième temps, nous avons analysé les processus de passage d'une unité à l'autre, identifié les questions sollicitant réellement la mobilisation des différentes unités chez les élèves et cerné l'influence de certaines caractéristiques des situations (notamment le contexte et les grandeurs observées).

Les groupes d'élèves ayant tous mobilisé à un moment ou à un autre les sept premières unités de raisonnement liées à une étude qualitative des variations concomitantes de deux grandeurs, l'un des résultats important obtenu est la description détaillée de ces unités (voir Tableau 3).

For example, unity U4 was initially described as qualifying the change of the increments of the dependent quantity for constant increments of the independent quantity. It has been further broken down into five sub-unities: U4a and U4b for considering the increment of each quantity, U4c for establishing concomitance of the related increments, U4d for describing the change of successive increments of the dependent quantity while the increments of the independent quantity are constant, and U4e for describing the global behaviour of the dependent quantity growth as the independent quantity increases.

Unité	Description finale des unités et sous-unités de raisonnement
U1	Identifier une relation fonctionnelle a) Identifier les deux grandeurs étudiées b) Établir la grandeur indépendante et la grandeur dépendante (existence d'une dépendance et sens de cette dépendance)
U2	Considérer une relation fonctionnelle sous l'angle de la variation a) Établir que la grandeur indépendante est variable b) Établir que la grandeur dépendante est variable c) Établir la concomitance entre les variations des deux grandeurs
U3	Décrire le comportement de la fonction a) Établir l'augmentation, la diminution ou la constance de la grandeur dépendante alors que la grandeur indépendante augmente b) Qualifier l'augmentation ou la diminution de la grandeur dépendante de manière intuitive et globale
U4	Décrire le comportement de la fonction dérivée a) Considérer des accroissements constants de la grandeur indépendante b) Considérer des accroissements de la grandeur dépendante c) Établir la concomitance entre des accroissements des deux grandeurs d) Décrire le changement d'accroissements successifs de la grandeur dépendante alors que les accroissements de la grandeur indépendante sont constants e) Décrire le comportement global de l'augmentation de la grandeur dépendante lorsque la grandeur indépendante augmente
U5	Considérer les changements de comportement de la fonction dérivée
U6	Considérer le comportement d'une fonction continue et monotone sur un intervalle (ou une phase) de plus en plus petit du domaine a) Généraliser intuitivement le comportement des accroissements de la grandeur dépendante lorsque les accroissements de la grandeur indépendante sont plus petits (le passage à des accroissements plus petits est induit par la question) b) Prendre des accroissements plus petits de la grandeur indépendante et observer le comportement des accroissements de la grandeur indépendante (le passage à des accroissements plus petits est spontané) c) Généraliser le comportement des accroissements de la grandeur dépendante lorsque les accroissements de la grandeur indépendante sont infiniment petits (passage à la limite)
U7	Interpréter le changement des accroissements en termes de taux de variation et nommer la grandeur associée selon le contexte (vitesse, débit, etc.)

Tableau 3. Description des unités de raisonnement à la suite de l'analyse du raisonnement covariationnel déployé par les élèves.

En ce qui concerne l'articulation des unités, nous avons observé un mouvement de va-et-vient, et ce, peu importe les unités sollicitées par le questionnement. Ainsi, les unités de raisonnement n'ont pas été mobilisées indépendamment et selon l'ordre dans lequel elles avaient été sollicitées. En fait, il semblerait que chacune des unités ait joué un rôle clé dans le déploiement d'un raisonnement covariationnel. En ce sens, nous avons décrit trois catégories d'unités à l'aide d'une métaphore comparant le déploiement d'un raisonnement covariationnel à la croissance d'un arbre. Les deux premières unités de raisonnement (U1 et U2), bien que peu sollicitées par le questionnement, apparaissaient de manière récurrente pour servir d'appui aux autres unités, nous les avons comparé aux *racines* d'un arbre. Les unités U3 et U4 sont apparues, quant à elles, constituer le cœur du raisonnement covariationnel. Occupant une grande place dans le discours, ces unités ont occupé un rôle de solidification à l'image du *tronc* de l'arbre. Les unités U5 à U7 ont, quant à elles, permis ponctuellement, de prolonger, de préciser, d'enrichir les unités troncs. Nous les voyons donc comme les *branches* de l'arbre. Nous avons constaté que la mobilisation des unités *branches* n'apparaissait que simultanément à la mobilisation des unités *troncs* qui, elles, s'appuyaient sur les unités *racines*.

Par ailleurs, l'influence de certaines caractéristiques des situations sur le déploiement du raisonnement covariationnel chez les élèves a été observée. Premièrement, nous avons constaté que l'articulation des unités était orientée par les questions qui se posaient aux élèves. En fait, les phénomènes réels étudiés et la disponibilité d'un matériel pour les simuler ont amené les élèves à se poser des questions en plus de celles posées. Par exemple, en S1, les élèves ont introduit la grandeur « diamètre du pichet » afin d'analyser le phénomène et de tenter de comprendre comment le niveau et le volume étaient liés. Ainsi, lorsque la question sur *comment bouge une grandeur lorsque l'autre change* s'est posée aux élèves, elle a été accompagnée d'un questionnement sur le lien qui unit ces grandeurs : Ces grandeurs sont-elles liées ? Comment sont-elles liées ? Comment fonctionne le phénomène ? Quelles autres grandeurs entrent en jeu ? Deuxièmement, les deux contextes n'ont pas mené à la même articulation des unités de raisonnement. Par exemple, dans le contexte du pichet, les élèves n'ont pas eu de difficulté à différencier et à lier les comportements du niveau de l'eau et de la vitesse du niveau de l'eau. À partir d'un raisonnement comme celui-ci : « lorsque le pichet s'élargit, le niveau de l'eau continue d'augmenter mais l'augmentation est de moins en moins grande, donc, la vitesse à laquelle monte le niveau de l'eau diminue », ils ont établi le lien entre le comportement de la fonction « niveau en fonction du temps » (f) et la fonction « vitesse du niveau en fonction du temps » (f'). Cette articulation de l'unité U4 (comportement des accroissements) mobilisée avec la fonction f puis U3 (comportement global de la fonction) mobilisée avec la fonction f' n'est pas apparue aussi clairement dans le contexte de l'échelle. La familiarité de la situation dans le contexte du pichet et la forme changeante de ce pichet qui implique plusieurs changements de comportements des fonctions étudiées ont ici influencé nettement l'articulation des unités de raisonnement et donc le déploiement d'un raisonnement covariationnel.

CONCLUSIONS

L'ensemble de nos résultats (description détaillée des unités de raisonnement, rôle de différentes catégories d'unités et influence des situations) permettent de mieux comprendre comment solliciter le déploiement d'un raisonnement covariationnel chez des élèves de 15 à 18 ans et, par conséquent, d'envisager la construction de situations favorisant une approche covariationnelle de la fonction avec ces élèves.

Nous avons montré comment l'étude successive de trois fonctions liées dans un contexte de modélisation d'une situation réelle et le travail sur les accroissements concomitants des grandeurs observées pouvaient permettre aux élèves de tisser des liens entre les comportements d'une fonction et de sa dérivée. Ainsi, nous pensons qu'une approche covariationnelle de la fonction, à travers des situations sollicitant le déploiement d'un raisonnement covariationnel tel que nous l'avons défini, peut favoriser le passage de la fonction à la dérivée dans la mesure où des allers-retours entre les unités de raisonnement sont favorisés par l'étude de ces deux fonctions en situation. De plus, l'analyse détaillée du raisonnement covariationnel ayant révélé la richesse et la complexité d'un tel raisonnement, nous considérons que l'étude covariationnelle devrait prendre place au sein de l'enseignement des fonctions à la fin du secondaire et au début du collégial (*from 10th to 12th grade*) à travers des activités de modélisation de diverses situations réelles.

Considérant les limites de notre travail, nous avons exploré de manière approfondie l'étude qualitative des accroissements mais il serait à présent important d'en faire de même avec l'étude quantitative et de réfléchir à la coordination de ces deux perspectives complémentaires.

ANNEXE A

UNITÉS DE RAISONNEMENT ASSOCIÉES AU DÉPLOIEMENT D'UN RAISONNEMENT COVARIATIONNEL ET QUESTIONNEMENT SUSCITANT *A PRIORI* LA MOBILISATION DE CES UNITÉS EN SITUATION

Unité	**Description de l'unité de raisonnement**	**Questionnement suscitant *a priori* la mobilisation de ces unités**
U1	Identifier la grandeur indépendante et la grandeur dépendante	Comment varie la grandeur dépendante lorsque la grandeur indépendante augmente ?
U2	Identifier la présence de variations concomitantes de deux grandeurs	
U3	Qualifier le changement de la grandeur dépendante lorsque la grandeur indépendante augmente	
U4	Qualifier le changement des accroissements de la grandeur dépendante pour des accroissements constants de la grandeur indépendante	Comment se comportent les accroissements de la grandeur dépendante lorsque les accroissements de la grandeur indépendante sont constants?
U5	Déterminer les différentes phases de variation (une phase est un intervalle de la grandeur indépendante sur lequel la « façon de varier » de la grandeur dépendante est la même)	
U6	Qualifier le changement des accroissements de la grandeur dépendante pour des accroissements constants **de plus en plus petits** de la grandeur indépendante	
U7	Interpréter le changement des accroissements en termes de taux de variation et nommer la grandeur associée selon le contexte (vitesse, débit, etc.)	
U8	Quantifier un accroissement de la grandeur dépendante pour un accroissement précis de la grandeur indépendante	Quelle est la valeur du rapport des accroissements concomitants des deux grandeurs pour un certain accroissement de la grandeur indépendante ?
U9	Quantifier un accroissement de la grandeur dépendante pour un accroissement unitaire de la grandeur indépendante	
U10	Quantifier le rapport entre l'accroissement correspondant (taux moyen) à un accroissement précis de la grandeur indépendante et ce dernier	
U11	Quantifier le rapport entre les accroissements correspondants (taux de variation moyen) à des accroissements de plus en plus petits de la grandeur indépendante et ces derniers	Quelle est la valeur du rapport des accroissements des deux grandeurs en une valeur de la grandeur indépendante ?
U12	Déterminer une valeur de la grandeur indépendante pour laquelle on connait la limite du rapport entre l'accroissement correspondant à un accroissement précis de la grandeur indépendante et ce dernier, lorsque l'accroissement de la grandeur indépendante tend vers 0 (taux de variation instantané)	
U13	Déterminer, pour une valeur de la grandeur indépendante donnée, la limite du rapport entre l'accroissement correspondant à un accroissement précis de la grandeur indépendante et ce dernier, lorsque l'accroissement de la grandeur indépendante tend vers 0 (taux de variation instantané)	

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CONCEPTUAL METAPHOR AND COHERENT INTEGRATION IN THE PHILOSOPHY OF MATHEMATICS

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INTRODUCTION

As my research progresses and my ongoing survey of the traditional debates in the philosophy of mathematics literature continues, I become increasingly convinced that the following description of the field is accurate:

1. There is no consensus in the philosophy of mathematics: all of the important traditional theories remain controversial and none are overwhelmingly accepted by the academic community.
2. The traditional foundational theories are usually understood as strongly incompatible with each other insofar as they involve inconsistent commitments.
3. Serious longstanding objections have been leveled against each of the traditional theories. Unless any of these criticisms can be definitively resolved, none of the traditional theories can be the correct one and the stalemate continues.
4. Despite its flaws, each traditional theory seems to accurately capture some important aspects of mathematics that its rivals have difficulty accommodating.
5. This sustained controversy in the philosophical foundations of mathematics cannot help but affect our mathematical practices and institutions. In particular, I postulate that it is a contributing factor in the confusion and frustration experienced by many mathematics students.

While many philosophers continue to endorse one of the traditional positions and refine their arguments in hopes of finally winning the decades-old debates, others attempt to develop genuinely novel approaches that bypass the perennial problems. One such is the theory of *embodied mathematics* put forward by George Lakoff and Rafael Núñez (2000) in *Where Mathematics Comes From*. While this theory has attracted positive attention from some mathematics educators, it tends to be either unpopular or unknown in both the mathematical and philosophical communities and a number of telling criticisms have been leveled against it in the literature. However, I contend that embodied mathematics—and, indeed, any theory of mathematics incorporating a conceptual understanding of metaphor—has an extremely promising asset: conceptual metaphor provides a mechanism whereby the best components of several theories of mathematics traditionally understood as inconsistent can be coherently integrated into a unified account.

CONTEXT AND MOTIVATION

Before proceeding any further, it is important that I provide some situating context for the reader. My PhD is an interdisciplinary degree that combined aspects of philosophy, mathematics, and math education, not an education degree proper. My dissertation has a predominantly theoretical and analytic character: my program did not involve any experimental methodologies. Naturally, this character is echoed in the current paper; while this may give my contribution a different flavour than some readers are used to, I have done my utmost to keep my discussion accessible to a broad audience.

My doctoral project was jointly motivated by three realizations that emerged from my experiences of learning, teaching, and researching mathematics during my bachelor's and master's degrees:

- I had become more interested in thinking about how mathematics is practiced than in practicing mathematics.
- In my experience, all three modalities of mathematical practice mentioned above (learning, teaching, researching) involve extensive but frequently unrecognized use of analogies and metaphors.
- Platonism (my favoured theory of mathematics at the time) is untenably mystical, and the traditional alternative theories are also unacceptably problematic.

My dissertation is a philosophical exploration of the connections between metaphor and mathematics. In that work, I do not present a novel philosophy of mathematics but instead argue in favour of a coherent patchwork of the best components of several leading theories bound together by conceptual metaphor. This paper focuses its attention on this one strand of my dissertation rather than aiming to summarize the entire work.[1]

TRADITIONAL THEORIES OF MATHEMATICS

A brief discussion of the foremost traditional theories of mathematics and some of the challenges they face provides necessary background for the remainder of the paper.

As the name suggests, versions of mathematical platonism have been around since at least the time of Plato, nearly two and a half millennia. Platonism is characterized by the belief that mathematical objects are real, abstract, and agent-independent (Linnebo, 2013); that is, according to a platonist, the fundamental difference between an octopus and an octahedron is that the former is concrete while the latter is abstract. The advantages of platonism are numerous and palpable: it fits with the experiences of both novices and professional mathematicians, it is consistent with the way we talk about mathematical entities, and it is compatible with standard theories of truth and semantics. However, there are at least two serious objections that platonists owe answers to. First, the principle of *parsimony* (popularly known as Occam's razor) tells us that, if an alternative to platonism exists with the same advantages and explanatory power but that does not depend on the assumed existence of abstract objects, we ought to prefer the alternative theory. This objection only succeeds if it can be coupled with a suitable alternative theory. The second objection is often referred to as the *access problem* (or Benacerraf's problem): how can we physical beings come to have knowledge of abstract objects that lack causal efficacy? (Brown, 2008). No widely accepted solution to the access problem has yet been proffered, with most attempts being troublingly non-naturalistic; for example, Plato's solution to the access problem given in the dialogues (notably the *Meno* and the *Phaedrus*) required the

[1] The full dissertation from which this paper is drawn can be found at https://ecommons.usask.ca/handle/10388/ETD-2014-04-1527

existence of immortal souls reincarnated into a series of physical bodies, a result that many philosophers are reluctant to accept (to say the least).

The first serious alternatives to platonism emerged during the latter half of the nineteenth century, a time when significant advancements were being made in a variety of mathematical fields—notably formal logic, set theory, and non-Euclidean geometry. *Logicism* is defined by the belief that all of mathematics is nontrivially reducible to logic. Because of this belief, logicist practices focus on proving mathematical results from logical axioms. It is hard to deny that logical methods make an essential contribution to mathematics, and that the combined efforts of math and logic have changed the world forever through the advent of the digital computer. Despite this, logicism does not seem to be a tenable philosophy of mathematics insofar as contemporary mathematics requires some rules and axioms that are non-logical, including the *axiom of infinity* and the *continuum hypothesis* (Ernest, 1998). The related theory, known as *formalism*, does not fall victim to this objection insofar as it views the axiomatic system of mathematics as a meaningless symbolic game rather than a towering tautological edifice propagating logical certainty. Accordingly, it is not the truth but the consistency of the formal system that is the central concern of the formalist. Formalism can be judged a considerable success insofar as its emphasis on symbols, rules, and systematic unification has increased clarity, unity, and rigor in mathematics. One common complaint about formalism is that it has problems explaining why math is so "unreasonably effective" when used by scientists to model the world (Wigner, 1960, p. 1). However, the most serious problem for formalism and logicism came to light in 1931 when Kurt Gödel published his legendary incompleteness theorems, demonstrating that any formal system subsuming ordinary arithmetic cannot prove its own consistency (Mac Lane, 1986). If a proof of the consistency of an arithmetic system is impossible then a robust axiomatic explanation of the philosophical foundations of mathematics seems unattainable.

Two varieties of *constructivism*—theories holding that mathematics is a human-dependent construction—round out this short overview of important traditional theories. *Intuitionism* arose approximately concurrently with logicism and formalism during the nineteenth century. Its central commitment is that, "mathematics is an essentially languageless activity of the mind having its origin in the perception of a move of time" (Brouwer, 1981, pp. 4-5); that is, intuitionists hold that our understanding of the integers arises from our ability to abstractly distinguish successive moments in time. Note that intuitionism differs from platonism in that it believes that the integers are a human-dependent construction, and differs from formalism in that one holds mathematics to be fundamentally symbolic while the other believes it is essentially languageless. The central commitments of intuitionism lead to some unusual conclusions, including a rejection of the law of the excluded middle and all notions of absolute infinity (Barrow, 1992). The primary objection to intuitionistic mathematics is that it is too restrictive, disallowing many important and useful results of classical mathematics that are pivotal to contemporary science. *Social constructivism* holds that mathematics is a cultural institution, with the strongest versions grounding mathematics solely in social convention. Most scholars I know admit that the development of mathematics can be influenced by historical tradition and social context (consider intuitionistic versus non-intuitionistic practices!). The strongest postmodern versions of social constructivism make mathematics objectionably arbitrary insofar as community consent is sufficient to make any assumption legitimate. Charges of relativism and arbitrariness are among the strongest objections against these constructivist theories, since mathematics is usually understood to be one of the most objective domains of human thought (Lakoff & Núñez, 2000). On the other hand, one key positive aspect of constructivism is its focus on mathematical practices, seeing the agents who engage in them and the historical and social contexts they take place in as noteworthy rather than irrelevant.

This discussion has barely scratched the surface of the philosophy of mathematics: not every traditional theory has been presented, and those discussed have been dramatically oversimplified. However, I believe it provides an adequate representation of the stalemate described in the introduction. What is wanted is a novel theory that can incorporate the benefits of all the traditional theories discussed while avoiding their flaws.

CONCEPTUAL METAPHOR AND EMBODIED MATHEMATICS

As I noted in the introduction, my doctoral research was partially motivated by an observation that the practice of mathematics frequently invokes unrecognized analogies and metaphors. During the early stages of my research, I had two important and unexpected realizations. First, like the philosophy of mathematics, the philosophy of metaphor is a vast, ancient, complicated, and unresolved discipline that has seen extensive development over the last fifty years. The first half of my dissertation is devoted to wading through these developments in an attempt to sort out a working understanding of metaphor. The second realization was that I was not the first scholar to posit a connection between metaphor and mathematics; though I had come to this idea without any knowledge of their work, it was inevitable that my research would have to react to Lakoff and Núñez's theory of mathematics. At the heart of this theory is an understanding of metaphor as a cognitive mechanism that George Lakoff and a cadre of collaborators have been developing for decades. A conceptual metaphor is "a grounded, unidirectional, inference-preserving cross-domain mapping—a neural mechanism that allows us to use the inferential structure of one conceptual domain to reason about another" (Lakoff & Núñez, 2000, p. 6). Metaphorical sentences are merely symptoms of these underlying conceptual metaphors. Conceptual metaphors are ubiquitous: nearly every concept we possess is at least partially metaphorically structured, though we are often unaware of this metaphoricity until we explicitly go looking for it. The more abstract a concept is, the more likely it is to have extensive metaphorical structure. This makes mathematics a prime candidate for explanation in terms of conceptual metaphor.

Lakoff and Núñez's *embodied mathematics* is a naturalistic theory with conceptual metaphor at its core. The foundations of the theory are biological: studies have shown humans have evolved to possess a suite of innate, preconceptual capacities related to quantity, order, spatiality, and grouping, from infancy. A key example of one such capacity is *subitizing*, the ability to immediately discern the size of a small collection of objects (Lakoff & Núñez, 2000). An infant's early interactions with the world include experiences of collecting objects together, constructing and disassembling objects, moving along trajectories, and other basic agential operations. As these primary experiences occur, subitizing simultaneously happens and becomes entangled with the former through neural conflation since "neurons that fire together wire together" (Lakoff & Johnson, 2003, p. 256). This entanglement between conceptual domains forms the experiential basis for *grounding metaphors* that extend and enrich the sparse, abstract domain of number by transferring inferential structure from rich, concrete domains, such as object collection (Lakoff & Núñez, 2000). Multiple grounding metaphors work in concert to transform our extremely limited preconceptual numerical capacities into the full-fledged number concepts we enjoy as adults. *Linking metaphors*, another kind of conceptual metaphor, map between two mathematical domains and yield indirectly grounded abstract ideas; the standard example of a linking metaphor is NUMBERS ARE POINTS ON A LINE.[2] Whereas grounding metaphors tend to arise naturally and unconsciously, the acquisition of linking

[2] Expressions of the form TARGET DOMAIN IS SOURCE DOMAIN are the standard way of referring to a conceptual metaphor. It is crucial to remember this is merely notation: conceptual metaphors are neural mappings.

metaphors requires significant amounts of instruction or a stroke of creative genius (Lakoff & Núñez, 2000). Lakoff and Núñez (2000) sum up their theory of mathematics as follows:

> *Where does mathematics come from? It comes from us! We create it, but it is not arbitrary—not a mere historically contingent social construction. What makes mathematics nonarbitrary is that it uses the basic conceptual mechanisms of the embodied human mind as it has evolved in the real world. Mathematics is a product of the neural capacities of our brains, the nature of our bodies, our evolution, our environment, and our long social and cultural history.* (p. 9)

The theory of embodied mathematics has been widely criticized. Many arguments critical of embodied mathematics fail insofar as they fundamentally misunderstand the position—by neglecting the central role of embodiment or by equivocally reverting to a traditional linguistic understanding of metaphor (for example Lakoff & Núñez, 2001; Postnikoff, 2008). Other criticisms of the theory are well-placed. The assumptions of the infant attention fixation experiments substantiating the subitization results at the base of embodied mathematics have been called into question (Bangu, 2016), and a considerable number of new findings about innate numerical capacities have been published over the last 20 years; the account given in *Where Mathematics Comes From* desperately needs an update to reflect the state of the art. Some critics have argued that a more dynamic and complex understanding of mathematical metaphor is needed to account for observed differences between individuals and among different kinds of mathematical practices (Presmeg, 2002; Schiralli & Sinclair, 2003). An important criticism of conceptual metaphor theory in general is that it fails to give an adequately detailed discussion of how symbols and language are related to concepts and conceptual metaphor, and that it unfairly deemphasizes the linguistic aspects of metaphor (Müller, 2008). It seems to me that none of these criticisms deal a fatal blow to embodied mathematics but only indicate directions for crucial further development of the theory. I believe that it is worth investing time and effort into improving embodied mathematics to overcome these criticisms, in order to help preserve what I claim is its primary strength: its ability to coherently integrate other theories of mathematics.

COHERENT INTEGRATION

Embodied mathematics is only one component of conceptual metaphor theory, a general philosophy of concepts and cognition developed by George Lakoff and several collaborators—notably Mark Johnson, Mark Turner, and Rafael Núñez—over the last 40 years. Because conceptual metaphor theory is a general account of concepts, it functions not only as a theory of mathematics but also as a theory of philosophies of mathematics. In this latter role, it is refreshingly inclusionary since conceptual metaphor provides a mechanism whereby the strongest aspects of traditionally inconsistent philosophies may be coherently integrated. Briefly, this is because every metaphor must involve *disanalogy*: some elements of the source domain remain unmapped to the target domain. This means that a devotee of embodied mathematics may employ a metaphorical mapping to partially conceptualize mathematics in terms of some other established philosophy without committing to its problematic assumptions.[3]

The following passage from Lakoff & Núñez (2000) may cause many readers to believe that the integration project is a non-starter: "Mind-based mathematics, as we describe it in this book, is not consistent with any of the existing philosophies of mathematics: Platonism, intuitionism, and formalism. Nor is it consistent with recent postmodernist accounts of mathematics as a

[3] While this approach is not feasible for other theories due to their strict antirelativist commitments, conceptual metaphor theory rejects agent-independent objectivity in favour of intersubjectivity and constrained relativism/perspectivism. See Lakoff and Johnson (1999), chapter 7 for more details.

purely social construction" (p. 341). However, note that this quote specifically refers to the consistency of philosophies, not the coherence. Other passages in *Where Mathematics Comes From* establish that the leading traditional theories have been metaphorically amalgamated with embodied mathematics from its origins. Embodied mathematics coheres with intuitionism and social constructivism insofar as it is also a constructivist position: it holds mathematics to be a human construct shaped by biological and cultural factors. However, Lakoff and Núñez reject both the intuitionist's strict insistence on constructive methods as well as the arbitrariness of radical social constructivism. The platonist metaphor, NUMBERS ARE THINGS IN THE WORLD, is induced by the four basic grounding metaphors of arithmetic. Lakoff and Núñez (2000) admit that this metaphor is an integral part of our mathematical practices, and that it is both natural and acceptable for mathematicians to speak about mathematical objects as though they exist objectively and independent of human agents. At the level of the philosophy of mathematics, however, they explicitly disallow the use of the platonist metaphor in making metaphysical inferences. Formalism is likewise embedded within embodied mathematics as the FORMAL REDUCTION METAPHOR, a collection of linking metaphors connecting the many disparate branches of mathematics through a set-theoretic hub. Seeing mathematical structures in set-theoretic terms can be amazingly productive for mathematicians. Philosophically, however, Lakoff and Núñez (2000) contend that an eliminative reduction of the diverse spectrum of mathematical practices to set theory would leave us conceptually impoverished. The success of these metaphorical integrations leads one to wonder whether other philosophies of mathematics might also be fruitfully added into the mix.

While there are many other philosophies of mathematics that could be mentioned here, I will restrict attention to three that I believe could significantly enhance embodied mathematics through metaphorical integration. First, Imre Lakatos (1976) presents a quasi-empirical theory of mathematics that focuses on the complicated and tumultuous historical negotiations whereby mathematical concepts arise, rather than the apparently timeless, refined definitions and proofs that are the end result of this history. Lakoff and Núñez (2000) acknowledge the historical dimension of mathematics, but focus most of their attention on the synchronic aspects of mathematical development. A dynamic synthesis of these synchronic and diachronic approaches could substantially enhance conceptual metaphor's account of mathematical development. Second, the *mathematical figuralism* of Stephen Yablo (2001) builds upon Kendall Walton's (1990) *theory of make-believe*, providing an account of mathematical language: mathematical statements are true insofar as they refer to an underlying real state of affairs indirectly by way of a game of make-believe (Yablo, 2001, p. 183). Given that figuralism and embodied mathematics are both mildly relativist, practice-oriented, metaphor-dependent accounts of mathematics, it seems obvious to explore the possibility of hybridizing the two theories; however, to my knowledge, this idea has received little to no attention to date. Embodied mathematics and figuralism seem well poised to complement each other insofar as the former provides a 'bottom up' account of conceptual development based on conceptual metaphors, whereas the latter explains the interpretation of mathematical sentences and expressions from the 'top down'. Third, coherently integrating a *semiotic perspective* with conceptual mathematics may provide resources for a more thorough explanation of how mathematical signs relate to mathematical concepts, bringing greater balance to Lakoff and Núñez's theory. The semiotic theory of Brian Rotman (2000) is a strong candidate for integration given its explicit aim to be coherent with traditional theories of mathematics:

> *[T]o have persisted so long each must encapsulate, however partially, an important facet of what is felt to be intrinsic to mathematical activity. Certainly, in some undeniable but obscure way, mathematics seems at the same time to be a meaningless game, a subjective construction, and a source of objective truth. The difficulty is to extract these part truths: the three accounts seem locked in an impasse which cannot be escaped from within the common terms that have allowed them to impinge on each other.* (p. 12)

Of course, speculatively noting coherences and synergies between philosophies of mathematics is nothing compared to the task of working out whether integration is actually possible and the details of how best to accomplish it. This is only the first step on a long and difficult path.

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DEVELOPING PRESERVICE TEACHERS' PROFESSIONAL NOTICING OF STUDENTS' LEARNING

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INTRODUCTION

Hiebert, Morris, Berk, and Jansen (2007) proposed a framework for teacher education that underscores the importance of preparing preservice teachers to learn from their teaching. Learning from teaching, or a specialized type of teacher noticing (Jacobs, Lamb, & Philipp, 2010), involves attending to students' learning and reflecting on how their teaching impacted what their students did or did not learn. Hiebert et al.'s *Learning from Teaching* model is comprised of four skills: (a) unpacking the learning goals of a lesson (Skill 1); (b) collecting evidence of student learning (Skill 2); (c) making conjectures about the cause-effect relationship between teaching and student learning (Skill 3); and (d) using the analysis to propose improvements and alternatives in teaching (Skill 4).

The research on developing these skills is thinly developed, and thus little is known regarding its role in teacher education programs (Morris, 2006; Santagata, Zannoni, & Stigler, 2007; Spitzer, Phelps, Beyers, Johnson, & Sieminski, 2011). What little research does exist has focused on the development of a select few of the four skills. Spitzer et al. (2011), for instance, did not include the development of Skills 1, 3 and 4 in their instruction to the preservice teachers because they were in the beginning of their program and the authors assumed that they lacked the pedagogical and subject matter knowledge necessary to apply them. Although there is validity to this argument (e.g., Morris, 2006), the focus on Skill 2 alone would only partially equip preservice teachers to learn from their teaching (Hiebert et al., 2007).

Further, the focus on Skill 2 alone assumes that the development of all four skills does not follow any particular sequence (Spitzer et al., 2011). This assumption is questionable, however, because according to Hiebert et al. (2007), each skill is theoretically linked to teaching activities that are deployed in a specific order: prior to, during, and following a lesson. That is, Skill 1 is linked with lesson planning, Skills 2 and 3 are used during the lesson (i.e., implementation), and Skill 4 is used to reflect on the lesson. Because in practice Skill 1 is applied before Skills 2, 3, and 4, it may benefit preservice teachers to receive professional development that follows this sequence to approximate teaching practice (Grossman et al., 2009).

In line with this reasoning, the present study was designed to explore the role of Skill 1 and to investigate whether its development would impact the other three skills in the model. Thus, the first goal of the current research was to understand the effects of explicit instruction on specifying learning goals (Skill 1) on preservice teachers' abilities to analyze what students learned (Skill 2), isolate the effects of teaching on students' learning (Skill 3), and revise a lesson accordingly (Skill 4). Furthermore, because the research has shown that teacher noticing skills are learned and do not develop naturally (Jacobs et al., 2010; Star & Strickland, 2008), the second objective was to examine the nature of Skill 1 acquisition as a function of the presence or absence of direct instruction on that skill.

DESIGN AND METHOD

The instruction took place in the context of an elementary mathematics methods course in a large urban university in Canada. A two-group pretest/post-test experimental design was used in the study. The study involved six phases: (a) an assessment of previously developed mathematical content knowledge for teaching and demographic information; (b) practice using an observation framework and review of key concepts related to teaching a lesson on the equal sign, (c) a pre-assessment of the skills addressed in both conditions (i.e., Skills 2, 3, and 4); (d) the experimental intervention; (e) a post-assessment of all four skills in Hiebert et al.'s (2007) framework; and (f) the administration of a post-interview to a subsample of the participants to explore one of the skills (i.e., Skill 1) in Hiebert et al.'s framework in more depth.

Both conditions received 6 hours of classroom instruction on Skills 2, 3, and 4, but only the Learning Goals condition received instruction on how to specify learning goals (2 hours and 30 minutes on Skill 1). The instruction, or Analysis of Learning sessions, involved a combination of three main activities: (a) whole class video analysis, (b) skill-based instruction delivered by the instructor, and (c) participants' framework development that took place in small groups. The video analysis component had two purposes: (a) to reflect on the meaning of a skill using an observation framework (occurred before the skill-based instruction), and (b) to practice the skills learned using an observation framework (following the skill-based instruction). The skill-based instruction included (a) instruction on one or more of Hiebert et al.'s (2007) skills, or (b) a review of skills previously addressed. The framework development activity involved working in small groups to modify the framework they had used while observing the video. Topics related to the development of children's algebraic reasoning were used during instruction (justifying conjectures, relational thinking, and eliciting conjectures) and during the pre- and post-measures (the meaning of the equal sign). Before the study began, both groups received a 30-minute review lesson in their methods class on children's thinking about the equal sign.

The level of preservice teachers' (N = 29) ability to engage in Skills 2, 3, and 4 were assessed prior to and following the instruction. I conducted individual interviews with a sub-sample of the preservice teachers from both conditions (n = 8) 9 to 13 weeks after the administration of the post-test to examine the nature of Skill 1 for preservice teachers who did and who did not receive explicit instruction on this skill. During the interview, the preservice teacher was asked to describe the learning goals and tasks she would include in her own lesson on the meaning of the equal sign. Next, the interviewer showed the preservice teacher a video of a fourth-grade lesson on the equal sign and asked her to compare it to the lesson she proposed with respect to learning goals and tasks. Finally, the interviewer asked the preservice teacher to explain her reasoning about the learning goals of the lesson observed in the video.

ANALYSIS AND SCORING

On the pre- and post-Analysis of Learning assessment, I included two types of questions, closed test items (i.e., multiple choice and true/false items) and open-ended items. The test items were assigned 0 points for an incorrect answer and 1 point for a correct answer. The number of correct responses on the test items were summed to calculate three different scores: (a) a Skill 1 test score (score ranged from 0 to 11; post-assessment only); (b) a Skill 2 Evidence score for items focused on student evidence related to the learning goals and revealing of student thinking or understanding (score ranged from 0 to 12); and (c) a Skill 2 Not Evidence score for items focused on student evidence not revealing of student thinking or understanding (score ranged from 0 to 7). For the Skill 2 Not Evidence score, lower scores indicated that less attention was paid to student behaviors that were not revealing of their thinking. I designed a rubric to code and score responses to the open-ended questions assessing Skills 1 (post-assessment only), 3, and 4. My rubric used the criteria outlined in Hiebert et al. (2007) and Santagata et al.'s (2007) method for scoring response quality.

All interviews were transcribed prior to analysis. I did not develop a coding rubric prior to coding the data because my goal was to provide a description of the participants' reflections and observations on teaching and learning, not compare them to a normative model from the literature. First, I generated a list of codes from all the interviews. Following that, I grouped these codes into the following superordinate categories: (a) types of learning goals in the planning context, (b) types of tasks in the planning context, (c) types of learning goals in the observation context (observation of the video), (d) quality of language used across both contexts, and (e) ways to reason about the learning goals.

Once the data set was double-coded, I prepared the data for analysis in two ways. First, I calculated the frequency of each quality-of-language code. Second, for each participant, I created a profile that consisted of a summary of her (a) lesson goals and tasks, (b) observations of learning goals and tasks included in the video lesson, and (c) reasoning about the learning goals in the video. My record of the participant's lesson on the equal sign indicated the unique collection of learning goal codes she identified in her lesson, the tasks she proposed to use, and whether her learning goals were or were not linked with a task to be used during the lesson. In particular, when she identified a learning goal in her lesson and did not link it with a task I recorded that she *identified* the learning goal. When she identified a learning goal in her lesson and linked it with a task, I recorded that she *specified* the learning goal. The same procedure was used to summarize each participant's observation of the learning goals in the lesson in the video. Finally, the participant's profile consisted of her unique collection of reasoning codes to describe what information from the video she used to discern the learning goals in the lesson.

Because my objective was to analyze group differences, I merged the participant profiles and created a profile for each group. The group profile was made up of all the learning goal, task, and reasoning codes for each participant in the group. For each learning goal, I indicated the number of participants in the group that linked the learning goal with a task (i.e., the number of participants that specified each learning goal included in the profile) and the number of participants that did not link the learning goal with a task (i.e., the number of participants in the group that identified each learning goal included in the profile). The same procedure was used to summarize the group's observation of the learning goals in the lesson from the video. Further, the group profile consisted of a collection of task codes unique to the participants in the group. Also, the group profile consisted of the unique collection of reasoning codes to describe what information participants in the group used to discern the learning goals addressed in the lesson.

RESULTS

The current study was designed to address three research questions. The first research question examined the effects of explicit instruction on specifying learning goals (Skill 1) on preservice teachers' abilities to collect evidence on what students learned (Skill 2), isolate the effects of teaching on students' learning (Skill 3), and revise a lesson to improve student learning (Skill 4). To address this question, I used the data from the pre- and post-assessment and examined group differences on Skills 2, 3, and 4 prior to and following instruction. Four 2 x 2 ANOVAs using group (Students Learning and Learning Goals) as the between-group factor and time as the within-group factor revealed significant main effects of time for Skill 2, $F(1,27) = 5.47$, $p = .027$; Skill 3 $F(1, 27) = 33.35$, $p < .001$; and Skill 4, $F(1, 27) = 8.31$, $p = .008$. No significant interactions were found.

The second research question focused on the effect of instruction on Skill 1 acquisition. It was hypothesized that, compared to the Students Learning group, the Learning Goals group would demonstrate a significantly higher score on Skill 1. I conducted an independent *t*-test using group (Students Learning and Learning Goals) as the between-group factor and Skill 1 Goals score as the dependent variable (range was from 2 to 21). On average, the Learning Goals group received a higher Skill 1 Goals score ($M = 13.47$, $SD = 2.61$) compared to the Students Learning group ($M = 12.14$, $SD = 2.68$), but this difference was not significant, $t(27) = 1.35$, $p = .19$.

The third question concerned the nature of Skill 1 with direct instruction on that skill and how it compares to Skill 1 without instruction. To address this question, data from interviews conducted with a sub-sample of the preservice teachers from both conditions ($N = 8$) were used to analyze the qualitative differences between both groups in terms of: (a) specifying and identifying learning goals in two contexts (*planning* a lesson and *observing* a lesson), (b) the types of tasks proposed when planning a lesson on the equal sign, (c) the reasoning used to identify the learning goal(s) when observing teaching, and (d) the frequency of high-quality language used when discussing learning goals and tasks.

The interview data revealed qualitative differences between both groups that were not captured by the quantitative analyses. First, compared to the planning context, the Students Learning group specified more learning goals in their discussions of learning goals observed in the video lesson. The greater number of specified learning goals in the observing context shows that the Students Learning group demonstrated greater Skill 1 abilities when learning goals were visible (i.e., observable in the video). Preservice teachers in the Learning Goals group demonstrated similar Skill 1 abilities in both contexts. Taken together, the results of my analysis across both contexts suggest that those who received instruction on Skill 1 used a similar strategy for specifying and identifying learning goals. On the contrary, the preservice teachers who did not receive Skill 1 instruction were skilled in specifying and identifying learning goals in a context where learning goals were more ‘visible’ (i.e., observed in the video).

The results suggested similar patterns for both groups for proposing tasks. Specifically, only a small number of preservice teachers in each group included tasks in their lessons that were not included in the video lesson. The majority of participants in both groups integrated the tasks from the video lesson.

With respect to the language used when proposing and identifying learning goals and tasks, the Learning Goals group provided more accurate and complete descriptions of the meaning of the equal sign, and more often provided complete and accurate explanations of children’s misconceptions. The Students Learning group, on the other hand, used a higher frequency of technical terms related to mathematical principles and the equal sign. Finally, the same number of participants in both groups focused on teacher behaviors in the video (e.g., prompts and questions, responses to students, representations and descriptions of concepts, and equations used) to reason about learning goals, but more participants in the Students Learning group attended to evidence of student learning, such as the statements they made during the lesson.

DISCUSSION

Contrary to my predictions, the results showed that learning how to identify and specify learning goals (Skill 1) did not support the development of Skills 2, 3, and 4. Thus, in line with

Spitzer et al. (2011), the development of learning from teaching may not be contingent on Skill 1 instruction. Although Hiebert et al. (2007) proposed that Skill 1 is tied to teaching activities that precede Skills 2, 3, and 4, the results from this study suggest that the order in which to introduce the skills during teacher preparation may not follow the order in which they play out in the classroom. My observations on the nature of Skill 1 after instruction may serve to explain why preservice teachers in both groups demonstrated similar Skill 1 abilities on the post-assessment.

The pre-assessment of Skill 2 revealed two things about the preservice teachers' ability to collect evidence of student learning. First, prior to instruction, the preservice teachers in both groups attended to student behaviors that do not support the analysis of student learning. Second, the preservice teachers were not skilled in collecting evidence revealing of student learning. Together, the pretest results show a greater tendency to attribute final answers as evidence of learning than student justifications, explanations, and key words (e.g., "it's backwards"). This finding is consistent with previous research demonstrating preservice teachers' difficulty understanding what constitutes evidence of student learning (e.g., Yeh & Santagata, 2015).

Following the instruction, the preservice teachers were more focused on student behaviors that support the analysis of student learning (i.e., reveal information about students' learning). At the same time, there was no significant change in their attention to student responses that are less informative of student learning. The preservice teachers' improvement in Skill 2 following instruction is consistent with previous research examining the use of video analysis in teacher education (Santagata & Angelici, 2010; Santagata et al., 2007; Star & Strickland, 2008). The design of my study cannot confirm the link between skill development and the analysis of video cases but, it is nevertheless possible that these activities improved the preservice teachers' observation (Star & Strickland, 2008) and analytic skills (Santagata & Angelici, 2010; Santagata et al., 2007), which in turn helped focus preservice teachers' attention to behaviors revealing of students' learning.

Constructing hypotheses, the third skill, to link teaching with student learning involves attending to behaviors from both teachers and students, *and* making meaningful connections between them. Moreover, expertise in this skill involves elaborating on the cause effect statement using observations from the lesson and pedagogical knowledge related to teaching and student learning. The pre-assessment results for Skill 3 demonstrated that the preservice teachers' initial hypothesis statements offered no analysis of how the teaching observed in the video impacted student learning. Following the instruction, the preservice teachers' ability to construct hypotheses statements in line with some of Hiebert et al.'s (2007) criteria improved. That is, the responses included statements that reflected their ability to attend to noteworthy teacher and student behaviors and an understanding of how to make connections between them.

Based on results from Yeh and Santagata (2015), the authors concluded preservice teachers' difficulties in constructing high-quality hypotheses was influenced by their difficulty reasoning about the instruction and recognizing what constitutes evidence of student learning. In line with this reasoning, it is plausible that the ability to collect evidence revealing of student learning (Skill 2) may serve as a prerequisite for developing skills in hypothesis construction (Skill 3). Moreover, because of the positive effects of guided video analysis on preservice teachers' ability to reason and evaluate teaching found in the literature (e.g., Santagata & Guarino, 2011), it is possible that this type of activity plays an important role in the development of this skill. With respect to my study then, I speculate that the preservice teachers' improvement in collecting evidence of student learning (Skill 2) in combination with the video analysis I included during instruction contributed to the preservice teachers' abilities to form hypotheses following instruction.

My study also assessed preservice teachers' skills in revising the lesson observed in the video (Skill 4). This particular skill is said to complete the learning from teaching cycle by connecting the analysis of student learning (Skill 2) and interpretations of the effect of teaching on student learning (Skill 3) with future lessons (i.e., specifying learning goals, or Skill 1; Yeh & Santagata, 2015). Prior to instruction, preservice teachers in the Learning Goals group were not skilled in proposing alternatives to improve the lesson observed in the video. Specifically, the results on the pre-assessment of Skill 4 indicated that this group failed to propose alternatives (a) grounded in evidence from the video, (b) that used pedagogical knowledge of teaching the equal sign, and (c) that elicited student thinking (Hiebert et al., 2007). Those in the Students Learning group were more skilled at proposing alternatives on the pre-assessment and most often used pedagogical knowledge of teaching the equal sign to justify how they would improve the lesson. On the post-assessment of Skill 4, the preservice teachers' ability to propose alternatives improved substantially. In particular, the majority of preservice teachers in both groups (i.e., more than half) proposed alternatives using observations from the video and pedagogical knowledge of teaching this topic.

The video analysis and framework development activity I included in the instruction may have played a role in the development of this particular skill. Each group designed an observation framework to help them record relevant information as they analyzed videos, guiding the preservice teachers to record important teaching behaviors (e.g., tasks, questions, key words), student responses, and form connections between them. It is possible that the preservice teachers in my study improved their ability to propose alternatives following the instruction because the activities during instruction supported their reflection of the teaching strategies observed in the video (Santagata & Angelici, 2010).

The results also indicated that the preservice teachers who did not receive direct instruction on specifying learning goals (Students Learning group) demonstrated Skill 1 abilities similar to those that did receive instruction on this skill (Learning Goals group). It is possible that the preservice teachers in the Students Learning group were able to identify the learning goals of the lesson without applying Skill 1. This assumption is based on evidence that preservice teachers without Skill 1 training can discern learning goals in situations Morris, Hiebert, and Stigler (2009) refer to as "supportive contexts". Morris et al. explained that in supportive contexts it is not necessary to examine elements of the lesson to identify learning goals, and thus the application of Skill 1 to identify learning goals can be avoided. That is, the key mathematics concepts of the lesson can be discerned from other sources of information, such as students' responses. Because of this, supportive contexts elicit a strategy whereby subject matter knowledge is used to interpret student responses; learning goals, then, are determined based on interpretations of students' responses, not the lesson itself. This strategy for identifying learning is problematic because it requires information (e.g., the students' responses to tasks) that is not available when a teacher plans a lesson. Moreover, Morris et al. showed that application of this strategy is limited to certain teaching situations. Situations referred to as "nonsupportive contexts" (Morris et al., 2009) are contexts where the mathematics concepts are not easily discerned from sources other than the elements of the lesson (e.g., tasks). Identifying learning goals in this context is more challenging for preservice teachers who have not received instruction on specifying learning goals (Morris et al., 2009).

Consistent with Morris et al. (2009), it would have been possible for the preservice teachers in the Students Learning group to identify learning goals of the video in absence of Skill 1 training had the video included 'supportive' information. The video I used during the assessment phases could be considered a supportive context for identifying learning goals because it is possible to discern the mathematics concepts of the lesson based on the students' responses. Although this cannot be confirmed, group differences may have emerged with a different video (nonsupportive context) or if the assessment involved planning a lesson on the equal sign (i.e., no video analysis).

The results from the interview support this notion. My predictions of the qualitative differences in the nature of this skill was borne out, but only in the planning context. That is, the Students Learning group's abilities to specify learning goals resembled those demonstrated by the Learning Goals group when they observed the lesson in the video. Further, the majority of preservice teachers in the Students Learning group used observations of student learning to discern the learning goals in the lesson. Contrary to this, one participant in the Learning Goals group relied on student responses when identifying the learning goals in the video. Based on this result, it may be assumed that those in the Students Learning group showed a greater tendency to focus on student behaviors when reflecting on the lesson. Although attention to student behaviors is necessary for collecting evidence of student learning (Skill 2), a strategy that emphasizes the role of student responses in discerning learning goals is limiting and is therefore less effective (Morris et al., 2009).

The results from my study indicate that three of the learning from teaching skills (Skills 2, 3, and 4) do not develop naturally and are learned. As such, the results have practical value for teacher educators. The results on the development of Skills 2, 3 and 4 lend support for Hiebert et al.'s (2007) contention that teacher training programs that incorporate instruction on these skills could enhance preservice teachers' analysis of teaching and the ability to reflect on the effectiveness of their own practice later on. Although the results indicated that Skills 2, 3, and 4 did not develop to the level of expert performance, the results nevertheless shed light on the development of learning from teaching skills during teacher training. In particular, developing more than one skill in the context of a methods course may be necessary for other skills, namely Skills 3 and 4. It is possible that the preservice teachers' developments in collecting evidence of student learning (Skill 2) contributed to their improvement in hypothesis construction (Skill 3) following instruction (Yeh & Santagata, 2015). I also speculate that the preservice teachers' skill in proposing alternatives (Skill 4) was impacted by their ability to reason and reflect deeply on teaching strategies (Santagata & Angelici, 2010), and that this reasoning was supported by the practice of constructing hypotheses (Skill 3). In addition, although the design of the study cannot confirm it, using video to develop these skills may only be effective when paired with practical activities that support the decomposition of teaching (Grossman et al., 2009).

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Ad hoc Session

THE CANADIAN JOURNAL OF SCIENCE, MATHEMATICS AND TECHNOLOGY EDUCATION (CJSMTE): MEET THE EDITORS

Egan J. Chernoff
University of Saskatchewan

The title of our *ad hoc* session, in my opinion, accurately sums up what took place during the session that Caroline Lajoie (French mathematics editor, CJSMTE) and I (English mathematics editor, CJSMTE) conducted at the (approximately) 40^{th} annual meeting of the Canadian Mathematics Education Study Group (CMESG): meet the editors of CJSMTE. In what follows, I will share a few key themes from our discussion.

To begin the session, I detailed particulars that I came to notice during my first year as editor. The one anomaly that definitely stood out for me during my tenure, thus far, was the lack of submissions from Canadian mathematics educators?! Parsing this notion a bit further, I explained how I understood that senior academics were inclined to publish their work in, for example, *Educational Studies in Mathematics*, *Mathematical Thinking and Learning* and the like; but, on that note, I shared with the attendees that the CJSMTE hits on all cylinders for senior academics but especially early-career academics. We considered that: CJSMTE is published both in print and online; articles are demarcated with a doi; the journal is housed with a reputable publisher; articles are peer reviewed; the journal has a strong editorial board; there are varied article types (e.g., book reviews); the turnaround time from submission to decision is one of the quickest amongst mathematics education journals. In other words, the CJSMTE should be on the radar of any and all young Canadian mathematics educators looking to publish their research. At the moment, though, it is not. My efforts to push contributions from Canadians led to a discussion on the history of the journal.

Thanks largely to Gila Hanna (founding editor of CJSMTE), Bernard Hodgson (early contributor to CJSMTE), and Natalie Sinclair (former French mathematics editor, CJSMTE), the connection between CJSMTE and CMESG, in a historical sense, was detailed for all in attendance. Did you know the CJSMTE was born at CMESG? Our discussion on the history of CMESG led, naturally, to a discussion on the state of the journal today. We debated not only the 'grade' that CJSMTE received in a recent edition of the newsletter of the International Group for the Psychology of Mathematics Education, but, also, whether or not that grade was influencing submissions to the journal. Based on the past and present day conversation regarding the journal, we discussed the potential of a more annual presence of CJSMTE at CMESG (modelled after *flm*) in some fashion.

I am pleased to say that our meet-the-editors CMESG session has already born fruit. There has been a marked increase in CJSMTE submissions from members of the Canadian mathematics education community. Yes, not all submissions have been published; however, due in large part to the high-quality work of our Canadian members, many submissions have resulted in publications. I am also pleased to report that our discussion of special issues of CJSMTE has resulted in a future special issue concerning the debate over the teaching and learning of mathematics that has raged on in our country for the last half decade. Stay tuned!

I wish, here, to thank all those individuals who attended our session (approximately a dozen) and definitely enriched our conversation. Thank you, all. Lastly, of course, follow the journal on Twitter, @CJSMTE.

Mathematics Gallery

Gallérie Mathématique

A PRELIMINARY ANALYSIS OF MATHEMATICS REQUIREMENTS IN ALBERTA UNIVERSITIES

Richelle Marynowski & Landry Forand
University of Lethbridge

The transition between secondary and post-secondary mathematics is a complex process for students. Some of the issues that students experience transitioning to post-secondary mathematics include "teaching and learning styles, type of mathematics taught, conceptual understanding, procedural knowledge required to advance through material, and changes in the amount of advanced mathematical thinking needed" (Hong et. al, 2009, p. 878). Additionally, Kajander and Lovric (2005) stated that "it seems that the transition in mathematics is by far the most serious and most problematic" (p. 149) of any discipline. With this in mind, understanding the relationship between secondary mathematics and university mathematics is key to ensure the smoothest possible transition for students.

In 2010, Alberta's high school mathematics program changed and was designed to provide students with the skills and competencies necessary to make a smooth transition from secondary mathematics studies to post-secondary programs and the world of work. The System Improvement Group report (2006) suggested two high school pathways that should satisfy stated admission requirements for the appropriate post-secondary programs as follows:

> 1. *Ensure that Pathway 1 satisfies the entrance requirements of the calculus-based (and similar) post-secondary programs with the fewest possible outcomes.*
>
> 2. *Ensure that Pathway 2 satisfies the entrance requirements of most of the remainder of the non-calculus-based programs.* (p. 57)

We analyzed data from four Alberta university calendars to determine if the intent of the courses was being recognized. Of the 50 programs at the four Alberta universities that have mathematics as an entrance requirement, we found:

- 22 programs require the Pathway 1 math course only (mathematics, science, or business),
- 26 programs require either the Pathway 1 or Pathway 2 course, (statistics or other mathematics content is deemed necessary), and
- 2 programs require the Pathway 2 course only (nursing and urban studies).

These findings are consistent with the intended purposes of the programs as stated by the System Improvement Group (2006).

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ASSESSING STUDENTS' STRATEGY UNDERSTANDING THROUGH A VIDEO EVALUATION TASK

Brittany Rappaport, Aryann Blondin, Nathalie Duponsel, & Helena P. Osana
Concordia University

Jérôme Proulx
Université du Québec à Montréal

This poster shared information about a study we conducted where we developed a new measure, *The Evaluation Task* (EvT), to evaluate students' conceptual understanding of place value.

As part of a larger project, second grade students (N = 60), from four classrooms, watched four videos of other children solving addition problems. We aimed to investigate how the students evaluated the children's work in the videos and we asked them to justify their evaluations. Half of the students were shown videos in which children used manipulatives to solve the problems, and the other half were shown videos in which written symbols (i.e., numbers) were used (see Figure 1). In addition, the children in half of the videos used a standard algorithm, but arrived at an incorrect answer. In the other half, the children used an idiosyncratic (i.e., invented) strategy and arrived at a correct solution. The students' evaluations of the videos provided insight into their procedural and conceptual knowledge of place value.

We found that when students evaluated the child's work appropriately, they used place value and regrouping terminology. For example, one student said, *"she should have written a 2 here, because 5 + 7 is 12, not 10*." When the students did not evaluate the work appropriately, they had more difficulties using mathematical terminology. For example, another student said, "*she put tens here and the ones here, so she's right*." Some students had difficulties identifying where the child in the video had made a mistake and could only compare their own strategy to the one in the video.

After conducting this study, we learned that students struggled more to interpret another child's solution as incorrect than when the solution was correct. In addition, the representation type (manipulatives or numerals) used in the video did not affect how the students interpreted the child's solution, but their mathematical difficulties may have influenced their ability to evaluate another child's work. As we continue our research in this domain, we know that the EvT needs some modifications. For example, correct and incorrect solutions should be included for both the standard algorithm and the idiosyncratic strategies.

The EvT has value in that it can still be considered a measure of students' mathematical understanding (e.g., conceptual, procedural) and can be used as a template for assessing other aspects of mathematical proficiency. One of the implications of this research is that it is valuable for teachers to encourage their students to interpret and evaluate the solutions of others, as this provides detailed information on the students' mathematical understanding, particularly when the solutions are incorrect.

Figure 1. Screenshots taken from the EvT. Left image shows manipulative representation, idiosyncratic strategy. Right image shows symbol representation, idiosyncratic strategy.

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USING ANALOGICAL REASONING AS A FRAMEWORK FOR INVESTIGATING TEACHING WITH MANIPULATIVES

Anna Tomaszewski, Aryann Blondin, Helena P. Osana, & Laura Iuhas
Concordia University

Manipulatives are concrete objects that can represent abstract mathematical concepts and are frequently used in elementary schools. We propose that instruction based on theories of analogical reasoning can support students' mathematics learning with manipulatives (Uttal, Liu, & DeLoache, 2006). The poster presented the Analogy-Based Evaluation Tool (ABET), which we developed to investigate the extent to which teachers use analogies when teaching mathematical concepts with manipulatives.

ANALOGY-BASED EVALUATION TOOL	
EVALUATION: VISUAL	
Source	How familiar is the source?
	Is the source visually presented?
	Is the source kept visible?
Target	Is the target visually presented?
	Is the target kept visible?
Visual Mappings	Does the teacher use gestures? What spatial cues does the teacher use? What images does the teacher refer to?
EVALUATION: VERBAL	
Verbal Explanations	
Source	What relevant features of the source does the teacher explain?
Target	What relevant features of the target does the teacher explain?
Referent	How does the teacher explain the mathematical concept central to the analogy?
Verbal Mappings	
Between Source and Target	Does the teacher describe structural similarities between the source and the target?
	Does the teacher describe surface similarities between the source and the target?
	Does the teacher describe differences between the source and the target?
Between Target and Referent	Does the teacher describe structural similarities between the target and the referent?
	Does the teacher describe differences between the target and the referent?

Figure 1. Analogy-Based Evaluation Tool (ABET).

We designed the ABET using theory and previous research in analogical reasoning. Since different manipulatives can represent the same mathematical idea (Uttal, Scudder, & DeLoache, 1997), they can be seen as analogs to each other (English, 2004). Structure mappings between analogs should be made explicit in mathematics instruction (Richland, Zur, & Holyoak, 2007). Because children tend to be distracted by surface similarities or differences (Begolli & Richland, 2016), they do not engage in structure mapping unless specifically cued to do so. Thus, the ABET captures the extent to which teachers engage in explicit supports for structure mapping. In addition, the ABET is sensitive to the different types of visual and verbal supports that are used explicitly by teachers to reduce the cognitive load of various representations during a lesson. Decreasing the demands of the analogy on the students' working memory facilitates structure mapping and learning (Richland & Simms, 2015). Observations of teachers' classroom practices with manipulatives using the ABET (for one study, see Blondin, Tomaszewski, & Osana, 2016) promise to contribute to the literature by describing Canadian teachers' practices and providing insight on future professional development needs.

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Appendices

Annexes

Appendix A / Annexe A

WORKING GROUPS AT EACH ANNUAL MEETING / GROUPES DE TRAVAIL DES RENCONTRES ANNUELLES

1977 *Queen's University, Kingston, Ontario*

- Teacher education programmes
- Undergraduate mathematics programmes and prospective teachers
- Research and mathematics education
- Learning and teaching mathematics

1978 *Queen's University, Kingston, Ontario*

- Mathematics courses for prospective elementary teachers
- Mathematization
- Research in mathematics education

1979 *Queen's University, Kingston, Ontario*

- Ratio and proportion: a study of a mathematical concept
- Minicalculators in the mathematics classroom
- Is there a mathematical method?
- Topics suitable for mathematics courses for elementary teachers

1980 *Université Laval, Québec, Québec*

- The teaching of calculus and analysis
- Applications of mathematics for high school students
- Geometry in the elementary and junior high school curriculum
- The diagnosis and remediation of common mathematical errors

1981 *University of Alberta, Edmonton, Alberta*

- Research and the classroom
- Computer education for teachers
- Issues in the teaching of calculus
- Revitalising mathematics in teacher education courses

1982 *Queen's University, Kingston, Ontario*

- The influence of computer science on undergraduate mathematics education
- Applications of research in mathematics education to teacher training programmes
- Problem solving in the curriculum

1983 *University of British Columbia, Vancouver, British Columbia*

- Developing statistical thinking
- Training in diagnosis and remediation of teachers
- Mathematics and language
- The influence of computer science on the mathematics curriculum

1984 *University of Waterloo, Waterloo, Ontario*

- Logo and the mathematics curriculum
- The impact of research and technology on school algebra
- Epistemology and mathematics
- Visual thinking in mathematics

1985 *Université Laval, Québec, Québec*

- Lessons from research about students' errors
- Logo activities for the high school
- Impact of symbolic manipulation software on the teaching of calculus

1986 *Memorial University of Newfoundland, St. John's, Newfoundland*

- The role of feelings in mathematics
- The problem of rigour in mathematics teaching
- Microcomputers in teacher education
- The role of microcomputers in developing statistical thinking

1987 *Queen's University, Kingston, Ontario*

- Methods courses for secondary teacher education
- The problem of formal reasoning in undergraduate programmes
- Small group work in the mathematics classroom

1988 *University of Manitoba, Winnipeg, Manitoba*

- Teacher education: what could it be?
- Natural learning and mathematics
- Using software for geometrical investigations
- A study of the remedial teaching of mathematics

1989 *Brock University, St. Catharines, Ontario*

- Using computers to investigate work with teachers
- Computers in the undergraduate mathematics curriculum
- Natural language and mathematical language
- Research strategies for pupils' conceptions in mathematics

1990 *Simon Fraser University, Vancouver, British Columbia*

- Reading and writing in the mathematics classroom
- The NCTM “Standards” and Canadian reality
- Explanatory models of children’s mathematics
- Chaos and fractal geometry for high school students

1991 *University of New Brunswick, Fredericton, New Brunswick*

- Fractal geometry in the curriculum
- Socio-cultural aspects of mathematics
- Technology and understanding mathematics
- Constructivism: implications for teacher education in mathematics

1992 *ICME–7, Université Laval, Québec, Québec*

1993 *York University, Toronto, Ontario*

- Research in undergraduate teaching and learning of mathematics
- New ideas in assessment
- Computers in the classroom: mathematical and social implications
- Gender and mathematics
- Training pre-service teachers for creating mathematical communities in the classroom

1994 *University of Regina, Regina, Saskatchewan*

- Theories of mathematics education
- Pre-service mathematics teachers as purposeful learners: issues of enculturation
- Popularizing mathematics

1995 *University of Western Ontario, London, Ontario*

- Autonomy and authority in the design and conduct of learning activity
- Expanding the conversation: trying to talk about what our theories don’t talk about
- Factors affecting the transition from high school to university mathematics
- Geometric proofs and knowledge without axioms

1996 *Mount Saint Vincent University, Halifax, Nova Scotia*

- Teacher education: challenges, opportunities and innovations
- Formation à l’enseignement des mathématiques au secondaire: nouvelles perspectives et défis
- What is dynamic algebra?
- The role of proof in post-secondary education

1997 *Lakehead University, Thunder Bay, Ontario*

- Awareness and expression of generality in teaching mathematics
- Communicating mathematics
- The crisis in school mathematics content

1998 *University of British Columbia, Vancouver, British Columbia*

- Assessing mathematical thinking
- From theory to observational data (and back again)
- Bringing Ethnomathematics into the classroom in a meaningful way
- Mathematical software for the undergraduate curriculum

1999 *Brock University, St. Catharines, Ontario*

- Information technology and mathematics education: What's out there and how can we use it?
- Applied mathematics in the secondary school curriculum
- Elementary mathematics
- Teaching practices and teacher education

2000 *Université du Québec à Montréal, Montréal, Québec*

- Des cours de mathématiques pour les futurs enseignants et enseignantes du primaire/Mathematics courses for prospective elementary teachers
- Crafting an algebraic mind: Intersections from history and the contemporary mathematics classroom
- Mathematics education et didactique des mathématiques : y a-t-il une raison pour vivre des vies séparées?/Mathematics education et didactique des mathématiques: Is there a reason for living separate lives?
- Teachers, technologies, and productive pedagogy

2001 *University of Alberta, Edmonton, Alberta*

- Considering how linear algebra is taught and learned
- Children's proving
- Inservice mathematics teacher education
- Where is the mathematics?

2002 *Queen's University, Kingston, Ontario*

- Mathematics and the arts
- Philosophy for children on mathematics
- The arithmetic/algebra interface: Implications for primary and secondary mathematics / Articulation arithmétique/algèbre: Implications pour l'enseignement des mathématiques au primaire et au secondaire
- Mathematics, the written and the drawn
- Des cours de mathématiques pour les futurs (et actuels) maîtres au secondaire / Types and characteristics desired of courses in mathematics programs for future (and in-service) teachers

2003 *Acadia University, Wolfville, Nova Scotia*

- L'histoire des mathématiques en tant que levier pédagogique au primaire et au secondaire / The history of mathematics as a pedagogic tool in Grades K–12
- Teacher research: An empowering practice?
- Images of undergraduate mathematics
- A mathematics curriculum manifesto

2004 *Université Laval, Québec, Québec*

- Learner generated examples as space for mathematical learning
- Transition to university mathematics
- Integrating applications and modeling in secondary and post secondary mathematics
- Elementary teacher education – Defining the crucial experiences
- A critical look at the language and practice of mathematics education technology

2005 *University of Ottawa, Ottawa, Ontario*

- Mathematics, education, society, and peace
- Learning mathematics in the early years (pre-K – 3)
- Discrete mathematics in secondary school curriculum
- Socio-cultural dimensions of mathematics learning

2006 *University of Calgary, Calgary, Alberta*

- Secondary mathematics teacher development
- Developing links between statistical and probabilistic thinking in school mathematics education
- Developing trust and respect when working with teachers of mathematics
- The body, the sense, and mathematics learning

2007 *University of New Brunswick, New Brunswick*

- Outreach in mathematics – Activities, engagement, & reflection
- Geometry, space, and technology: challenges for teachers and students
- The design and implementation of learning situations
- The multifaceted role of feedback in the teaching and learning of mathematics

2008 *Université de Sherbrooke, Sherbrooke, Québec*

- Mathematical reasoning of young children
- Mathematics-in-and-for-teaching (MifT): the case of algebra
- Mathematics and human alienation
- Communication and mathematical technology use throughout the post-secondary curriculum / Utilisation de technologies dans l'enseignement mathématique postsecondaire
- Cultures of generality and their associated pedagogies

2009 *York University, Toronto, Ontario*

- Mathematically gifted students / Les élèves doués et talentueux en mathématiques
- Mathematics and the life sciences
- Les méthodologies de recherches actuelles et émergentes en didactique des mathématiques / Contemporary and emergent research methodologies in mathematics education
- Reframing learning (mathematics) as collective action
- Étude des pratiques d'enseignement
- Mathematics as social (in)justice / Mathématiques citoyennes face à l'(in)justice sociale

2010 *Simon Fraser University, Burnaby, British Columbia*

- Teaching mathematics to special needs students: Who is at-risk?
- Attending to data analysis and visualizing data
- Recruitment, attrition, and retention in post-secondary mathematics
  Can we be thankful for mathematics? Mathematical thinking and aboriginal peoples
- Beauty in applied mathematics
- Noticing and engaging the mathematicians in our classrooms

2011 *Memorial University of Newfoundland, St. John's, Newfoundland*

- Mathematics teaching and climate change
- Meaningful procedural knowledge in mathematics learning
- Emergent methods for mathematics education research: Using data to develop theory / Méthodes émergentes pour les recherches en didactique des mathématiques: partir des données pour développer des théories
- Using simulation to develop students' mathematical competencies – Post secondary and teacher education
- Making art, doing mathematics / Créer de l'art; faire des maths
- Selecting tasks for future teachers in mathematics education

2012 *Université Laval, Québec City, Québec*

- Numeracy: Goals, affordances, and challenges
- Diversities in mathematics and their relation to equity
- Technology and mathematics teachers (K-16) / La technologie et l'enseignant mathématique (K-16)
- La preuve en mathématiques et en classe / Proof in mathematics and in schools
- The role of text/books in the mathematics classroom / Le rôle des manuels scolaires dans la classe de mathématiques
- Preparing teachers for the development of algebraic thinking at elementary and secondary levels / Préparer les enseignants au développement de la pensée algébrique au primaire et au secondaire

2013 *Brock University, St. Catharines, Ontario*

- MOOCs and online mathematics teaching and learning
- Exploring creativity: From the mathematics classroom to the mathematicians' mind / Explorer la créativité : de la classe de mathématiques à l'esprit des mathématiciens
- Mathematics of Planet Earth 2013: Education and communication / Mathématiques de la planète Terre 2013 : formation et communication (K-16)
- What does it mean to understand multiplicative ideas and processes? Designing strategies for teaching and learning
- Mathematics curriculum re-conceptualisation

2014 *University of Alberta, Edmonton, Alberta*

- Mathematical habits of mind / Modes de pensée mathématiques
- Formative assessment in mathematics: Developing understandings, sharing practice, and confronting dilemmas
- Texter mathematique / Texting mathematics
- Complex dynamical systems
- Role-playing and script-writing in mathematics education practice and research

2015 *Université de Moncton, Moncton, New Brunswick*

- Task design and problem posing
- Indigenous ways of knowing in mathematics
- Theoretical frameworks in mathematics education research / Les cadres théoriques dans la recherche en didactictique des mathématiques
- Early years teaching, learning and research: Tensions in adult-child interactions around mathematics
- Innovations in tertiary mathematics teaching, learning and research / Innovations au post-secondaire pour l'enseignement, l'apprentissage et la recherche

Appendix B / Annexe B

PLENARY LECTURES AT EACH ANNUAL MEETING / CONFÉRENCES PLÉNIÈRES DES RENCONTRES ANNUELLES

1977	A.J. COLEMAN	The objectives of mathematics education
	C. GAULIN	Innovations in teacher education programmes
	T.E. KIEREN	The state of research in mathematics education
1978	G.R. RISING	The mathematician's contribution to curriculum development
	A.I. WEINZWEIG	The mathematician's contribution to pedagogy
1979	J. AGASSI	The Lakatosian revolution
	J.A. EASLEY	Formal and informal research methods and the cultural status of school mathematics
1980	C. GATTEGNO	Reflections on forty years of thinking about the teaching of mathematics
	D. HAWKINS	Understanding understanding mathematics
1981	K. IVERSON	Mathematics and computers
	J. KILPATRICK	The reasonable effectiveness of research in mathematics education
1982	P.J. DAVIS	Towards a philosophy of computation
	G. VERGNAUD	Cognitive and developmental psychology and research in mathematics education
1983	S.I. BROWN	The nature of problem generation and the mathematics curriculum
	P.J. HILTON	The nature of mathematics today and implications for mathematics teaching

1984	A.J. BISHOP	The social construction of meaning: A significant development for mathematics education?
	L. HENKIN	Linguistic aspects of mathematics and mathematics instruction
1985	H. BAUERSFELD	Contributions to a fundamental theory of mathematics learning and teaching
	H.O. POLLAK	On the relation between the applications of mathematics and the teaching of mathematics
1986	R. FINNEY	Professional applications of undergraduate mathematics
	A.H. SCHOENFELD	Confessions of an accidental theorist
1987	P. NESHER	Formulating instructional theory: the role of students' misconceptions
	H.S. WILF	The calculator with a college education
1988	C. KEITEL	Mathematics education and technology
	L.A. STEEN	All one system
1989	N. BALACHEFF	Teaching mathematical proof: The relevance and complexity of a social approach
	D. SCHATTSNEIDER	Geometry is alive and well
1990	U. D'AMBROSIO	Values in mathematics education
	A. SIERPINSKA	On understanding mathematics
1991	J .J. KAPUT	Mathematics and technology: Multiple visions of multiple futures
	C. LABORDE	Approches théoriques et méthodologiques des recherches françaises en didactique des mathématiques
1992	ICME-7	
1993	G.G. JOSEPH	What is a square root? A study of geometrical representation in different mathematical traditions
	J CONFREY	Forging a revised theory of intellectual development: Piaget, Vygotsky and beyond
1994	A. SFARD	Understanding = Doing + Seeing ?
	K. DEVLIN	Mathematics for the twenty-first century
1995	M. ARTIGUE	The role of epistemological analysis in a didactic approach to the phenomenon of mathematics learning and teaching
	K. MILLETT	Teaching and making certain it counts
1996	C. HOYLES	Beyond the classroom: The curriculum as a key factor in students' approaches to proof
	D. HENDERSON	Alive mathematical reasoning

1997	R. BORASSI	What does it really mean to teach mathematics through inquiry?
	P. TAYLOR	The high school math curriculum
	T. KIEREN	Triple embodiment: Studies of mathematical understanding-in-interaction in my work and in the work of CMESG/GCEDM
1998	J. MASON	Structure of attention in teaching mathematics
	K. HEINRICH	Communicating mathematics or mathematics storytelling
1999	J. BORWEIN	The impact of technology on the doing of mathematics
	W. WHITELEY	The decline and rise of geometry in 20th century North America
	W. LANGFORD	Industrial mathematics for the 21st century
	J. ADLER	Learning to understand mathematics teacher development and change: Researching resource availability and use in the context of formalised INSET in South Africa
	B. BARTON	An archaeology of mathematical concepts: Sifting languages for mathematical meanings
2000	G. LABELLE	Manipulating combinatorial structures
	M. B. BUSSI	The theoretical dimension of mathematics: A challenge for didacticians
2001	O. SKOVSMOSE	Mathematics in action: A challenge for social theorising
	C. ROUSSEAU	Mathematics, a living discipline within science and technology
2002	D. BALL & H. BASS	Toward a practice-based theory of mathematical knowledge for teaching
	J. BORWEIN	The experimental mathematician: The pleasure of discovery and the role of proof
2003	T. ARCHIBALD	Using history of mathematics in the classroom: Prospects and problems
	A. SIERPINSKA	Research in mathematics education through a keyhole
2004	C. MARGOLINAS	La situation du professeur et les connaissances en jeu au cours de l'activité mathématique en classe
	N. BOULEAU	La personnalité d'Evariste Galois: le contexte psychologique d'un goût prononcé pour les mathématique abstraites
2005	S. LERMAN	Learning as developing identity in the mathematics classroom
	J. TAYLOR	Soap bubbles and crystals
2006	B. JAWORSKI	Developmental research in mathematics teaching and learning: Developing learning communities based on inquiry and design
	E. DOOLITTLE	Mathematics as medicine

2007	R. NÚÑEZ	Understanding abstraction in mathematics education: Meaning, language, gesture, and the human brain
	T. C. STEVENS	Mathematics departments, new faculty, and the future of collegiate mathematics
2008	A. DJEBBAR	Art, culture et mathématiques en pays d'Islam (IX^{e}-XV^{e} s.)
	A. WATSON	Adolescent learning and secondary mathematics
2009	M. BORBA	Humans-with-media and the production of mathematical knowledge in online environments
	G. de VRIES	Mathematical biology: A case study in interdisciplinarity
2010	W. BYERS	Ambiguity and mathematical thinking
	M. CIVIL	Learning from and with parents: Resources for equity in mathematics education
	B. HODGSON	Collaboration et échanges internationaux en éduction mathématique dans le cadre de la CIEM : regards selon une perspective canadienne / ICMI as a space for international collaboration and exchange in mathematics education: Some views from a Canadian perspective
	S. DAWSON	My journey across, through, over, and around academia: "...a path laid while walking..."
2011	C. K. PALMER	Pattern composition: Beyond the basics
	P. TSAMIR & D. TIROSH	The Pair-Dialogue approach in mathematics teacher education
2012	P. GERDES	Old and new mathematical ideas from Africa: Challenges for reflection
	M. WALSHAW	Towards an understanding of ethical practical action in mathematics education: Insights from contemporary inquiries
	W. HIGGINSON	Cooda, wooda, didda, shooda: Time series reflections on CMESG/GCEDM
2013	R. LEIKIN	On the relationships between mathematical creativity, excellence and giftedness
	B. RALPH	Are we teaching Roman numerals in a digital age?
	E. MULLER	Through a CMESG looking glass
2014	D. HEWITT	The economic use of time and effort in the teaching and learning of mathematics
	N. NIGAM	Mathematics in industry, mathematics in the classroom: Analogy and metaphor
2015	É. RODITI	Diversité, variabilité et convergence des pratiques enseignantes / Diversity, variability, and commonalities among teaching practices
	D. HUGHES HALLET	Connections: Mathematical, interdisciplinary, electronic, and personal

Appendix C / Annexe C

PROCEEDINGS OF ANNUAL MEETINGS / ACTES DES RENCONTRES ANNUELLES

Past proceedings of CMESG/GCEDM annual meetings have been deposited in the ERIC documentation system with call numbers as follows:

Proceedings of the 1980 Annual Meeting	ED 204120
Proceedings of the 1981 Annual Meeting	ED 234988
Proceedings of the 1982 Annual Meeting	ED 234989
Proceedings of the 1983 Annual Meeting	ED 243653
Proceedings of the 1984 Annual Meeting	ED 257640
Proceedings of the 1985 Annual Meeting	ED 277573
Proceedings of the 1986 Annual Meeting	ED 297966
Proceedings of the 1987 Annual Meeting	ED 295842
Proceedings of the 1988 Annual Meeting	ED 306259
Proceedings of the 1989 Annual Meeting	ED 319606
Proceedings of the 1990 Annual Meeting	ED 344746
Proceedings of the 1991 Annual Meeting	ED 350161
Proceedings of the 1993 Annual Meeting	ED 407243
Proceedings of the 1994 Annual Meeting	ED 407242

NOTE

There was no Annual Meeting in 1992 because Canada hosted the Seventh International Conference on Mathematical Education that year.